

FP3

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1 Hyperbolic Functions

The hyperbolic functions are analogs of the ordinary trigonometric, or circular functions. The basic hyperbolic functions are, as one might expect, analogous to sine and cosine; they are hyperbolic sine and hyperbolic cosine.

$$\sinh : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto \frac{e^x - e^{-x}}{2}$$

$$\cosh : \mathbb{R} \rightarrow \{x \mid x \in \mathbb{R}, x \geq 1\} : x \mapsto \frac{e^x + e^{-x}}{2}$$

From these one can derive the hyperbolic tangent, hyperbolic secant, hyperbolic cosecant and hyperbolic cotangent functions in much the same way as their circular counterparts.

$$\tanh : \mathbb{R} \rightarrow \{x \mid x \in \mathbb{R}, x \in [-1, 1]\} : x \mapsto \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{sech} : \mathbb{R} \rightarrow \{y \mid y \in \mathbb{R}, y \in [0, 1]\} : x \mapsto \frac{2}{e^x + e^{-x}}$$

$$\operatorname{csch} : \{x \mid x \in \mathbb{R}, x \neq 0\} \rightarrow \{y \mid y \in \mathbb{R}, y \neq 0\} : x \mapsto \frac{2}{e^x - e^{-x}}$$

$$\operatorname{coth} : \{x \mid x \in \mathbb{R}, x \neq 0\} \rightarrow \{y \mid y \in \mathbb{R}, y \notin [-1, 1]\} : x \mapsto \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

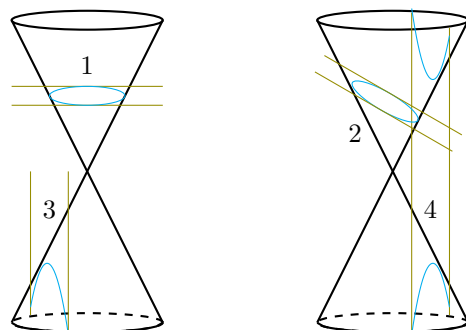
The inverse functions of hyperbolic sine, hyperbolic cosine and hyperbolic tangent are area hyperbolic sine, area hyperbolic cosine and area hyperbolic tangent;

$$\operatorname{arsinh} : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto \ln \left(x + \sqrt{x^2 + 1} \right)$$

$$\operatorname{arcosh} : \{x \mid x \in \mathbb{R}, x \geq 1\} \rightarrow \mathbb{R} : x \mapsto \ln \left(x + \sqrt{x^2 - 1} \right)$$

$$\operatorname{artanh} : \{x \mid x \in \mathbb{R}, x \in [-1, 1]\} \rightarrow \mathbb{R} : x \mapsto \frac{1}{2} \ln \left(\frac{x+1}{1-x} \right)$$

2 Conic Sections



1. Circle
2. Ellipse
3. Parabola
4. Hyperbola

All conic sections can be described in terms of loci; for any point P on a conic section,

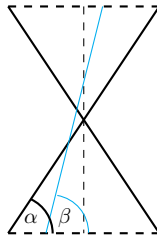
$$\frac{|PM|}{|PS|} = e$$

where e is the eccentricity, S is the focus and M is the closest point to P that lies on the directrix.

$0 \leq e < 1 \Rightarrow P$ describes an ellipse

$e = 1 \Rightarrow P$ describes a parabola

$e > 1 \Rightarrow P$ describes a hyperbola



The eccentricity can also be defined graphically, in terms of the cone and the intersection. Where e is the ratio between the angle of the cone and the intersection

$$e = \frac{\sin(\beta)}{\sin(\alpha)}$$

2.1 Ellipses

Ellipses are conic sections with eccentricity $e \in [0, 1)$. Considering their geometry, it is clear that, when expressed as a parametric equation, ellipses take the form;

$$x = a \cos \theta, y = b \sin \theta$$

Hence;

$$\left(\frac{x}{a}\right)^2 = \cos^2 \theta, \left(\frac{y}{b}\right)^2 = \sin^2 \theta$$

Making apparent an ellipse expressed in cartesian form;

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Theorem For $e \in [0, 1)$, an ellipse of focus $(ae, 0)$ and directrix $x = \frac{a}{e}$ is described by the equation;

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Proof

$$\frac{|PS|^2}{|PM|^2} = e \Rightarrow |PS|^2 = e^2 |PM|^2$$

By pythagoras,

$$|PS|^2 = (x - ae)^2 + y^2$$

$$|PM|^2 = \left(\frac{a}{e} - x\right)^2 + y^2$$

Thus

$$(x - ae)^2 + y^2 = \left(\frac{a}{e} - x\right)^2 e^2$$

The derivative of an ellipse can be found by implicitly differentiating its cartesian form or using the chain rule on its parametric form. By the former method;

$$\frac{d}{dx} \left(\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \right) = \frac{d}{dx} 1$$

$$\frac{2x}{a^2} + \frac{2y}{b^2} \times \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

And by the latter;

$$\frac{dx}{d\theta} = -a \sin \theta$$

$$\frac{dy}{d\theta} = b \cos \theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \left(\frac{dx}{d\theta}\right)^{-1} = -\frac{b \cos \theta}{a \sin \theta}$$

From this we can derive the general equation of a tangent to an ellipse at a point (i, j) :

$$y - j = -\frac{b^2 i}{a^2 j} (x - i)$$

$$a^2 j y + b^2 i x = (aj)^2 + (bi)^2$$

Similarly, we can derive the equation for a normal:

$$y - j = \frac{a^2 j}{b^2 i} (x - i)$$

$$b^2 i y + a^2 i j = a^2 j x + b^2 i j$$

2.2 Hyperbolae

A hyperbola is a conic section with $e > 1$.