EMSE 6035: Marketing of Technology

Uncertainty & Design of Experiments

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Background: Estimating Utility Model Coefficients Using Maximum Likelihood Estimation

$$\tilde{u}_j = \boldsymbol{\beta}'\mathbf{x}_j + \tilde{\varepsilon}_j$$

$$= \boxed{\beta_1} x_{j1} + \boxed{\beta_2} x_{j2} + \ldots + \tilde{\varepsilon}_j$$

Weights that denote the *relative* value of attributes $x_{j1}, x_{j2}, \ldots$

Estimate β_1 , β_2 , … , by minimizing the negative log-likelihood function:

minimize $-\ln(\mathcal{L}) = -\sum_{j=1}^{J} y_j \ln\left[P_j(\boldsymbol{\beta}|\mathbf{x})\right]$

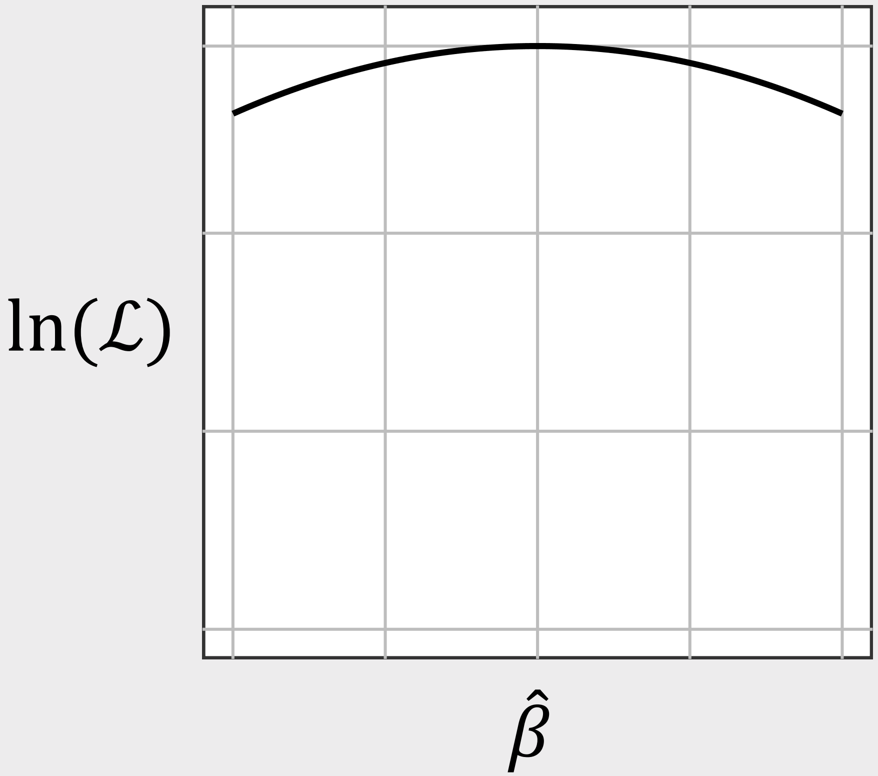
with respect to $\boldsymbol{\beta}$

$y_j = 1$ if alternative j was chosen
$y_j = 0$ if alternative j was not chosen

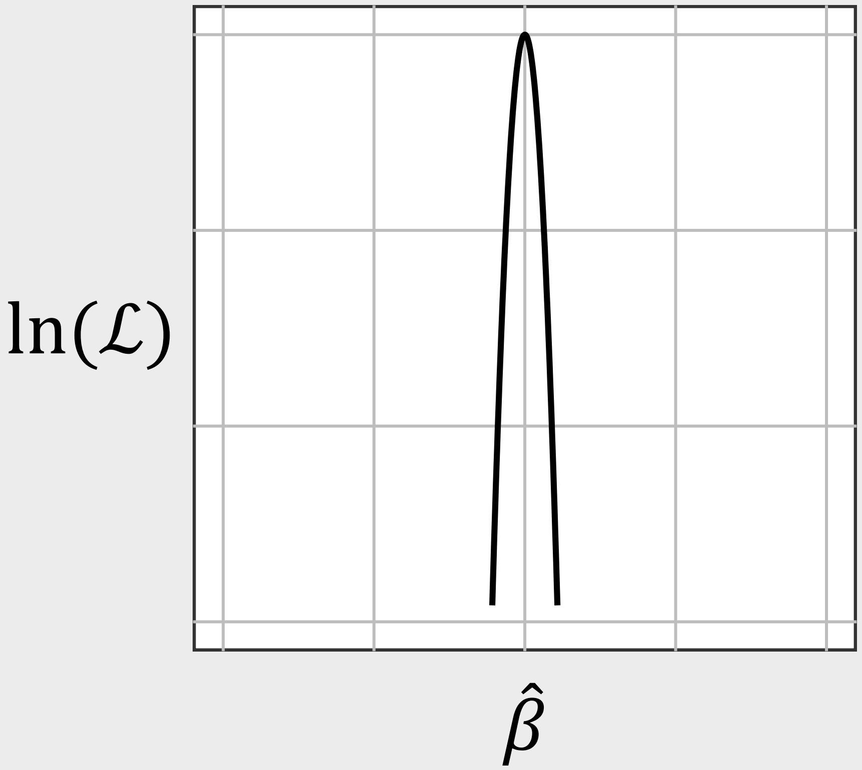
→ Produces point estimates: $\widehat{\boldsymbol{\beta}}$
…but these estimates are not precisely known

The certainty of $\widehat{\boldsymbol{\beta}}$ is inversely related to the curvature of the log-likelihood function

Greater variance in $\ln(\mathcal{L})$,
Less certainty in $\hat{\beta}$



Less variance in $\ln(\mathcal{L})$,
Greater certainty in $\hat{\beta}$



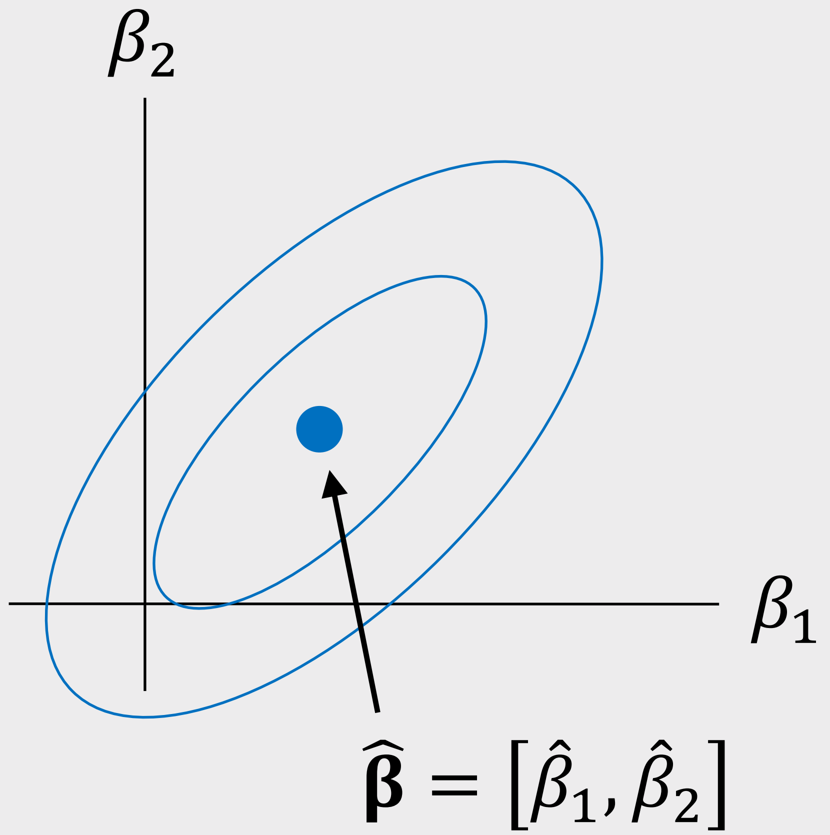
The certainty of $\widehat{\boldsymbol{\beta}}$ is inversely related to the curvature of the log-likelihood function

$$\sum\nolimits_{\boldsymbol{\beta}} = -\overbrace{\left[\nabla^2_{\boldsymbol{\beta}} \ln(\mathcal{L})\right]}^{\text{Hessian}}{}^{-1} = \begin{bmatrix} \sigma_{11}^2 & \cdots & \sigma_{m1}^2 \\ \vdots & \ddots & \vdots \\ \sigma_{1n}^2 & \cdots & \sigma_{mn}^2 \end{bmatrix}$$

Covariance of $\widehat{\boldsymbol{\beta}}$

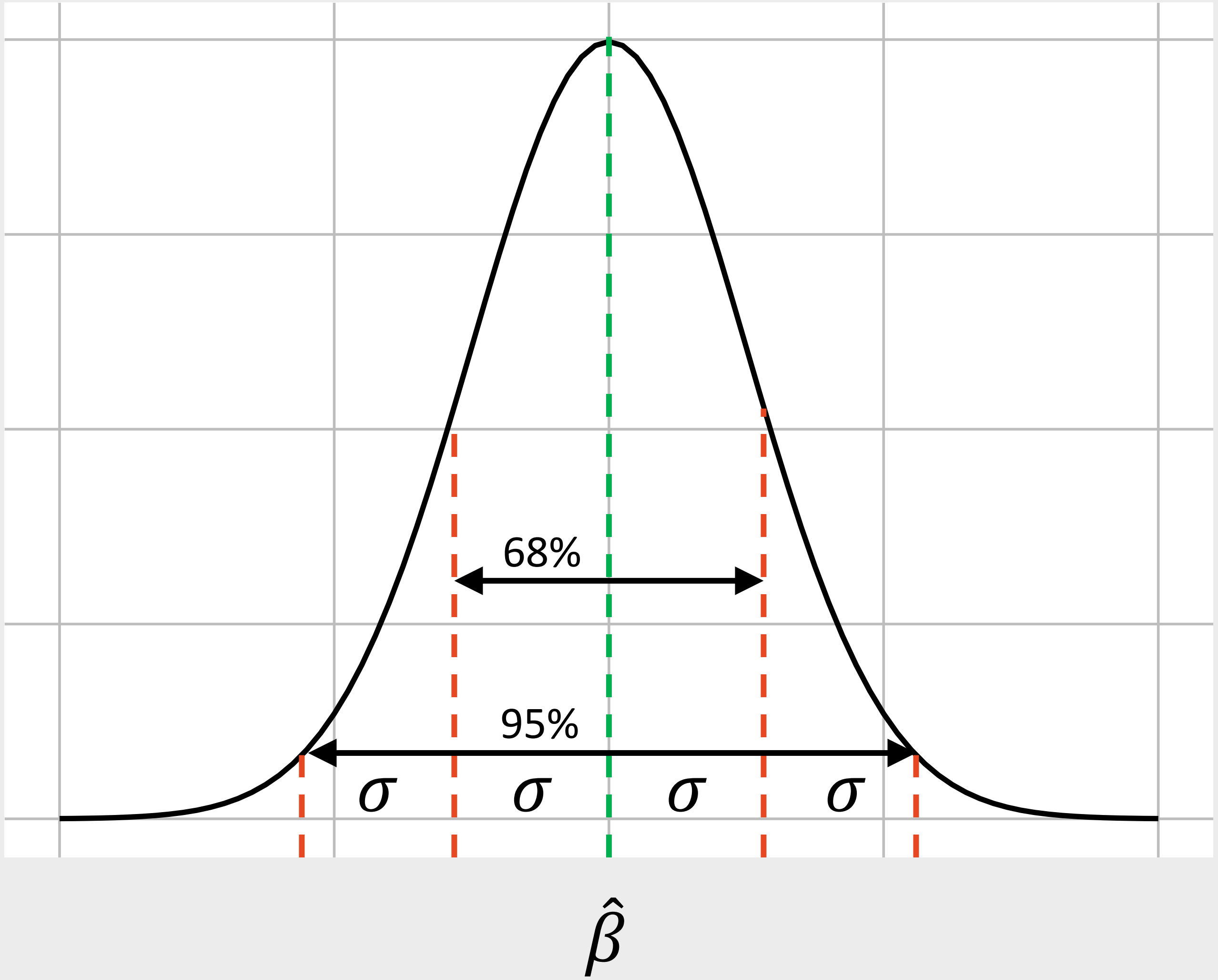
It is common to report $\widehat{\boldsymbol{\beta}}$ with its standard errors:

Est.	Std. Err.
$\hat{\beta}_1$	σ_1
$\hat{\beta}_2$	σ_2
$\vdots$	$\vdots$
$\hat{\beta}_m$	σ_m



We use standard errors to report uncertainty about $\widehat{\boldsymbol{\beta}}$

Est.	Std. Err.
$\hat{\beta}_1$	σ_1
$\hat{\beta}_2$	σ_2
$\vdots$	$\vdots$
$\hat{\beta}_m$	σ_m



A 95% confidence interval is approximately $[\hat{\beta} - 2\sigma, \hat{\beta} + 2\sigma]$

Practice Question 1

Suppose we estimate a model and get the following results:

$$\widehat{\boldsymbol{\beta}} = [-0.4, 0.5\,] \qquad \nabla^2_{\boldsymbol{\beta}} \ln(\mathcal{L}) = \begin{bmatrix} -6000 & 60 \\ 60 & -700 \end{bmatrix}$$

a) Use the hessian to compute the standard errors for $\widehat{\boldsymbol{\beta}}$.

b) Use the standard errors to compute a 95% confidence interval around $\widehat{\boldsymbol{\beta}}$.

Hints:

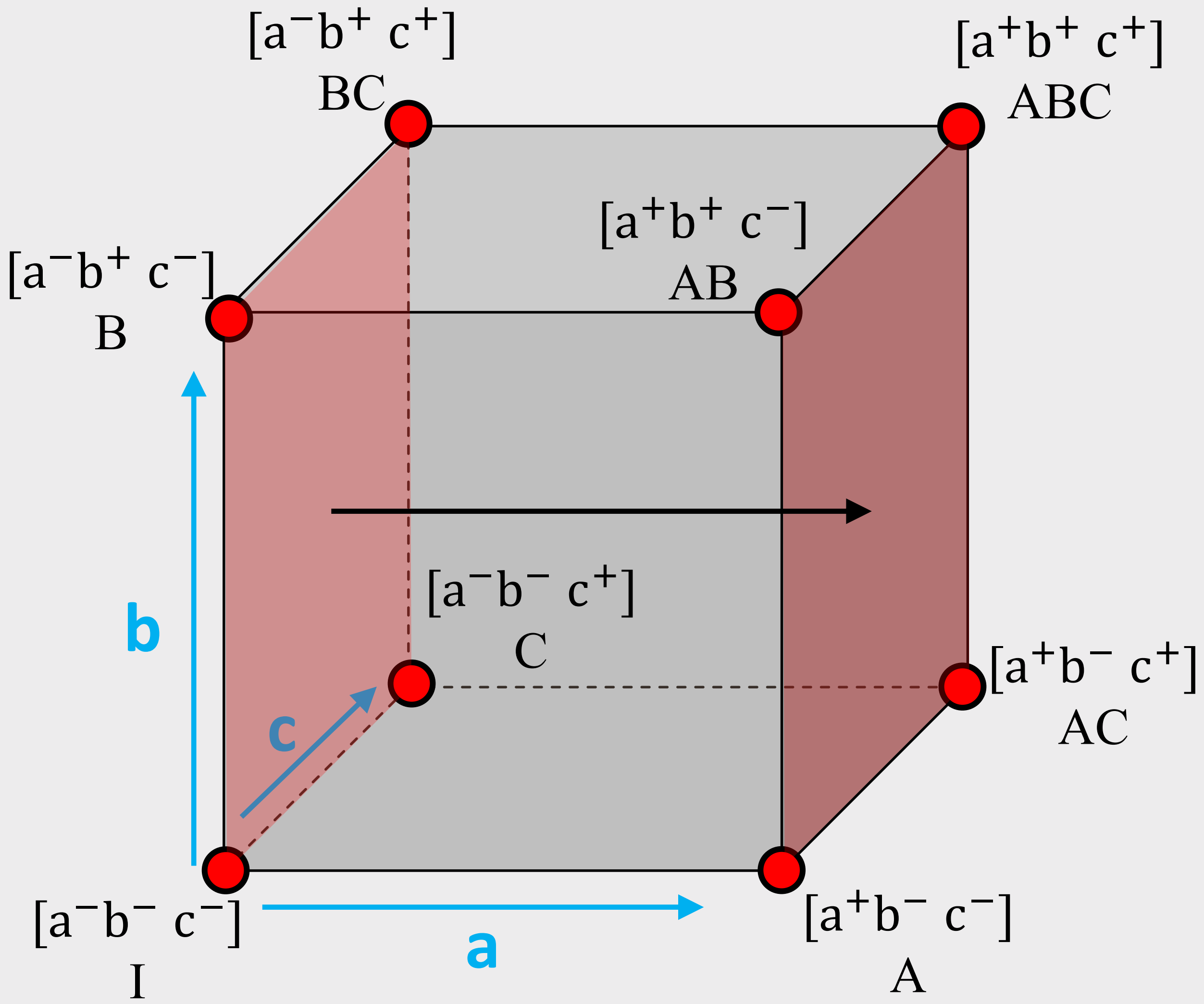
1. The covariance matrix is computed as $-\left[\nabla^2_{\boldsymbol{\beta}} \ln(\mathcal{L})\right]^{-1}$
2. Use the `matrix()` function to construct a matrix in *R*.
3. Use the `solve()` function to compute the inverse of a matrix in *R*.
4. Use the `diag()` function to get the numbers along the diagonal of a matrix in *R*.

Design of Experiments

Main Effects: Average change in the dependent variable associated with a change in an attribute level.

<u>Example</u>:

$$\mathrm{ME(a)} = \left(\frac{A + AB + AC + ABC}{4}\right) - \left(\frac{I + B + C + BC}{4}\right)$$



Main Effects: Average change in the dependent variable associated with a change in an attribute level.

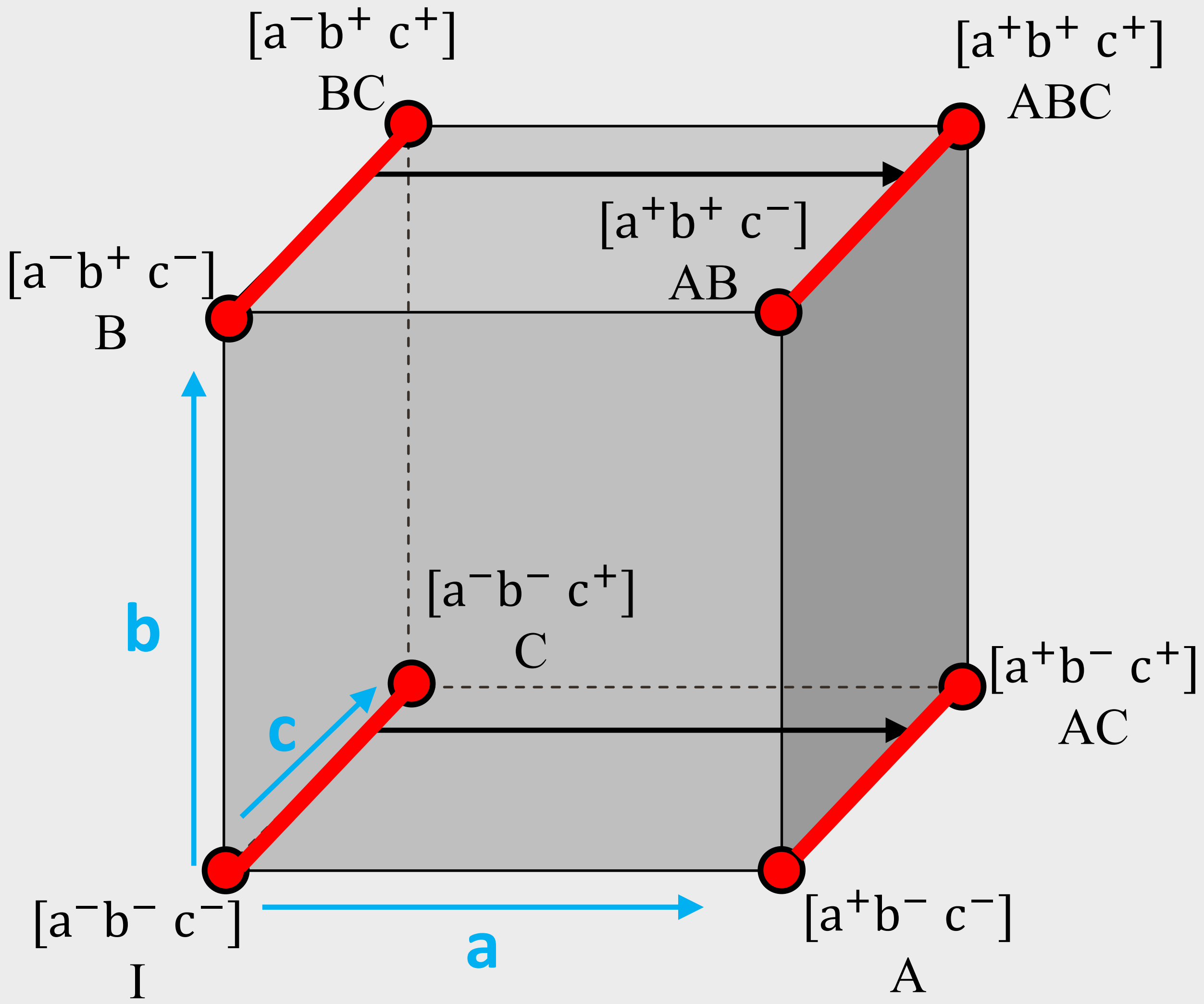
Example:

$$\mathrm{ME(a)} = \left(\frac{\mathrm{A + AB + AC + ABC}}{4}\right) - \left(\frac{\mathrm{I + B + C + BC}}{4}\right)$$

Interaction Effects: Difference in the main effect of one attribute based on the value of another attribute.

Example:

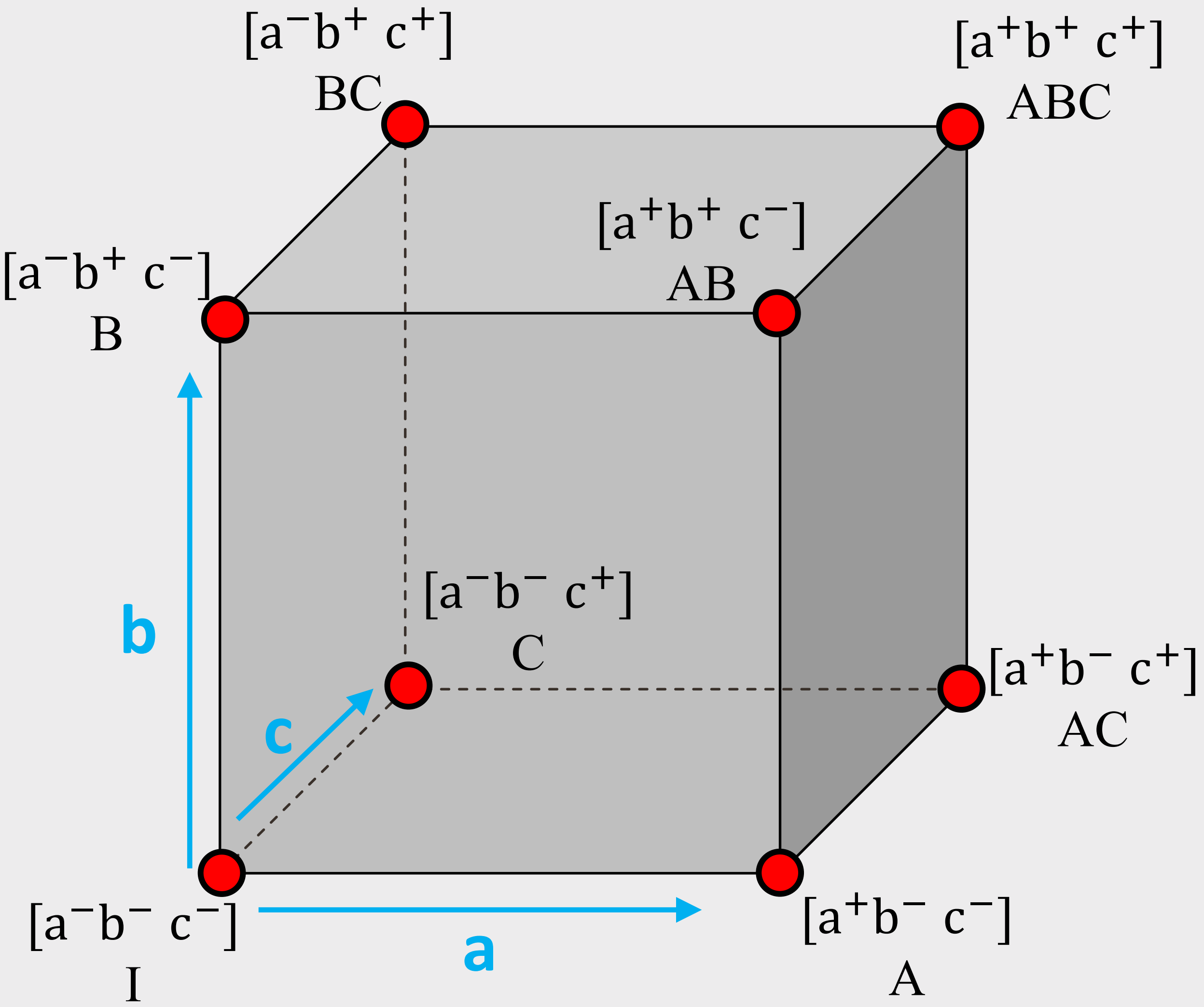
$$\mathrm{INT(ab)} = \frac{1}{2}\left[\left(\frac{\mathrm{AB + ABC}}{2}\right) - \left(\frac{\mathrm{B + BC}}{2}\right)\right] - \frac{1}{2}\left[\left(\frac{\mathrm{A + AC}}{2}\right) - \left(\frac{\mathrm{I + C}}{2}\right)\right]$$



Design of experiment affects amount of available information

Design: **Full Factorial**

a	b	c	Effect
-	-	-	I
+	-	-	A
-	+	-	B
-	-	+	C
+	+	-	AB
+	-	+	AC
-	+	+	BC
+	+	+	ABC



Balanced: For each attribute, all levels appear an equal number of times.

Orthogonal: For each pair of attributes, all pairs of levels appear together an equal number of times.

Fractional Factorial Designs

a	b	c	Effect
-	-	-	I
-	-	+	C
+	+	-	AB
+	+	+	ABC

Balanced? Yes

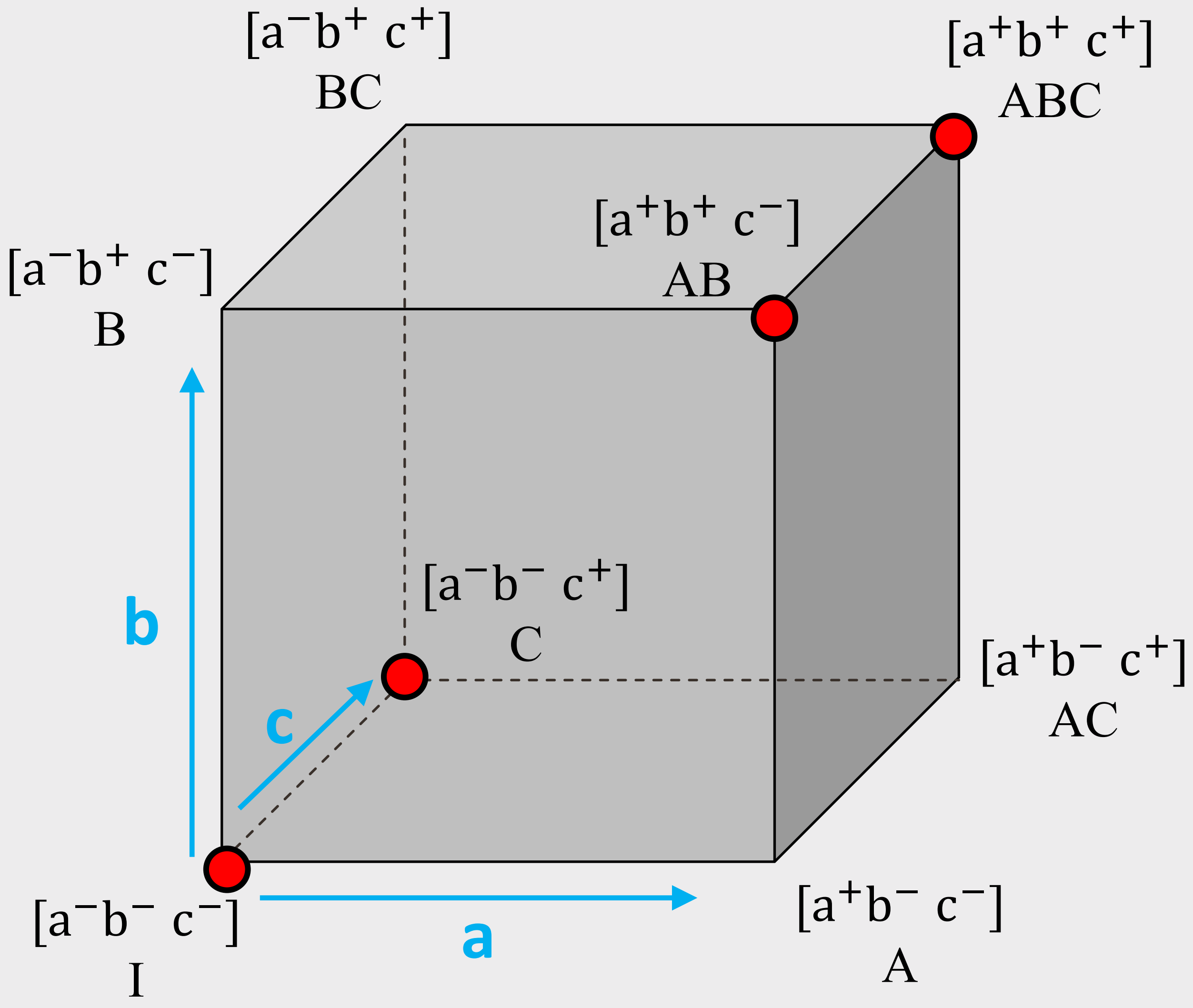
Orthogonal? No

Main effects of a and b are *confounded*

$$\mathrm{ME(a)} = \mathrm{ME(b)} = \left(\frac{AB + ABC}{2}\right) - \left(\frac{I + C}{2}\right)$$

To find other confounded effects, multiply by (a=b):

c(a=b)	ac = bc
b(a=b)	ab = I
ac(a=b)	c = abc

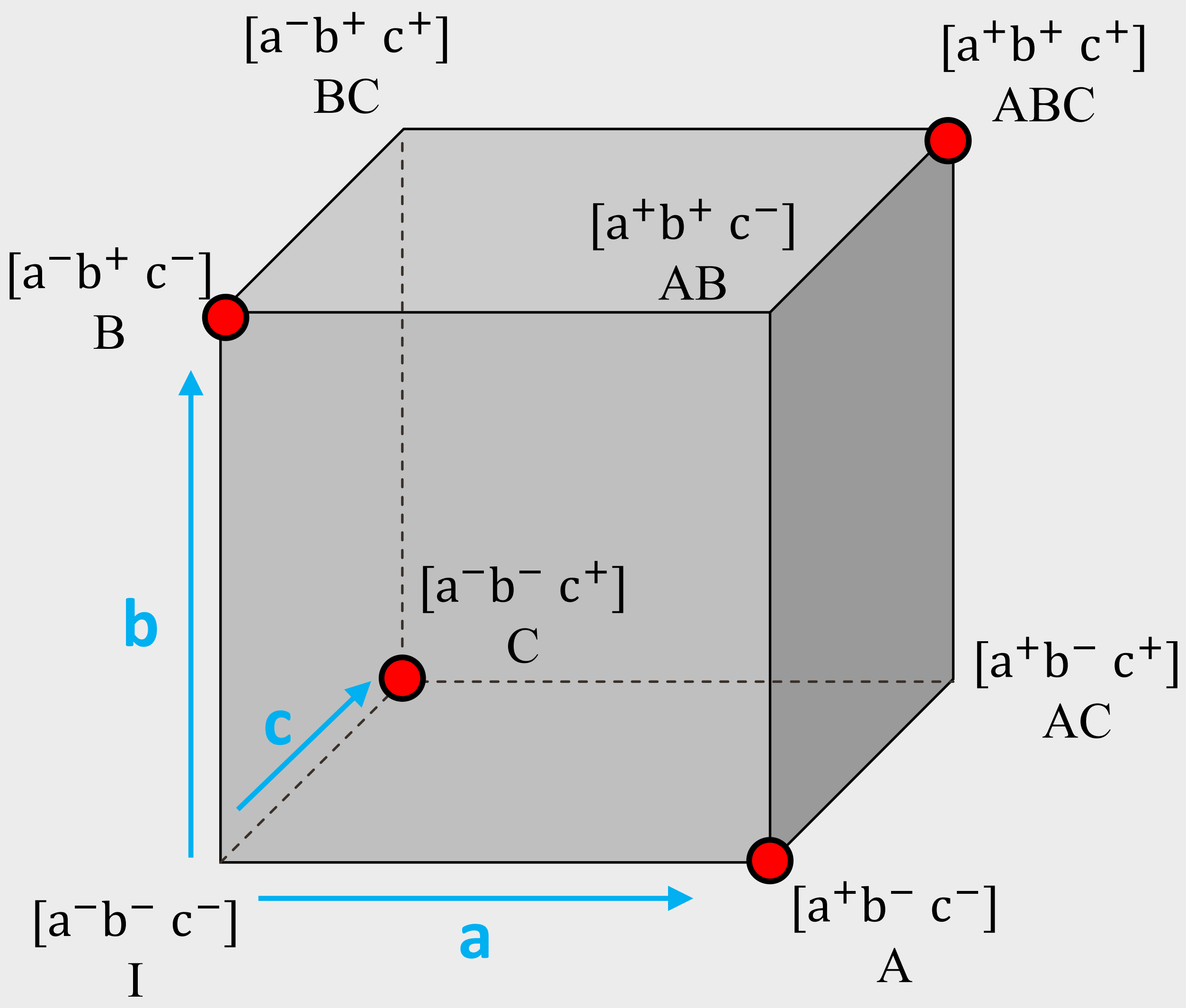


Fractional Factorial Designs

a	b	c	Effect
+	-	-	A
-	+	-	B
-	-	+	C
+	+	+	ABC

Balanced? Yes

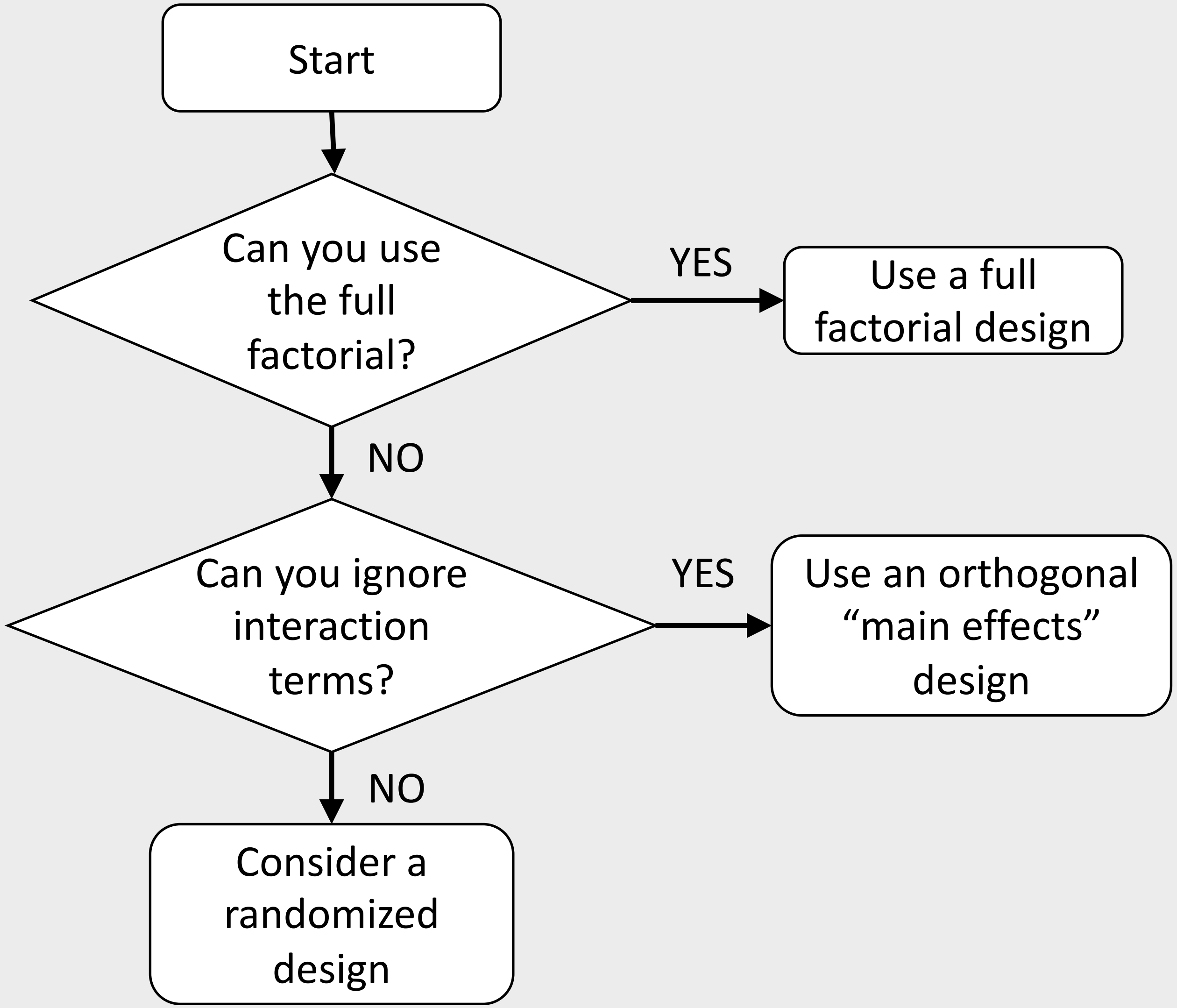
Orthogonal? Yes



None of the main effects are confounded, but each main effect is confounded with a two-way interaction:

a	bc
b	ac
c	ab
I	abc

Designing your experiment / conjoint survey



Practice Question 2

Consider the following experiment design:

a	b	c	Effect
+	-	-	A
-	+	-	B
+	-	+	AC
-	+	+	BC

a) Is the design balanced? Is it orthogonal?
b) Write out the equation to compute the main effect for a, b, and c.
c) Are any main effects confounded? If so, what are they confounded with?