FP3

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Contents

1 Hyperbolic Functions

The hyperbolic functions are analogs of the ordinary trigonometric, or circular functions. The basic hyperbolic functions are, as one might expect, analygous to sine and cosine; they are hyperbolic sine and hyperbolic cosine.

$$\sinh : \mathbb{R} \to \mathbb{R} : x \mapsto \frac{e^x - e^{-x}}{2}$$

$$\cosh : \mathbb{R} \to \{x \mid x \in \mathbb{R},\, x \geq 1\} : x \mapsto \frac{e^x + e^{-x}}{2}$$

From these one can derive the hyperbolic tangent, hyperbolic secant, hyperbolic cosecant and hyperbolic cotangent functions in much the same way as their circular counterparts.

$$\tanh : \mathbb{R} \to \{x \mid x \in \mathbb{R},\, x \in [-1,1]\} : x \mapsto \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{sech} : \mathbb{R} \to \{y \mid y \in \mathbb{R},\, y \in [0,1)\} : x \mapsto \frac{2}{e^x + e^{-x}}$$

$$\operatorname{csch} : \{x \mid x \in \mathbb{R},\, x \neq 0\} \to \{y \mid y \in \mathbb{R},\, y \neq 0\} : x \mapsto \frac{2}{e^x - e^{-x}}$$

$$\coth : \{x \mid x \in \mathbb{R},\, x \neq 0\} \to \{y \mid y \in \mathbb{R},\, y \notin [-1,1]\} : x \mapsto \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

The inverse functions of hyperbolic sine, hyperbolic cosine and hyperbolic tangent are area hyperbolic sine, area hyperbolic cosine and area hyperbolic tangent;

$$\operatorname{arsinh} : \mathbb{R} \to \mathbb{R} : x \mapsto \ln\left(x + \sqrt{x^2+1}\right)$$

$$\operatorname{arcosh} : \{x \mid x \in \mathbb{R},\, x \geq 1\} \to \mathbb{R} : x \mapsto \ln\left(x + \sqrt{x^2-1}\right)$$

$$\operatorname{artanh} : \{x \mid x \in \mathbb{R},\, x \in [-1,1]\} \to \mathbb{R} : x \mapsto \frac{1}{2}\ln\left(\frac{x+1}{1-x}\right)$$

2 Conic Sections

All conic sections can be described in terms of loci; for any point P on a conic section,

$$\frac{|PM|}{|PS|} = e$$

where e is the eccentricity, S is the focus and M is the closest point to P that lies on the directrix.

$$0 \leq e < 1 \Rightarrow P \text{ describes an ellipse}$$
$$e = 1 \Rightarrow P \text{ describes a parabola}$$
$$e > 1 \Rightarrow P \text{ describes a hyperbola}$$

2.1 Ellipses

Ellipses are conic sections with eccentricty $e \in [0, 1)$. Considering their geometry, it is clear that, when expressed as a parametric equation, ellipses take the form;

$$x = a\cos\theta,\ y = b\sin\theta$$

Hence;

$$\left(\frac{x}{a}\right)^2 = \cos^2\theta,\ \left(\frac{y}{b}\right)^2 = \sin^2\theta$$

Making apparent an ellipse expressed in cartesian form;

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Theorem For $e \in [0, 1)$, an ellipse of focus $(ae, 0)$ and directrix $x = \frac{a}{e}$ is described by the equation;

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Proof

$$\frac{|PS|^2}{|PM|^2} = e \Rightarrow |PS|^2 = e^2|PM|^2$$

By pythagoras,

$$|PS|^2 = (x - ae)^2 + y^2$$

$$|PM|^2 = \left(\frac{a}{e} - x\right)^2 + y^2$$

Thus

$$(x - ae)^2 + y^2 = \left(\frac{a}{e} - x\right)^2 e^2$$

The derivative of an ellipse can be found by implicitly differentiating its cartesian form or using the chain rule on its parametric form. By the former method;

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left(\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2\right) = \frac{\mathrm{d}}{\mathrm{d}\,x}1$$

$$\frac{2x}{a^2} + \frac{2y}{b^2} \times \frac{\mathrm{d}\,y}{\mathrm{d}\,x} = 0$$

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = -\frac{b^2x}{a^2y}$$

And by the latter;

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,\theta} = -a\sin\theta$$

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,\theta} = b\cos\theta$$

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = \frac{\mathrm{d}\,y}{\mathrm{d}\,\theta} \times \left(\frac{\mathrm{d}\,x}{\mathrm{d}\,\theta}\right)^{-1} = -\frac{b\cos\theta}{a\sin\theta}$$

From this we can derive the general equation of a tangent to an ellipse at a point $(i,\, j)$:

$$y - j = -\frac{b^2 i}{a^2 j}\,(x - i)$$

$$a^2 jy + b^2 ix = (aj)^2 + (bi)^2$$

Similarly, we can derive the equation for a normal:

$$y - j = \frac{a^2 j}{b^2 i}\,(x - i)$$

$$b^2 iy + a^2 ij = a^2 jx + b^2 ij$$