Edexcel Advanced Level GCE Mathematics FP2

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1 de Moivre's Theorem

de Moivre's theorem states:

$$\text{If } z = r\left(\cos\left(\theta\right) + i\sin\left(\theta\right)\right)$$

$$\text{Then } z^n = \left(r\left(\sin\left(\theta\right) + i\sin\left(\theta\right)\right)\right)^n = r^n\left(\cos\left(n\theta\right) + i\sin\left(n\theta\right)\right)$$

And in the Exponential form:

$$\text{If } z = re^{i\theta}$$

$$\text{Then } z^n = \left(re^{i\theta}\right)^n = r^n e^{i\theta n}$$

This can be proved through proof by induction by following the following framework:

- Prove for $n = 1$.
- Assume true for $n = k$
- Show true for $n = (k+1)$
- State conclusion.

Prove that: $\left(r\left(\cos\left(\theta\right) + i\sin\left(\theta\right)\right)\right)^n = r^n\left(\cos\left(n\theta\right) + i\sin\left(n\theta\right)\right)$

When $n = 1$

$$\text{LHS } = \left(r\left(\cos\left(\theta\right) + i\sin\left(\theta\right)\right)\right)^1 = r\left(\cos\left(\theta\right) + i\sin\left(\theta\right)\right)$$

$$\text{RHS } = r^1\left(\cos\left(\left(1\right)\theta\right) + i\sin\left(\left(1\right)\theta\right)\right) = r\left(\cos\left(\theta\right) + i\sin\left(\theta\right)\right)$$

$$\text{RHS } = \text{ LHS}$$

$$\rightarrow \text{True when } n = 1$$

Assume true for $n = k$

$$\therefore \quad \left(r\left(\cos\left(\theta\right) + i\sin\left(\theta\right)\right)\right)^k = r^k\left(\cos\left(k\theta\right) + i\sin\left(k\theta\right)\right)$$

When $n = (k+1)$

$$\left(r\left(\cos\left(\theta\right) + i\sin\left(\theta\right)\right)\right)^{k+1} =$$

$$= r\left(\cos\left(\theta\right) + i\sin\left(\theta\right)\right) \times r^k\left(\cos\left(k\theta\right) + i\sin\left(k\theta\right)\right) = r^{k+1}\left(\cos\left(\left(k+1\right)\theta\right) + i\sin\left(\left(k+1\right)\theta\right)\right)$$

Given it is true for $n = (k+1)$ when it is true for $n = k$ ad it is true for $n = 1$, by mathematical induction, it is true for all positive n.

The proof for negative integers is a little simpler as because we have proofed it for positive intergers, we can assume it works for them before we even begin!.

- Let $n = -m$
- Start with left hand side - rewrite as fraction - apply statement.
- Make the denominator real
- Simplify & rearange for the right hand side

$n = -m$

$$z^{-m} = \left(r\left(\cos(\theta) + i\sin(\theta)\right)\right)^{-m}$$

$$\text{LHS } = \frac{1}{\left(r\left(\cos(\theta) + i\sin(\theta)\right)\right)^{m}}$$

We have already proved that de Moivre's theorem works for positive integers so we can simplify tis further without having to explain anything:

$$\text{LHS } = \frac{1}{r^m\left(\cos(m\theta) + i\sin(m\theta)\right)}$$

Make the denominator real by multiplying by the complex conjugate

$$\text{LHS } \times \frac{\left(\cos(m\theta) - i\sin(m\theta)\right)}{\left(\cos(m\theta) - i\sin(m\theta)\right)} = \frac{\left(\cos(m\theta) - i\sin(m\theta)\right)}{r^m\left(\cos^2(m\theta) + \sin^2(m\theta)\right)}$$

Trigonometric identities tell us that:

$$\cos^2(m\theta) + \sin^2(m\theta) = 1$$

and

$$\cos(m\theta) = \cos(-m\theta)$$
$$-\sin(m\theta) = \sin(-m\theta)$$

Applying these to the LHS results in a fraction that is easily manipulated to our RHS:

$$\text{LHS } = \frac{\left(\cos(-m\theta) + i\sin(-m\theta)\right)}{r^m}$$

$$\text{LHS } = r^{-m}\left(\cos(-m\theta) + i\sin(-m\theta)\right) = \text{ RHS}$$

de Moivre's Theorem is very useful, as it can be applied to simplify complicated looking expressions:

Simplify:

$$\frac{\left(\cos\left(\frac{9\pi}{17}\right)+i\sin\left(\frac{9\pi}{17}\right)\right)^5}{\left(\cos\left(\frac{2\pi}{17}\right)-i\sin\left(\frac{2\pi}{17}\right)\right)^3}$$

First we must put the denominator into the correct polar form (with a + inbetween cos and sin), and then we can apply de Moivre's Theorem.

$$\frac{\left(\cos\left(\frac{9\pi}{17}\right)+i\sin\left(\frac{9\pi}{17}\right)\right)^5}{\left(\cos\left(\frac{-2\pi}{17}\right)+i\sin\left(\frac{-2\pi}{17}\right)\right)^3}$$

And now appying de Moivre's Theorem:

$$\frac{\cos\left(\frac{45\pi}{17}\right)+i\sin\left(\frac{45\pi}{17}\right)}{\cos\left(\frac{-6\pi}{17}\right)+i\sin\left(\frac{-6\pi}{17}\right)}$$

From this point we can easily simplify the complex number down.

$$\cos\left(\frac{51\pi}{17}\right)+i\sin\left(\frac{51\pi}{17}\right)$$

$$\cos\left(3\pi\right)+i\sin\left(3\pi\right)$$

$$\cos\left(\pi\right)+i\sin\left(\pi\right)=-1$$