Pricing II

Model Risk Question

Q1

(a) Describe the difference between a Bayesian Model Averaging and a Utility-based Worst-Case approach.

(b) Assume that you need to price a derivative instrument in an incomplete market situation. What is Cont's suggestion to measure model risk?

(c) Consider in a basic GBM world the pricing of a call option with maturity T. Assume alternative diffusion models

$$\mathbb{Q}_i \; : \; dS(t) = S(t)(rdt + \sigma_i(t)dW(t)) \tag{1}$$

where $\sigma_i : [0,t] \to [0,\infty[$ is a bounded deterministic volatility function. We observe prices of a traded European call (strike K, maturity T) with prices C^* and Σ the implied Black-Scholes volatility.

(i) Formulate the calibration condition for the given models.

(ii) Let $0 < T_1 < T$ and consider two models. The first on with constant volatility σ_1 and the second one with a volatility $\sigma_2 < \sigma_1$ for $0 \leq t < T_1$ and a constant volatility for $T_1 \leq t \leq T$. Specify the volatilities of the models in terms of the implied volatility Σ.

(iii) Consider the Δ at $t = 0$ and $t = T_1$ for both models for the European call with maturity T. Which is more sensitive to changes of the underlying at $t = 0$ and $t = T_1$?

(iv) Consider a European call with strike K and maturity T_1. What is Cont's measure of model risk for this option regarding the two models.

Solution

(a) Describe the difference between a Bayesian Model Averaging and a Utility-based Worst-Case approach.

Answer: The Bayesian Model is calculates model dependent quantities by averaging over expectation of the models. The worst-case approach has its foundations in the *MaxMin* approach as a robust version of expected utility. Assume U is a utility function, then the worst case approach considers $\max_X \min_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}(U(X))$.

(b) Assume that you need to price a derivative instrument in an incomplete market situation. What is Cont's suggestion to measure model risk?

Answer: For X a derivative we associate with $\Gamma(X)$ the ask price and with $-\Gamma(-X)$ its bid price. Cont's suggestion is

$$\Gamma^u(X) = \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}} \quad \text{and} \quad \Gamma^l(X) = -\Gamma^u(-X) = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}.$$

Cont measures the model risk as

$$\Gamma^u(X) - \Gamma^l(X).$$

(c) Consider in a basic GBM world the pricing of a call option with maturity T. Assume alternative diffusion models

$$\mathbb{Q}_i \; : \; dS(t) = S(t)(rdt + \sigma_i(t)dW(t)) \tag{2}$$

where $\sigma_i : [0,t] \to [0,\infty[$ is a bounded deterministic volatility function. We observe prices of a traded European call (strike K, maturity T) with prices C^* and Σ the implied Black-Scholes volatility.

(i) Formulate the calibration condition for the given models.
Answer:

$$\frac{1}{T}\int_0^T \sigma_i(s)^2 ds = \Sigma^2, \tag{3}$$

(ii) Let $0 < T_1 < T$ and consider two models. The first one with constant volatility σ_1 and the second one with a volatility $\sigma_2 < \sigma_1$ for $0 \le t < T_1$ and a constant volatility for $T_1 \le t \le T$. Specify the volatilities of the models in terms of the implied volatility Σ.
Answer:

$$\sigma_1 = \Sigma,$$

$$\sigma_2(t) = \sigma_2 \mathbf{1}_{[0,T_1]} + \sqrt{\frac{T\Sigma^2 - T_1\sigma_2^2}{T - T_1}}\mathbf{1}_{]T_1,T]}.$$

(iii) Consider the Δ at $t = 0$ and $t = T_1$ for both models for the European call with maturity T. Which is more sensitive to changes of the underlying at $t = 0$ and $t = T_1$?
Answer: We know

$$\Delta = \frac{\partial}{\partial S} Call_{BS}(S, K, \sigma, r, t, T) = \Phi(d_1)$$

where, as usual, Φ denotes the c.d.f. of the standard normal distribution and

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}}.$$

At $t = 0$ both models have the same Δ as they are both valued with the implied volatility Σ. At $t = T_1$ the Δ of model 2 is higher as the remaining volatility is higher. So it is more sensitive.

(iv) Consider a European call with strike K and maturity T_1. What is Cont's measure of model risk for this option regarding the two models.
Answer: By monotonicity of the BS-formula in terms of volatility the Cont model bounds are

$$\Gamma^u(X) = C^{BS}(K, T_1; \sigma_1) \quad \Gamma^l(X) = C^{BS}(K, T_1; \sigma_2).$$