Edexcel Advanced Level GCE Mathematics FP2

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Contents

1 de Moivre's Theorem

de Moivre's theorem states:

$$\text{If } z = r\left(\cos\left(\theta\right) + i\sin\left(\theta\right)\right)$$

$$\text{Then } z^n = \left(r\left(\sin\left(\theta\right) + i\sin\left(\theta\right)\right)\right)^n = r^n\left(\cos\left(n\theta\right) + i\sin\left(n\theta\right)\right)$$

And in the Exponential form:

$$\text{If } z = re^{i\theta}$$

$$\text{Then } z^n = \left(re^{i\theta}\right)^n = r^n e^{i\theta n}$$

This can be proved through proof by induction by following the following framework:

- Prove for $n = 1$.
- Assume true for $n = k$
- Show true for $n = (k+1)$
- State conclusion.

Prove that: $\left(r\left(\cos\left(\theta\right) + i\sin\left(\theta\right)\right)\right)^n = r^n\left(\cos\left(n\theta\right) + i\sin\left(n\theta\right)\right)$

When $n = 1$

$$\text{LHS } = \left(r\left(\cos\left(\theta\right) + i\sin\left(\theta\right)\right)\right)^1 = r\left(\cos\left(\theta\right) + i\sin\left(\theta\right)\right)$$

$$\text{RHS } = r^1\left(\cos\left(\left(1\right)\theta\right) + i\sin\left(\left(1\right)\theta\right)\right) = r\left(\cos\left(\theta\right) + i\sin\left(\theta\right)\right)$$

$$\text{RHS } = \text{ LHS}$$

$$\rightarrow \text{True when } n = 1$$

Assume true for $n = k$

$$\therefore \quad \left(r\left(\cos\left(\theta\right) + i\sin\left(\theta\right)\right)\right)^k = r^k\left(\cos\left(k\theta\right) + i\sin\left(k\theta\right)\right)$$

When $n = (k+1)$

$$\left(r\left(\cos\left(\theta\right) + i\sin\left(\theta\right)\right)\right)^{k+1} =$$

$$= r\left(\cos\left(\theta\right) + i\sin\left(\theta\right)\right) \times r^k\left(\cos\left(k\theta\right) + i\sin\left(k\theta\right)\right) = r^{k+1}\left(\cos\left(\left(k+1\right)\theta\right) + i\sin\left(\left(k+1\right)\theta\right)\right)$$

Given it is true for $n = (k+1)$ when it is true for $n = k$ ad it is true for $n = 1$, by mathematical induction, it is true for all positive n.

The proof for negative integers is a little simpler as because we have proofed it for positive intergers, we can assume it works for them before we even begin!.

- Let $n = -m$
- Start with left hand side - rewrite as fraction - apply statement.
- Make the denominator real
- Simplify & rearange for the right hand side

$n = -m$

$$z^{-m} = (r(\cos(\theta) + i\sin(\theta)))^{-m}$$

$$\text{LHS} = \frac{1}{(r(\cos(\theta) + i\sin(\theta)))^{m}}$$

We have already proved that de Moivre's theorem works for positive integers so we can simplify tis further without having to explain anything:

$$\text{LHS} = \frac{1}{r^m(\cos(m\theta) + i\sin(m\theta))}$$

Make the denominator real by multiplying by the complex conjugate

$$\text{LHS} \times \frac{(\cos(m\theta) - i\sin(m\theta))}{(\cos(m\theta) - i\sin(m\theta))} = \frac{(\cos(m\theta) - i\sin(m\theta))}{r^m\left(\cos^2(m\theta) + \sin^2(m\theta)\right)}$$

Trigonometric identities tell us that:

$$\cos^2(m\theta) + \sin^2(m\theta) = 1$$

and

$$\cos(m\theta) = \cos(-m\theta)$$
$$-\sin(m\theta) = \sin(-m\theta)$$

Applying these to the LHS results in a fraction that is easily manipulated to our RHS:

$$\text{LHS} = \frac{(\cos(-m\theta) + i\sin(-m\theta))}{r^m}$$

$$\text{LHS} = r^{-m}(\cos(-m\theta) + i\sin(-m\theta)) = \text{RHS}$$

de Moivre's Theorem is very useful, as it can be applied to simplify complicated looking expressions:

Simplify:

$$\frac{\left(\cos\left(\frac{9\pi}{17}\right)+i\sin\left(\frac{9\pi}{17}\right)\right)^5}{\left(\cos\left(\frac{2\pi}{17}\right)-i\sin\left(\frac{2\pi}{17}\right)\right)^3}$$

First we must put the denominator into the correct polar form (with a + inbetween cos and sin), and then we can apply de Moivre's Theorem.

$$\frac{\left(\cos\left(\frac{9\pi}{17}\right)+i\sin\left(\frac{9\pi}{17}\right)\right)^5}{\left(\cos\left(\frac{-2\pi}{17}\right)+i\sin\left(\frac{-2\pi}{17}\right)\right)^3}$$

And now appying de Moivre's Theorem:

$$\frac{\cos\left(\frac{45\pi}{17}\right)+i\sin\left(\frac{45\pi}{17}\right)}{\cos\left(\frac{-6\pi}{17}\right)+i\sin\left(\frac{-6\pi}{17}\right)}$$

From this point we can easily simplify the complex number down.

$$\cos\left(\frac{51\pi}{17}\right)+i\sin\left(\frac{51\pi}{17}\right)$$

$$\cos(3\pi)+i\sin(3\pi)$$

$$\cos(\pi)+i\sin(\pi)=-1$$

1.0.1 de Moivre's Theorem and Trigonometric Identities

de Moivre's Theorem can also be applied to problems consisting of trigonometric functions by using identites and binomial expansion. First though, it is worth noting the following, where Z is a complex number $r(\cos(\theta)+i\sin(\theta))$

$$Z+\tfrac{1}{Z} \;=\; 2\cos(\theta) \qquad Z^n+\tfrac{1}{Z^n} \;=\; 2\cos(n\theta)$$

$$Z-\tfrac{1}{Z} \;=\; 2i\sin(\theta) \qquad Z^n-\tfrac{1}{Z^n} \;=\; 2i\sin(\theta)$$

Trigonometric functions can be replaced with these arrangements of complex numbers and then manipulated far more easily. The following is an example of such a situation.

Express $\sin^3(\theta)$ in the form $d\cos(4\theta)+e\cos(2\theta)+f$

$$Z-\frac{1}{Z} \;=\; 2i\sin(\theta) \;\therefore\; \left(Z^4-\frac{1}{Z^4}\right)^4 \;=\; 16\sin(4\theta)$$

$$\begin{aligned}\left(Z^4-\frac{1}{Z^4}\right)^4 \;&=\; Z^4+4Z^3\left(\frac{-1}{Z}\right)+6Z^2\left(\frac{-1}{Z}\right)^2+4Z\left(\frac{-1}{Z}\right)^3+\left(\frac{-1}{Z}\right)^4\\ &\to Z^4+4Z^2+6-4Z^{-2}+Z^{-4}\\ &\to \left(Z^4+\frac{1}{Z^4}\right)-4\left(Z^2+\frac{1}{Z^2}\right)+6\end{aligned}$$

$$\therefore\ 16\sin^4(\theta) \ = \ 2\cos(4\theta) - 8\cos(2\theta) + 6$$
$$\therefore\ \sin^4(\theta) \ = \ \frac{1}{8}\cos(4\theta) - \frac{1}{2}\cos(2\theta) + \frac{3}{8}$$

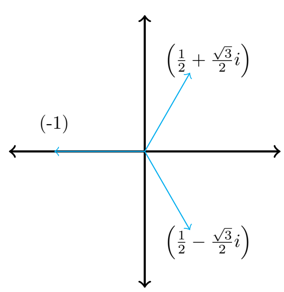
1.0.2 Finding the nth Root of a polynomial Using de Moivre's Theorem

Solve $Z^3 = 1$. Represent Solutions on an argand diagram.

$$|Z^3| = 1 \ \ arg(Z^3) = \pi$$
$$Z^3 = 1(\cos(\pi) + i\sin(\pi))$$
$$Z^3 = (\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi))$$
$$Z = (\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi))^{\frac{1}{3}}$$
$$Z = \cos\left(\frac{\pi + 2k\pi}{3}\right) + i\sin\left(\frac{\pi + 2k\pi}{3}\right)$$

We only need consider all values of k for which $\pi < \theta < \pi$

$$k = 0 \ \rightarrow \ Z = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$$
$$\therefore\ Z \ = \ \frac{1}{2} + \frac{\sqrt{3}}{2}i$$
$$k = 1 \ \rightarrow \ Z = \cos(\pi) + i\sin(\pi)$$
$$\therefore\ Z \ = \ -1$$
$$k = -1 \ \rightarrow \ Z = \cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)$$
$$\therefore\ Z \ = \ \frac{1}{2} - \frac{\sqrt{3}}{2}i$$



2 Polynomial Expansions

Maclaurin's expansion is a way to fit a polynomial to an arbitrary function. It states

$$f(x) \approx f(0) + xf'(0) + \frac{x^2 f''(0)}{2} + \frac{x^3 f'''(0)}{6} + \cdots + \frac{x^n f^n(0)}{n!} = \sum_{r=0}^{n} \frac{x^r f^r(0)}{r!}$$

Mapping this out to infinite terms gives an exact equivalence, called a Maclaurin series;

$$f(x) = \sum_{r_0}^{\infty} \frac{x^r f^r(0)}{r!}$$