Energy Trading

Lecture - Summer 2013

Professor Dr. Rüdiger Kiesel | Chair for Energy Trading and Finance | University of Duisburg-Essen

Agenda

Introduction
- Energy Markets
- Options, Forwards and Swaps

Stochastic Processes for Price Movements

Energy Derivatives

Energy Derivatives

Liberalisation

The German Electricity market went into Liberalization in April 1998.
The Pre - Liberalisation system was based on calculatory costs: the price was according to the 'cost-plus' rule

- Integrated value-chain: production, grid, distribution
- Electricity production to secure supply within a regional monopole
- Long-term supply contracts
- No liquid market on the whole sale market
- Regulated consumer prices, regulated investments

Liberalisation

The German Electricity market went into Liberalization in April 1998.
The Pre - Liberalisation system was based on calculatory costs: the price was according to the 'cost-plus' rule

- Integrated value-chain: production, grid, distribution
- Electricity production to secure supply within a regional monopole
- Long-term supply contracts
- No liquid market on the whole sale market
- Regulated consumer prices, regulated investments

Liberalisation

The German Electricity market went into Liberalization in April 1998.
The Pre - Liberalisation system was based on calculatory costs: the price was according to the 'cost-plus' rule

- Integrated value-chain: production, grid, distribution
- Electricity production to secure supply within a regional monopole
- Long-term supply contracts
- No liquid market on the whole sale market
- Regulated consumer prices, regulated investments

Liberalisation

The German Electricity market went into Liberalization in April 1998.
The Pre - Liberalisation system was based on calculatory costs: the price was according to the ’cost-plus’ rule

- Integrated value-chain: production, grid, distribution
- Electricity production to secure supply within a regional monopole
- Long-term supply contracts
- No liquid market on the whole sale market
- Regulated consumer prices, regulated investments

Liberalisation

The German Electricity market went into Liberalization in April 1998.
The Pre - Liberalisation system was based on calculatory costs: the price was according to the ’cost-plus’ rule

- Integrated value-chain: production, grid, distribution
- Electricity production to secure supply within a regional monopole
- Long-term supply contracts
- No liquid market on the whole sale market
- Regulated consumer prices, regulated investments

Liberalisation

Post - Liberalisation system based on forces of market: higher volatility of prices, flexibility has value.

- Unbundling of value-chain
- Power plants are used optimally – no obligation to secure supply
- New players and products
- Trading in Long- and Short-positions on a liquid whole sale market
- Investments based on market expectations

Liberalisation

Post - Liberalisation system based on forces of market: higher volatility of prices, flexibility has value.

- Unbundling of value-chain
- Power plants are used optimally – no obligation to secure supply
- New players and products
- Trading in Long- and Short-positions on a liquid whole sale market
- Investments based on market expectations

Liberalisation

Post - Liberalisation system based on forces of market: higher volatility of prices, flexibility has value.

- Unbundling of value-chain
- Power plants are used optimally – no obligation to secure supply
- New players and products
- Trading in Long- and Short-positions on a liquid whole sale market
- Investments based on market expectations

Liberalisation

Post - Liberalisation system based on forces of market: higher volatility of prices, flexibility has value.

- Unbundling of value-chain
- Power plants are used optimally – no obligation to secure supply
- New players and products
- Trading in Long- and Short-positions on a liquid whole sale market
- Investments based on market expectations

Liberalisation

Post - Liberalisation system based on forces of market: higher volatility of prices, flexibility has value.

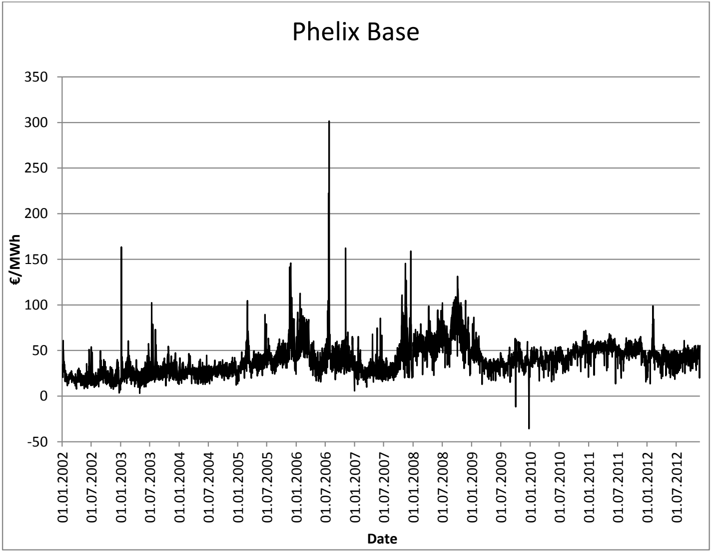
- Unbundling of value-chain
- Power plants are used optimally – no obligation to secure supply
- New players and products
- Trading in Long- and Short-positions on a liquid whole sale market
- Investments based on market expectations

Markets

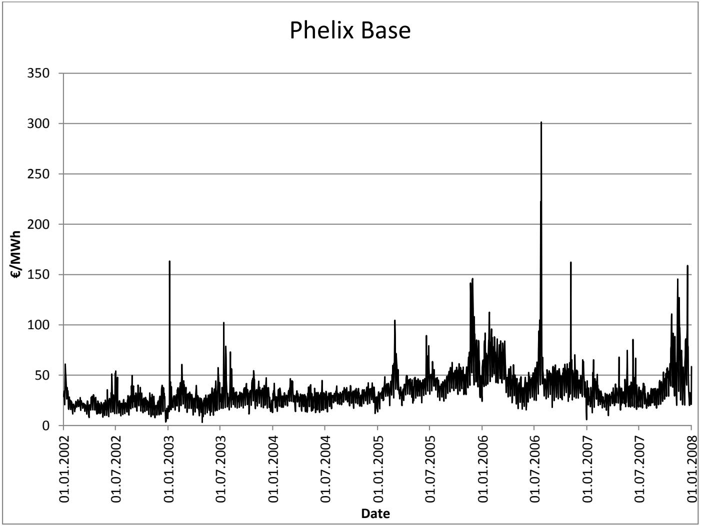
Since the deregulation of electricity markets in the end of the 1990s, power can be traded at exchanges like the Nordpool, http://www.nordpoolspot.com/ or the European Energy Exchange (EEX), http://www.eex.com/en. All exchanges have established spot and futures markets.

UNIVERSITÄT DUISBURG ESSEN

Spot prices

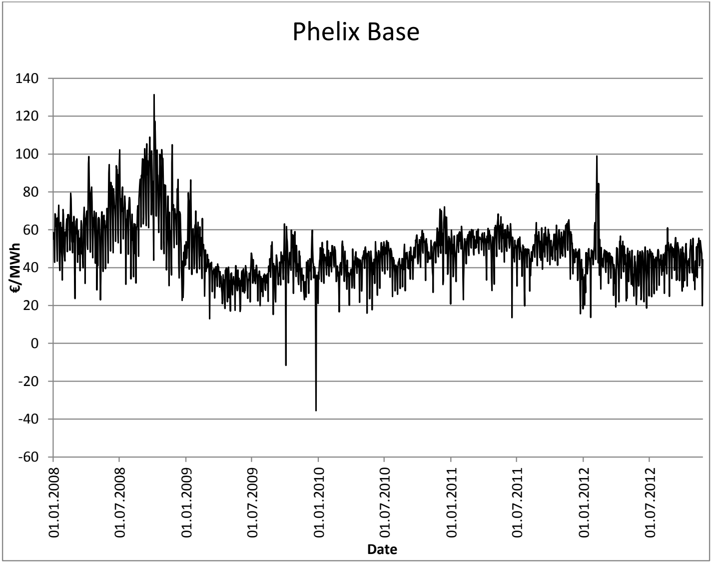


Spot prices


Phelix Base
€/MWh
350
300
250
200
150
100
50
0
01.01.2002
01.07.2002
01.01.2003
01.07.2003
01.01.2004
01.07.2004
01.01.2005
01.07.2005
01.01.2006
01.07.2006
01.01.2007
01.07.2007
01.01.2008
Date

UNIVERSITÄT DUISBURG ESSEN

Spot prices



Derivative Background

A derivative security, or contingent claim, is a financial contract whose value at expiration date T (more briefly, expiry) is determined exactly by the price (or prices within a prespecified time-interval) of the underlying financial assets (or instruments) at time T (within the time interval $[0, T]$).
Derivative securities can be grouped under three general headings: *Options, Forwards and Futures* and *Swaps*. During this lectures we will encounter all this structures and further variants.

Underlying Securities

- We will mainly use Commodities or Commodity Futures;
- Fixed income instruments: T-Bonds, Interest Rates (LIBOR, EURIBOR);
- Other classes are possible: (one or several) Stocks; Currencies (FX);
- Also Derivatives may be used as underlying for compound derivatives (call on call).

Underlying Securities

- We will mainly use Commodities or Commodity Futures;
- Fixed income instruments: T-Bonds, Interest Rates (LIBOR, EURIBOR);
- Other classes are possible: (one or several) Stocks; Currencies (FX);
- Also Derivatives may be used as underlying for compound derivatives (call on call).

Underlying Securities

- We will mainly use Commodities or Commodity Futures;
- Fixed income instruments: T-Bonds, Interest Rates (LIBOR, EURIBOR);
- Other classes are possible: (one or several) Stocks; Currencies (FX);
- Also Derivatives may be used as underlying for compound derivatives (call on call).

Underlying Securities

- We will mainly use Commodities or Commodity Futures;
- Fixed income instruments: T-Bonds, Interest Rates (LIBOR, EURIBOR);
- Other classes are possible: (one or several) Stocks; Currencies (FX);
- Also Derivatives may be used as underlying for compound derivatives (call on call).

UNIVERSITÄT DUISBURG ESSEN

Modelling Assumptions (Financial Markets)

We impose the following set of assumptions on the financial markets:

- *No market frictions:* No transaction costs, no bid/ask spread, no taxes, no margin requirements, no restrictions on short sales.
- *No default risk:* Implying same interest for borrowing and lending.
- *Competitive markets:* Market participants act as price takers.
- *Rational agents:* Market participants prefer more to less.

Modelling Assumptions (Financial Markets)

We impose the following set of assumptions on the financial markets:

- *No market frictions:* No transaction costs, no bid/ask spread, no taxes, no margin requirements, no restrictions on short sales.
- *No default risk:* Implying same interest for borrowing and lending.
- *Competitive markets:* Market participants act as price takers.
- *Rational agents:* Market participants prefer more to less.

Modelling Assumptions (Financial Markets)

We impose the following set of assumptions on the financial markets:

- *No market frictions:* No transaction costs, no bid/ask spread, no taxes, no margin requirements, no restrictions on short sales.
- *No default risk:* Implying same interest for borrowing and lending.
- *Competitive markets:* Market participants act as price takers.
- *Rational agents:* Market participants prefer more to less.

Modelling Assumptions (Financial Markets)

We impose the following set of assumptions on the financial markets:

- *No market frictions:* No transaction costs, no bid/ask spread, no taxes, no margin requirements, no restrictions on short sales.
- *No default risk:* Implying same interest for borrowing and lending.
- *Competitive markets:* Market participants act as price takers.
- *Rational agents:* Market participants prefer more to less.

Arbitrage

The concept of arbitrage lies at the centre of the relative pricing theory. All we need to assume additionally is that economic agents prefer more to less, or more precisely, an increase in consumption without any costs will always be accepted.
The essence of the technical sense of arbitrage is that it should not be possible to guarantee a profit without exposure to risk. Were it is possible to do so, arbitrageurs (we use the French spelling, as is customary) would do so, in unlimited quantity, using the market as a 'money-pump' to extract arbitrarily large quantities of riskless profit.
We assume that arbitrage opportunities do not exist!

Options

An option is a financial instrument giving one the *right but not the obligation* to make a specified transaction at (or by) a specified date at a specified price. *Call* options give one the right to buy. *Put* options give one the right to sell. *European* options give one the right to buy/sell on the specified date, the expiry date, on which the option expires or matures. *American* options give one the right to buy/sell at any time prior to or at expiry.

Options

The simplest call and put options are now so standard that they are called *vanilla* options.

Many kinds of options now exist, including so-called *exotic* options. Types include: *Asian* options, which depend on the *average* price over a period, *lookback* options, which depend on the *maximum* or *minimum* price over a period and *barrier* options, which depend on some price level being attained or not.

Terminology

The asset to which the option refers is called the *underlying asset* or the *underlying*. The price at which the transaction to buy/sell the underlying, on/by the expiry date (if exercised), is made, is called the *exercise price* or *strike price*. We shall usually use K for the strike price, time $t = 0$ for the initial time (when the contract between the buyer and the seller of the option is struck), time $t = T$ for the expiry or final time. Consider, say, a European call option, with strike price K; write $S(t)$ for the value (or price) of the underlying at time t. If $S(t) > K$, the option is *in the money*, if $S(t) = K$, the option is said to be *at the money* and if $S(t) < K$, the option is *out of the money*.

Payoff

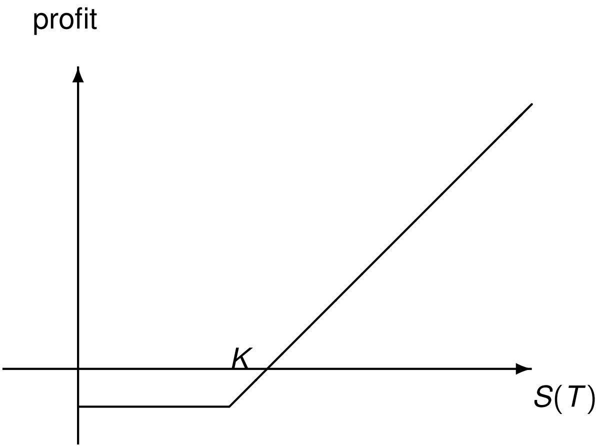
The payoff from the option is

$$S(T) - K \text{ if } S(T) > K \quad \text{and} \quad 0 \ \text{ otherwise}$$

(more briefly written as $(S(T) - K)^+$).
Taking into account the initial payment of an investor one obtains the profit diagram below.

Payoff

Profit diagram for a European call



Example: Options

A trader purchases a European call option maturing in 6 month for 100 Barrels crude oil with strike 82 USD/Barrel. He pays a premium of 2 USD/Barrel.

If the oil price rises to 87 USD/Barrel at maturity of the option, then the trader can exercise the call and buy 100 Barrels of crude oil for 82 USD/Barrel and sell them at 87 USD/Barrel in the market, making a profit of 300 USD.

If, however, the price of crude oil at maturity drops to 81 USD/Barrel below the strike price, then the trader would not exercise the option, making a loss (limited to the cost of the call premium) of 200 USD.

Arbitrage Relationship- Example

We now use the principle of no-arbitrage to obtain bounds for option prices. We focus on European options (puts and calls) with identical underlying (say a stock S), strike K and expiry date T. Furthermore we assume the existence of a risk-free bank account (bond) with constant interest rate r (continuously compounded) during the time interval $[0, T]$. We start with a fundamental relationship:

We have the following put-call parity between the prices of the underlying asset S and European call and put options on stocks that pay no dividends:

$$S_t + P_t - C_t = Ke^{-r(T-t)}. \tag{1}$$

Arbitrage Relationship - Example

Consider a portfolio consisting of one stock, one put and a short position in one call (the holder of the portfolio has written the call); write $V(t)$ for the value of this portfolio. Then

$$V(t) = S(t) + P(t) - C(t)$$

for all $t \in [0, T]$. At expiry we have

$$\begin{aligned} V(T) &= S(T) + (S(T) - K)^{-} - (S(T) - K)^{+} \\ &= S(T) + K - S(T) = K. \end{aligned}$$

This portfolio thus guarantees a payoff K at time T. Using the principle of no-arbitrage, the value of the portfolio must at any time t correspond to the value of a sure payoff K at T, that is $V(t) = Ke^{-r(T-t)}$. ■

European Call Price

For a European call $X = (S(T) - K)^+$ and we can evaluate the above expected value
The Black-Scholes price process of a European call is given by

$$
\begin{aligned}
C(t) &= S(t)\Phi(d_1(S(t), T-t)) \\
&\quad - Ke^{-r(T-t)}\Phi(d_2(S(t), T-t)).
\end{aligned}
$$

The functions $d_1(s,t)$ and $d_2(s,t)$ are given by

$$
\begin{aligned}
d_1(s,t) &= \frac{\log(s/K) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}, \\
d_2(s,t) &= \frac{\log(s/K) + (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}
\end{aligned}
$$

Forwards and Futures

- A *forward contract* is an agreement to buy or sell an asset S at a certain future date T for a certain price K.
- The agent who agrees to buy the underlying asset is said to have a *long* position, the other agent assumes a *short* position.
- The settlement date is called *delivery date* and the specified price is referred to as *delivery price*.

Forwards and Futures

- A *forward contract* is an agreement to buy or sell an asset S at a certain future date T for a certain price K.
- The agent who agrees to buy the underlying asset is said to have a *long* position, the other agent assumes a *short* position.
- The settlement date is called *delivery date* and the specified price is referred to as *delivery price*.

Forwards and Futures

- A *forward contract* is an agreement to buy or sell an asset S at a certain future date T for a certain price K.
- The agent who agrees to buy the underlying asset is said to have a *long* position, the other agent assumes a *short* position.
- The settlement date is called *delivery date* and the specified price is referred to as *delivery price*.

Forwards

- The *forward price* $F(t, T)$ is the delivery price which would make the contract have zero value at time t.
- At the time the contract is set up, $t = 0$, the forward price therefore equals the delivery price, hence $F(0, T) = K$.
- The forward prices $F(t, T)$ need not (and will not) necessarily be equal to the delivery price K during the life-time of the contract.

UNIVERSITÄT DUISBURG ESSEN

Forwards

- The *forward price* $F(t, T)$ is the delivery price which would make the contract have zero value at time t.
- At the time the contract is set up, $t = 0$, the forward price therefore equals the delivery price, hence $F(0, T) = K$.
- The forward prices $F(t, T)$ need not (and will not) necessarily be equal to the delivery price K during the life-time of the contract.

Forwards

- The *forward price* $F(t, T)$ is the delivery price which would make the contract have zero value at time t.
- At the time the contract is set up, $t = 0$, the forward price therefore equals the delivery price, hence $F(0, T) = K$.
- The forward prices $F(t, T)$ need not (and will not) necessarily be equal to the delivery price K during the life-time of the contract.

Forwards

- The payoff from a long position in a forward contract on one unit of an asset with price $S(T)$ at the maturity of the contract is
$$S(T) - K.$$
- Compared with a call option with the same maturity and strike price K we see that the investor now faces a downside risk, too. He has the obligation to buy the asset for price K.

Forwards

- The payoff from a long position in a forward contract on one unit of an asset with price $S(T)$ at the maturity of the contract is
$$S(T) - K.$$
- Compared with a call option with the same maturity and strike price K we see that the investor now faces a downside risk, too. He has the obligation to buy the asset for price K.

Futures

- Futures can be defined as standardized forward contracts traded at exchanges where a clearing house acts as a central counterparty for all transactions.
- Usually an initial margin is paid as a guarantee.
- Each trading day a settlement price is determined and gains or losses are immediately realized at a margin account.
- Thus credit risk is eliminated, but there is exposure to interest rate risk.

Futures

- Futures can be defined as standardized forward contracts traded at exchanges where a clearing house acts as a central counterparty for all transactions.
- Usually an initial margin is paid as a guarantee.
- Each trading day a settlement price is determined and gains or losses are immediately realized at a margin account.
- Thus credit risk is eliminated, but there is exposure to interest rate risk.

Futures

- Futures can be defined as standardized forward contracts traded at exchanges where a clearing house acts as a central counterparty for all transactions.
- Usually an initial margin is paid as a guarantee.
- Each trading day a settlement price is determined and gains or losses are immediately realized at a margin account.
- Thus credit risk is eliminated, but there is exposure to interest rate risk.

UNIVERSITÄT DUISBURG ESSEN

Futures

- Futures can be defined as standardized forward contracts traded at exchanges where a clearing house acts as a central counterparty for all transactions.
- Usually an initial margin is paid as a guarantee.
- Each trading day a settlement price is determined and gains or losses are immediately realized at a margin account.
- Thus credit risk is eliminated, but there is exposure to interest rate risk.

Black's Formula

We use the same notation - strike K, expiry T as in the spot case, and write Φ for the standard normal distribution function.

The arbitrage price C of a European futures call option is

$$C(t) = c(f(t), T - t),$$

where $c(f, t)$ is given by Black's futures options formula:

$$c(f, t) := e^{-rt}(f\Phi(\tilde{d}_1(f, t)) - K\Phi(\tilde{d}_2(f, t))),$$

where

$$\tilde{d}_{1,2}(f, t) := \frac{\log(f/K) \pm \frac{1}{2}\sigma^2 t}{\sigma\sqrt{t}}.$$

Swaps

A *swap* is an agreement whereby two parties undertake to exchange, at known dates in the future, various financial assets (or cash flows) according to a prearranged formula that depends on the value of one or more underlying assets. Examples are currency swaps (exchange currencies) and interest-rate swaps (exchange of fixed for floating set of interest payments).

Spread Options

Spread options can be used by owners of corresponding plants to manage market risk.
The pay off of a typical spread is

$$C^{(T)}_{\text{spread}} = \max(S_1(T) - S_2(T) - K, 0)$$

with S_i the underlyings, K the strike.

UNIVERSITÄT DUISBURG ESSEN

Spread Options

For $K = 0$ (exchange option) there is an analytic formula due to Margrabe (1978).

$$
\begin{aligned}
C_{\text{spread}}(t) &= e^{-r(T-t)}\left(S_1(t)\Phi(d_1) - S_2(t)\Phi(d_2)\right) \\
\text{where} \quad d_1 &= \frac{\log(S_1(t)/S_2(t)) + \sigma^2(T-t)/2}{\sqrt{\sigma^2(T-t)}} \quad , d_2 = d_1 - \sqrt{\sigma^2(T-t)} \\
\text{and} \quad \sigma &= \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}
\end{aligned}
$$

where ρ is the correlation between the two underlyings.
For $K \neq 0$ no easy analytic formula is available.

Agenda

Stock Price Return

- We wish to model the time evolution of a stock price $S(t)$ and consider how S will change in some small time-interval from the present time t to a time $t + dt$ in the near future.
- Writing $dS(t)$ for the change $S(t + dt) - S(t)$ in S, the *return* on S in this interval is $dS(t)/S(t)$. We decompose the return into two components, a *systematic* part and a *random* part.
- The systematic is modelled by μdt, where μ is some parameter representing the mean rate of return of the stock.
- The random part is modelled by $\sigma dW(t)$, where $dW(t)$ represents the stochastic noise term driving the stock price dynamics, and σ is a second parameter describing how much the stock price fluctuates. Thus σ governs how volatile the price is, and is called the *volatility* of the stock.

Stock Price Return

- We wish to model the time evolution of a stock price $S(t)$ and consider how S will change in some small time-interval from the present time t to a time $t + dt$ in the near future.
- Writing $dS(t)$ for the change $S(t + dt) - S(t)$ in S, the *return* on S in this interval is $dS(t)/S(t)$. We decompose the return into two components, a *systematic* part and a *random* part.
- The systematic is modelled by μdt, where μ is some parameter representing the mean rate of return of the stock.
- The random part is modelled by $\sigma dW(t)$, where $dW(t)$ represents the stochastic noise term driving the stock price dynamics, and σ is a second parameter describing how much the stock price fluctuates. Thus σ governs how volatile the price is, and is called the *volatility* of the stock.

Stock Price Return

- We wish to model the time evolution of a stock price $S(t)$ and consider how S will change in some small time-interval from the present time t to a time $t + dt$ in the near future.
- Writing $dS(t)$ for the change $S(t + dt) - S(t)$ in S, the *return* on S in this interval is $dS(t)/S(t)$. We decompose the return into two components, a *systematic* part and a *random* part.
- The systematic is modelled by μdt, where μ is some parameter representing the mean rate of return of the stock.
- The random part is modelled by $\sigma dW(t)$, where $dW(t)$ represents the stochastic noise term driving the stock price dynamics, and σ is a second parameter describing how much the stock price fluctuates. Thus σ governs how volatile the price is, and is called the *volatility* of the stock.

Stock Price Return

- We wish to model the time evolution of a stock price $S(t)$ and consider how S will change in some small time-interval from the present time t to a time $t + dt$ in the near future.
- Writing $dS(t)$ for the change $S(t + dt) - S(t)$ in S, the *return* on S in this interval is $dS(t)/S(t)$. We decompose the return into two components, a *systematic* part and a *random* part.
- The systematic is modelled by μdt, where μ is some parameter representing the mean rate of return of the stock.
- The random part is modelled by $\sigma dW(t)$, where $dW(t)$ represents the stochastic noise term driving the stock price dynamics, and σ is a second parameter describing how much the stock price fluctuates. Thus σ governs how volatile the price is, and is called the *volatility* of the stock.

Geometric Brownian Motion

Putting this together, we have the stochastic differential equation (SDE)

$$dS(t) = S(t)(\mu dt + \sigma dW(t)), \quad S(0) > 0, \tag{2}$$

due to Itô in 1944.

The economic importance of geometric Brownian motion was recognised by Paul A. Samuelson in his work, for which Samuelson received the Nobel Prize in Economics in 1970, and by Robert Merton, in work for which he was similarly honoured in 1997.

Brownian Motion I

- For the random noise we use Brownian Motion (introduced by the Botanist Robert Brown in 1828. It was introduced into finance by Louis Bachelier in 1900, and developed in physics by Albert Einstein in 1905. A mathematical theory was developed by Norbert Wiener) A stochastic process $X = (X(t))_{t \geq 0}$ is a standard Brownian motion, *BM*, if

(i) $X(0) = 0$ a.s.,

(ii) X has *independent increments*: $X(t+u) - X(t)$ is independent of $\sigma(X(s) : s \leq t)$ for $u \geq 0$,

(iii) X has *stationary increments*: the law of $X(t+u) - X(t)$ depends only on u,

and (iv), (v)

Brownian Motion II

A stochastic process $X = (X(t))_{t \geq 0}$ is a standard Brownian motion, *BM*, if (i) – (iii) and

(iv) X has *Gaussian increments*: $X(t+u) - X(t)$ is normally distributed with mean 0 and variance u,
$X(t+u) - X(t) \sim N(0, u)$,

(v) X has *continuous paths*: $X(t)$ is a continuous function of t.

Itô Processes

- We will use the following type of process expressed in terms of the stochastic differential equation

$$dX(t) = b(t)dt + \sigma(t)dW(t), \quad X(0) = x_0.$$

- For functions f we want to give meaning to the stochastic differential $df(X(t))$ of the process $f(X(t))$.
- This is done by the *Itô Formula*

$$\begin{aligned} df(X(t)) &= f'(X(t))dX(t) \\ &\quad + \frac{1}{2}f''(X(t))\sigma^2 dt. \end{aligned}$$

UNIVERSITÄT DUISBURG ESSEN

Itô Processes

- We will use the following type of process expressed in terms of the stochastic differential equation

$$dX(t) = b(t)dt + \sigma(t)dW(t), \quad X(0) = x_0.$$

- For functions f we want to give meaning to the stochastic differential $df(X(t))$ of the process $f(X(t))$.
- This is done by the *Itô Formula*

$$\begin{aligned} df(X(t)) &= f'(X(t))dX(t) \\ &\quad + \frac{1}{2}f''(X(t))\sigma^2 dt. \end{aligned}$$

Itô Processes

- We will use the following type of process expressed in terms of the stochastic differential equation

$$dX(t) = b(t)dt + \sigma(t)dW(t), \quad X(0) = x_0.$$

- For functions f we want to give meaning to the stochastic differential $df(X(t))$ of the process $f(X(t))$.
- This is done by the *Itô Formula*

$$\begin{aligned} df(X(t)) &= f'(X(t))dX(t) \\ &\quad + \frac{1}{2} f''(X(t))\sigma^2 dt. \end{aligned}$$

UNIVERSITÄT DUISBURG ESSEN

Multiplication rules

- The second term above corrects for special path properties of Brownian Motion and needs the quadratic variation of the process.
- We find

$$
\begin{aligned}
(dX)^2 &= (bdt + \sigma dW)^2 \\
&= \sigma^2 dt + 2b\sigma dt dW + b^2 (dt)^2 = \sigma^2 dt.
\end{aligned}
$$

- The quadratic variation of any Itô process can be calculated using the multiplication rules

	dt	dW
dt	0	0
dW	0	dt

Multiplication rules

- The second term above corrects for special path properties of Brownian Motion and needs the quadratic variation of the process.
- We find

$$
\begin{aligned}
(dX)^2 &= (bdt + \sigma dW)^2 \\
&= \sigma^2 dt + 2b\sigma dt dW + b^2 (dt)^2 = \sigma^2 dt.
\end{aligned}
$$

- The quadratic variation of any Itô process can be calculated using the multiplication rules

	dt	dW
dt	0	0
dW	0	dt

Multiplication rules

- The second term above corrects for special path properties of Brownian Motion and needs the quadratic variation of the process.
- We find

$$\begin{aligned}(dX)^2 &= (bdt + \sigma dW)^2 \\ &= \sigma^2 dt + 2b\sigma dt dW + b^2 (dt)^2 = \sigma^2 dt.\end{aligned}$$

- The quadratic variation of any Itô process can be calculated using the multiplication rules

	dt	dW
dt	0	0
dW	0	dt

General Itô Formula

If $X(t)$ is an Itô process and $f(t,x)$ a function with time and location variable, then $f = f(t, X(t))$ has stochastic differential

$$df = \left(f_t + bf_x + \frac{1}{2}\sigma^2 f_{xx} \right) dt + \sigma f_x dW.$$

Observe, that we left out all function arguments

Example: Geometric Brownian Motion

The SDE for GBM has the unique solution

$$S(t) = S(0)\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right\}.$$

Therefore, writing

$$f(t,x) := \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma x\right\},$$

we have

$$f_t = \left(\mu - \frac{1}{2}\sigma^2\right)f, \quad f_x = \sigma f, \quad f_{xx} = \sigma^2 f,$$

and with $x = W(t)$, one has

$$dx = dW(t), \quad (dx)^2 = dt.$$

Example: GBM

Thus Itô's lemma gives

$$
\begin{aligned}
df &= f_t dt + f_x dW + \frac{1}{2} f_{xx} (dW)^2 \\
&= f\left(\left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW + \frac{1}{2}\sigma^2 dt\right) \\
&= f(\mu dt + \sigma dW).
\end{aligned}
$$

Pricing Derivatives: Risk-Neutral Valuation

- A financial derivative is a function $f(S_T)$ of some underlying stock (or other financial asset) and as such a random variable.
- In a standard financial market model we can calculate the price of a derivative with the the risk-neutral valuation formula

$$\Pi(0) = e^{-rT}\mathbb{E}^*\left[f(S(T))\right],$$

where r is the interest rate and

- a special probability measure (or distribution for the underlying) has been used.

Pricing Derivatives: Risk-Neutral Valuation

- A financial derivative is a function $f(S_T)$ of some underlying stock (or other financial asset) and as such a random variable.
- In a standard financial market model we can calculate the price of a derivative with the the risk-neutral valuation formula

$$\Pi(0) = e^{-rT}\mathbb{E}^* \left[f(S(T))\right],$$

  where r is the interest rate and
- a special probability measure (or distribution for the underlying) has been used.

Pricing Derivatives: Risk-Neutral Valuation

- A financial derivative is a function $f(S_T)$ of some underlying stock (or other financial asset) and as such a random variable.
- In a standard financial market model we can calculate the price of a derivative with the the risk-neutral valuation formula

$$\Pi(0) = e^{-rT}\mathbb{E}^*\left[f(S(T))\right],$$

where r is the interest rate and
- a special probability measure (or distribution for the underlying) has been used.

UNIVERSITÄT DUISBURG ESSEN

Black-Scholes Model

The classical Black-Scholes model is

$$\begin{aligned} dB(t) &= rB(t)dt, & B(0) &= 1, \\ dS(t) &= S(t)\left(bdt + \sigma dW(t)\right), & S(0) &= p, \end{aligned}$$

with constant coefficients $b \in \mathbb{R},\ r, \sigma \in \mathbb{R}_+$.

Pricing a European Call

For a European call $C(T) = (S(T) - K)^+$ and we can evaluate the above expected value to obtain its Black-Scholes price process

$$\begin{aligned} C(t) &= S(t)\Phi(d_1(S(t), T-t)) \\ &\quad - Ke^{-r(T-t)}\Phi(d_2(S(t), T-t)). \end{aligned}$$

The functions $d_1(s,t)$ and $d_2(s,t)$ are given by

$$\begin{aligned} d_1(s,t) &= \frac{\log(s/K) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}, \\ d_2(s,t) &= \frac{\log(s/K) + (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}} \end{aligned}$$

Definition of the model

We assume a simple market model in which the price of the underlying commodity, S_t, follows a stochastic process which can be described as follows: Let $X_t = \ln S_t$ and

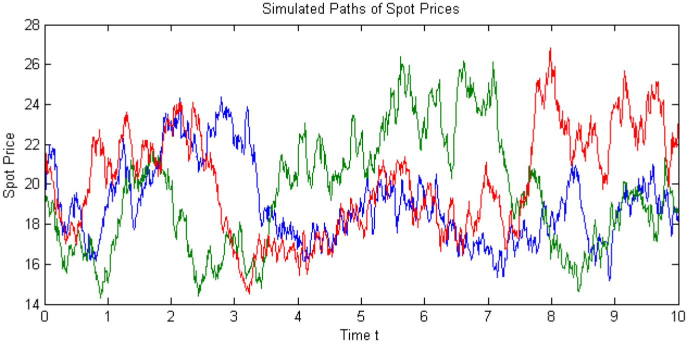
$$dX_t = \kappa(\ln\theta - X_t)dt + \sigma dW_t \quad , \quad X_0 = \ln(S_0)$$

Thus, the logarithm of the prices follow a mean reverting diffusion process, the so-called Ornstein-Uhlenbeck-Process.

- κ - Speed of mean reversion
- θ - Level of mean reversion
- σ - Volatility of the process
- dW_t - Brownian increments

Sample paths of the model

We simulate S_t (with $S_0 = 20, \theta = 20, \kappa = 1, \sigma = 0.2$) and get sample paths



UNIVERSITÄT DUISBURG ESSEN

Properties of the model

- Mean reverting
- Bounded volatility
- Continuous paths
- Relative price changes are normally distributed
- Analytic results for the forward-curve and option prices exist
- Calibration easily possible

Properties of the model

- Mean reverting
- Bounded volatility
- Continuous paths
- Relative price changes are normally distributed
- Analytic results for the forward-curve and option prices exist
- Calibration easily possible

Properties of the model

- Mean reverting
- Bounded volatility
- Continuous paths
- Relative price changes are normally distributed
- Analytic results for the forward-curve and option prices exist
- Calibration easily possible

Properties of the model

- Mean reverting
- Bounded volatility
- Continuous paths
- Relative price changes are normally distributed
- Analytic results for the forward-curve and option prices exist
- Calibration easily possible

Properties of the model

- Mean reverting
- Bounded volatility
- Continuous paths
- Relative price changes are normally distributed
- Analytic results for the forward-curve and option prices exist
- Calibration easily possible

Properties of the model

- Mean reverting
- Bounded volatility
- Continuous paths
- Relative price changes are normally distributed
- Analytic results for the forward-curve and option prices exist
- Calibration easily possible

Spot prices in the model

The spot price at any time t is

$$S_t = \exp\left(e^{-\kappa t}\ln S_0 + (1 - e^{-\kappa t})\ln\theta + \int_0^t \sigma e^{-\kappa(t-s)} dW_s\right).$$

Forward prices in the model

Using that the forward price is the expected value of future spot prices, $F(t, T) = \mathbb{E}^Q[S_T|\mathcal{F}_t]$, we get the formula for the forward price at time t for the forward expiring in T as

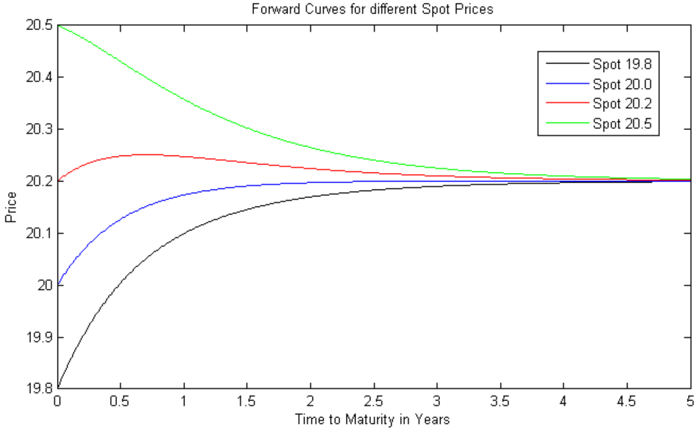
$$F(t, T) =$$

$$\exp\left(e^{-\kappa(T-t)}\ln S_t + (1 - e^{-\kappa(T-t)})\ln\theta + \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa(T-t)})\right).$$

We can see that the forward prices converge to the spot price with time to maturity tending to zero.

Forward curves in the model

Assuming the same parameters as before and varying S_t we get the forward curves:



Properties of the forward curve in the model

- $F(t, T) \to \exp(\ln\theta + \frac{\sigma^2}{4\kappa}) = \theta \exp\left(\frac{\sigma^2}{4\kappa}\right)$ as $T \to \infty$.
- If the spot price is low compared to the long term mean, the forward curve is upward sloping (contango).
- If the spot price is high compared to the long term mean, the forward curve is downward sloping (backwardation).
- If the spot price is close to the long term mean, the forward curve might be humped-shaped.

Properties of the forward curve in the model

- $F(t,T) \to \exp(\ln\theta + \frac{\sigma^2}{4\kappa}) = \theta \exp\left(\frac{\sigma^2}{4\kappa}\right)$ as $T \to \infty$.
- If the spot price is low compared to the long term mean, the forward curve is upward sloping (contango).
- If the spot price is high compared to the long term mean, the forward curve is downward sloping (backwardation).
- If the spot price is close to the long term mean, the forward curve might be humped-shaped.

Properties of the forward curve in the model

- $F(t, T) \to \exp(\ln \theta + \frac{\sigma^2}{4\kappa}) = \theta \exp(\frac{\sigma^2}{4\kappa})$ as $T \to \infty$.
- If the spot price is low compared to the long term mean, the forward curve is upward sloping (contango).
- If the spot price is high compared to the long term mean, the forward curve is downward sloping (backwardation).
- If the spot price is close to the long term mean, the forward curve might be humped-shaped.

Properties of the forward curve in the model

- $F(t,T) \rightarrow \exp(\ln\theta + \frac{\sigma^2}{4\kappa}) = \theta\exp(\frac{\sigma^2}{4\kappa})$ as $T \rightarrow \infty$.
- If the spot price is low compared to the long term mean, the forward curve is upward sloping (contango).
- If the spot price is high compared to the long term mean, the forward curve is downward sloping (backwardation).
- If the spot price is close to the long term mean, the forward curve might be humped-shaped.

Agenda

Storage, Inventory and Convenience Yield

- The theory of storage aims to explain the differences between spot and Futures (Forward) prices by analyzing why agents hold inventories.
- Inventories allow to meet unexpected demand, avoid the cost of frequent revisions in the production schedule and eliminate manufacturing disruption.
- This motivates the concept of convenience yield as a benefit, that accrues to the owner of the physical commodity but not to the holder of a forward contract.
- Thus the convenience yield is comparable to the dividend yield for stocks.
- A modern view is to view storage (inventory) as a timing option, that allows to put the commodity to the market when prices are high and hold it when the prices are low.

Storage, Inventory and Convenience Yield

- The theory of storage aims to explain the differences between spot and Futures (Forward) prices by analyzing why agents hold inventories.
- Inventories allow to meet unexpected demand, avoid the cost of frequent revisions in the production schedule and eliminate manufacturing disruption.
- This motivates the concept of convenience yield as a benefit, that accrues to the owner of the physical commodity but not to the holder of a forward contract.
- Thus the convenience yield is comparable to the dividend yield for stocks.
- A modern view is to view storage (inventory) as a timing option, that allows to put the commodity to the market when prices are high and hold it when the prices are low.

Storage, Inventory and Convenience Yield

- The theory of storage aims to explain the differences between spot and Futures (Forward) prices by analyzing why agents hold inventories.
- Inventories allow to meet unexpected demand, avoid the cost of frequent revisions in the production schedule and eliminate manufacturing disruption.
- This motivates the concept of convenience yield as a benefit, that accrues to the owner of the physical commodity but not to the holder of a forward contract.
- Thus the convenience yield is comparable to the dividend yield for stocks.
- A modern view is to view storage (inventory) as a timing option, that allows to put the commodity to the market when prices are high and hold it when the prices are low.

Storage, Inventory and Convenience Yield

- The theory of storage aims to explain the differences between spot and Futures (Forward) prices by analyzing why agents hold inventories.
- Inventories allow to meet unexpected demand, avoid the cost of frequent revisions in the production schedule and eliminate manufacturing disruption.
- This motivates the concept of convenience yield as a benefit, that accrues to the owner of the physical commodity but not to the holder of a forward contract.
- Thus the convenience yield is comparable to the dividend yield for stocks.
- A modern view is to view storage (inventory) as a timing option, that allows to put the commodity to the market when prices are high and hold it when the prices are low.

Storage, Inventory and Convenience Yield

- The theory of storage aims to explain the differences between spot and Futures (Forward) prices by analyzing why agents hold inventories.
- Inventories allow to meet unexpected demand, avoid the cost of frequent revisions in the production schedule and eliminate manufacturing disruption.
- This motivates the concept of convenience yield as a benefit, that accrues to the owner of the physical commodity but not to the holder of a forward contract.
- Thus the convenience yield is comparable to the dividend yield for stocks.
- A modern view is to view storage (inventory) as a timing option, that allows to put the commodity to the market when prices are high and hold it when the prices are low.

Spot-Forward Relationship in Commodity Markets

Under the no-arbitrage assumption we have

$$F(t,T) = S(t)e^{(r-y)(T-t)} \tag{3}$$

where r is the interest rate at time t for maturity T and y is the convenience yield.

Spot-Forward Relationship in Commodity Markets

Observe that (3) implies

- spot and forward are redundant (one can replace the other) and form a linear relationship (unlike options)
- with two forward prices we can derive the value of $S(t)$ and y
- knowledge of $S(t)$ and y allows us to construct the whole forward curve
- for $r - y < 0$ we have backwardation; for $r - y > 0$ we have contango.

UNIVERSITÄT DUISBURG ESSEN

Spot-Forward Relationship in Commodity Markets

Observe that (3) implies

- spot and forward are redundant (one can replace the other) and form a linear relationship (unlike options)
- with two forward prices we can derive the value of $S(t)$ and y
- knowledge of $S(t)$ and y allows us to construct the whole forward curve
- for $r - y < 0$ we have backwardation; for $r - y > 0$ we have contango.

Spot-Forward Relationship in Commodity Markets

Observe that (3) implies

- spot and forward are redundant (one can replace the other) and form a linear relationship (unlike options)
- with two forward prices we can derive the value of $S(t)$ and y
- knowledge of $S(t)$ and y allows us to construct the whole forward curve
- for $r - y < 0$ we have backwardation; for $r - y > 0$ we have contango.

Spot-Forward Relationship in Commodity Markets

Observe that (3) implies

- spot and forward are redundant (one can replace the other) and form a linear relationship (unlike options)
- with two forward prices we can derive the value of $S(t)$ and y
- knowledge of $S(t)$ and y allows us to construct the whole forward curve
- for $r - y < 0$ we have backwardation; for $r - y > 0$ we have contango.

Spot-Forward Relationship: Classical theory

- In a stochastic model we use

$$F(t,T) = \mathbb{E}_{\mathbb{Q}}(S(T)|\mathcal{F}_t)$$

  where $\mathcal{F}_t$ is the accumulated available market information (in most models the information generated by the spot price).

- $\mathbb{Q}$ is a risk-neutral probability
  - discounted spot price is a $\mathbb{Q}$-martingale
  - fixed by calibration to market prices or a market price of risk argument

Spot-Forward Relationship: Classical theory

- In a stochastic model we use

$$F(t,T) = \mathbb{E}_{\mathbb{Q}}(S(T)|\mathcal{F}_t)$$

  where $\mathcal{F}_t$ is the accumulated available market information (in most models the information generated by the spot price).
- $\mathbb{Q}$ is a risk-neutral probability
  - discounted spot price is a $\mathbb{Q}$-martingale
  - fixed by calibration to market prices or a market price of risk argument

Futures Prices and Expectation of Future Spot Prices

The rational expectation hypothesis (REH) states that the current futures price $f(t, T)$ for a commodity with delivery a time $T > t$ is the best estimator for the price $S(T)$ of the commodity. In mathematical terms

$$f(t, T) = \mathbb{E}[S(T)|\mathcal{F}_t]. \tag{4}$$

where $\mathcal{F}_t$ represents the information available at time t. The REH has been statistically tested in many studies for a wide range of commodities.

Futures Prices and Expectation of Future Spot Prices

When equality in (4) does not hold futures prices are biased estimators of future spot prices. If

- $>$ holds, then $f(t, T)$ is an up-ward biased estimate, then risk-aversion among market participants is such that buyers are willing to pay more than the expected spot price in order to secure access to the commodity at time T (political unrest);
- $<$ holds, then $f(t, T)$ is an down-ward biased estimate, this may reflect a perception of excess supply in the future.

Market Risk Premium

- The *market risk premium* or *forward bias* $\pi(t, T)$ relates forward and expected spot prices.
- It is defined as the difference, calculated at time t, between the forward $F(t, T)$ at time t with delivery at T and expected spot price:

$$\pi(t, T) = F(t, T) - \mathbb{E}^{\mathbb{P}}[S(T)|\mathcal{F}_t]. \tag{5}$$

Here $\mathbb{E}^{\mathbb{P}}$ is the expectation operator, under the historical measure $\mathbb{P}$, with information up until time t and $S(T)$ is the spot price at time T.

Market Risk Premium

- The *market risk premium* or *forward bias* $\pi(t, T)$ relates forward and expected spot prices.
- It is defined as the difference, calculated at time t, between the forward $F(t, T)$ at time t with delivery at T and expected spot price:

$$\pi(t, T) = F(t, T) - \mathbb{E}^{\mathbb{P}}[S(T)|\mathcal{F}_t]. \tag{5}$$

Here $\mathbb{E}^{\mathbb{P}}$ is the expectation operator, under the historical measure $\mathbb{P}$, with information up until time t and $S(T)$ is the spot price at time T.

Heath-Jarrow-Morton (HJM) model

The Heath-Jarrow-Morton model uses the entire forward rate curve as (infinite-dimensional) state variable. The dynamics of the forward rates $f(t, T)$ are *exogenously* given by

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW(t).$$

For any fixed maturity T, the initial condition of the stochastic differential equation is determined by the current value of the empirical (observed) forward rate for the future date T which prevails at time 0.

One-Factor GBM Specification

Here the volatility is

$$\sigma_1(t, T) = e^{-\kappa(T-t)}\sigma$$

and

$$dF(t, T) = F(t, T)\sigma_1(t, T)dW(t)$$

Two-Factor GBM Specification

Here the volatilities are

$$\sigma_1(t,T) = e^{-\kappa(T-t)}\sigma_1 \quad \text{and} \quad \sigma_2 > 0$$

and

$$\frac{dF(t,T)}{F(t,T)} = \sigma_1(t,T)dW_1(t) + \sigma_2 dW_2(t)$$

Modelling Approach

- We use the HJM-framework to model the forward dynamics directly.
- We distinguish between forward contracts with a fixed time delivery and forward contracts with a delivery period, called *swaps*.
- Since the HJM-framework cannot be applied to the swap dynamics literally, we differentiate between the *decomposable* swaps and the *atomic* swaps and apply the framework to the atomic swaps.

Modelling Approach

- We use the HJM-framework to model the forward dynamics directly.
- We distinguish between forward contracts with a fixed time delivery and forward contracts with a delivery period, called *swaps*.
- Since the HJM-framework cannot be applied to the swap dynamics literally, we differentiate between the *decomposable* swaps and the *atomic* swaps and apply the framework to the atomic swaps.

Modelling Approach

- We use the HJM-framework to model the forward dynamics directly.
- We distinguish between forward contracts with a fixed time delivery and forward contracts with a delivery period, called *swaps*.
- Since the HJM-framework cannot be applied to the swap dynamics literally, we differentiate between the *decomposable* swaps and the *atomic* swaps and apply the framework to the atomic swaps.

Modelling Approach

The dynamics of a decomposable swap with delivery period $[T_1, T_N]$ can than be obtained from $N - 1$ atomic swaps by

$$F(t, T_1, T_N) = \sum_{i=1}^{N-1} \frac{T_{i+1} - T_i}{T_N - T_1} F(t, T_i, T_{i+1}) \tag{6}$$

Modelling Approach

We discuss several lognormal dynamics of the swap price,

$$dF(t, T_1, T_2) = \Sigma(t, T_1, T_2)F(t, T_1, T_2)\, dW(t). \tag{7}$$

The only parameter in this model is the volatility function Σ which has to capture all movements of the swap price and especially the time to maturity effect.

Volatility Functions

We assume that the swap price dynamics *for all atomic swaps* is given by (7) where $\Sigma(t, T_1, T_2)$ is a continuously differentiable and positive function.

Starting out with a given volatility function for a fixed time forward contract we see that the volatility function Σ for the swap contract is given by

$$\Sigma(t, T_1, T_2) = \int_{T_1}^{T_2} \hat{w}(u, T_1, T_2)\sigma(t, u)\, du. \tag{8}$$

Schwartz

For the volatility function of the forward we use

$$\sigma(t, u) = ae^{-b(u-t)} \tag{9}$$

where $a, b > 0$ are constant.

Schwartz

The time to maturity effect is modeled by a negative exponential function.

- When the time to maturity tends to infinity the volatility function converges to zero.
- The exponential function causes that the volatility increases as the time to maturity decreases which leads to an increased volatility when the contract approaches the maturity.

Schwartz

The time to maturity effect is modeled by a negative exponential function.

- When the time to maturity tends to infinity the volatility function converges to zero.
- The exponential function causes that the volatility increases as the time to maturity decreases which leads to an increased volatility when the contract approaches the maturity.

Schwartz

Applying this forward volatility to (8) the swap volatility is:

$$\Sigma(t, T_1, T_2) = a\,\varphi(T_1, T_2) \tag{10}$$

where

$$\varphi(T_1, T_2) = \frac{e^{-b(T_1-t)} - e^{-b(T_2-t)}}{b(T_2 - T_1)} \tag{11}$$

The Black-76 specification of the forward volatility can be obtained if $\varphi(T_1, T_2) = 1$, that is $b = 0$ in (9).
The associated swap price volatility is then given by $\Sigma(t, T_1, T_2) = a$.

Model Summary

A summary of the different models presented above is listed in Table 1.

Model	$\Sigma(t, T_1, T_2)$
Black-76	a
Schwartz	$a\,\varphi(T_1, T_2)$
Fackler and Tien	$a(t)\,\varphi(T_1, T_2)$
Clewlow and Strickland	$a((1-c)\,\varphi(T_1, T_2) + c)$
Koekebakker and Lien	$a(t)((1-c)\,\varphi(T_1, T_2) + c)$
Benth and Koekebakker	$\hat{a}\,\varphi(T_1, T_2) + a(t)$

Table : The associated swap volatility models generated by (8) with $a \geq 0, b > 0$ and $0 \leq c \leq 1$ constants, $a(t)$ defined in (**??**) and $\varphi(T_1, T_2)$ is given by (11).

UNIVERSITÄT DUISBURG ESSEN

Agenda

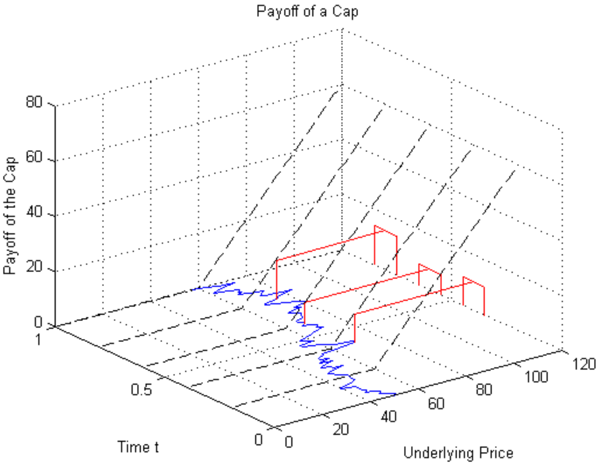
Caps

Buying a cap, the option holder has the right (but not the obligation) to buy a certain amount of energy at stipulated times $t_1, \dots, t_N$ during the delivery period at a fixed strike price K. It can be viewed as a strip of independent call options, for each time t_i the holder of the cap holds call options with maturity t_i and Strike K.

The static factors describing the cap are:

- times $t_1, \dots, t_N$ (how often? when?)
- strike K (price?)
- amount of the underlying (how much?)

Cap - Payoff


Payoff of a Cap
Payoff of the Cap
80
60
40
20
0
1
0.5
0
Time t
0
20
40
60
80
100
120
Underlying Price

Caps - Pricing

Whenever the price of the underlying exceeds the strike K at one of the dates $t_1, \dots, t_N$, the seller of the cap pays the holder of the cap the difference between the price of the underlying and the strike K or - in case one agreed on physical delivery - the underlying is delivered for the price K. Typically, the price of a cap is quoted as price per delivery hours to make different delivery periods comparable. In this case we get a price per MWh. The formula is

$$U_c(t) = \frac{1}{N} \sum_{i=1}^{N} e^{-r(t_i - t)} \mathbb{E}[\max(S(t_i) - K, 0)].$$

Caps - Hedging

The strike price K secures a maximum price for which the option holder is able to buy energy. A cap is used to cover a short position in the underlying (energy) against increasing market prices not only at a certain point in time but over the whole period covered by the exercising times $t_1, \ldots, t_N$. On the other hand, the option holder is still able to profit from low energy prices as he has the right but not the obligation to exercise the option at each time point.

Caps - Example

Assume you need 100 units of the underlying per day to run your business. Today it costs 100 Euro/unit. You can accept resource cost of up to 110 Euro/unit in order to beneficially run your business. You are afraid of rising prises and want to hedge against this risk but still have the chance to profit from low prices.
Thus, you ask for a cap with daily exercise up to the business horizon of 8 days with volume 100 units and, say, strike 108 Euro/unit which might cost 1600 Euro (2 Euro/unit). Then, your total cost is at most 108 Euro + 2 Euro = 110 Euro/unit but you still participate on low prices.

Caps - Example

The table shows one possible result of the cap on the profit of the company.

Day	Underlying	Cost without Cap	Cost with Cap
1	100	100	102
2	111	111	110
3	116	116	110
4	120	120	110
5	109	109	110
6	97	97	99
7	85	85	87
8	78	78	80
Average		102	101
S.d.		14.9	11.7

Floors

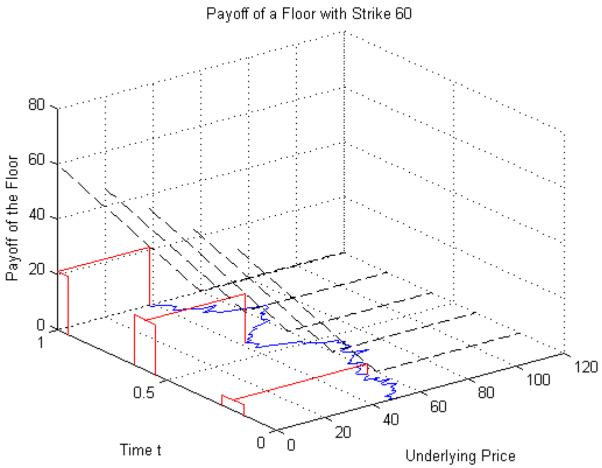
Buying a floor, the option holder has the right (but not the obligation) to sell a certain amount of energy at stipulated times $t_1, \dots, t_N$ during the delivery period at a fixed strike price K. It can be viewed as a strip of independent put options, for each time t_i the holder of the floor holds put options with maturity t_i and Strike K.

Similar to the case of a cap, the pricing formula is

$$U_f(t) = \frac{1}{N} \sum_{i=1}^{N} e^{-r(t_i - t)} \mathbb{E}[\max(K - S(t_i), 0)].$$

As with the cap, the price is quoted in Euro/MWh.

Floor - Payoff



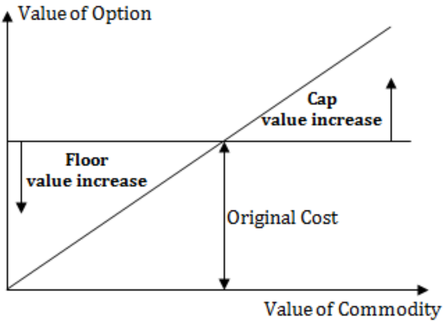
Floors - Hedging

The strike price K secures a minimum price for which the option holder is able to sell energy. A floor is used to cover a long position in the underlying (energy) against decreasing market prices not only at a certain point in time but over the whole period covered by the exercising times $t_1, \ldots, t_N$. On the other hand, the option holder is still able to profit from high energy prices as he has the right but not the obligation to exercise the option at each time point.

The holder of a short position might write a floor to produce liquidity upfront. The maximum gain from the short position is then limited to the strike K.

Example: Caps and Floors

For a fixed premium, a buyer of a cap (call) is protected on the market price becoming stronger, while a buyer of a floor (put) is protected on the market price becoming weaker.

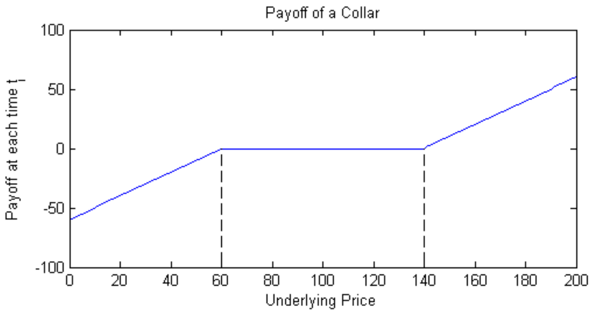


Collars

A collar is a combination of a cap and a floor such that variable prices are limited to a certain corridor. A long collar position consists of long one cap (with high strike K_2) and short one floor (with low strike K_1) - a short collar position is short one cap and long one floor. As long as the price of the underlying is between K_1 and K_2 at one of the dates t_i, no cash flows are exchanged. If the underlying is above K_2, the holder of the long collar position receives the difference of the actual price and K_2. If the underlying is below K_1, the short collar position receives the difference between K_1 and the actual price.

Collar - Payoff

As a long collar position is a strip of call options minus a strip of put options, the payoff of a collar at each time point t_i is the following:



UNIVERSITÄT DUISBURG ESSEN

Collar - Pricing

Collars might be seen as a strip of bear/bull spreads, or as a strip of call options minus a strip of put options in the case of a long collar position. Consequently, the pricing formula is just the combination of the formulas for the cap and the floor:

$$
\begin{aligned}
U_{collar}^{K_1,K_2}(t) &= U_{cap}^{K_2}(t) - U_{floor}^{K_1}(t) \\
&= \frac{1}{N}\sum_{i=1}^{N} e^{-r(t_i-t)}\mathbb{E}[(S(t_i)-K_2)^+ - (K_1 - S(t_i))^+]
\end{aligned}
$$

The price of a collar might be positive or negative - or even zero. In case the price is zero, the collar is called zero-cost collar.

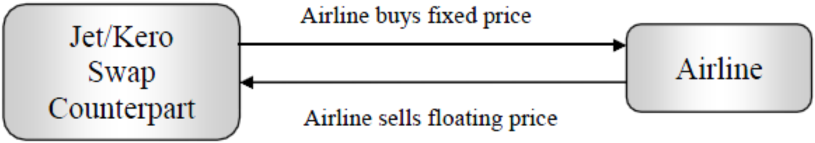
Collars - Hedging

The holder of a long position in a collar is protected against increases in the underlying price above K_2, but does not profit from falling underlying prices below K_1. Thus he is protected against rising prices with limited participation on downside prices. Having a short position in the underlying, a long collar ensures the ability to cover the short position for prices in the range of $[K_1, K_2]$. A short collar protects against falling prices. At the same time, the ability to participate on rising prices is limited to K_2. Having a long position in the underlying, a short collar ensures that the position can be closed for prices in the range of $[K_1, K_2]$.

Collars - Example

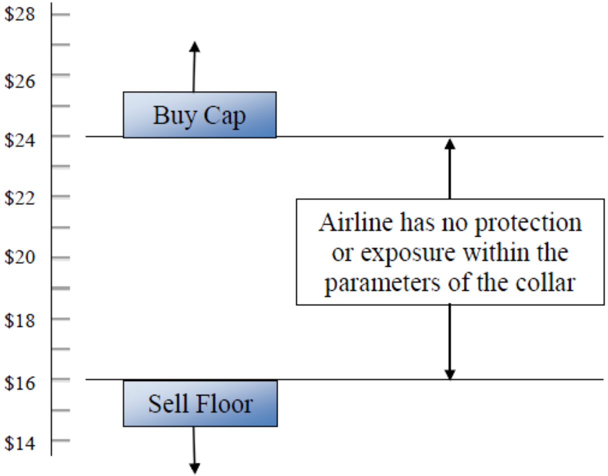
An energy consuming manufacturer bought the energy needed on the futures market. As its competitors did not, the manufacturer is now concerned about falling energy prices which would lead to a competitive disadvantage. Thus, the manufacturer tries to enter a short collar, protecting him against falling prices but leaving the risk of rising prices above K_2. This risk might be acceptable for the manufacturer as if prices rise too much, the manufacturer is able to stop its production and selling the energy already bought on the spot market - offsetting the losses of the collar.

Example: Jet Fuel Hedge by an Airline



- An airline buys a fixed-price swap from a bank or trader against its jet-fuel price exposure.
- Buying a swap it must lock in its minimum net price receivable at the current perceived swap value.

Jet Fuel Hedge by an Airline: Collar



Jet Fuel Hedge by an Airline: Collar

- Using a collar structure the airline can still protect itself from a price increase, but can keep its minimum net price receivable locked in at a lower rate than the current swap price.
- The purchase of the cap protects against jet-fuel prices rising above the strike of the cap.
- The sale of the floor reduces the cost of the premium in the purchase of the cap.
- A popular strategy is a zero-cost collar.

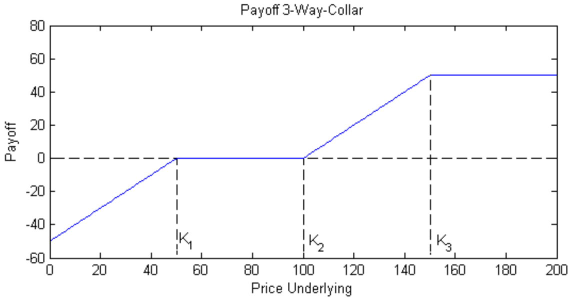
Collars - 3-way-collars

A long collar is short one floor with strike K_1, long one cap with higher strike K_2. A possible extension is to include a short position in one cap with strike $K_3 >> K_2$ in order to reduce the cost of the collar. This extension is called 3-way-collar. The price of a 3-way-collar is thus:

$$
\begin{aligned}
U_{3-way}^{K_1,K_2,K_3}(t) &= U_{cap}^{K_2}(t) - U_{cap}^{K_3}(t) - U_{floor}^{K_1}(t) \\
&= \frac{1}{N}\sum_{i=1}^{N} e^{-r(t_i-t)}\mathbb{E}[(S(t_i) - K_2)^+ \\
&\quad - (S(t_i) - K_3)^+ - (K_1 - S(t_i))^+]
\end{aligned}
$$

3-Way-Collar - Payoff

The holder of the 3-way-collar is protected against increases in the underlying price above K_2, but only till K_3. Afterwards, no protection exists anymore. This strategy might be a good choice if one wants to protect its buying costs but is able to stop its business if prices rally unexpectedly high (above K_3).



Swing Options

A swing option is similar to a cap or floor except that we have additional restrictions on the number of option exercises. Let $\phi_i \in \{0, 1\}$ be the decision whether to exercise ($\phi_i = 1$) or not to exercise ($\phi_i = 0$) the option at time t_i. The option's payoff at time t_i is given by

$$\phi_i(S(t_i) - K) \quad \text{call resp.} \quad \phi_i(K - S(t_i)) \quad \text{put.}$$

We now require that the number of exercises is between $E_{\min}$ and $E_{\max}$.

Swing Options

To determine the swing option value, we have to find an optimal exercise strategy $\Phi = (\phi_1, \ldots, \phi_N)$ maximising the expected payoff

$$\sum_{i=1}^{N} e^{-r(t_i - t)} \mathbb{E}[\phi_i (S(t_i) - K)] \quad \to \max$$

subject to

$$E_{\min} \leq \sum_{i=1}^{N} \phi_i \leq E_{\max}.$$

To calculate the option value various mathematical techniques are used.

Bounds for Swing Options

Strategy

For deterministic spot prices, we

- Calculate the discounted payoffs $P(t_i) = e^{-r(t_i-t)}(S(t_i) - K)$.
- Sort the discounted payoffs $P(t_i)$ in descending order.
- Take the first $E_{\min}$ payoffs regardless of their value and subsequent payoffs up to $E_{\max}$ until their sign become negative.

Bounds for Swing Options

For stochastic spot prices the MC-approach gives an upper bound, since information on the whole path is used, but in reality only information up to time t is available when deciding at time t.
A lower bound is given by the intrinsic value

$$\sum_{i=1}^{N} e^{-r(t_i-t)}\phi_i^F(F(t,t_i)-K) \quad \to \max$$

subject to

$$E_{\min} \leq \sum_{i=1}^{N} \phi_i^F \leq E_{\max}$$

where $\phi_i^F = \mathbf{1}_{\{F(t,t_i)>K\}}$, unless the restriction on $E_{\min}$ is in force.

UNIVERSITÄT DUISBURG ESSEN

Spread Options

Some market participants are exposed to the difference of commodity prices. Examples are

- the dark spread between power and coal (model for a coal-fired power plant)
- the spark spread between power and gas (model for a gas-fired power plant)
- the crack spread between different refinements of oil (model for a refinement plant)

Spread Options

Some market participants are exposed to the difference of commodity prices. Examples are

- the dark spread between power and coal (model for a coal-fired power plant)
- the spark spread between power and gas (model for a gas-fired power plant)
- the crack spread between different refinements of oil (model for a refinement plant)

Spread Options

Some market participants are exposed to the difference of commodity prices. Examples are

- the dark spread between power and coal (model for a coal-fired power plant)
- the spark spread between power and gas (model for a gas-fired power plant)
- the crack spread between different refinements of oil (model for a refinement plant)

Spread Trading

Spreads are used to describe power plants, refineries, storage facilities and transmission lines. Spread positions may be initiated in futures contracts

- for different, but related commodities,
- for different delivery month of the same commodity,
- for same commodity traded on different exchanges.

UNIVERSITÄT DUISBURG ESSEN

Spread Trading

Spreads are used to describe power plants, refineries, storage facilities and transmission lines. Spread positions may be initiated in futures contracts

- for different, but related commodities,
- for different delivery month of the same commodity,
- for same commodity traded on different exchanges.

Spread Trading

Spreads are used to describe power plants, refineries, storage facilities and transmission lines. Spread positions may be initiated in futures contracts

- for different, but related commodities,
- for different delivery month of the same commodity,
- for same commodity traded on different exchanges.

Spread Trading

Spread trading involves taking a long position in one futures contract and simultaneously taking a short position in another, related futures contract.

- Spread position neutralizes price risk.
- A profit or loss results only if the relative prices of the two contracts change.
- If spreads are expected to narrow, buy the lower-priced contract and sell the higher-priced contract.
- If spreads are expected to widen, buy the higher-priced contract and sell the lower-priced contract.

Spread Trading

Spread trading involves taking a long position in one futures contract and simultaneously taking a short position in another, related futures contract.

- Spread position neutralizes price risk.
- A profit or loss results only if the relative prices of the two contracts change.
- If spreads are expected to narrow, buy the lower-priced contract and sell the higher-priced contract.
- If spreads are expected to widen, buy the higher-priced contract and sell the lower-priced contract.

Spread Trading

Spread trading involves taking a long position in one futures contract and simultaneously taking a short position in another, related futures contract.

- Spread position neutralizes price risk.
- A profit or loss results only if the relative prices of the two contracts change.
- If spreads are expected to narrow, buy the lower-priced contract and sell the higher-priced contract.
- If spreads are expected to widen, buy the higher-priced contract and sell the lower-priced contract.

Spread Trading

Spread trading involves taking a long position in one futures contract and simultaneously taking a short position in another, related futures contract.

- Spread position neutralizes price risk.
- A profit or loss results only if the relative prices of the two contracts change.
- If spreads are expected to narrow, buy the lower-priced contract and sell the higher-priced contract.
- If spreads are expected to widen, buy the higher-priced contract and sell the lower-priced contract.

Spark Spread

- Differential between the price of electricity (output) and the price of natural gas (input).
- Can be used to financially replicate the physical reality of a gas-fired power plant: Short position in fuels and long position in electricity.
- Spark spreads are traded OTC.

Spark Spread

- Differential between the price of electricity (output) and the price of natural gas (input).
- Can be used to financially replicate the physical reality of a gas-fired power plant: Short position in fuels and long position in electricity.
- Spark spreads are traded OTC.

Spark Spread

- Differential between the price of electricity (output) and the price of natural gas (input).
- Can be used to financially replicate the physical reality of a gas-fired power plant: Short position in fuels and long position in electricity.
- Spark spreads are traded OTC.

Spark Spread

$$\text{Spark\_Spread} = \text{Power\_Price} - \text{Heat\_Rate} \cdot \text{Fuel\_Price}.$$

- Heat rate provides a conversion factor between fuels used to generate power and the power itself.
- Heat rate is the number of Btus needed to make 1kWh of electricity.
- In the absence of any inefficiency it takes 3412Btu to produce 1kWh of electricity.

Spark Spread

$$\text{Spark\_Spread} = \text{Power\_Price} - \text{Heat\_Rate} \cdot \text{Fuel\_Price}.$$

- Heat rate provides a conversion factor between fuels used to generate power and the power itself.
- Heat rate is the number of Btus needed to make 1kWh of electricity.
- In the absence of any inefficiency it takes 3412Btu to produce 1kWh of electricity.

Spark Spread

$$\text{Spark\_Spread} = \text{Power\_Price} - \text{Heat\_Rate} \cdot \text{Fuel\_Price}.$$

- Heat rate provides a conversion factor between fuels used to generate power and the power itself.
- Heat rate is the number of Btus needed to make 1kWh of electricity.
- In the absence of any inefficiency it takes 3412Btu to produce 1kWh of electricity.

Example: Spark Spread

The price of electricity is currently 42.69EUR/MWh, the price of natural gas is 4.86EUR/MMBtu and the heat rate is 8152Btu/kWh. The spark spread quoted in EUR/MWh is

$$\begin{aligned} \textit{Spread} &= 42.69\text{EUR/MWh} - 0.001 * 8152\text{Btu/kWh} * 4.86\text{EUR/MMBtu} \\ &= 3.07\text{EUR/MWh}. \end{aligned}$$

The positive spark spread means that it is economical to run the plant (without taking into account additional generating costs).

Clean Spreads

In countries covered by the European Union Emissions Trading Scheme, utilities have to consider also the cost of carbon dioxide emission allowances. Emission trading has started in the EU in January 2005.

- Clean spark spread represents the net revenue a gas-fired power plant makes from selling power, having bought gas and the required number of carbon allowances.
- Clean dark spread represents the net revenue a coal-fired power plant makes from selling power, having bought coal and the required number of carbon allowances.
- The difference between the clean dark spread and the clean spark spread is known as the climate spread.

Clean Spark Spread

Clean Spark Spread = Power Price - Heat Rate · Gas Price -
- Gas Emission Intensity Factor · Carbon Price

Clean Spark Spread reflects the cost of generating power from gas after taking into account gas and carbon allowance costs. A positive spread effectively means that it is profitable to generate electricity, while a negative spread means that generation would be a loss-making activity. However, it is important to note that the Clean Spark Spreads do not take into account additional generating charges beyond gas and carbon, such as operational costs.

Clean Dark Spread

Clean Dark Spread = Power Price - Heat Rate · Coal Price - - Coal Emission Intensity Factor · Carbon Price

Clean Dark Spread reflects the cost of generating power from coal after taking into account coal and carbon allowance costs. A positive spread effectively means that it is profitable to generate electricity for the period in question, while a negative spread means that generation would not be profitable. Clean Dark Spreads do not account for additional generating charges beyond coal and carbon.

Power Plant as a Clean Dark Spread

A coal-fired power plant can be viewed as a call option on the clean dark spread with the variable cost of running the plant (beyond coal and carbon) being the strike and the payoff equal to

$$\Pi = max\{P - HR \cdot Coal - I \cdot Carbon - V\}.$$

P: Power Price
HR: Heat Rate
Coal: Coal Price
I: Coal Emission Intensity Factor
Carbon: Carbon Price
V: Variable cost of running the plant (beyond coal and carbon)

Power Plant as a Clean Dark Spread

Indeed, the decision to run or not to run the power plant can be described as follows:

- If $P - HR \cdot Coal - I \cdot Carbon - V \geq 0$, then run the plant. In this case buying fuel and paying variable costs $(HR \cdot Coal + I \cdot Carbon + V)$ to run the plant and then selling the generated power for P results in the positive gain.
- If $P - HR \cdot Coal - I \cdot Carbon - V < 0$, then do not run the plant. In this case buying fuel and paying variable costs $(HR \cdot Coal + I \cdot Carbon + V)$ to run the plant will not be compensated by sold power.

Climate Spread

Climate Spread = Clean Dark Spread - Clean Spark Spread

In a carbon constrained economy a power producer in a geographic area where coal is currently the preferred method by which electricity is generated may eventually encounter a negative climate spread if carbon credit prices rise. This would mean that when taking into consideration the cost to produce (coal is on average 2.5 times as polluting as natural gas for the same MWh of electricity) the natural gas would be a better decision.

Example: Clean Spark Spread

The price of electricity is currently 42.69EUR/MWh, the price of natural gas is 4.86EUR/MMBtu, the carbon price is 12EUR/tCO_2, the heat rate is 8152Btu/kWh and the gas emission intensity factor is 0.11tCO_2/MWh. The clean spark spread quoted in EUR/MWh is

$$\begin{aligned}\text{Clean Spark Spread} &= 42.69\text{EUR/MWh}\\ &-0.001 * 8152\text{Btu/kWh} * 4.86\text{EUR/MMBtu}\\ &-0.11tCO_2/\text{MWh} * 12\text{EUR/t}CO_2\\ &= 1.75\text{EUR/MWh}.\end{aligned}$$

It is profitable to generate electricity, if additional generating charges beyond gas and carbon are lower than 1.75EUR/MWh.

Example: Clean Dark Spread

The price of electricity is currently 42.69EUR/MWh, the coal price is 95.04EUR/t or 3.96EUR/MMBtu (with heat content of 24MMBtu/t), the carbon price is 12EUR/tCO_2, the heat rate is 9500Btu/kWh and the coal emission intensity factor is 0.26tCO_2/MWh. The clean dark spread quoted in EUR/MWh is

$$\begin{aligned}\text{Clean Dark Spread} &= 42.69\text{EUR/MWh}\\ &-0.001 * 9500\text{Btu/kWh} * 3.96\text{EUR/MMBtu}\\ &-0.26tCO_2/\text{MWh} * 12\text{EUR/t}CO_2\\ &= 1.95\text{EUR/MWh}.\end{aligned}$$

It is profitable to generate electricity, if additional generating charges beyond coal and carbon are lower than 1.95EUR/MWh.

UNIVERSITÄT DUISBURG ESSEN

Example: Climate Spread

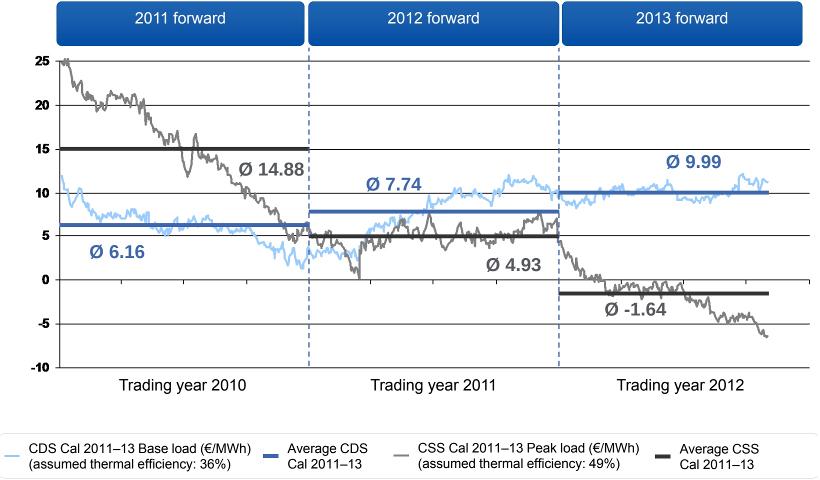
Suppose that the price of carbon rises to 19.6EUR/tCO_2.

Climate Spread = Clean Dark Spread - Clean Spark Spread

$$= -0.026\text{EUR/MWh} - 0.915\text{EUR/MWh}$$
$$= -0.941\text{EUR/MWh}.$$

The clean dark spread becomes negative (-0.026EUR/MWh), implying that electricity generation by a coal-fired power plant would be a loss-making activity, whereas the clean spark spread remains positive (0.915EUR/MWh), meaning that it is profitable to generate electricity by a gas-fired power plant, if additional generating charges beyond gas and carbon are lower than 0.915EUR/MWh.

Clean Spark Spread Forward



Gas Power Plant

Political Risk

- Installed capacity: 876 MW
- Variable cost ca. 60 EUR/MWh
- Profitable hours per year
  - 2010 (993),
  - 2011 (2309),
  - 2012 (737),

  with average profit 6.9 EUR per MWh.
- Typical assumption on investing
  - 3500 profitable hours
  - 10 EUR per MWh profit
- loss per year
  - 2010: (3500-993)*876*10=21961320 EUR,
  - 2011: (3500-2309)*876*10 = 10433160 EUR,
  - 2012: (3500-737)*876*10= 24203880 EUR.

Political Risk

- Installed capacity: 876 MW
- Variable cost ca. 60 EUR/MWh
- Profitable hours per year
  - 2010 (993),
  - 2011 (2309),
  - 2012 (737),

  with average profit 6.9 EUR per MWh.
- Typical assumption on investing
  - 3500 profitable hours
  - 10 EUR per MWh profit
- loss per year
  - 2010: (3500-993)*876*10=21961320 EUR,
  - 2011: (3500-2309)*876*10 = 10433160 EUR,
  - 2012: (3500-737)*876*10= 24203880 EUR.

Political Risk

- Installed capacity: 876 MW
- Variable cost ca. 60 EUR/MWh
- Profitable hours per year
  - 2010 (993),
  - 2011 (2309),
  - 2012 (737),

  with average profit 6.9 EUR per MWh.
- Typical assumption on investing
  - 3500 profitable hours
  - 10 EUR per MWh profit
- loss per year
  - 2010: (3500-993)*876*10=21961320 EUR,
  - 2011: (3500-2309)*876*10 = 10433160 EUR,
  - 2012: (3500-737)*876*10= 24203880 EUR.

Political Risk

- Installed capacity: 876 MW
- Variable cost ca. 60 EUR/MWh
- Profitable hours per year
  - 2010 (993),
  - 2011 (2309),
  - 2012 (737),

  with average profit 6.9 EUR per MWh.
- Typical assumption on investing
  - 3500 profitable hours
  - 10 EUR per MWh profit
- loss per year
  - 2010: (3500-993)*876*10=21961320 EUR,
  - 2011: (3500-2309)*876*10 = 10433160 EUR,
  - 2012: (3500-737)*876*10= 24203880 EUR.

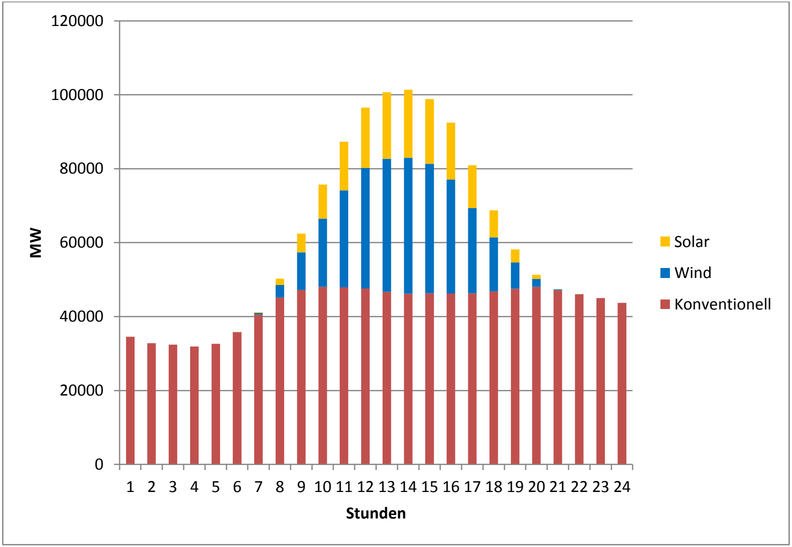
Political Risk

- Installed capacity: 876 MW
- Variable cost ca. 60 EUR/MWh
- Profitable hours per year
  - 2010 (993),
  - 2011 (2309),
  - 2012 (737),

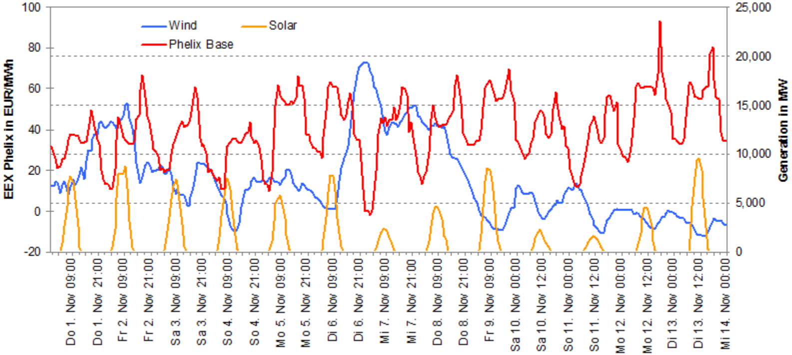
  with average profit 6.9 EUR per MWh.
- Typical assumption on investing
  - 3500 profitable hours
  - 10 EUR per MWh profit
- loss per year
  - 2010: (3500-993)*876*10=21961320 EUR,
  - 2011: (3500-2309)*876*10 = 10433160 EUR,
  - 2012: (3500-737)*876*10= 24203880 EUR.

UNIVERSITÄT DUISBURG ESSEN

A day in august



Wind, sun and electricity



Spread Options to Manage Market Risk

Spread options can be used by owners of corresponding plants to manage the market risk. Instead of spread trading with futures the owner of a power plant can directly purchase/sell a spread option.

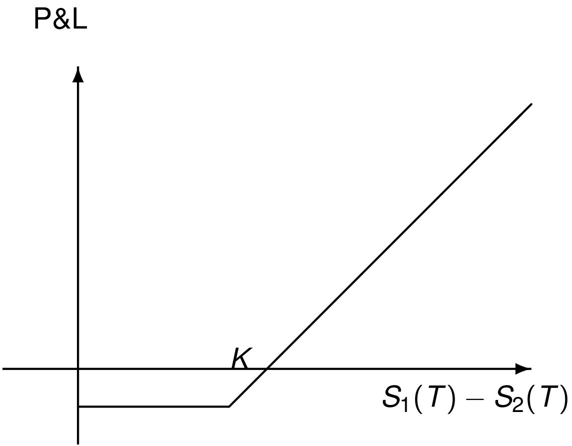
The pay off of a typical spread is

$$C^{(T)}_{\text{spread}} = \max(S_1(T) - S_2(T) - K, 0)$$

with S_i the underlyings, K the strike.

Spread Options to Manage Market Risk

P&L diagram of a typical spread



Example: Hedging with forward spark spread

An operator of a gas-fired power plant wants to protect himself from the fluctuation of gas and power prices during the future months of July. The total July power output is 100MWh and the plant's heat rate is 10MBtu/kWh. In order to insure the operational margins equal to the value of the forward spread the operator

- sells the financially settled forward spark spreads totalling 100MWh, i.e. sells 100MWh of power and buys 1.000MMBtu of gas.
- at maturity (end of June) of the forward spark spread contract, buys gas and sells electricity into the spot market for monthly delivery.

Example: Hedging with forward spark spread

An operator of a gas-fired power plant wants to protect himself from the fluctuation of gas and power prices during the future months of July. The total July power output is 100MWh and the plant's heat rate is 10MBtu/kWh. In order to insure the operational margins equal to the value of the forward spread the operator

- sells the financially settled forward spark spreads totalling 100MWh, i.e. sells 100MWh of power and buys 1.000MMBtu of gas.
- at maturity (end of June) of the forward spark spread contract, buys gas and sells electricity into the spot market for monthly delivery.

Example: Hedging with spark spread options

Since the option value always exceeds the value of the spread, the better strategy is to sell the option on the forward spark spread.

- Selling the option guarantees protection while providing higher margins than selling the forward spread.
- All obligations with respect to the option buyer are fulfilled through running the plant.

Example: Hedging with spark spread options

Since the option value always exceeds the value of the spread, the better strategy is to sell the option on the forward spark spread.

- Selling the option guarantees protection while providing higher margins than selling the forward spread.
- All obligations with respect to the option buyer are fulfilled through running the plant.

Example: Hedging with spark spread options

Suppose the spread value is 10USD/MWh and the spread option is sold for 12USD/MWh.

Scenario 1: The July spot spark spread is 5USD/MWh.	
Premium from selling the spark spread option	12USD/MWh
Payment to the spread option holder	-5USD/MWh
Buy 1.000MMBtu of gas and sell 100MWh of power at the spot market	5USD/MWh
Total	**12USD/MWh**

Scenario 2: The July spot spark spread is 15USD/MWh.	
Premium from selling the spark spread option	12USD/MWh
Payment to the spread option holder	-15USD/MWh
Buy 1.000MMBtu of gas and sell 100MWh of power at the spot market	15USD/MWh
Total	**12USD/MWh**

Basket Options

Assume underlyings

$$dS_i(t) = \mu_i S_i(t)dt + \sigma_i S_i(t)dW_i(t)$$

with $dW_i(t)dW_j(t) = \rho_{ij}dt$.
Here a basket of commodities is the underlying

$$B(t) = \sum_{i=1}^{m} w_i S_i(t).$$

Recall that the forward price are given by $F_i(t,T) = \mathbb{E}(S(T)|\mathcal{F}_t)$.

Basket Options – Example

Basket of energy prices related to power production

$$B(t) = 100 \times \left(30\% \frac{\text{oil}(t)}{\text{oil}(0)} + 30\% \frac{\text{coal}(t)}{\text{coal}(0)} + 40\% \frac{CO_2(T)}{CO_2(0)}\right).$$

At time 0 the basket is nomalized to 100. If the oil price increases by 100% the basket value increases by 30% to 130.

Basket Options

Pricing is not straightforward even in a BS-framework, since the sum of lognormals is not lognormal.
Typically a lognormal approximation is used, i.e.

$$\tilde{B}(T) = F_B \cdot \exp(-\frac{1}{2}\beta^2 + \beta N) \quad N \sim \mathcal{N}(0,1).$$

Now

$$\begin{aligned}
\mathbb{E}(B(T)) &= \sum_{i=1}^{n} w_i F_i(t,T) \\
\mathbb{E}[B^2(T)] &= \sum_{i,j=1}^{n} w_i w_j F_i(t,T) F_j(t,T) \exp\{\rho_{ij}\sigma_i\sigma_j(T-t)\} \\
(\text{use} \quad \mathbb{E}_t(S_i(T)) &= F_i(t,T)).
\end{aligned}$$

Basket Options

On the other hand

$$\mathbb{E}(\tilde{B}(T)) = F_B \quad \mathbb{E}(\tilde{B}^2(T)) = F_B^2 \exp(\beta^2).$$

Setting the first two moments equal

$$\begin{aligned} F_B &= \textstyle\sum_{i=1}^n w_i F_i(t,T) \\ \beta^2 &= \log\left(\frac{\sum_{i,j=1}^n w_i w_j F_i(t,T) F_j(t,T) \exp\{\rho_{ij}\sigma_i\sigma_j(T-t)\}}{F_B^2}\right) \end{aligned}$$

Now Black's formula can be use on $\tilde{B}$.

Basket Options- Example(continued)

With volatilities 0.3, 0.2 and 0.4 for Oil, Coal, CO2 and correlations $\varphi_{0C} = 0.1$, $\varphi_{0CO_2} = 0.6$, $\varphi_{CCO_2} = -0.2$ one gets for one year at the money calls

	price	imp vol.
MC	8,81	22.85%
Approx	8,95	23.16%

This implies a small pricing error, which s acceptable given the parameter uncertainty for σ_i and ρ_{ij}.