5.26(a). In practice, the results of restoration filtering are seldom this close to the original images. This example, and Example 5.12 in the next section, were idealized slightly to focus on the effects of noise on restoration algorithms.

5.9 CONSTRAINED LEAST SQUARES FILTERING

The problem of having to know something about the degradation function H is common to all methods discussed in this chapter. However, the Wiener filter presents an additional difficulty: the power spectra of the undegraded image and noise must be known also. We showed in the previous section that in some cases it is possible to achieve acceptable results using the approximation in Eq. (5-85), but a constant value for the ratio of the power spectra is not always a suitable solution.

The method discussed in this section requires knowledge of only the mean and variance of the noise. As discussed in Section 5.2, these parameters generally can be calculated from a given degraded image, so this is an important advantage. Another difference is that the Wiener filter is based on minimizing a statistical criterion and, as such, it is optimal in an average sense. The algorithm presented in this section has the notable feature that it yields an optimal result for each image to which it is applied. Of course, it is important to keep in mind that these optimality criteria, although they are comforting from a theoretical point of view, are not related to the dynamics of visual perception. As a result, the choice of one algorithm over the other will almost always be determined by the perceived visual quality of the resulting images.

By using the definition of convolution given in Eq. (4-94), and as explained in Section 2.6, we can express Eq. (5-64) in vector-matrix form:

$$\mathbf{g} = \mathbf{Hf} + \boldsymbol{\eta} \tag{5-86}$$

For example, suppose that $g(x, y)$ is of size $M \times N$. We can form the first N elements of vector $\mathbf{g}$ by using the image elements in the first row of $g(x, y)$, the next N elements from the second row, and so on. The dimensionality of the resulting vector will be $MN \times 1$. These are also the dimensions of $\mathbf{f}$ and $\boldsymbol{\eta}$, as these vectors are formed in the same manner. Matrix $\mathbf{H}$ then has dimensions $MN \times MN$. Its elements are given by the elements of the convolution in Eq. (4-94).

It would be reasonable to arrive at the conclusion that the restoration problem can now be reduced to simple matrix manipulations. Unfortunately, this is not the case. For instance, suppose that we are working with images of medium size, say $M = N = 512$. Then the vectors in Eq. (5-86) would be of dimension $262,144 \times 1$ and matrix $\mathbf{H}$ would be of dimension $262,144 \times 262,144$. Manipulating vectors and matrices of such sizes is not a trivial task. The problem is complicated further by the fact that $\mathbf{H}$ is highly sensitive to noise (after the experiences we had with the effect of noise in the previous two sections, this should not be a surprise). The key advantage of formulating the restoration problem in matrix form is that it facilitates derivation of restoration algorithms.

See Gonzalez and Woods [1992] for an entire chapter devoted to the topic of algebraic techniques for image restoration.

Although we do not fully derive the method of constrained least squares that we are about to present, this method has its roots in a matrix formulation. We give references at the end of the chapter to sources where derivations are covered in detail. Central to the method is the issue of the sensitivity of **H** to noise. One way to reduce the effects of noise sensitivity, is to base optimality of restoration on a measure of smoothness, such as the second derivative of an image (our old friend, the Laplacian). To be meaningful, the restoration must be constrained by the parameters of the problems at hand. Thus, what is desired is to find the minimum of a criterion function, C, defined as

$$C = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left[\nabla^2 f(x,y) \right]^2 \tag{5-87}$$

subject to the constraint

$$\left\| \mathbf{g} - \mathbf{H}\hat{\mathbf{f}} \right\|^2 = \left\| \boldsymbol{\eta} \right\|^2 \tag{5-88}$$

where $\|\mathbf{a}\|^2 \triangleq \mathbf{a}^T \mathbf{a}$ is the Euclidean norm (see Section 2.6), and $\hat{\mathbf{f}}$ is the estimate of the undegraded image. The Laplacian operator ∇^2 is defined in Eq. (3-50).

The frequency domain solution to this optimization problem is given by the expression

The quantity in brackets is the transfer function of the constrained least squares filter. Note that it reduces to the inverse filter transfer function when $\gamma = 0$.

$$\hat{F}(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + \gamma |P(u,v)|^2} \right] G(u,v) \tag{5-89}$$

where γ is a parameter that must be adjusted so that the constraint in Eq. (5-88) is satisfied, and $P(u,v)$ is the Fourier transform of the function

$$p(x,y) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix} \tag{5-90}$$

We recognize this function as a Laplacian kernel from Fig. 3.45. Note that Eq. (5-89) reduces to inverse filtering if $\gamma = 0$.

Functions $P(u,v)$ and $H(u,v)$ must be of the same size. If H is of size $M \times N$, this means that $p(x,y)$ must be embedded in the center of an $M \times N$ array of zeros. In order to preserve the even symmetry of $p(x,y)$, M and N must be even integers, as explained in Examples 4.10 and 4.15. If a given degraded image from which H is obtained is not of even dimensions, then a row and/or column, as appropriate, must be deleted before computing H for use in Eq. (5-89).

EXAMPLE 5.12: Comparison of deblurring by Wiener and constrained least squares filtering.

Figure 5.30 shows the result of processing Figs. 5.29(a), (d), and (g) with constrained least squares filters, in which the values of γ were selected manually to yield the best visual results. This is the same procedure we used to generate the Wiener filter results in Fig. 5.29(c), (f), and (i). By comparing the constrained least squares and Wiener results, we see that the former yielded better results (especially in terms of noise reduction) for the high- and medium-noise cases, with both filters generating essentially

a b c

FIGURE 5.30 Results of constrained least squares filtering. Compare (a), (b), and (c) with the Wiener filtering results in Figs. 5.29(c), (f), and (i), respectively.

equal results for the low-noise case. This is not surprising because parameter γ in Eq. (5-89) is a true scalar, whereas the value of K in Eq. (5-85) is a scalar approximation to the ratio of two unknown frequency domain *functions* of size $M \times N$. Thus, it stands to reason that a result based on manually selecting γ would be a more accurate estimate of the undegraded image. As in Example 5.11, the results in this example are better than one normally finds in practice. Our focus here was on the effects of noise blurring on restoration. As noted earlier, you will encounter situations in which the restoration solutions are not quite as close to the original images as we have shown in these two examples.

As discussed in the preceding example, it is possible to adjust the parameter γ interactively until acceptable results are achieved. However, if we are interested in mathematical optimality, then this parameter must be adjusted so that the constraint in Eq. (5-88) is satisfied. A procedure for computing γ by iteration is as follows.

Define a "residual" vector $\mathbf{r}$ as

$$\mathbf{r} = \mathbf{g} - \mathbf{H}\hat{\mathbf{f}} \tag{5-91}$$

From Eq. (5-89), we see that $\hat{F}(u,v)$ (and by implication $\hat{\mathbf{f}}$) is a function of γ. Then it follows that $\mathbf{r}$ also is a function of this parameter. It can be shown (Hunt [1973], Gonzalez and Woods [1992]) that

$$\begin{aligned} \phi(\gamma) &= \mathbf{r}^T\mathbf{r} \\ &= \|\mathbf{r}\|^2 \end{aligned} \tag{5-92}$$

is a monotonically increasing function of γ. What we want to do is adjust γ so that

$$\|\mathbf{r}\|^2 = \|\boldsymbol{\eta}\|^2 \pm \alpha \tag{5-93}$$

where α is an accuracy factor. In view of Eq. (5-91), if $\|\mathbf{r}\|^2 = \|\boldsymbol{\eta}\|^2$, the constraint in Eq. (5-88) will be strictly satisfied.

Because $\phi(\gamma)$ is monotonic, finding the desired value of γ is not difficult. One approach is to

1. Specify an initial value of γ.
2. Compute $\|\mathbf{r}\|^2$.
3. Stop if Eq. (5-93) is satisfied; otherwise return to Step 2 after increasing γ if $\|\mathbf{r}\|^2 < (\|\boldsymbol{\eta}\|^2 - \alpha)$ or decreasing γ if $\|\mathbf{r}\|^2 > (\|\boldsymbol{\eta}\|^2 + \alpha)$. Use the new value of γ in Eq. (5-89) to recompute the optimum estimate $\hat{F}(u,v)$.

Other procedures, such as a Newton–Raphson algorithm, can be used to improve the speed of convergence.

In order to use this algorithm, we need the quantities $\|\mathbf{r}\|^2$ and $\|\boldsymbol{\eta}\|^2$. To compute $\|\mathbf{r}\|^2$, we note from Eq. (5-91) that

$$R(u,v) = G(u,v) - H(u,v)F(u,v) \tag{5-94}$$

from which we obtain $r(x,y)$ by computing the inverse Fourier transform of $R(u,v)$. Then, from the definition of the Euclidean norm, it follows that

$$\|\mathbf{r}\|^2 = \mathbf{r}^T\mathbf{r} = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} r^2(x,y) \tag{5-95}$$

Computation of $\|\boldsymbol{\eta}\|^2$ leads to an interesting result. First, consider the variance of the noise over the entire image, which we estimate from the samples using the expression

$$\sigma_\eta^2 = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\left[\eta(x,y) - \overline{\eta}\right]^2 \tag{5-96}$$

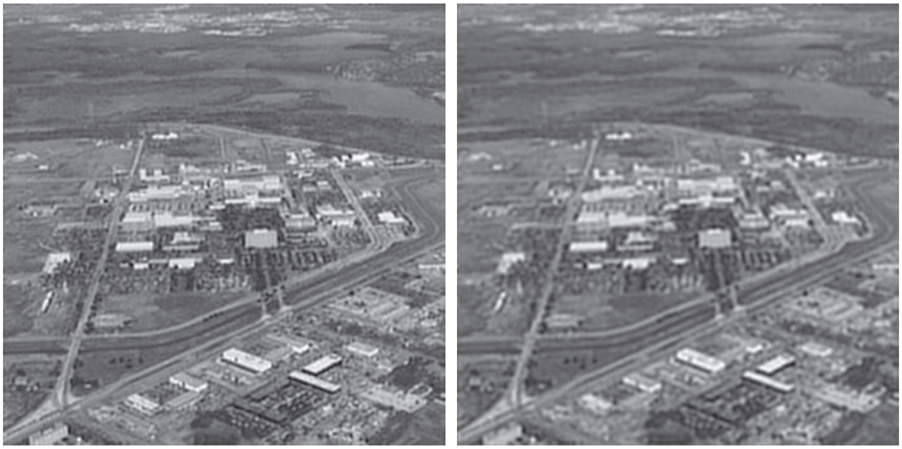
where

$$\overline{\eta} = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\eta(x,y) \tag{5-97}$$

is the sample mean. With reference to the *form* of Eq. (5-95), we note that the double summation in Eq. (5-96) is proportional to $\|\boldsymbol{\eta}\|^2$. This leads to the expression

$$\|\boldsymbol{\eta}\|^2 = MN\left[\sigma_\eta^2 + \overline{\eta}^2\right] \tag{5-98}$$

This is a most useful result. It tells us that we can estimate the unknown quantity $\|\boldsymbol{\eta}\|^2$ by having knowledge of only the mean and variance of the noise. These quantities are not difficult to estimate (see Section 5.2), assuming that the noise and image



a b

FIGURE 5.31 (a) Iteratively determined constrained least squares restoration of Fig. 5.25(b), using correct noise parameters. (b) Result obtained with wrong noise parameters.

intensity values are not correlated. This is an assumption of all the methods discussed in this chapter.

EXAMPLE 5.13: Iterative estimation of the optimum constrained least squares filter.

Figure 5.31(a) shows the result obtained using the algorithm just described to estimate the optimum filter for restoring Fig. 5.25(b). The initial value used for γ was 10^{-5}, the correction factor for adjusting γ was 10^{-6}, and the value for α was 0.25. The noise parameters specified were the same used to generate Fig. 5.25(a): a noise variance of 10^{-5}, and zero mean. The restored result is comparable to Fig. 5.28(c), which was obtained by Wiener filtering with K manually specified for best visual results. Figure 5.31(b) shows what can happen if the wrong estimate of noise parameters are used. In this case, the noise variance specified was 10^{-2} and the mean was left at 0. The result in this case is considerably more blurred.

5.10 GEOMETRIC MEAN FILTER

It is possible to generalize slightly the Wiener filter discussed in Section 5.8. The generalization is in the form of the so-called *geometric mean filter*:

$$\hat{F}(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2}\right]^{\alpha}\left[\frac{H^*(u,v)}{|H(u,v)|^2 + \beta\left[\dfrac{S_\eta(u,v)}{S_f(u,v)}\right]}\right]^{1-\alpha} G(u,v) \tag{5-99}$$

where α and β are nonnegative, real constants. The geometric mean filter transfer function consists of the two expressions in brackets raised to the powers α and $1-\alpha$, respectively.

When $\alpha = 1$ the geometric mean filter reduces to the inverse filter. With $\alpha = 0$ the filter becomes the so-called *parametric Wiener filter*, which reduces to the "standard"