# Having Fun with Lambert $\mathrm{W}(x)$ Function

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**Abstract**

This short note presents the Lambert $\mathrm{W}(x)$ function and its possible application in the framework of physics related to the Pierre Auger Observatory. The actual numerical implementation in C++ consists of Halley's and Fritsch's iteration with branch-point expansion, asymptotic series and rational fits as initial approximations.

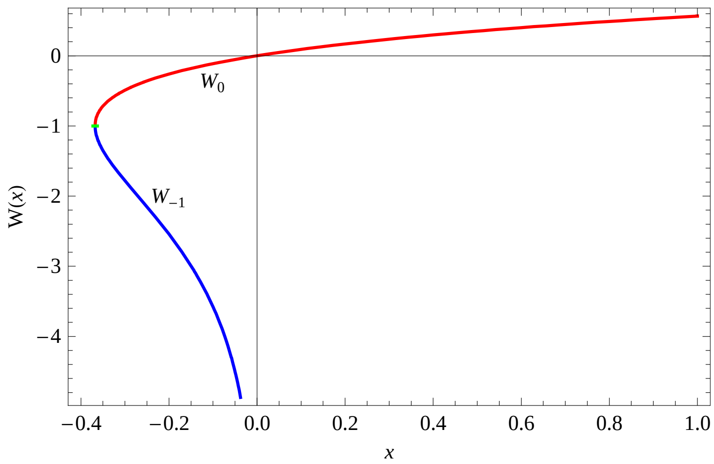


**Figure 1:** The two branches of the Lambert W function, $\mathrm{W}_{-1}(x)$ in blue and $W_0(x)$ in red. The branching point at $(-e^{-1}, -1)$ is denoted with a green dash.

# 1 Introduction

The Lambert $\mathrm{W}(x)$ function is defined as the inverse function of

$$y \exp y = x, \tag{1}$$

the solution being given by

$$y = \mathrm{W}(x), \tag{2}$$

or shortly

$$\mathrm{W}(x) \exp \mathrm{W}(x) = x. \tag{3}$$

Since the $x \mapsto x \exp x$ mapping is not injective, no unique inverse of the $x \exp x$ function exists. As can be seen in Fig. 1, the Lambert function has two real branches with a branching point located at $(-e^{-1}, -1)$. The bottom branch, $\mathrm{W}_{-1}(x)$, is defined in the interval $x \in [-e^{-1}, 0]$ and has a negative singularity for $x \to 0^-$. The upper branch is defined for $x \in [-e^{-1}, \infty]$.

The earliest mention of problem of Eq. (1) is attributed to Euler. However, Euler himself credited Lambert for his previous work in this subject. The $\mathrm{W}(x)$ function started to be named after Lambert only recently, in the last 10 years or so. The letter W was chosen by the first implementation of the $\mathrm{W}(x)$ function in the Maple computer software.

Recently, the $\mathrm{W}(x)$ function amassed quite a following in the mathematical community. Its most faithful proponents are suggesting to elevate it among the present set of elementary functions, such as $\sin(x)$, $\cos(x)$, $\ln(x)$, etc. The main argument for doing so is the fact that it is the root of the simplest exponential polynomial function.

While the Lambert W function is simply called W in the mathematics software tool *Maple*, in the *Mathematica* computer algebra framework this function is implemented under the name `ProductLog` (in the recent versions an alias `LambertW` is also supported).

There are numerous, well documented applications of $\mathrm{W}(x)$ in mathematics, physics, and computer science [1, 3]. Here we will give two examples that arise from the physics related to the Pierre Auger Observatory.

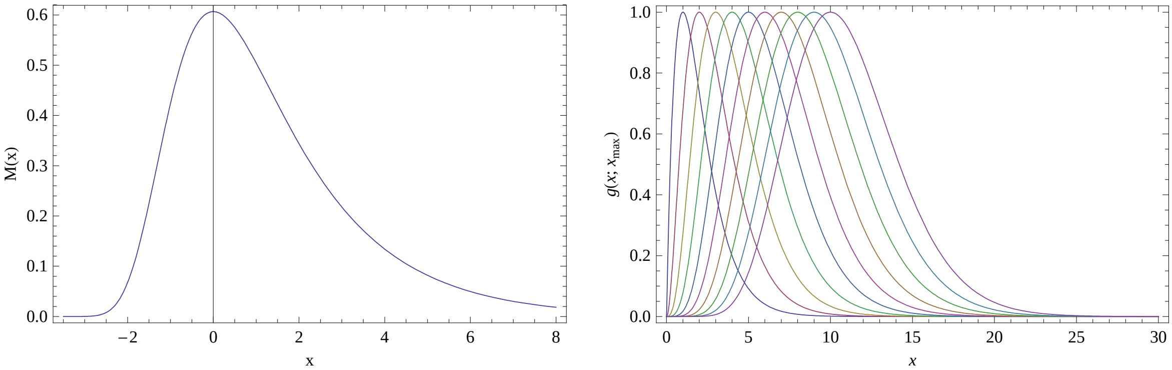


**Figure 2:** *Left:* The Moyal function $\mathrm{M}(x)$. *Right:* A family of one-parametric Gaisser-Hillas functions $g(x;\, x_{\text{max}})$ for $x_{\text{max}}$ in the range from 1 to 10 with step 1.

### 1.1 Moyal function

Moyal function is defined as

$$\mathrm{M}(x) = \exp\left(-\tfrac{1}{2}\left(x + \exp(-x)\right)\right). \tag{4}$$

Its inverse can be written in terms of the two branches of the Lambert W function,

$$\mathrm{M}_{\pm}^{-1}(x) = \mathrm{W}_{0,-1}(-x^2) - 2\ln x. \tag{5}$$

and can be seen in Fig. 2 (left).

Within the event reconstruction of the data taken by the Pierre Auger Observatory, the Moyal function is used for phenomenological recovery of the saturated signals from the photomultipliers.

### 1.2 Gaisser-Hillas function

In astrophysics the Gaisser-Hillas function is used to model the longitudinal particle density in a cosmic-ray air showers [4]. We can show that the inverse of the three-parametric Gaisser-Hillas function,

$$G(X;\, X_0, X_{\text{max}}, \lambda) = \left[\frac{X - X_0}{X_{\text{max}} - X_0}\right]^{\frac{X_{\text{max}} - X_0}{\lambda}} \exp\left(\frac{X_{\text{max}} - X}{\lambda}\right), \tag{6}$$

is intimately related to the Lambert W function. Using rescale substitutions,

$$x = \frac{X - X_0}{\lambda} \quad \text{and} \tag{7}$$

$$x_{\text{max}} = \frac{X_{\text{max}} - X_0}{\lambda}, \tag{8}$$

the Gaisser-Hillas function is modified into a function of one parameter only,

$$g(x;\, x_{\text{max}}) = \left[\frac{x}{x_{\text{max}}}\right]^{x_{\text{max}}} \exp(x_{\text{max}} - x). \tag{9}$$

The family of one-parametric Gaisser-Hillas functions is shown in Fig. 2 (right). The problem of finding an inverse,

$$g(x; x_{\max}) \equiv a \tag{10}$$

for $0 < a \leqslant 1$, can be rewritten into

$$-\frac{x}{x_{\max}} \exp\left(-\frac{x}{x_{\max}}\right) = -a^{1/x_{\max}} e^{-1}. \tag{11}$$

According to the definition (1), the two (real) solutions for $x$ are obtained from the two branches of the Lambert W function,

$$x_{1,2} = -x_{\max} \mathrm{W}_{0,-1}(-a^{1/x_{\max}} e^{-1}) = -x_{\max} \mathrm{W}_{0,-1}(-\sqrt[x_{\max}]{a}/e). \tag{12}$$

Note that the branch $-1$ or $0$ simply chooses the right or left side relative to the maximum, respectively.

## 2 Numerics

Before moving to the actual implementation let us review some of the possible nimerical and analytical approaches.

### 2.1 Recursion

For $x > 0$ and $\mathrm{W}(x) > 0$ we can take the natural logarithm of (3) and rearrange it,

$$\mathrm{W}(x) = \ln x - \ln \mathrm{W}(x). \tag{13}$$

It is clear, that a possible analytical expression for $\mathrm{W}(x)$ exhibits a degree of self similarity. The $\mathrm{W}(x)$ function has multiple branches in the complex domain. Due to the $x > 0$ and $\mathrm{W}(x) > 0$ conditions, the Eq. (13) represents the positive part of the $\mathrm{W}_0(x)$ principal branch, but as it turns out, in this form it is suitable for evaluation when $\mathrm{W}_0(x) > 1$, i.e. when $x > e$.

Unrolling the self-similarity (13) as a recursive relation, one obtains the following curious expression for $\mathrm{W}_0(x)$,

$$\mathrm{W}_0(x) = \ln x - \ln(\ln x - \ln(\ln x - \ldots)), \tag{14}$$

or in a shorthand of a continued logarithm,

$$\mathrm{W}_0(x) = \ln \frac{x}{\ln \frac{x}{\ln \frac{x}{\cdots}}}. \tag{15}$$

The above expression is clearly a form of successive approximation, the final result given by the limit, when it exists.

For $x < 0$ and $\mathrm{W}(x) < 0$ we can multiply both sides of Eq. (3) with $-1$, take logarithm, and rewrite it to get a similar expansion for the $\mathrm{W}_{-1}(x)$ branch,

$$\mathrm{W}(x) = \ln(-x) - \ln(-\mathrm{W}(x)). \tag{16}$$

Again, this leads to a similar recursive expression,

$$\mathrm{W}_{-1}(x) = \ln(-x) - \ln(-(\ln(-x) - \ln(-(\ln(-x) - \ldots)))), \tag{17}$$

or as a continued logarithm,

$$W_{-1}(x) = \ln \frac{-x}{-\ln \frac{-x}{-\ln \frac{-x}{\cdots}}}. \tag{18}$$

For this continued logarithm we will use the symbol $R^{[n]}_{-1}(x)$ where $n$ denotes the depth of the recursion.

Starting from yet another rearrangement of Eq. (3),

$$W(x) = \frac{x}{\exp W(x)}, \tag{19}$$

we can obtain a recursion relation for the $-e^{-1} < x < e$ part of the principal branch $W_0(x) < 1$,

$$W_0(x) = \frac{x}{\exp \frac{x}{\exp \frac{x}{\cdots}}}. \tag{20}$$

### 2.2 Halley's iteration

We can apply Halley's root-finding method [8] to derive an iteration scheme for $f(y) = W(y) - x$ by writing the second-order Taylor series

$$f(y) = f(y_n) + f'(y_n)\,(y - y_n) + \tfrac{1}{2} f''(y_n)\,(y - y_n)^2 + \cdots \tag{21}$$

Since root $y$ of $f(y)$ satisfies $f(y) = 0$ we can approximate the left-hand side of Eq. (21) with 0 and replace $y$ with $y_{n+1}$. Rewriting the obtained result into

$$y_{n+1} = y_n - \frac{f(y_n)}{f'(y_n) + \frac{1}{2} f''(y_n)\,(y_{n+1} - y_n)} \tag{22}$$

and using Newton's method $y_{n+1} - y_n = -f(y_n)/f''(y_n)$ on the last bracket, we arrive at the expression for the Halley's iteration for Lambert function

$$W_{n+1} = W_n + \frac{t_n}{t_n\, s_n - u_n}, \tag{23}$$

where

$$t_n = W_n \exp W_n - x, \tag{24}$$

$$s_n = \frac{W_n + 2}{2(W_n + 1)}, \tag{25}$$

$$u_n = (W_n + 1) \exp W_n. \tag{26}$$

This method is of the third order, i.e. having $W_n = W(x) + \mathcal{O}(\varepsilon)$ will give $W_{n+1} = W(x) + \mathcal{O}(\varepsilon^3)$. Supplying this iteration with sufficiently accurate first approximation of the order of $\mathcal{O}(10^{-4})$ will thus give a machine-size floating point precission $\mathcal{O}(10^{-16})$ in at least two iterations.

### 2.3 Fritsch's iteration

For both branches of Lambert function a more efficient iteration scheme exists [9],

$$W_{n+1} = W_n(1 + \varepsilon_n), \tag{27}$$

where $\varepsilon_n$ is the relative difference of successive approximations at iteration $n$,

$$\varepsilon_n = \frac{W_{n+1} - W_n}{W_n}. \tag{28}$$

The relative difference can be expressed as

$$\varepsilon_n = \left(\frac{z_n}{1 + W_n}\right)\left(\frac{q_n - z_n}{q_n - 2z_n}\right), \tag{29}$$

where

$$z_n = \ln\frac{x}{W_n} - W_n, \tag{30}$$

$$q_n = 2(1 + W_n)\left(1 + W_n + \tfrac{2}{3}z_n\right). \tag{31}$$

The error term in this iteration is of a fourth order, i.e. with $W_n = \mathrm{W}(x) + \mathcal{O}(\varepsilon_n)$ we get $W_{n+1} = \mathrm{W}(x) + \mathcal{O}(\varepsilon_n^4)$.

Supplying this iteration with a sufficiently reasonable first guess, accurate to the order of $\mathcal{O}(10^{-4})$, will therefore deliver machine-size floating point precission $\mathcal{O}(10^{-16})$ in only one iteration and excessive $\mathcal{O}(10^{-64})$ in two! We have to find reliable first order approximation that can be fed into the Fritsch iteration. Due to the lively landscape of the Lambert function properties, the approximations will have to be found in all the particular ranges of the function behavior.

## 3 Initial approximations

The following section deals with finding the appropriate initial approximations in the whole definition ranges of the two branches of the Lambert function.

### 3.1 Branch-point expansion

The inverse of the Lambert function, $\mathrm{W}^{-1}(y) = y \exp y$, has two extrema located at $\mathrm{W}^{-1}(-1) = -e^{-1}$ and $\mathrm{W}^{-1}(-\infty) = 0^-$. Expanding $\mathrm{W}^{-1}(y)$ to the second order around the minimum at $y = -1$ we obtain

$$\mathrm{W}^{-1}(y) \approx -\frac{1}{e} + \frac{(y+1)^2}{2e}. \tag{32}$$

The inverse $\mathrm{W}^{-1}(y)$ is thus in the lowest order approximated with a parabolic term implying that the Lambert function will have square-root behavior in the vicinity of the branch point $x = -e^{-1}$,

$$\mathrm{W}_{-1,0}(x) \approx -1 \mp \sqrt{2(1 + ex)}. \tag{33}$$

To obtain the additional terms in expression (33) we proceed by defining an inverse function, centered and rescaled around the minimum, i.e. $f(y) = 2(e\,\mathrm{W}^{-1}(y-1) + 1)$. Due

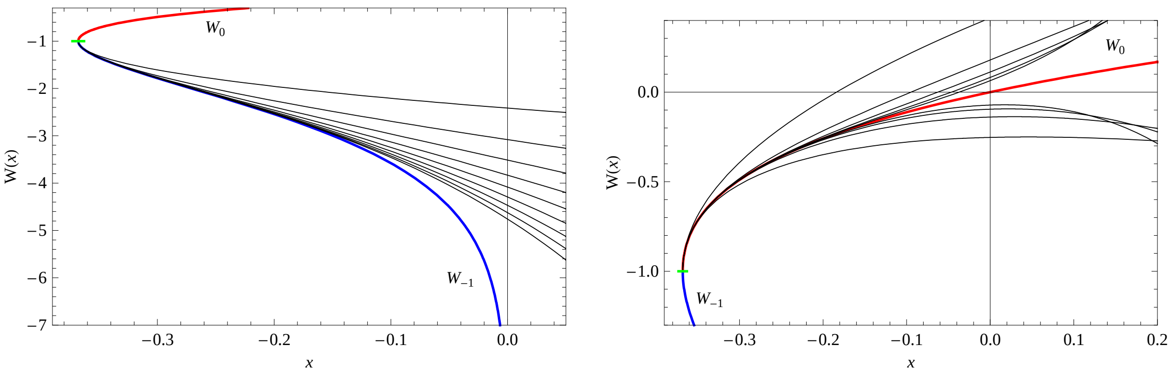


**Figure 3:** Successive orders of the branch-point expansion for the $W_{-1}(x)$ on the left and $W_0(x)$ on the right.

to the centering and rescaling the Taylor series of this function around $y = 0$ becomes particularly neat,

$$f(y) = y^2 + \tfrac{2}{3}y^3 + \tfrac{1}{4}y^4 + \tfrac{1}{15}y^5 + \cdots \tag{34}$$

Using this Taylor expansion we can derive coefficients [10] of the corresponding inverse function

$$f^{-1}(z) = 1 + \mathrm{W}\left(\frac{z-2}{2e}\right) = \tag{35}$$

$$= z^{1/2} - \tfrac{1}{3}z + \tfrac{11}{72}z^{3/2} - \tfrac{43}{540}z^2 + \cdots \tag{36}$$

From Eq. (35) we see that $z = 2(1 + ex)$. Using $p_\pm(x) = \pm\sqrt{2(1+ex)}$ as independent variable we can write this series expansion as

$$\mathrm{W}_{-1,0}(x) \approx B^{[n]}_{-1,0}(x) = \sum_{i=0}^{n} b_i p^i_\mp(x), \tag{37}$$

where the lowest few coefficients $b_i$ are

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $b_i$ | -1 | 1 | $-\frac{1}{3}$ | $\frac{11}{72}$ | $-\frac{43}{540}$ | $\frac{769}{17\,280}$ | $-\frac{221}{8\,505}$ | $\frac{680\,863}{43\,545\,600}$ | $-\frac{1963}{204\,120}$ | $\frac{226\,287\,557}{37\,623\,398\,400}$ |

## 3.2 Asymptotic series

Another useful tool is the asymptotic expansion [2] where using

$$A(a,b) = a - b + \sum_k \sum_m C_{km} a^{-k-m-1} b^{m+1} \tag{38}$$

where $C_{km}$ are related to the Stirling number of the first kind, the Lambert function can be expressed as

$$\mathrm{W}_{-1,0}(x) = A(\ln(\mp x), \ln(\mp \ln(\mp x))) \tag{39}$$

with $a = \ln x$, $b = \ln\ln x$ for the $\mathrm{W}_0$ branch and $a = \ln(-x)$, $b = \ln(-\ln(-x))$ for the $\mathrm{W}_{-1}$ branch. The first few terms are

$$A(a,b) = a - b + \frac{b}{a} + \frac{b(-2+b)}{2a^2} + \frac{b(6-9b+2b^2)}{6a^3} + \\ + \frac{b(-12+36b-22b^2+3b^3)}{12a^4} + \frac{b(60-300b+350b^2-125b^3+12b^4)}{60a^5} + \cdots \tag{40}$$

### 3.3 Rational fits

A useful quick-and-dirty approach to the functional approximation is to generate large enough sample of data points $\{w_i \exp w_i,\, w_i\}$. These points evidently lie on the Lambert function. Within some appropriately chosen range of $w_i$ values the points are fitted with a rational approximation

$$Q(x) = \frac{\sum_i a_i x^i}{\sum_i b_i x^i}, \tag{41}$$

varying the order of the polynomials in the nominator and denominator, and choosing the one that has the lowest maximal absolute residual in a particular interval of interest.

For the $\mathrm{W}_0(x)$ branch, the first set of equally-spaced $w_i$ component was chosen in a range that produced $w_i \exp w_i$ values in an interval $[-0.3, 0]$. The optimal rational fit turned out to be

$$Q_0(x) = x\frac{1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4}{1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4} \tag{42}$$

where the coefficients for this first approximation $Q_0^{[1]}(x)$ are

| $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $a_i$ | 5.931375839364438 | 11.392205505329132 | 7.338883399111118 | 0.653449016991959 |
| $b_i$ | 6.931373689597704 | 16.82349461388016 | 16.43072324143226 | 5.115235195211697 |

For the second fit of the $\mathrm{W}_0(x)$ branch a $w_i$ range was chosen so that the $w_i \exp w_i$ values were produced in the interval $[0.3, 2e]$ giving rise to the second optimal rational fit $Q_0^{[2]}(x)$ of the same form as in Eq. (42) but with coefficients

| $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $a_i$ | 2.445053070726557 | 1.343664225958226 | 0.148440055397592 | 0.0008047501729130 |
| $b_i$ | 3.444708986486002 | 3.292489857371952 | 0.916460018803122 | 0.0530686404483322 |

For the $\mathrm{W}_{-1}(x)$ branch one rational approximation of the form

$$Q_{-1}(x) = \frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5} \tag{43}$$

with the coefficients

| $i$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $a_i$ | -7.81417672390744 | 253.88810188892484 | 657.9493176902304 | |
| $b_i$ | | -60.43958713690808 | 99.9856708310761 | 682.6073999909428 |
| $i$ | 4 | 5 | | |
| $b_i$ | 962.1784396969866 | 1477.9341280760887 | | |

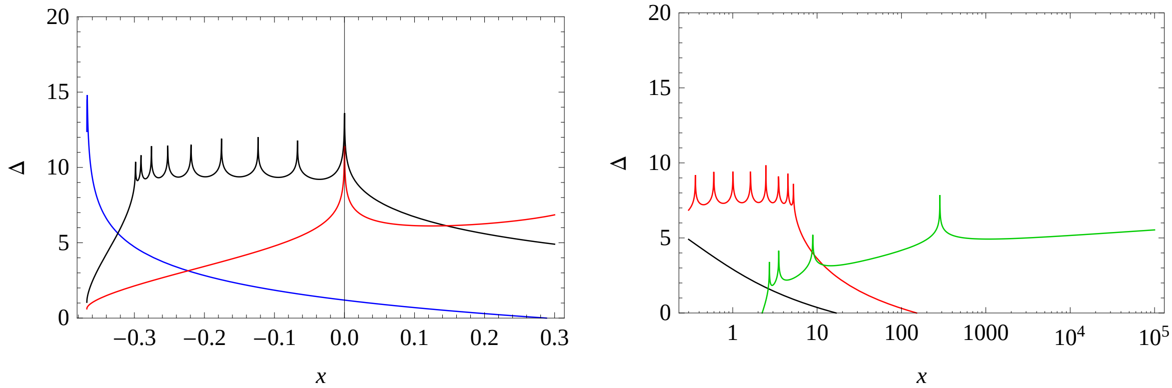


**Figure 4:** Combining different approximations of $W_0(x)$ into final piecewise function. The number of accurate decimal places $\Delta(x)$ is shown for two ranges, linear interval $[-e^{-1}, 0.3]$ on the left and logarithmic interval $[0.3, 10^5]$ on the right. The approximation are branch-point expansion $B_0^{[9]}(x)$ from Eq. (37) in blue, rational fits $Q_0^{[1]}(x)$ and $Q_0^{[2]}(x)$ from Eq. (42) in black and red, respectively, and asymptotic series $A_0(x)$ from Eq. (40) in green.

is enough.

## 4 Implementation

To quantify the accuracy of a particular approximation $\widetilde{W}(x)$ of the Lambert function $W(x)$ we can introduce a quantity $\Delta(x)$ defined as

$$\Delta(x) = -\log_{10} |\widetilde{W}(x) - W(x)|, \tag{44}$$

so that it gives us a number of correct decimal places the approximation $\widetilde{W}(x)$ is producing for some parameter $x$.

In Fig. 4 all mentioned approximations for the $W_0(x)$ are shown in the linear interval $[-e^{-1}, 0.3]$ on the left and logarithmic interval $[0.3, 10^5]$ on the right. For each of the approximations an use interval is chosen so that the number of accurate decimal places is maximized over the whole definition range. For the $W_0(x)$ branch the resulting piecewise approximation

$$\widetilde{W}_0(x) = \begin{cases} B_0^{[9]}(x) & ; -e^{-1} \leqslant x < -0.32358170806015724 \\ Q_0^{[1]}(x) & ; -0.32358170806015724 \leqslant x < 0.14546954290661823 \\ Q_0^{[2]}(x) & ; 0.14546954290661823 \leqslant x < 8.706658967856612 \\ A_0(x) & ; 8.706658967856612 \leqslant x < \infty \end{cases} \tag{45}$$

is accurate in the definition range $[-e^{-1}, 7]$ to at least 5 decimal places and to at least 3 decimal places in the whole definition range. The $B_0^{[9]}(x)$ is from Eq. (37), $Q_0^{[1]}(x)$ and $Q_0^{[2]}(x)$ are from Eq. (42), and $A_0(x)$ is from Eq. (40).

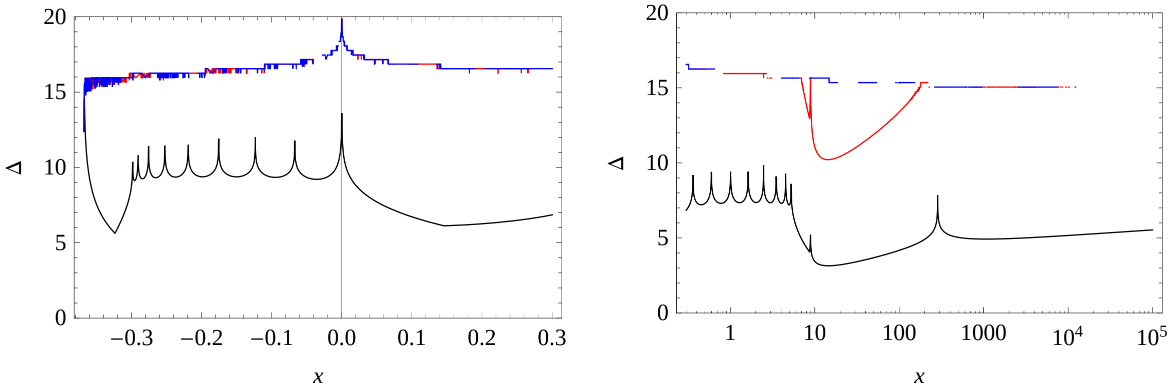


**Figure 5:** Final values of the combined approximation $\widetilde{W}_0(x)$ (black) from Fig. 4 after one step of the Halley iteration (red) and one step of the Fritsch iteration (blue).

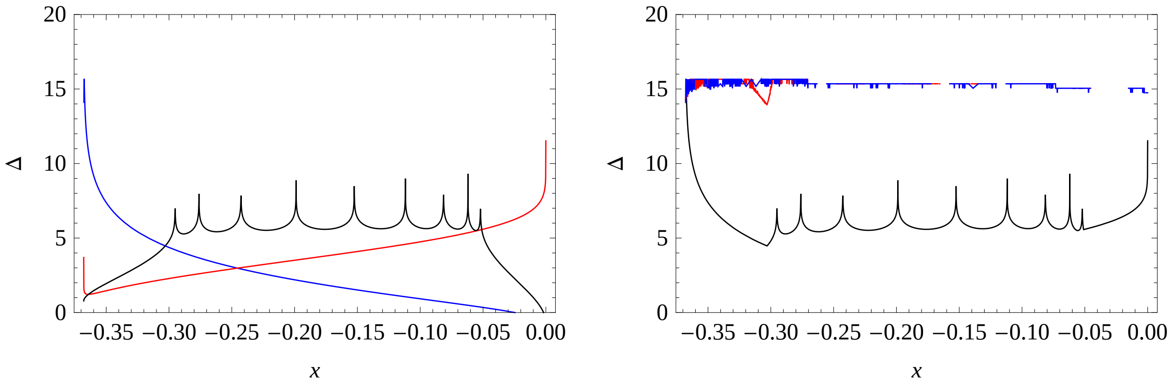


**Figure 6:** *Left:* Approximations of the $W_{-1}(x)$ branch. The branch point expansion $B_{-1}^{[9]}(x)$ is shown in blue, the rational approximation $Q_{-1}(x)$ in black, and the logarithmic recursion $R_{-1}^{[9]}$ in red. *Right:* Combined approximation is accurate to at least 5 decimal places in the whole definition range. The results after applying one step of the Halley iteration are shown in red and after one step of the Fritsch iteration in blue.

The final piecewise approximation $\widetilde{W}_0(x)$ is shown in Fig. 5 in black line. Using this approximation a single step of the Halley iteration (in red) and the Fritsch iteration (in blue) is performed and the resulting number of accurate decimal places is shown. As can be seen both iterations produce machine-size accurate floating point numbers in the whole definition interval except for the $[9, 110]$ interval where the Halley method requires another step of the iteration. For that reason we have decided to use only (one step of) the Fritsch iteration in the C++ implementation of the Lambert function.

In Fig. 6 (left) the same procedure is shown for the $W_{-1}(x)$ branch. The final approximation

$$\widetilde{W}_{-1}(x) = \begin{cases} B_{-1}^{[9]}(x) & ; -e^{-1} \leqslant x < -0.30298541769 \\ Q_{-1}(x) & ; -0.30298541769 \leqslant x < -0.051012917658221676 \\ R_{-1}^{[9]}(x) & ; -0.051012917658221676 \leqslant x < 0 \end{cases} \tag{46}$$

is accurate to at least 5 decimal places in the whole definition range $[-e^{-1}, 0]$ and where $B^{[9]}_{-1}(x)$ is from Eq. (37), $Q_{-1}(x)$ is from Eq. (43), and $R^{[9]}_{-1}(x)$ is from Eq. (18).

In Fig. 6 (right) the combined approximation $\widetilde{W}_{-1}(x)$ is shown in black line. The values after one step of the Halley iteration are shown in red and after one step of the Fritsch iteration in blue. Similarly as for the previous branch, the Fritsch iteration is superior, yielding machine-size accurate results in the whole definition range, while the Halley is accurate to at least 13 decimal places.

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# A Implementation in C++

Sources are available also from http://www.ung.si/~darko/LambertW.tar.gz

## A.1 Lambert.h

```
/*
  Implementation of Lambert W function

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  it under the terms of the GNU General Public License as published by
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  25 Jun 2009
*/

#ifndef _utl_LambertW_h_
#define _utl_LambertW_h_


/** Approximate Lambert W function
  Accuracy at least 5 decimal places in all definition range.
  See LambertW() for details.

  \param branch: valid values are 0 and -1
  \param x: real-valued argument \f$\geq-1/e\f$
  \ingroup math
*/

template<int branch>
double LambertWApproximation(const double x);


/** Lambert W function

  Lambert function \f$y={\rm W}(x)\f$ is defined as a solution
  to the \f$x=ye^y\f$ expression and is also known as
  "product logarithm". Since the inverse of \f$ye^y\f$ is not
  single-valued, the Lambert function has two real branches
  \f${\rm W}_0\f$ and \f${\rm W}_{-1}\f$.

  \f${\rm W}_0(x)\f$ has real values in the interval
  \f$[-1/e,\infty]\f$ and \f${\rm W}_{-1}(x)\f$ has real values
  in the interval \f$[-1/e,0]\f$.
  Accuracy is the nominal double type resolution
  (16 decimal places).

  \param branch: valid values are 0 and -1
  \param x: real-valued argument \f$\geq-1/e\f$ (range depends on
  branch)
*/

template<int branch>
double LambertW(const double x);


#endif
```

## A.2 Lambert.cc

```
/*
  Implementation of Lambert W function

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  (at your option) any later version.

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*/

#include <iostream>
#include <cmath>
#include <limits>
#include "LambertW.h"

using namespace std;


namespace LambertWDetail {


  const double kInvE = 1/M_E;


  template<int n>
  inline double BranchPointPolynomial(const double p);


  template<>
  inline
  double
  BranchPointPolynomial<1>(const double p)
  {
    return
      -1 + p;
  }


  template<>
  inline
  double
  BranchPointPolynomial<2>(const double p)
  {
    return
      -1 + p*(1 + p*(-1./3));
  }


  template<>
  inline
  double
  BranchPointPolynomial<3>(const double p)
  {
    return
      -1 + p*(1 + p*(-1./3 + p*(11./72)));
  }


  template<>
  inline
  double
  BranchPointPolynomial<4>(const double p)
  {
    return
      -1 + p*(1 + p*(-1./3 + p*(11./72 + p*(-43./540))));
  }


  template<>
  inline
  double
  BranchPointPolynomial<5>(const double p)
  {
    return
      -1 + p*(1 + p*(-1./3 + p*(11./72 + p*(-43./540 +
      p*(769./17280)))));
  }


  template<>
  inline
  double
  BranchPointPolynomial<6>(const double p)
  {
    return
      -1 + p*(1 + p*(-1./3 + p*(11./72 + p*(-43./540 + p*(769./17280
      + p*(-221./8505))))));
```

```
}


template<>
inline
double
BranchPointPolynomial<7>(const double p)
{
  return
    -1 + p*(1 + p*(-1./3 + p*(11./72 + p*(-43./540 + p*(769./17280
    + p*(-221./8505 + p*(680863./43545600)))))));
}


template<>
inline
double
BranchPointPolynomial<8>(const double p)
{
  return
    -1 + p*(1 + p*(-1./3 + p*(11./72 + p*(-43./540 + p*(769./17280
    + p*(-221./8505 + p*(680863./43545600 + p*(-1963./204120))))))));
}


template<>
inline
double
BranchPointPolynomial<9>(const double p)
{
  return
    -1 + p*(1 + p*(-1./3 + p*(11./72 + p*(-43./540 + p*(769./17280
    + p*(-221./8505 + p*(680863./43545600 + p*(-1963./204120
    + p*(226287557./37623398400.)))))))));
}


template<int order>
inline double AsymptoticExpansion(const double a, const double b);


template<>
inline
double
AsymptoticExpansion<0>(const double a, const double b)
{
  return a - b;
}


template<>
inline
double
AsymptoticExpansion<1>(const double a, const double b)
{
  return a - b + b / a;
}


template<>
inline
double
AsymptoticExpansion<2>(const double a, const double b)
{
  const double ia = 1 / a;
  return a - b + b / a * (1 + ia * 0.5*(-2 + b));
}


template<>
inline
double
AsymptoticExpansion<3>(const double a, const double b)
{
  const double ia = 1 / a;
  return a - b + b / a *
    (1 + ia *
      (0.5*(-2 + b) + ia *
         1/6.*(6 + b*(-9 + b*2))
      )
    );
}


template<>
inline
double
AsymptoticExpansion<4>(const double a, const double b)
{
  const double ia = 1 / a;
  return a - b + b / a *
    (1 + ia *
      (0.5*(-2 + b) + ia *
        (1/6.*(6 + b*(-9 + b*2)) + ia *
          1/12.*(-12 + b*(36 + b*(-22 + b*3)))
        )
      )
    );
}


template<>
inline
double
AsymptoticExpansion<5>(const double a, const double b)
{
  const double ia = 1 / a;
  return a - b + b / a *
    (1 + ia *
      (0.5*(-2 + b) + ia *
        (1/6.*(6 + b*(-9 + b*2)) + ia *
          (1/12.*(-12 + b*(36 + b*(-22 + b*3))) + ia *
            1/60.*(60 + b*(-300 + b*(350 + b*(-125 + b*12))))
          )
        )
      )
    );
}


template<int branch>
class Branch {

public:
  template<int order>
  static double BranchPointExpansion(const double x)
  { return BranchPointPolynomial<order>(eSign * sqrt(2*(M_E*x+1))); }

  // Asymptotic expansion
  // Corless et al. 1996, de Bruijn (1981)
  template<int order>
  static
  double
  AsymptoticExpansion(const double x)
  {
    const double logsx = log(eSign * x);
    const double logslogsx = log(eSign * logsx);
    return LambertWDetail::AsymptoticExpansion<order>(logsx, logslogsx);
  }

  template<int n>
  static inline double RationalApproximation(const double x);

  // Logarithmic recursion
  template<int order>
  static inline double LogRecursion(const double x)
  { return LogRecursionStep<order>(log(eSign * x)); }

  // generic approximation valid to at least 5 decimal places
  static inline double Approximation(const double x);

private:
  // enum { eSign = 2*branch + 1 }; this doesn't work on gcc 3.3.3
  static const int eSign = 2*branch + 1;

  template<int n>
  static inline double LogRecursionStep(const double logsx)
  { return logsx - log(eSign * LogRecursionStep<n-1>(logsx)); }
};


// Rational approximations

template<>
template<>
inline
double
Branch<0>::RationalApproximation<0>(const double x)
{
  return x*(60 + x*(114 + 17*x)) / (60 + x*(174 + 101*x));
}


template<>
template<>
inline
double
Branch<0>::RationalApproximation<1>(const double x)
{
  // branch 0, valid for [-0.31,0.3]
```

```
  return
    x * (1 + x *
      (5.931375839364438 + x *
        (11.392205505329132 + x *
          (7.338883399111118 + x*0.6534490169919599)
        )
      )
    ) /
    (1 + x *
      (6.931373689597704 + x *
        (16.82349461388016 + x *
          (16.43072324143226 + x*5.115235195211697)
        )
      )
    );
}


template<>
template<>
inline
double
Branch<0>::RationalApproximation<2>(const double x)
{
  // branch 0, valid for [-0.31,0.5]
  return
    x * (1 + x *
      (4.790423028527326 + x *
        (6.695945075293267 + x * 2.4243096805908033)
      )
    ) /
    (1 + x *
      (5.790432723810737 + x *
        (10.986445930034288 + x *
          (7.391303898769326 + x * 1.1414723648617864)
        )
      )
    );
}


template<>
template<>
inline
double
Branch<0>::RationalApproximation<3>(const double x)
{
  // branch 0, valid for [0.3,7]
  return
    x * (1 + x *
      (2.4450530707265568 + x *
        (1.3436642259582265 + x *
          (0.14844005539759195 + x * 0.0008047501729129999)
        )
      )
    ) /
    (1 + x *
      (3.4447089864860025 + x *
        (3.2924898573719523 + x *
          (0.9164600188031222 + x * 0.05306864044833221)
        )
      )
    );
}


template<>
template<>
inline
double
Branch<-1>::RationalApproximation<4>(const double x)
{
  // branch -1, valid for [-0.3,-0.05]
  return
    (-7.814176723907436 + x *
      (253.88810188892484 + x * 657.9493176902304)
    ) /
    (1 + x *
      (-60.43958713690808 + x *
        (99.98567083107612 + x *
          (682.6073999909428 + x *
            (962.1784396969866 + x * 1477.9341280760887)
          )
        )
      )
    );
}


// stopping conditions

template<>
template<>
inline
double
Branch<0>::LogRecursionStep<0>(const double logsx)
{
  return logsx;
}


template<>
template<>
inline
double
Branch<-1>::LogRecursionStep<0>(const double logsx)
{
  return logsx;
}


template<>
inline
double
Branch<0>::Approximation(const double x)
{
  if (x < -0.32358170806015724) {
    if (x < -kInvE)
      return numeric_limits<double>::quiet_NaN();
    else if (x < -kInvE+1e-5)
      return BranchPointExpansion<5>(x);
    else
      return BranchPointExpansion<9>(x);
  } else {
    if (x < 0.14546954290661823)
      return RationalApproximation<1>(x);
    else if (x < 8.706658967856612)
      return RationalApproximation<3>(x);
    else
      return AsymptoticExpansion<5>(x);
  }
}


template<>
inline
double
Branch<-1>::Approximation(const double x)
{
  if (x < -0.051012917658221676) {
    if (x < -kInvE+1e-5) {
      if (x < -kInvE)
        return numeric_limits<double>::quiet_NaN();
      else
        return BranchPointExpansion<5>(x);
    } else {
      if (x < -0.30298541769)
        return BranchPointExpansion<9>(x);
      else
        return RationalApproximation<4>(x);
    }
  } else {
    if (x < 0)
      return LogRecursion<9>(x);
    else if (x == 0)
    return -numeric_limits<double>::infinity();
    else
      return numeric_limits<double>::quiet_NaN();
  }
}


// iterations

inline
double
HalleyStep(const double x, const double w)
{
  const double ew = exp(w);
  const double wew = w * ew;
  const double wewx = wew - x;
  const double w1 = w + 1;
  return w - wewx / (ew * w1 - (w + 2) * wewx/(2*w1));
}


inline
double
FritschStep(const double x, const double w)
{
```

```
  const double z = log(x/w) - w;
  const double w1 = w + 1;
  const double q = 2 * w1 * (w1 + (2/3.)*z);
  const double eps = z / w1 * (q - z) / (q - 2*z);
  return w * (1 + eps);
}


template<
  double IterationStep(const double x, const double w)
>
inline
double
Iterate(const double x, double w, const double eps = 1e-6)
{
  for (int i = 0; i < 100; ++i) {
    const double ww = IterationStep(x, w);
    if (fabs(ww - w) <= eps)
      return ww;
    w = ww;
  }
  cerr << "convergence not reached." << endl;
  return w;
}


template<
  double IterationStep(const double x, const double w)
>
struct Iterator {

  static
  double
  Do(const int n, const double x, const double w)
  {
    for (int i = 0; i < n; ++i)
      w = IterationStep(x, w);
    return w;
  }

  template<int n>
  static
  double
  Do(const double x, const double w)
  {
    for (int i = 0; i < n; ++i)
      w = IterationStep(x, w);
    return w;
  }

  template<int n, class = void>
  struct Depth {
    static double Recurse(const double x, double w)
    { return Depth<n-1>::Recurse(x, IterationStep(x, w)); }
  };

  // stop condition
  template<class T>
  struct Depth<1, T> {
    static double Recurse(const double x, const double w)
    { return IterationStep(x, w); }
  };

  // identity
  template<class T>
  struct Depth<0, T> {
    static double Recurse(const double x, const double w)
    { return w; }
  };

};


} // LambertWDetail


template<int branch>
double
LambertWApproximation(const double x)
{
  return LambertWDetail::Branch<branch>::Approximation(x);
}

// instantiations
template double LambertWApproximation<0>(const double x);
template double LambertWApproximation<-1>(const double x);


template<int branch>
double LambertW(const double x);

template<>
double
LambertW<0>(const double x)
{
  if (fabs(x) > 1e-6 && x > -LambertWDetail::kInvE + 1e-5)
    return
      LambertWDetail::
        Iterator<LambertWDetail::FritschStep>::
          Depth<1>::
            Recurse(x, LambertWApproximation<0>(x));
  else
    return LambertWApproximation<0>(x);
}

template<>
double
LambertW<-1>(const double x)
{
  if (x > -LambertWDetail::kInvE + 1e-5)
    return
      LambertWDetail::
        Iterator<LambertWDetail::FritschStep>::
          Depth<1>::
            Recurse(x, LambertWApproximation<-1>(x));
  else
    return LambertWApproximation<-1>(x);
}

// instantiations
template double LambertW<0>(const double x);
template double LambertW<-1>(const double x);
```