Financial Mathematics

Lecture - Winter 2013/14

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Part I

Derivate

Basic Derivatives

- Options
- Forwards and Futures
- Swaps

Option Pricing

- Binomial Tree Models
- The Black-Scholes-Merton Model

Derivative Background

A derivative security, or contingent claim, is a financial contract whose value at expiration date T (more briefly, expiry) is determined exactly by the price (or prices within a prespecified time-interval) of the underlying financial assets (or instruments) at time T (within the time interval $[0, T]$).
Derivative securities can be grouped under three general headings: *Options, Forwards and Futures* and *Swaps*. During this lectures we will encounter all this structures and further variants.

Modelling Assumptions

We impose the following set of assumptions on the financial markets:

- *No market frictions:* No transaction costs, no bid/ask spread, no taxes, no margin requirements, no restrictions on short sales.
- *No default risk:* Implying same interest for borrowing and lending
- *Competitive markets:* Market participants act as price takers
- *Rational agents* Market participants prefer more to less

Arbitrage

The concept of arbitrage lies at the centre of the relative pricing theory. All we need to assume additionally is that economic agents prefer more to less, or more precisely, an increase in consumption without any costs will always be accepted.
The essence of the technical sense of arbitrage is that it should not be possible to guarantee a profit without exposure to risk. Were it possible to do so, arbitrageurs (we use the French spelling, as is customary) would do so, in unlimited quantity, using the market as a 'money-pump' to extract arbitrarily large quantities of riskless profit.
We assume that arbitrage opportunities do not exist!

Options

An option is a financial instrument giving one the *right but not the obligation* to make a specified transaction at (or by) a specified date at a specified price. *Call* options give one the right to buy. *Put* options give one the right to sell. *European* options give one the right to buy/sell on the specified date, the expiry date, on which the option expires or matures. *American* options give one the right to buy/sell at any time prior to or at expiry.

Options

The simplest call and put options are now so standard they are called *vanilla* options.
Many kinds of options now exist, including so-called *exotic* options. Types include: *Asian* options, which depend on the *average* price over a period, *lookback* options, which depend on the *maximum* or *minimum* price over a period and *barrier* options, which depend on some price level being attained or not.

Terminology

The asset to which the option refers is called the *underlying asset* or the *underlying*. The price at which the transaction to buy/sell the underlying, on/by the expiry date (if exercised), is made, is called the *exercise price* or *strike price*. We shall usually use K for the strike price, time $t = 0$ for the initial time (when the contract between the buyer and the seller of the option is struck), time $t = T$ for the expiry or final time. Consider, say, a European call option, with strike price K; write $S(t)$ for the value (or price) of the underlying at time t. If $S(t) > K$, the option is *in the money*, if $S(t) = K$, the option is said to be *at the money* and if $S(t) < K$, the option is *out of the money*.

Payoff

The payoff from the option is

$$S(T) - K \text{ if } S(T) > K \quad \text{and} \quad 0 \text{ otherwise}$$

(more briefly written as $(S(T) - K)^+$).
Taking into account the initial payment of an investor one obtains the profit diagram below.

Payoff

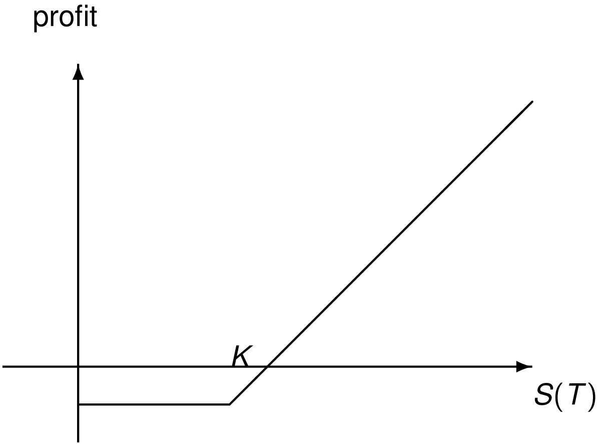


Figure : Profit diagram for a European call

Underlying Securities

- We will mainly use Commodities or Commodity Futures;
- Fixed income instruments: T-Bonds, Interest Rates (LIBOR, EURIBOR);
- Other classes are possible: (one or several) stocks; Currencies (FX);
- Also Derivatives may be used as underlying for compound derivatives (call on call).

Arbitrage Relationship- Example

We now use the principle of no-arbitrage to obtain bounds for option prices. We focus on European options (puts and calls) with identical underlying (say a stock S), strike K and expiry date T. Furthermore we assume the existence of a risk-free bank account (bond) with constant interest rate r (continuously compounded) during the time interval $[0, T]$. We start with a fundamental relationship:
We have the following put-call parity between the prices of the underlying asset S and European call and put options on stocks that pay no dividends:

$$S_t + P_t - C_t = Ke^{-r(T-t)}. \tag{1}$$

Arbitrage Relationship - Example

Consider a portfolio consisting of one stock, one put and a short position in one call (the holder of the portfolio has written the call); write $V(t)$ for the value of this portfolio. Then

$$V(t) = S(t) + P(t) - C(t)$$

for all $t \in [0, T]$. At expiry we have

$$\begin{aligned} V(T) &= S(T) + (S(T) - K)^- - (S(T) - K)^+ \\ &= S(T) + K - S(T) = K. \end{aligned}$$

This portfolio thus guarantees a payoff K at time T. Using the principle of no-arbitrage, the value of the portfolio must at any time t correspond to the value of a sure payoff K at T, that is $V(t) = Ke^{-r(T-t)}$. ■

Basic Structure

- A *forward contract* is an agreement to buy or sell an asset S at a certain future date T for a certain price K.
- The agent who agrees to buy the underlying asset is said to have a *long* position, the other agent assumes a *short* position.
- The settlement date is called *delivery date* and the specified price is referred to as *delivery price*.

Forwards

- The *forward price* $F(t, T)$ is the delivery price which would make the contract have zero value at time t.
- At the time the contract is set up, $t = 0$, the forward price therefore equals the delivery price, hence $F(0, T) = K$.
- The forward prices $F(t, T)$ need not (and will not) necessarily be equal to the delivery price K during the life-time of the contract.

Forwards

- The payoff from a long position in a forward contract on one unit of an asset with price $S(T)$ at the maturity of the contract is
$$S(T) - K.$$
- Compared with a call option with the same maturity and strike price K we see that the investor now faces a downside risk, too. He has the obligation to buy the asset for price K.

Futures

- Futures can be defined as standardized forward contracts traded at exchanges where a clearing house acts as a central counterparty for all transactions.
- Usually an initial margin is paid as a guarantee.
- Each trading day a settlement price is determined and gains or losses are immediately realized at a margin account.
- Thus credit risk is eliminated, but there is exposure to interest rate risk.

Swaps

A *swap* is an agreement whereby two parties undertake to exchange, at known dates in the future, various financial assets (or cash flows) according to a prearranged formula that depends on the value of one or more underlying assets. Examples are currency swaps (exchange currencies) and interest-rate swaps (exchange of fixed for floating set of interest payments).

Example: Fixed for Floating Interest Rate Swap

Consider the case of a *forward swap settled in arrears* characterized by:

- a fixed time t, the contract time,
- dates $T_0 < T_1, \ldots < T_n$, equally distanced $T_{i+1} - T_i = \delta$,
- R, a prespecified fixed rate of interest,
- K, a nominal amount.

Example: Fixed for Floating Interest Rate Swap

A swap contract S with K and R fixed for the period $T_0, \dots T_n$ is a sequence of payments, where the amount of money paid out at $T_{i+1},\ i = 0, \dots, n-1$ is defined by

$$X_{i+1} = K\delta(L(T_i, T_i) - R).$$

The floating rate over $[T_i, T_{i+1}]$ observed at T_i is a simple rate defined as

$$p(T_i, T_{i+1}) = \frac{1}{1 + \delta L(T_i, T_i)}.$$

A fundamental example

We consider a one-period model, i.e. we allow trading only at $t = 0$ and $t = T = 1$(say).

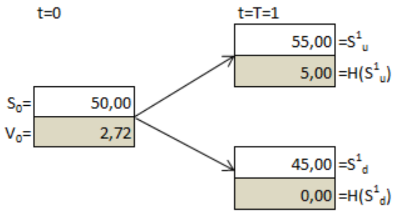
- Our aim is to value at $t = 0$ a European contingent claim on a stock S with maturity T.
- The payoff H of the instrument is a function f of the stock price at T. For a European call option with strike K, we would have $H = f(S_T) = (S_T - K)^+$.
- We assume that stocks do not pay dividends.
- Investors can borrow and lend at the risk-free rate r.

Replication: The Black-Scholes-Merton Approach

If we can find a *replicating portfolio* of instruments with known prices that has the same payoff as the contingent claim H in every state of the world, the value of the two positions has to be equal by the no arbitrage assumption.

An Example I

We calculate the price of a European call option on a stock S with strike $K = 50$ and maturity in $T = 1$ in a one-period model. The stock price today is 50 and can move up or down by 10%.



An Example II

In this example, we use $r = 1\%$. We try to replicate the option with investments of x in the stock and y in the risk-free bank account.

$$
\begin{aligned}
5 &= x \cdot 55 + y \cdot 1.01 \\
0 &= x \cdot 45 + y \cdot 1.01 \\
\Rightarrow x &= 0.5, y = -22.277
\end{aligned}
$$

The value of our investment today, and, by no arbitrage, the value of the option today, is

$$V_0 = 0.5 \cdot 50 - 22.277 \cdot 1 = 2.72$$

Risk-Neutral Valuation

- In the example above, we did not use any assumption about the risk preferences of investors and we did not need the probability with which the stock moves up or down.
- Idea: Pretend that investors are indifferent about risk. Then, every asset has to earn an expected return equal to the return of the risk-free investment (if there is no arbitrage). Find the probabilities for up- and down-movements of the stock in such a world and use them to find the price of the option.

An Example III

In the setup from our example, the risk-free rate of return is 1%. Therefore,

$$
\begin{aligned}
1.01 &\overset{!}{=} \text{prob}_{up} \cdot 1.1 + \text{prob}_{down} \cdot 0.9 \\
\Leftrightarrow 1.01 &\overset{!}{=} \text{prob}_{up} \cdot 1.1 + (1 - \text{prob}_{up}) \cdot 0.9 \\
\Leftrightarrow \text{prob}_{up} &= 55\%
\end{aligned}
$$

We get the price of the option

$$V_0 = (0.55 \cdot 5 + (1 - 0.55) \cdot 0)/1.01 = 2.72$$

Risk-Neutral Valuation for Options

The probabilities found in this way are called “risk-neutral” probabilities. They define a probability measure, the “risk-neutral” or “pricing” measure.

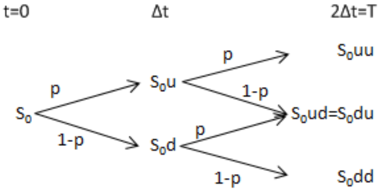
The price $V(t)$ at time t of an option with payoff $P(S_T)$ at time T is

- the expectation
- under the risk-neutral measure $\mathbb{Q}$
- of the discounted
- payoff.

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}P(S_T)\right]$$

The Cox-Ross-Rubinstein (CRR) Model

The Cox-Ross-Rubinstein model is a binomial tree model. The figure below shows a two-period tree, but it can easily be extended to multiple periods.



CRR Model: Choice of Parameters

The parameters u and d that control the movements of the stock in a binomial model can be chosen to match the volatility σ of the stock. In the CRR model, we have:

- $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$
- To match the expected return μ in the real world, the probability of an up movement has to be $p^* = \frac{e^{\mu\Delta t} - d}{u - d}$.
- For pricing, we need the risk neutral probabilities that can be derived as in the example. For an up movement, we get: $p = \frac{e^{r\Delta t} - d}{u - d}$.

Convergence

In a one-period binomial model, we only consider two points in time, $t = 0$ and $t = T$. This approximation is rather rough as it allows only two states of the world in $t = T$. We can add realism by dividing the interval $[0, T]$ into more steps. If we add more and more steps, the results get closer and closer to the results from the Black-Scholes model that we consider next. Mathematically speaking, the CRR model converges to the Black-Scholes model as the number of time steps in the interval increases to infinity.

The Black-Scholes-Merton Model

In 1973, Black and Scholes (1973) and Merton (1973) developed the Black-Scholes (or Black-Scholes-Merton) model which was a major breakthrough in option valuation. It can be derived in different ways:

- As the limit of the CRR model;
- Via the Black-Scholes partial differential equation (PDE);
- Via risk-neutral pricing.

For the second and third method, we need the additional assumption that stock prices follow a geometric Brownian motion with constant drift and volatility. This is not needed for the first method because there is an implicit distributional assumption in the CRR model.

The Black-Scholes Formula

The time-t price $c(t)$ of a European call option with strike K and maturity T on a non-dividend-paying stock S with volatility σ in the Black-Scholes model with risk-free interest rate r is

$$c(t) = S_t\Phi(d_1) - e^{-r(T-t)}K\Phi(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

where $\Phi(.)$ denotes the cumulative distribution function (cdf) of the standard normal distribution.

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Example: Option Prices in the Black Scholes Model

Find the price of a European call option in the BS model with strike $K = 100$, time to maturity 1 year, on a stock with time-0 price 100. The risk free rate is 5% p.a., the volatility of the stock is 20% p.a.

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{100}{100}\right) + \left(0.05 + \frac{0.2^2}{2}\right)(1-0)}{0.2\sqrt{1-0}} \\ &= 0.35 \\ d_2 &= 0.35 - 0.2 * \sqrt{1-0} \\ &= 0.15 \\ c(0) &= 100N(0.35) - e^{-0.05\cdot 1}100N(0.15) \\ &= 10.45. \end{aligned}$$

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Part II

Stochastic Analysis for Black-Scholes

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Brownian Motion - I

Definition

A stochastic process $X = (X(t))_{t \geq 0}$ is a standard Brownian motion, *BM*, if

(i) $X(0) = 0$ a.s.,

(ii) X has *independent increments*: $X(t+u) - X(t)$ is independent of $\sigma(X(s) : s \leq t)$ for $u \geq 0$,

(iii) X has *stationary increments*: the law of $X(t+u) - X(t)$ depends only on u,

and (iv), (v)

Brownian Motion - II

Definition

A stochastic process $X = (X(t))_{t \geq 0}$ is a standard Brownian motion, *BM*, if (i) – (iii) and

(iv) X has *Gaussian increments*: $X(t+u) - X(t)$ is normally distributed with mean 0 and variance u, $X(t+u) - X(t) \sim \Phi(0, u)$,

(v) X has *continuous paths*: $X(t)$ is a continuous function of t, i.e. $t \to X(t, \omega)$ is continuous in t for all $\omega \in \Omega$.

Brownian Motion

- We have Wiener's theorem:

Theorem (Wiener)

Brownian motion exists.

- We denote standard Brownian motion $BM(\mathbb{R})$ by $W = (W(t))$ (W for Wiener), though $B = (B(t))$ (B for Brown) is also common.
- Standard Brownian motion $BM(\mathbb{R}^d)$ in d dimensions is defined by $W(t) := (W_1(t), \ldots, W_d(t))$, where $W_1, \ldots, W_d$ are independent standard Brownian motions in one dimension (independent copies of $BM(\mathbb{R})$).

Geometric BM

- Consider how a stock $S(t)$ will change in some small time-interval from the present time t to a time $t + dt$ in the near future. With $dS(t) = S(t + dt) - S(t)$, the *return* on S in this interval is $dS(t)/S(t)$.
- It is economically reasonable to expect this return to decompose into two components, a *systematic* part and a *random* part. The systematic part could plausibly be modelled by μdt, where μ is some parameter representing the mean rate of return of the stock.
- The random part could plausibly be modelled by $\sigma dW(t)$, where $dW(t)$ represents the noise term driving the stock price dynamics, and σ is a second parameter describing how much effect this noise has - how much the stock price fluctuates. Thus σ governs how volatile the price is, and is called the *volatility* of the stock.

Geometric BM

Putting this together, we have the stochastic differential equation

$$dS(t) = S(t)(\mu dt + \sigma dW(t)), \quad S(0) > 0, \tag{2}$$

due to Itô in 1944. This corrects Bachelier's earlier attempt of 1900 (he did not have the factor $S(t)$ on the right - missing the interpretation in terms of returns, and leading to negative stock prices!). The economic importance of geometric Brownian motion was recognised by Paul A. Samuelson in his work, for which Samuelson received the Nobel Prize in Economics in 1970, and by Robert Merton, in work for which he was similarly honoured in 1997.

Quadratic Variation of BM

- We take a partition $\mathcal{P}$ of $[0, t]$ - a finite set of points $0 = t_0 < t_1 < \ldots < t_n = t$ with grid mesh $\|\mathcal{P}\| := \max |t_i - t_{i-1}|$ small.
- Writing $\Delta W(t_i) := W(t_i) - W(t_{i-1})$ and $\Delta t_i := t_i - t_{i-1}$,

$$\sum_{i=1}^{n} (\Delta W(t_i))^2$$

  will closely resemble

$$\sum_{i=1}^{n} \mathbb{E}((\Delta W(t_i))^2) = \sum_{i=1}^{n} \Delta t_i = \sum_{i=1}^{n} (t_i - t_{i-1}) = t.$$

- In fact the *quadratic variation* of W over $[0, t]$ is

$$\sum_{i=1}^{n} (\Delta W(t_i))^2 \to \sum_{i=1}^{n} \Delta t_i = t \quad (\max |t_i - t_{i-1}| \to 0).$$

Quadratic Variation of BM

Theorem (Lévy)

The quadratic variation of a Brownian path over $[0, t]$ *exists and equals* t*:*

$$\langle W \rangle_t = t.$$

Motivation

Stochastic integration was introduced by K. Itô in 1944, hence its name Itô calculus. It gives a meaning to

$$\int_0^t X dY = \int_0^t X(s,\omega) dY(s,\omega),$$

for suitable stochastic processes X and Y, the integrand and the integrator. We shall confine our attention here mainly to the basic case with integrator Brownian motion: $Y = W$.

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Indicators

- To define Itô integrals, we begin with the simplest possible integrands X, and extend successively in much the same way that we extended the measure-theoretic integral.
- If $X(t,\omega) = \mathbf{1}_{[a,b]}(t)$, there is exactly one plausible way to define $\int XdW$:

$$\int_0^t X(s,\omega)dW(s,\omega) := \begin{cases} 0 & \text{if } t \leq a, \\ W(t) - W(a) & \text{if } a \leq t \leq b, \\ W(b) - W(a) & \text{if } t \geq b. \end{cases}$$

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Simple Functions

Extend by linearity: if X is a linear combination of indicators, $X = \sum_{i=1}^{n} c_i \mathbf{1}_{[a_i,b_i]}$, we should define

$$\int_0^t X dW := \sum_{i=1}^{n} c_i \int_0^t \mathbf{1}_{[a_i,b_i]} dW.$$

Simple Processes

Call a stochastic process X *simple* if there is a partition $0 = t_0 < t_1 < \ldots < t_n = T < \infty$ and uniformly bounded $\mathcal{F}_{t_k}$-measurable (we know its value at time t_k) random variables ξ_k ($|\xi_k| \leq C$ for all $k = 0, \ldots, n$ and ω, for some C) and if $X(t, \omega)$ can be written in the form

$$X(t,\omega) = \xi_0(\omega)\mathbf{1}_{\{0\}}(t) + \sum_{i=0}^{n} \xi_i(\omega)\mathbf{1}_{(t_i,t_{i+1}]}(t) \quad (0 \leq t \leq T, \omega \in \Omega).$$

Simple Processes

Then if $t_k \leq t < t_{k+1}$,

$$
\begin{aligned}
I_t(X) := \int_0^t X dW &= \sum_{i=0}^{k-1} \xi_i (W(t_{i+1}) - W(t_i)) + \xi_k (W(t) - W(t_k)) \\
&= \sum_{i=0}^{n} \xi_i (W(t \wedge t_{i+1}) - W(t \wedge t_i)).
\end{aligned}
$$

Note that by definition $I_0(X) = 0 \quad \mathbb{P} - a.s.$.

Properties of the Stochastic Integral

We collect some properties of the stochastic integral:

Lemma

(i) $I_t(aX + bY) = aI_t(X) + bI_t(Y)$.

(ii) $\mathbb{E}(I_t(X)|\mathcal{F}_s) = I_s(X) \quad \mathbb{P} - a.s. \quad (0 \leq s < t < \infty)$, *hence* $I_t(X)$ *is a continuous martingale.*

Property (ii) also states the definition of a martingale. Loosely speaking the current value of the process is the best predictor based on current information of future values.

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Properties of the Stochastic Integral

We now can add further properties of the stochastic integral for simple functions.

Lemma

(i) We have the Itô isometry

$$\mathbb{E}\left((I_t(X))^2\right) = \mathbb{E}\left(\int_0^t X(s)^2 ds\right).$$

(ii) $\mathbb{E}\left((I_t(X) - I_s(X))^2 | \mathcal{F}_s\right) = \mathbb{E}\left(\int_s^t X(u)^2 du\right) \quad \mathbb{P} - a.s.$

Construction of the Stochastic Integral

We seek a class of integrands suitably approximable by simple integrands. It turns out that:

- We can define a suitable class of integrands X with (main property) $\int_0^t \mathbb{E}\left(X(u)^2\right) du < \infty$ for all $t > 0$.
- Each such X may be approximated by a sequence of simple integrands X_n so that the stochastic integral $I_t(X) = \int_0^t XdW$ may be defined as the limit of $I_t(X_n) = \int_0^t X_n dW$.
- The properties from both lemmas above remain true for the stochastic integral $\int_0^t XdW$.

Example

We calculate $\int W(u)dW(u)$. We start by approximating the integrand by a sequence of simple functions.

$$X_n(u) = \begin{cases} W(0) = 0 & \text{if} \quad 0 \leq u \leq t/n, \\ W(t/n) & \text{if} \quad t/n < u \leq 2t/n, \\ \vdots & \vdots \\ W\left(\frac{(n-1)t}{n}\right) & \text{if} \quad (n-1)t/n < u \leq t. \end{cases}$$

Example

By definition,

$$\int_0^t W(u)dW(u) = \lim_{n\to\infty} \sum_{k=0}^{n-1} W\left(\frac{kt}{n}\right)\left(W\left(\frac{(k+1)t}{n}\right) - W\left(\frac{kt}{n}\right)\right).$$

Rearranging terms, we obtain for the sum on the right

$$\begin{aligned}
&\sum_{k=0}^{n-1} W\left(\frac{kt}{n}\right)\left(W\left(\frac{(k+1)t}{n}\right) - W\left(\frac{kt}{n}\right)\right)\\
= \quad &\frac{1}{2}W(t)^2 - \frac{1}{2}\left[\sum_{k=0}^{n-1}\left(W\left(\frac{(k+1)t}{n}\right) - W\left(\frac{kt}{n}\right)\right)^2\right].
\end{aligned}$$

Example

Since the second term approximates the quadratic variation of W and hence tends to t for $n \to \infty$, we find

$$\int_0^t W(u)dW(u) = \frac{1}{2}W(t)^2 - \frac{1}{2}t. \tag{3}$$

Note the contrast with ordinary (Newton-Leibniz) calculus! Itô calculus requires the second term on the right – the Itô correction term – which arises from the quadratic variation of W.

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Quadratic Covariation.

We shall need to extend quadratic variation and quadratic covariation to stochastic integrals. The quadratic variation of

$$I_t(X) = \int_0^t X(u)dW(u) \quad \text{is} \quad \int_0^t X(u)^2 du.$$

This is proved in the same way as the case $X \equiv 1$, that W has quadratic variation process t.

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Itô Processes

$$X(t) := x_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dW(s)$$

defines a stochastic process X with $X(0) = x_0$.
We express such an equation symbolically in differential form, in terms of the stochastic differential equation

$$dX(t) = b(t)dt + \sigma(t)dW(t), \quad X(0) = x_0.$$

For $f \in C^2$ we want to give meaning to the stochastic differential $df(X(t))$ of the process $f(X(t))$.

Multiplication rules

These are just shorthand for the corresponding properties of the quadratic variations.

	dt	dW
dt	0	0
dW	0	dt

We find

$$
\begin{aligned}
d\langle X\rangle &= (b\,dt + \sigma\,dW)^2 \\
&= \sigma^2 dt + 2b\sigma\,dt\,dW + b^2(dt)^2 = \sigma^2 dt
\end{aligned}
$$

Basic Itô Formula

If X is a Itô Process and $f \in C^2$, then $f(X)$ has stochastic differential

$$\begin{aligned} df(X(t)) &= f'(X(t))dX(t) \\ &\quad + \frac{1}{2} f''(X(t))d\langle X\rangle(t), \end{aligned}$$

or writing out the integrals,

$$\begin{aligned} f(X(t)) &= f(x_0) + \int_0^t f'(X(u))dX(u) \\ &\quad + \frac{1}{2}\int_0^t f''(X(u))d\langle X\rangle(u). \end{aligned}$$

Itô Formula

If $X(t)$ is an Itô process and $f \in C^{1,2}$ then $f = f(t, X(t))$ has stochastic differential

$$df = \left(f_t + bf_x + \frac{1}{2}\sigma^2 f_{xx}\right) dt + \sigma f_x dW.$$

That is, writing f_0 for $f(0, x_0)$, the initial value of f,

$$f = f_0 + \int_0^t (f_t + bf_x + \frac{1}{2}\sigma^2 f_{xx}) dt + \int_0^t \sigma f_x dW.$$

Example: GBM

The SDE for GBM has the unique solution

$$S(t) = S(0)\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma dW(t)\right\}.$$

For, writing

$$f(t,x) := \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma x\right\},$$

we have

$$f_t = \left(\mu - \frac{1}{2}\sigma^2\right)f, \quad f_x = \sigma f, \quad f_{xx} = \sigma^2 f,$$

and with $x = W(t)$, one has

$$dx = dW(t), \quad (dx)^2 = dt.$$

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Example: GBM

Thus Itô's lemma gives

$$\begin{aligned} df &= f_t dt + f_x dW + \frac{1}{2} f_{xx} (dW)^2 \\ &= f\left(\left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW + \frac{1}{2}\sigma^2 dt\right) \\ &= f(\mu dt + \sigma dW). \end{aligned}$$

Girsanov's Theorem

Let $W = (W_1, \dots W_d)$ be a d-dimensional BM.

Theorem (Girsanov)

Define the equivalent probability measure $\tilde{\mathbb{P}}$ with Radon-Nikodým derivative

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = L(T) = \exp\left\{ -\int_0^T \gamma(s)' dW(s) - \frac{1}{2}\int_0^T \|\gamma(s)\|^2 \, ds \right\},$$

with $\gamma(t)$ a suitable d-dimensional process. Then define

$$\tilde{W}_i(t) := W_i(t) + \int_0^t \gamma_i(s) ds.$$

The process $\tilde{W} = (\tilde{W}_1, \dots, \tilde{W}_d)$ is d-dimensional BM.

Girsanov's Theorem

For $\gamma(t) = \gamma$, change of measure by the Radon-Nikodým derivative

$$\exp\left\{-\gamma W(t) - \frac{1}{2}\gamma^2 t\right\}$$

corresponds to a change of drift from c to $c - \gamma$.
In our setting any pair of equivalent probability measures $\mathbb{Q} \sim \mathbb{P}$ is a Girsanov pair, i.e.

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} = L$$

with L defined as above.

Financial Mathematics

Lecture - Winter 2013/14

Professor Dr. Rüdiger Kiesel | Chair for Energy Trading and Finance | University of Duisburg-Essen

Part III

Dynamic Financial Models and Black-Scholes

Financial Market Model

- $T > 0$ is a fixed a planning horizon.
- Uncertainty in the financial market is modelled by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an information set (filtration) $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$.
- There are $d + 1$ primary traded assets, whose price processes are given by stochastic processes $S_0, \dots, S_d$, which represent the prices of some traded assets.
- A numéraire is a price process $X(t)$ almost surely strictly positive for each $t \in [0, T]$.
- 'Historically' the money market account $B(t) = e^{rt}$ with a positive constant r was used as a numéraire.

Trading Strategies

- We call an $\mathbb{R}^{d+1}$-valued process

$$\varphi(t) = (\varphi_0(t), \ldots, \varphi_d(t)), \quad t \in [0, T]$$

  a trading strategy (or dynamic portfolio process).
- Here $\varphi_i(t)$ denotes the number of shares of asset i held in the portfolio at time t - to be determined on the basis of information available *before* time t; i.e. the investor selects his time t portfolio after observing the prices $S(t-)$.

Trading Strategies

- The value of the portfolio φ at time t is given by

$$V_\varphi(t) := \varphi(t) \cdot S(t) = \sum_{i=0}^{d} \varphi_i(t) S_i(t).$$

  $V_\varphi(t)$ is called the value process, or wealth process, of the trading strategy φ.
- The gains process $G_\varphi(t)$ is

$$G_\varphi(t) := \sum_{i=0}^{d} \int_0^t \varphi_i(u) dS_i(u).$$

- A trading strategy φ is called self-financing if the wealth process $V_\varphi(t)$ satisfies

$$V_\varphi(t) = V_\varphi(0) + G_\varphi(t) \quad \text{for all } t \in [0, T].$$

Discounted Processes

The discounted price process is

$$\tilde{S}(t) := \frac{S(t)}{S_0(t)} = (1, \tilde{S}_1(t), \dots \tilde{S}_d(t))$$

with $\tilde{S}_i(t) = S_i(t)/S_0(t),\ i = 1, 2, \dots, d$. The discounted wealth process $\tilde{V}_\varphi(t)$ is

$$\tilde{V}_\varphi(t) := \frac{V_\varphi(t)}{S_0(t)} = \varphi_0(t) + \sum_{i=1}^{d} \varphi_i(t)\tilde{S}_i(t)$$

and the discounted gains process $\tilde{G}_\varphi(t)$ is

$$\tilde{G}_\varphi(t) := \sum_{i=1}^{d} \int_0^t \varphi_i(t) d\tilde{S}_i(t).$$

Self-Financing

φ is self-financing if and only if

$$\tilde{V}_\varphi(t) = \tilde{V}_\varphi(0) + \tilde{G}_\varphi(t).$$

Thus a self-financing strategy is completely determined by its initial value and the components $\varphi_1, \dots, \varphi_d$. Any set of processes $\varphi_1, \dots, \varphi_d$ such that the stochastic integrals $\int \varphi_i d\tilde{S}_i$ exist can be uniquely extended to a self-financing strategy φ with specified initial value $\tilde{V}_\varphi(0) = v$ by setting the cash holding as

$$\varphi_0(t) = v + \sum_{i=1}^{d} \int_0^t \varphi_i(u) d\tilde{S}_i(u) - \sum_{i=1}^{d} \varphi_i(t) \tilde{S}_i.$$

Arbitrage Opportunities

A self-financing trading strategy φ is called an arbitrage opportunity if the wealth process V_φ satisfies the following set of conditions:

$$V_\varphi(0) = 0, \quad \mathbb{P}(V_\varphi(T) \geq 0) = 1,$$

and

$$\mathbb{P}(V_\varphi(T) > 0) > 0.$$

Martingale Measure

A probability measure $\mathbb{Q}$ defined on $(\Omega, \mathcal{F})$ is an equivalent martingale measure (EMM) if:

(i) $\mathbb{Q}$ is equivalent to $\mathbb{P}$,
(ii) the discounted price process $\tilde{S}$ is a $\mathbb{Q}$ martingale.
Assume $S_0(t) = B(t) = e^{rt}$, then $\mathbb{Q} \sim \mathbb{P}$ is a martingale measure if and only if every asset price process S_i has price dynamics under $\mathbb{Q}$ of the form

$$dS_i(t) = rS_i(t)dt + dM_i(t),$$

where M_i is a $\mathbb{Q}$-martingale.

EMMs and Arbitrage

Assume $\mathbb{Q}$ is an EMM. Then the market model contains no arbitrage opportunities.

Proof. Under $\mathbb{Q}$ we have that $\tilde{V}_\varphi(t)$ is a martingale. That is,

$$\mathbb{E}_\mathbb{Q}\left(\tilde{V}_\varphi(t)|\mathcal{F}_u\right) = \tilde{V}_\varphi(u), \text{ for all } \quad u \leq t \leq T.$$

For $\varphi \in \Phi$ to be an arbitrage opportunity we must have $\tilde{V}_\varphi(0) = V_\varphi(0) = 0$. Now

$$\mathbb{E}_\mathbb{Q}\left(\tilde{V}_\varphi(t)\right) = 0, \text{ for all } 0 \leq t \leq T.$$

EMMs and Arbitrage

Now $\tilde{V}_\varphi(t)$ is a martingale, so

$$\mathbb{E}_{\mathbb{Q}}\left(\tilde{V}_\varphi(t)\right) = 0, \; 0 \leq t \leq T,$$

in particular $\mathbb{E}_{\mathbb{Q}}\left(\tilde{V}_\varphi(T)\right) = 0.$
For an arbitrage opportunity φ we have $\mathbb{P}\left(V_\varphi(T) \geq 0\right) = 1$, and since $\mathbb{Q} \sim \mathbb{P}$, this means $\mathbb{Q}\left(V_\varphi(T) \geq 0\right) = 1$.
Both together yield

$$\mathbb{Q}\left(V_\varphi(T) > 0\right) = \mathbb{P}\left(V_\varphi(T) > 0\right) = 0,$$

and hence the result follows. ∎

Admissible Strategies

- A SF strategy φ is called ($\mathbb{P}^*$-) admissible if

$$\tilde{G}_\varphi(t) = \int_0^t \varphi(u) d\tilde{S}(u)$$

  is a ($\mathbb{P}^*$-) martingale.
- By definition $\tilde{S}$ is a martingale, and $\tilde{G}$ is the stochastic integral with respect to $\tilde{S}$.
- The financial market model $\mathcal{M}$ contains no arbitrage opportunities wrt admissible strategies.

Contingent Claims

A contingent claims X is a random variable with existing expected value.

- A contingent claim X is called attainable if there exists at least one admissible trading strategy such that

  $$V_{\varphi}(T) = X.$$

  We call such a trading strategy φ a replicating strategy for X.
- The financial market model $\mathcal{M}$ is said to be complete if any contingent claim is attainable.

No-Arbitrage Price

If a contingent claim X is attainable, X can be replicated by a portfolio $\varphi \in \Phi(\mathbb{P}^*)$. This means that holding the portfolio and holding the contingent claim are equivalent from a financial point of view. In the absence of arbitrage the (arbitrage) price process $\Pi_X(t)$ of the contingent claim must therefore satisfy

$$\Pi_X(t) = V_\varphi(t).$$

Risk-Neutral Valuation

The arbitrage price process of any attainable claim is given by the risk-neutral valuation formula

$$\Pi_X(t) = S_0(t)\mathbb{E}_{\mathbb{P}^*}\left[\frac{X}{S_0(T)}\middle|\mathcal{F}_t\right].$$

Thus, for any two replicating portfolios $\varphi, \psi \in \Phi(\mathbb{P}^*)$

$$V_\varphi(t) = V_\psi(t).$$

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Risk-Neutral Valuation

Proof. Since X is attainable, there exists a replicating strategy $\varphi \in \Phi(\mathbb{P}^*)$ such that $V_\varphi(T) = X$ and $\Pi_X(t) = V_\varphi(t)$. Since $\varphi \in \Phi(\mathbb{P}^*)$ the discounted value process $\tilde{V}_\varphi(t)$ is a martingale, and hence

$$\begin{aligned}
\Pi_X(t) &= V_\varphi(t) = S_0(t)\tilde{V}_\varphi(t) \\
&= S_0(t)\mathbb{E}_{\mathbb{P}^*}\left[\tilde{V}_\varphi(T)\middle|\mathcal{F}_t\right] \\
&= S_0(t)\mathbb{E}_{\mathbb{P}^*}\left[\frac{V_\varphi(T)}{S_0(T)}\middle|\mathcal{F}_t\right] \\
&= S_0(t)\mathbb{E}_{\mathbb{P}^*}\left[\frac{X}{S_0(T)}\middle|\mathcal{F}_t\right].
\end{aligned}$$

∎

Black-Scholes Model

The classical Black-Scholes model is

$$\begin{aligned} dB(t) &= rB(t)dt, & B(0) &= 1, \\ dS(t) &= S(t)\,(bdt + \sigma dW(t)), & S(0) &= p, \end{aligned}$$

with constant coefficients $b \in \mathbb{R},\ r, \sigma \in \mathbb{R}_+$.
The Black-Scholes price process of a European call is given by

$$\begin{aligned} C(t) &= S(t)\Phi(d_1(S(t), T-t)) \\ &\quad -Ke^{-r(T-t)}\Phi(d_2(S(t), T-t)). \end{aligned}$$

The functions $d_1(s,t)$ and $d_2(s,t)$ are given by

$$\begin{aligned} d_1(s,t) &= \frac{\log(s/K) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}, \\ d_2(s,t) &= \frac{\log(s/K) + (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}} \end{aligned}$$

Black-Scholes Model

The classical Black-Scholes model is

$$\begin{aligned} dB(t) &= rB(t)dt, & B(0) &= 1, \\ dS(t) &= S(t)\,(bdt + \sigma dW(t)), & S(0) &= p, \end{aligned}$$

with constant coefficients $b \in \mathbb{R},\ r, \sigma \in \mathbb{R}_+$.
We use the bank account being the natural numéraire and get from Itô's formula

$$d\tilde{S}(t) = \tilde{S}(t)\,\{(b-r)dt + \sigma dW(t)\}.$$

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EMM in BS-model

Any EMM is a Girsanov pair

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L(t)$$

with

$$L(t) = \exp\left\{-\int_0^t \gamma(s)dW(s) - \frac{1}{2}\int_0^t \gamma(s)^2 ds\right\}.$$

By Girsanov's theorem

$$dW(t) = d\tilde{W}(t) - \gamma(t)dt,$$

where $\tilde{W}$ is a $\mathbb{Q}$-BM. Thus the $\mathbb{Q}$-dynamics for $\tilde{S}$ are

$$d\tilde{S}(t) = \tilde{S}(t)\left\{(b - r - \sigma\gamma(t))dt + \sigma d\tilde{W}(t)\right\}.$$

EMM in BS-model

Since $\tilde{S}$ has to be a martingale under $\mathbb{Q}$ we must have

$$b - r - \sigma\gamma(t) = 0 \quad t \in [0, T],$$

and so we must choose

$$\gamma(t) \equiv \gamma = \frac{b - r}{\sigma},$$

this argument leads to a unique martingale measure. The $\mathbb{Q}$-dynamics of S are

$$dS(t) = S(t)\left\{rdt + \sigma d\tilde{W}\right\}.$$

Pricing Contingent Claims

By the risk-neutral valuation principle

$$\Pi_X(t) = e^{\{-r(T-t)\}}\mathbb{E}^*\left[X \middle| \mathcal{F}_t\right],$$

with $\mathbb{E}^*$ given via the Girsanov density

$$L(t) = \exp\left\{-\left(\frac{b-r}{\sigma}\right)W(t) - \frac{1}{2}\left(\frac{b-r}{\sigma}\right)^2 t\right\}.$$

Pricing Contingent Claims

For a European call $X = (S(T) - K)^+$ and we can evaluate the above expected value
The Black-Scholes price process of a European call is given by

$$\begin{aligned} C(t) &= S(t)\Phi(d_1(S(t), T-t)) \\ &\quad -Ke^{-r(T-t)}\Phi(d_2(S(t), T-t)). \end{aligned}$$

The functions $d_1(s,t)$ and $d_2(s,t)$ are given by

$$\begin{aligned} d_1(s,t) &= \frac{\log(s/K) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}, \\ d_2(s,t) &= \frac{\log(s/K) + (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}} \end{aligned}$$

BS by Arbitrage

Consider a self-financing portfolio which has dynamics

$$\begin{aligned} dV_\varphi(t) &= \varphi_0(t)dB(t) + \varphi_1(t)dS(t) \\ &= (\varphi_0 rB + \varphi_1 bS)dt + \varphi_1\sigma SdW. \end{aligned}$$

Assume that $V_\varphi(t) = V(t) = f(t, S(t))$. Then by Itô's formula

$$\begin{aligned} dV &= (f_t + f_x Sb + \frac{1}{2}S^2\sigma^2 f_{xx})dt \\ &\quad + f_x\sigma SdW. \end{aligned}$$

BS by Arbitrage

We match coefficients and find

$$\varphi_1 = f_x \text{ and } \varphi_0 = \frac{1}{rB}(f_t + \frac{1}{2}\sigma^2 S_t^2 f_{xx}).$$

So $f(t,x)$ must satisfy the Black-Scholes PDE

$$f_t + rxf_x + \frac{1}{2}\sigma^2 x^2 f_{xx} - rf = 0$$

and initial condition $f(T,x) = (x-K)^+$.

Option Dynamics

Let us now compute the dynamics of the call option's price $C(t)$ under the risk-neutral martingale measure $\mathbb{P}^*$. We find

$$dC(t) = rC(t)dt + \sigma\Phi(d_1(S(t), T-t))S(t)d\tilde{W}(t).$$

Defining the *elasticity coefficient* of the option's price as

$$\eta^c(t) = \frac{\Delta(S(t), T-t)S(t)}{C(t)} = \frac{\Phi(d_1(S(t), T-t))}{C(t)}$$

we can rewrite the dynamics as

$$dC(t) = rC(t)dt + \sigma\eta^c(t)C(t)d\tilde{W}(t).$$

So, as expected in the risk-neutral world, the appreciation rate of the call option equals the risk-free rate r. The volatility coefficient is $\sigma\eta^c$, and hence stochastic.

Greeks

We will now analyse the impact of the underlying parameters in the standard Black-Scholes model on the prices of call and put options. The Black-Scholes option values depend on the (current) stock price, the volatility, the time to maturity, the interest rate and the strike price. The sensitivities of the option price with respect to the first four parameters are called the *Greeks* and are widely used for hedging purposes. We can determine the impact of these parameters by taking partial derivatives.

BS-formula

Recall the Black-Scholes formula for a European call:

$$\pi^{call}(0) = C(S, T, K, r, \sigma) = S\Phi(d_1(S, T)) - Ke^{-rT}\Phi(d_2(S, T)),$$

with the functions $d_1(s, t)$ and $d_2(s, t)$ given by

$$\begin{aligned} d_1(s, t) &= \frac{\log(s/K) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}, \\ d_2(s, t) &= d_1(s, t) - \sigma\sqrt{t} = \frac{\log(s/K) + (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}. \end{aligned}$$

Greeks

$$
\begin{aligned}
\Delta &:= \frac{\partial C}{\partial S} &&= \Phi(d_1) > 0, \\
\mathcal{V} &:= \frac{\partial C}{\partial \sigma} &&= S\sqrt{T}\varphi(d_1) > 0, \\
\Theta &:= \frac{\partial C}{\partial T} &&= \frac{S\sigma}{2\sqrt{T}}\varphi(d_1) + Kre^{-rT}\Phi(d_2) > 0, \\
\rho &:= \frac{\partial C}{\partial r} &&= TKe^{-rT}\Phi(d_2) > 0, \\
\Gamma &:= \frac{\partial^2 C}{\partial S^2} &&= \frac{\varphi(d_1)}{S\sigma\sqrt{T}} > 0.
\end{aligned}
$$

Greeks

From the definitions it is clear that Δ – delta – measures the change in the value of the option compared with the change in the value of the underlying asset, $\mathcal{V}$ – vega – measures the change of the option compared with the change in the volatility of the underlying, and similar statements hold for Θ – theta – and ρ – rho

The Greeks

- Likewise, in order to quantify the risk associated with an instrument, one looks at *how much the price of the instrument changes if one of the underlying risk drivers changes* its value. Those risk measures are also called Greeks.
- Mathematically speaking, the Greeks are just the derivative of the price of the instrument with respect to the value of the risk driver.
- Once those Greeks are known for a portfolio, one can easily calculate how much the value of an option or a portfolio changes *marginally*, if one of the variables changes *marginally*, all others remain fixed.
- Again, the most important Greeks are Delta, Gamma, Vega, Rho, and Theta.

The Delta

- Delta is the derivative of the instrument price with respect to the price of the underlying.
- The delta of the underlying security is one.
- If the payoff is not linear (for example if the instrument is an option), Delta is not constant.
- As Delta is the derivative with respect to the underlying, it tells how much the value of the instrument changes if the value of the underlying changes marginally.
- Thus in a continuous time model, Delta is the amount of the underlying needed to be sold in order to offset the price change of the instrument. The ability to neutralize the trading book with respect to price changes in the underlying makes Delta the most important Greek.

Delta Hedging I

- Assume that you have an options position with delta Δ. This means that if the price of the underlying moves by a (very) small amount ϵ, the price of the option position moves approximately by $\Delta\epsilon$.
- A *delta hedge* consists of selling Δ units of the underlying (buying if $\Delta < 0$) and gives a portfolio that does not change its value if the price of the underlying changes by a marginal amount. The portfolio has a delta of zero and is called *delta neutral*.

Delta Hedging II

- BUT: Delta is not constant, so the portfolio has to be rebalanced.
- In the Black-Scholes model, a delta hedge with continuous rebalancing is a perfect hedge.
- In practice, continuous rebalancing is not possible so that the portfolio is not protected against larger market movements. Transaction costs and bid-ask spread also result in losses.

Delta in the Black-Scholes Model

In the Black-Scholes model, we have computed the Greeks explicitly for European call options. We have:

$$\Delta = \frac{\partial}{\partial S} Call_{BS}(S, K, \sigma, r, t, T) = \Phi(d_1)$$

where, as usual, Φ denotes the c.d.f. of the standard normal distribution and

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}}.$$

Delta of a Call Option in the Black-Scholes Model

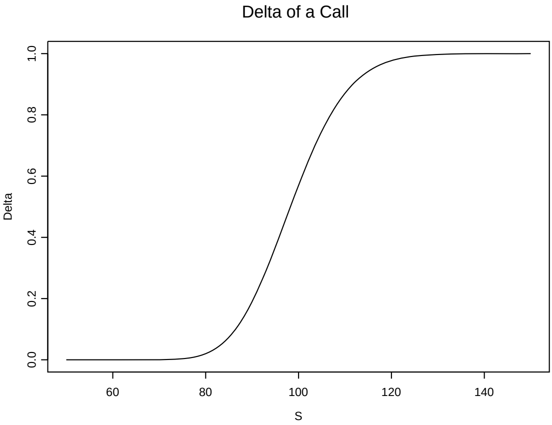

Figure : Delta for a European call in the BS model, $T = 1$, $r = 1\%$, $\sigma = 20\%$, $K = 100$.

Example: Delta Hedge

You are long USD $1,000$ in the 104 call. Interest rate is 5%, stock price today is 99, time to maturity 1 month, and implied volatility is 15.7%.

- How can you make your portfolio delta neutral by investing in the stock?
- You set up the delta neutral portfolio and the stock price jumps to USD100 immediately. What is your P/L for the portfolio?

Example: Delta Hedge

- Compute the price of the call with the BS formula:

$$Call_{BS} = 0.3858\text{USD}.$$

- The position consists of $N = 1,000/Call_{BS} = 2592$ call options.
- The delta of each option is

$$\begin{aligned}
\Delta = \Phi(d_1) &= \Phi\left(\frac{\log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= \Phi\left(\frac{\log(99/104) + (0.05 + 0.157^2/2)(1/12)}{0.157\sqrt{1/12}}\right) \\
&= 0.1654.
\end{aligned}$$

Example: Delta Hedge

- The delta of the position is long $\Delta_P = N \cdot \Delta = 428.70$.
- The stock has $\Delta = 1$.
- To make the position delta neutral, you have to enter a short position of 428.70 shares.

Example: Delta Hedge

- The loss from the short position in the stock is $428.70 \cdot 1 = 428.70$.
- To compute the gain from the long options position, we have to compute the option price for $S = 100$. Using the BS formula, we obtain

$$Call_{BS}(S = 100) = 0.5808.$$

- The gain from the options position is $2592 \cdot (0.5808 - 0.3858) = 505.37$.
- Our profit is $505.37 - 428.70 = 76.67$.

The Gamma

- Gamma is the second derivative of the instrument price with respect to the price of the underlying.
- If the instrument has a linear payoff, Gamma is zero.
- As delta is the first derivative of the option price with respect to the underlying, gamma is the derivative of delta with respect to the underlying and thus measures, how much Delta changes if the underlying changes.
- This is an important information in risk management as it tells the trader how much of the underlying he has to buy or sell if the underlying itself changes price.
- Geometrically, gamma might be seen as the slope of Delta.

Gamma Neutral Portfolios

As the delta of a portfolio changes with the price of the underlying, a delta hedge has to be rebalanced frequently.

- A delta hedge for a portfolio with a high (absolute) gamma has to be monitored more carefully than a delta hedge of a portfolio with gamma close to zero.
- To decrease the hedging error, a trader might want to make a delta neutral portfolio gamma neutral.
- The gamma cannot be changed by investing in the underlying because its gamma is zero.
- Strategy: Make the portfolio gamma neutral by investing in an option, then make it delta neutral by investing in the underlying.

Gamma in the Black-Scholes Model

The gamma of a European call option in the Black-Scholes model is given by

$$\Gamma = \frac{\partial^2}{\partial S^2} Call_{BS}(S, K, \sigma, r, t, T) = \frac{\varphi(d_1)}{S\sigma\sqrt{T-t}}.$$

Here, $\varphi(x)$ denotes the p.d.f. of the standard normal distribution.

Gamma of a Call Option in the Black-Scholes Model

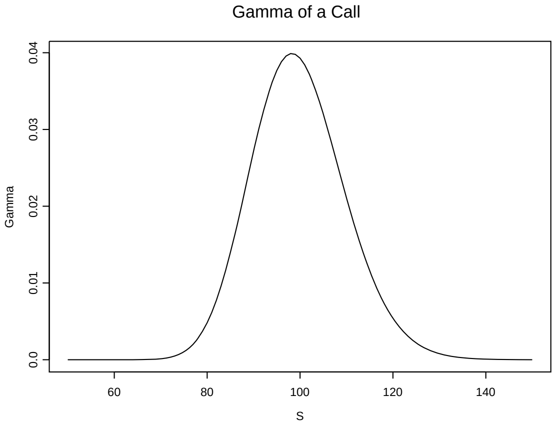

Figure : Gamma for a European call in the BS model, $T = 1$, $r = 1\%$, $\sigma = 20\%$, $K = 100$.

Gamma of a Call Option in the Black-Scholes Model

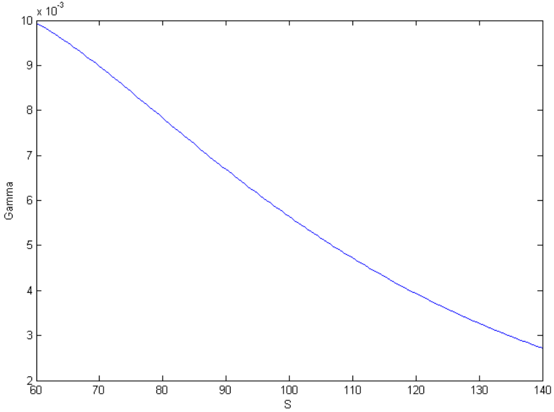

Figure : Gamma for a European call in the BS model, $T = 10$, $r = 1\%$, $\sigma = 20\%$, $K = 100$.

Example: Gamma Neutral Portfolio

You are in the same position as in the previous example. Additionally, you can trade in the 97 call.

- Put together a portfolio that is delta and gamma neutral.
- What is your P/L if the stock price jumps to 100?

Example: Gamma Neutral Portfolio

- The gamma of the 104 call is

$$\Gamma_{104C} = \frac{\varphi(d_1)}{S\sigma\sqrt{T-t}} = 0.0554.$$

- The gamma of the 97 call is

$$\Gamma_{97C} = \frac{\varphi(d_1)}{S\sigma\sqrt{T-t}} = 0.0758.$$

- To make the portfolio gamma neutral, we have to build up a position of n 97 calls so that $2592 \cdot 0.0554 + n \cdot 0.0758 = 0$.
- We find that $n = -1894.74$ makes the portfolio gamma neutral, i.e., we sell 1894.74 units of the 97 call.

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Example: Gamma Neutral Portfolio

- We compute the delta of the portfolio consisting of the two positions in the calls.
- The 97 call has a delta of $\Delta_{97C} = 0.7139$.
- The delta of the position is

$$\Delta_{P'} = 2592 \cdot 0.1654 - 1894.74 \cdot 0.7139 = -924.02$$

- We have to buy 924.02 units of the underlying to make the portfolio delta neutral. By buying the underlying, we do not change the gamma of the portfolio, it remains zero.

Example: Gamma Neutral Portfolio

- The price of the 97 call for $S = 99$ is $Call_{BS,97}(S = 99) = 3.2235$.
- The price of the 97 call for $S = 100$ is $Call_{BS,97}(S = 100) = 3.9735$.
- The P/L from the position in the 97 call is $-1894.97 * (3.9735 - 3.2235) = -1421.11$.
- The P/L from the position in the stock is 924.02.
- The portfolio P/L is $924.02 - 1421.11 + 505.37 = -8.28$.

The Vega

- Vega is the derivative of the instrument price with respect to implied volatility.
- Thus, vega indicates how much the option price changes if the implied volatility changes.
- Vega is not a Greek letter.

Vega in the Black-Scholes Model

The vega of a European call option in the Black-Scholes model is given by

$$\nu = \frac{\partial}{\partial \sigma} Call_{BS}(S, K, \sigma, r, t, T) = S\sqrt{T-t}\varphi(d_1).$$

Therefore, we have

$$\nu = S^2\sigma(T-t)\Gamma.$$

Vega of a Call Option in the Black-Scholes Model II

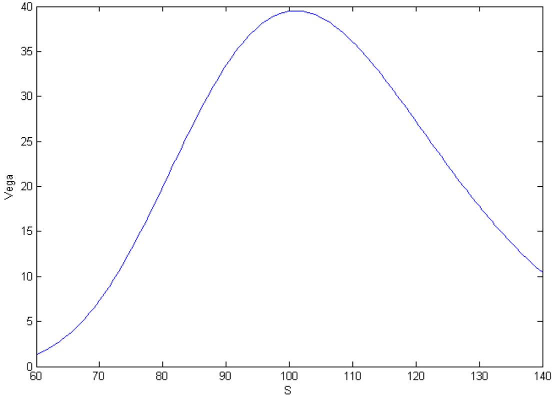

Figure : Vega for a European call in the BS model, $T = 1$, $r = 1\%$, $\sigma = 20\%$, $K = 100$.

Vega of a Call Option in the Black-Scholes Model III

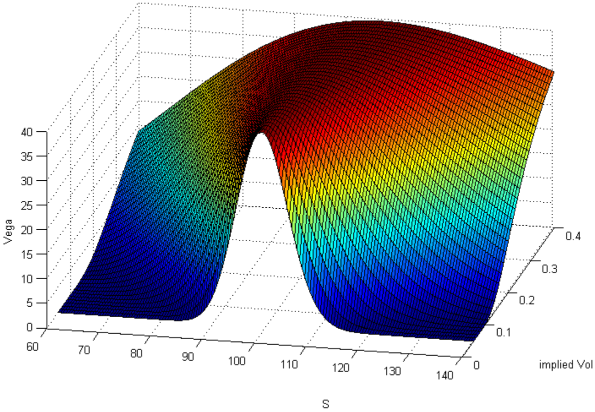

Figure : Vega for a European call in the BS model, $T = 1$, $r = 1\%$, $K = 100$.

The Theta

- Theta is the derivative of the instrument price with respect to time to maturity.
- Thus, theta indicates how much the option price changes as time moves closer to maturity.
- Theta is usually negative and therefore is also called *time decay* or *rent*.
- Note: The passage of time is deterministic. It does not make sense to hedge against these losses.

Theta of a Call Option in the Black-Scholes Model

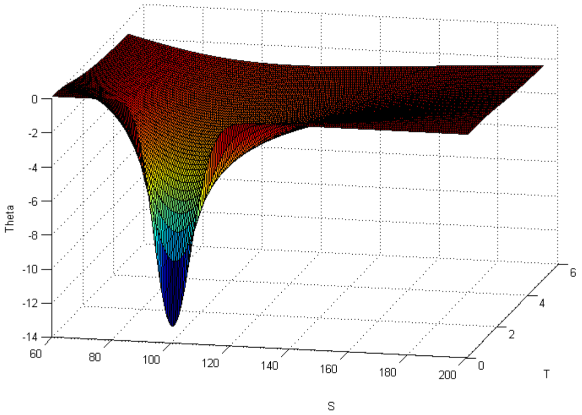

Figure : Theta for a European call in the BS model, $\sigma = 20\%$, $r = 1\%$, $K = 100$.

The Rho

- Rho is the derivative of the instrument price with respect to the risk free interest rate.
- It measures, how the price of the instrument changes if the interest rate changes.
- Rho is particular important for fixed-income portfolios. If the hedging portfolio of the instrument consists of a large portion of debt and only a small amount of initial capital, the influence of changes in the risk free rate might become quite big.

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Rho of a Call Option in the Black-Scholes Model

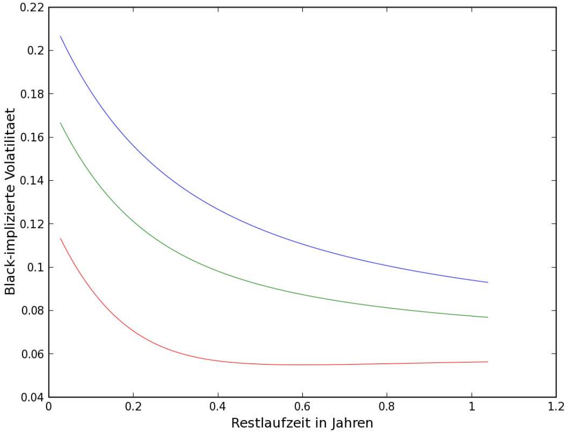

Figure : Rho for a European call in the BS model, $\sigma = 20\%$, $T = 1$, $K = 100$.

Vega

One of the main issues raised by the Black-Scholes formula is the question of modelling the volatility σ. Before we can implement the Black-Scholes formula to price options, we have to estimate σ.
Because the formula is explicit, we can, determine the $\mathcal{V}$ – the partial derivative

$$\mathcal{V} = \partial C / \partial \sigma,$$

finding

$$\mathcal{V} = S\sqrt{T}\varphi(d_1).$$

The important thing to note here is that vega is always positive.

Implied Volatility

Next, since vega is positive, C is a continuous – indeed, differentiable – strictly increasing function of σ. Turning this round, σ is a continuous (differentiable) strictly increasing function of C; indeed,

$$\mathcal{V} = \frac{\partial C}{\partial \sigma}, \quad \text{so} \quad \frac{1}{\mathcal{V}} = \frac{\partial \sigma}{\partial C}.$$

Thus the value $\sigma = \sigma(C)$ corresponding to the actual value $C = C(\sigma)$ at which call options are observed to be traded in the market can be read off. The value of σ obtained in this way is called the *implied volatility*.

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Dividend-Paying Assets

Let S_t be a dividend-paying stock with continuous-dividend rate ρ. To price a derivative with expiry T we set

$$X_t = e^{-\rho(T-t)} S_t.$$

Then X_t is a non-dividend paying asset and must have the dynamic

$$dX_t = rX_t dt + \sigma X_t dW_t.$$

We can thus compute the dynamics of S_t using Itô's formula (or the product rule)

$$dS_t = (r - \rho) S_t dt + \sigma S_t dW_t$$

Dividend-Paying Assets

The usual calculation give the European call option price

$$\begin{aligned} C(t) &= S(t)e^{-\rho(T-t)}\Phi(d_1(S(t), T-t)) \\ &\quad -Ke^{-r(T-t)}\Phi(d_2(S(t), T-t)). \end{aligned}$$

The functions $d_1(s,t)$ and $d_2(s,t)$ are given by

$$\begin{aligned} d_1(s,t) &= \frac{\log(s/K) + (r - \rho + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}, \\ d_2(s,t) &= \frac{\log(s/K) + (r - \rho - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}} \end{aligned}$$

Time-dependent Volatility

Assume

$$dS_t = rS_t dt + \sigma(t) S_t dW_t$$

We can solve this SDE an find

$$S_t = S_0 \exp\left\{\int_0^t \sigma(s) dW_s + \left(rt + \frac{1}{2}\int_0^t \sigma^2(s) ds\right)\right\}.$$

Now

$$\int_0^t \sigma(s) dW_s \sim N\left(0, \sqrt{\int_0^t \sigma^2(s) ds}\right)$$

and we set

$$\bar{\sigma}(t,T) = \sqrt{\frac{1}{T-t}\int_t^T \sigma^2(s) ds}.$$

Time-dependent Volatility

The usual calculation give the European call option price

$$\begin{aligned} C(t) &= S(t)\Phi(d_1(S(t), T-t)) \\ &\quad -Ke^{-r(T-t)}\Phi(d_2(S(t), T-t)). \end{aligned}$$

The functions $d_1(s,t)$ and $d_2(s,t)$ are given by (with appropriate time parameter)

$$\begin{aligned} d_1(s,t) &= \frac{\log(s/K) + (r + \frac{\bar{\sigma}^2}{2})t}{\bar{\sigma}\sqrt{t}}, \\ d_2(s,t) &= \frac{\log(s/K) + (r - \frac{\bar{\sigma}^2}{2})t}{\bar{\sigma}\sqrt{t}} \end{aligned}$$

Financial Mathematics

Lecture - Winter 2013/14

Professor Dr. Rüdiger Kiesel | Chair for Energy Trading and Finance | University of Duisburg-Essen

Part IV

Exotic Options

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Univariate Exotic Options

- Barrier Options
- Binary Options
- Compound Options

Multi-look Exotic Options

Exotic Options with more than one underlying

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Change of numéraire technique

Consider the option to exchange K assets Z_2 against asset Z_1 at time T, which gives one the right to the cash-flow $(Z_1(T) - KZ_2(T))^+$ (with $K = 1$). Using Z_2 as a numéraire, its price $\pi_{ex}(0)$ is such that

$$\frac{\pi_{ex}(0)}{Z_2(0)} = \mathbb{E}_{\mathbb{Q}_{Z_2}}\left[\left(\frac{Z_1(T)}{Z_2(T)} - K\right)^+\right],$$

or

$$\pi_{ex}(0) = Z_1(0)\mathbb{Q}_{Z_1}(A) - KZ_2(0)\mathbb{Q}_{Z_2}(A),$$

where $\mathbb{Q}_{Z_i}$ is the equivalent martingale measure with Z_i as numéraire and $A = \{\omega : Z_1(T,\omega) > KZ_2(T,\omega)\}$.

Barrier Options

One-barrier options specify a stock-price level, H say, such that the option pays ('knocks in') or not ('knocks out') according to whether or not level H is attained, from below ('up') or above ('down'). There are thus four possibilities: 'up and in', 'up and out', 'down and in' and 'down and out'. Since barrier options are path-dependent (they involve the behaviour of the path, rather than just the current price or price at expiry), they may be classified as exotic; alternatively, the four basic one-barrier types above may be regarded as 'vanilla barrier' options, with their more complicated variants as 'exotic barrier' options.

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Down-and-out Call

Consider a down-and-out call option with strike K and barrier H. The payoff is

$$
\begin{aligned}
& (S(T)-K)^{+}\mathbf{1}_{\{\min S(.)\geq H\}} \\
= \quad & (S(T)-K)\mathbf{1}_{\{S(T)\geq K, \min S(.)\geq H\}},
\end{aligned}
$$

so by risk-neutral pricing the value of the option $DOC_{K,H}$ is

$$
\mathbb{E}^{*}\left[e^{-rT}(S(T)-K)\mathbf{1}_{\{S(T)\geq K, \min S(.)\geq H\}}\right],
$$

where S is geometric Brownian motion.

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max and min of BM

Write $c := \frac{\left(b - \frac{1}{2}\sigma^2\right)}{\sigma}$; then

$$\min S(.) \geq H$$

iff

$$\min(ct + W(t)) \geq \sigma^{-1} \log(H/p_0).$$

Writing X for $X(t) := ct + W(t)$ – drifting Brownian motion with drift c, m^X, M^X for its minimum and maximum processes

$$m^X(t) := \min\{X(s) : s \in [0, t]\},$$

$$M^X(t) := \max\{X(s) : s \in [0, t]\},$$

the payoff function involves the bivariate process (X, m), and the option price involves the joint law of this process.

Reflection Principle

Consider $c = 0$. We require the joint law of standard BM and its maximum M (or minimum), (W, M).
We choose a level $b > 0$, and run the process until the *first-passage time* $\tau(b) := \inf\{t \geq 0 : W(t) \geq b\}$. This is a stopping time and by the strong Markov property the process begins afresh at level b, and by symmetry the probabilistic properties of its further evolution are invariant under *reflection* in the level b. This *reflection principle* leads to Lévy's joint density formula $(x < y)$

$$
\begin{aligned}
&\mathbb{P}_0\left(W(t) \in dx, M(t) \in dy\right)\\
= &\frac{2(2y - x)}{\sqrt{2\pi t^3}} \exp\left\{-\frac{1}{2}(2y - x)^2/t\right\}.
\end{aligned}
$$

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Density of $(X(t), M^X(t))$

Lévy's formula for the joint density of $(W(t), M(t))$ may be extended to the case of general drift c by the usual method for changing drift, Girsanov's theorem. The general result is

$$\begin{aligned}&\mathbb{P}_0\left(X(t) \in dx, M^X(t) \in dy\right)\\ = \quad &\frac{2(2y-x)}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(2y-x)^2}{2t} + cx - \frac{1}{2}c^2 t\right\}.\end{aligned}$$

Here as before $0 \leq x \leq y$.

Valuation Formula

It is convenient to decompose the price $DOC_{K,H}$ of the down-and-out call into the (Black-Scholes) price of the corresponding vanilla call, C_K say, and the *knockout discount*, $KOD_{K,H}$ say, by which the knockout barrier at H lowers the price:

$$DOC_{K,H} = C_K - KOD_{K,H}.$$

The option formula is, writing $\lambda := r - \frac{1}{2}\sigma^2$,

$$\begin{aligned} KOD_{K,H} &= p_0(H/p_0)^{2+2\lambda/\sigma^2}\Phi(c_1(p_0,T))) \\ &\quad -Ke^{-rT}(H/p_0)^{2\lambda/\sigma^2}\Phi(c_2(p_0,T)), \end{aligned}$$

where c_1, c_2 are given by

$$c_{1,2}(p,t) = \frac{\log(H^2/pK) + (r \pm \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}$$

Binary Options

We assume that the underlying model is a Black-Scholes model. Recall the payoffs for a European call resp. put

$$B^{call} = (S(T)-K)\mathbf{1}_{\{S(T)>K\}} \quad \text{resp.} \quad B^{put} = (K-S(T))\mathbf{1}_{\{S(T)<K\}}.$$

A binary (or digital) option is a contract whose payoff depends in a discontinuous way on the terminal price of the underlying asset.

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Binary Options

Simple Binary Option. The terminal payment at $t = T$ of the call resp. put are

$$B_d^{call} = \mathbf{1}_{\{S(T)>K\}} \quad \text{resp.} \quad B_d^{put} = \mathbf{1}_{\{S(T)<K\}}.$$

The valuation formula follows from the risk-neutral valuation principle and is

$$\begin{aligned} \pi_d^{call}(t) &= e^{-r(T-t)}\Phi(d_2(S,T)), && \text{"Digital call"} \\ \pi_d^{put}(t) &= e^{-r(T-t)}\Phi(-d_2(S,T)), && \text{"Digital put"} \end{aligned}$$

with

$$d_2(s,t) = \frac{\log(s/K) + (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}.$$

Binary Options

Cash-or-nothing options. Here the payoffs at expiry of the European call resp. put are given by

$$BC_d^{call} = C\mathbf{1}_{\{S(T)>K\}} \quad \text{resp.} \quad BC_d^{put} = C\mathbf{1}_{\{S(T)<K\}}.$$

Asset-or-nothing options. Here the corresponding payoffs are

$$BA_d^{call} = S(T)\mathbf{1}_{\{S(T)>K\}} \quad \text{resp.} \quad BA_d^{put} = S(T)\mathbf{1}_{\{S(T)<K\}}.$$

Observe that we have the decomposition

$$BA_d^{call} = B^{call} + KB_d^{call}$$

Gap Options

Gap options, with payoffs

$$B_{Gap}^{call} = (S(T)-C)\mathbf{1}_{\{S(T)>K\}} \quad \text{resp.} \quad B_{Gap}^{put} = (C-S(T))\mathbf{1}_{\{S(T)<K\}}.$$

Here the relevant decomposition for the call is

$$B_{Gap}^{call} = B^{call} - (C-K)B_d^{call}.$$

Super-share options, with payoffs

$$B_{SS}^{call} = \frac{S(T)}{K_1}\mathbf{1}_{\{K_1<S(T)<K_2\}}.$$

Paylater Options

Paylater options have final payoff

$$B_{PL}^{call} = (S(T) - (K + D^{call}))\mathbf{1}_{\{S(T)>K\}} \quad \text{resp.} \quad B_{PL}^{put} = ((K - D^{put}) - S(T))$$

where D^{call}, D^{put} have to be determined in such a way that the prices of the paylater options equal zero at $t = 0$.
The relevant decompositions are

$$B_{PL}^{call} = B^{call} - D^{call} B_d^{call} \quad \text{resp.} \quad B_{PL}^{put} = B^{put} - D^{put} B_d^{put},$$

so

$$D^{call} = \frac{\pi^{call}(0)}{\pi_d^{call}(0)} \quad \text{resp.} \quad D^{put} = \frac{\pi^{put}(0)}{\pi_d^{put}(0)}.$$

Compound Options

Compound Options give the right to buy (sell) at $t = T$ another option with maturity $T_1 \geq T$.

$$
\begin{aligned}
B_{com}^{CC} &= (\pi^{call}(T) - K)^+ && \text{"Call on a call"} \\
B_{com}^{CP} &= (\pi^{put}(T) - K)^+ && \text{"Call on a put"} \\
B_{com}^{PC} &= (K - \pi^{call}(T))^+ && \text{"Put on a call"} \\
B_{com}^{PP} &= (K - \pi^{put}(T))^+ && \text{"Put on a put".}
\end{aligned}
$$

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Valuation formula

There exists a uniquely determined $p^* > 0$ for $T \leq T_1$ such that for $S(T) = p^*$ we have

$$\pi^{call}(T, p^*) = K$$

Define

$$\begin{aligned} g_1(t) &= \frac{\log(S(t)/p^*) + (r + \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{(T - t)}}, \\ g_2(t) &= g_1(t) - \sigma\sqrt{T - t} \end{aligned}$$

and

$$\begin{aligned} h_1(t) &= \frac{\log(S(t)/K)_1 + (r + \frac{\sigma^2}{2})(T_1 - t)}{\sigma\sqrt{T_1 - t}}, \\ h_2(t) &= h_1(t) - \sigma\sqrt{T_1 - t}. \end{aligned}$$

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Valuation formula

Then the price of a call on a call is

$$\begin{aligned}\pi_{com}^{CC}(t) &= S(t)N^{(\rho_1)}(g_1(t), h_1(t)) \\ &-K_1 e^{-r(T_1-t)} N^{(\rho_1)}(g_2(t), h_2(t)) \\ &-Ke^{-r(T-t)}\Phi(g_2(t)),\end{aligned}$$

for $t \in [0, T]$, where $N^{(\rho)}(x, y)$ is the distribution of a bivariate standard normal distribution with correlation coefficient ρ and where $\rho_1 = \sqrt{\frac{T-t}{T_1-t}}$.

Asian Options

An Asian option is an option on a time average of the underlying asset.
Asian calls and puts have payoffs

$$(\bar{S} - K)^+$$

and

$$(\bar{S} - K)^-,$$

where the strike K is a constant and

$$\bar{S} = \frac{1}{n} \sum_{i=1}^{n} S(t_i)$$

is the average price of the underlying asset over the discrete set of monitoring dates
Thus: Asian option under discrete monitoring.

Asian Options

There are no exact formulas for the price of such options, because the distribution of $\bar{S}$ (average of lognormals) is intractable.

Variants are $(\bar{S} - S(t))^+$ and $(\bar{S} - S(t))^-$.

Asian Options

For Asian options under continuous monitoring

$$\bar{S} = \frac{1}{T - u} \int_u^T S(t)dt$$

over an interval $[u, T]$.
There are pricing approaches using transform analysis (Laplace transforms!).

Geometric average option

There the average $\bar{S}$ is replaced by

$$\left(\prod_{i=1}^{n} S(t_i)\right)^{1/n}$$

i.e. the geometric average of the underlying.
In the Black Scholes setting we have

$$\prod_{i=1}^{n} S(t_i)^{1/n} = S(0)\exp\left\{\left(r-\frac{1}{2}\sigma^2\right)\cdot\frac{1}{n}\sum_{i=1}^{n} t_i + \frac{\sigma}{n}\sum_{i=1}^{n} W(t_i)\right\}.$$

Geometric average option

We know $\mathbb{C}\mathrm{ov}(W(t_i)W(t_j)) = \min(t_i, t_j) = \sigma_{ij}$. Also the linear transformation stability of the normal distribution implies that for $X \sim \mathcal{N}(\mu, \Sigma)$, we have $a'X \sim \mathcal{N}(a'\mu, a'\Sigma a)$. So

$$\sum_{i=1}^{n} W(t_i) \sim \mathcal{N}\left(0, (1, \ldots, 1)(\sigma_{ij})_{n,n}\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}\right) = \mathcal{N}(0, \sum_{i=1}^{n}(2i-1)t_{n+1-i}).$$

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Geometric average option

So the geometric average of $S(t_1), \ldots, S(t_n)$ has the same distribution as the value of a time T $GBM(r - \delta, \bar{\sigma}^2)$ with

$$T = \frac{1}{n}\sum_{i=1}^{n} t_i, \qquad \bar{\sigma}^2 = \frac{\sigma^2}{n^2 T}\sum_{i=1}^{2}(2i-1)t_{n+1+j}, \qquad \delta = \frac{1}{2}\sigma^2 - \frac{1}{2}\bar{\sigma}^2.$$

Thus an option on the geometric average can be priced using the standard Black Scholes formula.

Pricing by simulation

Valuation of a European Call with maturity T strike K on S under constant interest rate r. By risk-neutral pricing

$$C(0) = \mathbb{E}(e^{-rT}(S(T) - K)^+).$$

For the Black-Scholes model

$$dS_t = S_t(rdt + \sigma dW_t)$$

so

$$S_T = S_0 \cdot \exp\left\{\left(r - \frac{1}{2}\sigma^2\right) T + \sigma W_T\right\}$$

with $W_T = \sqrt{T} \cdot Z \; Z \sim \mathcal{N}(0, 1)$.

Pricing by simulation

Also

$$
\begin{aligned}
C(0) &= S_0 \Phi\left(\frac{log(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
&\quad - e^{-rT} K \cdot \Phi\left(\frac{log(S/K) + (r - \frac{1}{2}\sigma^2)T)}{\sigma\sqrt{T}}\right).
\end{aligned}
$$

We can now compare the Monte-Carlo approach with the exact valuation formula.

Pricing by simulation

Algorithm:

for $i = 1, \ldots n$ *generate* $Z_i \sim \mathcal{N}(0,1)$ *independent*
set $S_i(T) = \exp((r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z_i)$
set $C_i = e^{-rT}(S_i(T) - K)^+$
set $\hat{C}_n = \frac{1}{n}(C_1 + \ldots + C_n)$

Pricing by simulation

We note that $\hat{C}_n$ is unbiased, i.e. $\mathbb{E}(\hat{C}_n) = C$ and strongly consistent: $\hat{C}_n - C \to 0$ a.s. $(n \to \infty)$. Also we can provide a confidence interval. Let

$$S_c = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(C_i - \hat{C}_n)^2}$$

denote the sample standard deviation and let z_δ denote the $1-\delta$ quantile of the standard normal $(\Phi(z_\delta) = 1-\delta)$. Then

$$\left[\hat{C}_n - z_{\delta/2}\frac{S_c}{\sqrt{n}}, \hat{C}_n + z_{\delta/2}\frac{S_c}{\sqrt{n}}\right]$$

is an asymptotically valid $1-\delta$ confidence interval for C.

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Pricing by simulation

Recall that $\hat{C}_n - \mathbb{C} \approx \mathcal{N}(0, S_c/\sqrt{n})$, then
$\Phi(-z_\delta) = 1 - \Phi(z_\delta) = \delta$

$$\begin{aligned} \mathbb{P}(-z_{\delta/2} \leq (C - \hat{C}_n)\tfrac{\sqrt{n}}{S_c} \leq z_{\delta/2}) &= 1 - \delta \\ \Leftrightarrow \mathbb{P}(\hat{C}_n - z_{\delta/2}\tfrac{S_c}{\sqrt{n}} \leq C \leq \hat{C}_n + z_{\delta/2}\tfrac{S_c}{\sqrt{n}}) &= 1 - \delta \end{aligned}$$

For $\delta = 0.05$ we have $z_{\delta/2} \approx 1.96$ for example.

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Pricing by simulation – Asian Options

Consider the payoff $(\bar{S} - K)^+$ with $\bar{S} = \frac{1}{m}\sum_{j=1}^{m} S(t_j)$ for a set of fixed dates $0 = t_0 < t_1 < \ldots < t_m \leq T$ with T the date at which the payoff is received.

Now we need the values of $S(t)$ at a number of intermediate points. However the basic problem is the same, simulate $S(t_{j+1})$ given $S(t_j)$, so

$$S(t_{j+1}) = S(t_j) \cdot \exp((r - \frac{1}{2}\sigma^2)(t_{j+1} - t_j) + \sigma\sqrt{t_{j+1} - t_j} Z_{j+1})$$

where $Z_1, \ldots, Z_m$ are independent $N(0, 1)$.

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Pricing by simulation – Asian Options

Algorithm

for $i = 1, \ldots, n$
for $j = 1, \ldots, m$ *generate* $Z_{ij} \sim \mathcal{N}(0, 1)$ *independent*
set $S_i(t_j) = S_i(t_{j-1}) \exp(\ldots Z_{ij})$
set $\bar{S}_i = \frac{1}{m}(S_i(t_1) + \ldots + S_i(t_m))$
set $C_i = e^{-rT}(\bar{S}_i - K)^+$
set $\hat{C}_n = (C_1 + \ldots + C_n)/n$

Indexed Options

Consider the following financial market model

$$
\begin{array}{lrcl}
\text{bank account} & dB(t) & = & rB(t)dt \\
\text{stock 1} & dS_1(t) & = & S_1(t)\left(b_1 dt + \sigma_{11} dW_1(t) + \sigma_{12} dW_2(t)\right); \\
\text{stock 2} & dS_2(t) & = & S_2(t)\left(b_2 dt + \sigma_{21} dW_1(t) + \sigma_{22} dW_2(t)\right)
\end{array}
$$

with all coefficients constant, and the volatility matrix $\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$ non-singular. An indexed option with parameters a_1, a_2 is given by the final payout

$$B_{ind} = (a_1 S_1(T) - a_2 S_2(T))^+$$

Indexed Options

Valuation can be done by using the change-of-numéraire formula outlined above. With the notation

$$\begin{aligned}\tilde{\sigma}_1^2 &= \sigma_{11}^2 + \sigma_{12}^2; \quad \tilde{\sigma}_2^2 = \sigma_{21}^2 + \sigma_{22}^2 \\ \rho &= \frac{\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}}{\tilde{\sigma}_1\tilde{\sigma}_2}\end{aligned}$$

we obtain

$$\begin{aligned}\pi_{ind}(0) &= a_1 s_1 \Phi\left(\frac{\log(s_1 a_1/(s_2 a_2) + (\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - 2\rho\tilde{\sigma}_1\tilde{\sigma}_2)T/2)}{\sqrt{(\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - 2\rho\tilde{\sigma}_1\tilde{\sigma}_2)T}}\right) \\ &\quad -a_2 s_2 \Phi\left(\frac{\log(s_1 a_1/(s_2 a_2) - (\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - 2\rho\tilde{\sigma}_1\tilde{\sigma}_2)T/2)}{\sqrt{(\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - 2\rho\tilde{\sigma}_1\tilde{\sigma}_2)T}}\right)\end{aligned}$$

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Options on the minimum/maximum of two stocks

The payoffs are given by

$$B^{Call}_{min} = (\min\{S_1(T), S_2(T)\} - K)^+ \qquad \text{"Call on minimum"}$$

$$B^{Call}_{max} = (\max\{S_1(T), S_2(T)\} - K)^+ \qquad \text{"Call on maximum"}$$

$$B^{Put}_{min} = (K - \min\{S_1(T), S_2(T)\})^+ \qquad \text{"Put on minimum"}$$

$$B^{Call}_{min} = (K - \max\{S_1(T), S_2(T)\})^+ \qquad \text{"Call on maximum"}$$

Options on the minimum/maximum of two stocks

We use the notation

$$
\begin{aligned}
\sigma^2 &= \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 - 2\rho\tilde{\sigma}_1\tilde{\sigma}_2 \\
d_1 &= \frac{log(s_1/K) + \left(r + \frac{1}{2}\tilde{\sigma}_1^2\right)T}{\tilde{\sigma}_1\sqrt{T}} \\
d_2 &= \frac{log(s_2/K) + \left(r + \frac{1}{2}\tilde{\sigma}_2^2\right)T}{\tilde{\sigma}_2\sqrt{T}} \\
d_3 &= \frac{log(s_1/s_2) - \left(r + \frac{1}{2}\sigma_1^2\right)T}{\sigma_1\sqrt{T}} \\
d_4 &= \frac{log(s_1/s_2) - \left(r + \frac{1}{2}\sigma_1^2\right)T}{\sigma_1\sqrt{T}}
\end{aligned}
$$

Options on the minimum/maximum of two stocks

The prices of the minimum/maximum options are given

$$
\begin{aligned}
X_{min}^{Call}(0) &= s_1\Phi^{(\tilde{\rho})}(d_1, d_3) + s_2\Phi^{(\tilde{\rho})}(d_2, d_4) \\
&\quad - Ke^{-rT}\Phi^{(\tilde{\rho})}(d_1 - \tilde{\sigma}_1\sqrt{T}, d_2 - \tilde{\sigma}_2\sqrt{T}) \\
X_{min}^{Put}(0) &= X_{min}^{Call}(0) + Ke^{-rT} - s_1\Phi(d_3) - s_2\Phi(d_4) \\
X_{max}^{Call}(0) &= X_{(1)}^{Call}(0) + X_{(2)}^{Call}(0) - X_{min}^{Call}(0) \\
X_{max}^{Put}(0) &= X_{(1)}^{Put}(0) + X_{(2)}^{Put}(0) - X_{min}^{Put}(0).
\end{aligned}
$$

Financial Mathematics

Lecture - Winter 2013/14

Professor Dr. Rüdiger Kiesel | Chair for Energy Trading and Finance | University of Duisburg-Essen

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Part V

Interest Rates

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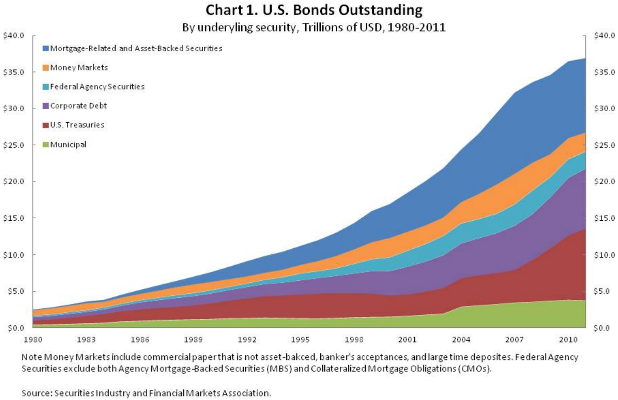
Development Bond Market



Basic Interest Rates

- Economic agents have to be rewarded for postponing consumption; in addition, there is a risk premium for the uncertainty of the size of future consumption.
- Investors, Firms, banks pay compensation for the willingness to postpone
- A common interest rate (equilibrium) emerges which allows to fulfill the aggregate liquidity demand.

Basic Interest Rates

Interest rates change with different maturities because

- *market segmentation hypothesis* different agents have different preferences for borrowing and lending in different segments of the yield curve. Agents can change their segment if the compensation for switching is high enough.
- *expectation hypothesis* today's interest rates (spot rates) are determined on the basis of the expected future rates plus a risk premium.
- *liquidity preference hypothesis* on average agents prefer to invest in short-term assets. Here Liquidity refers to 'closeness to maturity'. Agents are compensated through a risk premium for investing in longer dated assets.

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Yield Curves

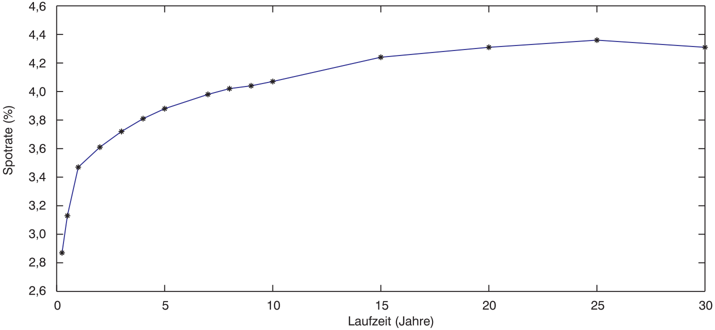

Abbildung 1.1: Aus dt. Bundeswertpapieren geschätzte Zinsstrukturkurve (Spotrates, jährliche Basis) vom 10.07.2006. Quelle: Bloomberg L.P.

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Yield Curves

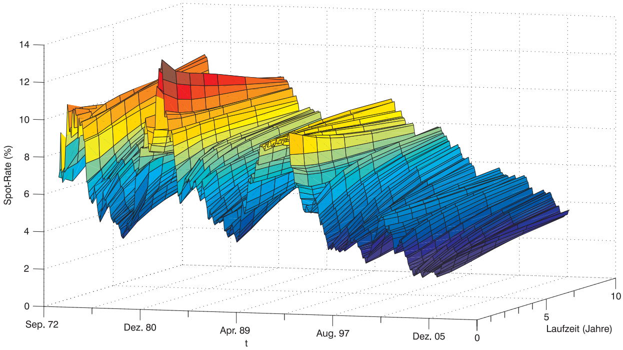
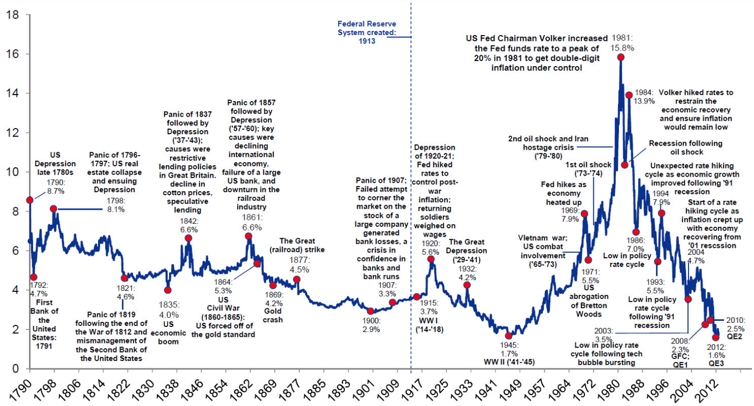


Abbildung 4.2: Entwicklung der deutschen Zinsstrukturkurve

Daten: Staatsanleihen mit 1,2,3,4,5,7 und 10 Jahren Laufzeit, monatlich, 30.9.72 bis 28.2.06.

10y yield US

The long history of long (10-year US treasuries) yields



Source: Global Financial Database, Goldman Sachs Global ECS Research. Special thanks to Jose Ursua.

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Notation

$p(t, T)$ denotes the price of a risk-free zero-coupon bond at time t that pays one unit of currency at time T.
We will use continuous compounding, i.e. a zero bond with interest rate $r(t, T)$ maturing at T will have the price

$$p(t, T) = e^{-r(t,T)(T-t)}.$$

Forward Rates

Given three dates $t < T_1 < T_2$ the basic question is: what is the risk-free rate of return, determined at the contract time t, over the interval $[T_1, T_2]$ of an investment of 1 at time T_1?

Time	t	T_1	T_2
	Sell T_1 bond Buy $\frac{p(t,T_1)}{p(t,T_2)}$ T_2 bonds	Pay out 1	Receive $\frac{p(t,T_1)}{p(t,T_2)}$
Net investment	0	-1	$+\frac{p(t,T_1)}{p(t,T_2)}$

Table : Arbitrage table for forward rates

Forward Rates

To exclude arbitrage opportunities, the equivalent constant rate of interest R over this period (we pay out 1 at time T_1 and receive $e^{R(T_2-T_1)}$ at T_2) has thus to be given by

$$e^{R(T_2-T_1)} = \frac{p(t,T_1)}{p(t,T_2)}.$$

Various Interest Rates

- The forward rate at time t for time period $[T_1, T_2]$ is defined as

$$R(t, T_1, T_2) = \frac{\log(p(t, T_1)) - \log(p(t, T_2))}{T_2 - T_1}$$

- The spot rate for the time period $[T_1, T_2]$ is defined as

$$R(T_1, T_2) = R(T_1, T_1, T_2)$$

- The instantaneous forward rate is

$$f(t, T) = -\frac{\partial \log(p(t, T))}{\partial T}$$

- The instantaneous spot rate is

$$r(t) = f(t, t)$$

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Rates

- The forward rate is the interest rate at which parties at time t agree to exchange K units of currency at time T_1 and give back $Ke^{R(t,T_1,T_2)(T_2-T_1)}$ units at time T_2. This means, one can lock in an interest rate for a future time period today.
- The spot rate $R(t, T_1)$ is the interest rate (continuous compounding) at which one can borrow money today and has to pay it back at T_1.
- The instantaneous forward and spot rate are the corresponding interest rates at which one can borrow money for an infinitesimal short period of time.
- The spot rate $R(t, T)$ as a function of T is referred to as the yield curve at time t.

Simple Relations

The money account process is defined by

$$B(t) = \exp\left\{\int_0^t r(s)ds\right\}.$$

The interpretation of the money market account is a strategy of instantaneously reinvesting at the current short rate.
For $t \leq s \leq T$ we have

$$p(t,T) = p(t,s)\exp\left\{-\int_s^T f(t,u)du\right\},$$

and in particular

$$p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}.$$

Simple Spot Rate

The simply-compounded spot interest rate prevailing at time t for the maturity T is denoted by $L(t, T)$ and is the constant rate at which an investment has to be made to produce an amount of one unit of currency at maturity, starting from $p(t, T)$ units of currency at time t, when accruing occurs proportionally to the investment time.

$$L(t, T) = \frac{1 - p(t, T)}{\tau(t, T)p(t, T)} \tag{4}$$

Here $\tau(t, T)$ is the daycount for the period $[t, T]$ (typically $T - t$).

Simple Spot Rate

- The bond price can be expressed as

$$p(t,T) = \frac{1}{1 + L(t,T)\tau(t,T)}.$$

  Other 'daycounts' denoted by $\tau(t,T)$ are possible.
- Notation is motivated by LIBOR rates (London InterBank Offered Rates).

Forward Rate Agreements

In order to introduce simply-compounded forward interest rates we consider forward-rate agreements (FRA). A FRA involves the current time t, the expiry time $T > t$ and the maturity time $S > T$. The contract gives its holder an interest-rate payment for the period between T and S. At maturity S, a fixed payment based on a fixed rate K is exchanged against a floating payment based on the spot rate $L(T, S)$ resetting in T with maturity S.

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Forward Rate Agreements

Formally, at time S one receives $\tau(T,S)K \cdot N$ units of currency and pays the amount $\tau(T,S)L(T,S) \cdot N$, where N is the contract nominal value. The value of the contract is therefore at S

$$N\tau(T,S)(K - L(T,S)). \tag{5}$$

We write this in terms of bond prices as

$$N\tau(T,S)\left(K - \frac{1-p(T,S)}{\tau(T,S)p(T,S)}\right) = N\left(K\tau(T,S) - \frac{1}{p(T,S)} + 1\right).$$

Forward Rate Agreements

Now we discount to obtain the value of this time S cashflow at t

$$
\begin{aligned}
& FRA(t, T, S, \tau(T, S), N, K) \\
= \quad & Np(t, S)\left(K\tau(T, S) - \frac{p(t,T)}{p(t,T)p(T,S)} + 1\right) \\
= \quad & N(Kp(t, S)\tau(T, S) - p(t, T) + p(t, S)).
\end{aligned}
$$

There is only one value of K that renders the contract value 0 at t. The resulting rate defines the simply-compounded forward rate.

Simply-Compounded Forward Interest Rate

The simply-compounded forward interest rate prevailing at time t for the expiry $T > t$ and maturity $S > T$ is denoted by $F(t; T, S)$ and is defined by

$$F(t; T, S) := \frac{1}{\tau(T, S)} \left[\frac{p(t, T)}{p(t, S)} - 1 \right]. \tag{6}$$

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Simply-Compounded Forward Interest Rate

- $FRA(\ldots) = Np(t,S)\tau(T,S)(K - F(t;T,S))$ is an equivalent definition.
- To value a FRA (typically with a different K) replace the LIBOR rate in (5) by the corresponding forward rate $F(t;T,S)$ and take the present value of the resulting quantity.

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Interest-Rate Swap

A generalisation of the FRA is the Interest-Rate Swap (IRS). A Payer (Forward-start) Interest-Rate Swap (PFS) is a contract that exchanges payments between two differently indexed legs, starting from a future time instant. At every instant T_i in a prespecified set of dates $T_{\alpha+1}, \ldots, T_\beta$ the fixed leg pays out the amount

$$N\tau_i \cdot K$$

corresponding to a fixed interest rate K, a nominal value N, and a year fraction τ_i between T_{i-1} and T_i, whereas the floating leg pays the amount

$$N\tau_i L(T_{i-1}, T_i).$$

Corresponding to the interest rate $L(T_{i-1}, T)$ resetting at the previous instant T_{i-1} for the maturity given by the current payment instant T_i, with T_α a given date.

Interest-Rate Swap

Set

$$\mathcal{T} := \{T_\alpha, \dots, T_\beta\} \quad \text{and} \quad \tau := \{\tau_{\alpha+1}, \dots, \tau_\beta\}.$$

Payers IRS(PFS): fixed leg is paid and floating leg is received
Receiver IRS (RFS): fixed leg is received and floating leg is paid.
The discounted payoff at time $t < T_\alpha$ of a PFS is

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i) N \tau_i (L(T_{i-1}, T_i) - K)$$

with $D(t, T)$ the discount factor (typically from bank account).
For a RFS we have

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i) N \tau_i (K - L(T_{i-1}, T_i)).$$

Interest-Rate Swap

We can view the last contract as a portfolio of FRAs and find

$$
\begin{aligned}
& RFS(t, \mathcal{T}, \tau, N, K) \\
= \quad & \sum_{i=\alpha+1}^{\beta} FRA(t, T_{i-1} T_i, \tau_i, N, K) \\
= \quad & N \sum_{i=\alpha+1}^{\beta} \tau_i p(t, T_i)(K - F(t, T_{i-1}, T_i)) \\
= \quad & -Np(t, T_\alpha) + Np(t, T_\beta) + N \textstyle\sum_{i=\alpha+1}^{\beta} \tau_i K p(t, T_i).
\end{aligned}
$$

The two legs of an IRS can be viewed as coupon-bearing bond (fixed leg) and floating rate note (floating leg).

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Interest-Rate Swap

A floating-rate note is a contract ensuring the payment at future times $T_{\alpha+1}, \dots, T_\beta$ of the LIBOR rates that reset at the previous instants $T_\alpha, \dots, T_{\beta-1}$. Moreover, the note pays a last cash flow consisting of the reimbursement of the notational value of the note at the final time T_β.

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Interest-Rate Swap

We can value the note by changing sign and setting $K = 0$ in the RFS formula and adding it to $Np(t, T_\beta)$, the present value of the cash flow N at T_β. So we see

$$\underbrace{-RFS(t, T, \tau, N, 0) + Np(t, T_\beta)}_{\text{value of note}} = \underbrace{Np(t, T_\alpha)}_{\text{from RFS formula}} \quad .$$

This implies that the note is always equivalent to N units at its first reset date T_α (the floating note trades at par).

Interest-Rate Swap

- We require the IRS to be fair at time t to obtain the forward swap rate.
- The forward swap rate $S_{\alpha,\beta}(t)$ at time t for the sets of time $\mathcal{T}$ and year fractions τ is the rate in the fixed leg of the above IRS that makes the IRS a fair contract at the present time, i.e. it is the K for which $RFS(t, T, \tau, N, K) = 0$.
- We obtain

$$S_{\alpha,\beta}(t) = \frac{p(t, T_\alpha) - p(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i p(t, T_i)}. \qquad (7)$$

Interest-Rate Swap

We write (7) in terms of forward rates. First divide the numerator and the denominator by $p(t, T_\alpha)$ and observe that

$$\frac{p(t, T_k)}{p(t, T_\alpha)} = \prod_{j=\alpha+1}^{k} \frac{p(t, T_j)}{p(t, T_{j-1})} = \prod_{j=\alpha+1}^{k} \frac{1}{1 + \tau_j F_j(t)}$$

with $F_j(t) := F(t, T_{j-1}; T_j)$. So (7) can be written as

$$S_{\alpha,\beta}(t) = \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1+\tau_j F_j(t)}}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j F_j(t)}}. \tag{8}$$

Caps

- A cap is a contract where the seller of the contract promises to pay a certain amount of cash to the holder of the contract if the interest rate exceeds a certain predetermined level (the cap rate) at a set of future dates.
- It can be viewed as a payer IRS where each exchange payment is executed only if it has positive value.
- The cap discounted payoff is

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i) N \tau_i (L(T_{i-1}, T_i) - K)^+.$$

- Each individual term is a caplet.

Floors

- A floor is equivalent to a receiver IRS where each exchange is executed only if it has positive value.
- The floor discounted payoff is

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i) N \tau_i (K - L(T_{i-1}, T_i))^+ .$$

- Each individual term is a floorlet.

Black's Formula for Caps

Pricing is done via Black's formula

$$Cap^{\text{Black}}(0, \mathcal{T}, \tau, N, K, \sigma_{\alpha,\beta}) = N \sum_{i=\alpha+1}^{\beta} p(0, T_i)\tau_i Bl(K, F(0, T_{i-1}, T_i), v_i, 1),$$

where

$$\begin{aligned}
Bl(K, F, v, \omega) &= F\omega\Phi(\omega d_i(K, F, v)) - K\omega\Phi(\omega d_2(K, F, v)) \\
d_1(K, F, v) &= \frac{\log(F/K) + v^2/2}{v} \\
d_2(K, F, v) &= \frac{\log(F/K) - v^2/2}{v} \\
v_i &= \sigma_{\alpha,\beta}\sqrt{T_{i-1}}
\end{aligned}$$

with the volatility parameter $\sigma_{\alpha,\beta}$ retrieved from market quotes.

Black's Formula for Floors

The corresponding floor is priced according to

$$
\begin{aligned}
& Flr^{\text{Black}}(0, \mathcal{T}, \tau, N, K, \sigma_{\alpha,\beta}) \\
= \quad & N \sum_{i=\alpha+1}^{\beta} p(0, T_i)\tau_i Bl(K, F(0, T_{i-1}, T_i), v_i, -1).
\end{aligned}
$$

Simple Properties

A cap (floor) is said to be at-the-money (ATM) if and only if

$$K = K_{ATM} := S_{\alpha,\beta}(0) = \frac{p(0, T_\alpha) - p(0, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i p(0, T_i)}.$$

The cap is instead said to be in-the-money (ITM) if $K < K_{ATM}$, and out-of-the-money (OTM) if $K > K_{ATM}$, with the converse holding for a floor.

Simple Properties Cap

- Simple protection against rising interest rates, but requires the payment of a premium.
- Strike is the maximal interest to be paid.
- Advantageous only if market expectation becomes true.

Swaptions

- Swap options or more commonly swaptions are options on an IRS. A European payer swaption is an option giving the right (and not the obligation) to enter a payer IRS at a given future time, the swaption maturity. Usually the swaption maturity coincides with the first reset date of the underlying IRS.
- The underlying-IRS length ($T_\beta - T_\alpha$) is called the tenor of the swap.
- The discounted payoff of a payer swaption can be written by considering the value of the underlying payer IRS at its first reset date T_α (also the maturity of the swaption)

$$N \sum_{i=\alpha+1}^{\beta} p(T_\alpha, T_i)\tau_i(F(T_\alpha; T_{i-1}, T_i) - K).$$

Swaptions Payoff

The option will be exercised only if this value is positive. So the current value is

$$ND(t, T_\alpha)\left(\sum_{i=\alpha+1}^{\beta} p(T_\alpha, T_i)\tau_i(F(T_\alpha; T_{i+1}, T_i) - K)\right)^{+}.$$

Swaptions Payoff

Since the positive part operator is a piece-wise linear and convex function we have

$$\left(\sum_{i=\alpha+1}^{\beta} p(T_\alpha, T_i)\tau_i(F(T_\alpha; T_{i-1}, T_i) - K)\right)^+$$
$$\leq \sum_{i=\alpha+1}^{\beta} p(T_\alpha, T_i)\tau_i(F(T_\alpha; T_{i-1}, T_i) - K)^+$$

with strict inequality in general. Thus an additive decomposition is not feasible.

Black's Formula for Swaptions

Swaptions are also valued with a Black-like formula

$$
\begin{aligned}
& PS^{\text{Black}}(0, \mathcal{T}, \tau, N, K, \sigma_{\alpha,\beta}) \\
= \quad & NBl(K, S_{\alpha,\beta}(0), \sigma_{\alpha,\beta}\sqrt{T_\alpha}, 1) \sum_{i=\alpha+1}^{\beta} \tau_i p(0, T_i)
\end{aligned}
$$

where $\sigma_{\alpha,\beta}$ is now a volatility parameter quoted in the market different from the corresponding $\sigma_{\alpha,\beta}$ in the cap/floor case.

Black's Formula for Swaptions

A receiver swaption gives the holder the right to enter at time T_α a receiver IRS with payment date in $\mathcal{T}$. Its Black-type valuation formula is

$$
\begin{aligned}
& RS^{\text{Black}}(0, \mathcal{T}, \tau, \mathcal{N}, \mathcal{K}, \sigma_{\alpha,\beta}) \\
= \quad & NBL(K, S_{\alpha,\beta}(0), \sigma_{\alpha,\beta}\sqrt{T_\alpha}, -1) \sum_{i=\alpha+1}^{\beta} \tau_i p(0, T_i).
\end{aligned}
$$

Swaptions Payoff

A swaption (either payer or receiver) is said to be at-the-money (ATM) if and only if

$$K = K_{ATM} = S_{\alpha,\beta}(0) = \frac{p(0,T_\alpha) - p(0,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i p(0,T_i)}.$$

The payer swaption is instead said to be in-the-money (ITM) if $K < K_{ATM}$, and out-of-the-money (OTM) if $K > K_{ATM}$. The receiver swaption is ITM if $K > K_{ATM}$, and OTM if $K < K_{ATM}$.

Derivatives

- Derivatives (contingent claims, options) are viewed a random variables, whose value depends on some underlying, i.e. $X = f(S)$.
- The (no-arbitrage) price process of a derivative is given by the risk-neutral valuation formula

$$\Pi_X(t) = D(0)\mathbb{E}^* \left[\frac{X}{D(T)} \right],$$

where $D(.)$ is some discount-factor (the numéraire.

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Example: The money market account as numéraire.

Assuming that a riskless asset (e.g. bank account) exists, it is natural to take it as a numéraire. Assume

$$B(t) = \exp\left\{\int_0^t r(u)du\right\}.$$

The discounted price process of a security with respect to the numéraire $B(t)$ is simply

$$\bar{S}(t) = \exp\left\{-\int_0^t r(u)du\right\} S(t).$$

'Historically' $B(t)$ was used as the numéraire $S_0(t)$ and then $\mathbb{Q}_B = \mathbb{P}^*$.

Change of numéraire technique – Concept

- With a given numéraire we use an equivalent martingale measure $\mathbb{P}^*$ such that the discounted basic security price processes are $\mathbb{P}^*$-martingales. We then calculate derivative prices as expectations under $\mathbb{P}^*$ probabilities.
- However, there may be situations where a different discount factor is more useful. Then we can change the numéraire and calculate prices of derivatives as expectations under new probabilities.
- So, prices can be calculated under any numéraire pair $X(t), \mathbb{Q}_X$, i.e. a process and a corresponding probability measure such the discounted basic security price processes are $\mathbb{Q}_X$-martingales.

Example: Zero-coupon bonds as numéraire

- A zero-coupon bond is the natural choice of numéraire if one looks at the time 0 price of an asset giving the right to a single cash-flow at a well-defined future time T. The simplest such asset is a zero-coupon bond with cash-flow 1 at time T. We denote its time t price by $p(t, T)$. We assume that the money-market account $B(t)$ (as defined above) was used to define $\mathbb{P}^*$.
- The corresponding probabilities are

$$\frac{d\mathbb{Q}^T}{d\mathbb{P}^*} = \frac{1}{p(0, T)B(t)} = \frac{1}{p(0, T)} \exp\left\{-\int_0^t r(u)du\right\}.$$

Example: Zero-coupon bonds as numéraire

The relative price process of a basic asset Z with respect to $p(t,T)$, i.e. $\bar{Z} = Z(t)/p(t,T)$, is called the forward price $F_Z(t)$ of the security Z, and given by

$$F_Z(t) = \mathbb{E}_{\mathbb{Q}^T}\left[Z(T)|\mathcal{F}_t\right],$$

implying that F_Z is a $\mathbb{Q}^T$ martingale.

Gaussian Forward Rates

- The dynamics of the forward rate are given under a risk-neutral martingale measure $\mathbb{P}^*$ by

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

  with deterministic forward rate volatility.
- Then the short-rate $r(t)$ as well as the forward rates $f(t,T)$ have Gaussian probability laws.

Zero-Coupon Bond Prices

Now $p(t, T)$ satisfies

$$dp(t,T) = p(t,T)\left\{\left(r(t) + A(t,T) + \frac{1}{2}S(t,T)^2\right)dt + S(t,T)dW(t)\right\},$$

where

$$A(t,T) = -\int_t^T \alpha(t,s)ds, \quad S(t,T) = -\int_t^T \sigma(t,s)ds.$$

So zero-coupon bond prices have a lognormal distribution.

Options on Bonds

Consider a European call C on a T^*-bond with maturity $T \leq T^*$ and strike K. So we consider the T-contingent claim

$$X = (p(T, T^*) - K)^+.$$

Its price at time $t = 0$ is

$$C(0) = p(0, T^*)\mathbb{Q}^*(A) - Kp(0, T)\mathbb{Q}^T(A),$$

with $A = \{p(T, T^*) > K\}$ and $\mathbb{Q}^T$ resp. $\mathbb{Q}^*$ the T- resp. T^*-forward risk-neutral measure.

Options on Bonds

$$\tilde{Z}(t,T) = \frac{p(t,T^*)}{p(t,T)}$$

has $\mathbb{Q}$-dynamics

$$d\tilde{Z} = \tilde{Z}\left\{S(S-S^*)dt - (S-S^*)dW(t)\right\},$$

so a deterministic variance coefficient. Now

$$\begin{aligned} & \mathbb{Q}^*(p(T,T^*) \geq K) \\ = \quad & \mathbb{Q}^*\left(\frac{p(T,T^*)}{p(T,T)} \geq K\right) \\ = \quad & \mathbb{Q}^*(\tilde{Z}(T,T) \geq K). \end{aligned}$$

Options on Bonds

Since $\tilde{Z}(t,T)$ is a $\mathbb{Q}^T$-martingale with $\mathbb{Q}^T$-dynamics

$$d\tilde{Z}(t,T) = -\tilde{Z}(t,T)(S(t,T) - S(t,T^*))dW^T(t),$$

we find that under $\mathbb{Q}^T$

$$\begin{aligned}\tilde{Z}(T,T) &= \frac{p(0,T^*)}{p(0,T)} \exp\left\{ -\int_0^T (S - S^*) dW_t^T \right\} \\ &\quad \times \exp\left\{ -\frac{1}{2}\int_0^T (S - S^*)^2 dt \right\}\end{aligned}$$

The stochastic integral in the exponential is Gaussian with zero mean and variance

$$\Sigma^2(T) = \int_0^T (S(t,T) - S(t,T^*))^2 dt.$$

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Options on Bonds

So

$$
\begin{aligned}
& \mathbb{Q}^T(p(T,T^*) \geq K) \\
= \quad & \mathbb{Q}^T(\tilde{Z}(T,T) \geq K) = \Phi(d_2)
\end{aligned}
$$

with

$$
d_2 = \frac{\log\left(\frac{p(0,T)}{Kp(0,T^*)}\right) - \frac{1}{2}\Sigma^2(T)}{\sqrt{\Sigma^2(T)}}.
$$

We need to repeat the argument to get the price of the call option.

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Price of Call Option

The price of the call option is given by

$$C(0) = p(0, T^*)\Phi(d_2) - Kp(0, T)\Phi(d_1),$$

with parameters given as above and

$$d_1 = \frac{\log\left(\frac{p(0,T)}{Kp(0,T^*)}\right) + \frac{1}{2}\Sigma^2(T)}{\sqrt{\Sigma^2(T)}}.$$

Options on Bonds - Simplified Formula

- We write the call option pricing formula as

$$C(0) = p(0,T)(F_B\Phi(d_2) - K\Phi(d_1)),$$

where

$$F_B = \frac{p(0,T^*)}{p(0,T)}$$

is the forward price of the T^*-bond.

- To use the formula for options on coupon-bonds also, we simply calculate the forward bond price and the forward bond price volatility σ_B and use it in the formula.

Call Option on Bonds - Simplified Formula

- F_B can be calculated using the formula

$$F_B = \frac{B_c(0) - I}{p(0, T)}$$

with $B_c(0)$ the bond price at time zero and I is the present value of the coupons that will be paid during the life of the options.

-

$$d_{1/2} = \frac{\log\left(\frac{F_B}{K}\right) \pm \frac{1}{2}\sigma_B^2}{\sigma_B^2 \sqrt{(T)}}.$$

Swaps

Consider the case of a *forward swap settled in arrears* characterized by:

- a fixed time t, the contract time,
- dates $T_0 < T_1, \ldots < T_n$, equally distanced $T_{i+1} - T_i = \delta$,
- R, a prespecified fixed rate of interest,
- K, a nominal amount.

Swaps

A swap contract S with K and R fixed for the period $T_0, \dots T_n$ is a sequence of payments, where the amount of money paid out at $T_{i+1},\ i = 0, \dots, n-1$ is defined by

$$X_{i+1} = K\delta(L(T_i, T_{i+1}) - R).$$

The floating rate over $[T_i, T_{i+1}]$ observed at T_i is a simple rate defined as

$$p(T_i, T_{i+1}) = \frac{1}{1 + \delta L(T_i, T_{i+1})}.$$

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Pricing Formula for Swaps

Using the risk-neutral pricing formula we obtain (use $K = 1$),

$$
\begin{aligned}
\Pi(t,S) &= \sum_{i=1}^{n} \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^{T_i} r(s)ds} \delta(L(T_{i-1},T_i) - R) \middle| \mathcal{F}_t \right] \\
&= \sum_{i=1}^{n} \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{T_{i-1}}^{T_i} r(s)ds} \middle| \mathcal{F}_{T_{i-1}} \right] \right. \\
&\qquad \left. \times\ e^{-\int_t^{T_{i-1}} r(s)ds} \left(\frac{1}{p(T_{i-1},T_i)} - (1+\delta R) \right) \middle| \mathcal{F}_t \right] \\
&= \sum_{i=1}^{n} \left(p(t,T_{i-1}) - (1+\delta R) p(t,T_i) \right) \\
&= p(t,T_0) - \sum_{i=1}^{n} c_i p(t,T_i),
\end{aligned}
$$

with $c_i = \delta R, i = 1, \ldots, n-1$ and $c_n = 1 + \delta R$. So we obtain the swap price as a linear combination of zero-coupon bond prices.

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Caps

- An interest cap is a contract where the seller of the contract promises to pay a certain amount of cash to the holder of the contract if the interest rate exceeds a certain predetermined level (the cap rate) at some future date. A cap can be broken down in a series of caplets.
- A caplet is a contract written at t, in force between $[T_0, T_1]$, $\delta = T_1 - T_0$, the nominal amount is K, the cap rate is denoted by R. The relevant interest rate (LIBOR, for instance) is observed in T_0 and defined by

$$p(T_0, T_1) = \frac{1}{1 + \delta L(T_0, T_1)}.$$

Caplets

A caplet C is a T_1-contingent claim with payoff

$$X = K\delta(L(T_0, T_1) - R)^+.$$

Writing $L = L(T_0, T_1), p = p(T_0, T_1), R^* = 1 + \delta R$, we have $L = (1 - p)/(\delta p)$, (assuming $K = 1$) and

$$\begin{aligned} X &= \delta(L - R)^+ = \delta \left(\frac{1-p}{\delta p} - R \right)^+ \\ &= \left(\frac{1}{p} - (1 + \delta R) \right)^+ = \left(\frac{1}{p} - R^* \right)^+. \end{aligned}$$

Risk-Neutral Formula for Caplets

$$
\begin{aligned}
\Pi_C(t) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^{T_1} r(s)ds} \left(\tfrac{1}{p} - R^*\right)^+ \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{T_0}^{T_1} r(s)ds} \middle| \mathcal{F}_{T_0} \right] e^{-\int_t^{T_0} r(s)ds} \left(\frac{1}{p} - R^*\right)^+ \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[p(T_0, T_1)\, e^{-\int_t^{T_0} r(s)ds} \left(\frac{1}{p} - R^*\right)^+ \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^{T_0} r(s)ds} \left(1 - pR^*\right)^+ \middle| \mathcal{F}_t \right] \\
&= R^* \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^{T_0} r(s)ds} \left(\frac{1}{R^*} - p\right)^+ \middle| \mathcal{F}_t \right].
\end{aligned}
$$

Risk-Neutral Formula for Caps

- This shows that a caplet is equivalent to R^* put options on a T_1-bond with maturity T_0 and strike $1/R^*$.
- Unfortunately, for pricing a cap we can not simply extend this formula.

Introduction

- Market models of LIBOR and Swap-rates are consistent with the market practice of pricing caps, floors and swaptions by means of the Black-formula.
- They provide a consistent and coherent framework for the joint modelling of a whole set of forward rates.
- They use the discretely compounded forward LIBOR and forward swap rates – both directly observable in the market – as fundamental quantities in the modelling process.

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Forward LIBOR-Rates

- Define the *tenor structure* $\mathcal{T} = \{T_0, \ldots, T_n\}$ as a set of maturities T_i with $0 = T_0 < T_1 < \cdots < T_n$, where T_n is the time horizon of our economy.
- A given tenor structure $\mathcal{T}$ is associated with a set of $\{\tau_1, \ldots, \tau_n\}$ of year fractions, where $\tau_i = T_i - T_{i-1}$, $i = 1, \ldots, n$.
- We assume that in the financial market under consideration, there exist zero-coupon bonds $p(\cdot, T_i)$ of all maturities T_i, $i = 1, \ldots, n$.
- The discretely compounded *forward LIBOR rate* prevailing at time t over the future period from T_{i-1} to T_i is defined by

$$L(t, T_{i-1}) = \frac{p(t, T_{i-1}) - p(t, T_i)}{\tau_i p(t, T_i)}, \quad 0 \leq t \leq T_{i-1}.$$

LIBOR Dynamics Under the FLM

- We call a $\mathbb{P}$-equivalent probability measure $\mathbb{Q}^{T_k}$ the *forward LIBOR measure (FLM) for the maturity T_k*, or more briefly *T_k forward measure,* if all bond price processes

$$\left(\frac{p(t, T_i)}{p(t, T_k)}\right)_{t\in[0,\min\{T_i,T_k\}]}, \quad i = 1, \dots, n,$$

relative to the numéraire $p(\cdot, T_k)$ are (local) martingales under $\mathbb{Q}^{T_k}$.

- So $(L(t, T_{k-1}))_{t\in[0,T_{k-1}]}$ of the forward LIBOR over the period from T_{k-1} to T_k is a martingale under $\mathbb{Q}^{T_k}$.

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LIBOR Dynamics Under the FLM

- In a diffusion setting, we can posit, for every $k \in \{1, \ldots, n\}$, the following driftless dynamics under the respective forward LIBOR measure $\mathbb{Q}^{T_k}$:

$$\begin{aligned} dL(t, T_{k-1}) &= L(t, T_{k-1})\sigma(t, T_{k-1}) \cdot dW^{T_k}(t) \\ &= L(t, T_{k-1}) \sum_{i=1}^{d} \sigma_i(t, T_{k-1}) dW_i^{T_k}(t), \end{aligned} \tag{9}$$

  where W^{T_k} is a d-dimensional Brownian motion with respect to $\mathbb{F}$ under the measure $\mathbb{Q}^{T_k}$ and has the instantaneous covariance matrix $\rho = (\rho_{ij})_{i,j=1,\ldots,d} \in \mathbb{R}^{d \times d}$.
- For simplicity, we assume that $\sigma : [0, T_{n-1}] \times \{T_1, \ldots, T_{n-1}\} \to \mathbb{R}_+^d$ is a bounded and deterministic function, with $\sigma(\cdot,\cdot) = (\sigma_1(\cdot,\cdot), \ldots, \sigma_d(\cdot,\cdot))$ a row vector.

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LIBOR Dynamics Under a FLM

Take $T_k \in \{T_1, \dots, T_n\}$ as fixed. The following relations for the **LIBOR dynamics under the forward measure** $\mathbb{Q}^{T_i}$, $T_i \in \mathcal{T}$, hold:

$$
\begin{aligned}
i < k \quad : \quad dL(t, T_{k-1}) = \quad & L(t, T_{k-1})\sigma(t, T_{k-1}) \\
& \cdot \left(\sum_{j=i+1}^{k} \frac{\tau_j L(t, T_{j-1})}{1 + \tau_j L(0, T_{j-1})} \rho\sigma(t, T_{j-1})' dt + dW^{T_i}(t) \right), \\
i > k \quad : \quad dL(t, T_{k-1}) = \quad & L(t, T_{k-1})\sigma(t, T_{k-1}) \\
& \cdot \left(- \sum_{j=k+1}^{i} \frac{\tau_j L(t, T_{j-1})}{1 + \tau_j L(0, T_{j-1})} \rho\sigma(t, T_{j-1})' dt + dW^{T_i}(t) \right),
\end{aligned}
$$

where $0 \leq t \leq \min\{T_i, T_{k-1}\}$ and W^{T_i} is a d-dimensional $(\mathbb{F}, \mathbb{Q}^{T_i})$-Brownian motion with instantaneous covariance matrix ρ.

LIBOR Dynamics Under a FLM

- So a forward LIBOR process $L(\cdot, T_{k-1})$ is a lognormal martingale only under its respective forward measure $\mathbb{Q}^{T_k}$.
- Put differently, there exists no measure under which all LIBORs are simultaneously lognormal.

Simulation of LIBOR Dynamics

- One has to resort to numerical methods such as Monte Carlo simulation when pricing certain complex derivatives that depend on the simultaneous realization of several LIBOR rates in the above setup.
- For simulation purposes *one* measure $\mathbb{Q}^{T_k}$ has to be chosen, under which all forward LIBOR rates have to be evolved simultaneously.
- The following relation between the Brownian motions W^{T_k} and $W^{T_{k-1}}$ under the respective measures $\mathbb{Q}^{T_k}$ and $\mathbb{Q}^{T_{k-1}}$ has to be used

$$dW^{T_{k-1}}(t) \quad = \quad dW^{T_k}(t) - \frac{\tau_k L(t, T_{k-1})}{1 + \tau_k L(0, T_{k-1})} \rho\sigma(t, T_k)' \, dt.$$

Valuation of Caplets in the LMM

- Recall that a *caplet with reset date T_k and maturity T_{k+1} and strike rate K*, or briefly a *T_k-caplet with strike K*, is a derivative that pays the holder

$$\tau_{k+1}(L(T_k, T_k) - K)^+$$

at time T_{k+1}.

- So a caplet can be regarded as a call option on a LIBOR rate.
- We work under the $\mathbb{Q}^{T_{k+1}}$-measure, then

$$L(t, T_k) = L(0, T_k) e^{\left(\int_0^t \sigma(s, T_k) \cdot dW^{T_k}(s) - \frac{1}{2} \int_0^t \|\sigma(s, T_k)\|^2 \, ds\right)}$$

for $0 \leq t \leq T_k$.

Valuation of Caplets in the LMM

So

$$\log L(T_k, T_k) \sim N\left(m, s^2\right),$$

with

$$m = \log L(0, T_k) - \frac{1}{2}\int_0^{T_k} \|\sigma(s, T_k)\|^2 \, ds$$

and

$$s^2 = \int_0^{T_k} \sigma(s, T_k)\rho\sigma(s, T_k)' \, ds.$$

Valuation of Caplets in the LMM

The caplet price, denoted by $C(0, T_K, K)$, is:

$$
\begin{aligned}
C(0, T_K, K) &= p(0, T_k)\mathbb{E}_{\mathbb{Q}_{T_k}}\left(\frac{\tau_{k+1}(L(T_k, T_k) - K)^+}{p(T_k, T_K)}\right) \\
&= \tau_{k+1} p(0, T_k)\mathbb{E}_{\mathbb{Q}_{T_k}}\left((L(T_k, T_k) - K)^+\right) \\
&= \tau_{k+1} p(0, T_k)\left(L(0, T_k)N(d_1) - KN(d_2)\right),
\end{aligned}
$$

with

$$
d_1 = \frac{\log(L(0, T_k)/K) + s^2/2}{s}
$$

and

$$
d_2 = \frac{\log(L(0, T_k)/K) - s^2/2}{s}.
$$

This is the *Black formula* for caplets.

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Interest Rate Swap

- Recall that an interest rate swap (IRS) is a contract to exchange fixed against floating payments, where the floating payments typically depend on LIBOR rates.
- An IRS is specified by its *reset-dates* $T_\alpha, T_{\alpha+1}, \ldots, T_{\beta-1}$, its *payment-dates* $T_{\alpha+1}, \ldots, T_\beta$, and the fixed rate K.
- At every $T_j \in \{T_{\alpha+1}, \ldots, T_\beta\}$, the fixed payment is $\tau_j K$ with $\tau_j = T_j - T_{j-1}$, while the floating payment is $\tau_j L(T_{j-1}, T_{j-1})$.

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Interest Rate Swap

The value of a swap in $t \leq T_\alpha$ can be determined without making any distributional assumptions on the LIBOR rates as

$$\sum_{i=\alpha+1}^{\beta} p(t, T_i)\tau_i(L(t, T_{i-1}) - K), \tag{10}$$

or alternatively

$$p(t, T_\alpha) - p(t, T_\beta) - \sum_{i=\alpha+1}^{\beta} p(t, T_i)\tau_i K, \tag{11}$$

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Forward Swap Rate

- The *forward swap rate* (FSR) at time t of the above IRS, which we denote by $S_{\alpha,\beta}(t)$, is the value for the fixed rate K that makes the t-value of the IRS zero.
- $S_{\alpha,\beta}(t)$ can thus be obtained by equating Expression (10) to zero and solving for K, which gives

$$S_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} w_i(t) L(t, T_{i-1}) \tag{12}$$

with

$$w_i(t) = \frac{\tau_i p(t, T_i)}{\sum_{j=\alpha+1}^{\beta} \tau_j p(t, T_j)},$$

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Forward Swap Rate

Equivalently equate (11) to zero, which gives

$$S_{\alpha,\beta}(t) = \frac{p(t, T_\alpha) - p(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i p(t, T_i)}.$$

Equation (12) shows that the FSR can be expressed as a suitably weighted average of the spanning forward LIBORs.

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Swaptions

- A European *payer swaption* gives the holder the right to enter a swap as fixed-rate payer at a fixed rate K (the *swaption strike*) at a future date that normally coincides with the first reset date T_α of the underlying swap.
- Similarly, a European *receiver swaption* gives the holder the right to enter a swap as fixed-rate receiver.
- The following relation holds for the t-value of a payer-swap: for $0 \le t \le T_\alpha$,

$$\begin{aligned} & \sum_{i=\alpha+1}^{\beta} p(t, T_i)\tau_i(L(t, T_{i-1}) - K) \qquad (13) \\ = \quad & (S_{\alpha,\beta}(t) - K) \sum_{i=\alpha+1}^{\beta} \tau_i p(t, T_i). \end{aligned}$$

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Swaptions

- The advantage of the expression on the right-hand side of Equation (13) over the expression on the left-hand side (which is our Formula (10)) is that one can instantly tell from $S_{\alpha,\beta}(t)$ if the t-value of the payer-swap is positive or negative.
- At the maturity date T_α, a payer swaption is exercised if and only if the value of the underlying swap is positive, which is the case if and only if $S_{\alpha,\beta}(T_\alpha) - K > 0$ holds.
- The payer-swaption-value in T_α is

$$(S_{\alpha,\beta}(T_\alpha) - K)^+ \sum_{i=\alpha+1}^{\beta} \tau_i p(T_\alpha, T_i),$$

and the receiver-swaption-value is

$$(K - S_{\alpha,\beta}(T_\alpha))^+ \sum_{i=\alpha+1}^{\beta} \tau_i p(T_\alpha, T_i).$$

Swap Rate Dynamics

- Observe that

$$A_{\alpha,\beta}(t) = \sum_{i=\alpha+1}^{\beta} \tau_i p(t, T_i)$$

is the t-price of a a portfolio of bonds (i.e. a traded asset).

- Thus, $A_{\alpha,\beta}(t)$, which if known as *accrual factor* or *present value of a basis point,* can be used as numéraire.
- Now note that

$$S_{\alpha,\beta}(t) = \frac{p(t, T_\alpha) - p(t, T_\beta)}{A_{\alpha,\beta}(t)},$$

where the numerator can be regarded as the price of a traded asset as well.

Swap Rate Dynamics

- In order for our model to be arbitrage-free, the swap rate $S_{\alpha,\beta}(\cdot)$ has to be a martingale under the numéraire pair $(\mathbb{Q}^{\alpha,\beta}, A_{\alpha,\beta}(\cdot))$.
- $\mathbb{Q}^{\alpha,\beta}$ is the so-called *forward swap measure*.

Swap Rate Dynamics

- We assume that $S_{\alpha,\beta}(\cdot)$ follows a lognormal martingale:

$$dS_{\alpha,\beta}(t) = \sigma(t)S_{\alpha,\beta}(t)dW_{\alpha,\beta}(t),$$

  where σ is a deterministic function and $W_{\alpha,\beta}(\cdot)$ is a standard $\mathbb{Q}^{\alpha,\beta}$-Brownian motion.
- The fact that the forward swap rate $S_{\alpha,\beta}(t)$ is lognormally distributed under $\mathbb{Q}^{\alpha,\beta}$ motivates the name *lognormal forward swap model.*

Black's Swaption Pricing Formula

The price of a payer swaption as specified above in a lognormal forward swap model is consistent with Black's formula for swaptions and is given by

$$PS(0, \{T_\alpha, T_\alpha, \ldots, T_\beta\}, K) \quad = \quad A_{\alpha,\beta}(0)\left(S_{\alpha,\beta}(0)N(d_1) - KN(d_2)\right)$$

with

$$d_1 = \frac{\log(S_{\alpha,\beta}(0)/K) + \frac{1}{2}\Sigma^2(T_\alpha)}{\Sigma(T_\alpha)}$$

and

$$d_2 = d_1 - \Sigma(T_\alpha) \quad \text{with} \quad \Sigma^2(T_\alpha) = \int_0^{T_\alpha} \sigma(s)^2\, ds.$$

The price of a receiver-swaption is given by

$$RS(0, \{T_\alpha, T_\alpha, \ldots, T_\beta\}, K) \quad = \quad A_{\alpha,\beta}(0)\left(KN(-d_2) - S_{\alpha,\beta}(0)N(-d_1)\right).$$

Swap Market Models

- The pricing of more complicated derivatives whose value depends on more than one swap-rate necessitates the modelling of the *simultaneous* evolution of a set of swap rates under one probability-measure.
- This leads us to the class of *Swap Market Models (SMMs),* which are arbitrage-free models of the joint evolution of a set of swap rates under a common probability measure.

Swap Market Models

Examples are:

1. The set of swap-rates $\left\{S_{\alpha,\alpha+1}(t), S_{\alpha+1,\alpha+2}(t), \ldots, S_{\beta-1,\beta}(t), \right\}$. As this is exactly the set of LIBOR rates
$$\left\{L(t, T_\alpha), L(t, T_{\alpha+1}), \ldots, L(t, T_{\beta-1})\right\},$$
this choice leads us back to the LMM framework.
2. The set of swap-rates $\left\{S_{\alpha,\alpha+1}(t), S_{\alpha,\alpha+2}(t), \ldots, S_{\alpha,\beta}(t)\right\}$.
3. The set of swap-rates $\left\{S_{\alpha,\beta}(t), S_{\alpha+1,\beta}(t), \ldots, S_{\beta-1,\beta}(t)\right\}$.

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The Relation Between LMM and SMM

- All products that can be priced in a LMM-framework can in principle also be priced in a SMM-framework and vice versa.
- This is due to the fact that swap rates can be expressed as weighted sums of LIBOR rates, and LIBOR rates can be expressed as functions of swap-rates.
- However, lognormal LMMs do not yield swaption-prices that agree with Black-prices, and lognormal SMMs do not give Black-consistent caplet prices.
- The incompatibility is mostly of a theoretical nature, as it can be shown (for example by simulation studies) that swap rates in the lognormal LMM are “almost” lognormal, and therefore swaption-prices in the lognormal LMM are very similar to those one would obtain by the Black formula.