Differential Equations Notes

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Contents

1 Bernoulli Differential Equation

$$6 \cdot (y)' - 2 \cdot y = x \cdot y^4$$

Divide each side by 6 to leave $(y)'$ alone

$$(y)' - \frac{y}{3} = \frac{x}{6} \cdot y^4$$

So it became a Bernoulli format: $(y)' + P(x) \cdot y = Q(x) \cdot y^n$

We need to make this transformation to solve this: $v = y^{(1-n)}$

$$v = y^{(-3)}$$

Let's take the derivative of v

$$(v)' = -3 \cdot y^{(-4)} \cdot (y)'$$

$(y)' - \dfrac{y}{3} = \dfrac{x}{6} \cdot y^4$ **where we multiply each side by** $-3 \cdot y^{(-4)}$ **to get** $(v)'$

$-3 \cdot y^{-4} \cdot (y)' + y^{-3} = -\dfrac{x}{2}$ **this is a linear differential equation**

$(y)' + P(x) \cdot y = Q(x)$ **looks like this**

So we can solve with the integration factor: $\mu = e^{\int 1 \, dx} = e^x$

$$(e^x \cdot v)' = -\frac{x}{2} \cdot e^x$$

Let's take the integral of both sides: $\displaystyle\int \left((e^x \cdot v)'\right) dx = \int \left(-\frac{x}{2} \cdot e^x\right) dx$

$$e^x \cdot v = -\frac{1}{2}\left(e^x \cdot x - x\right) + x$$

Here v **is left alone, then** v **is replaced by** y in the $v - y$ **equation and the solution is obtained.**

2 Homogeneous Differential Equations

$$x \cdot y \cdot (y)' + 4 \cdot x^2 + y^2 = 0$$

Divide each side by $x \cdot y$

$$\frac{dy}{dx} = -4 \cdot \frac{x}{y} - \frac{y}{x}$$

$u = \frac{y}{x}$ **transformation:**

$$y = u \cdot x$$

$$dy = u \cdot dx + x \cdot du$$

$$\frac{dy}{dx} = u + \frac{x \cdot du}{dy}$$

If we substitute them in the equation: $u + \frac{x \cdot du}{dx} = -\frac{4}{u} - u$

$$\frac{x \cdot \ du}{dx} = -\frac{4}{u} - 2u$$

$$\frac{x \cdot \ du}{dx} = -\frac{4 + 2u^2}{u}$$

$$\frac{dx}{-x} = \frac{u \cdot du}{4 + 2u^2}$$

$$\frac{1}{-x} \cdot dx = \frac{u}{4 + 2u^2} \cdot du$$

Let's take the integral of both sides: $\int \left(\frac{1}{-x} \right) dx = \int \left(\frac{u}{4 + 2u^2} \right) du$

$$-\ln|x| + \ln(c) = \frac{\ln\left|4 + 2u^2\right|}{4}$$

$$\ln\left(\frac{c}{x}\right) = \ln\left(4 + 2u^2\right)^{\frac{1}{4}}$$

$$\frac{c}{x^4} = 4 + 2u^2$$

$$\frac{c}{x^4} - 2 = u^2 = \left(\frac{y}{x}\right)^2$$

Where if y is left alone, the general solution is obtained.

3 Differential Equations That Can Be Homogenized

1. equation: $(7x + 3y - 1)\,dx + (3x - y + 1)\,dy = 0$

$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = (-7 \cdot -1) - (-3 \cdot 3) \neq 0$ **and therefore linearly independent.**

$$\frac{dy}{dx} = \frac{-7x - 3y + 1}{3x - y + 1}$$

$$\textbf{Let } x = \alpha + h, \quad y = \beta + k$$

$$\frac{d\beta}{d\alpha} = \frac{-7\alpha - 7h - 3\beta - 3k + 1}{3\alpha + 3h - \beta - k + 1}$$

$$-7h - 3k + 1 = 0$$

$$3h - k + 1 = 0$$

$$h = -\frac{1}{8}, \quad k = \frac{5}{8}$$

$$\frac{d\beta}{d\alpha} = \frac{-7\alpha - 3\beta}{3\alpha - \beta} = \frac{\alpha \cdot \left(-7 - \frac{3\beta}{\alpha}\right)}{\alpha \cdot \left(3 - \frac{\beta}{\alpha}\right)}$$

$$u = \frac{\beta}{\alpha} \textbf{ transformation:}$$

$$\beta = u \cdot \alpha$$

$$d\beta = u \cdot d\alpha + \alpha \cdot du$$

$$\frac{d\beta}{d\alpha} = u + \frac{\alpha \cdot du}{d\alpha}$$

Let's take the integral of both sides: $u + \dfrac{\alpha \cdot du}{d\alpha} = \dfrac{-7-3u}{3-u}$

$$\frac{\alpha \cdot du}{d\alpha} = \frac{-7-3u-3u+u^2}{3-u}$$

$$\frac{d\alpha}{\alpha} = \frac{3-u}{-7-6u+u^2} \cdot du$$

Let's take the integral of both sides: $\displaystyle\int \frac{1}{\alpha}\, d\alpha = \int \frac{3-u}{-7-6u+u^2}\, du$

The integral on the right can be solved using the transformation $m = -7-6u+u^2$ **.**

$$\ln|\alpha| + c = -\frac{1}{2}\ln|u^2-6u-7|$$

The function y can be found by substituting the necessary variables.

2. equation: $(x+5y+1)\ dx + (2x+10y+7)\ dy = 0$

$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = (1 \cdot 10) - (5 \cdot 2) = 0$ **and therefore linearly dependent.**

Let $x+5y=t$**.**

$$dx + 5dy = dt$$

$$dy = \frac{dt-dx}{5}$$

Let's put them in their places in the equation.

$$(t+1)\,dx + (2t+7) \cdot \left(\frac{dt-dx}{5}\right) = 0$$

Multiply each term of this equation by 5**.**

$$(5t+5)\,dx + (2t+7)\,(dt-dx) = 0$$

Let's organize the equation.

$$(5t+5)\,dx + (2t)\,dt - (2t)\,dx + 7dt - 7dx = 0$$

$$dx\,(5t+5-2t-7) + dt\,(2t+7) = 0$$

$$dx\,(3t-2) + dt\,(2t+7) = 0$$

$$dx = \frac{2t+7}{2-3t} \cdot dt$$

Let's integrate both sides of the equation.

$$\int dx = \int \frac{2t+7}{2-3t}\,dt$$

$$x + c = -\frac{2}{3} \cdot t - \frac{25}{3} \cdot \ln|2-3t|$$

When we substitute the necessary variables in the equation we find, we can reach the y function.

4 Bernoulli Differential Equation (Long Path)

$$(y)' + \frac{4}{x} \cdot y = x^3 y^2$$

This equation is a Bernoulli differential equation because it follows this format: $(y)' + P(x) \cdot y = Q(x) \cdot y^n$

We need to make this transformation to solve this: $v = y^{(1-n)}$

$$v = y^{(-1)}$$

$$(v)' = -y^{(-2)} \cdot (y)'$$

$$(y)' = -y^{(2)} \cdot (v)'$$

Let's replace $(y)'$ **in the question:** $-y^2 \cdot (v)' + \frac{4}{x} \cdot y = x^3 y^2$

Since $y = \frac{1}{v}$**, let's write the** y**'s in the equation in terms of** v

$$-\frac{(v)'}{v} + \frac{4}{x} = \frac{x^3}{v}$$

Multiply all terms by $-v$

$$(v)' - \frac{4}{x} \cdot v = -x^3$$

The final equation, $(y)' + P(x) \cdot y = Q(x)$ **is a linear differential equation since it fits this format.**

$$\mu = e^{\int -\frac{4}{x} dx} = e^{-4 \cdot ln|x|} = x^{-4}$$

$$\left(x^{-4} \cdot v\right)' = -x^3 \cdot x^{-4} = -x^{-1}$$

$$\int \left(x^{-4} \cdot v\right)' \, dx = \int -x^{-1} \, dx$$

$$x^{-4} \cdot v = -\ln|x| + c$$

$$v = -x^4 \cdot \ln|x| + c \cdot x^4$$

The function y **can be found by substituting the necessary variables.**

5 Exact Differential Equation

$$\left(2xy - 9x^2\right)\, dx + \left(2y + x^2 + 1\right)\, dy = 0$$

$$P(x, y) = 2xy - 9x^2, \quad Q(x, y) = 2y + x^2 + 1$$

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial x} = 2x$$

$$\int P(x, y)\, dx = \int (2xy - 9x^2)\, dx = x^2y - 3x^3 + R(y)$$

$$\int Q(x, y)\, dy = \int (2y + x^2 + 1)\, dy = y^2 + x^2y + y + R(x)$$

$$R(y) = y^2 + y, \; R(x) = -3x^3$$

Substituting the function R into any of the integral results yields the function y .

6 Clairaut Differential Equation

This is the general view of the Clairaut differential equation:

$$y = x \cdot (y)' + \psi\left((y)'\right)$$

$$y = x \cdot (y)' + \left((y)'\right)^2$$

Let $p = (y)'$ **.**

If we substitute p **in the equation:** $y = x \cdot p + p^2$

$$\frac{d}{dx} y = \frac{d}{dx}\left(x \cdot p + p^2\right)$$

$$(y)' = p + x \cdot (p)' + 2 \cdot p \cdot (p)'$$

If we substitute p **for** $(y)'$ **here,** $p = p + x \cdot (p)' + 2 \cdot p \cdot (p)'$

$$0 = x \cdot (p)' + 2 \cdot p \cdot (p)'$$

$$0 = (p)' \cdot (x + 2p)$$

$(p)' = 0$ **(general solution)**

$$\int (p)' \, dx = \int 0 \, dx$$

$$p = c$$

If we substitute the equation $p = c$ **in the equation** $y = x \cdot p + p^2$**:**

$$y = x \cdot c + c^2$$

$x + 2p = 0$ **(singular solution)**

$$p = -\frac{x}{2}$$

$$(y)' = -\frac{x}{2}$$

$$\int (y)' \, dx = \int \left(-\frac{x}{2}\right) dx$$

$$y = -\frac{x^2}{4} + c$$

Instead of integrating this solution, we can substitute p **into the equation** $y = x \cdot p + p^2$**.**

$$p = -\frac{x}{2},\ y = x \cdot \left(-\frac{x}{2}\right) + \left(-\frac{x}{2}\right)^2$$

7 Riccati Differential Equation

This is the general view of the Riccati differential equation:

$$(y)' = A(x) \cdot y^2 + B(x) \cdot y + C(x)$$

$$(y)' = y^2 - \frac{y}{x} - \frac{1}{x^2},\ y_1(x) = \frac{1}{x}$$

The transformation $y = v + y_1$ **is made.**

$$y = v + \frac{1}{x}$$

$$(y)' = (v)' - \frac{1}{x^2}$$

The obtained y and $(y)'$are written in their places in the question.

$$(v)' - \frac{1}{x^2} = \left(v + \frac{1}{x}\right)^2 - \frac{1}{x} \cdot \left(v + \frac{1}{x}\right) - \frac{1}{x^2}$$

Simplified version: $(v)' - \frac{1}{x} \cdot v = v^2$

This equation is a Bernoulli differential equation because it looks like this: $(y)' + P(x) \cdot y = Q(x) \cdot y^n$

Continuing from Bernoulli solution, the y function can be found.

8 Riccati Differential Equation (Short Path)

$$(y)' = y^2 - \frac{y}{x} - \frac{1}{x^2}, \; y_1(x) = \frac{1}{x}$$

The transformation $y = y_1 + \frac{1}{v}$ is made.

$$y = \frac{1}{x} + \frac{1}{v}$$

$$(y)' = -\frac{1}{x^2} - \frac{(v)'}{v^2}$$

The obtained y and $(y)'$are written in their places in the question.

$$-\frac{1}{x^2} - \frac{(v)'}{v^2} = \left(\frac{1}{x} + \frac{1}{v}\right)^2 - \frac{1}{x} \cdot \left(\frac{1}{x} + \frac{1}{v}\right) - \frac{1}{x^2}$$

Simplified version: $-\frac{(v)'}{v^2} = \frac{1}{x \cdot v} + \frac{1}{v^2}$

Multiply each side by $-v^2$**:** $(v)' = -\dfrac{v}{x} - 1$

$$(v)' + \frac{1}{x} \cdot v = -1$$

Since this equation is a linear differential equation, it can be solved by integration factor.

9 Solving Exact Differential Equation with Integration Factor

1. equation: $\left(x^2 + y^2\right) dx + (xy)\, dy = 0$

$$\frac{\partial P}{\partial y} = 2y \neq \frac{\partial Q}{\partial x} = y$$

Since the equality is not satisfied, the integration factor is used.

$$\ln(\lambda) = \int \left(\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} \right) dx\text{, \textbf{special case of being} } \lambda = \lambda\,(x)$$

$$ln\,(\lambda) = \int \left(\frac{2y - y}{x \cdot y} \right) dx = \int \left(\frac{1}{x} \right) dx = ln\,(x)$$

$$\lambda = x$$

Multiply both sides of the equation in the question by λ**.**

$$\left(x^3 + y^2 \cdot x\right) dx + \left(x^2 \cdot y\right) dy = 0$$

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial x} = 2xy$$

Since this equality is satisfied, it can be continued to be solved with the solution method of the exact differential equation.

2. equation: $\left(xy^2 + y\right) dx + (2y - x)\, dy = 0$

$$\frac{\partial P}{\partial x} = 2yx + 1 \neq \frac{\partial Q}{\partial x} = -1$$

Since the equality is not satisfied, the integration factor is used.

$$\ln(\lambda) = \int \left(\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} \right) dx$$

$$ln\,(\lambda) = \int \left(\frac{2xy+2}{2y-x} \right) dx$$

Is a y **dependent function since it does not depend only on** x **.** λ **must depend on y:**

$$\ln(\lambda) = \int \left(\frac{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}{P} \right) dy, \textbf{ special case of being } \lambda = \lambda\,(y)$$

$$\ln(\lambda) = \int \left(\frac{-2xy-2}{xy^2+y} \right) dy = \int \left(\frac{-2(xy+1)}{y(xy+1)} \right) dy = \int \left(-\frac{2}{y} \right) dy = \ln(y^{-2})$$

$$\lambda = y^{-2}$$

Multiply both sides of the question by λ**.**

$$\left(x + y^{-1}\right) dx + \left(2y^{-1} - xy^{-2}\right) dy = 0$$

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial x} = -\frac{1}{y^2}$$

Since this equality is satisfied, it can be continued to be solved with the solution method of the exact differential equation.

10 Separable Differential Equations

$$(y)' = (x+y+1)^2$$

Let $u = x + y + 1$**.**

Let's take the derivative of the two sides of the equation.

$$\frac{d}{dx}(u) = \frac{d}{dx}(x+y+1)$$

$$(u)' = 1 + (y)'$$

Let's find $(y)'$.

$$(y)' = (u)' - 1$$

Let's place the variables we found in their places in the equation.

$$(u)' = u^2 + 1$$

$$\frac{du}{dx} = u^2 + 1$$

Let's organize the equation.

$$dx = \frac{du}{u^2 + 1}$$

Let's integrate both sides of the equation.

$$\int dx = \int \frac{1}{u^2+1}\, du$$

$$x + c = \arctan(u)$$

When we substitute the necessary variables in the equation we find, we can reach the y function.

11 Some Trigonometric Derivatives and Integrals

$$\frac{d}{dx}(sin(x)) = cos(x)$$

$$\frac{d}{dx}(cos(x)) = -sin(x)$$

$$\frac{d}{dx}(tan(x)) = sec^2(x)$$

$$\frac{d}{dx}\left(cot(x)\right) = -cosec^2(x)$$

$$\frac{d}{dx}\left(sec(x)\right) = sec(x) \cdot tan(x)$$

$$\frac{d}{dx}\left(cosec(x)\right) = -cosec(x) \cdot cot(x)$$

$$\int sin(x)\, dx = -cos(x) + c$$

$$\int cos(x)\, dx = sin(x) + c$$

$$\int tan(x)\, dx = -ln\left(cos(x)\right) + c$$

$$\int cot(x)\, dx = ln\left(sin(x)\right) + c$$

$$\int sec(x)\, dx = ln\left(sec(x) + tan(x)\right) + c$$

$$\int cosec(x)\, dx = -ln\left(cosec(x) + cot(x)\right) + c$$

$$\int sec^2(x)\, dx = tan(x) + c$$

$$\int cosec^2(x)\, dx = -cot(x) + c$$

$$\frac{d}{dx}\left(arcsin(x)\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(arccos(x)\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(arctan(x)\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(arccot(x)\right) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(arccosec(x)\right) = -\frac{1}{|x| \cdot \sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(arcsec(x)\right) = \frac{1}{|x| \cdot \sqrt{x^2-1}}$$

$$\int \frac{1}{\sqrt{1-x^2}}\, dx = arcsin(x) + c$$

$$\int \frac{1}{1-x^2}\, dx = arctan(x) + c$$

$$\int \frac{1}{x\sqrt{x^2-1}}\, dx = arcsec|x| + c$$