



Week 8: *Optimization & MLE*

 EMSE 6035: Marketing Analytics for Design Decisions

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 October 18, 2023

Week 8: *Optimization & MLE*

1. Maximum likelihood estimation

2. Optimization (in general)

BREAK

3. Joins

4. Pilot data cleaning

Week 8: *Optimization & MLE*

1. Maximum likelihood estimation

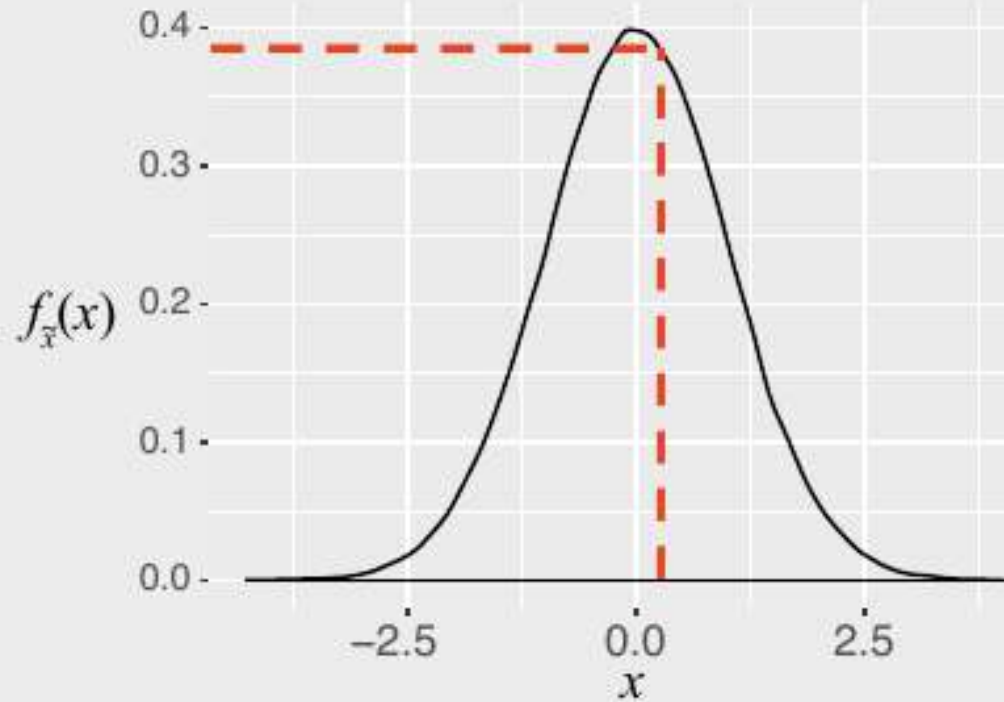
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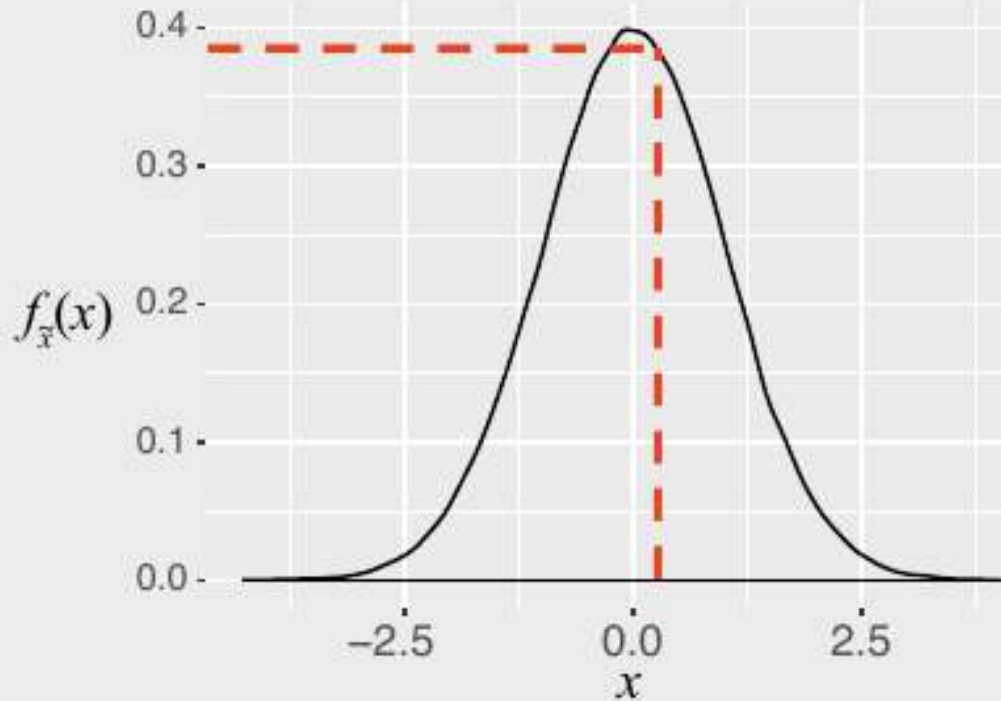
Computing the likelihood



x : an observation

$f(x)$: probability of observing x

Computing the likelihood



x : an observation

$f(x)$: probability of observing x

$\mathcal{L}(\theta|x)$: probability that θ are the true parameters, given that observed x

We want to estimate θ

We actually compute the *log*-likelihood
(converts multiplication to addition)

0.39	0.35	0.24	0.39	0.40	0.11	0.33	0.35	0.07	0.37
------	------	------	------	------	------	------	------	------	------

$$\mathcal{L}(\boldsymbol{\theta}|\mathbf{x}) = f_{\tilde{x}}(x_1) f_{\tilde{x}}(x_2) \dots f_{\tilde{x}}(x_n) = 1.63\text{e-}6$$

$$\log \mathcal{L}(\boldsymbol{\theta}|\mathbf{x}) = f_{\tilde{x}}(x_1) + f_{\tilde{x}}(x_2) + \dots + f_{\tilde{x}}(x_n) = 3$$

Practice Question 1

Observations - Height of students (inches):

```
#> [1] 65 69 66 67 68 72 68 69 63 70
```

a) Let's say we know that the height of students, \tilde{x} , in a classroom follows a normal distribution. A professor obtains the above height measurements students in her classroom. What is the log-likelihood that $\tilde{x} \sim \mathcal{N}(68, 4)$? In other words, compute $\ln \mathcal{L}(\mu = 68, \sigma = 4)$.

b) Compute the log-likelihood function using the same standard deviation ($\sigma = 4$) but with the following different values for the mean, μ : 66, 67, 68, 69, 70. How do the results compare? Which value for μ produces the highest log-likelihood?

Week 8: *Optimization & MLE*

1. Maximum likelihood estimation

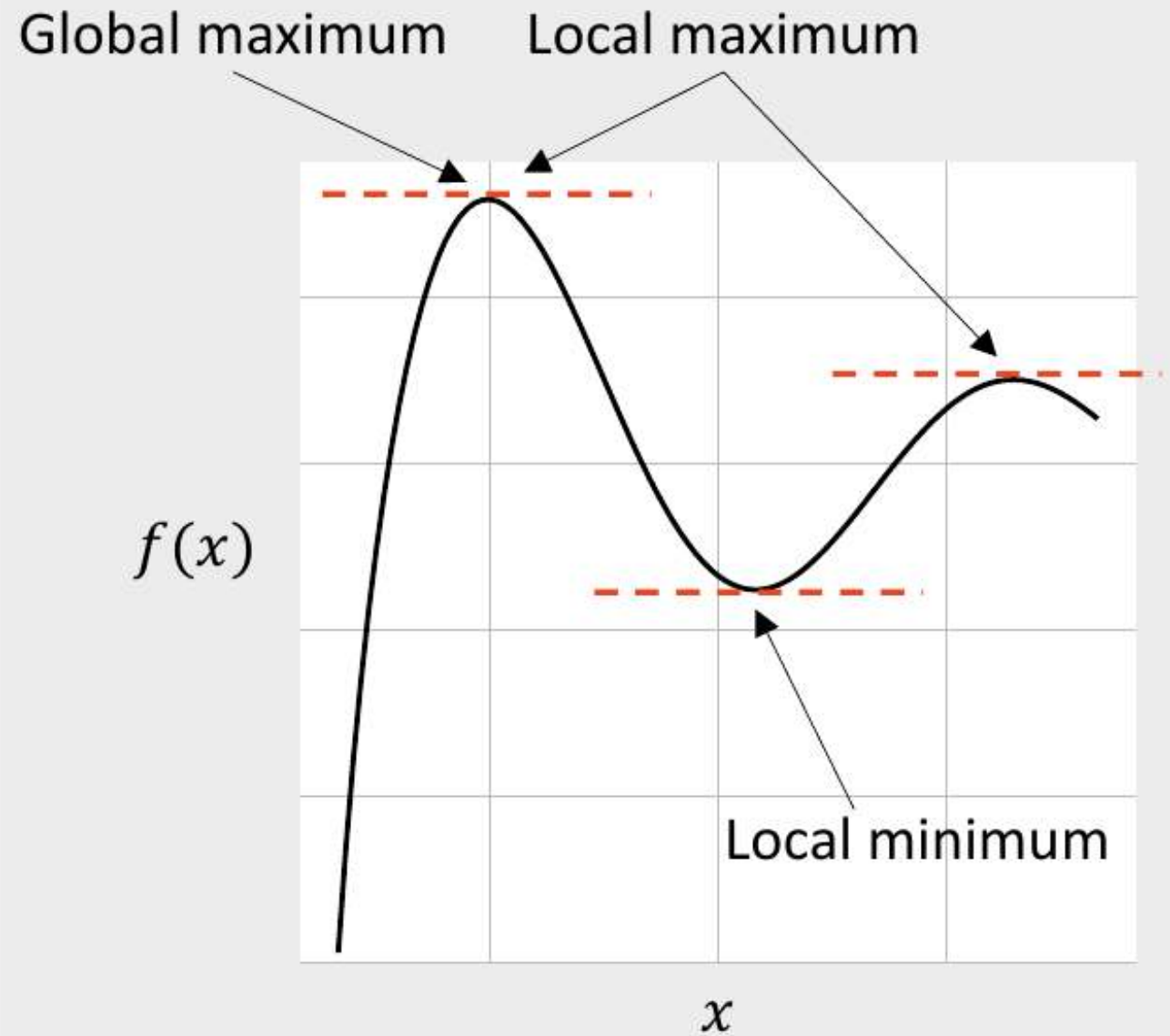
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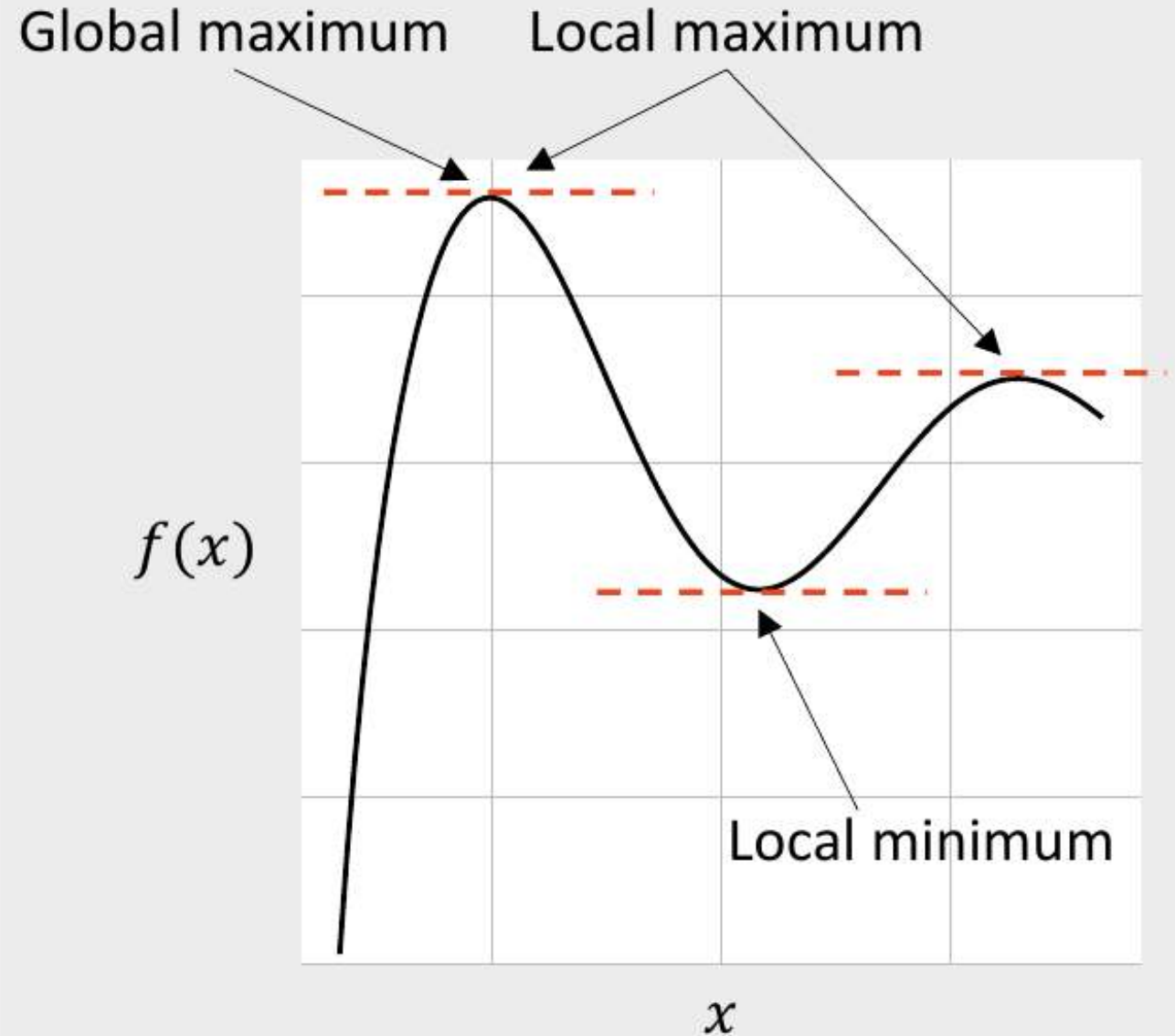
$$f(x)$$



First order necessary condition

x^* is a “stationary point” when

$$\frac{df(x^*)}{dx} = 0$$



First order necessary condition

x^* is a “stationary point” when

$$\frac{df(x^*)}{dx} = 0$$

Second order sufficiency condition

x^* is a local *maximum* when

$$\frac{d^2f(x^*)}{dx^2} < 0$$

x^* is a local *minimum* when

$$\frac{d^2f(x^*)}{dx^2} > 0$$

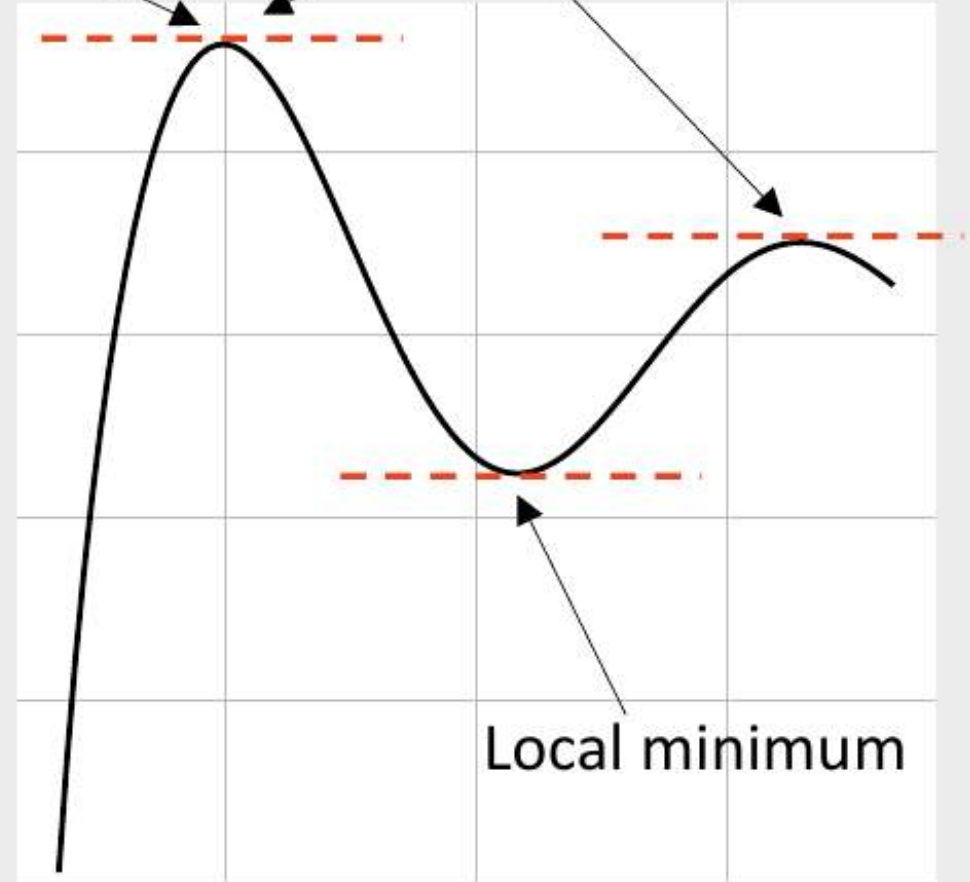
Global maximum

Local maximum

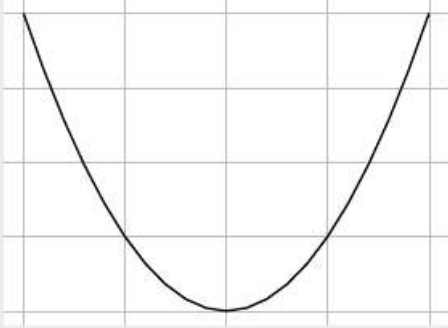
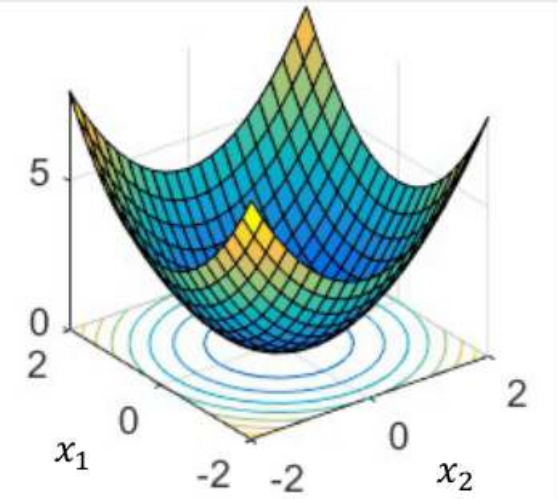
$f(x)$

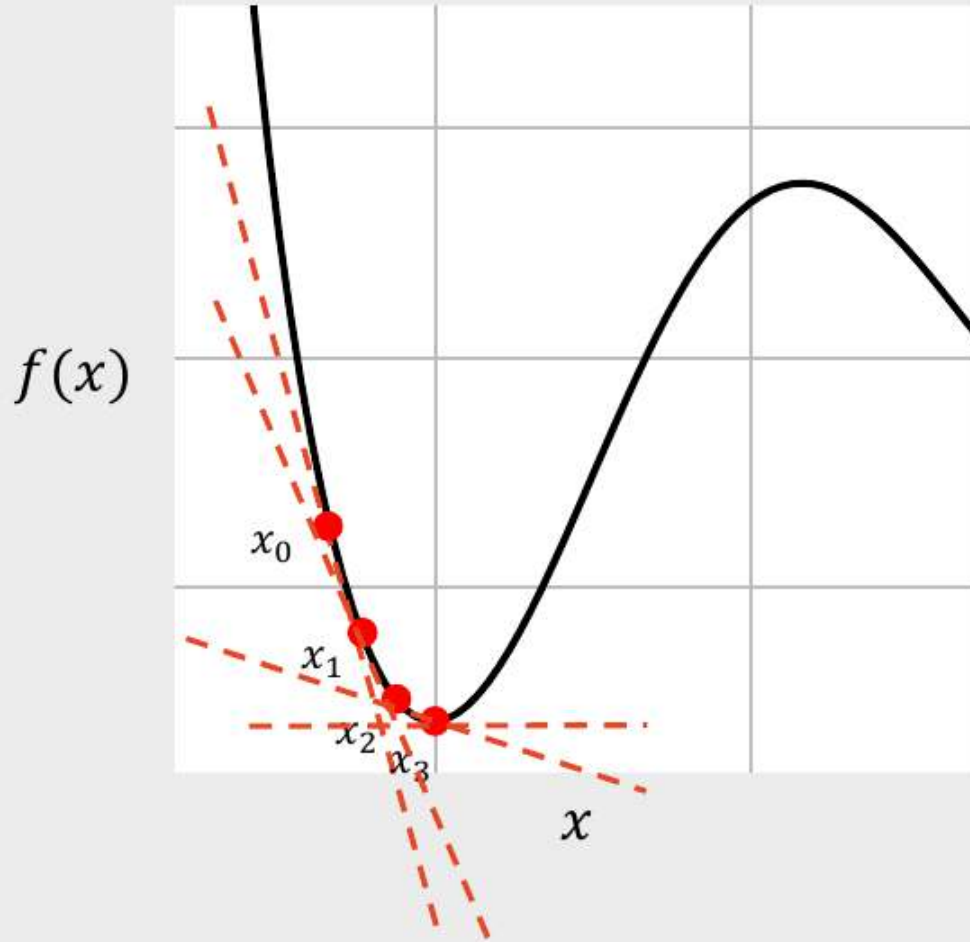
Local minimum

x



Optimality conditions for local **minimum**

Number of dimensions	First order condition	Second order condition	Example
One	$\frac{df(x^*)}{dx} = 0$	$\frac{d^2f(x^*)}{dx^2} > 0$	
Multiple	<p>“Gradient”</p> $\nabla f(x_1, x_2, \dots, x_n)$	<p>“Hessian”</p> $\nabla^2 f(x_1, x_2, \dots, x_n)$	
	$= \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right]$ $= [0, 0, \dots, 0]$	$= \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \dots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \dots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$ <p>Must be “positive definite”</p>	



Gradient Descent Method:

1. Choose a starting point, x_0
2. At that point, compute the gradient, $\nabla f(x_0)$
3. Compute the next point, with a step size γ :

$$x_{n+1} = x_n - \gamma \nabla f(x_n)$$

*Stop when $\nabla f(x_n) < \delta$ Very small number
or

*Stop when $(x_{n+1} - x_n) < \delta$

Practice Question 2

Consider the following function:

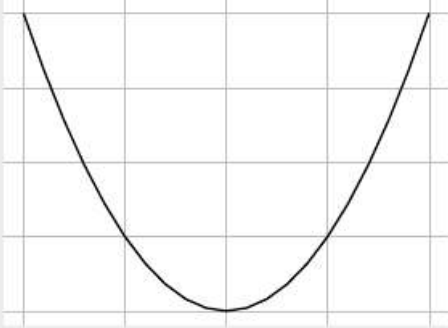
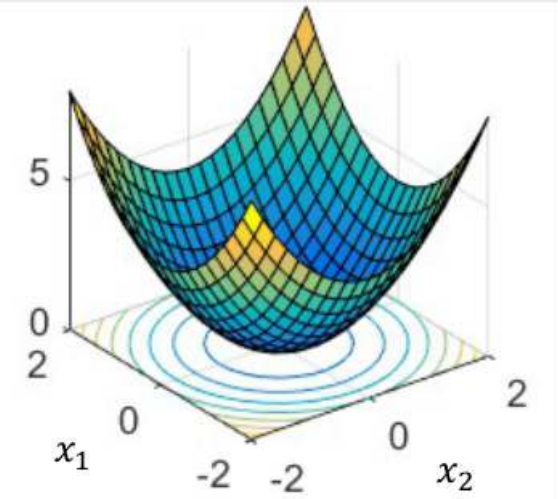
$$f(x) = x^2 - 6x$$

The gradient is:

$$\nabla f(x) = 2x - 6$$

Using the starting point $x = 1$ and the step size $\gamma = 0.3$, apply the gradient descent method to compute the next **three** points in the search algorithm.

Optimality conditions for local **minimum**

Number of dimensions	First order condition	Second order condition	Example
One	$\frac{df(x^*)}{dx} = 0$	$\frac{d^2f(x^*)}{dx^2} > 0$	
Multiple	<p>“Gradient”</p> $\nabla f(x_1, x_2, \dots, x_n)$	<p>“Hessian”</p> $\nabla^2 f(x_1, x_2, \dots, x_n)$	
	$= \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right]$ $= [0, 0, \dots, 0]$	$= \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \dots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \dots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$ <p>Must be “positive definite”</p>	

Practice Question 3

Consider the following function:

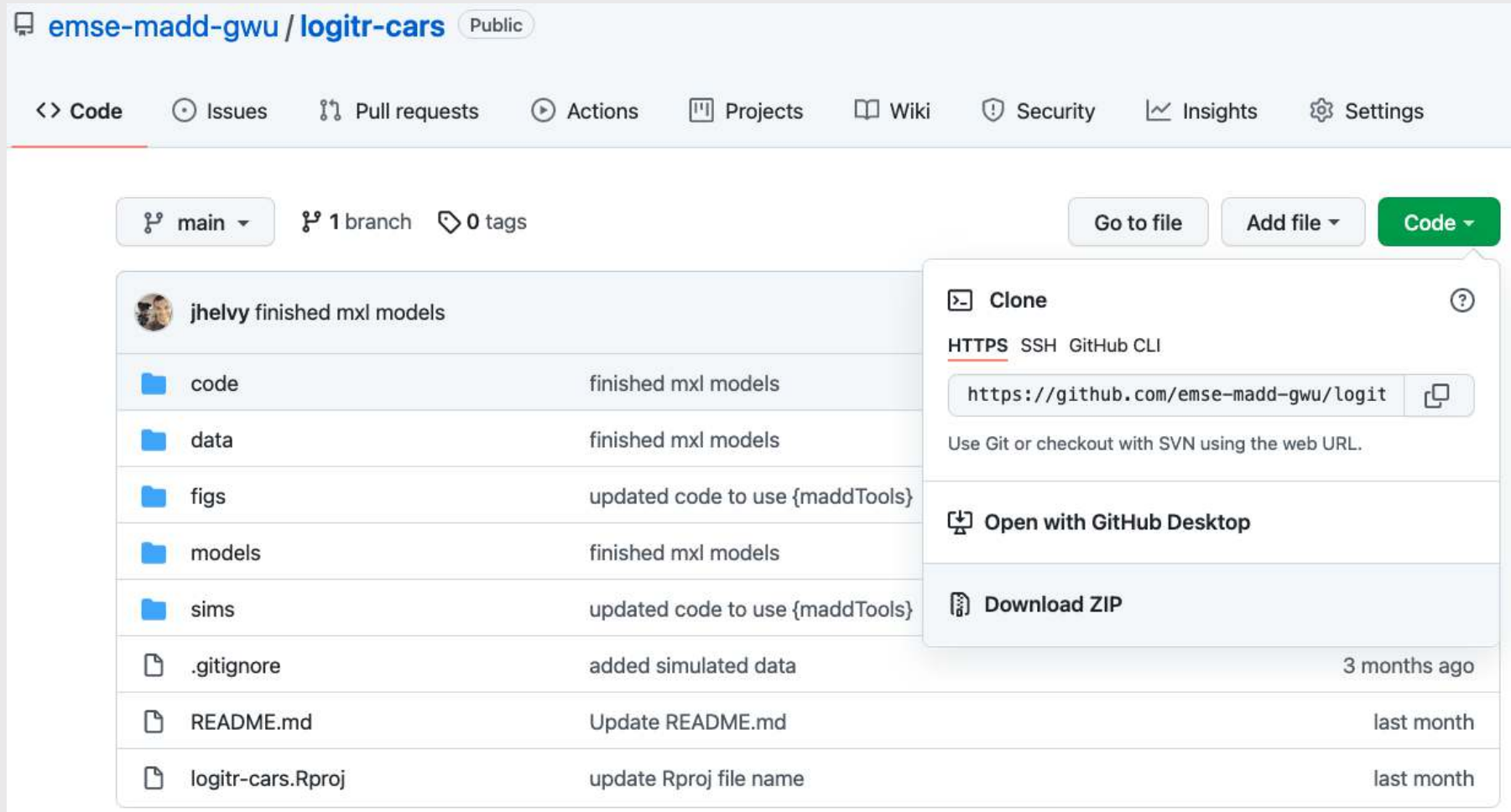
$$f(\underline{x}) = x_1^2 + 4x_2^2$$

The gradient is:

$$\nabla f(\underline{x}) = \begin{bmatrix} 2x_1 \\ 8x_2 \end{bmatrix}$$

Using the starting point $\underline{x}_0 = [1, 1]$ and the step size $\gamma = 0.15$, apply the gradient descent method to compute the next **three** points in the search algorithm.

Download the **logitr-cars** repo from GitHub



emse-madd-gwu / **logitr-cars** Public

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Go to file Add file Code

jhelvy finished mxl models

code	finished mxl models
data	finished mxl models
figs	updated code to use {maddTools}
models	finished mxl models
sims	updated code to use {maddTools}
.gitignore	added simulated data
README.md	Update README.md
logitr-cars.Rproj	update Rproj file name

Clone ?

HTTPS SSH GitHub CLI

<https://github.com/emse-madd-gwu/logitr-cars>

Use Git or checkout with SVN using the web URL.

Open with GitHub Desktop

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3 months ago

last month

last month

Estimating utility models

1. Open `logitr-cars.Rproj`
2. Open `code/3.1-model-mnl.R`

Maximum likelihood estimation

$$\begin{aligned}\tilde{u}_j &= v_j + \tilde{\varepsilon}_j \\ &= \beta_1 x_{j1} + \beta_2 x_{j2} + \dots + \tilde{\varepsilon}_j \\ &= \boldsymbol{\beta}' \mathbf{x}_j + \tilde{\varepsilon}_j\end{aligned}$$

Estimate $\boldsymbol{\beta} = [\beta_1, \beta_2, \dots, \beta_n]$
by maximizing the likelihood function

$$\begin{aligned}\text{minimize } -\log \mathcal{L} &= - \sum_{j=1}^J P_j (\boldsymbol{\beta} | \mathbf{x})^{y_j} \\ &\text{with respect to } \boldsymbol{\beta}\end{aligned}$$

$y_j = 1$ if alternative j was chosen

$y_j = 0$ if alternative j was not chosen

For logit model:

$$P_j = \frac{e^{v_j}}{\sum_{k=1}^J e^{v_k}} = \frac{e^{\boldsymbol{\beta}' \mathbf{x}_j}}{\sum_{k=1}^J e^{\boldsymbol{\beta}' \mathbf{x}_k}}$$

Break

05 : 00

Week 8: *Optimization & MLE*

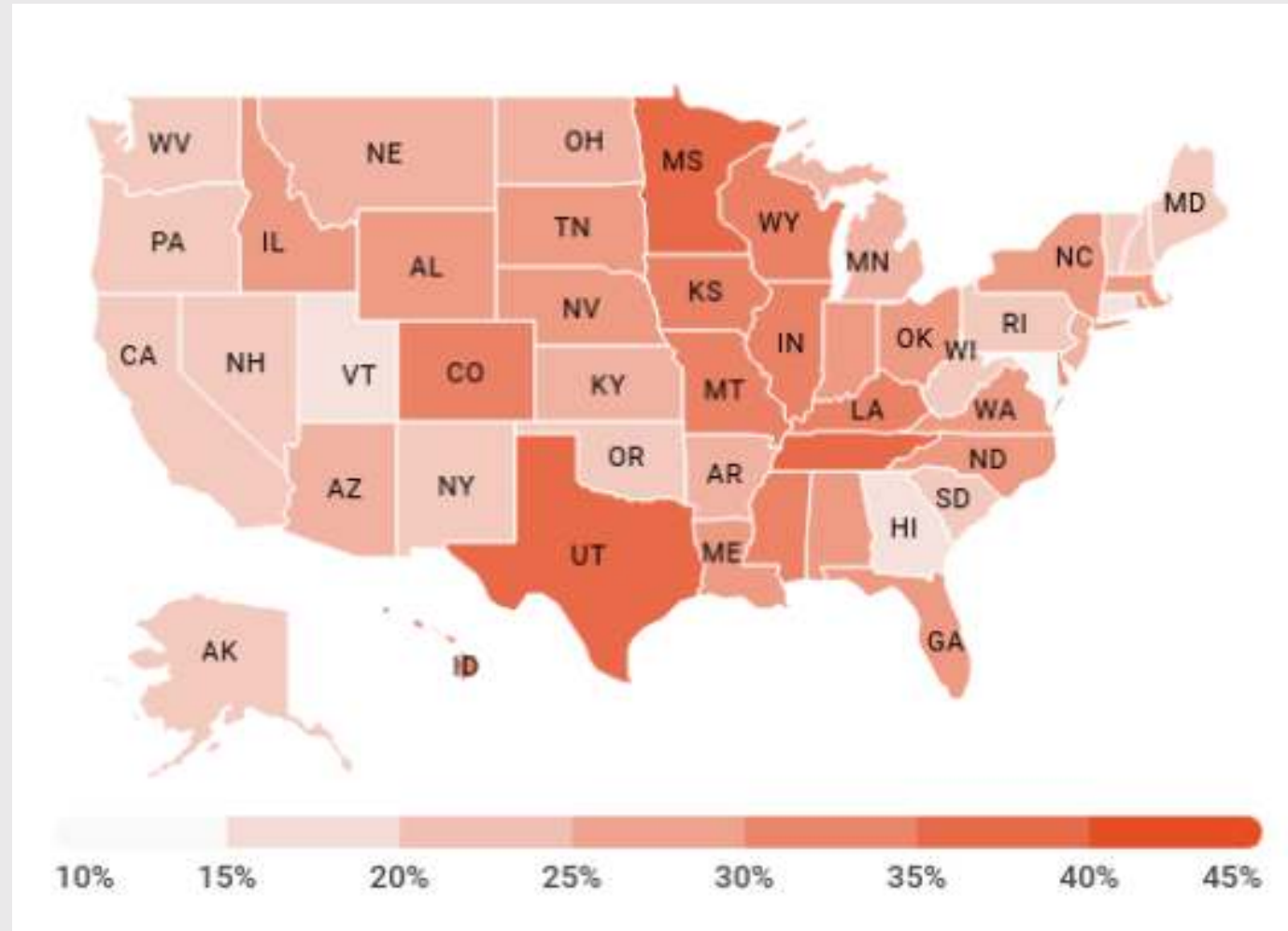
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BREAK

3. **Joins**

4. Pilot data cleaning

What's wrong with this map?



Likely culprit: Merging two columns

```
head(names)
```

```
#>           state_name
#> 1           Alabama
#> 2           Alaska
#> 3           Arizona
#> 4           Arkansas
#> 5 Armed Forces Africa
#> 6 Armed Forces Americas
```

```
head(abbs)
```

```
#> state_abb
#> 1      AA
#> 2      AE
#> 3      AE
#> 4      AE
#> 5      AE
#> 6      AK
```

```
result <- cbind(names, abbs)
head(result)
```

```
#>           state_name state_abb
#> 1           Alabama      AA
#> 2           Alaska      AE
#> 3           Arizona      AE
#> 4           Arkansas      AE
#> 5 Armed Forces Africa      AE
#> 6 Armed Forces Americas      AK
```

Joins

1. `inner_join()`
2. `left_join()` / `right_join()`
3. `full_join()`

Example: `band_members` & `band_instruments`

`band_members`

```
#> # A tibble: 3 × 2
#>   name band
#>   <chr> <chr>
#> 1 Mick  Stones
#> 2 John  Beatles
#> 3 Paul  Beatles
```

`band_instruments`

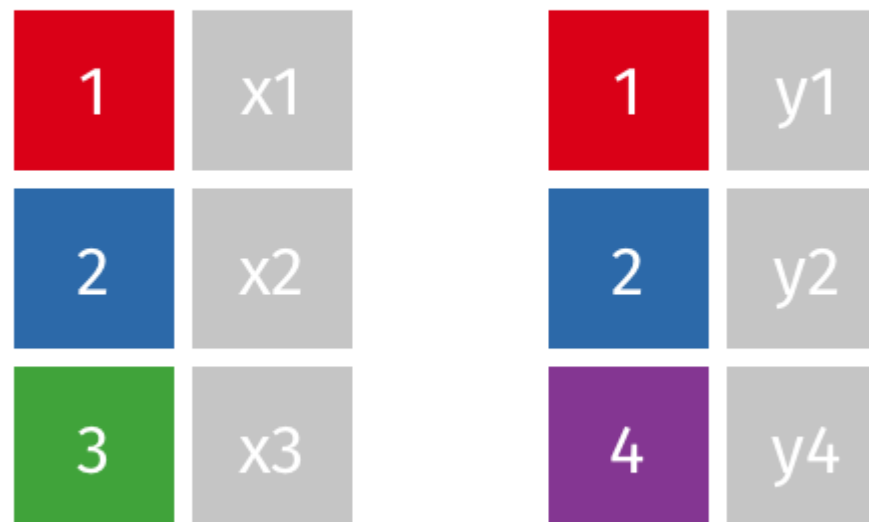
```
#> # A tibble: 3 × 2
#>   name plays
#>   <chr> <chr>
#> 1 John  guitar
#> 2 Paul  bass
#> 3 Keith guitar
```


inner_join()

```
band_members %>%  
  inner_join(band_instruments)
```

```
#> # A tibble: 2 × 3  
#>   name band    plays  
#>   <chr> <chr>  <chr>  
#> 1 John  Beatles guitar  
#> 2 Paul  Beatles  bass
```

inner_join(x, y)



full_join()

```
band_members %>%  
  full_join(band_instruments)
```

```
#> # A tibble: 4 × 3  
#>   name  band    plays  
#>   <chr> <chr>   <chr>  
#> 1 Mick  Stones <NA>  
#> 2 John  Beatles guitar  
#> 3 Paul  Beatles bass  
#> 4 Keith <NA>    guitar
```

full_join(x, y)

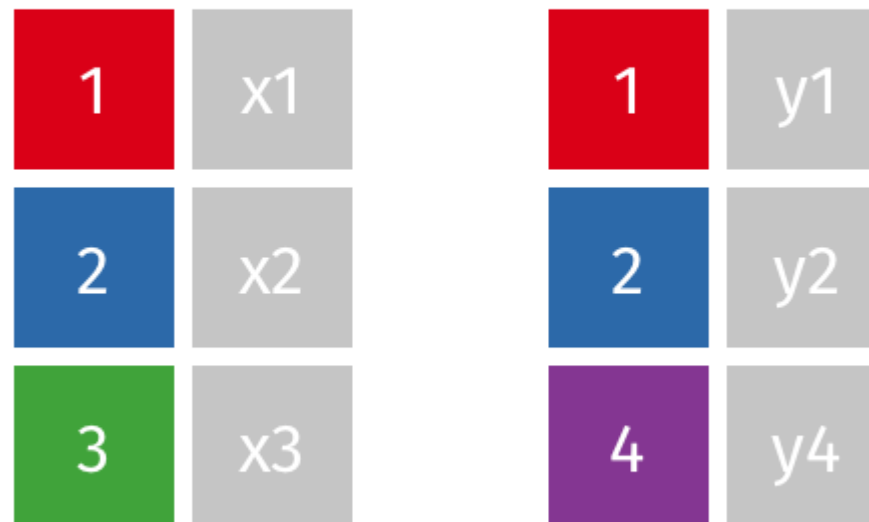
1	x1	1	y1
2	x2	2	y2
3	x3	4	y4

left_join()

```
band_members %>%  
  left_join(band_instruments)
```

```
#> # A tibble: 3 × 3  
#>   name  band  plays  
#>   <chr> <chr> <chr>  
#> 1 Mick  Stones <NA>  
#> 2 John  Beatles guitar  
#> 3 Paul  Beatles bass
```

left_join(x, y)

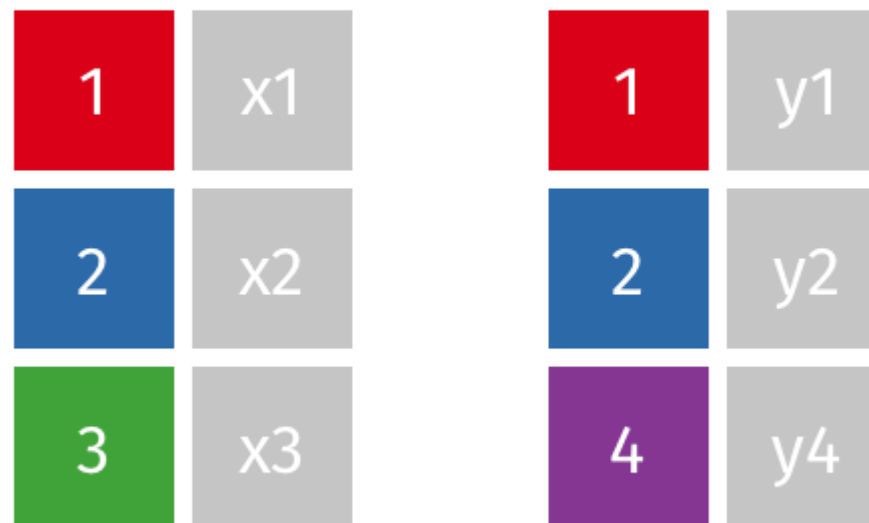


right_join()

```
band_members %>%  
  right_join(band_instruments)
```

```
#> # A tibble: 3 × 3  
#>   name  band  plays  
#>   <chr> <chr> <chr>  
#> 1 John  Beatles guitar  
#> 2 Paul  Beatles bass  
#> 3 Keith <NA>   guitar
```

right_join(x, y)



Specify the joining variable name

```
band_members %>%  
  left_join(band_instruments)
```

```
#> Joining with `by = join_by(name)`
```

```
#> # A tibble: 3 × 3  
#>   name band    plays  
#>   <chr> <chr>   <chr>  
#> 1 Mick  Stones  <NA>  
#> 2 John  Beatles guitar  
#> 3 Paul  Beatles bass
```

```
band_members %>%  
  left_join(band_instruments,  
            by = 'name')
```

```
#> # A tibble: 3 × 3  
#>   name band    plays  
#>   <chr> <chr>   <chr>  
#> 1 Mick  Stones  <NA>  
#> 2 John  Beatles guitar  
#> 3 Paul  Beatles bass
```

Specify the joining variable name

If the names differ, use `by = c("left_name" = "joining_name")`

```
band_members
```

```
#> # A tibble: 3 × 2  
#>   name band  
#>   <chr> <chr>  
#> 1 Mick  Stones  
#> 2 John  Beatles  
#> 3 Paul  Beatles
```

```
band_instruments2
```

```
#> # A tibble: 3 × 2  
#>   artist plays  
#>   <chr>   <chr>  
#> 1 John   guitar  
#> 2 Paul   bass  
#> 3 Keith  guitar
```

```
band_members %>%  
  left_join(band_instruments2,  
            by = c("name" = "artist"))
```

```
#> # A tibble: 3 × 3  
#>   name band   plays  
#>   <chr> <chr>   <chr>  
#> 1 Mick  Stones  <NA>  
#> 2 John  Beatles guitar  
#> 3 Paul  Beatles bass
```

Specify the joining variable name

Or just rename the joining variable in a pipe

```
band_members
```

```
#> # A tibble: 3 × 2  
#>   name band  
#>   <chr> <chr>  
#> 1 Mick  Stones  
#> 2 John  Beatles  
#> 3 Paul  Beatles
```

```
band_instruments2
```

```
#> # A tibble: 3 × 2  
#>   artist plays  
#>   <chr>   <chr>  
#> 1 John   guitar  
#> 2 Paul   bass  
#> 3 Keith  guitar
```

```
band_members %>%  
  rename(artist = name) %>%  
  left_join(band_instruments2,  
            by = "artist")
```

```
#> # A tibble: 3 × 3  
#>   artist band    plays  
#>   <chr>   <chr>   <chr>  
#> 1 Mick   Stones  <NA>  
#> 2 John   Beatles guitar  
#> 3 Paul   Beatles bass
```

Your turn

10:00

1. Write code to read in the `state_abbs.csv` and `state_regions.csv` data files in the "data" folder.
2. Create a new data frame called `states` by joining the two data frames `states_abbs` and `state_regions` together. The result should be a data frame with variables `region`, `name`, `abb`.

Your result should look like this:

```
head(states)
```

```
#> # A tibble: 6 × 3  
#>   region    name    abb  
#>   <chr>    <chr>   <chr>  
#> 1 Northeast Maine     ME  
#> 2 Northeast New Hampshire NH  
#> 3 Northeast Vermont    VT  
#> 4 Northeast Massachusetts MA  
#> 5 Northeast Rhode Island RI  
#> 6 Northeast Connecticut CT
```


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BREAK

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Download the **formr4conjoint** repo from GitHub

jhelvy / **formr4conjoint** Public

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jhelvy added package installs to readme

figs	added package installs to readme
survey	added consent form content in p1
.gitignore	Update .gitignore
LICENSE.md	Create LICENSE.md
README.Rmd	added package installs to readme
README.md	added package installs to readme 20 minutes ago
formr4conjoint.Rproj	Init 2 years ago

Clone ?

HTTPS SSH GitHub CLI

<https://github.com/jhelvy/formr4conjoint>

Use Git or checkout with SVN using the web URL.

Open with GitHub Desktop

Download ZIP

Cleaning formr survey data

1. Open `formr4conjoint.Rproj`
2. Open `code/data_cleaning.R`

Team time

For the rest of class, work with your team mates to start importing and cleaning your pilot survey data