Exercise 1:

Integration by substitution

a) $\int x \cos(x^2)\,dx$

$u = x^2$

$du = dx^2 = (x^2)'\,dx = 2 \cdot x\,dx$

$\boxed{\frac{du}{2} = x\,dx}$

$\int x \cos(x^2)\,dy = \int \cos u \cdot \frac{du}{2} = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C$

$= \frac{1}{2}\sin x^2 + C$

b) $\boxed{\int e^x \sin x \, dx}$ $\rightarrow$ $\boxed{\text{Integration by parts}}$

$= I$

$\boxed{u = e^x} \rightarrow du = de^x = e^x \, dx$ $(e^x)'$

$dv = \sin x \, dx \Rightarrow v = \int dv = \int \sin x \, dx =$

$= \boxed{-\cos x = v}$

In general:

$$\int u \, dv = u \cdot v - \int v \, du$$

$$\int e^x \sin x \, dx = e^x \cdot (-\cos x) - \int (-\cos x) \cdot e^x \, dx = -e^x \cos x + \int \cos x \, e^x \, dx$$

$$\boxed{\int \cos x\, e^x\, dx}$$

$$\boxed{u = e^x} \rightarrow \boxed{du = e^x\, dx}$$

$$\boxed{dv = \cos x\, dx} \rightarrow V = \int dv = \int \cos x\, dx = \boxed{\sin x = v}$$

$$\boxed{\int \cos x\, e^x\, dx} = e^x \cdot \sin x - \boxed{\int \sin x\, e^y\, dx} = I$$

$$I = -e^x \cos x + \int \cos x\, e^y\, dx = -e^x \cos x + \left(e^x \sin x - I\right)$$

$$I = -e^y \cos x + e^y \sin x - I \Rightarrow \boxed{2I = -e^y \cos x + e^x \sin x}$$

$$I = \int \sin x\, e^x\, dx = \frac{-e^x \cos x + e^x \sin x}{2} + C$$

$$\int e^x \sin x \, dx$$

$$\int e^x \cos x \, dx$$

Exercise 1:

c) $\int \frac{\ln(\arctan(x))}{1+x^2}\,dx$

$\int \ln(u)\cdot 1\cdot du$

In general

$\int \underset{k}{u}\, d\underset{l}{v} = u\cdot v - \int v\,du$

$u = \arctan(x)$

$du = d(\arctan(x)) = (\arctan(x))' \cdot dx = \boxed{\frac{dx}{1+x^2} = du}$

$k = \ln u \to dk = d(\ln u) = (\ln u)'\,du = \frac{1}{u}\,du$

$dl = 1\cdot du$

$l = \int dl = \int du = \boxed{u = l}$

$\int \ln u\,du = \ln(u)\cdot u - \int u\cdot\frac{du}{u} = \ln(u)\cdot u - \int du$

$$\int \ln u \, du = \ln u \cdot u - \int du = \ln u \cdot u - u + c$$

$$\int \frac{\ln(\arctan(x))}{1+x^2} \, dx = \ln(\arctan(x)) \cdot \arctan(x) - \arctan(x) + c$$

Exercise 2:

a) $\int x^2 \sin(x)\,dx$ → Integration by parts

$u = x^2 \rightarrow \boxed{du = 2x\,dx}$

$dv = \sin x\,dx \rightarrow v = \int dv = \int \sin x\,dx =$

$= -\cos x$

$\int x^2 \sin(x)\,dx = x^2 \cdot (-\cos x) - \int (-\cos x) \cdot 2x\,dx =$

$= -x^2 \cos x + \boxed{\int 2x \cos x\,dx}$

$u = 2x \rightarrow du = 2\,dx$

$dv = \cos x\,dx \rightarrow v = \int dv = \int \cos x\,dx = \sin x$

$$\int 2x\cos(x)\,dx = 2x\cdot\sin(x) - \int \sin(x)\cdot 2\,dx =$$

$$= 2x\cdot\sin(x) - 2\int \sin(x)\,dx = 2\cdot x\sin(x) - 2\cdot(-\cos x) + C$$

$$\int x^2 \sin(x)\,dx = -x^2\cdot\cos x + 2x\sin(x) + 2\cos(x) + C$$

Exercise 2 :

g) $\int x \ln(1+x)\, dx$

$u = 1 + x \Rightarrow x = u - 1$

$du = (1+x)'\, dx = dx$

$(1+x)' = 1' + x' = 0 + 1 = 1$

$\int x \ln(1+x)\, dx = \int (u-1) \ln u \cdot du$

$k = \ln u \longrightarrow dk = \frac{1}{u}\, du$

$dl = (u-1)\, du \rightarrow l = \int dl = \int (u-1)\, du = \int u\, du - \int 1 \cdot du = \frac{u^2}{2} - u = l$

$$\int (v-1)\ln(v)\,dv = \ln(v)\cdot\left(\frac{v^2}{2}-v\right) - \int\left(\frac{v^2}{2}-v\right)\frac{1}{v}\,dv$$

$$= \ln(v)\cdot\left(\frac{v^2}{2}-v\right) - \int\left(\frac{v^{\cancel{2}}}{2\cdot\cancel{v}} - \frac{\cancel{v}}{\cancel{v}}\right)dv =$$

$$= \ln(v)\cdot\left(\frac{v^2}{2}-v\right) - \int\left(\frac{v}{2}-1\right)dv = \ln(v)\cdot\left(\frac{v^2}{2}-v\right) -$$

$$- \int\frac{v}{2}\,dv + \int dv = \ln(v)\cdot\left(\frac{v^2}{2}-v\right) - \frac{1}{2}\cdot\frac{v^2}{2} + v + C$$

$$= \ln(x+1)\cdot\left(\frac{(x+1)^2}{2} - (x+1)\right) - \frac{1}{4}(x+1)^2 + (x+1) + C$$

$$\int x \ln(1+x)\,dx = \ln(x+1)\cdot\left(\frac{(x+1)^2}{2} - (x+1)\right)$$

$$-\frac{1}{4}(x+1)^2 + (x+1) + C$$

Exercise 4 :

$f(1) = 2$, $f(4) = 7$ $f'(1) = 5$

$f'(4) = 3$

$$\int_1^4 x f''(x)\,dx \qquad u = x \rightarrow \boxed{du = dx}$$

$$dv = f''(x)\,dx \rightarrow v = \int dv = \int f''(x)\,dx$$

$$= \boxed{f'(x) = v}$$

$$\int_1^4 x f''(x)\,dx = \left(x \cdot f'(x)\right)\Big|_1^4 - \int_1^4 f'(x)\,dx =$$

$$= \left(4 \cdot f'(4) - 1 \cdot f'(1)\right) - f(x)\Big|_1^4 = (4 \cdot 3 - 1 \cdot 5) - \left(f(4) - f(1)\right)$$

$$= (12 - 5) - (7 - 2) =$$

$$= 7 - 5 = \boxed{2}$$

$$\int_{0}^{0} x\,dx = \frac{x^2}{2} + C$$

$$\int_{1}^{2} x\,dx = \frac{x^2}{2}\Big|_1^2 = \left(\frac{2^2}{2} - \frac{1^2}{2}\right)$$

Exercise 3:

$$\int_0^3 x \sin(3-x)\,dx$$

$\boxed{l = 3 - x} \Rightarrow l - 3 = -x$

$x = 3 - l$

$dl = (3-x)'\,dx = -dx = dl$

multiply by (-1)

$\to x = 0 \to l = 3 - 0 = 3$

$\to x = 3 \to l = 3 - 3 = 0$

$$\int_3^0 (3-l)\sin l\,(-dl) = \int_0^3 (3-l)\sin(l)\,dl$$

Integration by parts

$$u = 3 - l \;\rightarrow\; du = -\,dl$$

$$dv = \sin l \;\rightarrow\; v = \int dv = \int \sin l \, dl = -\cos l$$

$$\int_0^3 (3-l)\sin l \, dl = (3-l)\cdot(-\cos l)\Big|_0^3 - \int_0^3 -\cos l \cdot -dl =$$

$$= \Bigg(\underbrace{(\overset{0}{\overset{\shortparallel}{3-3}}) \cdot (-\cos(3))}_{F(3)} - \underbrace{(3-0)\cdot(-\cos 0)}_{F(1)} \Bigg) - \int_0^3 \cos l \, dl =$$

$$= 3\cdot 1 - \sin l\Big|_0^3 = 3 - \left(\sin(3) - \overset{0}{\overset{\shortparallel}{\sin(0)}}\right) = 3 - \sin 3 + \overset{=0}{\sin 0}$$