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FinancialCrisis and quantitative methods: problems and solutions

The limits of existing models for correlation, rates and credit. Lessons from the crisis

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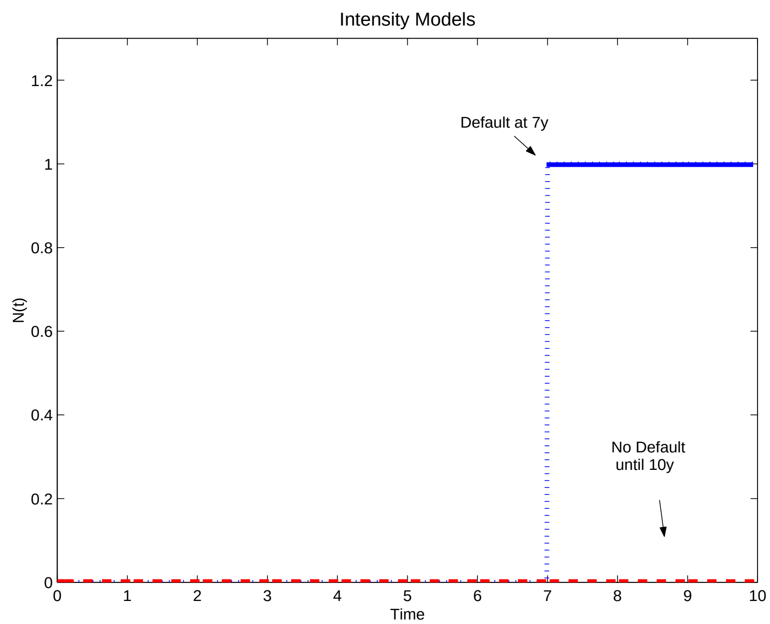
Realistic single name modelling for reliable multiname pricing

- Multiname products with intensity models. Risk of loss concentration missed by Gaussian Copula
- Limits of Base Correlation. Degenerate seniority-based correlations for capturing systemic risk
- Mapping for idiosyncratic and systemic Credit Risk. The limits underlined by intertemporal testing
- Case study: Financial's Credit Risk and liquidity in Libor Modelling after the subprime crisis
- The implications of high correlation on credit options. Market Price an Armageddon

Intensity Approach: Poisson Processes

Intensity Models (Default intensities are the fundamental modelling quantities. The models represent default as an exogenous, unpredictable event, the causes are not considered explicitly). Default of an individual name happens when a **jump process** N_t jumps for the first time



Intensity Models

Intensity Modelling

In single name modelling we will take τ^1, the first jump of the poisson process, as default time τ. Based on the above analysis, we can give the distribution of default time. In fact $Pr(N_T = 0) = e^{-\lambda T}$ amounts to say that

$$\begin{aligned} Pr(\tau > T) &= e^{-\lambda T}, \\ F_\tau(T) &= Pr(\tau \leq T) = 1 - e^{-\lambda T}, \end{aligned}$$

and the density is

$$f_\tau(T) = \lambda e^{-\lambda T}.$$

which is the density of a **(negative) exponential distribution**.

Fundamental results for simulation and multiname modelling

We know that simulation comes from transforming a uniform random variable U, since this is the distribution of any cumulative probability function. In this case

$$\boxed{\tau = F_\tau^{-1}(U) = -\tfrac{1}{\lambda}\ln(1-U).}$$

By the way, notice that, if we set $\tau\lambda = \varepsilon$, we see ε is now a unit exponential rv, since

$$F_\varepsilon(z) = \Pr(\varepsilon \le z) = F_\tau\left(\frac{z}{\lambda}\right) = 1 - e^{-z}.$$

So I can also write

$$\tau = \frac{\varepsilon}{\lambda},$$

so that τ is a function of a unit exponential random variable. In turn, $\varepsilon = -\ln(1-U)$.

Time-dependent credit spreads

With time inhomogeneous Poisson process we define

$$\Lambda\left(T\right)=\int_{0}^{T}\lambda\left(s\right)ds$$

For τ, we have that we can write it as

$$\tau=\Lambda^{-1}\left(\varepsilon\right)=\Lambda^{-1}\left(-\ln(1-U)\right)$$

with ε a unit exponential rv and U a uniform rv.

How can we relate defaults in this context?

Linking defaults in intensity setting

$$\tau_1 = \Lambda_1^{-1}(\varepsilon_1), \tau_2 = \Lambda_2^{-1}(\varepsilon_2), \dots, \tau_n = \Lambda_n^{-1}(\varepsilon_n)$$

$$flat\ \lambda_i: \quad \tau_1 = \frac{\varepsilon_1}{\lambda_1},\ \tau_2 = \frac{\varepsilon_2}{\lambda_2},\ \dots,\ \tau_n = \frac{\varepsilon_n}{\lambda_n}$$

Stochasticity of τ_i can come from two different sources: surely stochasticity of ε_i, and possibly stochasticity of the Λ_i.

We can s**et dependency among the ε_i of the different names and keep the intensities independent or even deterministic.** This is the framework that has been the standard for correlation products in the market, using a copula function, so that

$$\mathbf{F}_{1,2,\dots,n}^{Joint} = \mathbf{Copula}(\mathbf{F}_1^{M\arg inal}, \mathbf{F}_2^{M\arg inal}, \dots, \mathbf{F}_n^{M\arg inal}).$$

The copula separates information on the dependency from information on the individual distributions. **The Copula contains all information on the dependency without information on the Individual distributions. Given the Copula, we can put the Individual distributions together to find the Joint distribution**.

1-Factor Gaussian Copula

The copula used in the market is based on an n-variate Gaussian distribution as follows

$$X_i = \sqrt{\rho_i} M + \sqrt{1 - \rho_i} Y_i \quad i = 1, \ldots, n \tag{1}$$

where Y_i, M are standard independent gaussian.

The distribution of X_i is Gaussian, with $\mathbb{E}[X_i] = 0$ and $Var[X_i] = \rho_i + 1 - \rho_i = 1$. Moreover

$$Corr(X_i X_j) = \sqrt{\rho_i \rho_j}$$

so the distribution is a simplified multivariate gaussian

$$\Pr(X_1 \leq x_1, \ldots, X_n \leq x_n) =: \Phi_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$$

The copula is

$$C_{X_1,\ldots,X_n}(u_1, \ldots, u_n) = \Phi_{X_1,\ldots,X_n}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n))$$

Last to Default

In a portfolio of n names we indicate: $\tau^{(j)}$ =time of the j-th default in the portfolio

Protection Buyer	$\longrightarrow$ rate S at $T_{a+1}, \ldots, T_b$ before $\tau^{(n)}$ $\longrightarrow$	Protection Seller
	$\longleftarrow$ Protection $\mathrm{L{\scriptstyle GD}}^{(n)}$ at $\tau^{(n)}$ if $T_a < \tau^{(n)} \leq T_b$ $\longleftarrow$	

In a gaussian copula, deterministic intensity setting:

- Correlation 1: S is the same as the spread paid on CDS written on the least risky name

- Correlation 0: S is around the product of the individual spreads, usually very low

Last to Default

In fact, in an intensity modelling setting, recall we set $\int_0^\tau \lambda_i(s)\,ds = \Lambda_i(\tau) = \varepsilon_i$ or in homogeneous case $\lambda_i \tau = \Lambda_i(\tau) = \varepsilon_i$. Given n names

$$
\begin{aligned}
\tau_1 &= \Lambda_1^{-1}(\varepsilon_1), \tau_2 = \Lambda_2^{-1}(\varepsilon_2), \ldots, \tau_n = \Lambda_n^{-1}(\varepsilon_n) \\
{}^{flat\ \lambda_i:}\tau_1 &= \frac{\varepsilon_1}{\lambda_1},\ \tau_2 = \frac{\varepsilon_2}{\lambda_2},\ \ldots,\ \tau_n = \frac{\varepsilon_n}{\lambda_n}
\end{aligned}
$$

if $\rho = 1$ this means simply that all ε are the same ($\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_n$). The τ's are different, with the last default time (highest) being the one associated to lowest λ_i. The copula approach for different ρ inherits this separation between the determinitic model for single name default risk and exogenous correlation assumption.

Is such a behaviour possible in all models? Does it make sense? Can it lead to counterintuitive behaviour? Let us see an example...

A paradox in market copula: Forward-start Last to Default

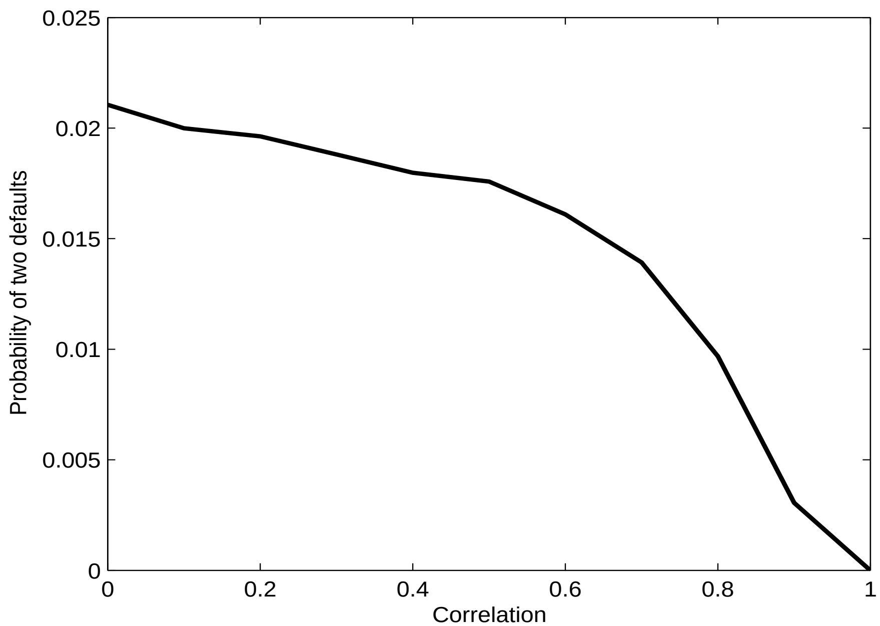
Consider we are exposed to losses from default of two names (or two different investments). We are not so much worried about the probability that they default, but we are very worried about the probability that they **both** default in a particular, **quite short period of time**, because we know that in that period we may run short of liquidity...It does not sound such a bizarre worry, in 2008.

For example, we may be worried about the period between 3y and 5y from now, and we are exposed to two investments with the following default intensities:

$$\lambda_1 = 0.08, \ \lambda_2 = 0.32$$

How may we protect ourselves against this risk? We may buy protection on a forward start Last-to-Default, paying us only if both names default in the specific period of time that worries us. Clearly the value of this protection depends on the probability of the two names to default together exactly in that short period of time. We may expect this probability to be higher when correlation is higher, as it always happens with last to default...

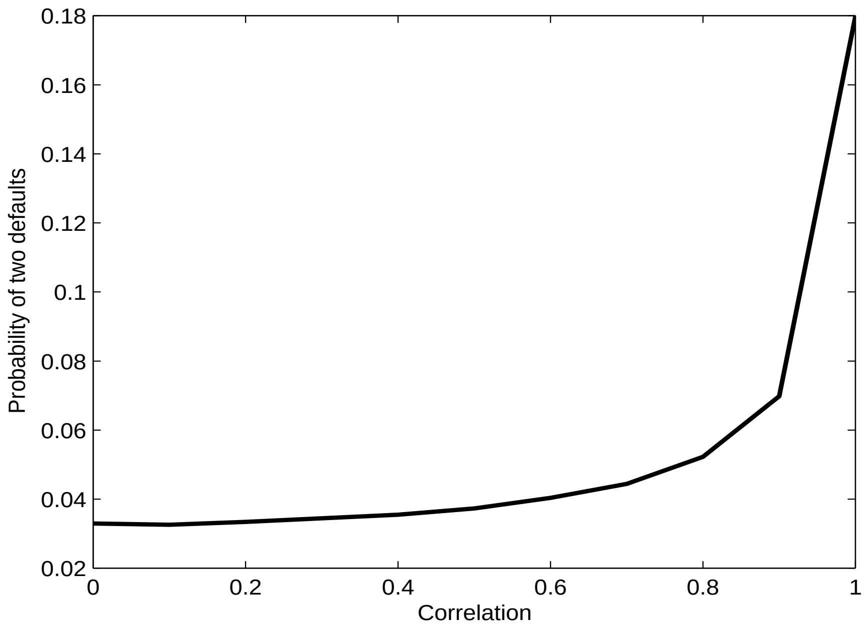
A paradox in market copula: missing the correlation - liquidity risk



The behaviour is opposite than expected. The point is that in standard gaussian copula, which is a static model, the correlation has little to do with the risk related to the timing of defaults.

A paradox in market copula: Forward start Last to Default

Supposing that, while average default risk does not change, we think these two name to have the same risk, $\lambda_1 = \lambda_2 = 0.2$



So we see that the temporal aspect of correlation risk seems to be related more to changes in credit spreads than to gaussian copula ρ.

CDO: protection on tranched loss

The payoff of a tranche $[A, B]$ can be written as the difference of two equity tranches, normalized by the size of the $[A, B]$ tranche.

Equity-Tranches-based definition. Define first the Tranched Loss of an equity tranche with detachment X, exposed to losses in $[0, X]$,

$$
\begin{aligned}
L_X(t) &= \frac{1}{X}\left[(L(t))\, 1_{\{L(t)\leq X\}} + X 1_{\{L(t)>X\}}\right] \\
& \frac{1}{X}\left[L(t) - (L(t) - X)^+\right]
\end{aligned}
$$

Then

$$
L_{A,B}(t) = \frac{1}{B-A}\left[B L_B(t) - A L_A(t)\right]
$$

Base correlation

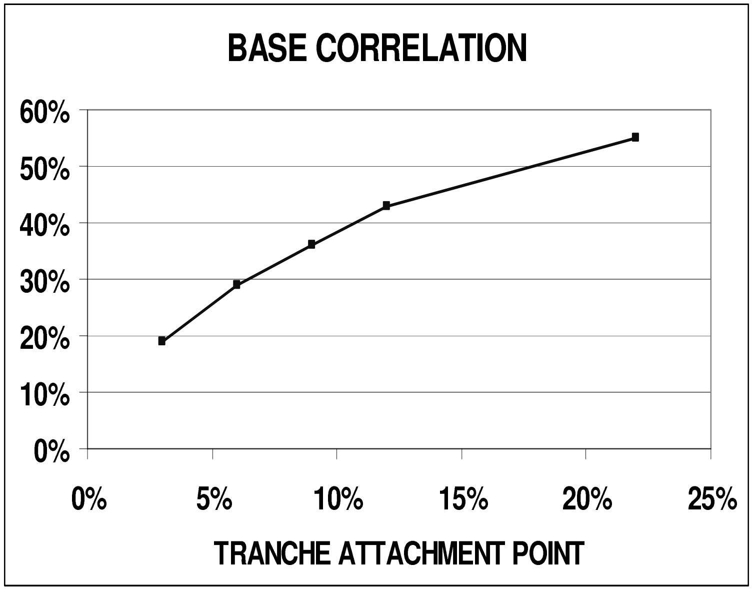
Using Gaussian copulas with **Base Correlation:** for tranches$[0, A]$ $[A, B]$ $[B, C]$ one can give

$$\boxed{\rho_A}: \mathbb{E}\left[L_{0,A}(t)\right] = \frac{1}{A-0}\left[A \; GL_A\left(\boxed{\rho_A}\right) - 0 \; GL_0(-)\right] = \mathbb{E}^{Mkt}\left[L_{0,A}(t)\right]$$

$$\underline{\rho_B}: \mathbb{E}\left[L_{A,B}(t)\right] = \frac{1}{B-A}\left[B \; GL_B\left(\underline{\rho_B}\right) - A \; GL_A\left(\boxed{\rho_A}\right)\right] = \mathbb{E}^{Mkt}\left[L_{A,B}(t)\right]$$

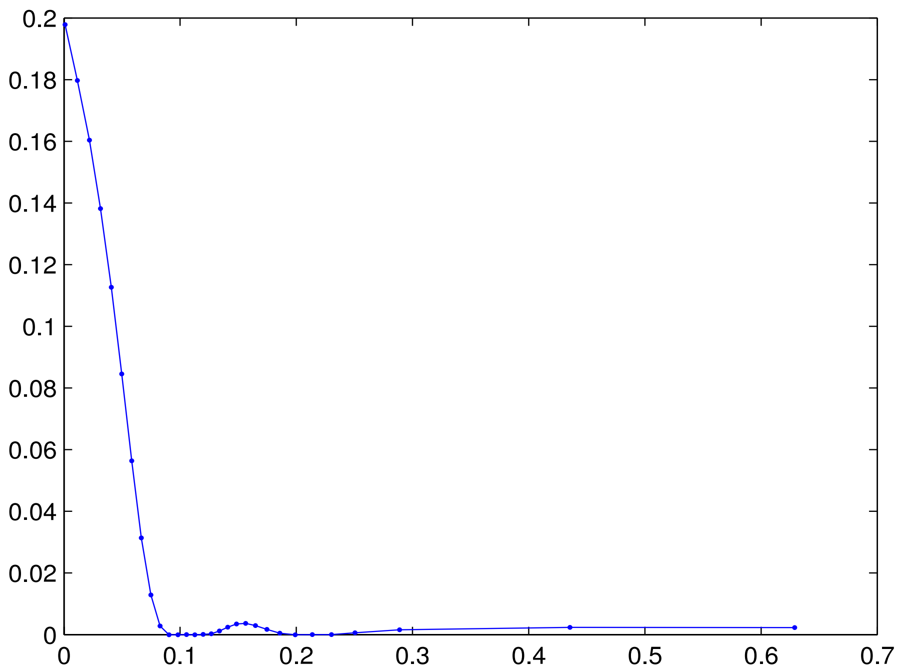
$$\overline{\rho_C}: \mathbb{E}\left[L_{B,C}(t)\right] = \frac{1}{C-B}\left[C \; GL_C\left(\overline{\rho_C}\right) - B \; GL_B\left(\underline{\rho_B}\right)\right] = \mathbb{E}^{Mkt}\left[L_{B,C}(t)\right]$$

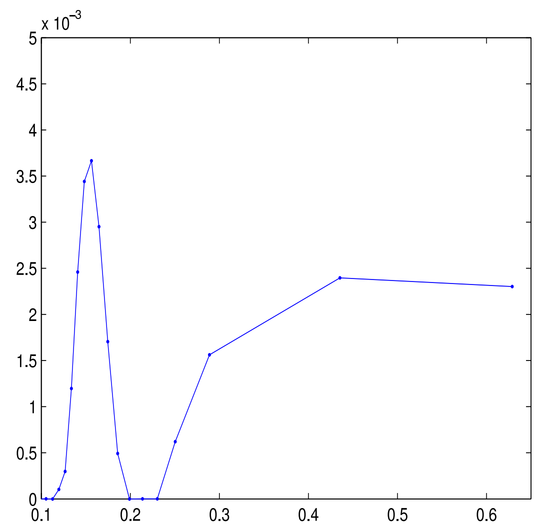
Base Correlation



Scenario Density

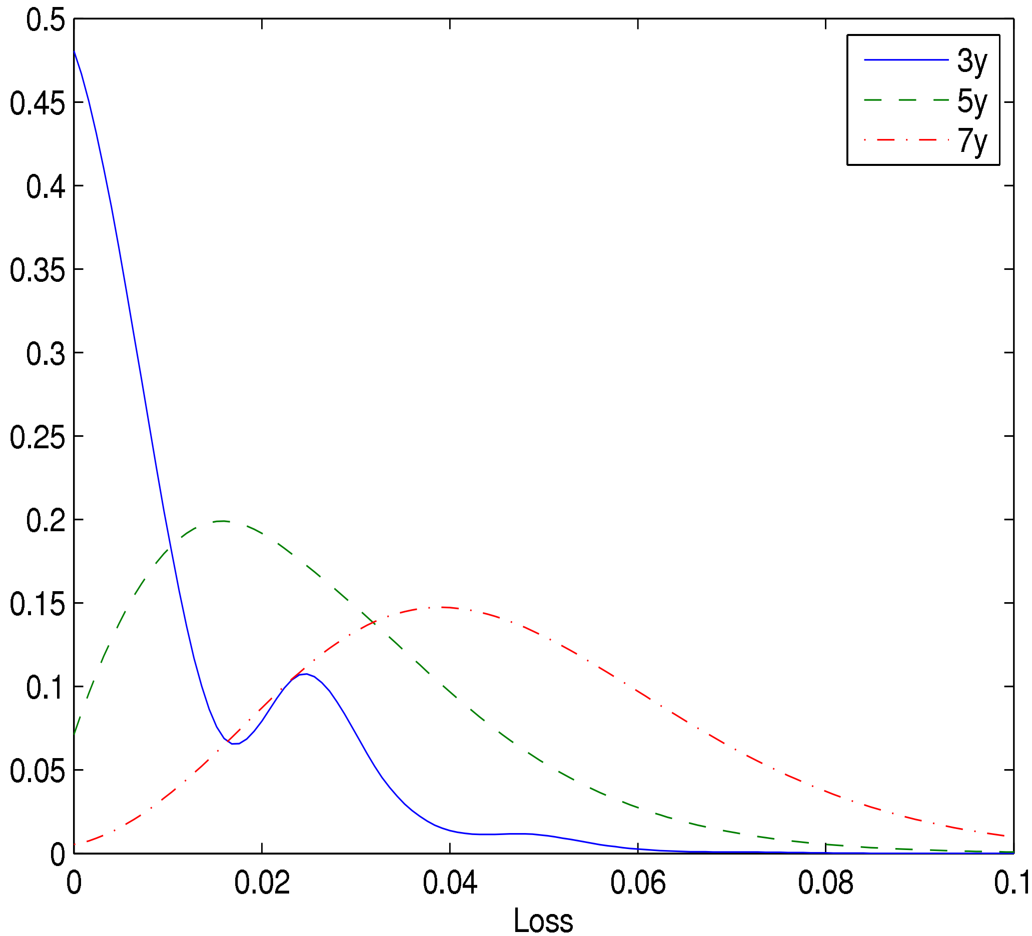
An approach with flat correlation can only fit the market changing inconsistently the correlation assumptions for different tranches. Complex models that fit the correlation skew with a consistent set of parameters (Hull and White Perefect Copula, Brigo et al. GPL...) show that the density of a market portfolio Loss has some distinctive features.

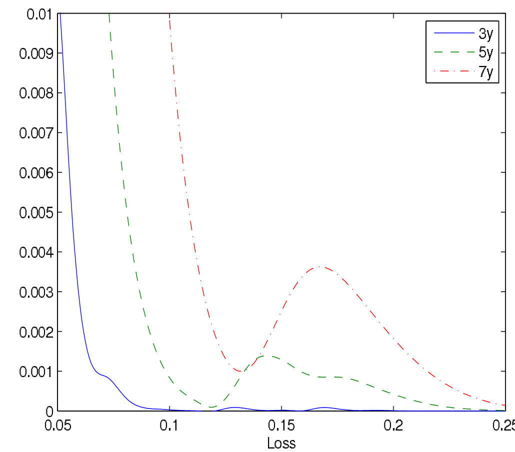


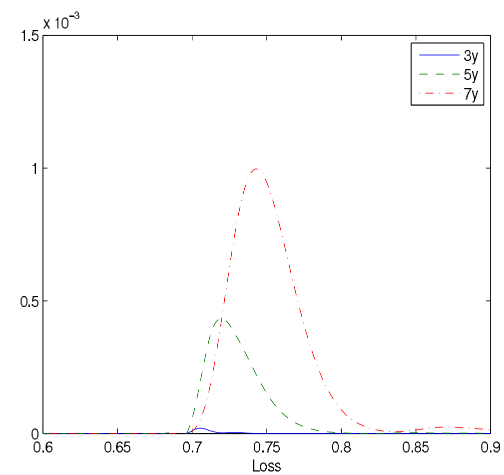


Plot of the scenario distribution of the individual default probabilities. Zoom on the right.

GPL: Implied Loss density

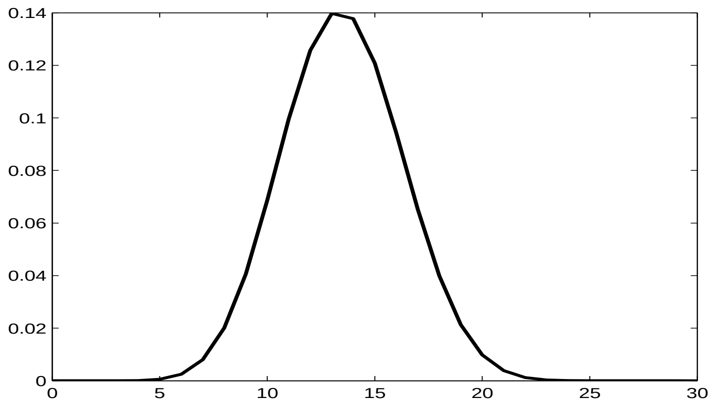


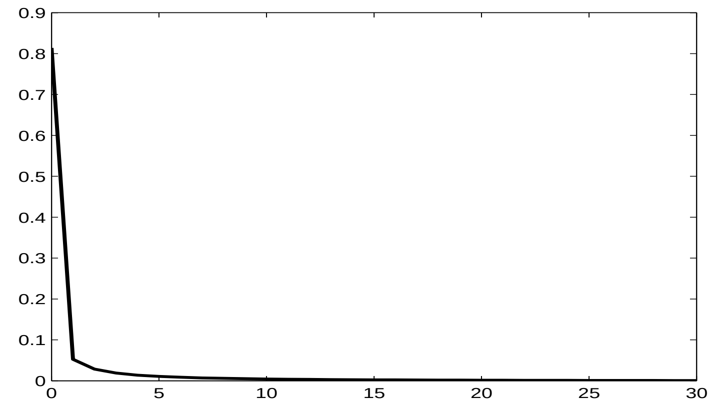




The Multimodal Loss Distribution: Fat Tails in Credit

We notice the characteristic multimodality of the implied density found, which is a regularity in the considered time serie. Is this incompatible with gaussian copula? It seems incompatible with the gaussian copula as it is implemented in market practice, where, even when one does not resort to the homogeneous pool assumption, one still uses a single copula input. With one single copula input, it is hard to model a realistic loss. See below two different cases, $\rho = 5\%$ $(\lambda = 0.15)$ and $\rho = 70\%$ $(\lambda = 0.01)$.





How can we incorporate multimodal losses sticking to one single gaussian copula? We have to abandon the standard market quotation system.

An heterogeneous correlation structure

Can we relate, at least in part, the existence of the smile to the most strikingly unrealistic implication of the homogeneous pool assumption, which is the assumption of flat correlation $\rho_{ij} = \rho$?

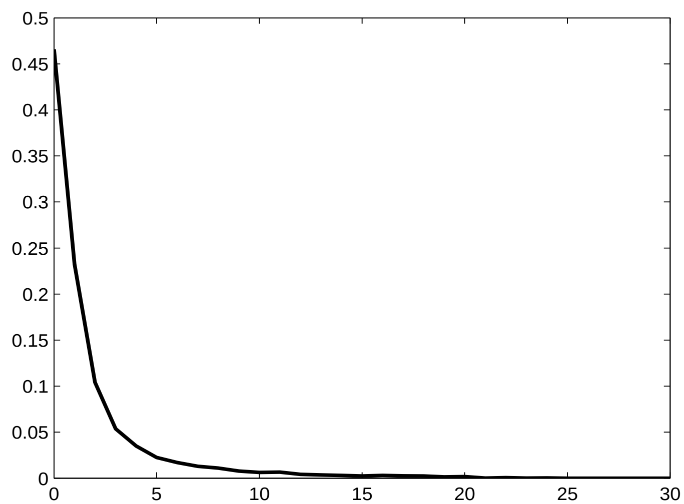
A model with a flat correlation around an average level tends to miss:
A) Some pairs of names are characterized by a **very high correlation** ($\rho_{ij} = 1$ in the most extreme case)
B) Some pairs of names are characterized by a **very low correlation** ($\rho_{ij} = 0$ in the most extreme case)

Notice that clusters of high correlations make **big losses and small losses** (far from average) **more likely**, while clusters of low correlations make **losses around the expected loss more likely**

This means increasing kurtosis and generate **fat tails effect**.

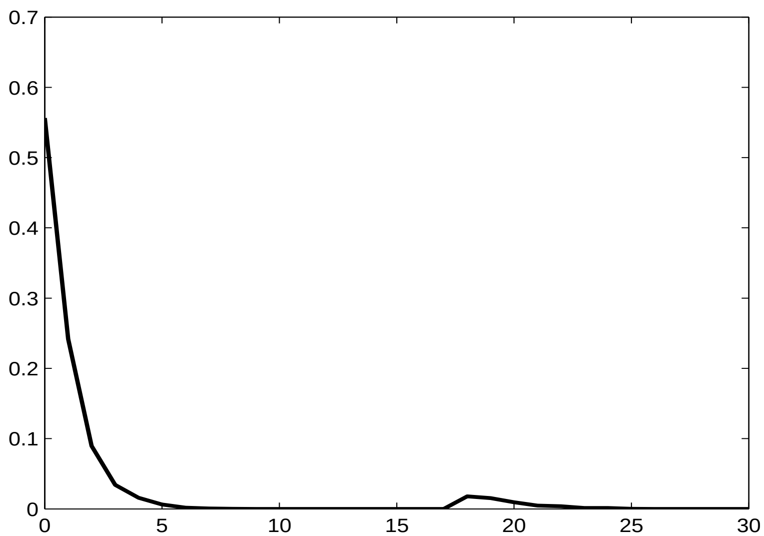
Heterogeneous correlations

We now consider the same issue in relation with result of Brigo et al. (2006) on multimodal loss distribution. We consider a simple non homogeneous pool with two classes of names: a smaller group (40%) of riskier names, $\lambda = 0.015$, quite loosely related to any other (their risk is mostly idiosyncratic), and larger group of more senior names, $\lambda = 0.005$, that we expect to default only in case of a more generalized systemic (or sector-wide) crisis. We may decide to assign a high 70% correlation to the group of senior names, leaving a 20% correlation for all other couples of names:



Heterogeneous correlations with systemic shock

Although there are, zooming the in the tail, some irregularities compared to flat correlation case, these are not the clear bumps pointed out by market analysis. Going back to the above example, notice that we have related the riskiness of the most senior names only to a possible systemic or sector-wide crisis. An assumption, more extreme than the one above, but more consistent with this interpretation, would be to assign to these names perfect correlation $\rho = 1$. In this case, we obtain



Heterogeneous correlations with systemic shock

The Gaussian copula with heterogeneous correlations, where a subset of senior names are assumed to be perfectly correlated (with low default probability) gives rise to multimodal loss densities, featuring bumps on the far tail. This appears related to the fact that $\rho = 1$ is the only case when Gaussian copula features a tail dependency.

Comparing this second solution (heterogeneous correlation) to the previous one (scenarios on copulas), we see that the interpretation of the configurations that give rise to multimodal densities is different. In particular, for scenarios on copulas, it is the low correlation scenario which is responsible for the bump in the tail. For heterogeneous correlation, we have an opposite (and probably more reasonable) explanation for bumps in the tail.

Can we use this trick of an heterogenous correlation to fit the market? Let us see a simple solution.

Heterogeneous correlation as a function of credit risk

In the previous example not only we used a heterogenous correlation, but we made correlation a function of the spreads. This corresponds to the idea that names with different **levels** of credit risk are often associated to different **types** of credit risk: the risk of subinvestment grade names usually comes from idiosyncratic, or firm-specific, risk factors, while the risk of senior names usually comes from the risk of more systemic crisis, such as the credit crunch liquidity crisis. While there are a number of exceptions (fallen angels, business links), this appears a first ingredient to put in the correlation matrix: correlation must be a decreasing function of a name's default risk.

We take the i-Traxx spreads as from February 2008, and we set us in the simplest possible gaussian correlation framework: One Factor Gaussian Copula.

Heterogeneous correlation as a function of credit risk

We define the n-variate Gaussian distribution as usual

$$X_i = \sqrt{\rho_i} M + \sqrt{1-\rho_i} Y_i \quad i = 1, \ldots, n$$

so that

$$\rho_{ij} = Corr\left(X_i X_j\right) = \sqrt{\rho_i \rho_j}$$

However, at this point we do not assume that the individual correlation parameters ρ_i are all the same, but instead we make them a function of the individual intensities or spreads:

$$\rho_i = f\left(Spread_i\right)$$

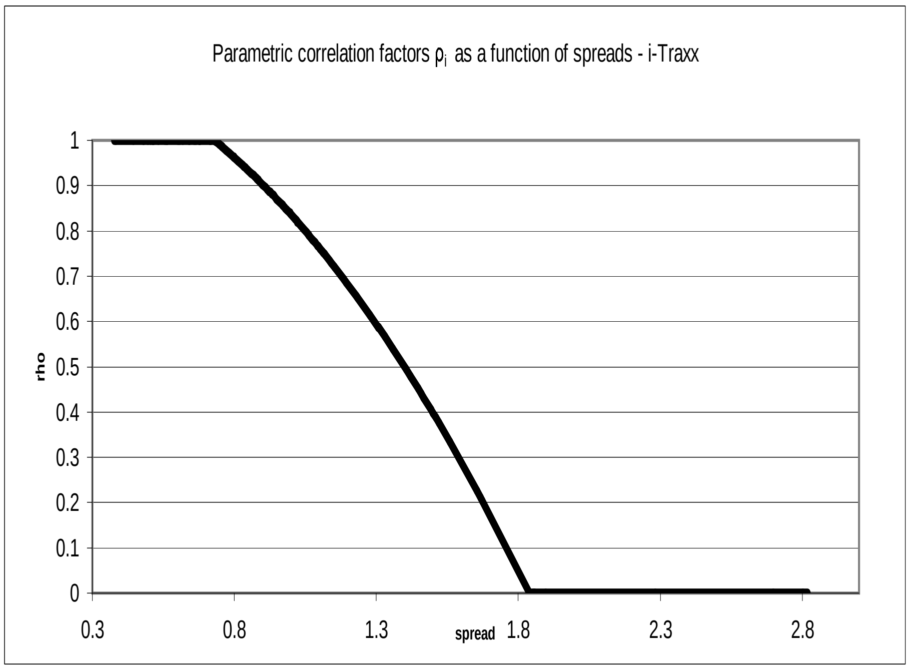
How this influences resulting correlation? If ρ_i, ρ_j are high, then ρ_{ij} will also be high, while if either ρ_i, ρ_j is low the resulting correlation of the two names will be lower, and very low when both $\rho_i \approx 0$, $\rho_j \approx 0$.

Parameterizing Correlations

Even in this simple framework, we can make correlation of more senior names higher by setting $\rho_i = f\,(Spread_i) > \rho_j = f\,(Spread_j)$ when $Spread_i < Spread_j$. So we have to choose a **decreasing** parameterization, and we also want it to be sufficiently general to allow for correlation factors as high as $\rho_i = 1$ and as low as $\rho_i = 0$. We would also like the parameterization to show some smoothness (is this really required?). A possible parameterization of this kind is a 3-parameter constrained polinomial (see Duminuco and Morini (2008)), as follows.

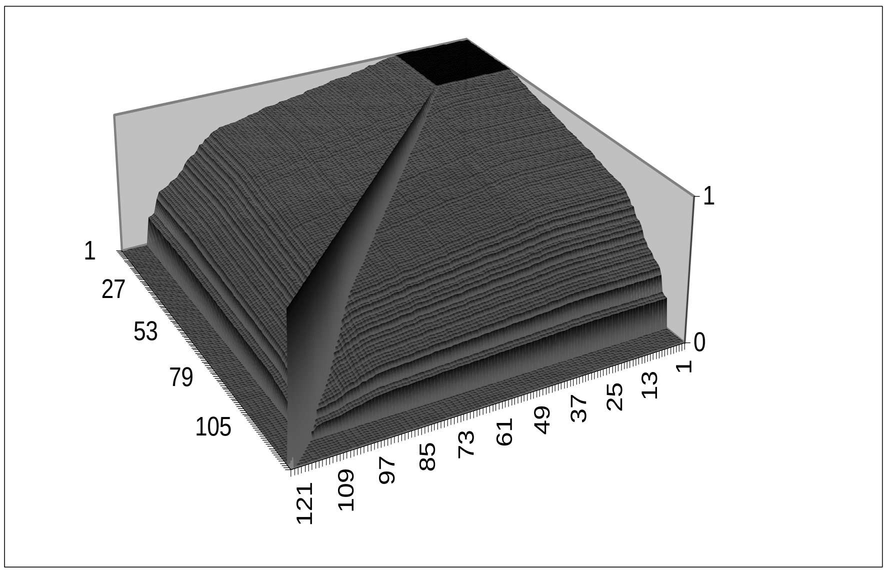
Parametric Correlation for a correlation skew in a Gaussian Copula

Setting the parameters based on market spreads and correlations of February 15, 2008, this results in the following dependence of the correlation factors ρ_i on the spreads of the i-Traxx names:



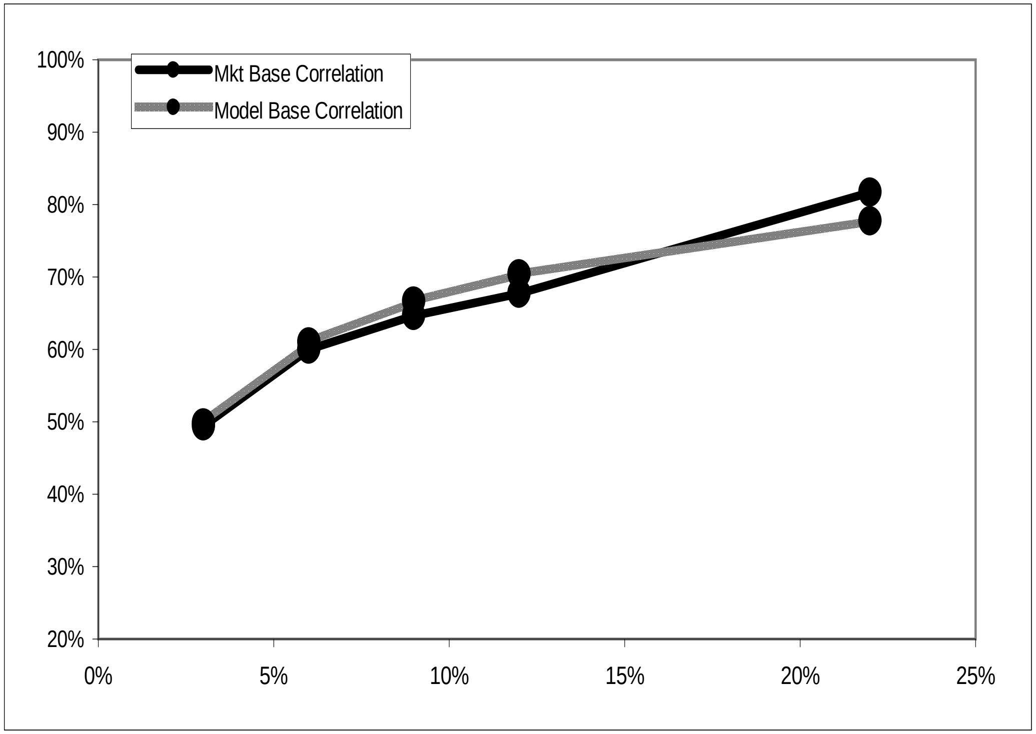
Parametric Correlation: the resulting Matrix

This reminds of **local correlation**: models that make a flat correlation be a (mainly increasing) function of the loss $L(t)$, so as to change it when approaching different detachments. Here we do it implicitly: when more risky names default through idiosyncratic events, and the loss approaches more senior tranches, the remaining names have a higher average correlation. However here we remain in a standard gaussian copula, with a more realistic non-flat correlation. In particular, it leads to the heterogenous correlation:



Parametric Correlation: Implied Correlation Skew

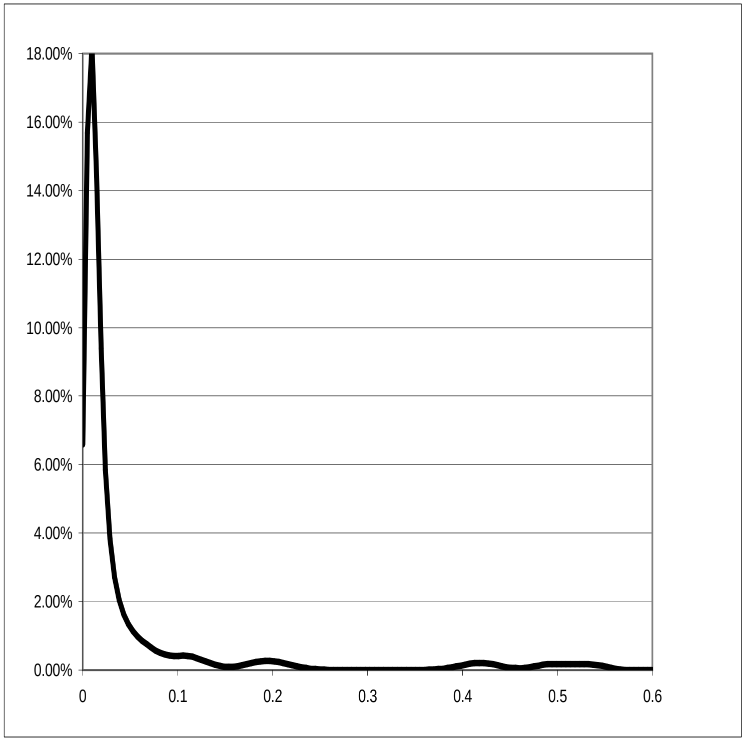
This correlation implies a rather realistic correlation skew, consistent with market correlation skew (February 15, 2008, 5y correlation skew)



One can take more factors (e.g. the financial sector factor) or scenarios, to increases the flexibility in fitting correlation skews.

Parametric Correlation: the implied Loss Distribution

The implied Loss distribution can easily computed, and it has the desired multimodal structure.



Setting Correlation for a Bespoke portfolio: Mapping?

The above analysis regards explaining market quotes for portfolio credit risk. What if we have no market quotes for a given portoflio? This is the most common cas in the market, and in 2006-2007 it led to a characteristic market practice, that was named Mapping. In the following we describe this market technique and we devise a way on testing it on market data.

What is the idea behind Mapping? If we want to stick to Gaussian Copula with average correlation (the market model), we know that correlation is not a characteristic of the portfolio, but of the position of one tranche in the capital structure. When we move to a Bespoke portfolio, we often have no liquid information on correlations, but only single name spreads. Is there a way, from Bespoke single name spreads and from Index correlation information, to understand which correlation one should give to the different tranches of the Bespoke?

According to mapping, the answer is yes.

Setting Correlation for a Bespoke portfolio: Mapping?

According to mapping, the answer is yes. A level of correlation ρ should corresponds to some fundamental characteristic of one tranche, so that we can first use Index information to detect the map,

$$\rho_K^{Index} = f\left(Invariant_K^{Index}\right)$$

and then we can apply this map to the bespoke, finding bespoke correlations as

$$\rho_{\bar{K}}^{Bespoke} = f\left(Invariant_{\bar{K}}^{Bespoke}\right)$$

What could be this invariant?

From the Index to Bespokes: Mapping Methods

1) No Mapping

In the simplest case, the invariant is just the strike K. This means that

$$\rho_K^{Index} = f^{NM}(K), \quad \rho_{\bar{K}}^{Bespoke} = f^{NM}(\bar{K})$$

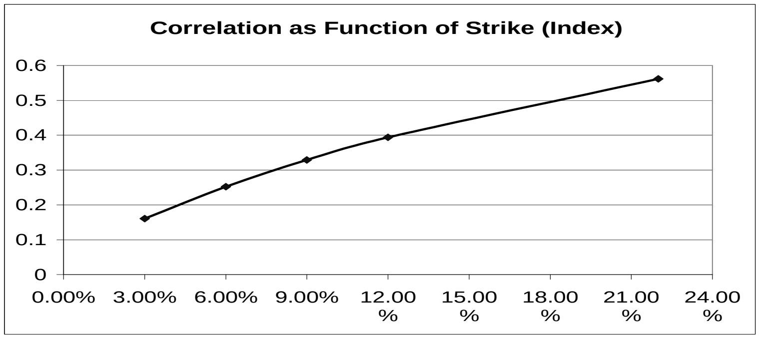
2) Expected Loss

Another solution considers not K, but $\frac{K}{\mathbb{E}[L]}$. Now one must detect from the index the map

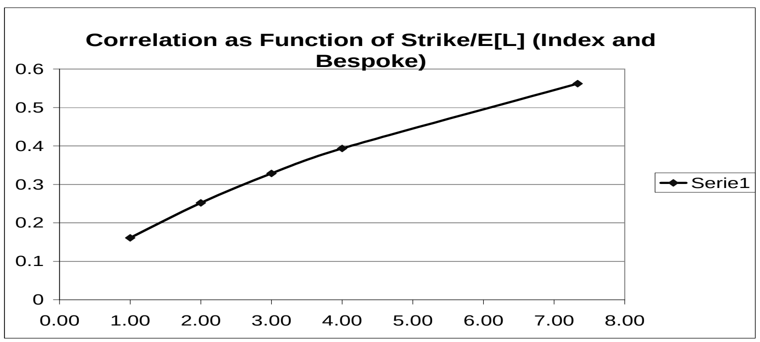
$$\rho_K^{Index} = f^{EL}\left(\frac{K}{\mathbb{E}[L]}\right)$$

and then one can apply it to the bespoke, so that

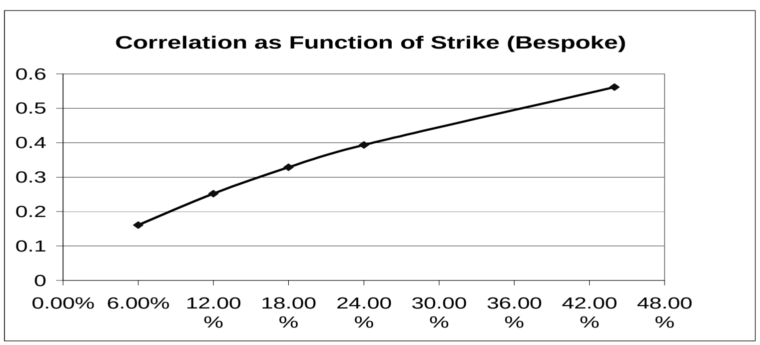
$$\rho_{\bar{K}}^{Bespoke} = f^{EL}\left(\frac{\bar{K}}{\mathbb{E}[L^{Bespoke}]}\right).$$



$E[L] = 3\%$. We rewrite the skew in terms of $K/E[L]$



This map is what DOES NOT change . We translate it back into a skew using $E\left[L^{Bespoke}\right] = 6\%$



Expected Tranched Loss

A more refined alternative, that also takes into account spread dispersion, uses as invariant the normalized expected tranched loss $\mathbb{E}[L_{0,K}]$, where $L_{0,K}$ is the loss of a $[0\%, K\%]$ tranche

$$\frac{\mathbb{E}[L_{0,K}]}{\mathbb{E}[L]}$$

Now it is more difficult to write explicitly the map, since tranched loss itself depends on correlation. So one starts from a correlation ρ_K^{Index} in the index, and tries to find the detachment $\bar{K}$ in the bespoke such that, if we use ρ_K^{Index} for the Bespoke $\bar{K}$-tranche, we have

$$\frac{\mathbb{E}\left[L_{0,\bar{K}}^{Bespoke}; \rho_K^{Index}\right]}{\mathbb{E}\left[L^{Bespoke}\right]} = \frac{\mathbb{E}\left[L_{0,K}; \rho_K^{Index}\right]}{\mathbb{E}[L]}$$

Expected Tranched Loss

In this sense, $\bar{K}$ in the bespoke is *equivalent* to K in the Index. The equivalent strikes, through extrapolation/interpolation, provide the correlations for the bespoke.

If it is true that correlation only depends on the invariant, one can for example guess the correlation to apply to iTraxx Main or to CDX High Yield. based on observing correlations on CDX Investment Grade. Let us see results of Lehman (2007), for 5y tranches, and then some tests of our own.

Lehman tests: Mapping across regions

CDX.NA.IG$\mapsto$iTraxx Main : Tranches 5y NPV

	Market	NM	EL	ETL
0-3%	10.53	10.83	9.58	10.35
3-6%	42.2	38.5	36	42.9
6-9%	12.3	7.2	6.4	10.3
9-12%	5.6	2.5	1	4.7
12-22%	2.2	0.8	0	1.9

In mapping to a portfolio of similar credit quality, although with strong regional differences, using ETL appears better than any other methods.

Lehman tests: Mapping across credit quality

CDX.NA.IG$\mapsto$CDX.NA.HY : Tranches 5y NPV

	Market	NM	EL	ETL
0-10%	68.75	61.73	74.92	74.79
10-15%	26.07	19.31	28.85	22.78
15-25%	225.7	230.2	155.2	136.7
25-35%	56.1	134.2	21.3	28.1

Instead, we see that, moving to a portfolio with a relevant difference in riskiness, mapping methods all perform badly. In particular, EL and ETL put too much risk in the equity tranche and too little in the in the senior tranche. Lehman (2007) suggest this might be related to the quotation features of the HY compared to the IG. Now we see some tests of ours, trying to clarify these issues.

Mapping: what is it, really?

We start from the following considerations.

1) Mapping methods simply try to capture the relation between changes in the spreads of a portfolio, and changes in the quotation numbers of tranches, the base correlation. They can work when the changes in correlation can be captured based on the change of the spreads of the components. On the other hand, if one tests mapping on portfolios too different from each other, this fundamental assumption could not hold, which makes it difficult to compare the methods.

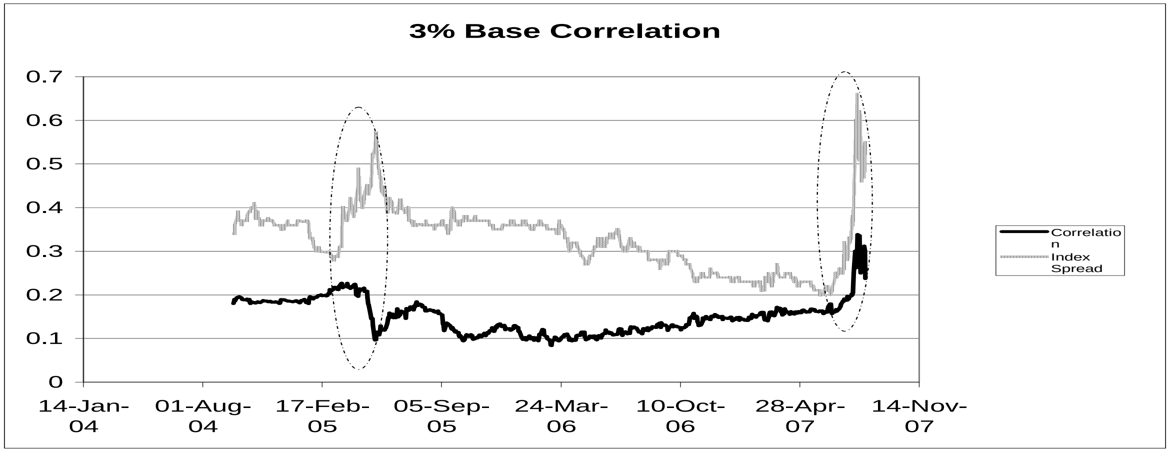
Therefore we would like to test mapping for portfolios that are really similar, apart from some well detected characteristics expressed by portfolio spreads. A good idea appears to take the same portfolio but before and after some well detected shocks that changed its structure. So we will see if a Mapping method actually captures the effect of this change in the underlying portfolio on the correlation, without the analysis being spoiled by too many difficulties in detecting what are the fundamental differences between two portfolios. If a mapping method works well in this simplified test, it may work also in a real Bespoke mapping. If it does not work here, it is hopeless to apply it to more complex situations.

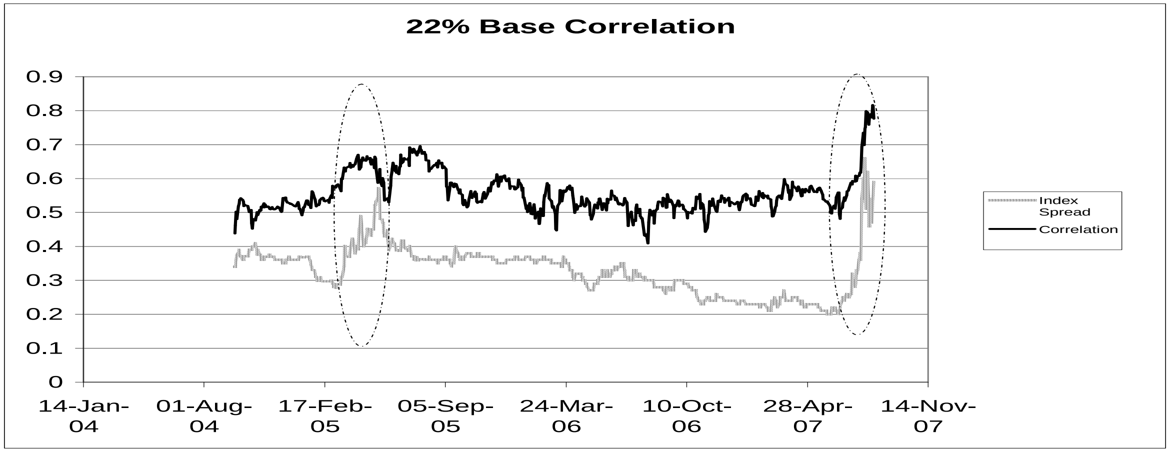
Mapping: can it also help in hedging?

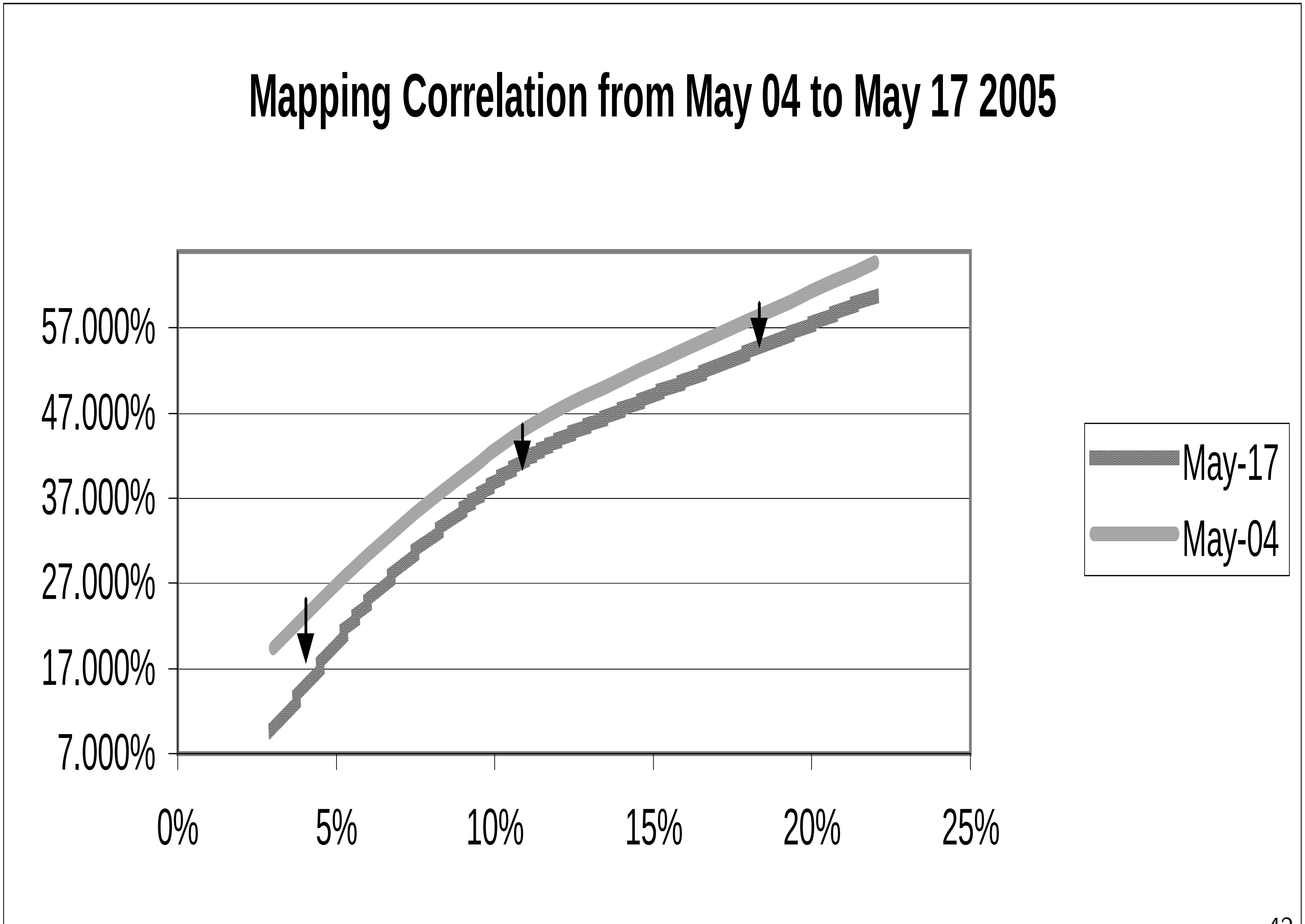
2) If we are able to understand this relationship between changes in spreads and changes in correlation, this is useful also for risk-management of Tranches: in fact, if a given change in the underlying spreads tends to be associated to a specific change in correlation, it is stupid to delta-hedge (compute sensitivities wrt changes in spreads) without taking this into account. Hedging would be much more efficient if we can capture the change in correlation usually associated to change in spreads (analogous to the sticky-strike/sticky-delta issue in equity trading). But this means to perform a mapping from one day to a different subsequent day. So we analyze the behaviour of the above mapping methods of the iTraxx Index and Tranches by checking if they can explain the association spreads-correlation in the history of i-Traxx.

Let us see the history of i-Traxx, to understand which changes happened in the portfolio features. Then we see if correlation mapping can capture these regularities, on a iTraxx portfolio.

History: i-Traxx and Correlations






Mapping Correlation from May 04 to May 17 2005
57.000%
47.000%
37.000%
27.000%
17.000%
7.000%
0%
5%
10%
15%
20%
25%
May-17
May-04

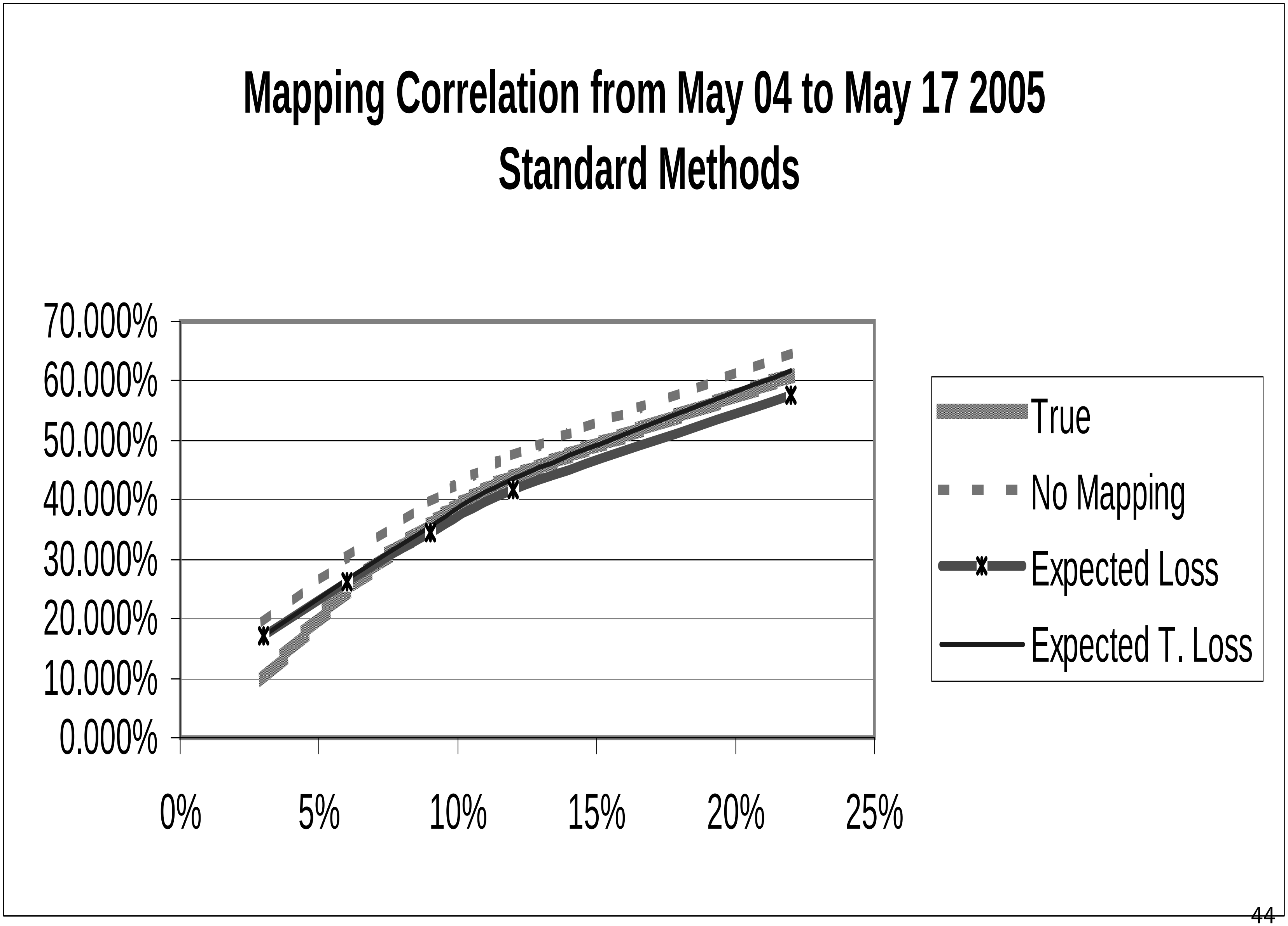
The 2005 idiosyncratic shock: Mapping when there is more idiosyncratic risk

We consider a lag of few days in May 05 characterized by an increase in portfolio riskiness with a big change in correlation (**decrease of correlation**, thus here the increase in risk is **idiosyncratic**)

Initial ρ	19.38%	30.43%	39.75%	47.45%	64.72%
Final ρ	9.66%	24.73%	35.74%	43.78%	60.88%
Diff	-9.72%	-5.71%	-4.00%	-3.68%	-3.85%

Let us see how one can predict it by mapping

Correlation	**3%**	**6%**	**9%**	**12%**	**22%**
No Mapping	19.38%	30.43%	39.75%	47.45%	64.72%
Mapping (EL)	17.22%	26.22%	34.55%	41.72%	57.66%
Mapping (ETL)	17.17%	26.62%	35.58%	43.47%	61.65%

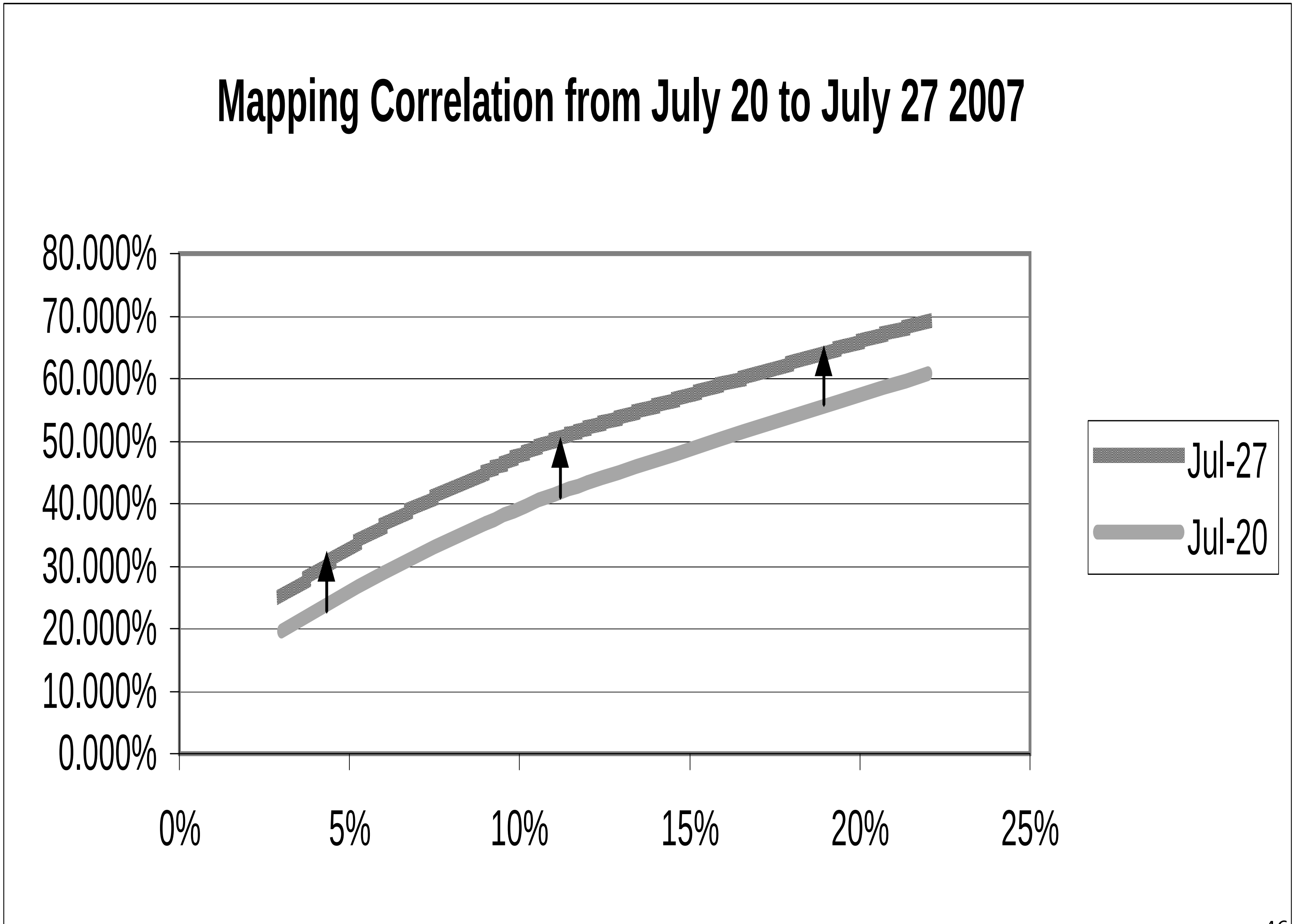

Mapping Correlation from May 04 to May 17 2005
Standard Methods
70.000%
60.000%
50.000%
40.000%
30.000%
20.000%
10.000%
0.000%
0%
5%
10%
15%
20%
25%
True
No Mapping
Expected Loss
Expected T. Loss

The 2005 idiosyncratic shock: Mapping when there is more idiosyncratic risk

Initial ρ	19.38%	30.43%	39.75%	47.45%	64.72%
Final ρ	9.66%	24.73%	35.74%	43.78%	60.88%
Diff	-9.72%	-5.71%	-4.00%	-3.68%	-3.85%

Err.on ρ	3%	6%	9%	12%	22%
No Mapping	-9.7%	-5.7%	-4.0%	-3.7%	-3.8%
Mapping (EL)	-7.6%	-1.5%	1.2%	2.1%	3.2%
Mapping (ETL)	-7.5%	-1.9%	0.2%	0.3%	-0.8%

We see that NM here would induce a bigger error compared to use Mapping. Among the Mapping methods, ETL is slightly better.


Mapping Correlation from July 20 to July 27 2007
80.000%
70.000%
60.000%
50.000%
40.000%
30.000%
20.000%
10.000%
0.000%
0%
5%
10%
15%
20%
25%
Jul-27
Jul-20

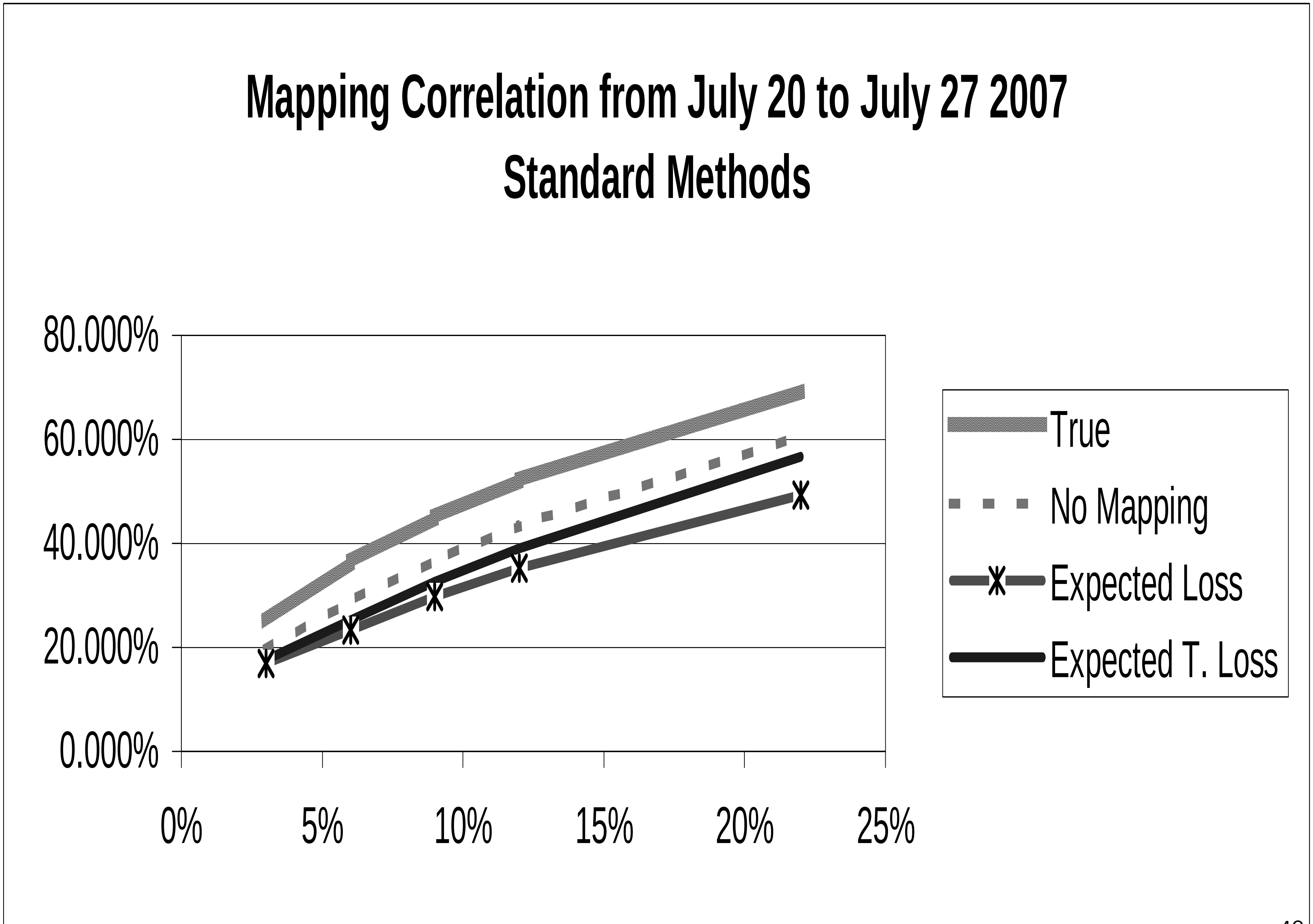
The 2007 systemic shock: Mapping when there is more systemic risk

We consider a lag of few days in July-August 07 characterized by an increase in portfolio riskiness with a big change in correlation, comparable to the one in the previous test. However this time it is an **increase of correlation**, here the increase in risk is **systemic**.

Initial ρ	19.61%	28.93%	36.76%	43.38%	60.70%
Final ρ	24.92%	36.37%	44.93%	52.09%	69.34%
Diff	5.31%	7.44%	8.17%	8.71%	8.64%

Let us see how one can predict it by mapping.

Correlation	**3%**	**6%**	**9%**	**12%**	**22%**
No Mapping	19.61%	28.93%	36.76%	43.38%	60.70%
Mapping (EL)	16.81%	23.33%	29.71%	35.18%	49.26%
Mapping (ETL)	17.61%	25.26%	32.57%	39.11%	56.61%


Mapping Correlation from July 20 to July 27 2007
Standard Methods
80.000%
60.000%
40.000%
20.000%
0.000%
0%
5%
10%
15%
20%
25%
True
No Mapping
Expected Loss
Expected T. Loss

The 2007 systemic shock: Mapping when there is more systemic risk

Initial ρ	19.61%	28.93%	36.76%	43.38%	60.70%
Final ρ	24.92%	36.37%	44.93%	52.09%	69.34%
Diff	5.31%	7.44%	8.17%	8.71%	8.64%

Err.on ρ	**3%**	**6%**	**9%**	**12%**	**22%**
No Mapping	5.31%	7.44%	8.17%	8.71%	8.64%
Mapping (EL)	8.11%	13.04%	15.22%	16.91%	20.08%
Mapping (ETL)	7.31%	11.11%	12.36%	12.98%	12.73%

We see that here No Mapping appears better than Mapping, which is moving us in the wrong direction (although among the mapping methods ETL is slightly better)..

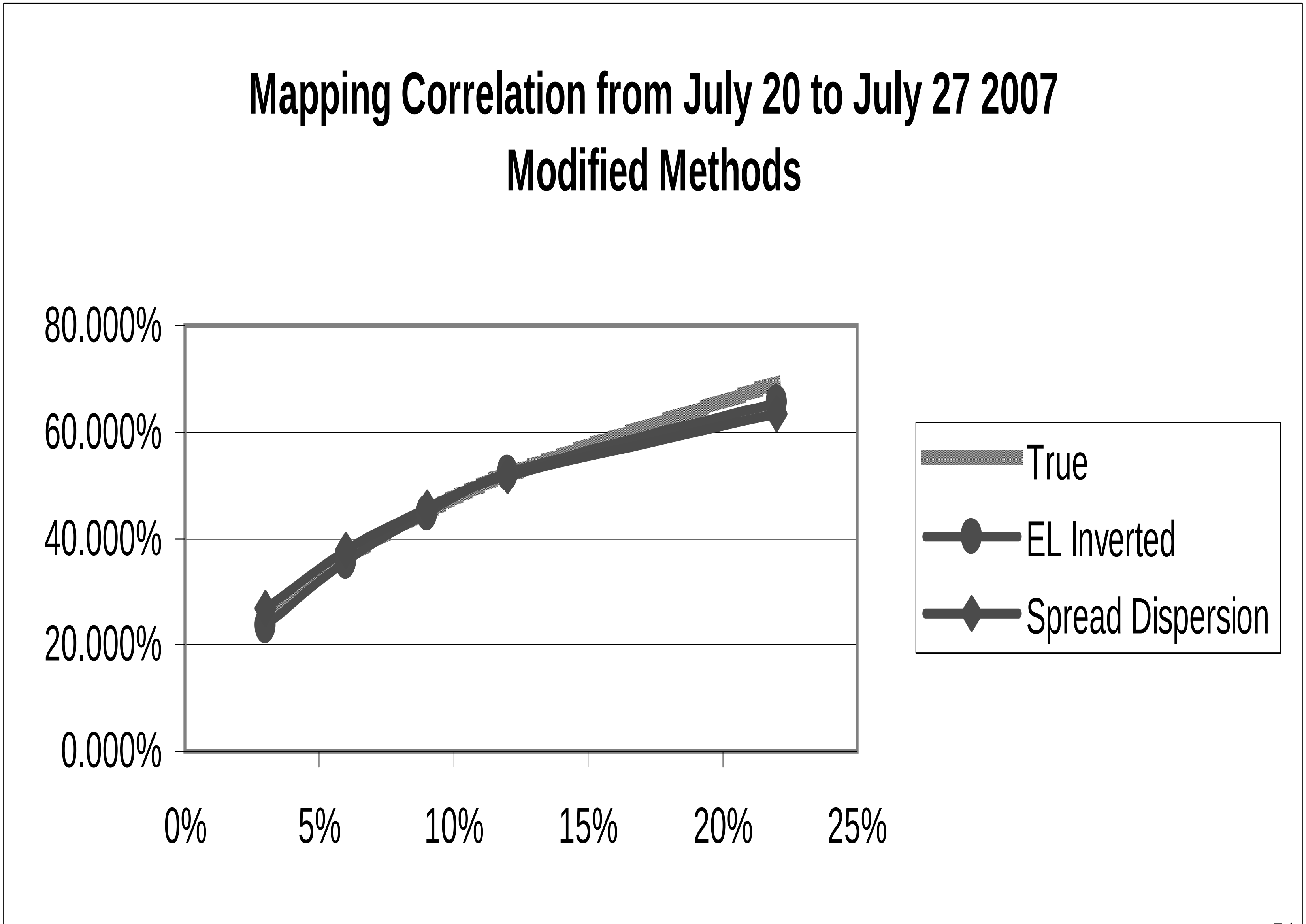
The 2007 systemic shock: Mapping when there is more systemic risk

What if we introduce a mechanism of correlation update with a behavior opposite to EL? We can assume that correlation moves proportionally with increase in Expected Loss (seen as a mapping method, the invariant is $K \times \mathbb{E}[L]$, we call this Expected Loss Inverted). We can also observe that, looking at the spreads of the two portfolios, the dispersion index has moved from 11.28% to 8.40%. We can map taking as invariant $K^{Dispersion}$ (Spread Dispersion)

Correlation	**3%**	**6%**	**9%**	**12%**	**22%**
No Mapping	19.61%	28.93%	36.76%	43.38%	60.70%
Mapping (EL)	16.81%	23.33%	29.71%	35.18%	49.26%
Mapping (ETL)	17.61%	25.26%	32.57%	39.11%	56.61%

True ρ	24.92%	36.37%	44.93%	52.09%	69.34%

Correlation	**3%**	**6%**	**9%**	**12%**	**22%**
ELI	23.61%	35.65%	44.87%	52.30%	65.46%
SD	26.76%	37.88%	45.61%	51.78%	63.36%


Mapping Correlation from July 20 to July 27 2007
Modified Methods
80.000%
60.000%
40.000%
20.000%
0.000%
0%
5%
10%
15%
20%
25%
True
EL Inverted
Spread Dispersion

The 2007 systemic shock: Mapping when there is more systemic risk

Err.on ρ	3%	6%	9%	12%	22%
No Mapping	14.21%	19.49%	19.98%	20.29%	19.17%
Mapping (EL)	17.18%	25.42%	27.46%	29.05%	31.30%

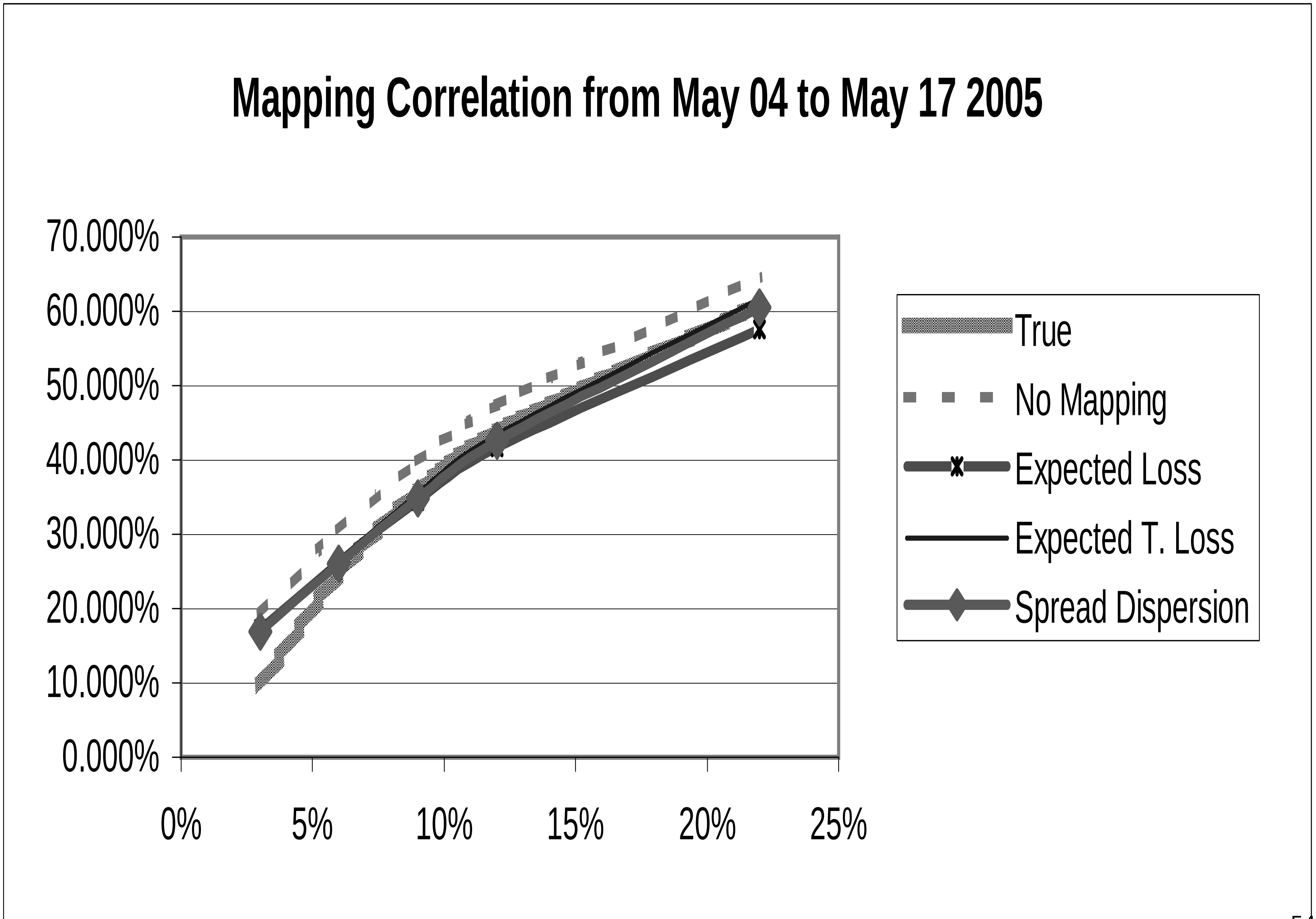
Err.on ρ	3%	6%	9%	12%	22%
ELI	1.31%	0.72%	0.06%	-0.21%	3.88%
SD	-1.84%	-1.51%	-0.68%	0.31%	5.98%

Intertemporal Mapping

The two simple methods appear better. Also for the idiosyncratic 2005 crisis the simple Dispersion method captures market regularity (dispersion moved from 5.70% to 6.15%)

Err.on ρ	**3%**	**6%**	**9%**	**12%**	**22%**
No Mapping	-9.7%	-5.7%	-4.0%	-3.7%	-3.9%
Mapping (EL)	-7.6%	-1.5%	1.2%	2.1%	3.2%

Err.on ρ	**3%**	**6%**	**9%**	**12%**	**22%**
SD	-7.19%	-1.31%	0.86%	1.12%	0.43%


Mapping Correlation from May 04 to May 17 2005
70.000%
60.000%
50.000%
40.000%
30.000%
20.000%
10.000%
0.000%
0%
5%
10%
15%
20%
25%
True
No Mapping
Expected Loss
Expected T. Loss
Spread Dispersion

Intertemporal Mapping

In the intertemporal tests, the standard mapping method works well when an increase in portfolio riskiness is associated to an increase of idiosyncratic risk, while it appears not to work well when the increase in riskiness is systemic. In the latter case, in fact, we should capture a movement which is in the opposite direction compared to standard mapping. This may affect also the mapping to bespoke portfolio: in fact, bespokes can have more sector or regional concentration than an Index, generating more systemic risk at a portfolio level.

The puzzle in the interest rate curve: counterparty risk?

A problem emerged in the fixed income market with the burst of the subprime crisis. Take the simplest possible swap, a Forward Rate Agreement (FRA). The payoff of a FRA at T_i is

$$\alpha_i(L(T_{i-1}, T_i) - K)$$

where $L(T_{i-1}, T_i)$ is the value of a Libor rate. The FRA is quoted through its equilibrium rate, which is easy to find since a FRA can be priced, without any model, through a simple replication procedure.

$$\begin{aligned} & \alpha_i(L(T_{i-1}, T_i) - K) = \\ = \quad & \frac{1}{P\left(T_{i-1}, T_i\right)} - 1 - K\alpha_i. \end{aligned}$$

$(-1 - K\alpha_i)$ is a known amount of money at T_i. $\dfrac{1}{P\left(T_{i-1}, T_i\right)}$ is equivalent to a known unitary amount of money at T_{i-1}.

Forward Rate Agreements

Fixing $K = K^{FRA}$ setting this price to zero we introduce the **Forward Libor Rate** $F(t; T_{i-1}, T)$ with expiry (fixing time) T_{i-1} and maturity (payment time) T_i:

$$K^{FRA} = F(t; T_{i-1}, T_i) = \frac{1}{\alpha_i}\left[\frac{P(t, T_{i-1})}{P(t, T_i)} - 1\right].$$

By change of numeraire, we would have reached the same result:

$$\begin{aligned} FRA_t(K) &= \mathbb{E}\left[D(t, T_i)\,\alpha_i\,(L(T_{i-1}, T_i) - K)\right] \\ &= P(t, T_i)\,\alpha_i \mathbb{E}^i\left[(L(T_{i-1}, T_i) - K)\right] \end{aligned}$$

Forward Rate Agreements

Setting to 0,

$$K^{FRA} = \mathbb{E}^i\left[L(T_{i-1}, T_i)\right] = \frac{1}{\alpha_i}\mathbb{E}^i\left[\frac{1}{P(T_{i-1}, T_i)} - 1\right] = \frac{1}{\alpha_i}\mathbb{E}^i\left[\frac{1}{P(T_{i-1}, T_i)}\right] - \frac{1}{\alpha_i}$$

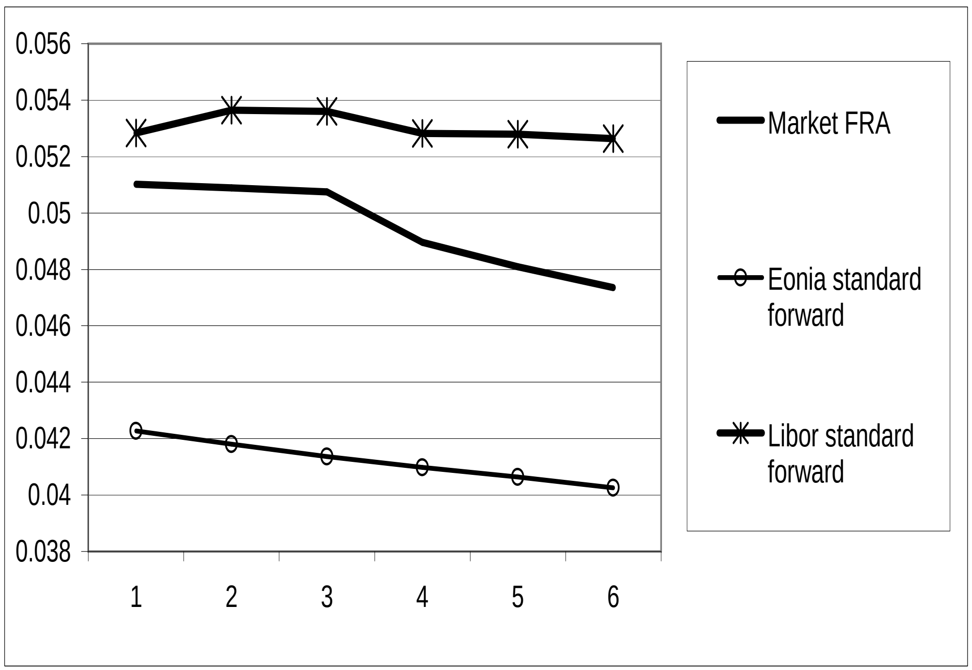
Taking $\frac{1}{P(T_{i-1}, T_i)}$ as a payoff at the observation date T_{i-1}, it is $\frac{P(T_{i-1}, T_{i-1})}{P(T_{i-1}, T_i)}$. But this is clearly a martingale under Q^i, so

$$K^{FRA} = \frac{1}{\alpha_i}\mathbb{E}^i\left[\frac{P(T_{i-1}, T_{i-1})}{P(T_{i-1}, T_i)}\right] - \frac{1}{\alpha_i} = \frac{1}{\alpha_i}\left[\frac{P(t, T_{i-1})}{P(t, T_i)} - 1\right]$$

which is the same $F(t; T_{i-1}, T_i)$ described above. This shows that the standard “trivial” forward is expectation of the future Libor rate under the forward measure associated to the bond price used to define the Libor rate itself.

The puzzle in the interest rate curve: counterparty risk?

Taking market spot Libor quotes as $L(t, T_i)$, and computing the Libor standard forward $F(T_{i-1}; T_{i-1}, T_i)$ one would expect this to correspond to the Market FRA equilibrium rate K^{FRA}. In fact, historically the difference was around few basis points. In August 2007, the difference has increased dramatically, reaching a peak in September 2008, and the error in setting $K^{FRA} = F(t; T_{i-1}, T)$ has become no more negligible:



The puzzle in the interest rate curve: counterparty risk?

Now the FRA cannot be priced as a trivial forward on a Libor rate, as it used to be, at least approximately. One possible explanation is the increased perception of bank-vs-bank counterparty/liquidity risk after the burst of the subprime crisis. The presence of counterparty/liquidity risk in Libor quotes is often estimated based on the difference between Libor and rates theoretically free of such risk, such as Eonia (whose trivial forward is reported in the above picture). Thus now the risk free rate $L(T_{i-1}, T_i) = \frac{1}{\alpha_i}\left[\frac{1}{P(t,T_i)} - 1\right]$ is taken to be Eonia, while Libor would be a different default risky rate $\bar{L}(T_{i-1}, T_i)$.

The puzzle in the interest rate curve: counterparty risk?

With deterministic default probabilities, the fundamental no-arbitrage relationship defining the spot Libor rate is now

$$
\begin{aligned}
1_{\{\tau>t\}} &= 1_{\{\tau>t\}} P\left(t,T_i\right)\left[Rec+\Pr\left(t;\tau>T_i\right)Lgd\right] * \left[1+\bar{L}(t,T_i)\alpha\left(t,T_i\right)\right] \\
1_{\{\tau>t\}} &= 1_{\{\tau>t\}} P\left(t,T_i\right)\left[Rec+1_{\{\tau>t\}}\Pr\left(\tau>T_i|\tau>t\right)Lgd\right] * \left[1+\bar{L}(t,T_i)\alpha\left(t,T_i\right)\right.
\end{aligned}
$$

thus

$$
\bar{L}(t,T_i)=\frac{1}{\alpha\left(t,T_i\right)}\left(\frac{1}{P\left(t,T_i\right)\left[Rec+\Pr_t\left(\tau>T_i\right)Lgd\right]}-1\right)
$$

Since $P\left(t,T_i\right)\left[Rec-\Pr_t\left(\tau>T_i\right)Lgd\right]$ is a natural definition for a Libor-based risky bond, we call it $\bar{P}\left(t,T_i\right)$.

The puzzle in the interest rate curve: counterparty risk?

If this is a viable interpretation, we can try to analyze this situation within a simple counterparty risk framework (assuming for now that we can implicitly include liquidity risk in our explicit assessment of counterparty risk. In dealing with global market quotes that are a result of thousands of deals, we cannot distinguishing between payers and receiver, so we assume they have symmetric default risk, like in Duffie and Singleton (1997). It corresponds to assuming a common curve of default intensity. We also set ourselves in a simplified context assuming default payments at maturity and $\alpha_i = 1$, assumptions that we relax later but that are useful for the general analysis.

The puzzle in the interest rate curve: counterparty risk?

$$FRA_0^D(K) =$$
$$\mathbb{E}\left[D(0,T_i)\left(\bar{L}(T_{i-1},T_i)-K\right)\right] - Lgd\mathbb{E}\left[D(0,T_i)\,1_{\{\tau^{You}<T_i\}}\left(\bar{L}(T_{i-1},T_i)-K\right)^+\right]+$$
$$+Lgd\mathbb{E}\left[D(0,T_i)\,1_{\{\tau^{Me}<T_i\}}\left(K-\bar{L}(T_{i-1},T_i)\right)^+\right]$$

where default risk of both Me (payer) and the Counterparty can default. Using law of iterated expectations and due to the assumption of symmetric default risk,

$$\Pr^{H_i}(\tau^{me}<T_i) = \Pr^{H_i}\left(\tau^{You}<T_i\right) =: \Pr^{H_i}(\tau<T_i)$$

we obtain $FRA_0^D(K) = \mathbb{E}\left[D(0,T_i)\left(Rec+1_{\{\tau>T_i\}}Lgd\right)\left(\bar{L}(T_{i-1},T_i)-K\right)\right]$. Thus, a swap with symmetric counterparty risk can be treated as a simple defaultable payoff.

The puzzle in the interest rate curve: counterparty risk?

This equivalent payoff $\left(Rec + 1_{\{\tau > T_i\}} Lgd\right)\left(\bar{L}(T_{i-1}, T_i) - K\right)$ can be priced through replication,

$$\begin{aligned} FRA_0^D(K) &= P(0, T_{i-1})\left[Rec + \Pr(\tau > T_{i-1}) Lgd\right] + \\ &\quad - P(0, T_i)\left[Rec + \Pr(\tau > T_i) Lgd\right](1 + K\alpha_i) \\ K^{eq} &= \left(\frac{P(0, T_{i-1})\left[Rec + \Pr(\tau > T_{i-1}) Lgd\right]}{P(0, T_i)\left[Rec + \Pr(\tau > T_i) Lgd\right]} - 1\right)\frac{1}{\alpha_i} \end{aligned}$$

How does this relates with the trivial forward of the above pictures?

The puzzle in the interest rate curve: counterparty risk?

We have

$$
\begin{aligned}
\bar{L}(t,T_i) &= \frac{1}{\alpha\left(t,T_i\right)}\left[\frac{1}{\bar{P}\left(t,T_i\right)}-1\right] \\
K^{eq} &= \bar{F}\left(t;T_{i-1},T_i\right)=\frac{1}{\alpha_i}\left[\frac{\bar{P}\left(t,T_{i-1}\right)}{\bar{P}\left(t,T_i\right)}-1\right] \\
&= \frac{1}{\alpha_i}\left[\frac{P\left(t,T_{i-1}\right)\left[Rec+\Pr_t\left(\tau>T_{i-1}\right)Lgd\right]}{P\left(t,T_i\right)\left[Rec+\Pr_t\left(\tau>T_i\right)Lgd\right]}-1\right]
\end{aligned}
$$

Thus, nothing has changed introducing risk of default!

The puzzle in the interest rate curve: counterparty risk?

Going back to the change of numeraire pricing approach, we see that

$$FRA_0^D(K) = P(0, T_i)[Rec + \Pr(\tau > T_i) Lgd] \mathbb{E}^i \left[\left(\bar{L}(T_{i-1}, T_i) - K \right) \right]$$

In setting to zero, we see that

$$\begin{aligned} P(0, T_i)[Rec + \Pr(\tau > T_i) Lgd] \mathbb{E}^i \left[\left(\bar{L}(T_{i-1}, T_i) - K^{eq} \right) \right] &= 0, \\ K^{eq} &= \mathbb{E}^i \left[\bar{L}(T_{i-1}, T_i) \right] \end{aligned}$$

Since we know $K^{eq} = \bar{F}(0; T_{i-1}, T_i)$, this implies that $\bar{F}(0; T_{i-1}, T_i)$ is the expectation of $\bar{L}(T_{i-1}, T_i)$ under the T_i forward measure.

The puzzle in the interest rate curve: counterparty risk?

Would things change by using stochastic default probabilities? We can use change of numeraire without this assumption. We take $\bar{P}(t,T_i) = \mathbb{E}\left[D(t,T_i)\left[Rec + 1_{\{\tau>T_i\}} Lgd\right]\right]$ as numeraire. Notice that again $\bar{L}(t,T_i) = \frac{1}{\alpha(t,T_i)}\left[1/\bar{P}(t,T_i) - 1\right]$. We have

$$
\begin{aligned}
FRA_t^D(K) &= \mathbb{E}\left[D(t,T_i)\left[Rec + 1_{\{\tau>T_i\}} Lgd\right]\alpha_i\left(\bar{L}(T_{i-1},T_i) - K\right)\right] \\
&= \bar{P}(t,T_i)\,\alpha_i \mathbb{E}^{\bar{P}}\left[\frac{\left[Rec + 1_{\{\tau>T_i\}} Lgd\right]\left(\bar{L}(T_{i-1},T_i) - K\right)}{\bar{P}(T_i,T_i)}\right] \\
&= \bar{P}(t,T_i)\,\alpha_i \mathbb{E}^{\bar{P}}\left[\frac{\left[Rec + 1_{\{\tau>T_i\}} Lgd\right]\left(\bar{L}(T_{i-1},T_i) - K\right)}{\left[Rec + 1_{\{\tau>T_i\}} Lgd\right]}\right] \\
&= \bar{P}(t,T_i)\,\alpha_i \mathbb{E}^{\bar{P}}\left[\left(\bar{L}(T_{i-1},T_i) - K\right)\right]
\end{aligned}
$$

The puzzle in the interest rate curve: counterparty risk?

Now we can write

$$FRA_t^D(K) = \bar{P}(t,T_i)\,\alpha_i \mathbb{E}^{\bar{P}}\left[\left(\bar{F}(T_{i-1};T_{i-1},T_i) - K\right)\right]$$

with $\bar{F}(t;T_{i-1},T_i) = \frac{1}{\alpha_i}\left[\frac{\bar{P}\left(t,T_{i-1}\right)}{\bar{P}\left(t,T_i\right)} - 1\right]$. This is a tradable asset divided by the numeraire, so it is a martingale under this pricing measure, leading to

$$\begin{aligned} FRA_t^D(K) &= \bar{P}(t,T_i)\,\alpha_i\left(\bar{F}(t;T_{i-1},T_i) - K\right) \\ K^{eq} &= \bar{F}(t;T_{i-1},T_i) \end{aligned}$$

Again, we stick to the trivial forward, and we have not been able to explain market anomalies.

A Measure mismatch: different markets, different risk

What can be missing?

One hidden assumption is that we are assuming that the **default/liquidity risk in Libor quotes is the appropriate one to use for the FRA market**. **This may be wrong** for a number of reasons: Libor could be an underestimation of the actual risk (*Risk Magazine*, June 08), the credit risk of the counterparties or the liquidity conditions in the FRA market could be different compared to the Libor/Deposit world, the presence of collaterals can make a difference...

What happens if this conditions do not hold? We would have two default curves, one associated again to Libor default event τ, giving again $\bar{L}(t, T_i) = \frac{1}{\alpha}\left[\frac{1}{\bar{P}(t,T_i)} - 1\right]$, but then for the risk of default of the FRA payoff we would have a different curve associated to a different event τ^{FRA}.

A Measure mismatch: different markets, different risk

Under deterministic default probabilities, this means moving from

$$P\left(0,T_i\right)\alpha_i\left[Rec+\Pr\left(\tau>T_i\right)Lgd\right]\mathbb{E}^i\left[\left(\bar{L}(T_{i-1},T_i)-K^{eq}\right)\right] \;=\; 0$$

to

$$P\left(0,T_i\right)\alpha_i\left[Rec^{FRA}+\Pr\left(\tau^{FRA}>T_i\right)Lgd\right]\mathbb{E}^i\left[\left(\bar{L}(T_{i-1},T_i)-K^{eq}\right)\right] \;=\; 0$$

Does it make any difference to the equilibrium rate? No.

Measure mismatch and risky credit risk

Again, we have to check if the assumption of deterministic default probabilities is relevant. In case of stochastic default probability,

$$FRA_t^D(K) = \mathbb{E}\left[D(t,T_i)\,\alpha_i\left[Rec^{FRA} + 1_{\left\{\tau^{FRA}>T_i\right\}}Lgd\right]\left(\bar{L}(T_{i-1},T_i) - K\right)\right]$$

We take as numeraire $\bar{P}^{FRA}(t,T_i) = \mathbb{E}\left[D(t,T_i)\left[Rec^{FRA} + 1_{\left\{\tau^{FRA}>T_i\right\}}Lgd\right]\right]$ and we arrive to

$$\begin{aligned} FRA_t^D(K) &= \bar{P}^{FRA}(t,T_i)\,\alpha_i\mathbb{E}^{\bar{P}\_FRA}\left[\left(\bar{L}(T_{i-1},T_i) - K\right)\right] \\ &= \bar{P}^{FRA}(t,T_i)\,\alpha_i\mathbb{E}^{\bar{P}\_FRA}\left[\left(\bar{F}(T_{i-1};T_{i-1},T_i) - K\right)\right] \end{aligned}$$

Are there any reasons, now, to assume that $\bar{F}(t;T_{i-1},T_i)$ is a martingale under this measure? Not, it won't. Thus it is now possible

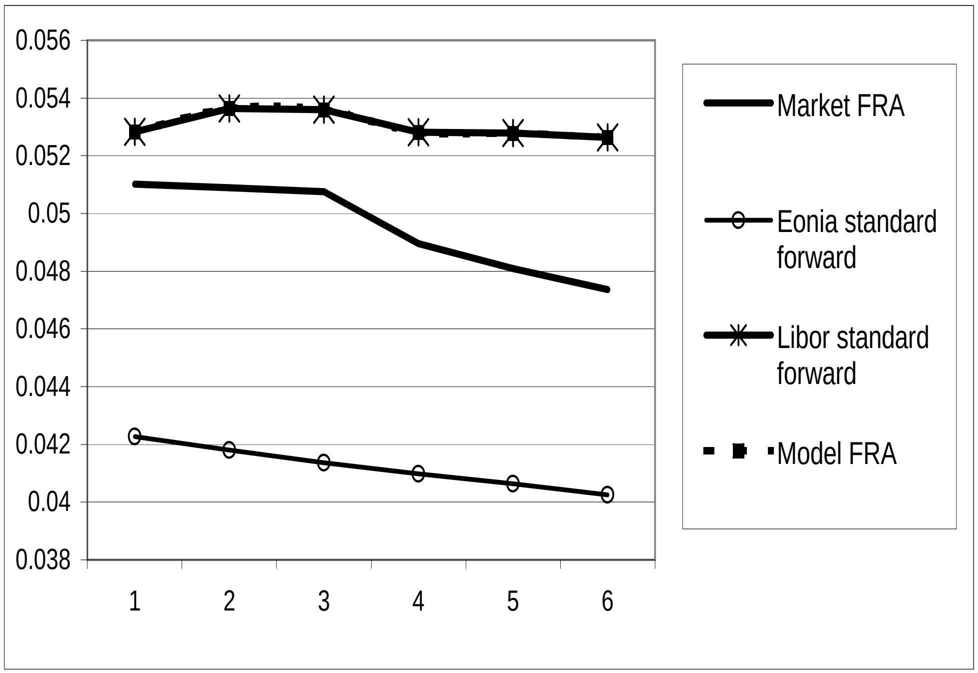
$$K^{eq} \neq \bar{F}(t;T_{i-1},T_i).$$

Measure mismatch and risky credit risk

Thus we see that, without an explicit assumption of regular asymmetry between payer and receiver, we can have a FRA equilibrium rate different from the trivial forward only if we assume both that the FRA market has credit conditions not equal to those expressed by Libor fixings, and that these credit conditions can vary stochastically. Notice this means that the curves can even be the same now, but they are not perfectly correlated. The simplest possible choice to have a flavour of this effect on the FRA equilibrium rate is to assume different scenarios for default probabilities in FRA and Libor markets, from being negligible (in this scenario the crisis is ended) to being much higher than what they are now (in this scenario the crisis is going to get worse). We will actually have 4 scenarios, since we decouple FRA and Libor markets to assess the effect of different correlation assumptions (correlation 1 means that the optimistic scenario for Libor coincides with the optimistic scenario for FRA, correlation -1 means that the optimistic scenario for Libor always corresponds to the pessimistic one for FRA. In each scenario, we exploit above results for the deterministic probability case.

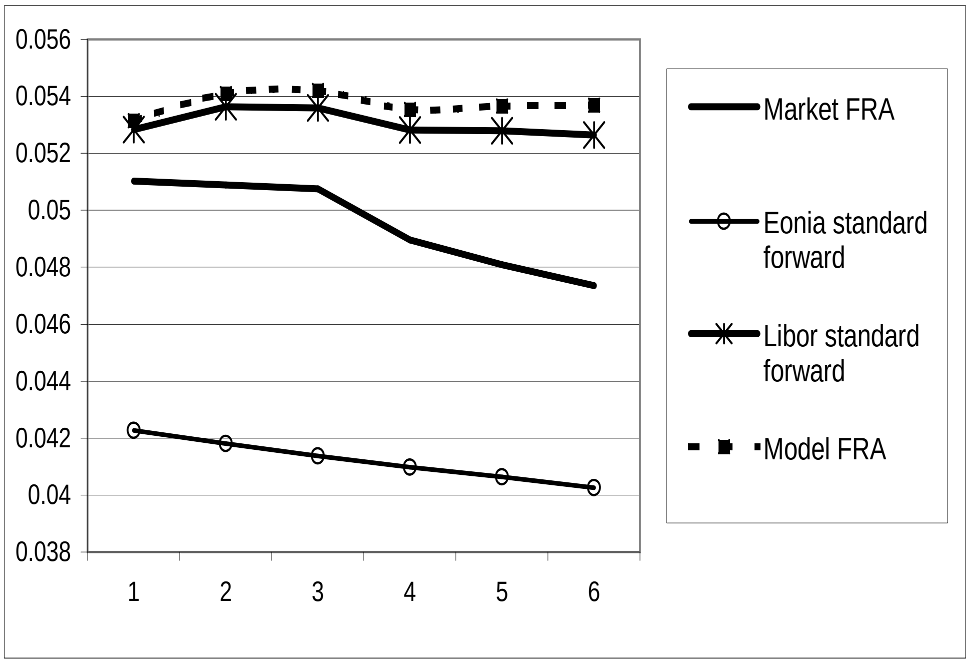
Measure mismatch and risky credit risk

Confirming what we have seen analytically, the chart below shows that if the curves are the same now and they are perfectly correlated, the FRA equilibrium rate coincides with the standard forward, no matter the volatility.



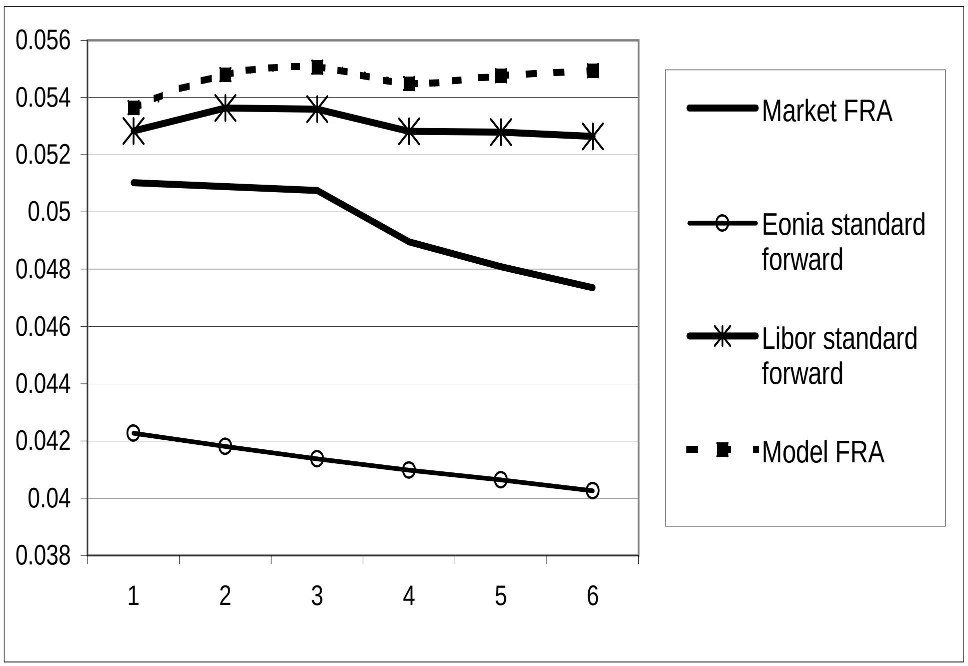
Measure mismatch and risky credit risk

If we assume the default risk today in the two markets is the same, but we reduce the correlation between Libor and FRA default risks, the FRA rate increases and gets even higher than the trivial forward, opposite to market evidence.



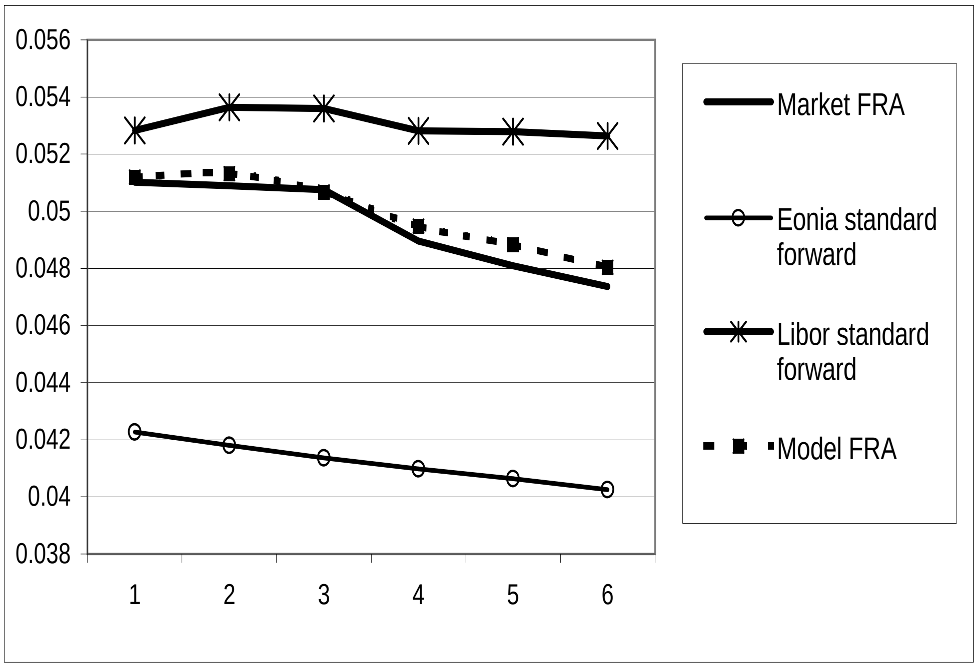
Measure mismatch and risky credit risk

We keep a high correlation, but now eventually we assume that the risk in the FRA market is lower than the risk in the Libor market. Again this increases the FRA equilibrium rate, opposite to market evidence.



Measure mismatch and risky credit risk

Now we assume that the risk in the FRA market is higher than the risk in the Libor market. This reduces the FRA equilibrium rate, leading us to possible configurations (in particular high volatility and correlation) which resemble the market pattern[1].



[1] If we took the volatility to be zero (no stochasticity in the default risk), we would go back to the trivial forward, in spite of setting higher FRA market default risk.

Measure mismatch and risky credit risk

These results are confirmed by change of numeraire and Girsanov theorem. The anomalies in the market pattern are influenced by the dynamics of $\bar{F}_i(t)$ under the measure associated to $\bar{P}^{FRA}(t, T_i)$. We know $\bar{F}(T_{i-1}; T_{i-1}, T_i) = \frac{1}{\alpha_i}\left[\frac{\bar{P}\left(t,T_{i-1}\right)}{\bar{P}\left(t,T_i\right)} - 1\right] =: \bar{F}_i(t)$ and we can define an analogous rate associated to credit conditions in the FRA market $\bar{F}_i^{FRA}(t) = \frac{1}{\alpha_i}\left[\frac{\bar{P}^{FRA}\left(t,T_{i-1}\right)}{\bar{P}^{FRA}\left(t,T_i\right)} - 1\right]$. Starting from the martingale dynamics of $\bar{F}(t, T_i)$ under the $\bar{P}(t, T_i)$ associated measure,

$$d\bar{F}_i(t) = \sigma_i^L(t)\bar{F}_i(t)\, dW_L^L(t)$$

and $\bar{F}_i^{FRA}(t)$ under the $\bar{P}^{FRA}(t, T_i)$ associated measure,

$$d\bar{F}_i^{FRA}(t) = \sigma_i^{FRA}(t)\bar{F}_i^{FRA}(t)\, dW_{FRA}^{FRA}(t),$$

Measure mismatch and risky credit risk

thus assuming a total vector diffusion under $\bar{P}(t, T_i)$

$$dW^L(t) = \begin{bmatrix} dW_L^L(t) \\ dW_{FRA}^L(t) \end{bmatrix}$$

the dynamics of $\bar{F}_i(t)$ under the $\bar{P}^{FRA}(t, T_i)$ associated measure is regulated by

$$dW^L(t) = dW^{FRA} - \rho \mathrm{DC}\left(\ln\left(\frac{\bar{P}(t, T_i)}{\bar{P}^{FRA}(t, T_i)}\right)\right)' dt \qquad (2)$$

Measure mismatch and risky credit risk

$$
\begin{aligned}
& \mathrm{DC}\left(\ln\left(\frac{\bar{P}\left(t,T_i\right)}{\bar{P}^{FRA}\left(t,T_i\right)}\right)\right) = \mathrm{DC}\left(\ln\left(\frac{1+\alpha\bar{F}_i^{FRA}(t)}{1+\alpha\bar{F}_i(t)}\right)\right) = \\
& \mathrm{DC}\left(\ln\left(1+\alpha\bar{F}_i^{FRA}(t)\right)\right) - \mathrm{DC}\left(\ln\left(1+\alpha\bar{F}_i(t)\right)\right) \\
= \quad & \frac{\alpha\mathrm{DC}\left(\bar{F}_i^{FRA}(t)\right)}{1+\alpha\bar{F}_i^{FRA}(t)} - \frac{\alpha\mathrm{DC}\left(\bar{F}_i(t)\right)}{1+\alpha\bar{F}_i(t)} \\
= \quad & \left[0,\frac{\alpha\sigma_i^{FRA}(t)\bar{F}_i^{FRA}(t)}{1+\alpha\bar{F}_i^{FRA}(t)}\right] - \left[\frac{\alpha\sigma_i^{L}(t)\bar{F}_i(t)}{1+\alpha\bar{F}_i(t)},0\right] \\
= \quad & \left[-\frac{\alpha\sigma_i^{L}(t)\bar{F}_i(t)}{1+\alpha\bar{F}_i(t)},\frac{\alpha\sigma_i^{FRA}(t)\bar{F}_i^{FRA}(t)}{1+\alpha\bar{F}_i^{FRA}(t)}\right]
\end{aligned}
$$

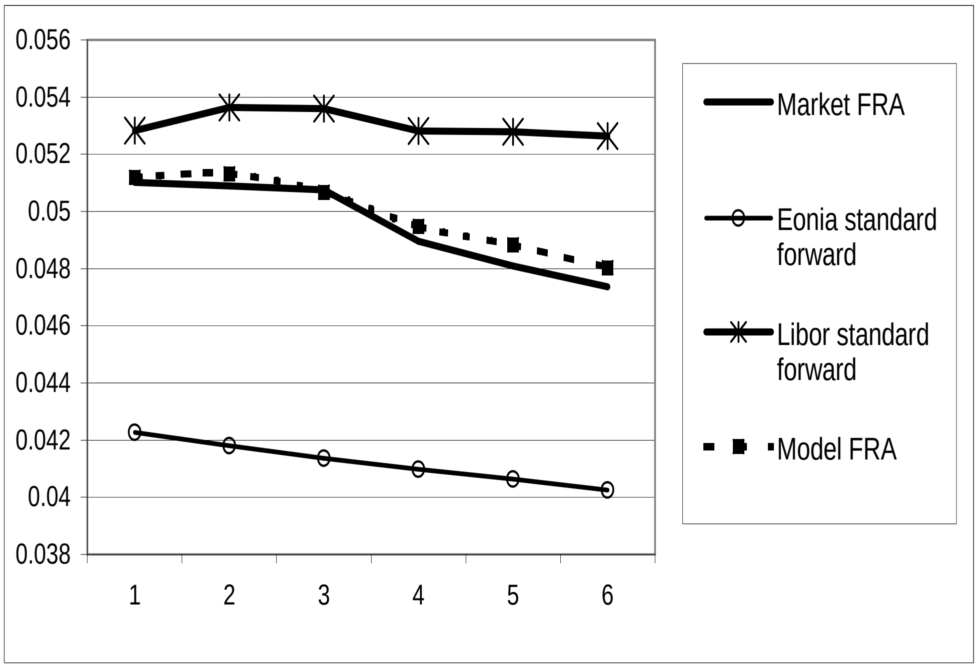
Measure mismatch and risky credit risk

$$
\begin{aligned}
dW^L(t) &= dW^{FRA} - \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} -\frac{\alpha\sigma_i^L(t)\bar{F}_i(t)}{1+\alpha\bar{F}_i(t)} \\ \frac{\alpha\sigma_i^{FRA}(t)\bar{F}_i^{FRA}(t)}{1+\alpha\bar{F}_i^{FRA}(t)} \end{bmatrix} dt \\
dW_L^L(t) &= dW_L^{FRA} + \frac{\alpha\sigma_i^L(t)\bar{F}_i(t)}{1+\alpha\bar{F}_i(t)} - \rho\frac{\alpha\sigma_i^{FRA}(t)\bar{F}_i^{FRA}(t)}{1+\alpha\bar{F}_i^{FRA}(t)}
\end{aligned}
$$

$$
d\bar{F}_i(t) = \sigma_i^L(t)\bar{F}_i(t)\left[\frac{\alpha\sigma_i^L(t)\bar{F}_i(t)}{1+\alpha\bar{F}_i(t)} - \rho\frac{\alpha\sigma_i^{FRA}(t)\bar{F}_i^{FRA}(t)}{1+\alpha\bar{F}_i^{FRA}(t)}\right] + \sigma_i^L(t)\bar{F}_i(t)\ dW_L^{FRA}
$$

Measure mismatch and risky credit risk

When $\rho \approx 1, \sigma_i^L(t) \approx \sigma_i^{FRA}(t)$, then $\bar{F}_i^{FRA}(t) > \bar{F}_i(t)$ implies a negative drift for Libor when in the FRA measure, leading to a FRA equilibrium rate $(= \mathbb{E}^{\bar{P}\_FRA}\left[\bar{L}(T_{i-1}, T_i)\right])$ lower than the trivial forward $(= \mathbb{E}^{\bar{P}}\left[\bar{L}(T_{i-1}, T_i)\right])$.



The Credit Index

We start from a portfolio of n names, with an initial portfolio notional $N(0) = 1$. Each name has a notional $\frac{1}{n}$ and a flat recovery rate R. Calling τ_i the default times of the single names, the total loss is

$$L(t) = \frac{1}{n}(1-R)\sum_{i=1}^{n} 1_{\{\tau_i<t\}}$$

In a credit index the discounted payoffs of the Premium and Protection Legs are

$$Prot_t^{T_A,T_M} = \int_{T_A}^{T_M} D(t,u)\, dL(u) \quad Prem_t^{T_A,T_M} = \left\{ \sum_{j=A+1}^{M} D(t,T_j) \int_{T_{j-1}}^{T_j} N(t)\, dt \right\} K$$

By $D(t,T)$ we indicate the discount factor from T to t. Its expectation is the corresponding bond price, $P(t,T) = \mathbb{E}^Q\left[D(t,T)\mathbf{1} \middle| \mathcal{F}_t\right]$.

Evaluating Premium and Protection Legs

The value of the two legs is computed by expectation under the risk neutral probability measure $\mathbb{Q}$. We use the notation $\Pi\left(X_t\right) = \mathbb{E}\left[\left. X_t \right| \mathcal{F}_t\right]$. For the value of the two legs we have

$$\begin{aligned}
\Pi\left(Prot_t^{T_A,T_M}\right) &: = \mathbb{E}\left[\left. Prot_t^{T_A,T_M} \right| \mathcal{F}_t\right], \\
\Pi\left(Prem_t^{T_A,T_M}\left(K\right)\right) &: = \mathbb{E}\left[\left. Prem_t^{T_A,T_M}\left(K\right) \right| \mathcal{F}_t\right] = \Pi\left(DV_t^{T_A,T_M}\right) K
\end{aligned}$$

where $\Pi\left(DV_t^{T_A,T_M}\right)$ is the sensitivity of the Index to changes in the spread. The Payer Forward Index starting at T_A and lasting until T_M has a price given by

$$\Pi\left(I_t^{T_A,T_M}\left(K\right)\right) := \mathbb{E}\left[\left. I_t^{T_A,T_M}\left(K\right) \right| \mathcal{F}_t\right] := \Pi\left(Prot_t^{T_A,T_M}\right) - \Pi\left(DV_t^{T_A,T_M}\right) K$$

In the simplest definition, the equilibrium spread at time t is the value of the spread K

that sets the value of the forward index to zero at time t:

$$S_t^{T_A,T_M} = \frac{\Pi\left(Prot_t^{T_A,T_M}\right)}{\Pi\left(DV_t^{T_A,T_M}\right)}.$$

allowing to write the index value as

$$\Pi\left(I_t^{T_A,T_M}(K)\right) = \Pi\left(DV_t^{T_A,T_M}\right)\left(S_t^{T_A,T_M} - K\right)$$

Index Options

A **payer** Index Option with inception 0, strike K and exercise date T_A, written on an index with maturity T_M, is a contract giving the right to *enter at T_A into the running Index with final payment at T_M* **as protection buyer** *paying a fixed rate K*. The purpose of a Credit Index Option is to allow the protection buyer to lock in a particular spread K, that the protection buyer has the right (and not the obligation) to make effective at a future time. However, if the above simple payoff was considered, there would be an additional side-effect: the option buyer would give away protection from inception 0 to maturity T_A. In order to attract investors, standard Credit Index Option payoff includes the payment of the losses from the option inception to T_A, the so-called **front end protection**

$$\begin{aligned} F_t^{T_A} &= D(t, T_A) L(T_A) \\ \Pi\left(F_t^{T_A}\right) &= \mathbb{E}\left[D(t, T_A) L(T_A) \right| \mathcal{F}_t\right]. \end{aligned}$$

The rough approach

In the roughest approach, it is evaluated by the decomposition

$$\mathbb{E}\left[D\left(t, T_A\right) \Pi\left(DV_{T_A}^{T_A,T_M}\right)\left(S_{T_A}^{T_A,T_M} - K\right)^{+} \middle| \mathcal{F}_t\right] + \mathbb{E}\left[F_t^{T_A} \middle| \mathcal{F}_t\right]$$

and then by expressing the first component through a standard Black formula

$$\Pi\left(DV_t^{T_A,T_M}\right) Black\left(S_t^{T_A,T_M}, K, \sigma^{T_A,T_M}\sqrt{T_A - t}\right) + \Pi\left(F_t^{T_A}\right),$$

The above formula neglects the fact that the front end protection is received only upon exercise, so the correct payoff is NOT

$$\left(\Pi\left(DV_{T_A}^{T_A,T_M}\right)\left(S_{T_A}^{T_A,T_M} - K\right)\right)^{+} + F_{T_A}^{T_A}$$

BUT

$$\left(\Pi\left(DV_{T_A}^{T_A,T_M}\right)\left(S_{T_A}^{T_A,T_M} - K\right) + F_{T_A}^{T_A}\right)^{+}$$

The market approach

One defines a *Loss-Adjusted* index spread by taking Front End Protection into account directly,

$$\tilde{I}_t^{T_A,T_M}(K) = Prot_t^{T_A,T_M} - Prem_t^{T_A,T_M}(K) + F_t^{T_A}.$$

It is natural to give a new spread definition, setting to zero $\Pi\left(\tilde{I}_t^{T_A,T_M}(K)\right)$ rather than $\Pi\left(I_t^{T_A,T_M}(K)\right)$. This leads to the following *Loss-Adjusted Market Index Spread*

$$\tilde{S}_t^{T_A,T_M} = \frac{\Pi\left(Prot_t^{T_A,T_M}\right) + \Pi\left(F_t^{T_A}\right)}{\Pi\left(DV_t^{T_A,T_M}\right)}$$

that allows to write the payoff as

$$D(t,T_A)\left(\Pi\left(DV_{T_A}^{T_A,T_M}\right)\left(\tilde{S}_{T_A}^{T_A,T_M} - K\right)\right)^+.$$

The market approach

$$\mathbb{E}\left[D(t,T_A)\left(\Pi\left(DV_{T_A}^{T_A,T_M}\right)\left(\tilde{S}_{T_A}^{T_A,T_M}-K\right)\right)^+ \middle| \mathcal{F}_t \right],$$

one can think of taking $\Pi\left(DV_{T_A}^{T_A,T_M}\right)$ as numeraire and $\tilde{S}_t^{T_A,T_M}$ as lognormal underlying variable so as to price the option with the

Market Credit Index Option Formula:

$$\Pi\left(DV_t^{T_A,T_M}\right) Black\left(\tilde{S}_t^{T_A,T_M}, K, \tilde{\sigma}^{T_A,T_M}\sqrt{T_A}\right). \qquad (3)$$

The flaws of the Standard Formula

However, the market approach has **three problems**:.

1. The definition of the spread $\tilde{S}_t^{T_A,T_M}$ is regular only when denominator

$$\Pi\left(DV_t^{T_A,T_M}\right) = \sum_{j=A+1}^{M} \mathbb{E}\left[D(t,T_j)\,\alpha_j \left(1 - \frac{L(T_j)}{(1-R)}\right) \middle| \mathcal{F}_t \right]$$

   is different from zero. $\Pi\left(DV_t^{T_A,T_M}\right)$ is the price of a portfolio of defaultable assets, and goes to zero when $L(t) = (1-R)$.

2. When $\Pi\left(DV_t^{T_A,T_M}\right) = 0$ the pricing formula (3) is undefined, while it is clear than in this case one receives the FE protection.

3. Since it is not strictly positive, $\Pi\left(DV_t^{T_A,T_M}\right)$ as a numeraire would lead to the definition of a pricing measure not equivalent to the standard risk-neutral measure.

Subfiltration Pricing

Jamshidian (2004) uses a *subfiltration structure*, separating default free information from information containing the default event $\mathcal{F}_t = \mathcal{H}_t \vee \mathcal{J}_t$

- $\mathcal{F}_t =$ all available information up to t
- $\mathcal{J}_t = \sigma(\{\tau < u\}, u \leq t) =$ information up to t on the default event: if it has already happened or not, and in the former case the exact time τ of default
- $\mathcal{H}_t =$information up to t on economic quantities which affect default probability, but no specific information on happening of default

Subfiltrations allow to use a **result by Jeanblanc and Rutkowski (2000)** for defaultable payoffs $\mathbf{Y}_t^T = \mathbf{1}_{\{\tau > t\}} \mathbf{Y}_t^T$

$$\boxed{\mathbb{E}\left[\mathbf{Y}_t^T | \mathcal{F}_t\right] = \frac{\mathbf{1}_{\{\tau > t\}}}{\mathbb{Q}(\tau > t | \mathcal{H}_t)} \mathbb{E}\left[\mathbf{Y}_t^T | \mathcal{H}_t\right]}. \tag{4}$$

so pricing formulas are written in terms of conditional survival probability $\mathbb{Q}\left(\tau > t | \mathcal{H}_t\right)$ which never go to zero

The Armageddon Event

In a multiname setting we have a plurality of possible subfiltrations,

$$
\begin{aligned}
\mathcal{F}_t &= \mathcal{J}_t^i \vee \mathcal{H}_t^i, \qquad (5) \\
\mathcal{J}_t^i &= \sigma\left(\{\tau_i > u\}, u \leq t\right),
\end{aligned}
$$

Pricing under any $\mathcal{H}_t^i$ would avoid the multiname Index spread $\tilde{S}_t^{T_A,T_M}$ to have an irregular behaviour (jump to infinity) at default of the entire pool. However, the choice would be arbitrary, and, more importantly, using **Jeanblanc and Rutkowski formula** (4) requires that the payoff Y_t^T goes to zero when $\tau_i \leq t$, which does not happen with the above portfolio.

Pricing without Armageddon

The most effective possibility for using (4) in this context is the following. Define a new stopping time

$$\hat{\tau} = \max(\tau_1, \tau_2, \ldots, \tau_n)$$

and define a new filtration $\hat{\mathcal{H}}_t$ such that

$$\begin{aligned} \mathcal{F}_t &= \hat{\mathcal{J}}_t \vee \hat{\mathcal{H}}_t \\ \hat{\mathcal{J}}_t &= \sigma\left(\{\hat{\tau} > u\}, u \leq t\right), \end{aligned}$$

so that $\hat{\mathcal{H}}_t$ excludes, from the total flow of market information, the information on the happening of a so-called portfolio "armageddon event".

A DV01 without Armageddon

Define

$$\hat{\Pi}\left(DV_t^{T_A,T_M}\right) := \mathbb{E}\left[DV_t^{T_A,T_M} | \hat{\mathcal{H}}_t\right].$$

Exploiting that $DV_t^{T_A,T_M} = \mathbf{1}_{\{\hat{\tau}>T_A\}} DV_t^{T_A,T_M}$, we have the equivalence

$$\begin{aligned}
\Pi\left(DV_t^{T_A,T_M}\right) &= \boxed{\mathbb{E}\left[DV_t^{T_A,T_M} | \mathcal{F}_t\right] = \frac{\mathbf{1}_{\{\hat{\tau}>t\}}}{\mathbb{Q}\left(\hat{\tau}>t|\hat{\mathcal{H}}_t\right)} \mathbb{E}\left[DV_t^{T_A,T_M} | \hat{\mathcal{H}}_t\right]} \qquad (6) \\
&= \frac{\mathbf{1}_{\{\hat{\tau}>t\}}}{\mathbb{Q}\left(\hat{\tau} > t|\hat{\mathcal{H}}_t\right)} \hat{\Pi}\left(DV_t^{T_A,T_M}\right).
\end{aligned}$$

The quantity $\hat{\Pi}\left(DV_t^{T_A,T_M}\right)$ is never null, and we will see that it is what we need for an effective definition of the Index Spread and of an equivalent pricing measure for Index Options.

The Arbitrage-free Index Spread

We now apply the above formula to the Loss-Adjusted index payoff . We obtain

$$\begin{aligned}\Pi\left(\tilde{I}_t^{T_A,T_M}(K)\right) &= \qquad (7\\ = &\frac{\mathbf{1}_{\{\hat{\tau}>t\}}}{\mathbb{Q}\left(\hat{\tau}>t|\hat{\mathcal{H}}_t\right)}\left\{\hat{\Pi}\left(Prot_t^{T_A,T_M}\right)-\hat{\Pi}\left(Prem_t^{T_A,T_M}(K)\right)+\mathbb{E}\left[\mathbf{1}_{\{\hat{\tau}>T_A\}}F_t^{T_A}|\hat{\mathcal{H}}_t\right]\right\}\\ &+\frac{\mathbf{1}_{\{\hat{\tau}>t\}}(1-R)}{\mathbb{Q}\left(\hat{\tau}>t|\hat{\mathcal{H}}_t\right)}\times\mathbb{E}\left[\mathbf{1}_{\{t<\hat{\tau}\le T_A\}}D(t,T_A)|\hat{\mathcal{H}}_t\right]+\\ &+\mathbf{1}_{\{\hat{\tau}\le t\}}(1-R)P(t,T_A)\end{aligned}$$

The formula shows the actual **components of the value of the Loss-Adjusted index**. The last two components of the Loss-Adjusted index value do not depend on the index spread K. Only in those scenarios where some names survive until maturity the payoff actually depends on K. The financially meaningful definition of the Index Spread, that makes it a martingale under a natural pricing measure, considers the level of K setting the Index value to zero in all scenarios where some names survive until maturity.

The Arbitrage-free Index Spread

The financially meaningful definition of the Index Spread, that makes it a martingale under a natural pricing measure, considers the level of K setting the Index value to zero in all scenarios where some names survive until maturity. Only in such scenarios, in fact, the payoff actually depends on the Index Spread. This corresponds to setting to zero only the first component of the index value (7),

$$\frac{\mathbf{1}_{\{\hat{\tau}>t\}}}{\mathbb{Q}\left(\hat{\tau}>t|\hat{\mathcal{H}}_t\right)}\left\{\hat{\Pi}\left(Prot_t^{T_A,T_M}\right)-\hat{\Pi}\left(Prem_t^{T_A,T_M}(K)\right)+\mathbb{E}\left[\mathbf{1}_{\{\hat{\tau}>T_A\}}F_t^{T_A}|\hat{\mathcal{H}}_t\right]\right\},$$

which is the price of an armageddon-knock out tradable asset. We obtain the following definition of the equilibrium *Arbitrage-free Index Spread*

$$\hat{S}_t^{T_A,T_M}=\frac{\hat{\Pi}\left(Prot_t^{T_A,T_M}\right)+\mathbb{E}\left[\mathbf{1}_{\{\hat{\tau}>T_A\}}F_t^{T_A}|\hat{\mathcal{H}}_t\right]}{\hat{\Pi}\left(DV_t^{T_A,T_M}\right)} \qquad (8)$$

This definition of the index spread is both regular, since $\hat{\Pi}\left(DV_t^{T_A,T_M}\right)$ is bounded away from zero, and has a reasonable financial meaning.

Implementing the Index Spread

How do we actually implement it in practice? At time 0, under independence of default risk and interest rates, and using $\mathbb{E}\left[\mathbf{1}_{\{\hat{\tau}>T_A\}} F_0^{T_A}\right] = \mathbb{E}\left[F_0^{T_A}\right] - \mathbb{E}\left[\mathbf{1}_{\{\hat{\tau}\le T_A\}} F_0^{T_A}\right]$,

$$\begin{aligned}\hat{S}_0^{T_A,T_M} &= \frac{\mathbb{E}\left[Prot_0^{T_A,T_M}\right] + \mathbb{E}\left[F_0^{T_A}\right] - \mathbb{E}\left[\mathbf{1}_{\{\hat{\tau}\le T_A\}} F_0^{T_A}\right]}{\mathbb{E}\left[DV_0^{T_A,T_M}\right]} \\ &= \tilde{S}_0^{T_A,T_M} - \frac{(1-R)\,P(0,T_A)\,\mathbb{Q}(\hat{\tau}\le T_A)}{\mathbb{E}\left[DV_0^{T_A,T_M}\right]}\end{aligned}$$

The market Standard Formula

Pricing the option requires the definition of a viable change of measure, therefore it means solving also Problem 3. Although this is technically the most demanding of the three problems, the preceding analysis and in particular the introduction of an appropriate subfiltration already gives us the correct tools to deal with this issue. Now it is natural to take the quantity

$$\hat{\Pi}\left(DV_t^{T_A,T_M}\right) = \mathbb{E}\left[DV_t^{T_A,T_M}|\hat{\mathcal{H}}_t\right] = \mathbb{E}\left[\sum_{j=A+1}^{M} D\left(t,T_j\right)\alpha_j\left(1-\frac{L\left(T_j\right)}{\left(1-R\right)}\right)|\hat{\mathcal{H}}_t\right]$$

to define a probability measure $\hat{\mathbb{Q}}^{T_A,T_M}$ allowing to simplify the computation. Differently from $\Pi\left(DV_t^{T_A,T_M}\right)$ that one should select if subfiltrations had not been introduced, $\hat{\Pi}\left(DV_t^{T_A,T_M}\right)$ is strictly positive.

The No-Armageddon Pricing Measure

We define the T_A, T_M-*no-armageddon pricing measure* $\hat{\mathbb{Q}}^{T_A,T_M}$ through definition of the Radon-Nykodim derivative of this measure with respect to $\mathbb{Q}$

$$Z_{T_A} = \left.\frac{d\hat{\mathbb{Q}}^{T_A,T_M}}{d\mathbb{Q}}\right|_{\hat{\mathcal{H}}_{T_A}} = \frac{B_0 \; \hat{\Pi}\left(DV_{T_A}^{T_A,T_M}\right)}{\hat{\Pi}\left(DV_0^{T_A,T_M}\right)\; B_{T_A}}.$$

and we can compute that

$$Z_t = \mathbb{E}\left[\left. Z_{T_A}\right| \hat{\mathcal{H}}_t\right] = \mathbb{E}\left[\left.\left.\frac{d\hat{\mathbb{Q}}^{T_A,T_M}}{d\mathbb{Q}}\right|_{\hat{\mathcal{H}}_{T_A}}\right| \hat{\mathcal{H}}_t\right] = \frac{B_0\hat{\Pi}\left(DV_t^{T_A,T_M}\right)}{\hat{\Pi}\left(DV_0^{T_A,T_M}\right) B_t}$$

Thus also the Radon-Nykodim derivative restricted to all $\hat{\mathcal{H}}_t$, $t \leq T_A$, can be expressed in closed form through market quantities. This is sufficient to apply the Bayes rule for conditional change of measure.

Arbitrage-free Credit Index Option Formula

One can choose any different martingale (smile) dynamics. We stick to lognormality for consistency with market standard formulas. The new **Arbitrage-Free Formula** can be written as

$$\Pi\left(Option_0^{T_A,T_M}(K)\right) =$$

$$\Pi\left(DV_0^{T_A,T_M}\right) Black\left(\hat{S}_0^{T_A,T_M}, K, \hat{\sigma}^{T_A,T_M}\sqrt{T_A}\right) + \boxed{(1-R)\,P(0,T_A)\,\mathbb{Q}(\hat{\tau} \leq T_A)}$$

$$= \Pi\left(DV_0^{T_A,T_M}\right) \left[Black\left(\tilde{S}_t^{T_A,T_M} - \frac{\boxed{(1-R)\,P(0,T_A)\,\mathbb{Q}(\hat{\tau} \leq T_A)}}{\Pi\left(DV_t^{T_A,T_M}\right)}, K, \hat{\sigma}^{T_A,T_M} \right) \right.$$

$$\left. + \frac{\boxed{(1-R)\,P(0,T_A)\,\mathbb{Q}(\hat{\tau} \leq T_A)}}{\Pi\left(DV_0^{T_A,T_M}\right)} \right]$$

vs the **Market Formula**

$$\Pi\left(DV_t^{T_A,T_M}\right) Black\left(\tilde{S}_t^{T_A,T_M}, K, \sigma^{T_A,T_M}\sqrt{T_A - t}\right)$$

Options on i-Traxx Europe Main - 2007 vs 2008

In the next table you see market inputs in March 2007 and in March 2008.

	March-09-07	March-11-08
Spot Spread 5y: $S_0^{9m,5y}$	**22.50 bp**	**154.50 bp**
Implied Volatility, $K = \tilde{S}_0^{9m,5y} \times 0.9$	52%	108%
Implied Volatility, $K = \tilde{S}_0^{9m,5y} \times 1.1$	54%	113%
Correlation 22% I-Traxx Main: $\rho_{0.22}^{I}$	**0.545**	**0.912**
Correlation 30% CDX IG: $\rho_{0.3}^{C}$	**0.701**	**0.999**

They include information on Index Tranches, in fact the new formula introduces an explicit dependence on default correlation information, because of the spread redefinition and of the price component corresponding to evaluating the $\hat{\mathcal{H}}_t$-probability of a portfolio armageddon in $(t, T_A]$.

Credit Index Options before and after 2007 subprime crisis

Computing $\hat{\tau}$ probabilities corresponds to price the most senior tranche possible,

$$\begin{aligned}\mathbb{Q}\left(\hat{\tau} \leq T_A\right) &= \mathbb{E}\left[1_{\{\hat{\tau} \leq T_A\}} \Big| \right] = \mathbb{E}\left[L_{a,b}\left(t\right) \Big| \hat{\mathcal{H}}_0 = \mathcal{F}_0 \right] \\ \left[a, b\right] &= \left[\frac{n-1}{n}\left(1-R\right), \left(1-R\right)\right]\end{aligned}$$

so one needs the correlations associated to the most senior tranche possible at a short maturity.

For the 125 name Main Index, with 40% recovery, these detachment points correspond to$[59.52\%; 60\%]$.

For the 50 name Crossover Index, with 40% recovery, these detachment points correspond to$[58.8\%; 60\%]$.

It is market common agreement that correlation increases with seniority and decreases (less markedly) with maturity. We may expect a correlation even higher than the highest level quoted by the CDX market, $\rho_{30\%}$.

Credit Index Options before and after 2007 subprime crisis

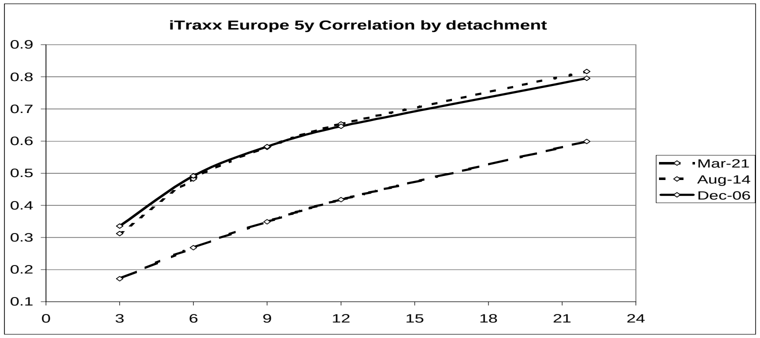


Figure 1: Base Correlation by Detachment, iTraxx Main.

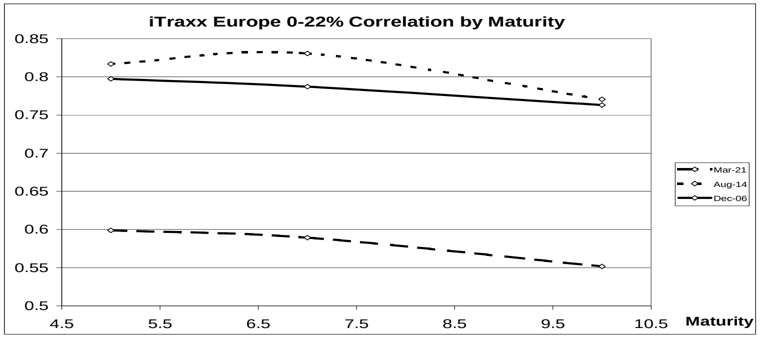


Figure 2: Base Correlation by Maturity, iTraxx Main.

Credit Index Options before and after 2007 subprime crisis

However, rather than extrapolating as in the market standard, we just consider a range of equally spaced correlations in-between i-Traxx and CDX most seniors. Generally this underestimates the probability of $\hat{\tau}$, and consequently also the relevance of the new formula.

Options on i-Traxx Europe Main - March 2007

Strike (Call)	26	21
Market Formula	**23.289**	**11.619**
No-Arb. Form. $\rho = 0.545$	23.289	11.619
No-Arb. Form. $\rho = 0.597$	23.289	11.619
No-Arb. Form. $\rho = 0.649$	23.289	11.618
No-Arb. Form. $\rho = 0.701$	23.286	11.614
Strike (Put)	26	21
Market Formula	**13.840**	**21.069**
No-Arb. Form. $\rho = 0.545$	13.840	21.069
No-Arb. Form. $\rho = 0.597$	13.840	21.069
No-Arb. Form. $\rho = 0.649$	13.840	21.069
No-Arb. Form. $\rho = 0.701$	13.843	21.071

March-09-07 Options on i-Traxx 5y, Maturity 9m

The bid-offer spread for options in March 07 was more than 1bp.

Options on i-Traxx Europe Main - March 08

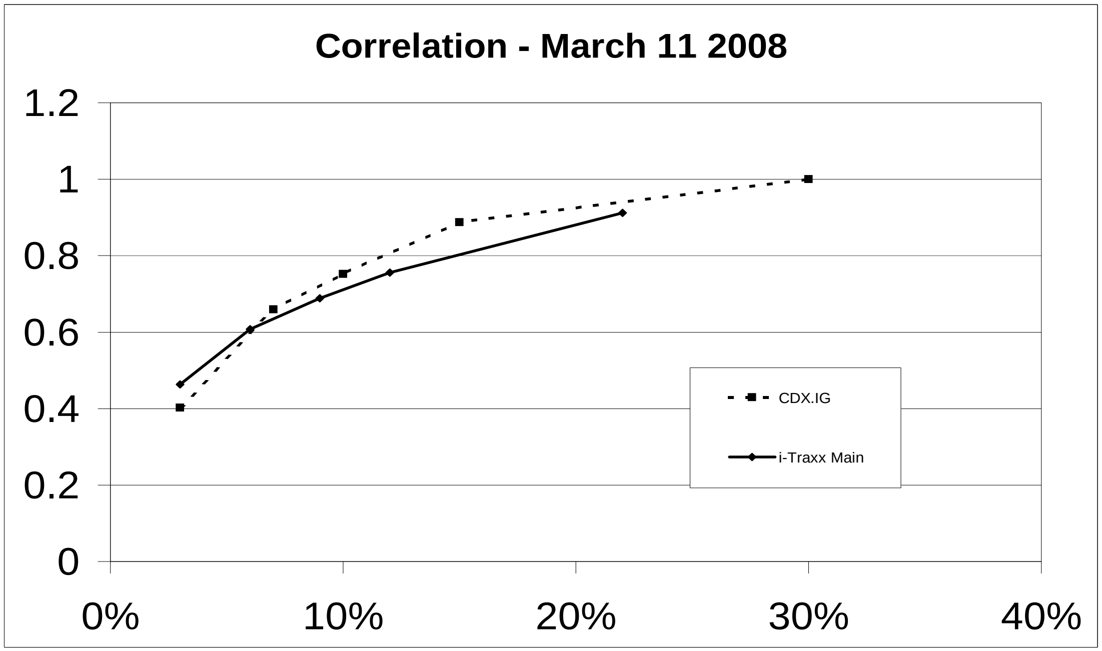


Figure 3: JPM Base Correlation for i-Traxx Main and CDX.IG

Options on i-Traxx Europe Main - March 2008

Strike (Call)	180	147
Market Formula	**286.241**	**189.076**
No-Arb. Form. $\rho = 0.912$	277.668	179.624
Difference	**8.573**	**9.453**
No-Arb. Form. $\rho = 0.941$	271.460	172.769
Difference	**14.781**	**16.307**
No-Arb. Form. $\rho = 0.970$	258.887	158.862
Difference	**27.354**	**30.215**
No-Arb. Form. $\rho = 0.999$	212.867	107.630
Difference	**73.374**	**81.447**

March-11-08 Options on i-Traxx 5y, Maturity 9m

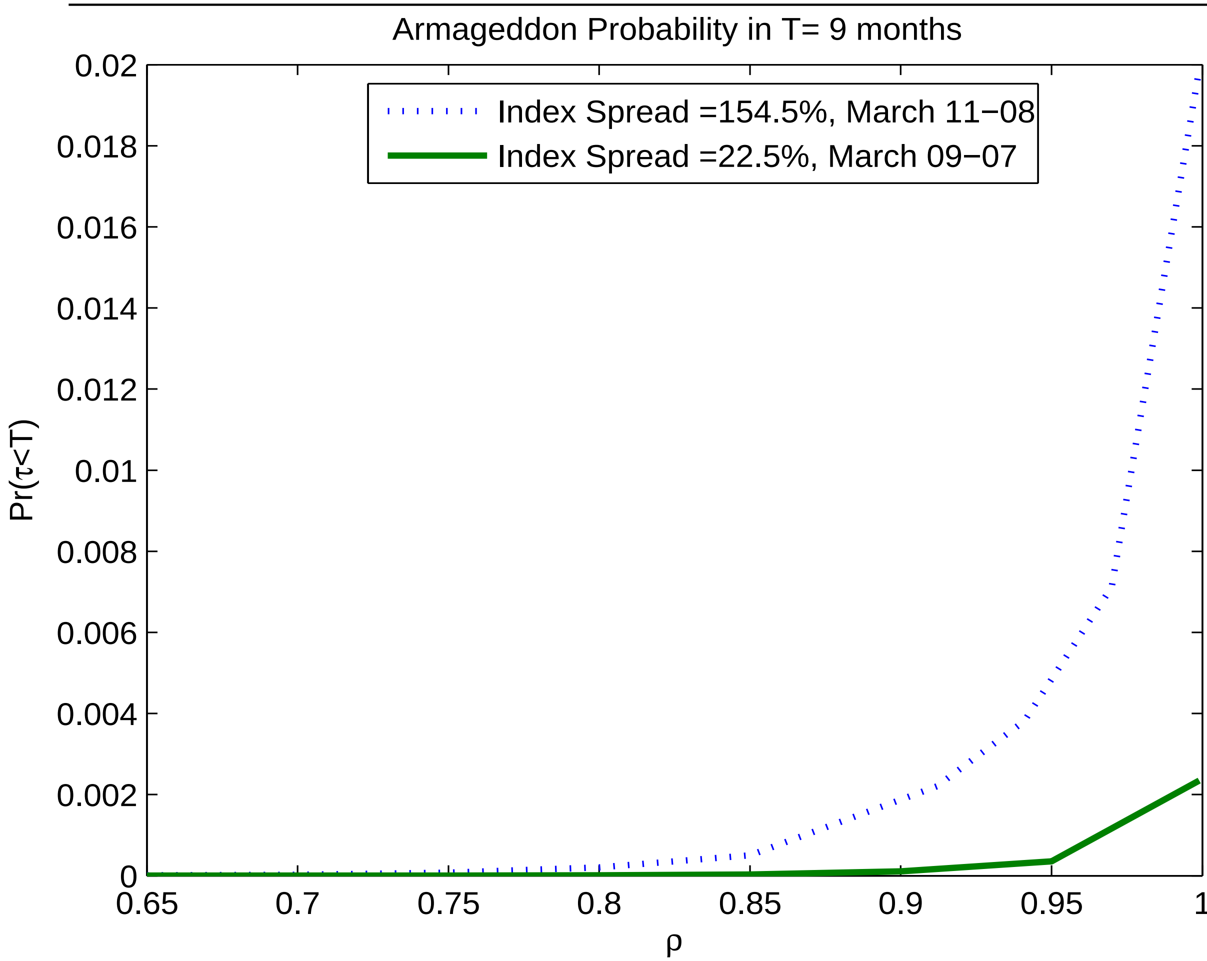
The bid-offer spread for options in March 08 was in the range 5-8 bps.

Options on i-Traxx Europe Main - March 2008

Strike (Put)	180	147
Market Formula	**222.242**	**253.076**
No-Arb. Form. $\rho = 0.912$	226.871	256.826
Difference	**4.629**	**3.750**
No-Arb. Form. $\rho = 0.941$	230.326	259.634
Difference	**8.084**	**6.559**
No-Arb. Form. $\rho = 0.970$	237.606	265.580
Difference	**15.365**	**12.504**
No-Arb. Form. $\rho = 0.999$	268.186	290.948
Difference	**45.945**	**37.872**

March-11-08 Options on i-Traxx 5y, Maturity 9m

Risk Neutral probability of an Armageddon - Main i-Traxx



Figure

Portfolio Armageddon and Tranche options

Therefore taking correctly into account the possibility of portfolio armageddon is not only an issue that allows the definition of the spread and of the pricing measure to be regular under a mathematical point of view, but for options on the Main and Crossover Indices it is also of financial relevance in market situations similar to the current one.

The **historical probability** of total portfolio default appears clearly negligible when we are considering large portfolios of investment grade issuers. However there is evidence in the literature that also this risk is priced, so that its **risk-neutral probability** is not negligible. In Brigo et al. (2006), the GPL model needs to include a jump process associated to an armageddon event to price correctly market tranches of different seniority and maturity.