MSc Financial Engineering – Pricing II

Lecture Series Spring 2014

Professor Dr. Rüdiger Kiesel | Birkbeck College | University of Duisburg-Essen

Part I

A Theory of Model Risk

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What is Model Risk?

Tools to Assess Model Risk

Risk, Uncertainty, and Ambiguity

Risk Measures

Stochastic Models of Financial Markets

Model Setting

Arbitrage and Equivalent Martingale Measures

Black-Scholes Model

Quantitative Frameworks for Model Risk

Parametric Model Risk

Risk Capturing Functional

AVaR_α induced risk capturing functional

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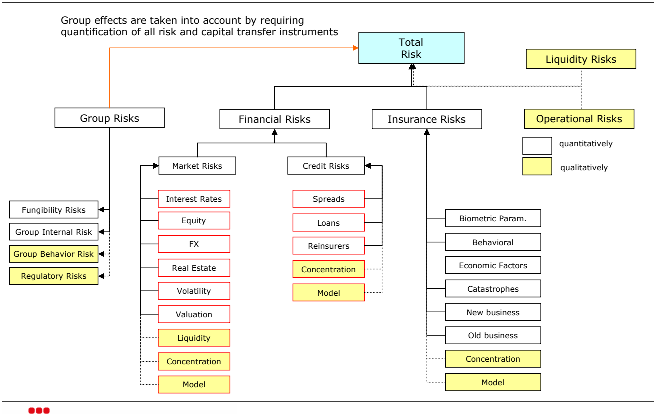
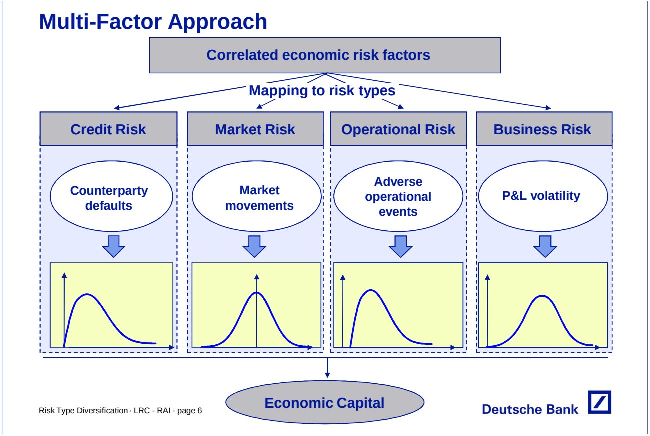
SST Classification of Risk Types



Deutsche Bank Risk Types



Deutsche Bank Annual Report 2012

- Credit risk, market risk and operational risk attract regulatory capital.
- Liquidity risk is the risk arising from our potential inability to meet all payment obligations when they come due or only being able to meet these obligations at excessive costs.
- Business risk describes the risk we assume due to potential changes in general business conditions, such as our market environment, client behavior and technological progress.
- Reputational risk is the risk that publicity concerning a transaction, counterparty or business practice involving a client will negatively impact the public's trust in our organization.
- Furthermore, they mention Insurance specific risk and risk concentration.

Basel Regulation I

Regulators formulate some principles for models

- Uncertainty is specific to the instrument and the point in time the valuation is effected.
- A bank is expected to consider all relevant market information likely to have a material effect on an instrument's fair value when selecting the appropriate inputs.
- Observable inputs or transactions may not be relevant. In such cases, the observable data should be considered, but may not be determinative.

Basel Regulation II

- There are also operative recommendations
  - Validation includes evaluation of the model's theoretical soundness and mathematical integrity and the appropriateness of model assumptions, including consistency with the market.
  - A bank should use a range of approaches and cross-check validations.
  - Valuation models should be tested and reviewed under possible stress conditions. There should be defined triggers for such reviews.
- Basel III introduces a risk-invariant leverage ratio of 3 percent in recognition that models may be biased.

Motivation

- Model risk has been recognized as one of the fundamental reasons for financial distress for banks and insurance companies. Recently, a number of authors addressed this issue:
  - Schoutens et. al. (2004): A perfect calibration - now what?
  - Cont (2006): Model uncertainty and its impact on the pricing of derivative instruments.
  - Bannör, Scherer (2011): Quantifying the degree of parameter uncertainty in complex stochastic models
  - Morini, M. (2011): Understanding and Managing Model Risk, Wiley.
- Important questions:
  - How sensitive is the value of a given derivative to the choice of the pricing model (parametric setting)?
  - Can one quantify a provision for model risk (as for market and credit risk)?

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Schoutens - CallFit, NIGCIR

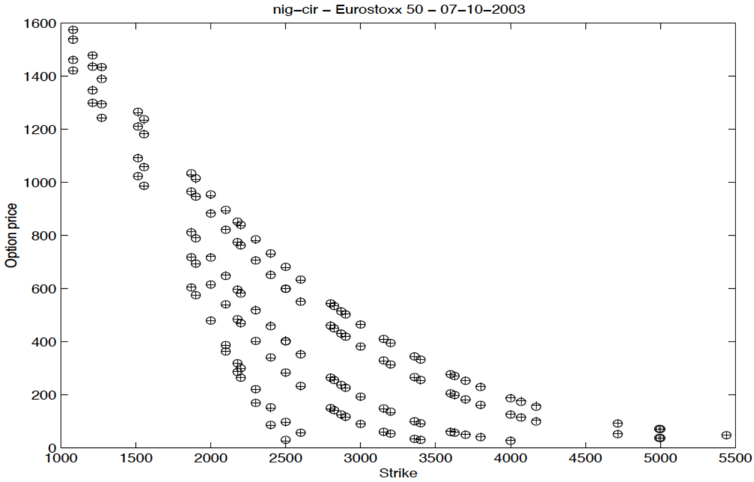

Figure 1: Calibration of NIG-CIR Model

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Schoutens - CallFit, BNS

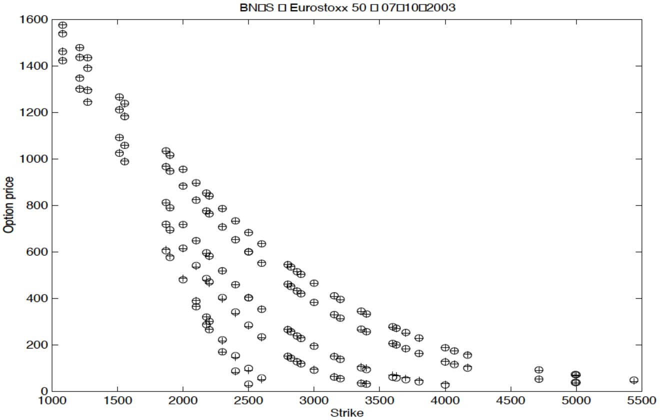
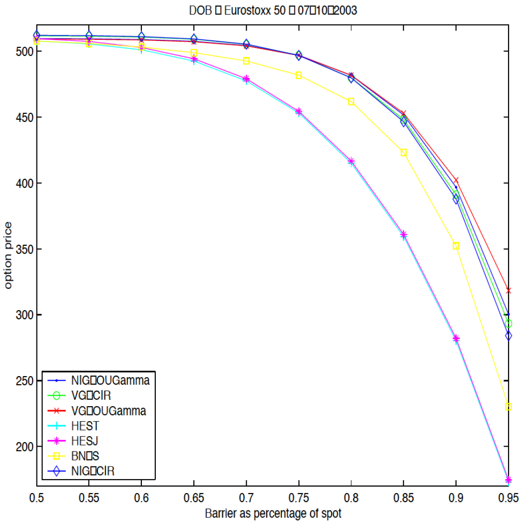


Figure 2: Calibration of Barndorff-Nielsen-Shephard Model

Schoutens - Exotic Fit



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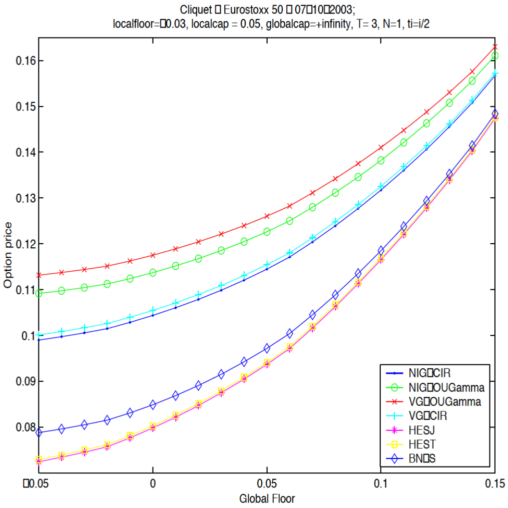
Schoutens - Clique Fit



How to define model risk?

- *Simple Idea:* Model risk is the possibility that a (financial) institution suffers losses due to mistakes in the development and application of (validation) models.

Derman (1996) – Value Approach

- Is the payoff accurately described?
- Is the software reliable?
- Has the model been appropriately calibrated to the prices of the simpler, liquid constituents that comprise the derivative?
- Does the model provide a realistic description of the factors that affect the derivative's value?

Last point: Residual risk relating to the factors – model risk?

Rebonato (2003) – Price Approach

- Model risk is the risk of occurrence of a significant difference between the mark-to-model value of a complex and/or illiquid instrument and the price at which the same instrument is revealed to have traded in the market.
- Thus, models can be counterintuitive, unreasonable, maybe not even free of arbitrage, as long as the market agrees with the valuation.
- So, large losses can only happen when a sudden gap opens between market prices and model valuation.

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J. Hull: Risk Management and Financial Institutions

For pricing models are used to

- observe model prices for similar instruments that trade
- imply model parameters and interpolate as appropriate
- value new instrument
- perform reverse engineering from prices observed from counterparty quotes to understand their models

Model Risk can lead to ..

- Incorrect price at time product is bought or sold
- Incorrect hedging

Marking Prices of an Instrument to Market

- Use price quoted by market maker (usually financial institutions mark to mid of bid and offer)
- Use price at which financial institution has traded product
- Use interdealer broker prices
- Use interdealer price indications
- Use model (marking to model)
- Contact member of the trading community

Products

- Linear products: Very little uncertainty about the right model, but mistakes do happen
- Standard Products
  - We do not need usually a model to know the price of an actively traded product. The market tells us the price.
  - The model is a communication tool (e.g., implied volatilities are quoted for options)
  - It is also an interpolation tool (e.g., a tool for interpolating between strike prices and maturities)
- Non-Standard Products
  - In the case of nonstandard products models play a key role in both pricing and hedging
  - It is a good idea to use more that one model whenever possible

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Hull: Model Risk

- Dangers in Model Building
  - Overfitting
  - Overparametrization
- Detecting Model Problems
  - Monitor types of trading a financial institution is doing with other financial institutions
  - Monitor profits being recorded from trading of different products (over- reap. under-pricing).
  - Use Model Audit Group
    - Check that a model has been implemented correctly
    - Examine whether there is a sound rationale for the model
    - Compare the model with others that can accomplish the same task
    - Specify limitations of model
    - Assess uncertainties in prices and hedge parameters given by model

Definitions

- Typically a stochastic model $(\Omega, \mathcal{F})$ defining future scenarios and a probability measure $\mathbb{P}$ on these outcomes.
- We have to distinguish between
  - *Risk:* we are able to specify a unique $\mathbb{P}$
  - *(Knightian) Uncertainty:* we are not able to specify a precise $\mathbb{P}$
  - *Ambiguity:* we are facing several possible specifications $\mathbb{P}_1, \mathbb{P}_2, \ldots$
- Approaches to deal with ambiguity are "averaging" and "worst-case".

Bayesian Model Averaging

- Let $\mathcal{M} = \{M_1, \ldots, M_J\}$ be a family of models and $\{\Theta_1, \ldots, \Theta_J\}$ the corresponding parameter spaces.
- A Bayesian observer has
  - a prior on model parameters $p(\theta_j | M_j)$
  - a prior on model weights $\mathbb{P}(M_j)$
- Given a set of observations y the posterior probabilities are

$$\mathbb{P}(M_j|y) = \frac{p(y|M_j)\mathbb{P}(M_j)}{\sum_{k=1}^{J} p(y|M_k)\mathbb{P}(M_k)}$$

with

$$p(y|M_j) = \int_{\Theta_j} \mathbb{P}(y|\theta_j, M_j) p(\theta_j|M_j) d\theta_j.$$

Bayesian Model Averaging – Valuation

- If we want to compute a model dependent quantity X given by an expectation we average over models.
- Given a set of observations y the posterior expectation is

$$\mathbb{E}(X|y) = \sum_{j=1}^{J} \mathbb{E}(X|y, M_j)\mathbb{P}(M_j|y).$$

- Problems:
  - how to specify priors (stock vol vs jump diffusion?)
  - computational intensive

Worst-Case Approach

- The worst-case approach has its foundations in the "MaxMin" approach as a robust version of expected utility.
- Assume U is a utility function, then

$$\max_{X} \min_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}(U(X)).$$

- Separation of risk-aversion and aversion to ambiguity
    - risk aversion is captured in U
    - aversion to ambiguity is captured in min

Convex Risk Measures

Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathcal{X} \subset \mathcal{L}^0(\Omega)$ a vector space. $\mathcal{Y} \subset \mathcal{X}$ be a sub-vector space and $\pi \in \mathcal{Y}^*$.

$$\rho : \mathcal{X} \to \mathbb{R} \tag{1}$$

is a convex risk measure with π translation invariance iff

- ρ is monotone:

$$X \leq Y \implies \rho(X) \geq \rho(Y).$$

- ρ is convex:

$$\forall \lambda \in [0,1] : \rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y).$$

- ρ is π-translation invariant:

$$\forall Y \in \mathcal{Y} : \rho(X + Y) = \rho(X) + \pi(Y).$$

Coherent Risk Measures

A coherent risk measure is a function ρ on abstract risk positions such that

- *Monotonicity*: if $X \geq 0$, then $\rho(X) \leq 0$;
- *Translation invariance*: if $k \in \mathbb{R}$, then $\rho(X + k) = \rho(X) - k$;
- *Homogeneity*: if $\lambda \geq 0$ in $\mathbb{R}$, then $\rho(\lambda X) = \lambda \rho(X)$;
- *Subadditivity*: $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

Convex Risk Measures – Properties

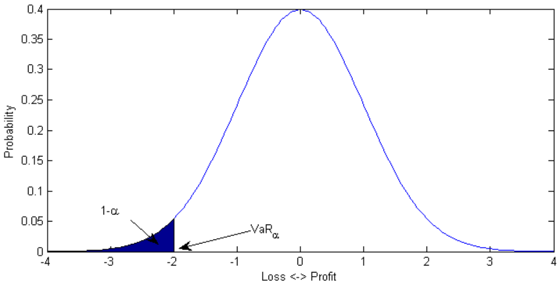
- ρ is coherent $\Leftrightarrow \ \rho(cX) = c\rho(X), \ \forall c > 0$.
- ρ is normalized $\Leftrightarrow \ \rho(0) = 0$.
- Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$.
  ρ is $\mathbb{P}$-law invariant $\Leftrightarrow \ \mathbb{P}^X = \mathbb{P}^Y$ implies $\rho(X) = \rho(Y)$.

Examples: VaR and AVaR

- general probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\beta \in (0,1]$, $X \in L^1(\mathbb{P})$, then
$$VaR_\beta(X) = q^{\mathbb{P}}_{-X}(1-\beta).$$
- VaR is not a coherent risk measure as it is not subadditive in general.
- the average value at risk at level $\alpha \in (0,1]$ is
$$AVaR_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha VaR_\beta(X)d\beta.$$
- AVaR$_\alpha$ is a convex risk measure (coherent and law-invariant).

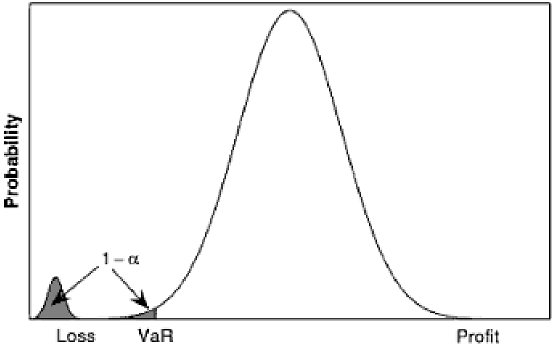
Value at Risk

The worst $1-\alpha$ scenarios are below the $-VaR_{\alpha}$, all others are below.



Value at Risk

The figure illustrates a portfolio with a low VaR compared to its risk.



Average Value-at-Risk (AVaR)

- The AVaR specifies the expectation of the loss in case the loss is above the VaR.
- AVaR is also called Tail Value-at-Risk (TVaR) or Conditional Value-at-Risk (CVaR) or Expected Shortfall (ES).

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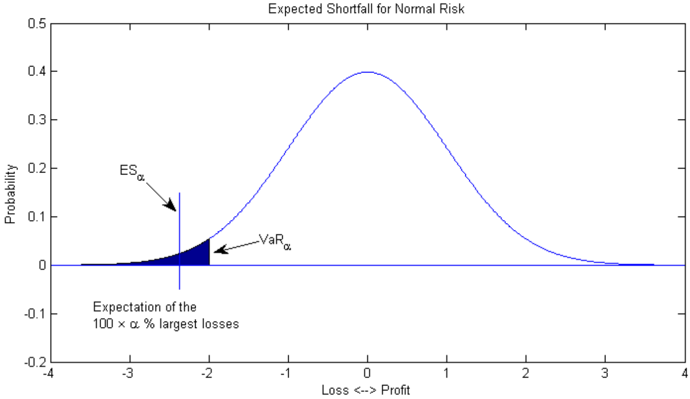
Illustration of AVaR



Financial Market Model

- $T > 0$ is a fixed a planning horizon.
- Uncertainty in the financial market is modelled by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an information set (filtration) $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$.
- There are $d + 1$ primary traded assets, whose price processes are given by stochastic processes $S_0, \ldots, S_d$, which represent the prices of some traded assets.
- A numéraire is a price process $X(t)$ almost surely strictly positive for each $t \in [0, T]$.
- 'Historically' the money market account $B(t) = e^{rt}$ with a positive constant r was used as a numéraire.

Trading Strategies

- We call an $\mathbb{R}^{d+1}$-valued process

$$\varphi(t) = (\varphi_0(t), \ldots, \varphi_d(t)), \quad t \in [0, T]$$

  a trading strategy (or dynamic portfolio process).
- Here $\varphi_i(t)$ denotes the number of shares of asset i held in the portfolio at time t - to be determined on the basis of information available *before* time t; i.e. the investor selects his time t portfolio after observing the prices $S(t-)$.

Trading Strategies

- The value of the portfolio φ at time t is given by

$$V_\varphi(t) := \varphi(t) \cdot S(t) = \sum_{i=0}^{d} \varphi_i(t) S_i(t).$$

  $V_\varphi(t)$ is called the value process, or wealth process, of the trading strategy φ.

- The gains process $G_\varphi(t)$ is

$$G_\varphi(t) := \sum_{i=0}^{d} \int_0^t \varphi_i(u) dS_i(u).$$

- A trading strategy φ is called self-financing if the wealth process $V_\varphi(t)$ satisfies

$$V_\varphi(t) = V_\varphi(0) + G_\varphi(t) \quad \text{for all } t \in [0, T].$$

Discounted Processes

The discounted price process is

$$\tilde{S}(t) := \frac{S(t)}{S_0(t)} = (1, \tilde{S}_1(t), \dots \tilde{S}_d(t))$$

with $\tilde{S}_i(t) = S_i(t)/S_0(t),\ i = 1, 2, \dots, d$. The discounted wealth process $\tilde{V}_\varphi(t)$ is

$$\tilde{V}_\varphi(t) := \frac{V_\varphi(t)}{S_0(t)} = \varphi_0(t) + \sum_{i=1}^{d} \varphi_i(t)\tilde{S}_i(t)$$

and the discounted gains process $\tilde{G}_\varphi(t)$ is

$$\tilde{G}_\varphi(t) := \sum_{i=1}^{d} \int_0^t \varphi_i(t) d\tilde{S}_i(t).$$

Self-Financing

φ is self-financing if and only if

$$\tilde{V}_\varphi(t) = \tilde{V}_\varphi(0) + \tilde{G}_\varphi(t).$$

Thus a self-financing strategy is completely determined by its initial value and the components $\varphi_1, \ldots, \varphi_d$. Any set of processes $\varphi_1, \ldots, \varphi_d$ such that the stochastic integrals $\int \varphi_i d\tilde{S}_i$ exist can be uniquely extended to a self-financing strategy φ with specified initial value $\tilde{V}_\varphi(0) = v$ by setting the cash holding as

$$\varphi_0(t) = v + \sum_{i=1}^{d} \int_0^t \varphi_i(u) d\tilde{S}_i(u) - \sum_{i=1}^{d} \varphi_i(t)\tilde{S}_i.$$

Arbitrage Opportunities

A self-financing trading strategy φ is called an arbitrage opportunity if the wealth process V_φ satisfies the following set of conditions:

$$V_\varphi(0) = 0, \quad \mathbb{P}(V_\varphi(T) \geq 0) = 1,$$

and

$$\mathbb{P}(V_\varphi(T) > 0) > 0.$$

Martingale Measure

A probability measure $\mathbb{Q}$ defined on $(\Omega, \mathcal{F})$ is an equivalent martingale measure (EMM) if:

(i) $\mathbb{Q}$ is equivalent to $\mathbb{P}$,
(ii) the discounted price process $\tilde{S}$ is a $\mathbb{Q}$ martingale.
Assume $S_0(t) = B(t) = e^{rt}$, then $\mathbb{Q} \sim \mathbb{P}$ is a martingale measure if and only if every asset price process S_i has price dynamics under $\mathbb{Q}$ of the form

$$dS_i(t) = rS_i(t)dt + dM_i(t),$$

where M_i is a $\mathbb{Q}$-martingale.

EMMs and Arbitrage

Assume $\mathbb{Q}$ is an EMM. Then the market model contains no arbitrage opportunities.
Proof. Under $\mathbb{Q}$ we have that $\tilde{V}_\varphi(t)$ is a martingale. That is,

$$\mathbb{E}_{\mathbb{Q}}\left(\tilde{V}_\varphi(t)|\mathcal{F}_u\right) = \tilde{V}_\varphi(u), \text{ for all } \quad u \leq t \leq T.$$

For $\varphi \in \Phi$ to be an arbitrage opportunity we must have $\tilde{V}_\varphi(0) = V_\varphi(0) = 0$. Now

$$\mathbb{E}_{\mathbb{Q}}\left(\tilde{V}_\varphi(t)\right) = 0, \text{ for all } 0 \leq t \leq T.$$

EMMs and Arbitrage

Now $\tilde{V}_\varphi(t)$ is a martingale, so

$$\mathbb{E}_\mathbb{Q}\left(\tilde{V}_\varphi(t)\right) = 0,\ 0 \leq t \leq T,$$

in particular $\mathbb{E}_\mathbb{Q}\left(\tilde{V}_\varphi(T)\right) = 0$.
For an arbitrage opportunity φ we have $\mathbb{P}\left(V_\varphi(T) \geq 0\right) = 1$, and since $\mathbb{Q} \sim \mathbb{P}$, this means $\mathbb{Q}\left(V_\varphi(T) \geq 0\right) = 1$.
Both together yield

$$\mathbb{Q}\left(V_\varphi(T) > 0\right) = \mathbb{P}\left(V_\varphi(T) > 0\right) = 0,$$

and hence the result follows. ∎

Contingent Claims

A contingent claims X is a random variable with existing expected value.

- A contingent claim X is called attainable if there exists at least one admissible trading strategy such that

  $$V_{\varphi}(T) = X.$$

  We call such a trading strategy φ a replicating strategy for X.
- The financial market model $\mathcal{M}$ is said to be complete if any contingent claim is attainable.

No-Arbitrage Price

If a contingent claim X is attainable, X can be replicated by a portfolio $\varphi \in \Phi(\mathbb{P}^*)$. This means that holding the portfolio and holding the contingent claim are equivalent from a financial point of view. In the absence of arbitrage the (arbitrage) price process $\Pi_X(t)$ of the contingent claim must therefore satisfy

$$\Pi_X(t) = V_\varphi(t).$$

Risk-Neutral Valuation

The arbitrage price process of any attainable claim is given by the risk-neutral valuation formula

$$\Pi_X(t) = S_0(t)\mathbb{E}_{\mathbb{P}^*}\left[\frac{X}{S_0(T)}\middle|\mathcal{F}_t\right].$$

Thus, for any two replicating portfolios $\varphi, \psi \in \Phi(\mathbb{P}^*)$

$$V_\varphi(t) = V_\psi(t).$$

Risk-Neutral Valuation

Proof. Since X is attainable, there exists a replicating strategy $\varphi \in \Phi(\mathbb{P}^*)$ such that $V_\varphi(T) = X$ and $\Pi_X(t) = V_\varphi(t)$. Since $\varphi \in \Phi(\mathbb{P}^*)$ the discounted value process $\tilde{V}_\varphi(t)$ is a martingale, and hence

$$\begin{aligned}
\Pi_X(t) &= V_\varphi(t) = S_0(t)\tilde{V}_\varphi(t) \\
&= S_0(t)\mathbb{E}_{\mathbb{P}^*}\left[\tilde{V}_\varphi(T)\middle|\mathcal{F}_t\right] \\
&= S_0(t)\mathbb{E}_{\mathbb{P}^*}\left[\frac{V_\varphi(T)}{S_0(T)}\middle|\mathcal{F}_t\right] \\
&= S_0(t)\mathbb{E}_{\mathbb{P}^*}\left[\frac{X}{S_0(T)}\middle|\mathcal{F}_t\right].
\end{aligned}$$

∎

Black-Scholes Model

The classical Black-Scholes model is

$$
\begin{aligned}
dB(t) &= rB(t)dt, & B(0) &= 1, \\
dS(t) &= S(t)\,(bdt + \sigma dW(t)), & S(0) &= p,
\end{aligned}
$$

with constant coefficients $b \in \mathbb{R},\ r, \sigma \in \mathbb{R}_+$.
The Black-Scholes price process of a European call is given by

$$
\begin{aligned}
C(t) &= S(t)\Phi(d_1(S(t), T-t)) \\
&\quad -Ke^{-r(T-t)}\Phi(d_2(S(t), T-t)).
\end{aligned}
$$

The functions $d_1(s,t)$ and $d_2(s,t)$ are given by

$$
\begin{aligned}
d_1(s,t) &= \frac{\log(s/K) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}, \\
d_2(s,t) &= \frac{\log(s/K) + (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}
\end{aligned}
$$

Parameter uncertainty

- $(\Omega, \mathcal{F}, \mathbb{F})$ filtered measurable space
- $S = (S_t)$ basic instruments, contingent claim $X = F(S)$
- parametrized family of (martingale) measures $(\mathbb{Q}_\theta)_{\theta \in \Theta}$ on $(\Omega, \mathcal{F})$.
- parameter $\theta \in \Theta$, (risk neutral) value of contingent claim is

$$\theta \to \mathbb{E}_\theta(X) := \mathbb{E}_{\mathbb{Q}_\theta}(X).$$

Parameter Uncertainty

To use models we need to specify the parameters

- estimation
  - some estimator $\hat{\vartheta}$ is used instead the true parameter ϑ
  - bias and volatility of the estimator have to be considered
- calibration
  - search for parameter that minimizes some pricing error condition, e.g.

$$\vartheta_c = \underset{\vartheta}{\text{argmin}} \left| \sum_{\text{set of derivatives}} \text{model price}(\vartheta) - \text{market price} \right|$$

  - parameters may not be uniquely identified
- Both approaches
  - produce parameter uncertainty,
  - may disregard information.

Bannör-Scherer Approach

- distribution R for likelihood of parameter on parameter space Θ available
- convex risk measures gauge extent of parameter risk
- this allows to calculate parameter risk-induced spreads
- Advantages
  - parameter's distribution is exploited
  - risk aversion can be incorporated without being maximally conservative
  - Cont's (2006, Math. Finance, 16(3), 519 -547) suggestion is an extreme points

Risk Capturing Functionals

We denote the space of all derivatives by

$$\mathcal{D} := \bigcap_{\theta \in \Theta} L^1(\mathbb{Q}_\theta) \tag{2}$$

We call

$$\Gamma : \mathcal{D} \to \mathbb{R}$$

a risk-capturing functional with properties

- order preservation $X \geq Y \Rightarrow \Gamma(X) \geq \Gamma(Y)$
- diversification
  $\forall\ \lambda \in [0,1] : \ \Gamma(\lambda X + (1-\lambda)Y) \leq \lambda \Gamma(X) + (1-\lambda)\Gamma(Y)$.
- parameter independence consistency

$$\theta \to \mathbb{E}_\theta(X) \equiv \text{constant} \ \Rightarrow \ \Gamma(X) = \mathbb{E}_\theta(X).$$

Model Risk – Cont's Suggestion

- For X a derivative we associate with $\Gamma(X)$ the ask price and with $-\Gamma(-X)$ its bid price.
- Cont's suggestion

$$\Gamma^u(X) = \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}} \text{ and } \Gamma^l(X) = -\Gamma^u(-X) = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}.$$

This approach produces typically a wide bid-ask spread.

Construction of Risk Capturing Functionals

- R a probability measure on Θ
- Let $\mathcal{A} \subset L^0(R)$ be a vector space of measurable functions containing the constants

$$\mathcal{D}^{\mathcal{A}} := \left\{ X \in \bigcap_{\theta \in \Theta} L^1(\mathbb{Q}_\theta) : \theta \to \mathbb{E}_\theta(X) \in \mathcal{A} \right\} \tag{3}$$

- $\rho : \mathcal{A} \to \mathbb{R}$ be convex risk measure (normalized, law-invariant)
- Define the parameter risk capturing function

$$\Gamma : \mathcal{D}^{\mathcal{A}} \to \mathbb{R}, \ \Gamma(X) = \rho\left(\theta \to \mathbb{E}_\theta(X)\right) \tag{4}$$

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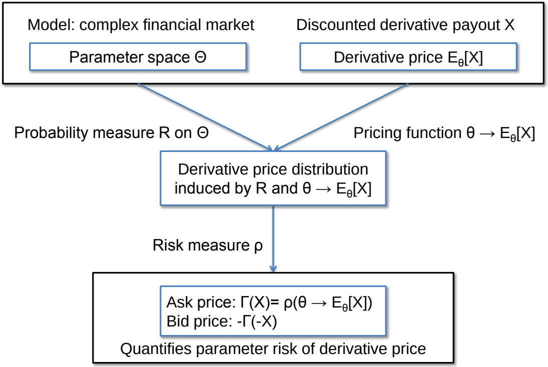
Parameter Risk-Capturing Valuation



Definition AVaR

- general probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\beta \in (0,1]$, $X \in L^1(\mathbb{P})$, then

$$VaR_\beta(X) = q^{\mathbb{P}}_{-X}(1-\beta).$$

- the average value at risk at level $\alpha \in (0,1]$ is

$$AVaR_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha VaR_\beta(X) d\beta.$$

- AVaR_α is a convex risk measure (coherent and law-invariant).

Definition AVaR risk capturing functional

- Assume a parametrized family of (martingale) measures $\mathcal{Q}_\Theta = (\mathbb{Q}_\theta)_{\theta\in\Theta}$.
- Let R be a distribution on Θ.
- Consider the $\mathcal{L}^1(R)$ admissible functionals, so $AVaR_\alpha : \mathcal{L}^1(R) \to \mathbb{R}$.
- Define the AVaR$_\alpha$ risk-capturing functional $R \star AVaR_\alpha : \mathcal{C}^{\mathcal{L}^1(R)} \to \mathbb{R}$ as

$$R \star AVaR_\alpha(X) := AVaR_\alpha\left(\theta \to \mathbb{E}_\theta(X)\right).$$

Convergence Property of AVaR

- Assume $R_N \to R_0,\ (N \to \infty)$ weakly on $\mathcal{Q}_\Theta$;
- ρ_N a sequence of convex risk measures with ρ_N is R_N invariant;
- A sequence Γ_N with $\Gamma_N = \rho_N\left(\mathcal{Q}_\Theta \to \mathbb{E}_\theta(X)\right)$ has the convergence property (CP) if and only if

$$\lim_{N\to\infty} \Gamma_N(X) = \Gamma_0(X) = \rho_0\left(\mathcal{Q}_\Theta \to \mathbb{E}_\theta(X)\right)\ \forall X \in C^{\mathcal{A}}.$$

- AVaR -induced risk-capturing functionals fulfill (CP) for Θ compact.

Using asymptotic distributions

- (CP) allows us, if the parameter distribution R is complicated to calculate or even unknown, to use a parameter distribution $\tilde{R}$ which is “close” to the original distribution R (in the sense of weak convergence, like, e.g., some asymptotic distribution) and calculate the risk-captured price with the parameter distribution $\tilde{R}$ instead.
- In particular, if the distribution R is propagated from an estimator $\hat{\theta}_N$ and the asymptotic distribution of the estimator $\hat{\theta}_N$ is known (let us, e.g., denote the asymptotic distribution by R_∞), we can use the distribution R_∞ instead, if the sample size $N \in \mathbb{N}$ is large enough.

Calculating AVaR

- Assume $(\theta_N)_{N\in\mathbb{N}}$ is an asymptotically normal sequence of estimators for the true parameter $\theta_0 \in \Theta \subset \mathbb{R}_m$ with positive definite covariance matrix Σ, so

$$\sqrt{N}\,(\theta_N - \theta_0) \to \mathcal{N}_m\,(0, \Sigma)\,.$$

- If $\theta \mapsto \mathbb{E}_\theta(X)$ is continuously differentiable and $\nabla\mathbb{E}_{\theta_0} \neq 0$, then

$$\sqrt{N}\,(\mathbb{E}_{\theta_N}(X) - \mathbb{E}_{\theta_0}(X)) \to \mathcal{N}\left(0, (\nabla\mathbb{E}_{\theta_0})'\,\Sigma\nabla\mathbb{E}_{\theta_0}\right)$$

- For $\theta_N \star AVaR_\alpha(X)$ we calculate the AVaR as for a normally distributed variable

$$\theta_N \star AVaR_\alpha(X) \approx \mathbb{E}_{\theta_0}(X) + \frac{\varphi\left(\Phi^{-1}(\alpha)\right)}{\alpha\sqrt{N}}\sqrt{(\nabla\mathbb{E}_{\theta_0})'\,\Sigma\nabla\mathbb{E}_{\theta_0}},$$

MSc Financial Engineering – Pricing II

Lecture Series Spring 2014

Professor Dr. Rüdiger Kiesel | Birkbeck College | University of Duisburg-Essen

Part II

Examples of Model Risk

Model Risk for Equity Derivatives

- Uncertain Volatility
- Local Volatility vs Jumps
- Robustness of Hedging Strategies

Model Risk For Interest Rates

Model Risk For Energy Derivatives

- Spread Options
- Commodity Models
- Energy Model Risk Results
- Political Risk

Problem Setting

- Consider in a basic GBM world the pricing of a call option with maturity T.
- Assume alternative diffusion models

$$\mathbb{Q}_i \ : \ dS(t) = S(t)(rdt + \sigma_i(t)dW(t)) \tag{5}$$

where $\sigma_i : [0, t] \to [0, \infty[$ is a bounded deterministic volatility function.
- Assume traded European calls with prices C^*, then we calibrate the models such that

$$\frac{1}{T}\int_0^T \sigma_i(s)^2 ds = \Sigma^2, \tag{6}$$

where Σ is the implied Black-Scholes volatility.

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Calibration

- Clearly (6) has many solutions, e.g. piecewise constant or piecewise linear functions.
- $$\sigma_1(t) = \Sigma,$$
- $$\sigma_i(t) = a_i \mathbf{1}_{[0,T_1]} + \sqrt{\frac{T\Sigma^2 - T_1 a_i^2}{T - T_1}} \mathbf{1}_{]T_1,T]}, \quad i = 2, \dots n,$$

  with $\Sigma < a_i < \Sigma\sqrt{T/T_1}$ for $i = 2, \dots n$.
- Set $a_1 = \Sigma$ and

  $$\bar{a} = \max\{a_i, i = 1, \dots n\}, \quad \underline{a} = \min\{a_i, i = 1, \dots n\}.$$

Model Risk

- Consider a call X with maturity $T_1 < T$. For $i = 1, \dots, n$ the models $(\Omega, \mathcal{F}, \mathbb{F}^S, \mathbb{Q}_i)$ are complete, so the call can be perfectly hedged. There is no market risk!
- However, the corresponding $\Delta-$ hedging strategy depends on the volatility – it is not model-free. It is even a random variable for each $\mathbb{Q}_j, j \neq i$.
- The Cont model bounds are

$$\Gamma^u(X) = C^{BS}(K, T_1; \bar{a}) \quad \Gamma^l(X) = C^{BS}(K, T_1; \underline{a}).$$

Jump-Diffusion Model

Consider a jump-diffusion model

$$\mathbb{Q}_1 \quad : \quad S(t) = S(0) \exp \left\{ \mu t + \sigma W(t) + \sum_{i=1}^{N_t} \epsilon_i \right\} \tag{7}$$

where

- $\mu, \sigma > 0$ are a constant;
- $W(t)$ is a standard Brownian motion;
- N_t is a Poisson process with intensity λ;
- (ϵ_i) is a family of independent random variables with distribution F (shown in (1)), such that $V_i = 1 + \epsilon_i$ are log-normally distributed and with $k = \int xF(dx)$.

Review: Classical Construction JD-model

If $1+\epsilon$ is log-normally distributed with parameters

$$\mathbb{E}\log(1+\epsilon) = \gamma - \frac{\delta^2}{2}, \quad \mathbb{V}\mathrm{ar}\log(1+\epsilon) = \delta^2, \quad \mathbb{E}\epsilon = k = e^{\gamma} - 1$$

under the historical measure $\mathbb{P}$, one can find an equivalent martingale measure $\mathbb{P}^*$ such that the intensity of N is $\tilde{\lambda} > 0$ and the $(1+\epsilon_i)$ are log-normally distributed with parameters

$$\mathbb{E}^*\log(1+\epsilon) = \tilde{\gamma} - \frac{\delta^2}{2}, \quad \mathbb{V}\mathrm{ar}^*\log(1+\epsilon) = \delta^2, \quad \mathbb{E}^*\epsilon = \tilde{k} = e^{\tilde{\gamma}} - 1$$

under $\mathbb{P}^*$.

Review: Option Pricing JD-model

We find for the price of a European call

$$C(S,1,T,\tilde{\lambda},\tilde{\sigma}) = \sum_{n=0}^{\infty} \frac{(\lambda' T)^n}{n!} \exp\{-\lambda' T\} C_{BS}(S,1,\tilde{r}_n,T,\tilde{\sigma}_n),$$

with C_{BS} the Black-Scholes call price and parameters

$$\lambda' = \tilde{\lambda}(1+\tilde{k}), \quad \tilde{r}_n = \frac{n\tilde{\gamma}}{T} - \tilde{\lambda}\tilde{k}, \quad \tilde{\sigma}_n^2 = \frac{1}{T}\left(\sigma^2 T + \frac{n\delta^2}{2}\right).$$

Local Volatility Model

- $$\mathbb{Q}_2 \ : \ dS(t) = S(t)(rdt + \sigma(t, S(t))dW(t)) \qquad (8)$$
  where $\sigma(t, S)$ is calibrated to the implied volatilities in figure (2)
- The resulting volatility function is in figure (1)

Jump Sizes and Local Volatiliy

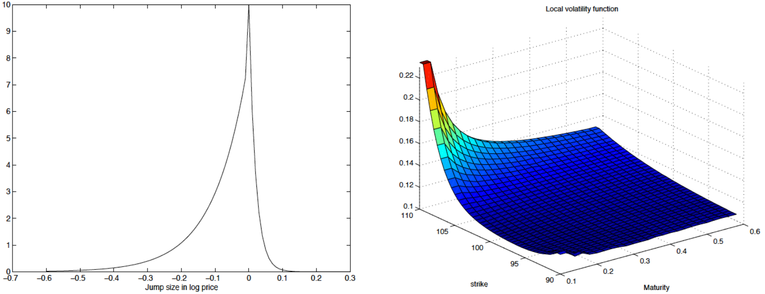

Figure : Jump Sizes and Local Volatiliy, from Cont (2006)

Implied Volatility (Both Models)

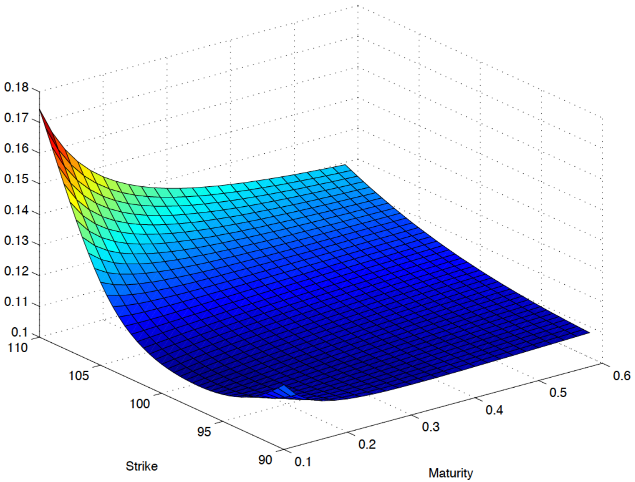

Figure : Implied Volatility, from Cont (2006)

Sample Paths

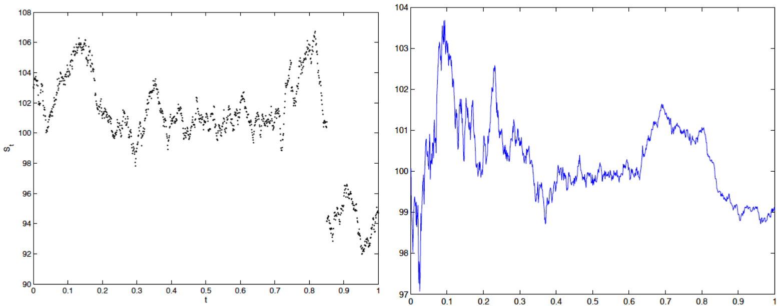

Figure : Sample Path, from Cont (2006)

Pricing a Barrier Option

- Consider a knock-out call with strike at the money, maturity $T = 0.2$ and knock-out barrier $B = 110$.
- Due to the high-short end volatiles in the local-vol model, it will produce higher barrier prices.

	Local Vol	Jump-Diffusion
Call	3.5408	3.5408
Barrier	2.73	1.63

- The value of Cont's measure for model risk is then 1.1.

Delta in the Black-Scholes Model

In the Black-Scholes model, we have computed the Greeks explicitly for European call options. We have:

$$\Delta = \frac{\partial}{\partial S} Call_{BS}(S, K, \sigma, r, t, T) = \Phi(d_1)$$

where, as usual, Φ denotes the c.d.f. of the standard normal distribution and

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}}.$$

Delta of a Call Option in the Black-Scholes Model

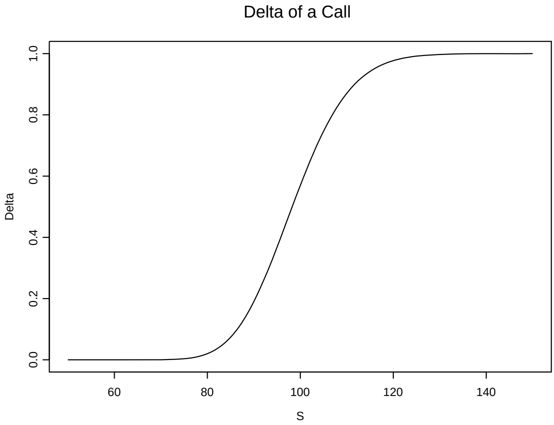

Figure : Delta for a European call in the BS model, $T = 1$, $r = 1\%$, $\sigma = 20\%$, $K = 100$.

Example: Delta Hedge

You are long USD $1,000$ in the 104 call. Interest rate is 5%, stock price today is 99, time to maturity 1 month, and implied volatility is 15.7%.

- How can you make your portfolio delta neutral by investing in the stock?
- You set up the delta neutral portfolio and the stock price jumps to USD100 immediately. What is your P/L for the portfolio?

Example: Delta Hedge

- Compute the price of the call with the BS formula:

$$Call_{BS} = 0.3858\text{USD}.$$

- The position consists of $N = 1,000/Call_{BS} = 2592$ call options.
- The delta of each option is

$$\begin{aligned}\Delta = \Phi(d_1) &= \Phi\left(\frac{\log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)\\ &= \Phi\left(\frac{\log(99/104) + (0.05 + 0.157^2/2)(1/12)}{0.157\sqrt{1/12}}\right)\\ &= 0.1654.\end{aligned}$$

Example: Delta Hedge

- The delta of the position is long $\Delta_P = N \cdot \Delta = 428.70$.
- The stock has $\Delta = 1$.
- To make the position delta neutral, you have to enter a short position of 428.70 shares.

Example: Delta Hedge

- The loss from the short position in the stock is $428.70 \cdot 1 = 428.70$.
- To compute the gain from the long options position, we have to compute the option price for $S = 100$. Using the BS formula, we obtain

$$Call_{BS}(S = 100) = 0.5808.$$

- The gain from the options position is $2592 \cdot (0.5808 - 0.3858) = 505.37$.
- Our profit is $505.37 - 428.70 = 76.67$.

$\Delta-$ Hedging Strategies

- Consider a European call which is hedges according to a simple Black-Scholes Delta hedge.
- Use a jump-diffusion model to generate the underlying

$$S(t) = S(0)\exp\left\{\mu t + \sigma W(t) + \sum_{i=1}^{N_t} \epsilon_i\right\}$$

- Calculate $\Delta-$Hedges according to a Black-Scholes model with (updated) implied volatilities.
- Figure (5) shows the distribution of the P&L of such hedges as a percentage of the option price at inception.

Black-Scholes Delta Hedge

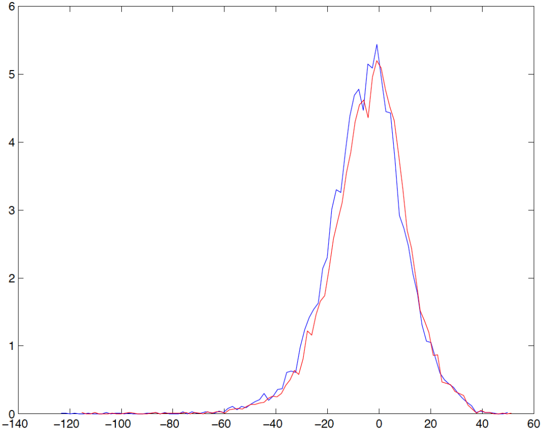

Figure : Delta Hedges with static and updated implied volatility, from Cont (2006)

Swaps

A swap contract S with K and R pays at every instant T_i in a prespecified set of dates $T_{\alpha+1}, \dots, T_\beta$ the amount of money

$$X_{i+1} = K\tau(L(T_i, T_{i+1}) - R).$$

The floating rate over $[T_i, T_{i+1}]$ observed at T_i is a simple rate defined as

$$p(T_i, T_{i+1}) = \frac{1}{1 + \tau L(T_i, T_{i+1})}.$$

Pricing Formula for Swaps

Using the risk-neutral pricing formula we obtain (use $K = 1$),

$$
\begin{aligned}
\Pi(t,S) &= \sum_{i=1}^{n} \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^{T_i} r(s)ds}\tau(L(T_{i-1},T_i)-R) \middle| \mathcal{F}_t \right] \\
&= \sum_{i=1}^{n} \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{T_{i-1}}^{T_i} r(s)ds} \middle| \mathcal{F}_{T_{i-1}} \right] \right. \\
&\qquad \left. \times\ e^{-\int_t^{T_{i-1}} r(s)ds}\left(\frac{1}{p(T_{i-1},T_i)} - (1+\tau R) \right) \middle| \mathcal{F}_t \right] \\
&= \sum_{i=1}^{n} \left(p(t,T_{i-1}) - (1+\tau R)p(t,T_i) \right) \\
&= p(t,T_0) - \sum_{i=1}^{n} c_i p(t,T_i),
\end{aligned}
$$

with $c_i = \tau R, i = 1, \ldots, n-1$ and $c_n = 1 + \tau R$. So we obtain the swap price as a linear combination of zero-coupon bond prices.

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Interest-Rate Swap

- We require the IRS to be fair at time t to obtain the forward swap rate.
- The forward swap rate $S_{\alpha,\beta}(t)$ at time t for the sets of time $\mathcal{T}$ and year fractions τ is the rate in the fixed leg of the above IRS that makes the IRS a fair contract at the present time, i.e. it is the K for which it has the value 0.
- We obtain

$$S_{\alpha,\beta}(t) = \frac{p(t, T_\alpha) - p(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i p(t, T_i)}. \tag{9}$$

Swaptions

- Swap options or more commonly swaptions are options on an IRS. A European payer swaption is an option giving the right (and not the obligation) to enter a payer IRS at a given future time, the swaption maturity. Usually the swaption maturity coincides with the first reset date of the underlying IRS.
- A Bermudan swaption allows to enter into a swap at any time T_{ex} in $T_\alpha, \ldots, T_\beta$.
- The value of a swaption can be written as

$$S^{ex,b}_{T_{ex}}(K) = \sum_{i=ex+1}^{\beta} p(T_{ex}, T_i)\tau_i(S_{\alpha,\beta}(T_{ex}) - K).$$

Vasicek model:

- The dynamic for the short rate is

$$dr(t) = (\alpha - \beta r(t))dt + \gamma dW(t)$$

- Bond prices are

$$p(t,T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(u)du} 1 \middle| \mathcal{F}_t \right].$$

Swap Market Models (SMM)

- We assume that $S_{\alpha,\beta}(\cdot)$ follows a lognormal martingale:

$$dS_{\alpha,\beta}(t) = \sigma(t)S_{\alpha,\beta}(t)dW_{\alpha,\beta}(t),$$

  where σ is a deterministic function and $W_{\alpha,\beta}(\cdot)$ is a standard $\mathbb{Q}^{\alpha,\beta}$-Brownian motion.
- The fact that the forward swap rate $S_{\alpha,\beta}(t)$ is lognormally distributed under $\mathbb{Q}^{\alpha,\beta}$motivates the name *lognormal forward swap model.*
- For a Bermudan swaption we would consider all swap rates $S_{\alpha,\beta}, S_{\alpha+1,\beta}, \dots, S_{\beta-1,\beta}$ and their correlations $\rho_{i,j}$ simultanously.

Bermudan Pricing

- If the correlation of the swap rates is high, then the payoffs behaviour simultaneously and there is not much value in the right to exercise in the future.
- If the correlation of the swap rates is low (or even negative), then the payoff structure may change and the right to exercise in the future may become valuable.
- One-factor models, such as the Vasicek model, do not allow for a correlation structure. On the other hand SMM allow to model a more realistic correlation structure.

Problem Setting

- Model risk has hardly been discussed in the context of energy markets (to our knowledge).
- A topical question is the need for reinvestment (replacement investments and building more capacity) in the power plant park. The financial streams of such an investment can be generated on the market for energy derivatives in terms of spread options.
- We use the Bannör, Scherer (2011) approach to discuss the model risk in such a valuation problem.

Gas Power Plant

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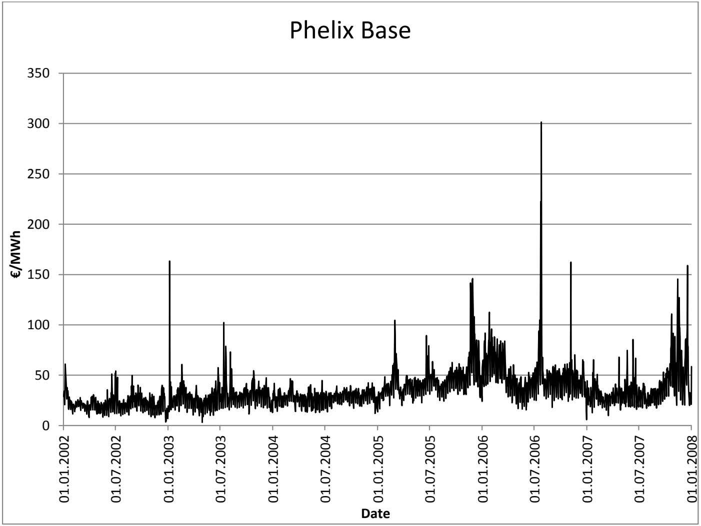
Phelix Base 2002-2008



Spread Options

Market participants are exposed to the difference of commodity prices. Examples are

- the dark spread between power and coal (model for a coal-fired power plant)
- the spark spread between power and gas (model for a gas-fired power plant)
- In countries covered by the European Union Emissions Trading Scheme, utilities have to consider also the cost of carbon dioxide emission allowances. Emission trading has started in the EU in January 2005.

Clean Spark Spread

$$CSS_\tau = P_\tau - h\,G_\tau - c_E\,E_\tau, \tag{10}$$

where P_τ is the power price, G_τ is the gas price, E_τ is the carbon certificate price at time τ, h is the heat rate, c_E emission conversion rate.

- The clean spark spread reflects the profit/loss of generating power from gas after taking into account gas and carbon allowance costs.
- A positive spread effectively means that it is profitable to generate electricity, while a negative spread means that generation would be a loss-making activity.
- Note that the clean spark spreads do not take into account additional generating charges beyond gas and carbon.

Spread Options to Manage Market Risk

- Spread options can be used by owners of corresponding plants to manage the market risk. Instead of spread trading with futures the owner of a power plant can directly purchase/sell a spread option.
- The payoff of a typical spread option with maturity τ is

$$C_{\text{spread}}(\tau) = \max(S_1(\tau) - S_2(\tau) - K, 0)$$

with S_i the underlyings, K the strike.

Valuation of Spread Options

In the Black-Scholes world there is an analytic formula for $K = 0$ (exchange option) due to Margrabe (1978).

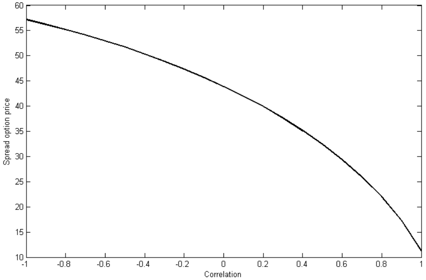
$$
\begin{aligned}
C_{\text{spread}}(t) &= (S_1(t)\Phi(d_1) - S_2(t)\Phi(d_2)) \\
P_{\text{spread}}(t) &= (S_2(t)\Phi(-d_2) - S_1(t)\Phi(-d_1)) \\
\text{where} \quad d_1 &= \frac{\log(S_1(t)/S_2(t)) + \sigma^2(\tau - t)/2}{\sqrt{\sigma^2(\tau - t)}}, \quad d_2 = d_1 - \sqrt{\sigma^2(\tau - t)} \\
\text{and} \quad \sigma &= \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}
\end{aligned}
$$

where ρ is the correlation between the two underlyings.
For $K \neq 0$ no easy analytic formula is available.

Spread Option Value and Correlation

The value of a spread option depends strongly on the correlation between the two underlyings.

$S_1 = S_2 = 100$, $\tau = 3$, $r = 0.02$, $\sigma_1 = 0.6$, $\sigma_2 = 0.4$.



- The higher the correlation between the two underlyings the lower is the volatility of the spread and hence the value of the spread option.

Approximative Spread Option Valuation

- A very good reference is Carmona, Durrleman (2003), Siam Review 45 (4), 627-685.
- There is also a survey by Krekel, de Kok, Korn, Man in Wilmott Magazine (2004) available.

Approximation by Kirk's Formula (3 Assets)

An accurate approximation formula for the three asset case is also given in E.Alos, A.Eydeland and P.Laurence, Energy Risk, (2011). Again for $r = 0$ we have for τ small the formula

$$C_{\mathrm{K3}}(S_1(t), S_2(t), S_3(t), K, \tau) \approx C_{\mathrm{BS}}(S_1(t), S_2(t) + S_3(t) + K, \sigma_S, \tau) \tag{11}$$

with

$$\begin{aligned} \sigma_S &= \sqrt{\sigma_1^2 + b_2^2\sigma_2^2 + b_3^2\sigma_3^2 - 2\rho_{12}\sigma_1\sigma_2 b_2 - 2\rho_{13}\sigma_1\sigma_3 b_3 + 2\rho_{23}\sigma_2\sigma_3 b_2 b_3} \\ b_2 &= \frac{S_2(t)}{S_2(t)+S_3(t)+K} \quad \text{and} \quad b_3 = \frac{S_3(t)}{S_2(t)+S_3(t)+K} \end{aligned}$$

and ρ_{ij} is the correlation between the underlying i, j.

Present Value of a Power Plant

- The operator acts on the spot market. The specific daily configuration of the power plant is not traded, so we use historical probabilities. We use a strip of clean spark spreads.
- R.Carmona, M. Coulon, D. Schwarz (2012) present a valuation approach using a full structural model
  - the difference between reduced form models (which we use) and the structural model is relatively small for high-efficiency gas plants, but reduced-form overprices for low-efficiency plants
  - we also define the power price exogeneously.
- We aim to study the model risk within a simulation approach.

Emission Certificates

We model the emission price as a geometric Brownian motion

$$\mathrm{d}E_t = \alpha^E \, E_t \, \mathrm{d}t + \sigma^E \, E_t \, \mathrm{d}W_t^E, \tag{12}$$

Gas Price

- We model the gas price as a mean-reverting process

$$\begin{aligned} G_t &= e^{g(t)+Z_t}, \\ \mathrm{d}Z_t &= -\alpha^G Z_t\,\mathrm{d}t + \sigma^G\,\mathrm{d}W_t^G, \end{aligned} \tag{13}$$

- α^G is the speed of mean-reversion for gas prices.

Power Price

- We model the power price as a sum of two mean-reverting processes

$$\begin{aligned} P_t &= e^{f(t)+X_t+Y_t}, \\ \mathrm{d}X_t &= -\alpha^P X_t \,\mathrm{d}t + \sigma^P \,\mathrm{d}W_t^P, \\ \mathrm{d}Y_t &= -\beta\, Y_t \,\mathrm{d}t + J_t \,\mathrm{d}N_t, \end{aligned} \tag{14}$$

- α^P and β are speeds of mean-reversion for the smooth and the jump component of power prices.
- N is a Poisson process with intensity λ.
- J_t are independent identically distributed (i.i.d) random variables representing the jump size.

Seasonal components

$g(t)$ and $f(t)$ are seasonal trend components for gas and power, respectively, defined as

$$\begin{aligned} f(t) &= a_1 + a_2\,t + a_3\cos(a_5 + 2\pi t) + a_4\cos(a_6 + 4\pi t), \\ g(t) &= b_1 + b_2\,t + b_3\cos(b_5 + 2\pi t) + b_4\cos(b_6 + 4\pi t), \end{aligned} \tag{15}$$

where a_1 and b_1 may be viewed as production expenses, a_2 and b_2 are the slopes of increase in these costs. The rest of the parameters are responsible for two seasonal changes in summer and winter respectively.

Dependence Structure

In the current setting we also assume that W^E, W^G and N are mutually independent processes, but there is some correlation between W^P and W^G

$$\mathrm{d}W_t^P \, \mathrm{d}W_t^G = \rho \, \mathrm{d}t. \tag{16}$$

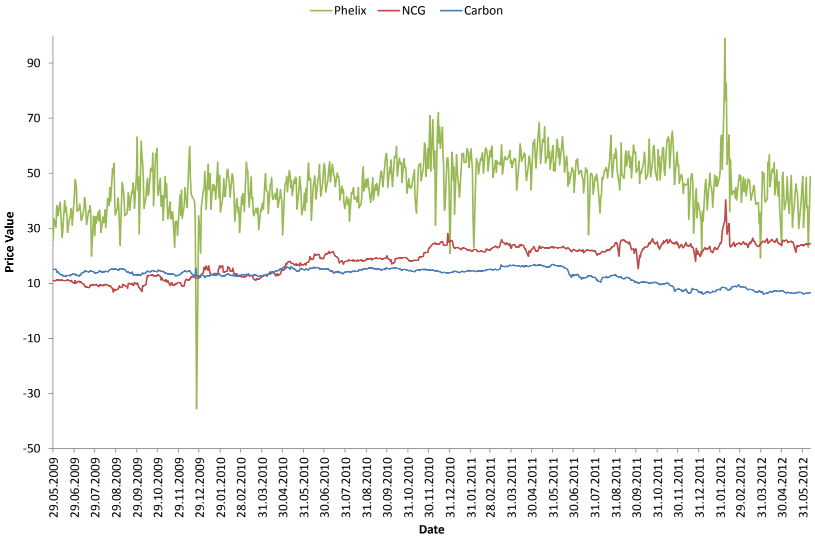
Parameter Uncertainty

- The total set of parameters includes $\{\alpha^E, \sigma^E, g(t), \alpha^G, \sigma^G, f(t), \alpha^P, \beta, \sigma^P, \lambda, \mathbb{E}[J], \mathbb{E}[J^2], \rho\}$.
- Hence, the hybrid model we have chosen for modelling the clean spark spread is not parsimonious and allows for several degrees of freedom.
- Consequently, the risk of determining parameters in a wrong way is considerable.

Data sources

- Phelix Day Base: It is the average price of the hours 1 to 24 for electricity traded on the spot market. It is calculated for all calendar days of the year as the simple average of the auction prices for the hours 1 to 24 in the market area Germany/Austria. (EUR/MWh),
- NCG: Delivery is possible at the virtual trading hub in the market areas of NetConnect Germany GmbH & Co KG. daily price (EUR/MWh),
- Emission certificate daily price: One EU emission allowance confers the right to emit one tonne of carbon dioxide or one tonne of carbon dioxide equivalent. (EUR/EUA).
- We cover the last three years: 25.09.2009 - 08.06.2012.

Price Paths, 25.09.2009 - 08.06.2012.



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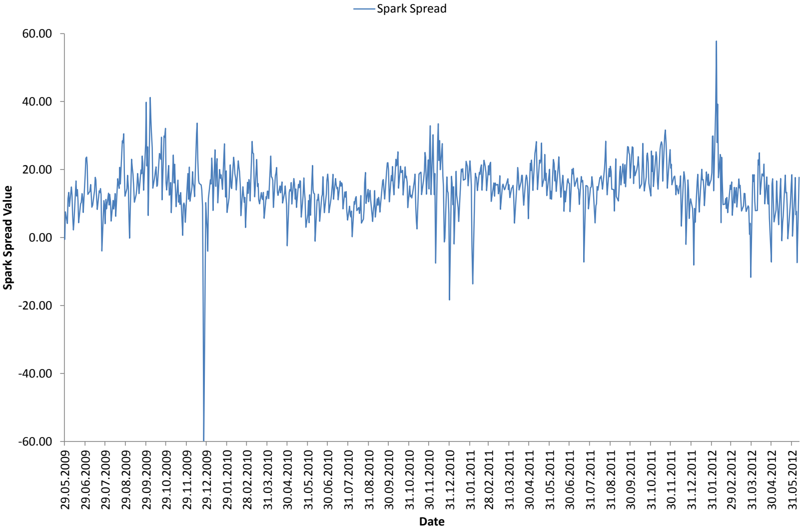
Clean Spark Spread, 25.09.2009 - 08.06.2012.



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Emissions and Gas

- Apply a standard procedure to de-seasonalize gas (don't change notation).
- $\log E_t$ and $\log G_t$ are normally distributed.
- Thus, we can use standard Maximum Likelihood Methods.

Power I

The estimation procedure for the power price includes several steps:

- Estimation of the seasonal trend and deseasonalisation.
- With an iterative procedure we filter out returns with absolute values greater than three times the standard deviation of the returns of the series at the current iteration. The process is repeated until no further outliers can be found.
- As a result we obtain a standard deviation of the jumps, σ_j, and a cumulative frequency of jumps, l. The latter is defined as the total number of filtered jumps divided by the annualised number of observations.

Power II

- Once we have filtered the X_t process, we can identify it as a first order autoregressive model in continuous time, i.e. so-called AR(1) process. Discretizing the process and estimating it by maximum likelihood method (MLE) yields the estimates.

Estimation Results

Estimation Step	Product	Estimates	Method
GBM	Emissions	$\alpha^E = -0.2843$, $\sigma^E = 0.4079$	MLE
Seasonal trend	Power	$a_1 = 3.6716$, $a_2 = 0.0980$, $a_3 = -0.0274$ $a_4 = 0.0368$, $a_5 = 0.6524$, $a_6 = 0.9530$	OLS
Seasonal trend	Gas	$b_1 = 2.3420$, $b_2 = 0.3503$, $b_3 = 0.0218$ $b_4 = -0.0445$, $b_5 = 0.7829$, $b_6 = 1.6126$	OLS
Filtering	Power		$3\times$Std.Dev rule
Base process Base process	Gas Power	$\alpha^G = 13.5827$, $\sigma^G = 0.7768$ $\alpha^P = 121.8684$, $\sigma^P = 2.5943$, $\rho = 0.1247$	Multivariate normal regression
Spike mean-reversion Spike intensity Spike size (Laplace) Spike size (normal)	Power Power Power Power	$\beta = 243.7240$ $\lambda = 13.4936$ $\mu_s(median) = 0.3975$, $\sigma_s(scale) = 0.6175$ $\mu_s(mean) = 0.0863$, $\sigma_s(variance) = 0.5857$	 Annual frequency MLE MLE
Heat rate	Gas	$h = 2.5$	
Interest rate		$r = 3\%$	

We will be capturing model risk in

- Jump size distribution;
- Correlation;
- Gas alone;
- Gas and power base signal;
- Gas, power and emissions (all the parameters, except of jump size).

General Procedure

- We reduce the problem here by considering the distributions of the single parameters separately (e.g. the correlation coefficient, the jump size distribution parameters). Hence, we do some kind of "sensitivity analysis" w.r.t. different parameters, disregarding the remaining parameter risk.
- Each parameter θ_j is to be estimated by an estimator $\hat{\theta}_j(X_1, \dots, X_N)$ under the real-world measure and we assume the other parameters $\theta_1, \dots, \theta_{j-1}, \theta_{j+1}, \theta_N$ to be known. We use plug-in estimators as the true values and figure out the asymptotic distribution of the estimators.
- We calculate the parameter risk-captured prices which are generated by the Average-Value-at-Risk (AVaR) w.r.t. different significance levels $\alpha \in (0, 1]$.

Spark Spread Analysis I

In our investigation we will focus on the clean spark spread to model the value of virtual gas power plant. We will use spot price processes in order to assess the day-by-day risk position of such a plant. Thus, we will model the daily profit (or loss) of a power plant as

$$V_t = \max\{P_t - h\,G_t - c_E\,E_t, 0\}, \tag{17}$$

where P_t is the power price, G_t is the gas price, E_t is the carbon certificate price, h is the heat rate, c_E emission conversion rate.

Spark Spread Analysis II

- We compute the spark spread value V_t given in (17) for every day t for a time period of three years.
- The general formula is

$$VPP(t,T) = \int_t^T e^{-r(s-t)}\, V(s)\,\mathrm{d}s,$$

with *i* referring to the simulation run.

- Then, by fixing all the parameters except of one (e.g. correlation) and setting the shift value (e.g. 1%), we compute shifted up and down spark spread values, i.e. V_t^{up} and V_t^{down}.

Power Plant Analysis I

We compute the value of the power plant (VPP) by means of Monte Carlo simulations. For a fixed large number N and a fixed period $T = 3$ years we have

$$VPP(t, T) = \frac{1}{N} \sum_{i=1}^{N} VPP_i(t, T),$$

where

$$VPP_i(t, T) = \sum_{s=t}^{T} e^{-r(T-s)} V_i(s).$$

Power Plant Analysis II

- We also compute shifted both up and down power plant values, i.e. $VPP^{up}(t,T)$ and $VPP^{down}(t,T)$ (i.e. w.r.t. shifted spark spread values) and calculate the sensitivity

$$sVPP(\theta_0) = \frac{VPP^{up}(t,T) - VPP^{down}(t,T)}{2 \cdot shift}.$$

- Finally, we compute the bid and ask prices, i.e. we use the closed formula for AVaR to get the risk-captured prices by subtracting and adding risk-adjustment value to $VPP(t,T)$ respectively.
- For a specified significance level $\alpha \in (0,1)$ this risk-adjustment value is computed as

$$\frac{\varphi(\Phi^{-1}(\alpha))}{\alpha}\sqrt{\frac{sVPP(\theta_0)' \cdot \Sigma \cdot sVPP(\theta_0)}{N}}.$$

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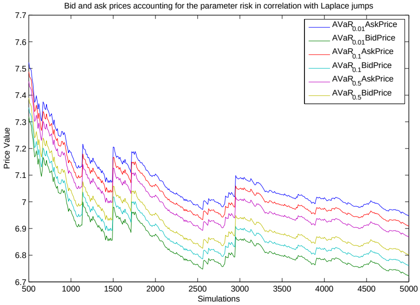
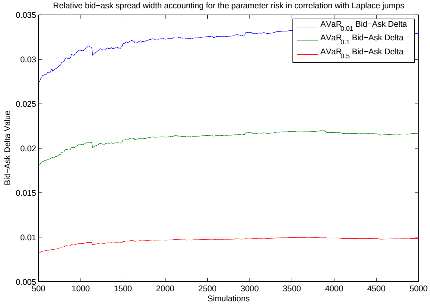
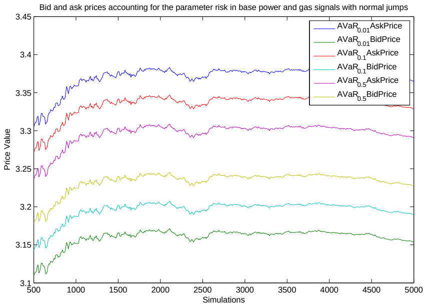
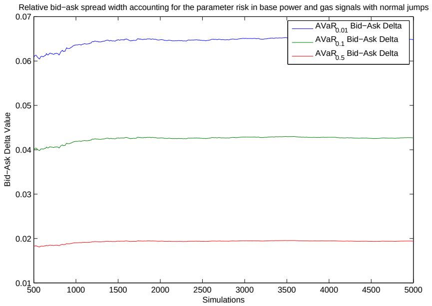
Parameter-risk implied bid-ask spread w.r.t. correlation parameter, Laplace jumps.



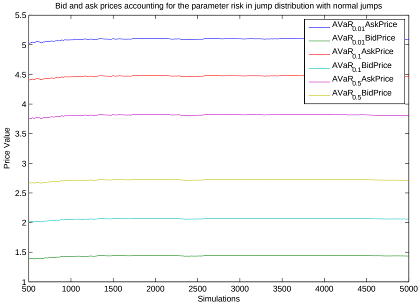


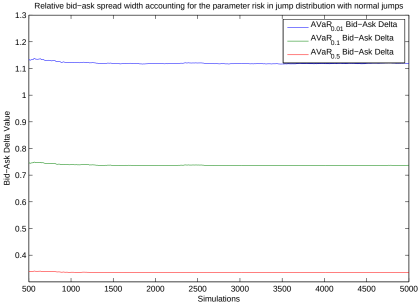
Parameter-risk implied bid-ask spread w.r.t. the gas and power base processes, Gaussian jumps.



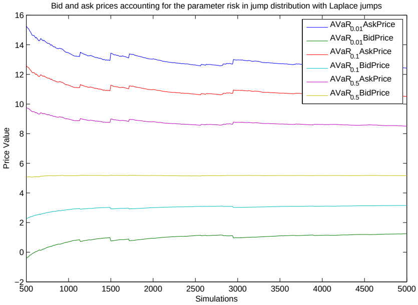


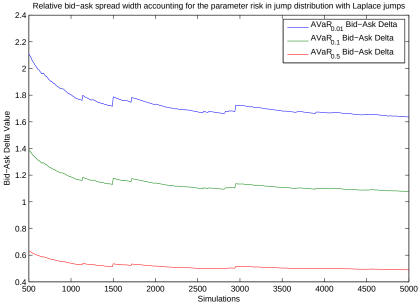
Parameter-risk implied bid-ask spread w.r.t. jump size distribution: Gaussian.





Parameter-risk implied bid-ask spread w.r.t. jump size distribution: Laplace.



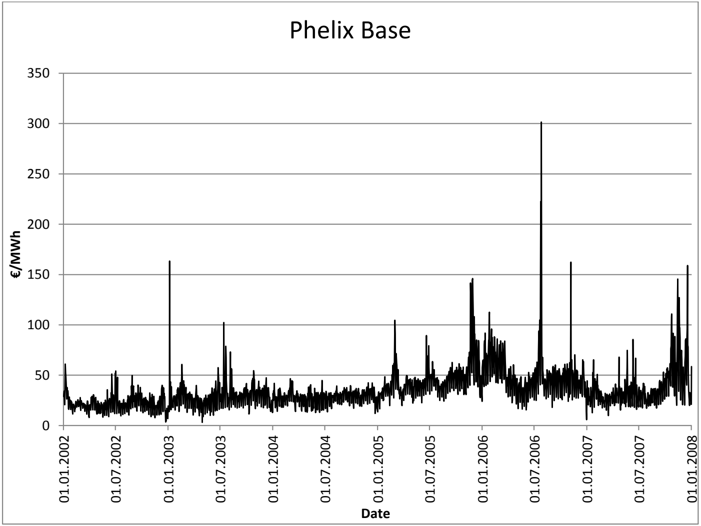


Resulting values for the relative width of the bid-ask spread for various model risk sources. α_1 = 0.01, α_2 = 0.1, α_3= 0.5.

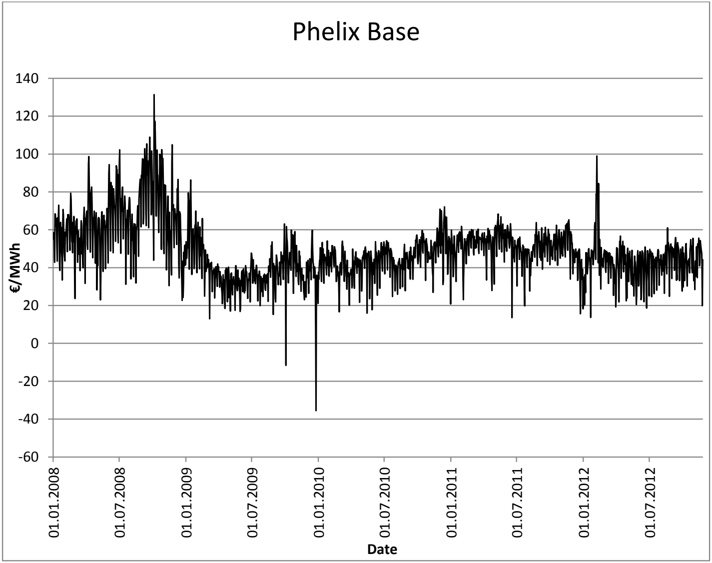
	Jumps size distribution					
	Gaussian			Laplace		
Model Risk	α_1	α_2	α_3	α_1	α_2	α_3
Jumps	111.9%	73.71%	33.51%	163.5%	107.7%	48.96%
Correlation	6.95%	4.58%	2.08%	3.29%	2.17%	0.99%
Gas and power base	6.48%	4.27%	1.94%	3.07%	2.02%	0.92%
Gas	6.11%	4.03%	1.83%	2.89%	1.91%	0.87%
Gas, power and carbon	8.21%	5.41%	2.46%	3.83%	2.52%	1.15%

Gas Power Plant

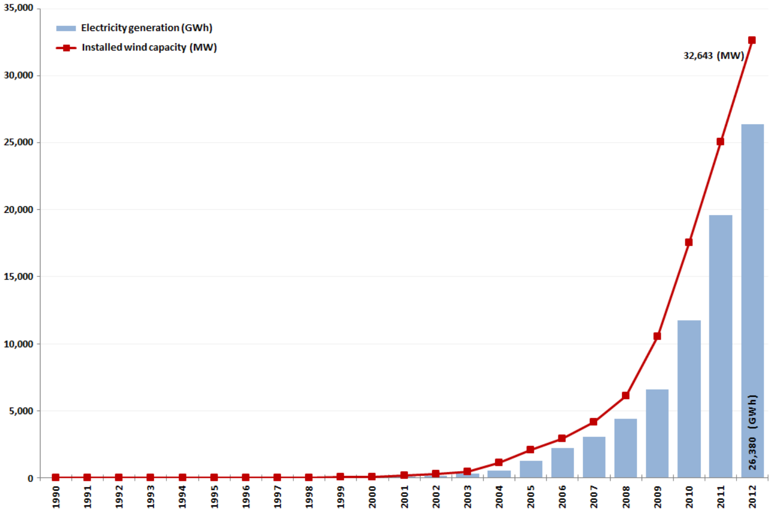
Phelix Base 2002-2008



Phelix Base 2008-2012

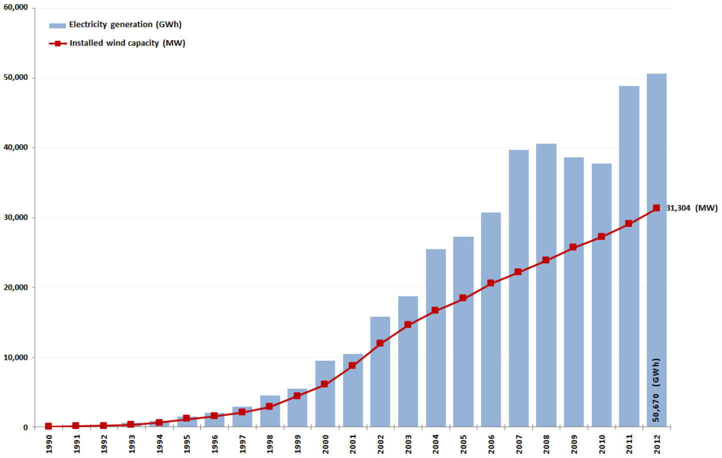


Photovoltaik



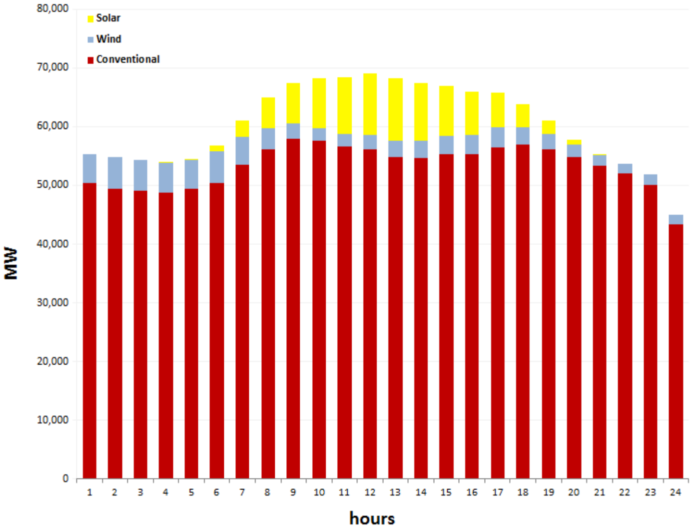
Source: Federal Ministry for the Environment, Nature Conservation and Nuclear Safety, 2013.

Wind



Source: Federal Ministry for the Environment, Nature Conservation and Nuclear Safety, 2013.

A day in July


Solar
Wind
Conventional
80,000
70,000
60,000
50,000
40,000
30,000
20,000
10,000
0
MW
1
2
3
4
5
6
7
8
9
10
11
12
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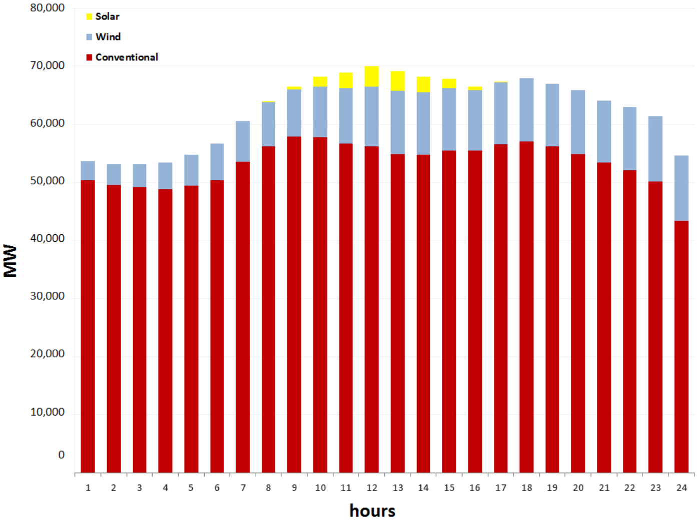
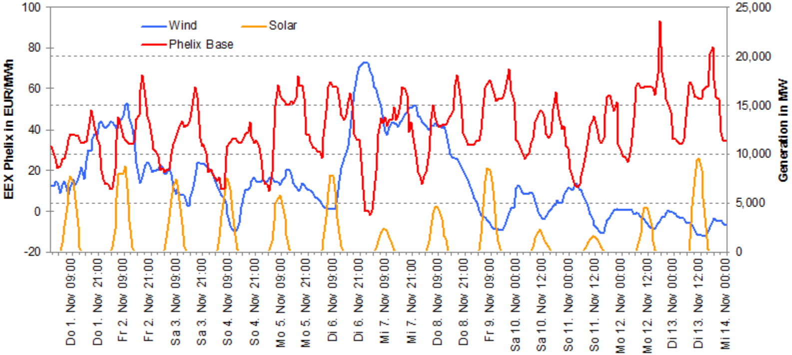
15
16
17
18
19
20
21
22
23
24
hours

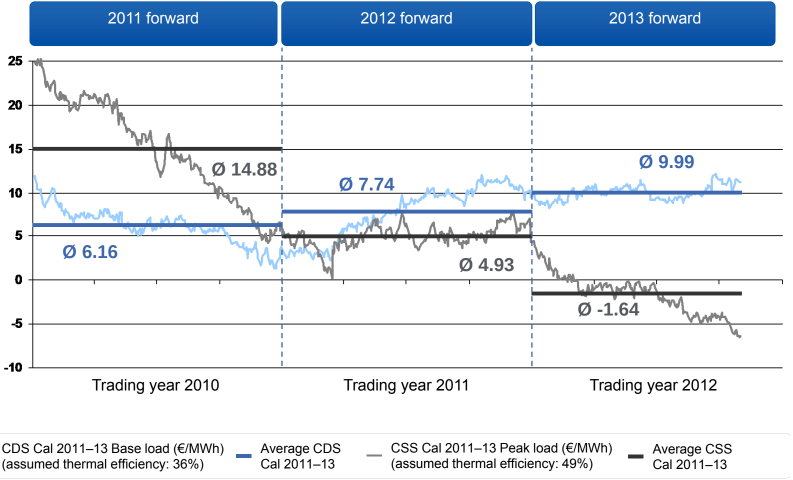
A day in December



Wind, sun and electricity

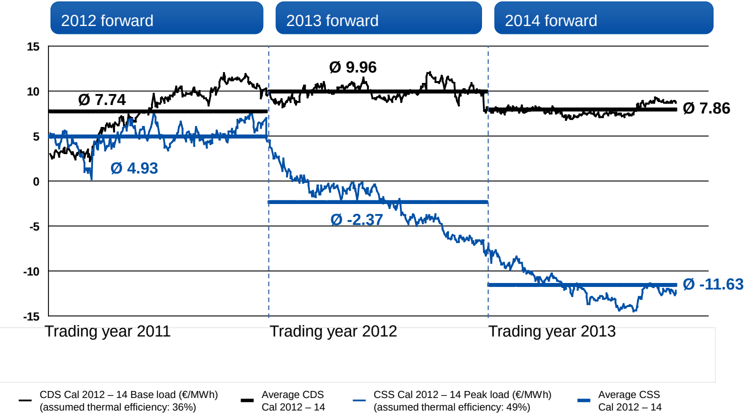


Does it get better?



..or worse?

Germany: Clean Dark (CDS) and Spark Spreads (CSS)



Source: RWE Supply & Trading, prices until 08 November 2013

RWE Response 14.August 2013

Decision on capacity measures

Measure	Plant	MW[1]	Fuel	Location	Date
Decom-missioning	Amer 8	610	Hard coal	NL	Q1-2016[2]
Long-term mothballing	Moerdijk 2	430	Gas	NL	Q4-2013
	Gersteinwerk F	355	Gas – steam turbine	DE	Q3-2013
	Gersteinwerk G	355	Gas – steam turbine	DE	Q2-2014
	Weisweiler H	270	Topping gas turbine[3]	DE	Q3-2013
	Weisweiler G	270	Topping gas turbine[3]	DE	Q3-2013
	2 mid-size units	85	Gas	NL	Q1-2013
Summer mothballing	Emsland B	360	Gas – steam turbine	DE	Q2-2014
	Emsland C	360	Gas – steam turbine	DE	Q2-2014
Termination of 3 contracts	Confidential	1,170	Hard coal	DE	Q4-2013 – Q4-2014
Total		**4,265 MW**			

[1] Net nominal capacity | [2] Depending on the final decision on the Dutch “Energieakkoord”, with a decision expected by the end of August 2013 | [3] At a lignite plant

RWE The energy to lead

RWE AG | H1 2013 Conference Call | 14 August 2013 5

MSc Financial Engineering – Pricing II

Lecture Series Spring 2014

Professor Dr. Rüdiger Kiesel | Birkbeck College | University of Duisburg-Essen

Part III

Model Validation and Model Comparison

Model Comparison

Credit Risk Models

- Credit Default Swap (CDS)
- Structural Models
- Reduced Form Models

Assessing the Gap Risk

How to compare models?

We may have two "equally sound" but different models, which give considerably different prices to derivatives (see Schoutens).

- Reasonableness: the models should capture the relevant factors which are relevant for the payoff, model assumptions should be realistic.
- Calibration: use liquid markets which contain relevant information for pricing the payoff (Schoutens: plain vanilla options). Models should produce results consistent with market practice.

Comparing equally sound models

What are the crucial risk factors for the payoff to be priced (these are more important than the relevant factors)?

- Reality Check: analyse the crucial risk factors to decide which ones will be important for future price movements. This is not so important in case we do not have a liquid market and intend to keep a derivative until maturity.
- Market Intelligence: perform reverse engineering on prices from other market participants we observe in order to judge the market opinion.

Example: Leveraged Credit Default Note

- The banks sells credit default protection to a client (notational 1).
- For a leveraged credit default note the banks sells protection for a notational $l_v > 1$.
- Bank pays $l_v \times S_T(0)$, with $S_T(0)$ the spread for the reference entity for maturity at T.
- At maturity clients receives notational in case of no default. Otherwise, the client will receive the notational minus the leveraged default loss (nothing if this is below 0).

Gap Risk

- Which fee should the client pay to the bank?
- In case of default the clients loss is not larger than the notational 1, but the loss may be higher $l_v \times l_d$, with l_d the loss-given-default.
- The bank has to cover a gap $l_v \times l_d - 1$.
- Pricing the gap risk by risk neutral valuation leads to

$$\mathbb{E}\left[\mathbf{1}_{\{\tau_d < T\}} p(0, \tau_d)(l_v \times l_d - 1)^+\right],$$

with τ_d the default time.

Specifying a Trigger

- To mitigate the gap risk and thus reduce the price of the leveraged note a trigger can be introduced: the contract is terminated early if $S_T(t)$, the spread of the CDS of the reference entity touches a certain level s_{tr}.
- We set $\tau_{tr} = \inf\{t : S_T(t) \geq s_{tr}\}$, then the leveraged note is unwind at $\tau_{term} = \min\{\tau_{tr}, \tau_d\}$.
- The bank can try to set s_{tr} in such a way that the cost of unwinding the leveraged note is just the notational, i.e. there should be no gap risk.

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Definition

A credit default swap is an exchange of a periodic payment against a one-off contingent payment if some credit event occurs on a reference asset.

$$\text{Protection Buyer} \quad \underset{\text{periodic fee}}{\overset{\text{contingent payment}}{\begin{matrix}\longleftarrow \\ \longrightarrow\end{matrix}}} \quad \text{Protection Seller}$$

Structure

The ingredients of the basic structure are the specification of

- maturity T: usually from one to ten years,
- underlying: corporate or sovereign,
- credit event: default, bankruptcy, downgrade.

Valuation

- Assume a deterministic term structure of interest rates with short rate $r > 0$.
- Let $S_T(0)$ be the fixed coupon that the protection buyer pays at coupon dates $t_i, i = 1, \ldots, n$; $t_n = T$.
- The payment continues until either default or maturity. In case of default, assume that the payment from the protection seller to the protection buyer is equal to the difference between the notional amount of the bond and the recovery value δ, i.e. the loss given default is paid.
- The fixed side of the payment is set so that contract value is zero at initiation.

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Valuation

Thus, since the cash flow at coupon date t_i for the protection buyer is $S_T(0)\mathbf{1}_{\{\tau>t_i\}}$ and the payment for the protection seller at time of default τ is $(1-\delta)\mathbf{1}_{\{\tau\leq T\}}$, we obtain

$$(1-\delta)\mathbb{E}^*\left(e^{-r\tau}\mathbf{1}_{\{\tau\leq T\}}\right)=\sum_{i=1}^{n}e^{-rt_i}S_T(0)\mathbb{E}^*\left(\mathbf{1}_{\{\tau>t_i\}}\right),$$

and so

$$S_T(0)=\frac{(1-\delta)\mathbb{E}^*\left(e^{-r\tau}\mathbf{1}_{\{\tau\leq T\}}\right)}{\sum_{i=1}^{n}e^{-rt_i}\mathbb{E}^*\left(\mathbf{1}_{\{\tau>t_i\}}\right)}.$$

Spread Calculation

- At a time $t_{j-1} < t \leq t_j$ we have

$$S_T(t) = \frac{(1-\delta)\mathbb{E}^*\left(e^{-r\tau}\mathbf{1}_{\{\tau \leq T\}}|\{\tau > t\}\right)}{\sum_{i=j}^{n} e^{-rt_i}\mathbb{E}^*\left(\mathbf{1}_{\{\tau > t_i\}}|\{\tau > t\}\right)}.$$

- We need to obtain

$$\mathbb{P}^*\left(\tau \leq T \,|\tau > t\right)$$

and

$$\mathbb{P}^*\left(\tau > t_i|\tau > t\right)$$

within our models.

- So, to evaluate the gap risk these quantities are crucial.

Basic Merton Model

Merton (1974) assumes that a firm is financed by equity and a single zero-coupon bond with notational amount (face value) F and maturity T.
The firm's value is given by

$$dV(t) = (r - \delta)V(t)dt + \sigma V(t)dW(t)$$

under an equivalent martingale (pricing) measure $\mathbb{P}^*$, with r, σ constant, W Brownian motion and constant payout (dividend) rate δ, which may be negative (i.e. pay-in).

Basic Merton Model

Default is only possible at maturity. There are two possibilities:

$$V_T \geq F, \quad \text{thus} \quad p^d(T,T) = F$$

or

$$V_T < F, \quad \text{thus} \quad p^d(T,T) = V_T.$$

Equity Owners

For equity owners the payoff is

$$S_T = \max\{V_T - F, 0\}$$

thus stocks can be viewed as call options on the value of the firm with ($\bar{V}_t = e^{-\delta(T-t)} V_t$)

$$S_t = C_E(\bar{V}_t, F) = e^{-\delta(T-t)} V_t \Phi(d_1) - F e^{-r(T-t)} \Phi(d_2)$$

and

$$d_1 = \frac{\log(\bar{V}_t/F) + (r - \delta + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_2 + \sigma\sqrt{T-t}.$$

Bond Owners

For bond owners the payoff is

$$p^d(T,T) = F - \max\{F - V_T, 0\}.$$

This can be viewed as the difference of a risk-free payment and a put option on the value of the firm with

$$p^d(t,T) = Fe^{-r(T-t)} - P_E(\bar{V}_t, F),$$

where

$$P_E(\bar{V}_t, F) = -V_t e^{-\delta(T-t)}\Phi(-d_1(\bar{V}_t, T-t)) + Fe^{-r(T-t)}\Phi(-d_2(\bar{V}_t, T-t))$$

So

$$p^d(t,T) = V_t e^{-\delta(T-t)}\Phi(-d_1(\bar{V}_t, T-t)) + Fe^{-r(T-t)}\Phi(d_2(\bar{V}_t, T-t)).$$

Bond Credit Spreads

For spreads $S(0,T)$ we use $p^d(0,T) = e^{-(r+S(0,T))T}$ and find

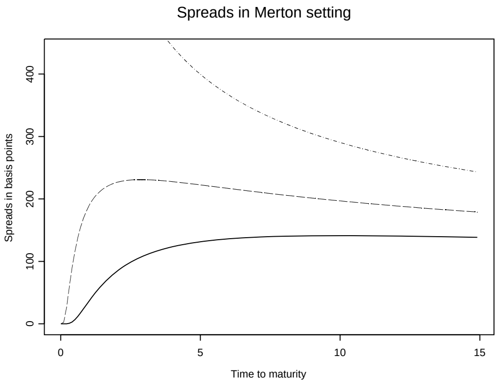
$$S(0,T) = \frac{1}{T} \log \left\{ \frac{1}{l_0} \Phi(-d_1) + \Phi(d_2) \right\},$$

with

$$l_t = \frac{Fe^{-rt}}{V_t}$$

the leverage ratio.

Calculated Bond Credit Spreads



Counting Process

Define the stopping time (= time of default)

$$\tau = \inf\{t : X(t) = D\}$$

and the counting process (Poisson)

$$N(t) = \begin{cases} 0 & \text{if } \tau > t \\ 1 & \text{if } \tau \leq t \end{cases}$$

$N(t)$ is the number of defaults up to and including time t.

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Calculation in Standard Model

Let r_t be the (stochastic) short rate, δ the (stochastic, independent) recovery rate. The valuation formula for a corporate bond is

$$p^d(t,T) = \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \left(\mathbf{1}_{\{\tau > T\}} + \delta \mathbf{1}_{\{\tau \le T\}} \right) \Big| \mathcal{F}_t \right]$$

($\mathcal{F}_t$ is the financial market filtration with all relevant information)

$$= \mathbb{E}^* \left[e^{-\int_t^T r_s ds} \left(\delta + (1-\delta) \mathbf{1}_{\{\tau > T\}} \right) \Big| \mathcal{F}_t \right]$$

using independence and the properties of the counting process

$$= \mathbb{E}^* \left[\delta e^{-\int_t^T r_s ds} \Big| \mathcal{F}_t \right] + \mathbb{E}^* \left[(1-\delta) e^{-\int_t^T (r_s + \lambda_s) ds} \Big| \mathcal{F}_t \right]$$

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Calculation in Standard Model II

Now $\delta = 0$ i.e. zero recovery

$$p^d(t, T) = \mathbb{E}^* \left[e^{-\int_t^T (r_s + \lambda_s) ds} \middle| \mathcal{F}_t \right]$$

with $r_s + \lambda_s$ a default-adjusted rate.
Constant (or independent) interest rate gives

$$p^d(t, T) = p(t, T) \mathbb{E}^* \left[e^{-\int_t^T \lambda_s ds} \middle| \mathcal{F}_t \right].$$

Survival Probability

The standard "memory-less" property of exponential random variables gives for deterministic intensity

$$\mathbb{P}(\tau > T | \tau > s) = e^{-\int_s^T \lambda_u du}.$$

This coincides with the forward survival probabilities at time $t = 0$

$$\frac{\mathbb{P}(\tau > T)}{\mathbb{P}(\tau > s)} = e^{-\int_s^T \lambda_u du}.$$

Structural Model

- In Merton's model the survival probability is dependent on the firm's value process and the barrier level (induced by the trigger).

$$V(t) \quad \downarrow \quad \Rightarrow \mathbb{P}(\tau > s|V(t)) \quad \downarrow .$$

- So $S_T(t)$ increases as

$$S_T(t) = \frac{l_d e^{-r\tau}\mathbb{P}^*\left(\tau \leq T|\tau > t\right)}{\sum_{i=1}^{n} e^{-rt_i}\mathbb{P}^*\left(\tau > t_i|\tau > t\right)} = \frac{l_d e^{-r\tau}\mathbb{P}^*\left(\tau \leq T|V_t\right)}{\sum_{i=1}^{n} e^{-rt_i}\mathbb{P}^*\left(\tau > t_i|V_t\right)}.$$

- No gap risk

Reduced Form Model

- For deterministic intensity we know the behaviour of the spread in advance!
- Even with stochastic intensity the default event and the spread level show little correlation.
- So, in any case – maximal gap risk!