Universitatea Babeş-Bolyai, Facultatea de Matematică şi Informatică

Secţia: Informatică engleză, Curs: Dynamical Systems

Laboratory 4.

1. Introduce the matrix $A = [[0, -2, 0], [1, -2, 0], [0, 0, -2]]$.

(a) Check that the column vector $u_1 = (0, 0, 1)$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = -2$, i.e. check that $Au_1 - \lambda_1 u_1$ is the null vector.

(b) Check that the column vector $u_2 = (1 + i, 1, 0)$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = -1 + i$, i.e. check that $Au_2 - \lambda_2 u_2$ is the null vector.

(c) Check that the column vector $u_3 = (1 - i, 1, 0)$ is an eigenvector corresponding to the eigenvalue $\lambda_3 = -1 - i$, i.e. check that $Au_3 - \lambda_3 u_3$ is the null vector.

(d) Introduce the matrix P whose columns are u_1, u_2, u_3 in this order.

(e) Introduce the diagonal matrix J whose elements on the main diagonal are $\lambda_1, \lambda_2, \lambda_3$ in this order.

(f) Check that $A = PJP^{-1}$, i.e. $A - PJP^{-1}$ is the null matrix. Of course, A is not diagonalizable over $\mathbb{R}$, but, due to this relation, it is said that A is diagonalizable over $\mathbb{C}$.

(g) Compute e^{tJ} and e^{tA}. The matrix e^{tJ} has complex entries with nonzero imaginary part because J has complex entries with nonzero imaginary part. The matrix A has real entries and we know that, by definition, e^{tA} has also real entries.

(h) Compute the limit as $t \to \infty$ for each entry of e^{tA}.

(i) True/False:

"Each solution of the differential system $X' = AX$ satisfies $\lim\limits_{t\to\infty} X(t) = 0_3$."

Here 0_3 is the null column vector in $\mathbb{R}^3$.

2. (a) Find a 4×4 real matrix A with the eigenvalues $2, 2, -1, 0$ which is not diagonal but it is diagonalizable over $\mathbb{R}$.

Hint: First find an invertible 4×4 real matrix P (there are many!), then introduce the diagonal matrix J with $2, 2, -1, 0$ on the main diagonal. Take $A = PJP^{-1}$. From this relation we deduce that A is similar to the diagonal matrix J, thus, by definition, it is diagonalizable.

(b) Find the determinant and the characteristic polynomial of A from (a). Find the eigenvectors of A. Find the Jordan form of A. If it is invertible, find the inverse of A.

3. Introduce the matrix (a) $A = [[2, 1, 0], [0, 2, 1], [0, 0, 2]]$.

(b) $A = [[0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1], [-1, 0, -2, 0]]$.

Find the eigenvectors and the Jordan form. Note that A is not diagonalizable.

Find e^{tA} and the general solution of $X' = AX$.

4. We consider $x' = 1 - x^2$.

(i) Find its equilibrium points. Find the expression of $\phi(t,-1)$, $\phi(t,1)$.
I think this is faster without Maple or Sage. In fact, Sage will not solve the IVPs $x' = 1 - x^2,\quad x(0) = -1$ and $x' = 1 - x^2,\quad x(0) = 1$.

(ii) Find the expression of each of the solutions $\varphi(t,-2)$, $\varphi(t,0)$, $\varphi(t,2)$.

Note that $\varphi(t,-2) = \frac{e^{2t}+3}{e^{2t}-3}, \quad \varphi(t,0) = \frac{e^{2t}-1}{e^{2t}+1}, \quad \varphi(t,2) = \frac{e^{2t}+1/3}{e^{2t}-1/3}$.
In Maple these expressions have other forms. Use
convert(convert(tanh(t-arctanh(2)),exp),exp) to obtain the expression $\frac{e^{2t}+3}{e^{2t}-3}$.

(iii) Represent the graph of $\varphi(t,-2)$, $\varphi(t,0)$, $\varphi(t,2)$. Note that their maximal interval of definition has the form $I_{-2} = (-\infty, \beta_{-2})$, $I_0 = \mathbb{R}$ and $I_2 = (\alpha_2, +\infty)$, where $\beta_{-2}, \alpha_2 \in \mathbb{R}$. Pay attention that we talk about the interval of definition! And $\mathbb{R} \setminus \{a\}$ is not an interval, it is a union of two intervals.
If you want, find the exact values $\beta_{-2} = \ln \sqrt{3}$ and $\alpha_2 = -\ln \sqrt{3}$.

(iv) Find $\lim\limits_{t\to-\infty} \varphi(t,-2)$, $\lim\limits_{t\to-\infty} \varphi(t,0)$, $\lim\limits_{t\to+\infty} \varphi(t,0)$, $\lim\limits_{t\to+\infty} \varphi(t,2)$.

(v) Specify the monotonicity of the functions found at (ii) looking at their graph. Find the image of each function. More exactly, prove that $\gamma_{-2} = (-\infty,-1)$, $\gamma_0 = (-1,1)$, $\gamma_2 = (1,\infty)$.

(vi) Finally, in your notebooks represent the phase portrait of $x' = 1 - x^2$ and confirm (using the theory presented in the lecture) the properties you found.