PIGEONS & PIGEONHOLES

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The Dirichlet principle, more commonly called "the pigeonhole principle," is one of the simplest mathematical ideas formulated. Yet its simplicity belies its surprising applications. In essence, the principle states: "Suppose we have four pigeons but only three pigeonholes. No matter how we assign a hole to each pigeon, *at least two* pigeons will have to share the same hole." Applying this principle, we can draw conclusions that would be difficult—if not impossible—to verify in practice, but are nonetheless true. For example:

> In any group of people there are *at least two* who have the same number of acquaintances (within the group).
>
> On any day there are *at least two* Rexburg residents who begin their breakfasts at the same moment.
>
> In Eastern Idaho there are *at least two* people who have the same number of hairs on their heads.

Conclusions based on this principle are statements of certainty not of chance.

These follow from the pigeonhole principle. Let me emphasize, however, that we are not dealing with probability or likelihood. Conclusions based on this principle are statements of certainty not of chance. For as the principle is described above, there *will be* at least two pigeons in one of the three holes—this is a statement of fact not of probability.

As a first step in applying the pigeonhole principle, consider the assertion: Among 13 people at least two have the same birthday month. What are the pigeons and what are the pigeonholes? The pigeons are the 13 people and the pigeonholes the 12 months of the year. Distinct months cannot be assigned to all 13 individuals. Hence, at least two share the same birthday month. Similarly, among 366 people at least two have the same birthday. The pigeons here are the 366 people and the pigeonholes the 365 days of the year.[1] Again, two or more of these individuals must share the same day as their birthday. Finally, among 11 people—each with a primary phone number—at least two have phone numbers ending in the same digit. The pigeonhole principle guarantees this because there are only 10 possible ending digits (i.e., 0, 1, 2, 3, 4, 5, 6, 7, 8, 9).

Next, imagine we have seven pigeons but only three pigeonholes. Suppose these pigeons will tolerate one roommate, but would rather not have two roommates squeezed in with them. Can this be done? Picture placing the seven pigeons (one by one) into the three holes so as to limit multiple occupancy. The first three pigeons can be placed into distinct holes; the next three (at best) can be evenly distributed among the holes, so that now each pigeonhole houses two pigeons. Hapless pigeon number seven is doomed to live with two disgruntled roommates. Conclusion: *At least three* pigeons will have to share the same hole. From this extension of the pigeonhole principle, we can conclude that among 25 people at least three have the same birthday month; among 731 people at least three have the same birthday; among 21 people at least three have phone numbers ending in the same digit. (Calculations: $25 = 12 \times 2 + 1$; $731 = 365 \times 2 + 1$; $21 = 10 \times 2 + 1$.)

Going one step further, given ten pigeons and three pigeonholes *at least four* pigeons will have to share the same hole. Analogously, among 37 people at least four have the same birthday month; among 1096 people at least four have the same birthday; among 31 people at least four have phone numbers ending in the same digit. (Calculations: $37 = 12 \times 3 + 1$; $1096 = 365 \times 3 + 1$; $31 = 10 \times 3 + 1$.) Why stop here? Among 49 people *at least five* have the same birthday month; among 1461 people at least five have the same birthday; among 41 people at least five have phone numbers ending in the same digit. (Calculations: $49 = 12 \times 4 + 1$; $1461 = 365 \times 4 + 1$; $41 = 10 \times 4 + 1$.)

The pigeonhole principle empowers me to state with complete confidence, while I sit through a sellout performance in the Hart auditorium, that *at least 12* people in the audience have the same birthday—regardless of who is in attendance. Though I cannot tell you at that moment—or perhaps ever—who they are, such individuals must exist. My conviction rests on the fact that the Hart holds 4,200 people and $4{,}200 = 365 \times 11 + 185$. Therefore, 184 people could leave the auditorium and my statement would still be correct. Moreover, during a packed devotional in the Hart, there will be *at least two* attendees who not only have the same birthday, but are also the same age. This does not require rifling through the personal records of BYU-I students, but is based on the assumption that 3,651 (or more) of the 4,200 in attendance are students whose ages fall within a 10-year range. Since there are 3,650 days in a 10-year span, at least two attendees must be exactly the same age—to the day. Under similar assumptions for a BYU devotional in the Marriott Center, there will be at least two attendees *who were born within four hours of each other.* The number of four-hour periods over a 10-year span is $10 \times 365 \times 6 = 21{,}900$ and the Marriott seats more than 23,000 people.

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We can now tackle the three problems at the beginning. First, "In any group of people there are *at least two* who have the same number of acquaintances (within the group)." Choose any group size. To be specific, say we choose 50. If one of these individuals has no acquaintances within the group, then each of the others may have 0, or 1, or 2, or 3, . . . , or 48 acquaintances in the group. Therefore, we have 49 "holes" (which could be numbered 0, 1, 2, 3, . . . , 48) and 50 "pigeons" (people). In this case, we conclude that at least two people have the same number of acquaintances within the group. Next, suppose every person in the group has an acquaintance. Again there are 49 holes (which could be numbered 1, 2, 3, . . . , 49) and 50 people, leading to the same result.

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Second, "On any day there are *at least two* Rexburg residents who begin their breakfasts at the same moment." We need to make two assumptions (neither one farfetched) to work this problem. The first is that more than 18,000 residents of Rexburg have breakfast—throwing into the pool of "residents" BYU-I students for good measure. The second assumption is that the breakfast window is five hours, perhaps from 6 a.m. to 11 a.m. Calculation shows there are 5 × 60 × 60 = 18,000 seconds in a five-hour period. Consequently, under our assumptions at least two Rexburg residents begin their breakfasts at the same moment (second).

Third, "In Eastern Idaho there are *at least two* people who have the same number of hairs on their heads." This problem seems particularly fanciful. Yet it follows immediately from the pigeonhole principle, once a few facts are known. On average, a person's head has approximately 100,000 hair follicles; some people have as many as 150,000 (see *www.pg.com/science/haircare*). The population of Eastern Idaho exceeds 272,000. Even if we overestimate the number of hair follicles on the human head to be 250,000, we can comfortably conclude that in Eastern Idaho there are at least two people with the same number of hairs on their heads.

Andrew Wiles at Princeton University applied the pigeonhole principle in his celebrated proof of Fermat's last theorem—a famous conjecture by Pierre de Fermat (1601–1665). Scientific applications, however, are seldom based exclusively on the pigeonhole principle. This principle is primarily used as an analytic tool. For example, the most efficient method of searching for data stored on a computer (whether it be locating specific words in a text file, searching for a file on a hard drive, or finding information on the Internet) is called "hashing." The pigeonhole principle is indispensable in analyzing and explaining the remarkable efficiency of this method. Pigeonhole applications seem limited only by our imagination. I leave it to the reader to verify the calculations for a final (albeit challenging) exercise, one that requires no special assumptions:

> *At least two* people in Idaho have the same initials (including middle initial, if any) and the following characteristics: either all of them *or* none of them are LDS; all of them *or* none of them like cats; all of them *or* none of them were born before 1967; all of them *or* none of them wear contacts; all of them *or* none of them bite their fingernails; all of them *or* none of them live within 3 ½ miles of a McDonald's.

A philosophical aspect of the pigeonhole principle has yet to be addressed. Recall this principle only guarantees that at least one of the pigeonholes will contain two or more pigeons. It neither specifies a particular hole nor identifies which pigeons are in the group of "two or more." In theory, there is no way to know either of these things, since the pigeonhole principle is an assertion of existence not of explicit solution. Again, while sitting through a full-house performance in the Hart, I know with certainty that at least 12 in attendance have the same birthday. As I scan the audience, looking for clues, I will be none the wiser in guessing who the 12 (or more) might be nor the birthday they share. For this reason, the pigeonhole principle is called an "Existence theorem": It confirms the existence of a solution but gives no method for finding it. Existence theorems in mathematics—and by extension in the applied and theoretical sciences—are extremely important. If one hopes to solve a difficult problem, which *a priori* may or may not have a solution, it is very helpful to know whether a solution exists in the first place. Searching for the nonexistent is frustrating.

When introducing the pigeonhole principle to my students, I point out that in mathematics it is possible to know that a question has an answer without anyone knowing what the answer is. This is remarkable. (Might there be epistemological analogues in religious matters? Are there spiritual avenues of *knowing* that afford no more than an adumbrated view of the object of one's conviction—an object that in truth exists?) Knowing that a problem has a solution is wonderful, but equally satisfying is knowing there is *only* one solution. In mathematical parlance, this is called "uniqueness of solution." A Uniqueness theorem states that *if* there is a solution, it is the *only* solution. The pigeonhole principle is not so endowed because there may be two or more pigeons in multiple holes. Back to our example in the Hart, there could be several groups of "12 or more people" in the audience who have the same birthday as those in their group. The pigeonhole principle in no way asserts there is only one such group but that there is at least one.

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The marriage of existence and uniqueness qualities in a mathematical result is cause for celebration. Ordinary & Partial Differential Equations (Math 371 & 472) showcase several "Existence and Uniqueness theorems" (e.g., the existence and uniqueness of solution to a Dirichlet boundary value problem for Laplace's equation). This means, in essence, we can know that a question has *exactly* one answer without anyone knowing what the answer is. Amazing. We are not only certain there is a solution, but also confident that once a solution is found we can end our search—for there are no others.

From the simple to the complex, from the concrete to the abstract, human thought takes flight. During your next visit to Trafalgar Square or some other popular pigeon-gathering spot, consider how interesting ideas often hide among ordinary things—even pigeons vying for compartments within the eaves of weatherworn buildings.

1 For crispness of presentation, leap years are ignored.