Approximating real Pochhammer products: A comparison with powers

ARTICLE *in* CENTRAL EUROPEAN JOURNAL OF MATHEMATICS · SEPTEMBER 2009

Impact Factor: 0.58 · DOI: 10.2478/s11533-009-0036-1

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Approximating real Pochhammer products: a comparison with powers

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Abstract

Accurate estimates of real Pochhammer products, lower (falling) and upper (rising), are presented. Double inequalities comparing the Pochhammer products with powers are given. Several examples showing how to use the established approximations are stated.

Key words: approximation, double inequality, falling factorial, lower factorial, Pochhammer product, rising factorial, sequential product, upper factorial.

MSC 2000 primary: 41A80
MSC 2000 secondary: 26D15, 33F05, 40A99

1 Introduction.

In the theory of special functions the rising sequential product,

$$x^{(n)} := \prod_{i=0}^{n-1}(x+i) = x(x+1)\cdot\ldots\cdot(x+n-1), \tag{1}$$

is considered, known also under the name rising factorial power or the Pochhammer[1] upper factorial. However, in combinatorics the falling sequential product,

$$x_{(n)} := \prod_{i=0}^{n-1}(x-i) = x(x-1)\cdot\ldots\cdot(x-n+1), \tag{2}$$

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[1]German mathematician, 1840–1920

is used, known as the Pochhammer lower factorial. The Pochhammer factorials are closely linked with each other. Obviously, the fallowing relations hold:

$$x^{(n)} = \frac{x}{x+n} \cdot \big(x+n\big)_{(n)} \quad \text{and} \quad x_{(n)} = \frac{x}{x-n} \cdot (x-n)^{(n)}. \tag{3}$$

In addition we have

$$x^{(n)} = x^{(m)} \cdot (x+m)^{(n-m)} \tag{4}$$

for integers $n > m \geq 0$[2]. Similarly,

$$x_{(n)} = x_{(n-m)} \cdot (x-n+m)_{(m)} \tag{5}$$

for (large) $x > 0$ and integers m and n such that $0 \leq m < n < x$[3].

Recently [10, p. 64] it was supposed that $m_{(k-1)} \cdot m^{1-k} = 1 + O\big(\frac{1}{m}\big)$ for integers $m \geq k \geq 1$. This formula is correct for fixed k, but it is wrong if k is allowed to depend on m. For example, $\lim\limits_{m\to\infty} m_{(k-1)} \cdot m^{1-k} = 0$ for $k = m, m-1, m-2, \ldots$ as it is evident using Stirling's factorial formula [1, 6.1.38; p. 257]. Challenged by this paper we would like to make an accurate estimate of the falling product. We take interest in the question of how the sequence of Pochhammer upper factorials $n \mapsto x^{(n)}$ grows for positive constant x. Similarly, we take the interest in the question how the sequences of Pochhammer lower factorials $n \mapsto x_{(n)}$ grows for $n < x$ and given large positive constant x. Obviously, we have $x_{(n)} < x^n$ for $n < x$ and $x \geq 2$. Moreover, we would like to find positive sequence $n \mapsto c_n(x)$ such that $x_{(n)} \geq c_n(x) \cdot x^n$ for $n < x$. Furthermore, we wish to find some good approximation for decreasing sequence

$$n \longmapsto \frac{x_{(n)}}{x^n} \equiv \prod_{i=0}^{n-1} \left(1 - \frac{i}{x}\right).$$

---

[2] $x^{(0)} := 1$

[3] $x_{(0)} := 1$

Indeed, using recurrence and Stirling's formula for the Gamma function [1, 6.1.15, 6.1.38; pp. 256–257], we have

$$
\begin{aligned}
x_{(n)} &= \frac{\Gamma(x+1)}{\Gamma(x-n+1)} \\
&= \sqrt{2\pi x}\left(\frac{x}{e}\right)^{x} \exp\left(\frac{\vartheta_1}{12x}\right) \\
&\quad \cdot \left[\sqrt{2\pi(x-n)}\left(\frac{x-n}{e}\right)^{x-n} \exp\left(\frac{\vartheta_2}{12(x-n)}\right)\right]^{-1} \\
&= x^n \cdot e^{-n}\left(\frac{x}{x-n}\right)^{x-n+\frac{1}{2}} \exp\left(\frac{\vartheta_1}{12x} - \frac{\vartheta_2}{12(x-n)}\right) \qquad (6)
\end{aligned}
$$

for some ϑ_1, $\vartheta_2 \in (0,1)$. Similar result we get for $x^{(n)}$. But, we wish to obtain for $x_{(n)}$, and for $x^{(n)}$ as well, more accurate formula "including only one theta". To do this we first connect the Pochhammer rising factorial with the sum,

$$
\ln x^{(n)} = -\ln(x+n) + \sum_{i=0}^{n} \ln(x+i), \qquad (7)
$$

for positive (large) constant x. Now, we are in the position to use an adequate summation formula.

2 Hermite's summation formula.

For four times continuously differentiable function $f : [0,\infty) \to \mathbb{R}$, keeping the sign of its fourth derivative, the Euler-Maclaurin formula [8, (21a), (21b); p. 117] of order four produces Hermite's summation formula [9, (1.3), (1.4), (4.1), (4.4); pp. 312, 316]

$$
\sum_{j=0}^{n} f(j) = \int_0^n f(t)\,\mathrm{d}t + \frac{1}{2}\left[f(0)+f(n)\right] + \frac{1}{12}\left[f'(n)-f'(0)\right] + \rho_n, \qquad (8)
$$

where

$$\rho_n = -\int_0^n W(x)\, f^{(4)}(t)\,\mathrm{d}t = W(\tau)\left[\, f^{(3)}(0) - f^{(3)}(n)\right], \tag{9}$$

for some $\tau = \tau(n) \in [0,1]$. Here, $W \in C^3(\mathbb{R})$ is $1-$periodic function, defined as

$$\begin{array}{lll} \text{(i)} & W(t) := \frac{1}{24}\left[t(1-t)\right]^2 & \text{if } 0 \le t < 1 \\ \text{(ii)} & W(t+1) := W(t) & \text{for } t \in \mathbb{R} \end{array}$$

and estimated as

$$0 \le W(t) \le \frac{1}{384}\,, \qquad t \in \mathbb{R}. \tag{10}$$

3 Approximating Pochhammer factorials.

The sum in (7) results from the function $f(t,x) \equiv \ln(t+x)$. To apply (8) and (9) we need the derivatives,

$$\begin{aligned} \frac{\partial f}{\partial t}(t,x) &\equiv \frac{1}{t+x} > 0, \\ \frac{\partial^3 f}{\partial t^3}(t,x) &\equiv \frac{2}{(t+x)^3} > 0 \\ \frac{\partial^4 f}{\partial t^4}(t,x) &\equiv -\frac{6}{(t+x)^3} < 0\,. \end{aligned} \tag{11}$$

According to (7), (8) and (11), we have

$$\begin{aligned} \ln x^{(n)} &= -\ln(n+x) + \int_0^n \ln(t+x)\,\mathrm{d}t \\ &\quad + \frac{1}{2}\left[\ln a + \ln(n+x)\right] - \frac{n}{12x(n+x)} + R_n(x) \\ &= \left[\ln\left(\frac{t+x}{e}\right)^{t+x}\right]_0^n + \ln\left(\frac{n+x}{x}\right)^{-\frac{1}{2}} - \frac{n}{12x(n+x)} + R_n(x) \\ &= \ln\left[x^n \cdot \left(\frac{n+x}{x}\right)^{n+x-\frac{1}{2}} \exp\left(-n - \frac{n}{12x(n+x)}\right)\right] + R_n(x), \end{aligned} \tag{12}$$

where, using (9) and (10), we estimate

$$0 \le R_n(x) = W(\xi) \cdot \left(\frac{2}{x^3} - \frac{2}{(n+x)^3} \right) \le \frac{1}{384} \cdot \frac{2}{x^3} = \frac{1}{192x^3} \, .$$

Thus, for any positive integer n,

$$0 \le R_n(x) = \frac{\vartheta}{192x^3} \tag{13}$$

for some $\vartheta \in [0,1]$. Therefore, invoking (12),

$$x^{(n)} = x^n \cdot \left(\frac{n+x}{x} \right)^{n+x-\frac{1}{2}} \exp\left(-n - \frac{n}{12x(n+x)} \right) \cdot \exp\left(\frac{\vartheta}{192x^3} \right) \tag{14}$$

for some $\vartheta \in [0,1]$. Consequently, for positive integer n and real $x > n$, referring to (3), we also get

$$x_{(n)} = \frac{x}{x-n} \cdot (x-n)^n \left(\frac{x}{x-n} \right)^{x-\frac{1}{2}} \exp\left(-n - \frac{n}{12(x-n)x} + \frac{\theta}{192(x-n)^3} \right).$$

Hence,

$$x_{(n)} = x^n \cdot \left(\frac{x}{x-n} \right)^{x-n+\frac{1}{2}} \exp\left(-n - \frac{n}{12x(x-n)} \right) \cdot \exp\left(\frac{\theta}{192(x-n)^3} \right), \tag{15}$$

for some $\theta \in [0,1]$. Obviously, the precision of this formula is higher than that of (6) and it increases if x grows.

Introducing the function

$$P(n,x) := x^{1/2-x}(x+n)^{x+n-\frac{1}{2}} \exp\left(-n - \frac{n}{12x(x+n)} \right) \tag{16}$$

into (14) we find that for any $x > 0$ and for any positive integer n there exists a function $I(n,x)$ such that

$$x^{(n)} = P(n,x) \cdot I(n,x), \tag{17}$$

where holds the uniform estimate

$$1 \leq I(n,x) \leq \exp\left(\frac{1}{192x^3}\right). \tag{18}$$

This relation, together with (4), enables us to achieve more accurate estimate of $x^{(n)}$. For example, for integers $n > m \geq 1$ we obtain

$$\begin{aligned} x^{(n)} &= x^{(m)} \cdot (x+m)^{(n-m)} \\ &= x^{(m)} \cdot P(x+m, n-m) \cdot I(x+m, n-m), \end{aligned}$$

where

$$1 \leq I(x+m, n-m) \leq \exp\left(\frac{1}{192(x+m)^3}\right).$$

For instance, we have

$$(1/\pi)^{(n)} = \Big[(1/\pi)^{(18)} \cdot P(1/\pi + 18, n-18)\Big] \cdot I(1/\pi + 18, n-18),$$

where

$$1 \leq I(1/\pi + 18, n-18) < 1 + 10^{-6} \qquad \text{for } n \geq 19.$$

Thus, concretizing

$$\begin{aligned} \left(\frac{1}{\pi}\right)^{(n)} &= \left(\frac{1}{\pi}\right)^{(18)} \cdot \left(\frac{1}{\pi} + 18\right)^{1/\pi - 35/2} \left(\frac{1}{\pi} + n\right)^{1/\pi + n - 1/2} \\ &\quad \cdot \exp\left((18-n)\Big(1 + \frac{1}{12(18+1/\pi)(n+1/\pi)}\Big)\right) \cdot K(n), \end{aligned}$$

where $1 \leq K(n) < 1 + 10^{-6}$ for $n \geq 19$. The accuracy of this estimate illustrates Figure 1 showing the graph of the sequence $\big(K(n)\big)_{n\in\mathbb{N}}$.

Concerning the lower factorials, using the function

$$Q(n,x) := x^{1/2+x}(x-n)^{-(x-n)-\frac{1}{2}} \exp\left(-n - \frac{n}{12x(x-n)}\right), \tag{19}$$

in (15), we conclude that for any (large) $x > 0$ and any positive integers m and n with $m < n < x$ there exists a function $J(n,x)$ such that

$$x_{(n)} = Q(n,x) \cdot J(n,x), \tag{20}$$

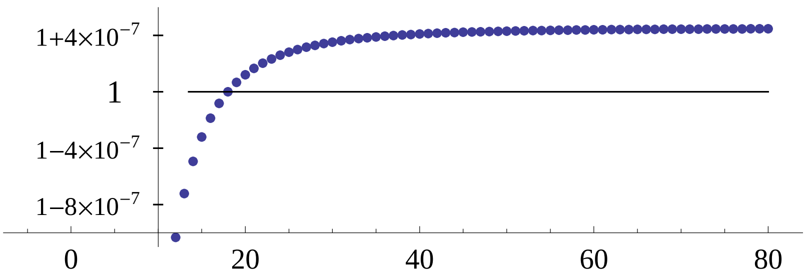

Figure 1: The graph of the sequence $n \mapsto K(n)$.

where we have
$$1 \le J(n,x) \le \exp\left(\frac{1}{192(x-n)^3}\right). \tag{21}$$
This relation, together with (5), enables us to estimate $x_{(n)}$ more accurately. For example, for integers m and n such that $1 \le m < n < x$ we have
$$\begin{aligned} x_{(n)} =& x_{(n-m)} \cdot \big(x-n+m\big)_{(m)} \\ &= \big[(x-n+m)_{(m)} \cdot Q(n-m,x)\big] \cdot J(n-m,x) \end{aligned} \tag{22}$$
with the estimate
$$1 \le J(n-m,x) \le \exp\left(\frac{1}{192(x-n+m)^3}\right) < \exp\left(\frac{1}{192m^3}\right). \tag{23}$$
This way we find, for example,
$$\left(e^{10}\right)_{(n)} = \left[\left(e^{10}-n+26\right)_{(26)} \cdot Q\left(n-26,e^{10}\right)\right] \cdot J\left(n-26,e^{10}\right),$$
where
$$1 \le J\left(n-26,e^{10}\right) < 1+10^{-6} \qquad \text{if} \quad 19 \le n < e^{10}.$$

We note that, for integers $n > m \ge 1$, referring to (14), we have
$$\begin{aligned} n! &= (m-1)! \cdot m^{(n-m)} \cdot n \\ &= (m-1)! \cdot m^{1/2-m} n^{n-1/2} \cdot \exp\left(-(n-m) - \frac{n-m}{12m\,n} + \frac{\vartheta}{192m^3}\right) \cdot n. \end{aligned}$$
Hence, we can formulate the next proposition similar to Stirling's factorial formula.

Proposition 1. *For integers $n > m \geq 1$ the equality*

$$n! = \alpha(m) \cdot \beta(n) \cdot \gamma(m,n)$$

holds true with

$$\alpha(m) := (m-1)! \cdot \left(\frac{e}{m}\right)^m \sqrt{m} \cdot \exp\left(-\frac{1}{12m}\right),$$
$$\beta(n) := \left(\frac{n}{e}\right)^n \sqrt{n} \cdot \exp\left(\frac{1}{12n}\right),$$

and

$$1 \leq \gamma(m,n) \leq \gamma^*(m) := \exp\left(\frac{1}{192m^3}\right).$$

This proposition is quite applicable since, for example, we have $\gamma^*(1) < 1.01$, $\gamma^*(2) < 1.001$ and $\gamma^*(10) < 1.00001$.

4 Comparing factorials with powers.

The formulas (14) and (15) make possible to compare factorials and powers. So, we would like to study these formulas closer. It is evident that in both formulas the main role plays the variable $t := \frac{n}{x}$. Looking at (14) and (15) we introduce two "lower-L" and two "upper-U" functions:

$$L(s,x) := \exp\left(-x + \frac{1}{12x}\right) \cdot \frac{1}{\sqrt{s}} \left(\frac{s}{e}\right)^{-sx} \exp\left(-\frac{1}{12sx}\right), \tag{24}$$
$$U(s,x) := \exp\left(+x - \frac{1}{12x}\right) \cdot \frac{1}{\sqrt{s}} \left(\frac{s}{e}\right)^{+sx} \exp\left(+\frac{1}{12sx}\right), \tag{25}$$
$$L^*(t,x) := L(1-t,x)$$
$$\equiv (1-t)^{tx-\frac{1}{2}-x} \exp\left(-tx\Big(1 + \frac{1}{12x^2(1-t)}\Big)\right), \tag{26}$$
$$U^*(t,x) := U(1+t,x)$$
$$\equiv (1+t)^{tx-\frac{1}{2}+x} \exp\left(-tx\Big(1 + \frac{1}{12x^2(1+t)}\Big)\right). \tag{27}$$

The functions $L^*(t,x)$ and $U^*(t,x)$ enable to formulate (below) the approximation theorems for rising (upper)/falling (lower) factorials.

4.1 Rising Pochhammer factorial.

Referring to (14) and (27) we obtain the following theorem comparing upper factorials and powers.

Theorem 1. *For a real $x > 0$ and an integer $n \geq 1$ the following estimate holds:*

$$U^*\left(\tfrac{n}{x}, x\right) \leq \frac{x^{(n)}}{x^n} \leq U^{**}\left(\tfrac{n}{x}, x\right) := U^*\left(\tfrac{n}{x}, x\right) \cdot \exp\left(\frac{1}{192x^3}\right). \tag{28}$$

Theorem 1 makes it possible, using (4), to approximate upper factorials quite accurately. Figures 2 and 3 illustrate, for $x = \pi$ and $x = 10\pi$, the graphs of the sequences $n \mapsto \frac{x^{(n)}}{x^n}/U^*\left(\frac{n}{x}, x\right)$ and $n \mapsto \frac{x^{(n)}}{x^n}/U^{**}\left(\frac{n}{x}, x\right)$, respectively.

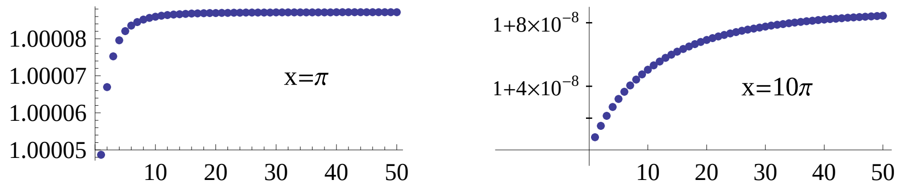

Figure 2: The graph of the sequence $n \mapsto \frac{x^{(n)}}{x^n}/U^*\left(\frac{n}{x}, x\right)$.

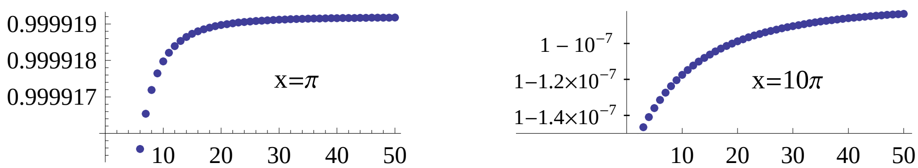

Figure 3: The graph of the sequence $n \mapsto \frac{x^{(n)}}{x^n}/U^{**}\left(\frac{n}{x}, x\right)$.

For positive x, the sequence $n \mapsto \frac{x^{(n)}}{x^n} \equiv \prod_{i=0}^{n-1}\left(1 + \frac{i}{x}\right)$ is strictly increasing. Moreover, the function $s \mapsto U(s,x)$ is also increasing for sufficiently large x, however not for all positive x. More precisely, we have the result:

Proposition 2. *For $x \geq 1$, the function $s \mapsto U(s,x)$ is strictly increasing on the interval $\left[1+\frac{1}{x}, \infty\right)$.*

Proof. Under the supposition $x \geq 1$, we consider the function $\varphi_x : \mathbb{R}^+ \to \mathbb{R}$,

$$\varphi_x(s) := \ln U(s,x) \equiv \left(x - \frac{1}{12x}\right) - \frac{1}{2}\ln s + sx(\ln s - 1) + \frac{1}{12sx}\,, \tag{29}$$

having the derivatives

$$\varphi'_x(s) \equiv x \ln s - \frac{1}{2s} - \frac{1}{12s^2x} \tag{30}$$

$$\varphi^{(2)}_x(s) \equiv \frac{x}{s} + \frac{1}{2s^2} + \frac{1}{6s^3x} > 0\,. \tag{31}$$

The auxiliary function $g(x) := \varphi'_x\left(1+\frac{1}{x}\right)$ has the derivatives

$$g'(x) \equiv \ln\left(1+\frac{1}{x}\right) - \frac{12x^2+29x+19}{12(x+1)^3} \tag{32}$$

$$g^{(2)}(x) \equiv -\frac{x^2+4x+6}{6x(x+1)^4} < 0\,. \tag{33}$$

Therefore, the function $x \mapsto g'(x)$ is decreasing on $\mathbb{R}^+$. But, since $\lim\limits_{x\to\infty} g'(x) = 0$, the derivative $g'(x)$ is positive on $\mathbb{R}^+$, i.e. $g(x)$ is strictly increasing on $\mathbb{R}^+$. Consequently, bearing in mind the definition of $g(x)$ above, we have

$$g(x) > g(1) = \varphi'_1(2) = \ln 2 - \frac{1}{4} - \frac{1}{48} > 0,$$

for $x > 1$. Hence, for $x \geq 1$ and $s > 1 + \frac{1}{x}$, we obtain, considering (31),

$$\varphi'_x(s) > \varphi'_x\left(1+\frac{1}{x}\right) = g(x) > 0.$$

Thus, for $x \geq 1$, the function $s \mapsto \varphi_x(s)$ and therefore also the function $s \mapsto U(s,x)$, referring to (29), are strictly increasing on the interval $\left[1+\frac{1}{x}, \infty\right)$. ■

We remark that, putting $x = 1$ and $n = m - 1$ (m being positive integer) into (28) we obtain, using the identity $m! = (m-1)! \cdot m = 1^{(m-1)} \cdot m$, the estimate

$$F_1(m) := e^{11/12} \cdot F(m) \leq m! \leq F_2(m) := e^{\frac{1}{192}} \cdot F_1(m),$$

where

$$F(m) \equiv \sqrt{m} \left(\frac{m}{e}\right)^m \cdot \exp\left(\frac{1}{12m}\right).$$

Thus, for example,

$$2.50 \cdot F(m) \leq m! \leq 2.52 \cdot F(m),$$

for any positive integer m. The sequences $m \mapsto m!/F_1(m)$ and $m \mapsto m!/F_2(m)$ are depicted in Figure 4.

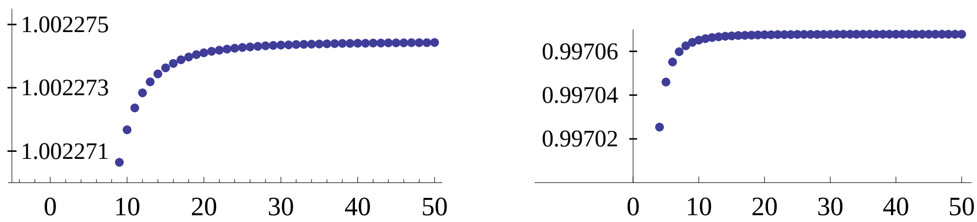

Figure 4: The graphs of the sequences $m \mapsto m!/F_1(m)$ and $m \mapsto m!/F_2(m)$.

The obtained formula can be improved taking some (fixed) positive integer m_0 and considering integers $n \geq m_0 + 1$. Setting $x = m_0$ and $n = m - m_0$ into (28) and using the identity $m! = (m_0 - 1)! \cdot m_0^{(m-m_0)} \cdot m$, we obtain, for $m \geq m_0 + 1$, the estimate

$$K_1(m_0) \cdot F(m) \leq m! \leq K_2(m_0) \cdot F(m)$$

where

$$K_1(m_0) \equiv (m_0 - 1)! \cdot m_0^{1/2 - m_0} \cdot \exp\left(m_0 - \frac{1}{12m_0}\right),$$

and

$$K_2(m_0) \equiv K_1(m_0) \cdot \exp\left(\frac{1}{192m_0^3}\right).$$

For example, we have

$$2.50662 \cdot F(m) < m! < 2.50664 \cdot F(m)$$

for $m \geq 11$.

Proposition 3. *For integers $n > m \geq 0$ we have the estimates*

$$\begin{aligned}
(2n-1)!! &\geq \left[\left(\tfrac{1}{2}\right)^{(m)} (m+\tfrac{1}{2})^{-m} \exp\left(m - \tfrac{1}{6(2m+1)}\right)\right] \\
&\quad \cdot \left(\frac{2n+1}{e}\right)^n \exp\left(\tfrac{1}{6(2n+1)}\right) \\
(2n-1)!! &\leq \left[\left(\tfrac{1}{2}\right)^{(m)} (m+\tfrac{1}{2})^{-m} \exp\left(m - \tfrac{1}{6(2m+1)} + \tfrac{1}{24(2m+1)^3}\right)\right] \\
&\quad \cdot \left(\frac{2n+1}{e}\right)^n \exp\left(\tfrac{1}{6(2n+1)}\right).
\end{aligned}$$

Thus, (setting $m = 0$), we have

$$G^*(n) := \left(\tfrac{2n+1}{e}\right)^n \cdot \exp\left(-\tfrac{n}{3(2n+1)}\right) \leq (2n-1)!! \leq \left(\tfrac{2n+1}{e}\right)^n \exp\left(\tfrac{1}{24}\right) =: G^{**}(n),$$

for $n \geq 1$. Moreover, for $n \geq 10$ (putting $m = 9$), we obtain

$$8032.3 \times \left(\tfrac{2n+1}{e}\right)^n < (2n-1)!! < 8032.4 \times \left(\tfrac{2n+1}{e}\right)^n.$$

Proof. Choosing integers $n > m \geq 0$ and considering (4), we have

$$(2n-1)!! := \prod_{i=0}^{n-1} (2i+1) = 2^n \left(\frac{1}{2}\right)^{(n)} = 2^n \left(\frac{1}{2}\right)^{(m)} (\tfrac{1}{2}+m)^{(n-m)}, \tag{34}$$

where, referring to (28), (27) and (25), we see that there exists a $\vartheta \in [0,1]$

such that

$$\begin{aligned}
(\tfrac{1}{2}+m)^{(n-m)} =& (\tfrac{1}{2}+m)^{n-m} \cdot U\big(1+\tfrac{n-m}{1/2+m}, \tfrac{1}{2}+m\big) \cdot \exp\left(\frac{\vartheta}{192(\frac{1}{2}+m)^3}\right) \\
=& (\tfrac{1}{2}+m)^{n-m} \cdot U\big(\tfrac{2n+1}{2m+1}, \tfrac{2m+1}{2}\big) \cdot \exp\left(\frac{\vartheta}{24(2m+1)^3}\right) \\
=& (\tfrac{1}{2}+m)^{n-m} \cdot \exp\big(\tfrac{2m+1}{2} - \tfrac{1}{6(2m+1)}\big) \cdot \frac{1}{\sqrt{\frac{2n+1}{2m+1}}} \\
& \cdot \left(\frac{2n+1}{(2m+1)e}\right)^{(2n+1)/2} \exp\left(\frac{1}{12\left(\frac{2n+1}{2}\right)}\right) \exp\left(\frac{\vartheta}{24(2m+1)^3}\right) \\
&= 2^{-n}\big(m+\tfrac{1}{2}\big)^{-m} \exp\left(m+\tfrac{1}{2} - \frac{1}{6(2m+1)} + \frac{1}{6(2n+1)}\right) \\
& \cdot \exp\left(\frac{\vartheta}{24(2m+1)^3}\right).
\end{aligned}$$

This last equation, together with (34), verifies the proposition. ■

The sequences $n \mapsto (2n-1)!!/G^*(n)$ and $n \mapsto (2n-1)!!/G^{**}(n)$ are illustrated in Figure 5.

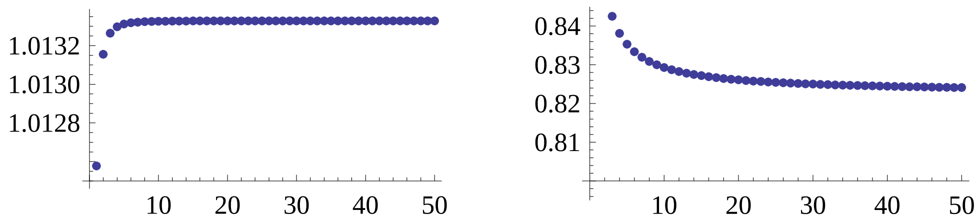

Figure 5: The graphs of the sequences $n \mapsto (2n-1)!!/G^*(n)$ (left) and $n \mapsto (2n-1)!!/G^{**}(n)$ (right).

4.2 Falling Pochhammer factorial.

Referring to (16) and (26), we can verify the next theorem comparing the lower factorials and powers.

Theorem 2. *For real (large) $x > 0$ and an integer n such that $1 \le n < x$ there holds the estimate*

$$L^*\left(\tfrac{n}{x}, x\right) \le \frac{x_{(n)}}{x^n} \le L^{**}\left(\tfrac{n}{x}, x\right) := L^*\left(\tfrac{n}{x}, x\right) \cdot \exp\left(\frac{1}{192(x-n)^3}\right). \qquad (35)$$

Theorem 2, together with (5), enables us to obtain a good approximations of lower factorials. In Figures 6 and 7 the sequences $n \mapsto \frac{x_{(n)}}{x^n}/L^*(n/x, x)$ and $n \mapsto \frac{x_{(n)}}{x^n}/L^{**}(n/x, x)$, respectively, are depicted.

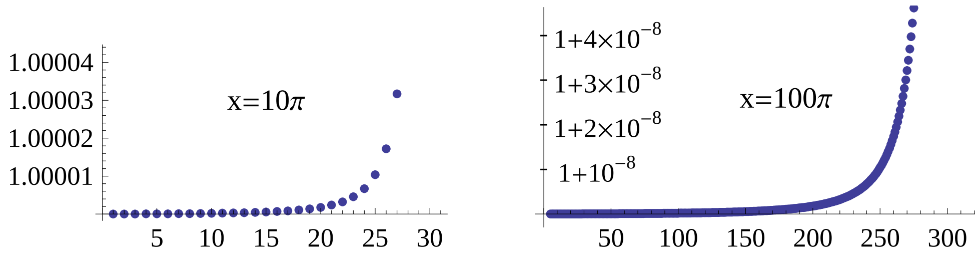

Figure 6: The graphs of the sequences $n \mapsto \frac{x_{(n)}}{x^n}/L^*(n/x, x)$.

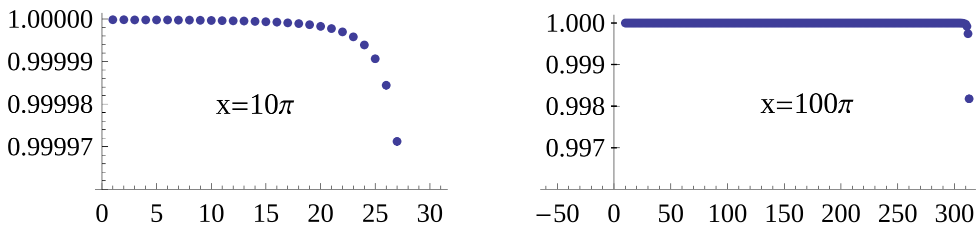

Figure 7: The graphs of the sequences $n \mapsto \frac{x_{(n)}}{x^n}/L^{**}(n/x, x)$.

In this approximation the main role is played by the function $L^*(t, x) \equiv L(1 + t, x)$. Since the sequence $n \mapsto \frac{x_{(n)}}{x^n} \equiv \prod_{i=0}^{n-1}\left(1 - \frac{i}{x}\right) \approx L^*(\frac{n}{x}, x)$ is decreasing, for $x > n \ge 1$, we expect the same behavior for the function $t \mapsto L^*(t, x)$. However, for $t \ge 0$, the mapping $t \mapsto L^*(t, x)$ is not decreasing as it is shown in Figure 8 where the function $t \mapsto L^*(t, x)$ is plotted.

Proposition 4. *For $x > 1$, the function $t \mapsto L^*(t, x)$ is strictly decreasing on the interval $\left[\frac{1}{x}, 1\right)$ and the estimate $0 < L^*(t, x) < 1$ holds for all $t \in [0, 1)$*

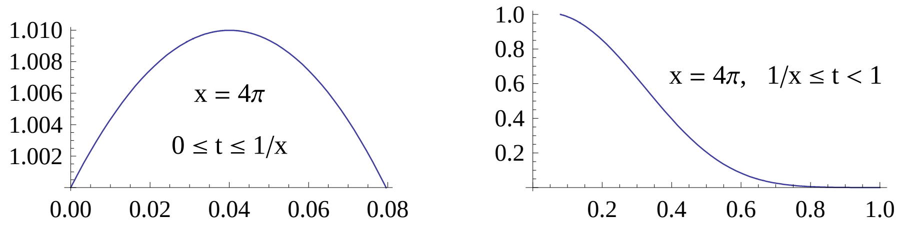

Figure 8: The graph of the function $t \mapsto L^*(t,x)$.

where $\lim\limits_{t\uparrow 1} L^*(t,x) = 0$. *Consequently, the finite sequence* $n \mapsto L^*\left(\frac{n}{x}, x\right)$ *is strictly decreasing and all its terms lie inside the interval* $(0,1)$.

Proof. To estimate $L^*(t,x)$, we consider its logarithm,

$$\begin{aligned}\psi_x(t) &:= \ln\big(L^*(t,x)\big)\\ &\equiv \left(tx - \frac{1}{2} - x\right)\ln(1-t) - tx\left(1 + \frac{1}{12x^2(1-t)}\right).\end{aligned} \tag{36}$$

The derivatives of this function with respect to t (x being parameter) we express in terms of the variable $u := \frac{1}{1-t} > 1$ as

$$\psi_x'(t) \equiv -x\ln u + \frac{u}{2} - \frac{u^2}{12x}\,, \tag{37}$$

$$\psi_x^{(2)}(t) \equiv -\frac{u}{6x}\left[\left(u - \frac{3x}{2}\right)^2 + \frac{15}{4}x^2\right] < 0\,, \tag{38}$$

$$\psi_x^{(3)}(t) \equiv -\frac{u^2}{2x}\left[(u-x)^2 + x^2\right] < 0\,, \tag{39}$$

$$\psi_x^{(4)}(t) \equiv -2\frac{u^3}{x}\left[\left(u - \frac{3x}{4}\right)^2 + \frac{7x^2}{16}\right] < 0\,, \tag{40}$$

$$\psi_x^{(5)}(t) \equiv -10\frac{u^4}{a}\left[\left(u - \frac{3x}{5}\right)^2 + \frac{6x^2}{25}\right] < 0\,, \tag{41}$$

$$\begin{aligned}\psi_x^{(6)}(t) &\equiv -\frac{24x}{(1-t)^5} + \frac{60}{(1-t)^6} - \frac{60}{a(1-t)^7}\\ &\equiv -60\frac{u^5}{a}\left[\left(u - \frac{x}{2}\right)^2 + \frac{3x^2}{20}\right] < 0\,.\end{aligned} \tag{42}$$

Therefore, using the Maclaurin's formula of order five, we obtain

$$\psi_x(t) = \psi_x^*(t) + r_x(t), \tag{43}$$

where

$$\begin{aligned}\psi_x^*(t) = {} & \left(\frac{1}{2} - \frac{1}{12x}\right) t - \left(\frac{x}{2} + \frac{1}{12x} - \frac{1}{4}\right) t^2 - \left(\frac{x}{6} + \frac{1}{12x} - \frac{1}{6}\right) t^3 \\ & - \left(\frac{x}{12} + \frac{1}{12x} - \frac{1}{8}\right) t^4 - \left(\frac{x}{20} + \frac{1}{12x} - \frac{1}{10}\right) t^5,\end{aligned} \tag{44}$$

and

$$-\frac{t^6}{6(1-t)^5}\left(\frac{x}{5} + \frac{1}{2x(1-t)^2}\right) < r_x(t) < 0, \tag{45}$$

for $0 < t < 1$.

Considering (44), we calculate

$$\psi_x^*\left(\tfrac{1}{x}\right) = -\frac{x^2 - 2x + 10}{120x^6} < 0. \tag{46}$$

Thus, according to (43) and (45), we have

$$\psi_x\left(\tfrac{1}{x}\right) < 0. \tag{47}$$

Similarly, by fourth-order Maclaurin's formula, we have

$$\psi_x'(t) = \psi_x^{**}(t) + \rho_x(t), \tag{48}$$

with

$$\begin{aligned}\psi_x^{**}(t) = {} & \left(\frac{1}{2} - \frac{1}{12x}\right) - \left(x + \frac{1}{6x} - \frac{1}{2}\right) t - \left(\frac{x}{2} + \frac{1}{4x} - \frac{1}{2}\right) t^2 \\ & - \left(\frac{x}{3} + \frac{1}{3x} - \frac{1}{2}\right) t^3 - \left(\frac{x}{4} + \frac{5}{12x} - \frac{1}{2}\right) t^4,\end{aligned} \tag{49}$$

and

$$\rho_x(t) < 0 \quad \text{for } \ 0 < t < 1. \tag{50}$$

Now, using (49), we yield

$$\psi_x^{**}\left(\tfrac{1}{x}\right) = -\frac{6x^5 + x^4 - 2x + 5}{12x^5} < -\frac{x^2 - 2x + 5}{12x^5} < 0, \quad \text{for } x \geq 1. \tag{51}$$

Hence, from (48) and (50), we obtain

$$\psi_x'\left(\tfrac{1}{x}\right) < 0, \quad \text{for } x \geq 1. \tag{52}$$

But, since the function $t \mapsto \psi_x'(t)$ is decreasing, due to (38), we have

$$\psi_x'(t) < 0, \quad \text{for } t \geq \frac{1}{x} \text{ and } x \geq 1. \tag{53}$$

Therefore, the mapping $t \mapsto \psi_x(t)$ is strictly decreasing on the interval $\left[\frac{1}{x}, 1\right)$ and, due to (47), $\psi_x(t) < 0$ for $t \geq \frac{1}{x}$ and $x \geq 1$. Consequently, considering (36), the function $t \mapsto L^*(t,x)$ is strictly decreasing too, on the interval $\left[\frac{1}{x}, 1\right)$, and the estimate $0 < L^*(t,x) < 1$ holds for $t \in \left[\frac{1}{x}, 1\right)$ and $x > 1$. An easy calculation shows that $\lim_{t\uparrow 1} L^*(t,x) = 0$. This means that the sequence $n \mapsto L^*\left(\frac{n}{x}, x\right)$ is strictly decreasing and all its terms lie inside the interval $(0,1)$. ■

The function $t \mapsto L^*(t,x)$ does not differ very much from the function $t \mapsto \widetilde{L}^*(t,x) := \exp\left(-xt^2/2\right)$ as it is seen from the Figure 9, where both functions are plotted together.

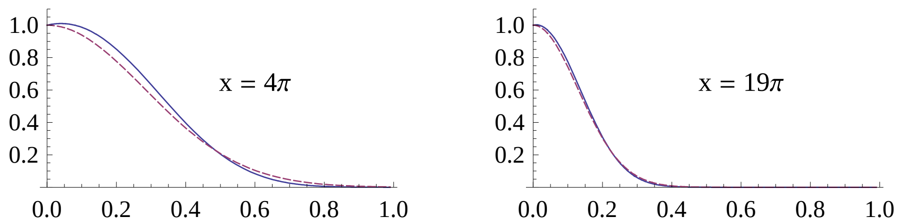

Figure 9: The graphs of the functions $t \mapsto L^*(t,x)$ (continuous line) and $t \mapsto \widetilde{L}^*(t,x)$ (dashed line).

Proposition 5. *The estimate*

$$\Psi_x^*(t) < L^*(t,x) < \Psi_x^{**}(t) \tag{54}$$

holds true for $x > 1$ and $t \in (0,1)$, where

$$\Psi_x^*(t) := \exp\left(-\frac{x}{2}\,t^2 + \Big(-\frac{x}{(1-t)^2} + \frac{1}{(1-t)^3} - \frac{1}{2x(1-t)^4}\Big)\frac{t^3}{6}\right) \tag{55}$$

$$\Psi_x^{**}(t) := \exp\left(-\frac{x}{2}\,t^2 + \frac{1}{2}\Big(1 - \frac{1}{6x}\Big)t + \frac{1}{4}\Big(1 - \frac{1}{3x}\Big)t^2\right). \tag{56}$$

Proof. Using the Maclaurin formula of order two and considering (36)–(38), we obtain

$$\begin{aligned}\psi_x(t) &= \left(\frac{1}{2} - \frac{1}{12x}\right)t + \frac{1}{2}\left[-x + \left(\frac{1}{2} - \frac{1}{6x}\right)\right]t^2 + \frac{1}{6}\,\psi_x^{(3)}(\tau)t^3 \\ &= -\,\frac{x}{2}\,t^2 + \left[\frac{1}{2}\left(1 - \frac{1}{6x}\right)t + \frac{1}{4}\left(1 - \frac{1}{3x}\right)t^2\right] + \frac{1}{6}\,\psi_x^{(3)}(t^*)t^3, \end{aligned} \tag{57}$$

for $t \in (0,1)$ and some t^* between 0 and t. But, according to (39) we have

$$\psi_x^{(3)}(t) \equiv -\frac{x}{(1-t)^2} + \frac{1}{(1-t)^3} - \frac{1}{2x(1-t)^4}, \tag{58}$$

where, referring to (40), the function $t \mapsto \psi_x^{(3)}(t)$ is strictly decreasing. Thus, $\psi_x^{(3)}(t^*) > \psi_x^{(3)}(t)$. Consequently, using (57) and (58), we get the estimate (54). ■

The relation (54) is illustrated in Figure 10 where are depicted the functions $t \mapsto \Psi_x^*(t)$, $t \mapsto L^*(t,x)$ and $t \mapsto \Psi_x^{**}(t)$.

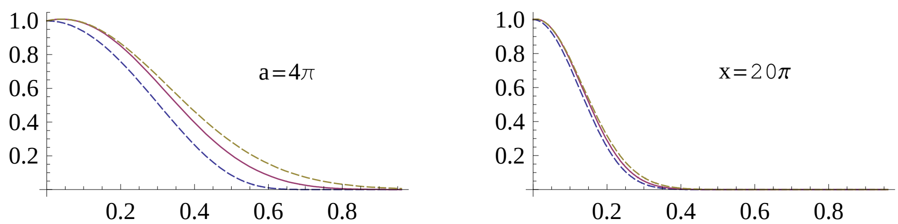

Figure 10: The graphs of the functions $t \mapsto \Psi_x^*(t)$, $t \mapsto L^*(t,x)$ and $t \mapsto \Psi_x^{**}(t)$.

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