FP3

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1 Hyperbolic Functions

The hyperbolic functions are analogs of the ordinary trigonometric, or circular functions. The basic hyperbolic functions are, as one might expect, analygous to sine and cosine; they are hyperbolic sine and hyperbolic cosine.

$$\sinh : \mathbb{R} \to \mathbb{R} : x \mapsto \frac{e^x - e^{-x}}{2}$$

$$\cosh : \mathbb{R} \to \{x \mid x \in \mathbb{R},\, x \geq 1\} : x \mapsto \frac{e^x + e^{-x}}{2}$$

From these one can derive the hyperbolic tangent, hyperbolic secant, hyperbolic cosecant and hyperbolic cotangent functions in much the same way as their circular counterparts.

$$\tanh : \mathbb{R} \to \{x \mid x \in \mathbb{R},\, x \in [-1,1]\} : x \mapsto \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\text{sech} : \mathbb{R} \to \{y \mid y \in \mathbb{R},\, y \in [0,1)\} : x \mapsto \frac{2}{e^x + e^{-x}}$$

$$\text{csch} : \{x \mid x \in \mathbb{R},\, x \neq 0\} \to \{y \mid y \in \mathbb{R},\, y \neq 0\} : x \mapsto \frac{2}{e^x - e^{-x}}$$

$$\coth : \{x \mid x \in \mathbb{R},\, x \neq 0\} \to \{y \mid y \in \mathbb{R},\, y \notin [-1,1]\} : x \mapsto \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

The inverse functions of hyperbolic sine, hyperbolic cosine and hyperbolic tangent are area hyperbolic sine, area hyperbolic cosine and area hyperbolic tangent;

$$\text{arsinh} : \mathbb{R} \to \mathbb{R} : x \mapsto \ln\left(x + \sqrt{x^2+1}\right)$$

$$\text{arcosh} : \{x \mid x \in \mathbb{R},\, x \geq 1\} \to \mathbb{R} : x \mapsto \ln\left(x + \sqrt{x^2-1}\right)$$

$$\text{artanh} : \{x \mid x \in \mathbb{R},\, x \in [-1,1]\} \to \mathbb{R} : x \mapsto \frac{1}{2}\ln\left(\frac{x+1}{1-x}\right)$$

2 Ellipses

Ellipses are curves that can be simultaneously described as loci or as conic sections. When expressed as a parametric equation, ellipses take the form;

$$x = a\cos\theta,\ y = b\sin\theta$$

Hence;

$$\left(\frac{x}{a}\right)^2 = \cos^2\theta,\ \left(\frac{y}{b}\right)^2 = \sin^2\theta$$

Making apparent an ellipse expressed in cartesian form;

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

2.1 Tangents and Normals

The derivative of an ellipse can be found by implicitly differentiating its cartesian form or using the chain rule on its parametric form. By the former method;

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\left(\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2\right)=\frac{\mathrm{d}}{\mathrm{d}\,x}1$$

$$\frac{2x}{a^2}+\frac{2y}{b^2}\times\frac{\mathrm{d}\,y}{\mathrm{d}\,x}=0$$

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x}=-\frac{b^2x}{a^2y}$$

And by the latter;

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,\theta}=-a\sin\theta$$

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,\theta}=b\cos\theta$$

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x}=\frac{\mathrm{d}\,y}{\mathrm{d}\,\theta}\times\left(\frac{\mathrm{d}\,x}{\mathrm{d}\,\theta}\right)^{-1}=-\frac{b\cos\theta}{a\sin\theta}$$

From this we can derive the general equation of a tangent to an ellipse at a point $(i,\, j)$:

$$y-j=-\frac{b^2i}{a^2j}\left(x-i\right)$$

$$a^2jy+b^2ix=\left(aj\right)^2+\left(bi\right)^2$$

Similarly, we can derive the equation for a normal:

$$y-j=\frac{a^2j}{b^2i}\left(x-i\right)$$

$$b^2iy+a^2ij=a^2jx+b^2ij$$