# Futures, Options and Derivative Instruments

Lecture - Summer 2013

Professor Dr. Rüdiger Kiesel | Chair for Energy Trading and Finance | University of Duisburg-Essen

UNIVERSITÄT DUISBURG ESSEN

# Literatur

UNIVERSITÄT DUISBURG ESSEN

# Literature

- **Main References:**
  - Hull, J. (2009). Options, Futures, and Other Derivatives, Pearson.
  - Burger, M., B. Graeber, et al. (2008). Managing Energy Risk: An Integrated View on Power and Other Energy Markets, John Wiley & Sons.

UNIVERSITÄT DUISBURG ESSEN

# Agenda

# The History of Derivatives

# Derivatives Quotes

"We view them as time bombs both for the parties that deal in them and the economic system ... In our view ... derivatives are financial weapons of mass destruction, carrying dangers that, while now latent, are potentially lethal."

Warren Buffett in his Chairman's Letter in the Berkshire Hathaway 2002 Annual Report

# Derivatives Quotes

"The use of a growing array of derivatives and the related application of more-sophisticated approaches to measuring and managing risk are key factors underpinning the greater resili-ence of our largest financial institutions .... Derivatives have permitted the unbundling of financial risks."

Alan Greenspan, May 2005

# Derivatives Quotes

"Derivatives don’t kill people, people kill people."

Clifford W. Smith, Jr., Professor, University of Rochester - Risk, March, 1994, p. 6

# Derivatives Quotes

"Blaming derivatives for financial losses is akin to blaming cars for drunk driving fatalities."

Christopher L. Culp; MediaNomics, April, 1995, p. 4

# Derivatives Quotes

"'Derivatives.' That's the 11-letter four-letter word."

Richard Syron, Chairman, American Stock Exchange; Fortune, March 20, 1995, p. 50

UNIVERSITÄT DUISBURG ESSEN

# History of Derivatives

**Late 1960s - Black-Scholes Formula**

- Fischer Black and Myron Scholes tackle the problem of determining how much an option is worth. Robert Merton joins them in 1970.

**1993 - Metallgesellschaft bankrupt**

- Metallgesellschaft AG, formely one of Germany's largest industrial conglomerates with over 20.000 employees, looses 1.3 billion dollars after speculating for a rise in the futures market. Oil prices dropped and left the company buying oil at a price substantially over the market price.

# History of Derivatives (2)

**1995 - Barings Bank Disaster**

- Nick Leeson loses $1.4 billion, speculating that the Nikkei 225 index of leading Japanese company shares would not move materially from its normal trading range (short straddle). That assumption was shattered by the Kobe earthquake on the 17th January 1995 after which Leeson attempted to conceal his losses, which reached twice the bank’s available trading capital. Barings was declared insolvent the 26th Feburary 1995

**1997 - Nobel Prize in Economics awarded to Robert Merton and Myron Scholes**

- "for a new method to determine the value of derivatives."

UNIVERSITÄT DUISBURG ESSEN

# History of Derivatives (3)

**1998 - Long Term Credit Management Bailout**

- The hedge fund, among the board of directors Myron Samuel Scholes und Robert C. Merton, needs to be rescued at a cost of $3.5 billion. The fund had speculated on spreads of govern-ment bonds, i.e. the interest spread between 30 and 10 years US treasury bonds.

**2001 - Enron goes Bankrupt**

- The 7th largest company in the US and the world's largest energy trader made extensive use of energy and credit derivatives but becomes the biggest firm to go bankrupt in American history after systematically attempting to conceal huge losses.

# History of Derivatives (4)

**2006 - Amaranth Advisors**

- The multi strategy hedge fund loses more than $ 6 billion from trading in natural gas futures. The hedge fund collapses.

# History of Derivatives(5)

## 2008 - Societe Generale trading scandal

- Jerume Kerviel, a French trader from Societe Generale , causes a loss of EUR 4.9 billion. Kerviel was assigned to arbitrage discrepancies between equity derivatives and cash equity prices, but he manipulated risk management systems and exceeded his authority to engage in un-authorized trades totaling as much as EUR 49.9 billion, which is higher than the bank's total market capitalization. When Societe Generale tried to close out open positions built up by Kerviel, European stock markets suffered heavy losses of about 6 %.

## 2011 - UBS trading scandal

- A UBS trader causes losses in a dimension of EUR 2 billion through unauthorized trading of stock index futures.

# Financial Derivatives Timeline (6)

**2012 - Whale of London**

- Bruno Michael Iksli, nicknamed the London Whale, causes $ 2 billion trading losses for JPMorgan with aggressive trading.

**2012 - Libor Manipulation**

- In a series of fraudulent actions, the Libor (London Interbank Offered Rate), which underpins approximately $350 trillion in derivatives, is manipulated by several parties.

UNIVERSITÄT DUISBURG ESSEN

# Agenda

# Basic Derivatives

UNIVERSITÄT DUISBURG ESSEN

# Derivative Background (1)

- A derivative security, or contingent claim, is a financial contract whose value
  - at expiration date $T$ (more briefly, expiry) is determined exactly by the price (or prices within a prespecified time-interval) of
  - the underlying financial assets (or instruments) at time $T$ (within the time interval $[0, T]$).

# Derivative Background (2)

- Derivative securities can be grouped under three general headings: *Options, Forwards and Futures* and *Swaps*.
- During this lectures we will encounter all this structures and further variants.

# Modelling Assumptions (1)

We impose the following set of assumptions on the financial markets:

- *No market frictions:* No transaction costs, no bid/ask spread, no taxes, no margin requirements, no restrictions on short sales.
- *No default risk:* Implying same interest for borrowing and lending

# Modelling Assumptions (2)

- *Competitive markets:* Market participants act as price takers, infinite number of participants
- *Rational agents* Market participants prefer more to less

# Arbitrage (1)

- The concept of arbitrage lies at the centre of the relative pricing theory. All we need to assume additionally is that economic agents prefer more to less, or more precisely, an increase in consumption without any costs will always be accepted.

# Arbitrage (2)

- The essence of the technical sense of arbitrage is that it should not be possible to guarantee a profit without exposure to risk. Were it possible to do so, arbitrageurs would do so, in unlimited quantity, using the market as a ‘money-pump’ to extract arbitrarily large quantities of riskless profit.
- *We assume that arbitrage opportunities do not exist!*

# Continuous and Discrete Compounding

- Discrete compounding applies when we consider discrete time points, i.e. interest payments at the end of a period.
  - $S_t = S_0(1+r)^t$
- Continuous compounding applies when interest payments are done at a constant rate through the time period.
  - $S_t = S_0 e^{rt}$

# Derivate Markets

- Financial derivatives are basically traded in two ways: on organized exchanges and over-the counter (OTC).
- Products at exchanges are standardized contracts that are defined by the exchange. Important exchanges for commodities are the Chicago Mercantile Exchange (CME), the Intercontinental Exchange (ICE) in London or the European Energy Exchange (EEX) in Leipzig.
- OTC trading takes place via computers and phones between various commercials and investment banks. OTC contracts are non-standardized and can be flexible adjusted to the demand of the parties.

# Underlying Securities

- Stocks (one or several);
- Fixed income instruments: T-Bonds, Interest Rates (LIBOR, EURIBOR);
- Commodities or Commodity Futures;
- Currencies (FX);
- Also Derivatives may be used as underlying for compound derivatives (call on call).

# Options

- An option is a financial instrument giving one the *right, but not the obligation* to make a specified transaction at (or by) a specified date at a specified price.
- *Call* options give one the right to buy. *Put* options give one the right to sell.
- *European* options give one the right to buy/sell on the specified date, the expiry date, on which the option expires or matures. *American* options give one the right to buy/sell at any time prior to or at expiry.

# Exotic Options

Many kinds of options now exist, including so-called *exotic* options. Types include:

- *Asian* options, which depend on the *average* price over a period,
- *barrier* options, which depend on some price level being attained or not.
- *lookback* options, which depend on the *maximum* or *minimum* price over a period.

# Options - Terminology (1)

- The asset to which the option refers is called the *underlying asset* or the *underlying*.
- The price at which the transaction to buy/sell the underlying, on/by the expiry date (if exercised), is made, is called the *exercise price* or *strike price*.
- We shall usually use $K$ for the strike price, time $t = 0$ for the initial time (when the contract between the buyer and the seller of the option is struck), time $t = T$ for the expiry or final time.

# Options - Terminology (2)

Consider, say, a European call option, with strike price $K$; write $S(t)$ for the value (or price) of the underlying at time $t$.

- If $S(t) > K$, the option is *in the money*,
- if $S(t) = K$, the option is *at the money*,
- if $S(t) < K$, the option is *out of the money*.

# Options - Payoff

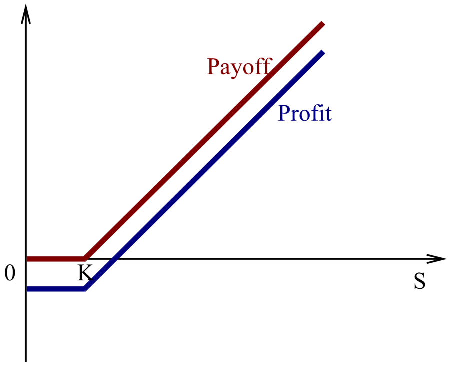
- The payoff from a call option is

$$S(T) - K \text{ if } S(T) > K \quad \text{and} \quad 0 \text{ otherwise}$$

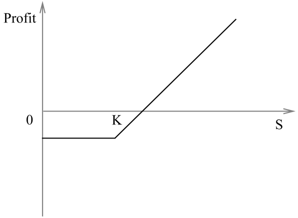
  (more briefly written as $(S(T) - K)^+$).
- The profit from a call option is the payoff $(S(T) - K)^+$ minus the call premium $c$.

# Options - Payoff/Profit diagram

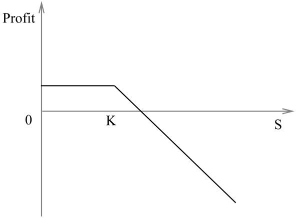
Considering only the option payoff, we obtain the payoff diagram, taking into account the initial payment of an investor one obtains the profit diagram below.



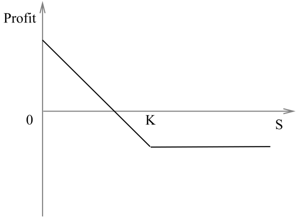
# Options - Profit diagrams of vanilla options



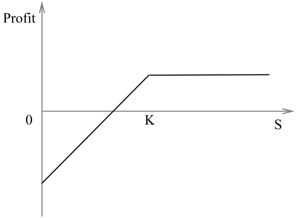

(a) long call




(b) short call




(c) long put




(d) short put

# Forwards - Basic Structure

- A *forward contract* is an agreement to buy or sell an asset $S$ at a certain future date $T$ for a certain price $K$.
- The agent who agrees to buy the underlying asset is said to have a *long* position, the other agent assumes a *short* position.
- The settlement date is called *delivery date* and the specified price is referred to as *delivery price*.

# Forwards

- The *forward price* $F(t, T)$ is the delivery price which would make the contract have zero value at time $t$.
- At the time the contract is set up, $t = 0$, the forward price therefore equals the delivery price, hence $F(0, T) = K$.
- The forward prices $F(t, T)$ need not (and will not) necessarily be equal to the delivery price $K$ during the life-time of the contract.

# Forwards

- The payoff from a long position in a forward contract on one unit of an asset with price $S(T)$ at the maturity of the contract is
$$S(T) - K.$$
- Compared with a call option with the same maturity and strike price $K$ we see that the investor now faces a downside risk, too. He has the obligation to buy the asset for price $K$.

# Spot-Forward Relationship

Under the no-arbitrage assumption we have

| | $t$ | $T$ |
|---|---|---|
| buy stock | $-S(t)$ | delivery |
| borrow to finance | $S(t)$ | $-S(t)e^{r(T-t)}$ |
| sell forward on S | | $F(t,T)$ |

All quantities are known at $t$, the time $t$ cashflow is zero, so the cashflow at T needs to be zero so we have

$$F(t,T) = S(t)e^{r(T-T)}$$

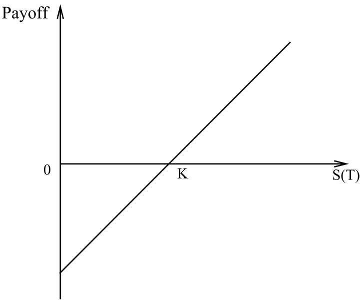
# Forwards

| | $t_0$ | $t_1$ | $T$ |
|---|---|---|---|
| | Forward Price as Such as to make Forward-Value 0<br>$K = F(t_0, T) = e^{-r(T-t_0)} S_{t_0}$ | Forward Value different than 0 possible | Forward Price is Price of the Underlying |
| **Standard Model** | | | |
| Value | 0 | $e^{-r(T-t_1)} S_{t_1}$-K | $S_T - K$ |
| Price | $F(t_o, T) = e^{-r(T-t_0)} S_{t_0}$ | $F(t_1, T) = e^{-r(T-t_1)} S_{t_1}$ | $F(T, T) = S_T$ |

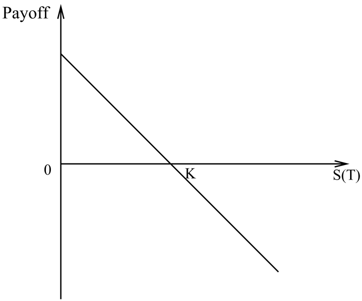
# Futures

- Futures can be defined as standardised forward contracts traded at exchanges where a clearing house acts as a central counterparty for all transactions.
- Usually an initial margin is paid as a guarantee.
- Each trading day a settlement price is determined and gains or losses are immediately realized at a margin account.
- Thus credit risk is eliminated, but there is exposure to interest rate risk.

# Payoff from a forward/futures contract




(e) long position




(f) short position

# Swaps

- A *swap* is an agreement whereby two parties undertake to exchange, at known dates in the future, various financial assets (or cash flows) according to a prearranged formula that depends on the value of one or more underlying assets.
- Examples are currency swaps (exchange currencies) and interest-rate swaps (exchange of fixed for floating set of interest payments).

# Agenda

# Arbitrage and Valuation

# Valuation by Replication

- We now want to find the fair price of a derivative, i.e. we want to find the option premium
- We consider a one-period model, i.e. we allow trading only at $t = 0$ and $t = T$(say).
- Our aim is to value at $t = 0$ a European derivative on a stock $S$ with maturity $T$.

# Valuation by Replication - Idea

- If it is possible to duplicate the payoff $H$ of a derivative using a portfolio $V$ of underlying (basic) securities, the price of the portfolio at $t = 0$ must equal the price of the derivative at $t = 0$.

# Valuation by Replication - Example (1)

- We have a stock $S$ with $S(0) = 20$ and a European Call option with strike $K = 21$ and maturity $T$.
- In $T$, there are two possible states: an up- and a down state

- up: $20(1 + u) = 22 \Rightarrow u = 0.1$
- down: $20(1 + d) = 18 \Rightarrow d = -0.1$

# Valuation by Replication - Example (2)

- The key idea now is to try to find a portfolio combining bond and stock, which synthesizes the cash flow of the option.
- If such a portfolio exists, holding this portfolio today would be equivalent to holding the option – they would produce the same cash flow in the future.
- Therefore the price of the option $C(0)$ should be the same as the price of constructing the portfolio $V(0)$, otherwise investors could just restructure their holdings in the assets and obtain a riskfree profit today.

# Valuation by Replication - Example (3)

We construct the portfolio from

- a riskfree bond (bank account) with $B(0) = 1$ and $B(T) = B(t) \cdot (1 + r) = 1.05$, that is the interest rate $r = 0.05$
- a the risky stock $S$ with $S(0) = 20$ and two possible values at $t = T$,
  - $S_1(u) = 22$
  - $S_1(d) = 18$

# Valuation by Replication - Example (4)

- we invest $\theta_0$ in the bank account and buy
- $\theta_1$ stocks
- the value of our portfolio today is given as
  - $V(0) = \theta_0 + \theta_1 \cdot S(0)$.

# Valuation by Replication - Example (5)

- at expiry date $t = T$, the value of the portfolio depends on the value of the stock $S(T)$, which can be 18 or 22 \$.
  - $V(T) = \theta_0 \cdot (1 + r) + \theta_1 \cdot S(T)$
- the Payoff from the call option in $t = T$ is given with 1 or 0.

# Valuation by Replication - Example (6)

- We can reproduce the call-option with the stock and the bank account if we find the values $\theta_0$ and $\theta_1$ which replicate the payoff from the call-option.
- The problem is a linear system with two equations and two unknown variables
  - $\theta_0 \cdot 1.05 + \theta_1 \cdot 22 = 1$
  - $\theta_0 \cdot 1.05 + \theta_1 \cdot 18 = 0.$
- The solution of this equations is $\theta_0 = -4.286$ and $\theta_1 = 0.25$

# Valuation by Replication - Example (7)

- To replicate the option payoff in $t$, we need $-4.286 + 0.25 \cdot 20 = 0.714\$$. This is the value of the call-option in $t = 0$, where $C(0) = V(0)$
- V(0) is called the no-arbitrage price. Every other price allows a riskless profit, since if the option is too cheap, buy it and finance yourself by selling short the above portfolio (i.e. sell the portfolio without possessing it and promise to deliver it at time T = 1 – this is riskfree because you own the option). If on the other hand the option is too dear, write it (i.e. sell it in the market) and cover yourself by setting up the above portfolio.

# No-Arbitrage and Expectation (1)

- Let us now assume the probability of the stock rising is $p$, the probability of the stock falling is $1 - p$
- The probability $p$ is unkown to the market because the future development of the stock is unknown

- Still, the market has a perception of the probability and it will be called $q$
- And of course, under what market participants think, the market must be free of Arbitrage!

# No-Arbitrage and Expectation (2)

- This means that the following equation must hold:

$$q(1+u)S_0 + (1-q)(1+d)S_0 = S_0(1+r)$$

- We solve this equation for $q$:

$$\begin{aligned} q(1+u) + (1-q)(1+d) &= 1+r \\ q(1+u) + (1+d) - q(1+d) &= 1+r \\ q(1+u-1-d) &= 1+r-1-d \\ q &= \frac{r-d}{u-d} \end{aligned}$$

- This is the so-called Arbitrage-free Risk-neutral probability!
- In our example it takes the value $q = \frac{0.05+0.1}{0.1+0.1} = 0.75$ and $1-q = 0.25$

# No-Arbitrage and Expectation - General Principle

- Using the risk-neutral probabilities we state the following general pricing principle:
- The Arbitrage-free price of a derivative $H$ on an asset $S$ in $t$ with maturity in $T$ is given by

$$H_t = \frac{B_t}{B_T}\mathbb{E}[H_T]$$

- This is the so-called Risk-Neutral Valuation Formula
- We calculate the price of the example:

$$\begin{aligned} H_0 &= \frac{B_t}{B_T}\mathbb{E}[H_T] = \frac{1}{1+r}(q(1+u)H_0 + (1-q)(1+d)H_0) \\ &= \frac{1}{1.05}(0.75 \cdot 1 + 0.25 \cdot 0) = 0.714 \end{aligned}$$

- The same price as before!

UNIVERSITÄT DUISBURG ESSEN

# The Cox-Ross-Rubinstein model

- Our model consists of two basic securities. Recall that the essence of the relative pricing theory is to take the price processes of these basic securities as given and price secondary securities in such a way that no arbitrage is possible.
- Our time horizon is $T$ and the set of dates in our financial market model is $t = 0, 1, \ldots, T$. Assume that the first of our given basic securities is a (riskless) bond or bank account $B$, which yields a riskless rate of return $r > 0$ in each time interval $[t, t+1]$, i.e.
  - $B(t+1) = (1+r)B(t), \quad B(0) = 1.$
- So its price process is $B(t) = (1+r)^t, \quad t = 0, 1, \ldots, T.$

# The CRR Model

- Furthermore, we have a risky asset (stock) $S$ with price process
  - $$S(t+1) = \begin{cases} (1+u)S(t) & \text{with prob} & q, \\ (1+d)S(t) & \text{with prob} & 1-q, \end{cases}$$

    with $-1 < d < u$, $S_0 \in \mathbb{R}_0^+$ and for $t = 0, 1, \ldots, T-1$.

# The CRR Model

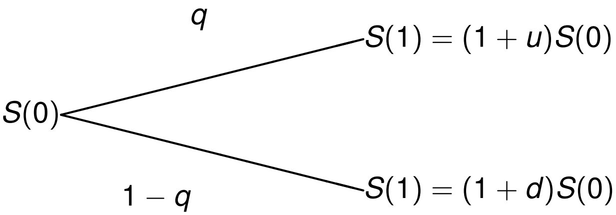



Figure : One-step tree diagram

# The CRR Model

- This construction emphasises again that a multi-period model can be viewed as a sequence of single-period models. Indeed, in the Cox-Ross-Rubinstein case we use identical and independent single-period models.
- As we will see in the sequel this will make the construction of equivalent martingale measures relatively easy. Unfortunately we can hardly defend the assumption of independent and identically distributed price movements at each time period in practical applications.

# Pricing in the CRR Model

- We now turn to the pricing of derivative assets in the Cox-Ross-Rubinstein market model. To do so we first have to discuss whether the Cox-Ross-Rubinstein model is arbitrage-free and complete.
- To answer these questions we have, according to our fundamental theorems, to understand the structure of equivalent martingale measures in the Cox-Ross-Rubinstein model. In trying to do this we use (as is quite natural and customary) the bond price process $B(t)$ as numéraire.

# Pricing measure

(i) A pricing probabilities for the discounted stock price $\tilde{S}$ exists if and only if

$$d < r < u. \tag{1}$$

(ii) If inequality (1) holds true, then these probabilities are given by

$$q = \frac{r - d}{u - d}. \tag{2}$$

# Pricing Formula

- We can now use the risk-neutral valuation formula to price *every* contingent claim in the Cox-Ross-Rubinstein model.
- The arbitrage price process of a contingent claim $X$ in the Cox-Ross-Rubinstein model is given by

$$\pi_X(0) = B(t)\mathbb{E}^{\mathbb{Q}}\left(X/B(T)\right)$$

where $\mathbb{E}^{\mathbb{Q}}$ is the expectation with respect to the unique probabilities $q = (r-d)/(u-d)$.

# Pricing Formula

- We now give simple formulas for pricing (and hedging) of European contingent claims $X = f(S_T)$ for suitable functions $f$ (in this simple framework all functions $f : \mathbb{R} \to \mathbb{R}$). We use the notation

$$
\begin{aligned}
& F_\tau(x, q) \\
:= \quad & \sum_{j=0}^{\tau} \binom{\tau}{j} q^j (1-q)^{\tau-j} f\left(x(1+u)^j(1+d)^{\tau-j}\right)
\end{aligned}
\tag{3}
$$

- Observe that this is just an evaluation of $f(S(j))$ along the probability-weighted paths of the price process. Accordingly, $j, \tau - j$ are the numbers of times $Z(i)$ takes the two possible values $d, u$.

# European claims

- Consider a European contingent claim with expiry $T$ given by $X = f(S_T)$. The arbitrage price process $\pi_X(t),\ t = 0, 1, \dots, T$ of the contingent claim is given by (set $\tau = T - t$)

$$\pi_X(t) = (1 + r)^{-\tau} F_\tau(S_t, q). \tag{4}$$

UNIVERSITÄT DUISBURG ESSEN

# European call

- Consider a European call option with expiry $T$ and strike price $K$ written on (one share of) the stock $S$. The arbitrage price process $\Pi_C(t),\ t = 0, 1, \ldots, T$ of the option is given by (set $\tau = T - t$)

$$\begin{aligned} & \Pi_C(t) \\ = \quad & (1+r)^{-\tau} \sum_{j=0}^{\tau} \binom{\tau}{j} q^j (1-q)^{\tau-j} \\ & (S(t)(1+u)^j(1+d)^{\tau-j} - K)^+ . \end{aligned} \tag{5}$$

- For a European put option, we can either argue similarly or use put-call parity.

UNIVERSITÄT DUISBURG ESSEN

# A Three-period Example (1)

- We consider a continuous-time model with one-year risk-free interest rate (continuously compounded) $\rho = 0.06$ and the volatility of the stock is 20%, so $\sigma = 0.2$.
- The corresponding 3-step CRR-quantities are $\Delta = 1/3$
  - the up and down movements of the stock price

$$1+u = e^{\sigma\sqrt{\Delta}} = 1.1224, \quad 1+d = (1+u)^{-1} = e^{-\sigma\sqrt{\Delta}} = 0.8910,$$

  - the discrete interest rate follows from

$$(1+r)^3 = e^{\rho}.$$

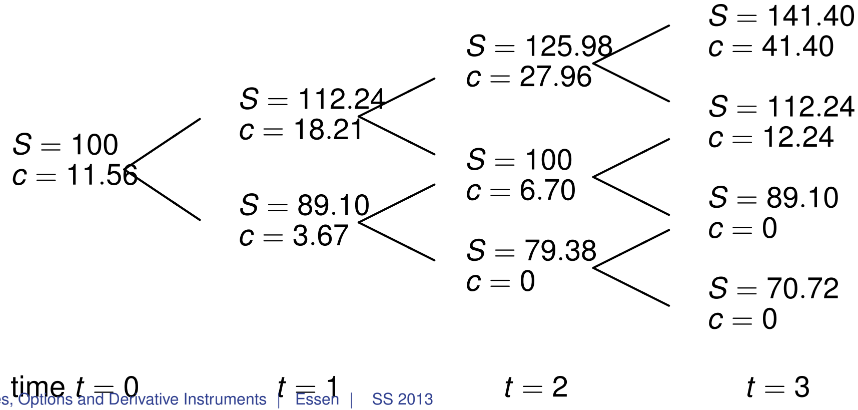
  - the risk-neutral probabilities

$$q = \frac{r-d}{u-d} = 0.5584.$$

UNIVERSITÄT DUISBURG ESSEN

# A Three-period Example (2)

- We assume that $S(0) = 100$. Prices of the stock and the call with strike $K = 100$ are given below.



# A Three-period Example (3)

- To price a European call option with maturity one year ($N = 3$) and strike $K = 100$) we can either use the explicit valuation formula or work our way backwards through the tree.
- One can implement the simple evaluation formulae for the CRR- and the BS-models and compare the values. The figure (84) below is for $S = 100, K = 90, \rho = 0.06, \sigma = 0.2, T = 1$.

# Delta of an option

- The delta $\Delta$ of an option is the ration of the change in the price of the option to the change of the price of the underlying.
- It is the number of units of the underlying we should hold for each option short in order to create a riskfree portfolio – the delta hedge.
- The delta of a call is positive, whereas the delta of a put is negative.

# Example of the Delta

- $$\Delta_{0,1} = \frac{112.24 - 89.10}{18.21 - 3.67} = 1.5915$$
- $$\Delta^{u}_{1,2} = \frac{125.89 - 100}{27.96 - 6.70} = 1.2178$$
- $$\Delta^{d}_{1,2} = \frac{100 - 79.38}{6.70 - 0} = 3.0776$$
- Observe that the delta is time and state dependent.

# Binomial Approximations (1)

- Suppose we observe financial assets during a continuous time period $[0, T]$.
- To construct a stochastic model of the price processes of these assets (to, e.g. value contingent claims) one basically has two choices:
  - one could model the processes as continuous-time stochastic processes (for which the theory of stochastic calculus is needed)
  - one could construct a sequence of discrete-time models in which the continuous-time price processes are approximated by discrete-time stochastic processes in a suitable sense.
- We follow the second approach and obtain the asymptotics of a sequence of Cox-Ross-Rubinstein models.

# Binomial Approximations (2)

- With $B^{n,p}$ the Binomial cumulative distribution function of $\bar{B}^{n,p} = 1 - B^{n,p}$, we find in the $n$th Cox-Ross-Rubinstein model for the price of a European call at time $t = 0$ the following formula

$$\begin{aligned} \Pi_C^{(n)}(0) &= S_n(0)\bar{B}^{k_n,\hat{p}_n}(a_n) \\ &\quad - K(1+r_n)^{-k_n}\bar{B}^{k_n,p_n^*}(a_n). \end{aligned} \tag{6}$$

- We have the following limit relation:

$$\lim_{n\to\infty} \Pi_C^{(n)}(0) = \Pi_C^{BS}(0)$$

with $\Pi_C^{BS}(0)$ given by the Black-Scholes formula.

# The Black-Scholes Formula

- The Black-Schole Formula for the price $\Pi_C^{BS}(0)$ of a European call (we use $S = S(0)$ to ease the notation) is

$$\Pi_C^{BS}(0) = S\Phi(d_1(S,T)) - Ke^{-rT}\Phi(d_2(S,T)). \tag{7}$$

  with $\Phi(.)$ the standard Normal cumulative distribution function.
- The functions $d_1(s,t)$ and $d_2(s,t)$ are given by

$$\begin{aligned} d_1(s,t) &= \frac{\log(s/K) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}, \\ d_2(s,t) &= d_1(s,t) - \sigma\sqrt{t} \\ &= \frac{\log(s/K) + (r - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}} \end{aligned}$$

# American Options in the CRR model (1)

- We now consider how to evaluate an American put option in a standard CRR model.
- We assume that the time interval $[0, T]$ is divided into $N$ equal subintervals of length $\Delta$ say.
- Assuming the risk-free rate of interest $r$ (over [0,T]) as given, we have $1 + r = e^{\rho\Delta}$ (where we denote the risk-free rate of interest in each subinterval by $r$).

# American Options in the CRR model (2)

- The remaining degrees of freedom are resolved by choosing $u$ and $d$ as follows:

$$1+u=e^{\sigma\sqrt{\Delta}}, \quad \text{and} \quad 1+d=(1+u)^{-1}=e^{-\sigma\sqrt{\Delta}}.$$

- The risk-neutral probabilities for the corresponding single period models are given by

$$p^*=\frac{r-d}{u-d}=\frac{e^{\rho\Delta}-e^{-\sigma\sqrt{\Delta}}}{e^{\sigma\sqrt{\Delta}}-e^{-\sigma\sqrt{\Delta}}}.$$

# American Options in the CRR model (3)

- Thus the stock with initial value $S = S(0)$ is worth $S(1+u)^i(1+d)^j$ after $i$ steps up and $j$ steps down.
- Consequently, after $N$ steps, there are $N+1$ possible prices, $S(1+u)^i(1+d)^{N-i}$ $(i = 0, \ldots, N)$. There are $2^N$ possible paths through the tree.

# American Options in the CRR model (4)

- It is common to take $N$ of the order of 30, for two reasons:
  - (i) typical lengths of time to expiry of options are measured in months (9 months, say); this gives a time step around the corresponding number of days,
  - (ii) $2^{30}$ paths is about the order of magnitude that can be comfortably handled by computers (recall that $2^{10} = 1,024$, so $2^{30}$ is somewhat over a billion).

# American Options in the CRR model (5)

- We can now calculate both the value of an American put option and the optimal exercise strategy by working backwards through the tree
- This method of backward recursion in time is a form of the dynamic programming (DP) technique, due to Richard Bellman, which is important in many areas of optimisation and Operational Research.

# American Options in the CRR model - Step (1)

1. Draw a binary tree showing the initial stock value and having the right number, $N$, of time intervals.
2. Fill in the stock prices: after one time interval, these are $S(1+u)$ (upper) and $S(1+d)$ (lower); after two time intervals, $S(1+u)^2$, $S$ and $S(1+d)^2 = S/(1+u)^2$; after $i$ time intervals, these are $S(1+u)^j(1+d)^{i-j} = S(1+u)^{2j-i}$ at the node with $j$ 'up' steps and $i-j$ 'down' steps (the '$(i,j)$' node).
3. Using the strike price $K$ and the prices at the terminal nodes, fill in the payoffs $f^A_{N,j} = \max\{K - S(1+u)^j(1+d)^{N-j}, 0\}$ from the option at the terminal nodes underneath the terminal prices.

# American Options in the CRR model - Step (2)

4. Work back down the tree, from right to left. The no-exercise values $f_{ij}$ of the option at the $(i, j)$ node are given in terms of those of its upper and lower right neighbours in the usual way, as discounted expected values under the risk-neutral measure:

$$f_{ij} = e^{-\rho\Delta}[q f^A_{i+1,j+1} + (1-q) f^A_{i+1,j}].$$

The intrinsic (or early-exercise) value of the American put at the $(i, j)$ node – the value there if it is exercised early – is

$$K - S(1+u)^j(1+d)^{i-j}$$

(when this is non-negative, and so has any value).

# American Options in the CRR model - Step (3)

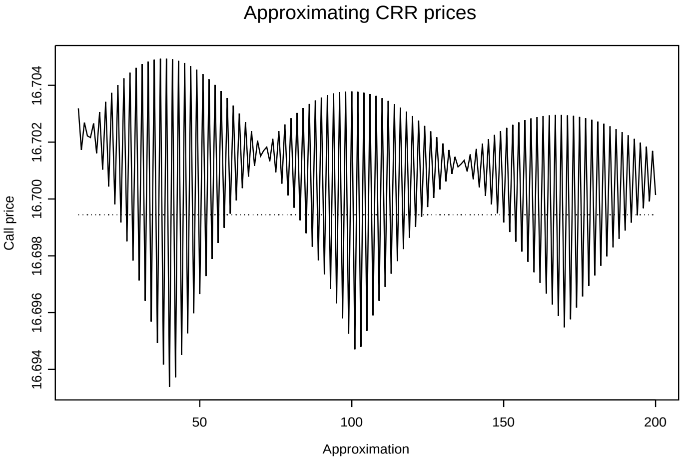
- The value of the American put is the higher of these:

$$
\begin{aligned}
& f_{ij}^A \\
&= \max\{f_{ij}, K - S(1+u)^j(1+d)^{i-j}\} \\
&= \max\Big\{e^{-\rho\Delta}(q f_{i+1,j+1}^A + (1-q) f_{i+1,j}^A)\ , \\
&\qquad\qquad K - S(1+u)^j(1+d)^{i-j}\Big\}.
\end{aligned}
$$

# American Options in the CRR model - Step (4)

5. The initial value of the option is the value $f_0^A$ filled in at the root of the tree.
6. At each node, it is optimal to exercise early if the early-exercise value there exceeds the value $f_{ij}$ there of expected discounted future payoff.

# A Three-period Example (4)



# A Three-period Example (5)

- To price a European put, with price process denoted by $p(t)$, and an American put, $P(t)$, (maturity $N = 3$, strike 100), we can for the European put either use the put-call parity, the risk-neutral pricing formula, or work backwards through the tree. For the prices of the American put we use the technique outlined above.
- We indicate the early exercise times of the American put in bold type. Recall that the discrete-time rule is to exercise if the intrinsic value $K - S(t)$ is larger than the value of the corresponding European put.

UNIVERSITÄT DUISBURG ESSEN

# A Three-period Example (6)

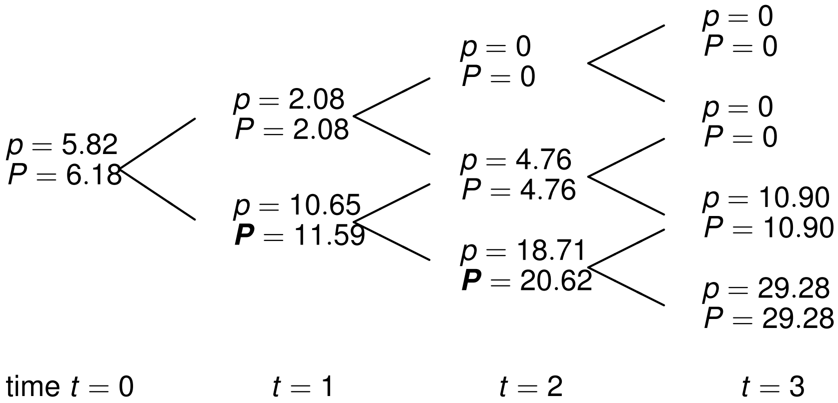



Figure : European $p(.)$ and American $P(.)$ put prices

# Agenda

# The Black-Scholes Model

# The Black-Scholes Formula for a European Call

(a) Myron Scholes

(b) Robert C. Merton

(c) Fisher Black

# European Call Price

For a European call $X = (S(T) - K)^+$ and we can evaluate the above expected value
The Black-Scholes price process of a European call is given by

$$\begin{aligned} C(t) &= S(t)\Phi(d_1(S(t), T-t)) \\ &\quad -Ke^{-r(T-t)}\Phi(d_2(S(t), T-t)). \end{aligned}$$

The functions $d_1(s,t)$ and $d_2(s,t)$ are given by

$$\begin{aligned} d_1(s,\tau) &= \frac{\log(s/K) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}, \\ d_2(s,\tau) &= \frac{\log(s/K) + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \end{aligned}$$

# Greeks (1)

- We will now analyse the impact of the underlying parameters in the standard Black-Scholes model on the prices of call and put options.
- The Black-Scholes option values depend on the
  1. (current) stock price,
  2. the volatility,
  3. the time to maturity,
  4. the interest rate
  5. the strike price.

# Greeks (2)

- The sensitivities of the option price with respect to the first four parameters are called the *Greeks* and are widely used for hedging purposes.
- We can determine the impact of these parameters by taking partial derivatives.

# Greeks

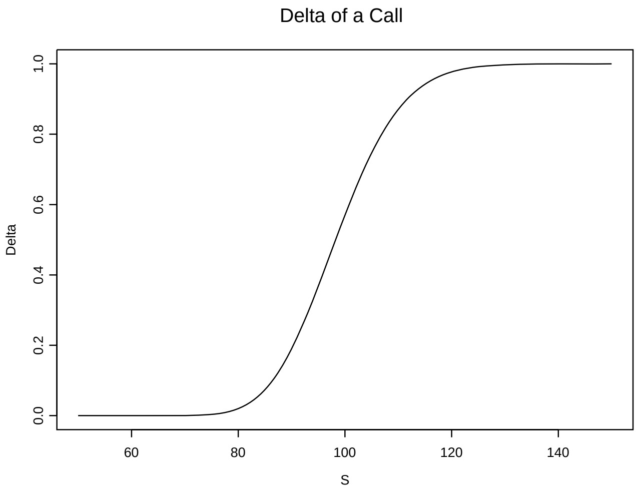
$$
\begin{aligned}
\Delta &:= \frac{\partial C}{\partial S} &= \Phi(d_1) > 0, \\
\mathcal{V} &:= \frac{\partial C}{\partial \sigma} &= S\sqrt{\tau}\varphi(d_1) > 0, \\
\Theta &:= \frac{\partial C}{\partial \tau} &= \frac{S\sigma}{2\sqrt{\tau}}\varphi(d_1) + Kre^{-r\tau}\Phi(d_2) > 0, \\
\rho &:= \frac{\partial C}{\partial r} &= \tau K e^{-r\tau}\Phi(d_2) > 0, \\
\Gamma &:= \frac{\partial^2 C}{\partial S^2} &= \frac{\varphi(d_1)}{S\sigma\sqrt{\tau}} > 0.
\end{aligned}
$$

# Greeks

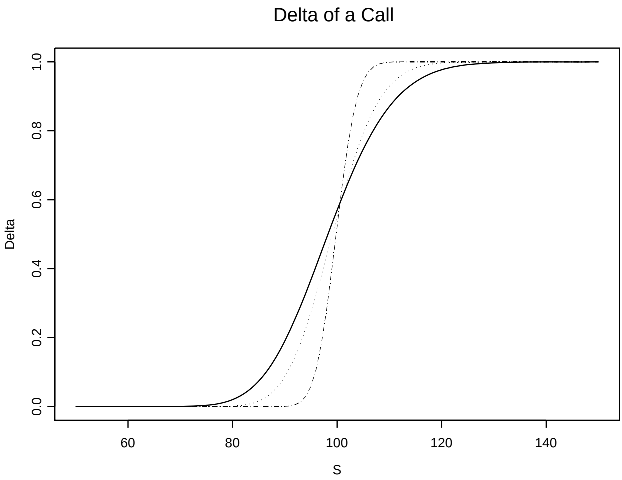
From the definitions it is clear that

- $\Delta$ – delta – measures the change in the value of the option compared with the change in the value of the underlying asset,
- $\mathcal{V}$ – vega – measures the change of the option compared with the change in the volatility of the underlying,
- similar statements hold for $\Theta$ – theta – and $\rho$ – rho

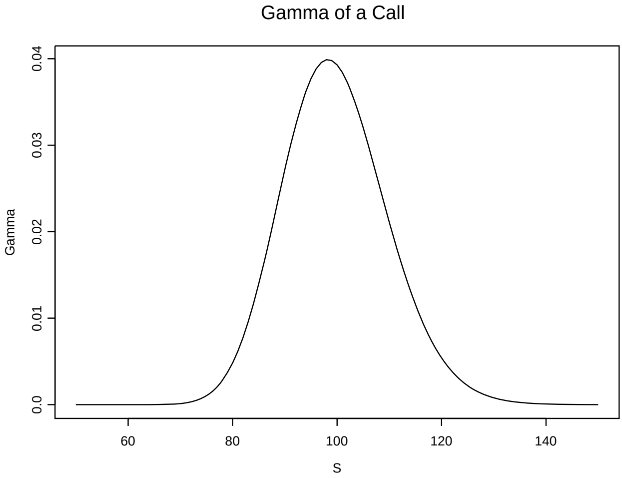
# Greeks



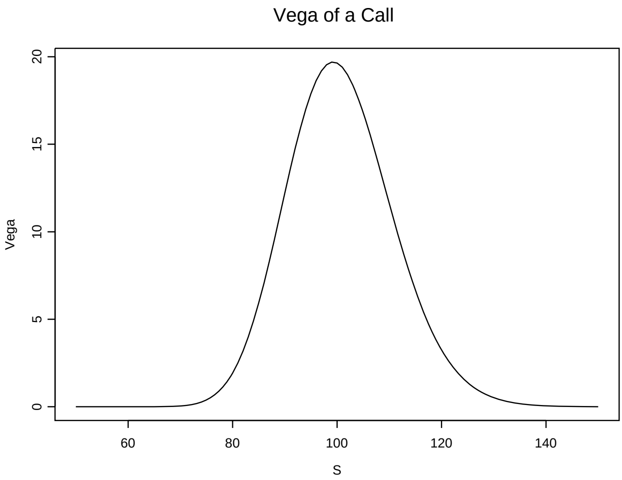
# Greeks



# Greeks



# Greeks



# Greeks

- The Black-Scholes partial differential equation can be used to obtain the relation between the Greeks,
- $$rC = \frac{1}{2}s^2\sigma^2\Gamma + rs\Delta - \Theta.$$

# Vega

- Before we can implement the Black-Scholes formula to price options, we have to estimate $\sigma$.
- Because the formula is explicit, we can, determine the $\mathcal{V}$ – the partial derivative

$$\mathcal{V} = \partial C / \partial \sigma,$$

finding

$$\mathcal{V} = S\sqrt{T}\Phi(d_1).$$

- Note here is that vega is always positive.

UNIVERSITÄT DUISBURG ESSEN

# Implied Volatility

- Since vega is positive, $C$ is a continuous – indeed, differentiable – strictly increasing function of $\sigma$.
- Turning this round, $\sigma$ is a continuous (differentiable) strictly increasing function of $C$; indeed,

$$\mathcal{V} = \frac{\partial C}{\partial \sigma}, \quad \text{so} \quad \frac{1}{\mathcal{V}} = \frac{\partial \sigma}{\partial C}.$$

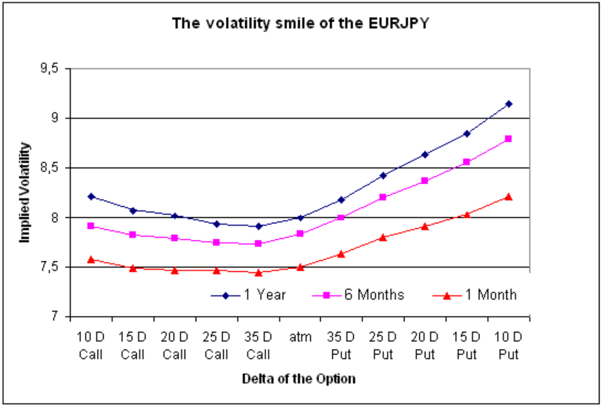
- Thus the value $\sigma = \sigma(C)$ corresponding to the actual value $C = C(\sigma)$ at which call options are observed to be traded in the market can be read off. The value of $\sigma$ obtained in this way is called the *implied volatility*.

# The Black-Scholes Implied Volatility

- The Black-Scholes model assumes a constant volatility over all maturities and strikes.
- One can calculate the implicit volatility given the market price of options.
- The implicit volatility is changing over time and maturity, so contradicting the modelling assumptions.
- However, it can be used to predict future volatility.
- Furthermore, volatility can be made an own "asset class".

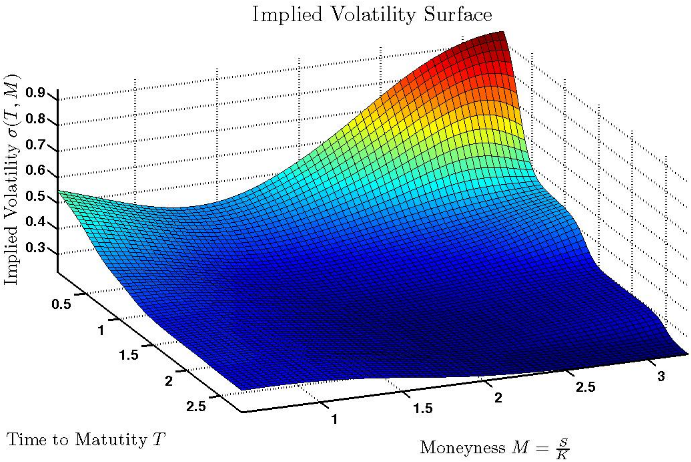
# Volatility Smile

Volatility Index VDAX-NEW (in percentage points)



# Volatility Surface

Volatility Index VDAX-NEW (in percentage points)



# VDAX-NEW: Definition

- The volatility index VDAX-NEW was developed by Deutsche Börse and Goldman Sachs. It tracks the degree of fluctuation expected by the derivatives market, i.e. the implied volatility, for the DAX index. The index expresses in percentage terms what degree of volatility is to be expected for the following 30 days.
- VDAX-NEW started on 20 April 2005 and will replace VDAX in the medium-term.

# VDAX-NEW: Examples

Examples:

- A DAX of 4,000 and a VDAX-NEW of 10 indicate that the DAX stock index is expected to fluctuate between 3,885 and 4,115 over the next thirty days:

$$4000 \pm 4000 \times 0.1 \times \sqrt{\frac{30}{365}} \approx 4000 \pm 115.$$

- A DAX of 4,000 and a VDAX-NEW of 20 indicate that the DAX stock index is expected to fluctuate between 3,770 and 4,230 over the next thirty days:

$$4000 \pm 4000 \times 0.2 \times \sqrt{\frac{30}{365}} \approx 4000 \pm 230.$$

# VDAX-NEW

Volatility Index VDAX-NEW (in percentage points)

# VDAX-NEW: Calculation

- Index is based on 8 sub-indices (option series) which include DAX Options from 2-24 months expiration.
- The main rolling index is calculated 30 days to expiration on linear interpolation of the two sub-indices closest to the 30 days expiration.
- In addition to at-the-money options (VDAX), out-of-the-money options are also considered (VDAX-NEW).
- Calculation frequency: Once a minute on every trading day at Eurex between 8.50am and 5.30pm CET.

# VDAX-NEW: Features

- Expresses market expectation of the amplitude of fluctuation in DAX.
- Index is able to react only to changes in volatility.
- Allows better replication for derivatives and structured products.
- Due to ATM and OTM options VDAX-NEW captures more of the volatility skew than VDAX.
- Index establishes volatiliy as a tradable and separate asset class for investors.

# VDAX-NEW: Use

- Investment/Trading: Speculation on future levels of volatility.
- Hedging: Cover short volatility positions.
- Diversification of German equity portfolios (correlation to DAX is -0.5689, correlation to MDAX is -0.4668).
- Benchmark for German equity volatility.

# VDAX-NEW: Diversification

- Portfolio of 80% DAX and 20% VDAX-NEW between 1992 and 2004 would have generated a 3.8 percentage points on average higher yield than investments in DAX only. And this with lower risk!

UNIVERSITÄT DUISBURG ESSEN

# Agenda

Introduction – The History of Derivatives

Basic Derivatives

Arbitrage and Valuation

The Black-Scholes Model

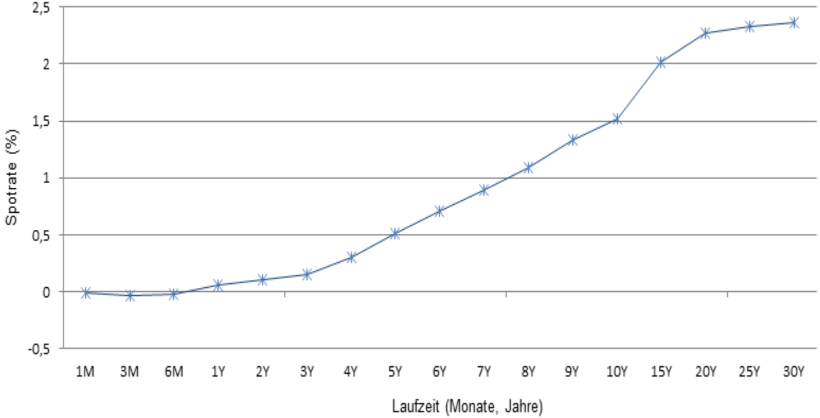
Interest Rates
- Basic Interest Rates
- Market Rates
- Interest Rate Derivatives
- Valuation of Structured Products

UNIVERSITÄT DUISBURG ESSEN

# Basic Interest Rates

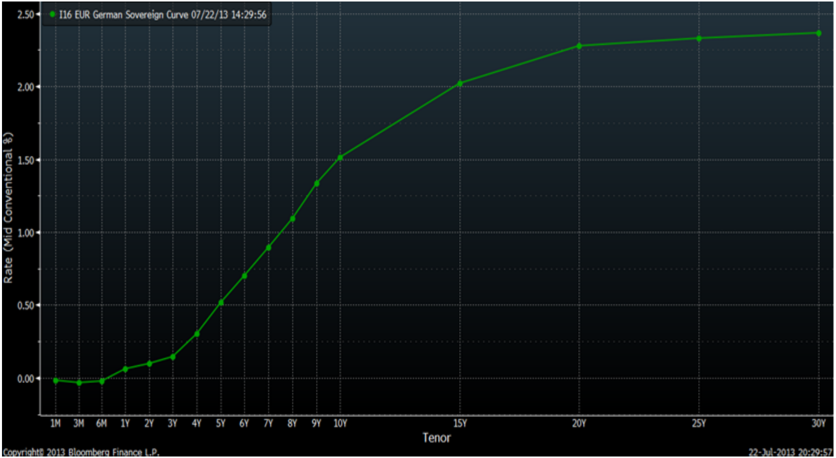
- Economic agents have to be rewarded for postponing consumption; in addition, there is a risk premium for the uncertainty of the size of future consumption.
- Investors, Firms, banks pay compensation for the willingness to postpone
- A common interest rate (equilibrium) emerges which allows to fulfill the aggregate liquidity demand.

# Yield Curves



UNIVERSITÄT DUISBURG ESSEN

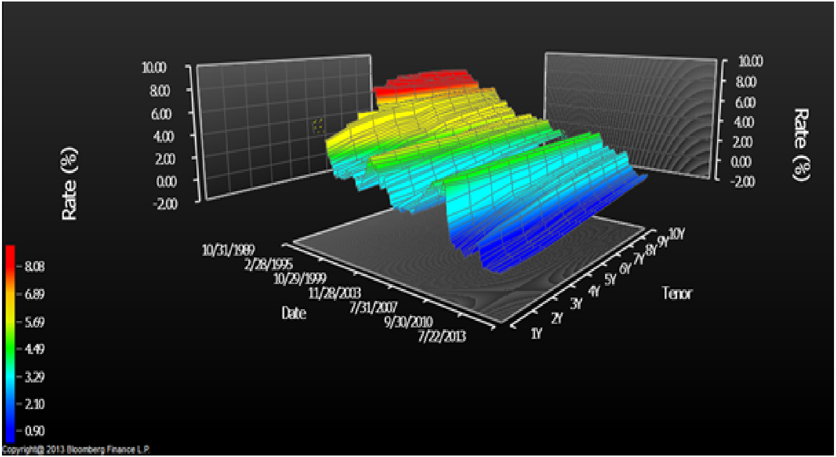
# Yield Curves



UNIVERSITÄT DUISBURG ESSEN

# Yield Curves

1988-2013.png




Daten: Staatsanleihen July 2013.

# Fixed-rate Bond

With a fixed-rate bond the seller promises the buyer to pay fixed coupons $C$ over time, until the bond matures, and when it matures the seller will repay the principal amount borrowed. The price of a bond is determined by its cashflow and the discount factor ($T = t + n$ maturity, $N$ notional value)

$$
\begin{aligned}
p_c(t, T) &= \frac{C}{(1 + r_1)^1} + \frac{C}{(1 + r_2)^2} + \ldots + \frac{C + N}{(1 + r_n)^n} \\
&= \sum_{i=1}^{n} \frac{C}{(1 + r_i)^i} + \frac{N}{(1 + r_n)^n}
\end{aligned}
$$

# Yield to Maturity

The yield to maturity $y$ can be calculated from the coupon bond prices

$$p_c(t,T) = \frac{C}{(1+y)^1} + \frac{C}{(1+y)^2} + \ldots + \frac{C+N}{(1+y)^n}$$

# Forward Rates

Forward rates cover the transaction where money is borrowed or lent between two future dates, on terms that are agreed upon today. Under the no-arbitrage assumption forward rates can be uniquely determined from spot rates and vice versa. For a two-year investment there are two strategies available

- invest in a two-year bond
- invest in a one-year bond for one year and in the forward contract from year 1 and year 2

By no arbitrage

$$(1 + r_2)^2 = (1 + r_1) \times (1 + f_{1,2})$$

UNIVERSITÄT DUISBURG ESSEN

# Notation

$p(t, T)$ denotes the price of a risk-free zero-coupon bond at time $t$ that pays one unit of currency at time $T$.
We will use continuous compounding, i.e. a zero bond with interest rate $r(t, T)$ maturing at $T$ will have the price

$$p(t, T) = e^{-r(t,T)(T-t)}.$$

# Forward Rates

Given three dates $t < T_1 < T_2$ the basic question is: what is the risk-free rate of return, determined at the contract time $t$, over the interval $[T_1, T_2]$ of an investment of 1 at time $T_1$?

| Time | $t$ | $T_1$ | $T_2$ |
|---|---|---|---|
| | Sell $T_1$ bond<br>Buy $\frac{p(t,T_1)}{p(t,T_2)}$ $T_2$ bonds | Pay out 1 | Receive $\frac{p(t,T_1)}{p(t,T_2)}$ |
| Net investment | 0 | $-1$ | $+\frac{p(t,T_1)}{p(t,T_2)}$ |

Table : Arbitrage table for forward rates

# Forward Rates

To exclude arbitrage opportunities, the equivalent constant rate of interest $R$ over this period (we pay out 1 at time $T_1$ and receive $e^{R(T_2-T_1)}$ at $T_2$) has thus to be given by

$$e^{R(T_2-T_1)} = \frac{p(t,T_1)}{p(t,T_2)}.$$

# Various Interest Rates

- The forward rate at time $t$ for time period $[T_1, T_2]$ is defined as

$$R(t, T_1, T_2) = \frac{\log(p(t, T_1)) - \log(p(t, T_2))}{T_2 - T_1}$$

- The spot rate for the time period $[T_1, T_2]$ is defined as

$$R(T_1, T_2) = R(T_1, T_1, T_2)$$

- The instantaneous forward rate is

$$f(t, T) = -\frac{\partial \log(p(t, T))}{\partial T}$$

- The instantaneous spot rate is

$$r(t) = f(t, t)$$

# Rates

- The forward rate is the interest rate at which parties at time $t$ agree to exchange $K$ units of currency at time $T_1$ and give back $Ke^{R(t,T_1,T_2)(T_2-T_1)}$ units at time $T_2$. This means, one can lock in an interest rate for a future time period today.
- The spot rate $R(t,T_1)$ is the interest rate (continuous compounding) at which one can borrow money today and has to pay it back at $T_1$.
- The instantaneous forward and spot rate are the corresponding interest rates at which one can borrow money for an infinitesimal short period of time.

# Simple Relations

The money account process is defined by

$$B(t) = \exp\left\{\int_0^t r(s)ds\right\}.$$

The interpretation of the money market account is a strategy of instantaneously reinvesting at the current short rate.
For $t \leq s \leq T$ we have

$$p(t,T) = p(t,s)\exp\left\{-\int_s^T f(t,u)du\right\},$$

and in particular

$$p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}.$$

# Simple Spot Rate

The simply-compounded spot interest rate prevailing at time $t$ for the maturity $T$ is denoted by $L(t, T)$ and is the constant rate at which an investment has to be made to produce an amount of one unit of currency at maturity, starting from $p(t, T)$ units of currency at time $t$, when accruing occurs proportionally to the investment time.

$$L(t, T) = \frac{1 - p(t, T)}{\tau(t, T)p(t, T)} \tag{8}$$

Here $\tau(t, T)$ is the daycount for the period $[t, T]$ (typically $T - t$).

UNIVERSITÄT DUISBURG ESSEN

## Simple Spot Rate

- The bond price can be expressed as

$$p(t,T) = \frac{1}{1 + L(t,T)\tau(t,T)}.$$

  Other 'daycounts' denoted by $\tau(t,T)$ are possible.
- Notation is motivated by LIBOR rates (London InterBank Offered Rates).

# Forward Rate Agreements

In order to introduce simply-compounded forward interest rates we consider forward-rate agreements (FRA). A FRA involves the current time $t$, the expiry time $T > t$ and the maturity time $S > T$. The contract gives its holder an interest-rate payment for the period between $T$ and $S$. At maturity $S$, a fixed payment based on a fixed rate $K$ is exchanged against a floating payment based on the spot rate $L(T, S)$ resetting in $T$ with maturity $S$.

# Forward Rate Agreements

Formally, at time $S$ one receives $\tau(T,S)K \cdot N$ units of currency and pays the amount $\tau(T,S)L(T,S) \cdot N$, where $N$ is the contract nominal value. The value of the contract is therefore at $S$

$$N\tau(T,S)(K - L(T,S)). \tag{9}$$

Using (8) we write this in terms of bond prices as

$$N\tau(T,S)\left(K - \frac{1 - p(T,S)}{\tau(T,S)p(T,S)}\right) = N\left(K\tau(T,S) - \frac{1}{p(T,S)} + 1\right).$$

UNIVERSITÄT DUISBURG ESSEN

# Forward Rate Agreements

Now we discount to obtain the value of this time $S$ cashflow at $t$

$$
\begin{aligned}
& FRA(t, T, S, \tau(T, S), N, K) \\
= \; & Np(t, S) \left( K\tau(T, S) - \frac{p(t,T)}{p(t,T)p(T,S)} + 1 \right) \\
= \; & N(Kp(t, S)\tau(T, S) - p(t, T) + p(t, S)).
\end{aligned}
$$

There is only one value of $K$ that renders the contract value 0 at $t$. The resulting rate defines the simply-compounded forward rate.

# Simply-Compounded Forward Interest Rate

The simply-compounded forward interest rate prevailing at time $t$ for the expiry $T > t$ and maturity $S > T$ is denoted by $F(t;T,S)$ and is defined by

$$F(t;T,S) := \frac{1}{\tau(T,S)}\left[\frac{p(t,T)}{p(t,S)} - 1\right]. \tag{10}$$

# Simply-Compounded Forward Interest Rate

- $FRA(\ldots) = Np(t, S)\tau(T, S)(K - F(t; T, S))$ is an equivalent definition.
- To value a FRA (typically with a different $K$) replace the LIBOR rate in (9) by the corresponding forward rate $F(t; T, S)$ and take the present value of the resulting quantity.

# Interest-Rate Swap

A generalisation of the FRA is the Interest-Rate Swap.(IRS). A Payer (Forward-start) Interest-Rate Swap (PFS) is a contract that exchanges payments between two differently indexed legs, starting from a future time instant. At every instant $T_i$ in a prespecified set of dates $T_{\alpha+1}, \ldots, T_\beta$ the fixed leg pays out the amount

$$N\tau_i \cdot K$$

corresponding to a fixed interest rate $K$, a nominal value $N$, and a year fraction $\tau_i$ between $T_{i-1}$ and $T_i$, whereas the floating leg pays the amount

$$N\tau_i L(T_{i-1}, T_i).$$

Corresponding to the interest rate $L(T_{i-1}, T)$ resetting at the previous instant $T_{i-1}$ for the maturity given by the current payment instant $T_i$, with $T_\alpha$ a given date.

# Interest-Rate Swap

Set

$$\mathcal{T} := \{T_\alpha, \ldots, T_\beta\} \quad \text{and} \quad \tau := \{\tau_{\alpha+1}, \ldots, \tau_\beta\}.$$

Payers IRS(PFS): fixed leg is paid and floating leg is received
Receiver IRS (RFS): fixed leg is received and floating leg is paid.
The discounted payoff at time $t < T_\alpha$ of a PFS is

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i) N \tau_i (L(T_{i-1}, T_i) - K)$$

with $D(t, T)$ the discount factor (typically from bank account).
For a RFS we have

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i) N \tau_i (K - L(T_{i-1}, T_i)).$$

# Interest-Rate Swap

We can view the last contract as a portfolio of FRAs and find

$$
\begin{aligned}
& RFS(t, \mathcal{T}, \tau, N, K) \\
= \quad & \sum_{i=\alpha+1}^{\beta} FRA(t, T_{i-1} T_i, \tau_i, N, K) \\
= \quad & N \sum_{i=\alpha+1}^{\beta} \tau_i p(t, T_i)(K - F(t, T_{i-1}, T_i)) \\
= \quad & -Np(t, T_\alpha) + Np(t, T_\beta) + N \textstyle\sum_{i=\alpha+1}^{\beta} \tau_i K p(t, T_i).
\end{aligned}
$$

The two legs of an IRS can be viewed as coupon-bearing bond (fixed leg) and floating rate note (floating leg).

# Interest-Rate Swap

A floating-rate note is a contract ensuring the payment at future times $T_{\alpha+1}, \dots, T_\beta$ of the LIBOR rates that reset at the previous instants $T_\alpha, \dots, T_{\beta-1}$. Moreover, the note pays a last cash flow consisting of the reimbursement of the notational value of the note at the final time $T_\beta$.

# Interest-Rate Swap

We can value the note by changing sign and setting $K = 0$ in the RFS formula and adding it to $Np(t, T_\beta)$, the present value of the cash flow $N$ at $T_\beta$. So we see

$$\underbrace{-RFS(t, T, \tau, N, 0) + Np(t, T_\beta)}_{\text{value of note}} = \underbrace{Np(t, T_\alpha)}_{\text{from RFS formula}} .$$

# Interest-Rate Swap

This implies that the note is always equivalent to $N$ units at its first reset date $T_\alpha$ (the floating note trades at par). We require the IRS to be fair at time $t$ to obtain the forward swap rate. The forward swap rate $S_{\alpha,\beta}(t)$ at time $t$ for the sets of time $\mathcal{T}$ and year fractions $\tau$ is the rate in the fixed leg of the above IRS that makes the IRS a fair contract at the present time, i.e. it is the fixed rate $K$ for which $RFS(t, T, \tau, N, K) = 0$. We obtain

$$S_{\alpha,\beta}(t) = \frac{p(t, T_\alpha) - p(t, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i p(t, T_i)}. \tag{11}$$

# Interest-Rate Swap

We write (11) in terms of forward rates. First divide numerator and denominator by $p(t, T_\alpha)$ and observe that

$$\frac{p(t,T_k)}{p(t,T_\alpha)} = \prod_{j=\alpha+1}^{k} \frac{p(t,T_j)}{p(t,T_{j-1})} = \prod_{j=\alpha+1}^{k} \frac{1}{1+\tau_j F_j(t)}$$

with $F_j(t) := F(t, T_{j-1}; T_j)$. So (11) can be written as

$$S_{\alpha,\beta}(t) = \frac{1 - \prod_{j=\alpha+1}^{\beta} \frac{1}{1+\tau_j F_j(t)}}{\sum_{i=\alpha+1}^{\beta} \tau_i \prod_{j=\alpha+1}^{i} \frac{1}{1+\tau_j F_j(t)}}. \tag{12}$$

# Caps

- A cap is a contract where the seller of the contract promises to pay a certain amount of cash to the holder of the contract if the interest rate exceeds a certain predetermined level (the cap rate) at a set of future dates.
- It can be viewed as a payer IRS where each exchange payment is executed only if it has positive value.
- The cap discounted payoff is

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i) N\tau_i (L(T_{i-1}, T_i) - K)^+.$$

- Each individual term is a caplet.

# Floors

- A floor is equivalent to a receiver IRS where each exchange is executed only if it has positive value.
- The floor discounted payoff is

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i) N \tau_i (K - L(T_{i-1}, T_i))^+.$$

- Each individual term is a floorlet.

# Simple Properties

A cap (floor) is said to be at-the-money (ATM) if and only if

$$K = K_{ATM} := S_{\alpha,\beta}(0) = \frac{p(0,T_\alpha) - p(0,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i p(0,T_i)}.$$

The cap is instead said to be in-the-money (ITM) if $K < K_{ATM}$, and out-of-the-money (OTM) if $K > K_{ATM}$, with the converse holding for a floor.

# Simple Properties Cap

- Simple protection against rising interest rates, but requires the payment of a premium (of course)
- Strike is the maximal interest to be paid
- Advantageous only if market expectation becomes true

# Swaptions

- Swap options or more commonly swaptions are options on an IRS. A European payer swaption is an option giving the right (and not the obligation ) to enter a payer IRS at a given future time, the swaption maturity. Usually the swaption maturity coincides with the first reset date of the underlying IRS.
- The underlying-IRS length ( $T_\beta - T_\alpha$) is called the tenor of the swap.
- The discounted payoff of a payer swaption can be written by considering the value of the underlying payer IRS at its first reset date $T_\alpha$ (also the maturity of the swaption)

$$N \sum_{i=\alpha+1}^{\beta} p(T_\alpha, T_i)\tau_i(F(T_\alpha; T_{i-1}, T_i) - K).$$

# Swaptions Payoff

The option will be exercised only if this value is positive. So the current value is

$$ND(t, T_\alpha)\left(\sum_{i=\alpha+1}^{\beta} p(T_\alpha, T_i)\tau_i(F(T_\alpha; T_{i+1}, T_i) - K)\right)^+ .$$

UNIVERSITÄT DUISBURG ESSEN

# Swaptions Payoff

Since the positive part operator is a piece-wise linear and convex function we have

$$\left(\sum_{i=\alpha+1}^{\beta} p(T_\alpha, T_i)\tau_i(F(T_\alpha; T_{i-1}, T_i) - K)\right)^+$$

$$\leq \sum_{i=\alpha+1}^{\beta} p(T_\alpha, T_i)\tau_i(F(T_\alpha; T_{i-1}, T_i) - K)^+$$

with strict inequality in general. Thus an additive decomposition is not feasible.

# Swaptions Payoff

A swaption (either payer or receiver) is said to be at-the-money (ATM) if and only if

$$K = K_{ATM} = S_{\alpha,\beta}(0) = \frac{p(0, T_\alpha) - p(0, T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i p(0, T_i)}.$$

The payer swaption is instead said to be in-the-money (ITM) if $K < K_{ATM}$, and out-of-the-money (OTM) if $K > K_{ATM}$. The receiver swaption is ITM if $K > K_{ATM}$, and OTM if $K < K_{ATM}$.

# Product Buyer

- (Institutional) Investor buys product for certain nominal value $N$
- Receives coupons at prespecified time points.
- At terminal date (maturity) the nominal is paid back.
- Investor wants to receive as high as possible coupon payments. Therefore Investor is willing to take a point of view towards market development.

# Product Buyer

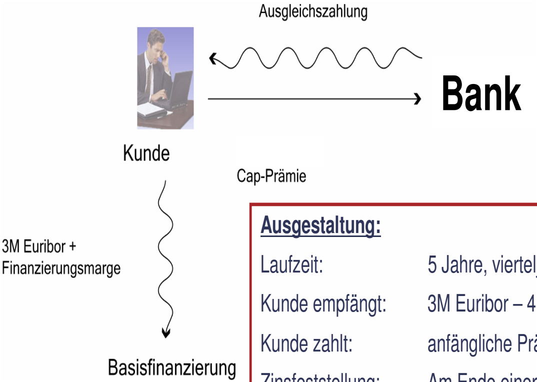
- Investor has a liability and has to pay floating or fixed interest
- Investor can enter a swap which pays the liability cash-flow
- Investor pays coupons on a structured product in return, which (in case that the market view of the investor becomes true) are cheaper than the original cash flow

# Cap

- Simple protection against rising interest rates, but requires the payment of a premium (of course)
- Strike is the maximal interest to be paid
- Advantageous only if market expectation becomes true

UNIVERSITÄT DUISBURG ESSEN

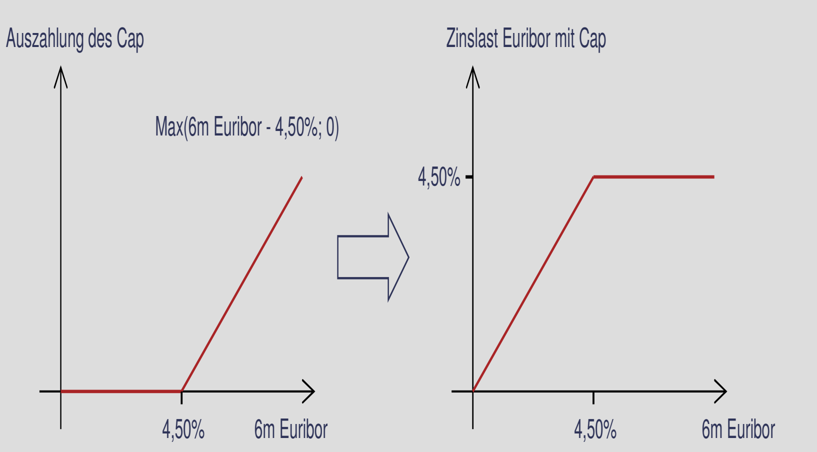
# Cap




**Ausgestaltung:**

| | |
|---|---|
| Laufzeit: | 5 Jahre, vierteljährlich, Act/360 |
| Kunde empfängt: | 3M Euribor – 4,50%, wenn 3M Euribor > 4,50% |
| Kunde zahlt: | anfängliche Prämie |
| Zinsfeststellung: | Am Ende einer Zinsperiode (in Arrears) |

# Cap


Auszahlung des Cap
Max(6m Euribor - 4,50%; 0)
4,50%
6m Euribor
Zinslast Euribor mit Cap
4,50%
4,50%
6m Euribor

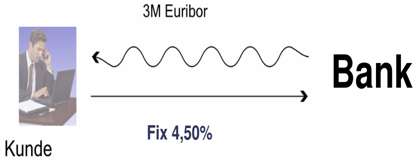
# Instruments for Interest Rate Management

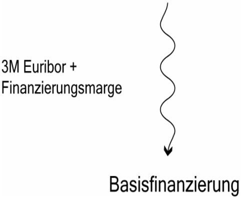
- Agreements to exchange rates
  - Exchange of rate payments
  - fixed tenor
  - no costs to enter the contract
  - Examples: FRA, Swaps
- Insurance against rates movements
  - Option on payment
  - fixed maturity
  - buyer pays premium
  - Examples: Caps, Floors, Swaptions

# Interest Rate Swap

- Buyer of a swap
  - receives fixed swap rate $S(0, T)$
  - no initial payment (since this is the fair rate)
  - expects an increase of rates above the swap rate $S(0, T)$ during life-time of the contract
  - to be profitable the increase must be higher than suggested by current swap rate
- Benefit: Insurance against rising rates, no initial payment
- Risk: No participation if interest rates fall

# Interest Rate Swap




**<u>Ausgestaltung:</u>**

| | |
|---|---|
| Laufzeit: | 8 Jahre, vierteljährlich, Act/360 |
| Kunde empfängt: | 3M Euribor |
| Kunde zahlt: | 4,50 % |
| Zinsfeststellung: | Am Anfang einer Zinsperiode (in Advance) |

UNIVERSITÄT DUISBURG ESSEN

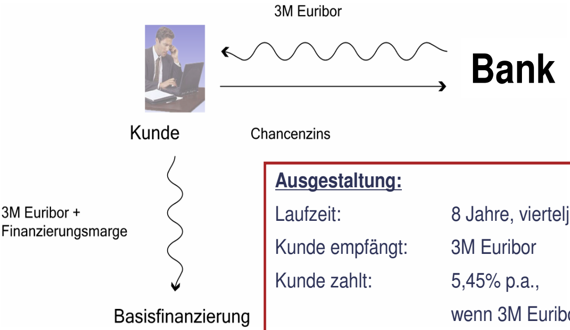
# Constant Maturity Swap

- For a constant maturity swap (CMS) one of the reference rates is a variable market rate
- Example: 3 -Month Euribor vs 5 year swap rate
- For standard term structure CMS has a positive value, so Euribor + spread is paid

# Constant Maturity Swap

- Advantages: It is possible to take advantage of favourable movements of the term structure; starts with lower costs
- Disadvantages: Unfavourable movements of term structure generate losses; losses are potentially unlimited

# Interest Rate Swap with Optionality

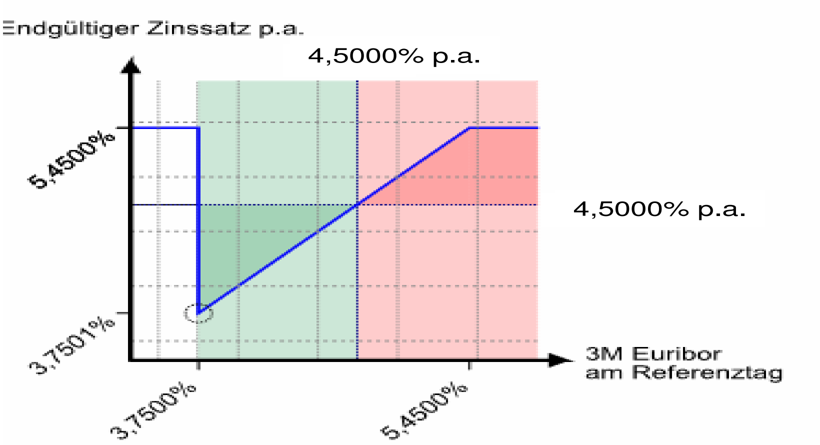



**Ausgestaltung:**

| | |
|---|---|
| Laufzeit: | 8 Jahre, vierteljährlich, Act/360 |
| Kunde empfängt: | 3M Euribor |
| Kunde zahlt: | 5,45% p.a., |
| | wenn 3M Euribor > 5,45% oder |
| | wenn 3M Euribor < 3,75%; |
| | sonst 3M Euribor |
| Zinsfeststellung: | Am Anfang einer Zinsperiode (in Advance) |

UNIVERSITÄT DUISBURG ESSEN

# IRS with Optionality: Risk Profile



UNIVERSITÄT DUISBURG ESSEN

# Interest Rate Swap with Optionality

- Chance
  - Rates are capped at 5,45 % for the next 8 years
  - Participation on low rates is still possible
- Risk
  - Participation on lower than 3,75 % rates is not possible
  - In case rates are lower than 3,75 % Rates a high rate (5,45 %) has to be paid
- Chance/Risk the swap can be traded (i.e. be sold).