3.4. Involutions and non-singular conics

As we have seen that conics are parametrizable by projective lines, the Möbius transformations may have some nice representations on them. One example is given by the projection from points outside the conic.

Theorem 3.1 (Frégier)**.** *Given a point p outside a conic $C \subset \mathbb{P}^2$, then for every point x on C, the residual intersection x' of C with the line (px) defines an involution on C. Conversely, every involution occurs in this way.*

Proof. That p does define an involution is clear. (It remains to show that this involution is indeed a Möbius transformation.) Conversely given two non-homologous points of some involution ι on C, they define two lines meeting in a point p. The involution defined by the projection from p coincides with ι on four points, hence it has to be identical with ι. □

We call the point p in Theorem **??** the Fregier point of the involution. Note that this result explains Exercise 3.5.2. The two fixed points of an involution now have a nice geometric interpretation: they simply correspond to the two tangency points of the two tangents to C through its Fregier point p. We can then state the following corollary.

Corollary 3.2. *The two points of intersection of a conic with a line through a point p are conjugate with respect to the two tangency points of the two tangents to C from p.*

Proof. Apply Frégier's Theorem and (c) of Proposition 3.2. □

Of course, all of these must be suitably modified in the case of chararteristic 2 (see Exercise 3.5.3).

Two involutions have exactly one homologous pair of points in common. This common pair is nothing but the two fixed points of their product (see Exercise 3.5.4). Geometrically, we get it by intersecting C with the line joining the two corresponding Fregier points. This also gives a geometric way of factoring an arbitrary Möbius transformation into a product of two involutions. In fact a Möbius transformation M is determined by its two fixed points and a pair (z, Mz). The fixed points may or may not be defined over the field, but the line F joining them is! Now, we may choose any point p on F as the Fregier point of one of the factors; the other factor—its Frégier point q—is constructed accordingly (see Fig. 16):

- join p with z, and let z' be the residual intersection
- form the line $(z'\,Mz)$ and let q be its intersection with F.

Automorphisms of the projective line can also be viewed externally. More specifically, let $\mathscr{L}$ and $\mathscr{M}$ be two lines meeting in A, and let O be a point outside the lines. Then O sets up a one-to-one correspondence between $\mathscr{L}$ and $\mathscr{M}$ through a perspectivity, i.e. the points L and M on $\mathscr{L}$ and $\mathscr{M}$ respectively are associated if L, M, and O are collinear.