

Project Part 2

Pattern Recognition

ECE 759

Kudiyar Orazymbetov (korazym@ncsu.edu)
Nico Casale (ncasale@ncsu.edu)

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1 Introduction

In this report, we outline the process we undertook to implement two classification algorithms: linear discriminant analysis (LDA) and Decision Trees. We tested our generated classifiers on the Yale Extended Face Dataset B and MNIST [2] datasets. To complete this, we experimented with two feature generation methods: principle components analysis (PCA) and LDA itself (for use in decision trees.) These were primarily useful in training our decision trees, which require a thorough exploration of the feature space to grow. Note that throughout this document, we use the terms '*features*' and '*attributes*' interchangeably.

After presenting an explanation of our algorithms' implementations, we will discuss our test results and exploration of the hyperparameter space. The hyperparameters are adjustments that variously control the accuracy and computational complexity of our classifiers. We utilized cross validation to verify our classifiers' stability and our optimal choice of hyperparameters.

Code for this project is included in our project submission. In addition, it's available on GitHub at <https://github.com/n-casale/ece759-project>. The Git repository includes a more thorough representation of the path our code took to its current state.

2 Feature Selection

2.1 Decision Tree and Extra-Tree Feature Generation

In working with the decision trees, we utilized dimensionality reduction techniques including linear discriminant analysis (LDA) and principle components analysis (PCA) for training and testing classical decision trees, which pass over all features to consider which decision nodes to generate. This improved the computational requirements of the algorithm, as features in lower dimensions require less processing to distinguish.

LDA reduces data with K classes in large dimensions (on the order of the number of pixels) to K dimensions which captures the most energy of the data. Implementing LDA as dimensionality reduction, MNIST and Extended YaleB datasets are reduced to 10 and 38 variables respectively (for each data point.) We then worked with this transformed data as the input to our decision tree classifier for training and testing.

In a separate experiment, we considered the effects of utilizing the raw pixel data as features in *extra-trees* which are ensembles of decision trees trained on randomly chosen features and thresholds. When testing, the mode of the predicted classes is chosen as the global prediction of the ensemble of trees. This method is shown to be more accurate than our classical decision trees trained on PCA- and LDA-generated features in Section (??).

2.2 LDA vs PCA

LDA and PCA linearly transform the data to reduce its dimensionality. Dimensionality reduction is helpful in many algorithms because it eases computational requirements and improves the generalizability of the classifier. This latter benefit is conferred by the manner in which dimensionality reduction smoothes out the individual variations in a particular class. A good dimensionality reduction algorithm should maximize the 'distance' between classes in the transformed (reduced) space, while minimizing the variance, or spread of those classes in the transformed space.

The difference between the two techniques is that LDA is supervised whereas PCA is unsupervised. PCA finds orthogonal projections which capture the most variance of each feature, thereby expressing the data along *principle*

components that are the most distinct. We can visually exemplify the PCA technique as in the figure below.

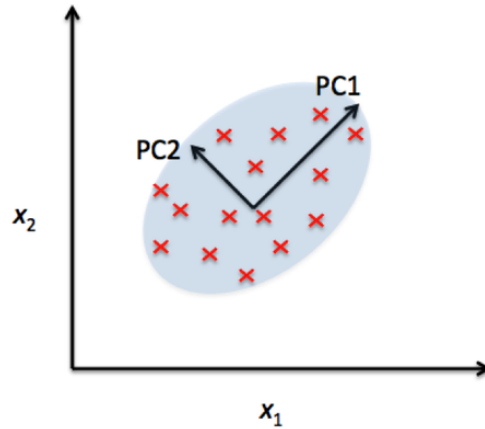


Figure 2.1: PCA implemented on 2-dimensional data [1].

In our PCA implementation, we rely on the *singular value decomposition* (SVD) to generate our features, conventionally described as *scores*. To perform the singular value decomposition and obtain our features, we must first pre-process the data. We arrange it so that the rows of our data X are the individual images in the dataset, with each column corresponding to a pixel value. Then, we subtract the column-wise empirical mean from each column to render each column zero-mean. Then, we utilize MATLAB's built-in `svd(.)` to obtain the matrices $[U, S, V]$. Our features are represented by $U * S$. Taking the transpose of this matrix, and extracting the first n columns, where n is the number of features we seek to utilize, we have a matrix of features corresponding to each image in our dataset. This method was useful, although not as effective in training decision trees as LDA, which we now discuss.

LDA maximizes the class separability and can be represented visually as in the figure below.

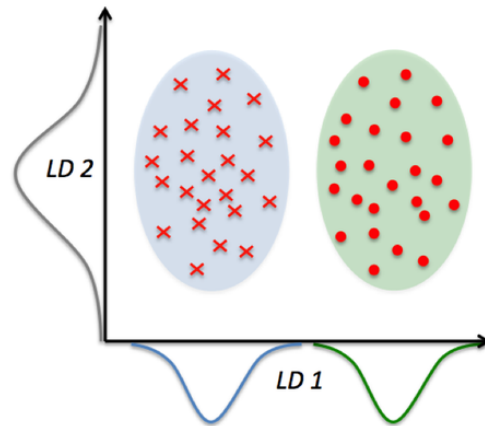


Figure 2.2: LDA visualization [1].

The underlying concept of LDA is taking the eigenvectors of $\frac{\Sigma_b}{\Sigma_w}$ where Σ_w is within-class scatter matrix and Σ_b is between-class scatter matrix. This division of matrices separates the classes away from one another. This is distinct from PCA, which is unsupervised, thereby only capturing the data in a minimum number of variables irrespective of class. Please see Section (3.3) for more information on LDA.

3 Algorithm Implementations

3.1 Decision Tree Algorithm

A binary decision tree is a hierarchical structure that takes input data at its root and propagates it to one of many leaves. Each *leaf* of the tree represents a class designation. To reach a leaf, the features of the data are utilized at *nodes* to make a binary decision: to proceed down the left or right *branch* of the tree? To answer this question, the node also carries a *threshold* that the feature value of the test data is compared against. If the test feature is less than the threshold, we proceed down the left branch. Otherwise, the right. An illustration of a simple decision tree is pictured below.

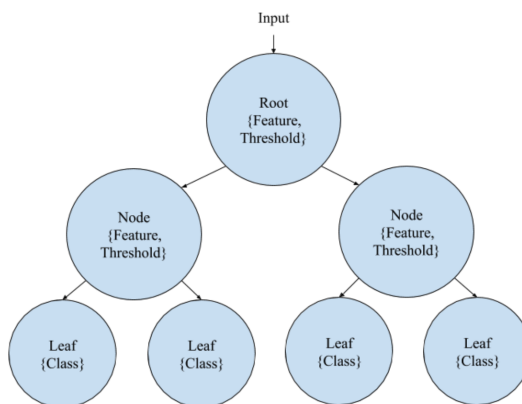


Figure 3.1: An example decision tree.

This decision tree structure needs to be generated before it can be used with test data. To train a decision tree that appropriately classifies our test data according to the features we generated, we employ a recursive function. The function signature is

```
tree = trainDecisionTree(set)
```

Where **set** is the training set, which is a MATLAB structure that contains the class labels and generated features. **tree** is the returned structure that can be used during testing. It is essentially a nested structure that contains two types of elements: nodes and leaves. At each node of the tree, a feature and threshold are specified. If a test sample's value at that particular feature is less than the threshold, the sample is passed down the left branch of the node. Similarly, if the sample's feature is greater than the threshold, it goes through the right branch. This is repeated until we reach a leaf node, which specifies a class membership.

The decision tree training algorithm has a few major steps, and proceeds by evaluating a metric called *information gain* at various configurations. For now, suffice it to say that information gain is a scalar that represents the improvement in prediction as we narrow down the set (by growing the tree) to find appropriate leaves. Our implementation most closely follows that of the ID3 and C4.5 algorithms developed by Ross Quinlan in the late 80s and early 90s [3] [4].

In general, the decision tree training algorithm is described qualitatively as follows:

1. Check stopping conditions, which generate leaves.
 - If there are no more features to split on, return a leaf with the class mode of the set.
 - The set is smaller than **minLeaf**, which is a tuning parameter that is meant to reduce overfitting of the training data. If this condition is met, return a leaf with the class mode of the set.

- If all samples in the set belong to the same class, return a leaf with the class.
 - If no feature yields an improvement to the information gain (discussed below), then return a leaf with the class mode of the set. Note that this condition is only evaluated after step 2.
2. Iterate over each feature. Sort the set along the current feature. We utilize a threshold that splits the set between adjacent feature values such that the two subsets are composed of training samples whose features are less than and greater than the threshold respectively. Because the information gain across thresholds is convex on the whole (see Fig. 3.2), we use a line search that approximates the highest information gain for each threshold. This serves to reduce the computational complexity of our training algorithm.
- Let `attributeBest` and `indBest` be the feature and index that yield the highest information gain. Since the set is sorted, we can simply split the set at the index given by `indBest` for the recursion.
3. Recur over the two subsets given by `indBest` to find the next attribute that yields the highest information gain. Note that we exclude `attributeBest` in the recursion of `trainDecisionTree(.)` so that the same feature isn't chosen in the subset. In this way, the decision tree is grown so that it makes the most improvements to information gain at the nodes which are closest to the root.

The algorithm is also reproduced in pseudocode below.

Algorithm 3.1: `trainDecisionTree`

Data: *set* of training samples with class labels (*set*(1)) and attributes (features) (*set*(2)).

minLeaf, an integer specifying the minimum number of elements in *set* required to make a splitting node.

Result: *tree*, a structure containing nodes and leaves.

```

1 begin
2   check for base cases:
3   if set(2) =  $\emptyset$  then
4     no more attributes (features) to split on.
5     return leaf with mode of set(1) (class labels)
6   if length(set(1)) < minLeaf then
7     return leaf with mode of set(1) (class labels)
8   thisSetEntropy  $\leftarrow$  getEntropy(set)
9   if not thisSetEntropy then
10    all samples are in the same class.
11    return leaf with first element of set(1)
12  instantiate tracking variables:
13  attributeBest, thresholdBest, infoGainBest, indexBest  $\leftarrow$  0
14  for attribute  $\in$  range(# of features left in set) do
15    sort set along attribute.
16    perform line search heuristic to approximate max information gain by splitting set at various indices.
17    thisInfoGain  $\leftarrow$  lineSearch(set)
18    if thisInfoGain > infoGainBest then
19      attributeBest  $\leftarrow$  attribute
20      infoGainBest  $\leftarrow$  thisInfoGain
21      indexBest  $\leftarrow$  index given by line search
22      thresholdBest  $\leftarrow$  attribute value midway between indexBest and indexBest + 1
23  if not infoGainBest or not attributeBest then
24    no attribute provides an information gain.
25    return leaf with mode of set(1) (class labels)
26  re-sort set along attributeBest.
27  subsets  $\leftarrow$  getSubsets(set, attributeBest, indexBest)
28  subtree1  $\leftarrow$  trainDecisionTree(subsets(1), minLeaf)
29  subtree2  $\leftarrow$  trainDecisionTree(subsets(2), minLeaf)
30  return node with attributeBest, thresholdBest, subtree1, and subtree2

```

Once we have trained the decision tree, we can begin to test it by passing the features generated from the test set through the tree. As the features are utilized in propagating through the tree, we eventually reach a leaf node and assign its associated class to the test vector. Testing results can be found in Section (??). The testing algorithm is reproduced in psuedocode below.

Algorithm 3.2: testDecisionTree

Data: *set* of test samples with class labels (*set*(1)) and attributes (features) (*set*(2)).

tree, the struct of structs that represents the trained decision tree.

Result: *set*, the input structure modified to include predicted class membership.

```

1 begin
2   for each sample  $\in$  set do
3     treeWalked  $\leftarrow$  tree
4     classified  $\leftarrow$  false
5     attributes  $\leftarrow$  features corresponding to this sample
6     while not classified do
7       if treeWalked is a node then
8         thisAttribute  $\leftarrow$  attribute given by treeWalked
9         thisAttributeValue  $\leftarrow$  value of the sample's feature at thisAttribute
10        thisThreshold  $\leftarrow$  threshold given by treeWalked
11        remove thisAttribute from the sample's feature vector.
12        compare thisAttributeValue to thisThreshold:
13        if thisAttributeValue < thisThreshold then
14          choose left branch.
15          treeWalked  $\leftarrow$  left branch given by current treeWalked.
16        else
17          choose right branch.
18          treeWalked  $\leftarrow$  right branch given by current treeWalked.
19      else
20        treeWalked is a leaf.
21        append predicted label to set(2) at this sample.
22      classified  $\leftarrow$  true

```

With these two algorithms in place, we can train and test decision trees with varying depth and accuracy. Next, we discuss an adjacent concept that is integral to the decision tree algorithm.

3.1.1 Entropy and Information Gain

When growing the decision tree, the training algorithm utilizes a metric called *information gain* to optimize the predictive power of the tree. Before we can define information gain, we must first understand *entropy*. Entropy is an information theoretic concept that represents the amount of uncertainty in a given set of data. It is defined as

$$H(X) = - \sum_{i=1}^n P(x_i) \log_2 P(x_i) \quad (3.1)$$

Where $P(x_i)$ is the proportion of the number of elements with class x_i to the number of elements in the set X , and there are n classes in the set. Note that if all samples belong to class i , $P(x_i) = 1$ and $\log_2 P(x_i) = 0$, so the entropy is zero.

In calculating the information gain of splitting the set into two subsets, we utilize the entropy of the parent set and

subtract from it a weighted entropy of each subset [5]. This is better expressed by the following equation:

$$\text{IG}(X) = H(X) - \sum_{i=1}^2 \frac{|S_i|}{|X|} H(S_i) \quad (3.2)$$

Where $|\cdot|$ is the cardinality, or number of elements in the set, and S_i are the two subsets composed of the elements of X partitioned across a given threshold for a given feature. The figure below is a plot of the information gain across splitting indices. So, after partitioning the set across a single attribute, (all elements of S_1 are less than the threshold, and all elements in S_2 are greater), we have a scalar value of information gain to decide which splitting threshold and feature would most improve the predictive ability of the decision tree. The plot below represents just one feature, but this calculation is required across all currently available features in the set. This quickly becomes computationally taxing for a large dataset, so we utilize a line search heuristic to improve the computational time without sacrificing a significant amount of accuracy.

The line search starts in the middle of the set and splits it, computing an information gain (IG). Then, we compare this middle IG to the IGs obtained by splitting the set at the 25% and 75% indices. If either of these reveals a larger IG, we set it to the new 'middle' index and consider the IGs which emerge from splitting the set half-way between the old 'middle' and half-way between the old 'left' or 'right' (depending on which one yielded a larger IG.) This allows us to approximate the maximum value of the information gain for a given feature without a brute-force technique of considering each threshold. We ensured that the line search incorporates a variety of locations to coax it to approaching the global maximum, rather than getting caught in the minor variations between adjacent indices. Please consult `code/decisionTree/trainDecisionTree.m` for further details.

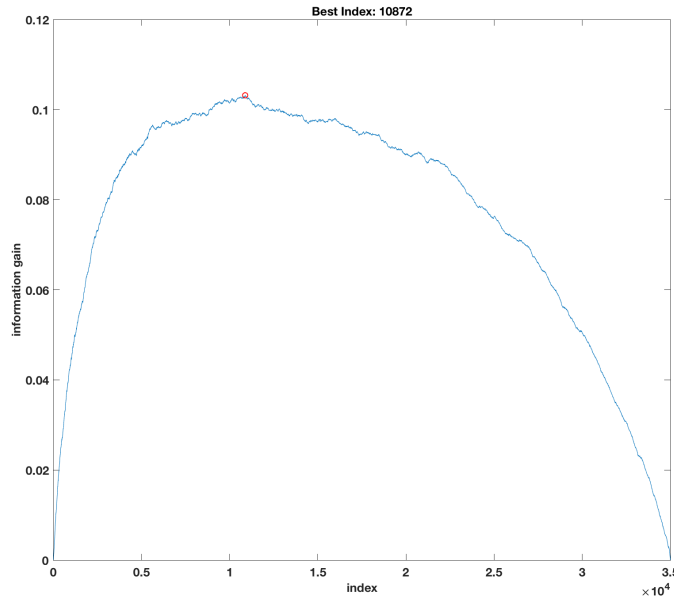


Figure 3.2: Information Gain across all possible thresholds.

3.2 Extra-Trees

The relative inadequacy of decision trees in classifying the samples of MNIST and Yale B led us to explore other tree-like classifiers. The first alternative that we considered is called *extra-trees*, which are ensembles of binary decision trees that are grown in a stochastic manner. In testing, these ensembles *vote* on the class by propagating the test sample down each tree until a leaf is found. The mode of all the votes is assigned to the global class prediction. See Section (??) for the result of this experiment.

Training extra-trees requires less computational and theoretical effort than classical decision trees. The reduced steps are certainly tangential to a basic decision tree, but incorporate less rigour. By utilizing the ensemble of trees, we can overcome the inaccuracies of a single random decision tree and make a strong prediction.

The steps to train an extra-tree are as follows:

1. Check stopping conditions, which generate leaves.
 - If there are no more features to split on, return a leaf with the class mode of the set.
 - The set is smaller than `minLeaf`, which is a tuning parameter that is meant to reduce overfitting of the training data. If this condition is met, return a leaf with the class mode of the set.
 - If all samples in the set belong to the same class, return a leaf with the class.
2. Choose a random feature. In extra-trees, we don't need to generate features, we can simply use the raw pixels as features.
3. Find the mean and variance of this feature across all samples in the set. Generate a random value from a normal distribution with this mean and variance.
4. Recur these steps on the subsets obtained by splitting the parent set on the randomly chosen feature and threshold, where the first subset contains samples whose feature is less than the threshold, and the second subset contains those which are greater than the threshold.

For MNIST, we generated 100 extra-trees, trained on half of the dataset. In testing, the majority-vote of the trees was used to assign a predicted class to the test vectors. Likewise, for Yale B, we utilized half the dataset and 100 extra-trees.

3.3 Linear Discriminant Analysis (LDA)

Our classification success criterion is the extent to which the error rate is minimized. For LDA, we employ the Bayes Rule to classify our test points after training the classifier. We assign the test point to the class with the highest conditional probability (*i.e.* $P(w_i|x)$, where w_i is the class and x the test vector). In practice, it is not feasible to get conditional probability for a given point unless we have a large amount of data. So we estimate the distribution and calculate the probabilities from there.

LDA relies on the assumption of a normal distribution for each class. Linear discriminant analysis generally achieves good performance in the tasks of face and object recognition, even though the assumptions of common covariance matrix among groups and normality are often violated [6].

Since the MNIST and Yale datasets are high-dimensional, we cannot check the normality of individual pixels. Instead, we reduce the dimensions via orthogonal projection. several classes (10 for MNIST and 38 for Yale B.) The class separation in a direction \mathbf{w} in this case is given by the ratio [7].

$$S = \frac{\mathbf{w}^T \Sigma_b \mathbf{w}}{\mathbf{w} \Sigma \mathbf{w}} \quad (3.3)$$

In the case of two classes, this reduces to the ratio of between-class variance and within-class variance.

3.3.1 Algorithm steps

Below, we describe the steps necessary to construct the LDA classifier. We begin by instantiating some variables with notation:

n is the number of classes.

N is the total number of training data samples.

N_i is the number of points in each class i .

μ_i and μ are mean vectors for each class and global mean for the data.

The steps that we follow in our LDA algorithm are as follows.

1. We calculate within-class scatter matrix for each class

$$\Sigma_i = \frac{1}{N_i - 1} \sum_{\mathbf{x} \in D_i} (\mathbf{x} - \mu_i) (\mathbf{x} - \mu_i)^T \quad (3.4)$$

then sum them to obtain

$$\Sigma_W = \sum_{i=1}^n (N_i - 1) \Sigma_i \quad (3.5)$$

2. We then find the average within-class scatter matrix by calculating

$$\Sigma = \frac{\Sigma_W}{N} \quad (3.6)$$

3. We also calculate the between-class scatter matrix by

$$\Sigma_B = \sum_{i=1}^n \frac{N_i}{N} (\mu_i - \mu) (\mu_i - \mu)^T \quad (3.7)$$

4. We need to find eigenvectors and eigenvalues of $\Sigma^{-1} \Sigma_B$.
5. We then sort the eigenvectors depending on the magnitude of eigenvalues.
6. The number of highest eigenvalues will be $c - 1$ which will be 9 and 37 respectively for MNIST and Extended Yale datasets.
7. Project our data onto the subspace (constructed by the eigenvectors of the highest eigenvalues).
8. Since we have reduced dimensions of our data, we can easily apply a discriminant function for a test vector to see which class it belongs
9. The discriminant function for each class is

$$f_i(x_k) = \mu_i^T w_a^{-1} x_k - \frac{1}{2} \mu_i^T w_a^{-1} \mu_i + \ln(P_i) \quad (3.8)$$

where P_i is the probability of each class.

10. Then we select the class with highest discriminant function evaluation.

3.3.2 Helpful Functions

In the above pseudo code, we used our own supplementary functions. After steps 1-7, there is a function `classify_comparison`, which applies the discriminant function in order to classify the test data. We also tried to calculate if the algorithm is effective in just using the distances from the centers of each class using `classify_from_centroid`. It did not work very well on MNIST dataset, but it does perform well with the Extended Yale B dataset.

3.3.3 Derivation of discriminant function

We classify the data point as being in class i if

$$P(\mathbf{x}|i)P(i) > P(\mathbf{x}|j)P(j), \forall j \neq i.$$

Assuming all covariances are equal, *i.e.* $\Sigma = \Sigma_i = \Sigma_j$, and that they follow a Multivariate Normal distribution,

$$P(\mathbf{x}|i) = \frac{1}{(2\pi)^{n/2}|\Sigma_i|^{1/2}} \exp(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_i)^T \Sigma_i^{-1} (\mathbf{x} - \boldsymbol{\mu}_i)),$$

the above condition becomes

$$\frac{P(i)}{(2\pi)^{n/2}|\Sigma_i|^{1/2}} \exp(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_i)^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}_i)) > \frac{P(j)}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_j)^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}_j))$$

Taking the logarithm of both sides,

$$\ln(|\Sigma|) - 2\ln(P(i)) + (\mathbf{x} - \boldsymbol{\mu}_i)^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}_i) < \ln(|\Sigma|) - 2\ln(P(j)) + (\mathbf{x} - \boldsymbol{\mu}_j)^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}_j)$$

After some algebraic manipulation, we have

$$\ln(P(i)) + \boldsymbol{\mu}_i \Sigma^{-1} \mathbf{x}^T - \frac{1}{2} \boldsymbol{\mu}_i \Sigma^{-1} \boldsymbol{\mu}_i^T > \ln(P(j)) + \boldsymbol{\mu}_j \Sigma^{-1} \mathbf{x}^T - \frac{1}{2} \boldsymbol{\mu}_j \Sigma^{-1} \boldsymbol{\mu}_j^T$$

Let $f_i = \boldsymbol{\mu}_i \Sigma^{-1} \mathbf{x}^T - \frac{1}{2} \boldsymbol{\mu}_i \Sigma^{-1} \boldsymbol{\mu}_i^T + \ln(P(i))$. Then we can write

$$f_i > f_j, \forall j \neq i$$

as a measure of the class membership given the discriminant function.

4 Test Results

The table below illustrates the lowest classification errors we were able to achieve with our decision tree, extra-tree, and LDA classifiers on the MNIST and Yale Extended B datasets. In LDA, we split the datasets in half so we can use the other half for test. Using decision trees on MNIST, we utilized half of the dataset for training and half for testing. However, since the Yale B dataset is so small, we were only able to get lower error rates by using ~95% of the set for training and ~5% for testing (on decision trees.) For our best extra-tree result on MNIST, we split the dataset in half and trained 100 trees. The Yale B result comes from one of our 5-fold cross validations with 50 trees.

Algorithm	MNIST	Yale B
LDA	13.5%	6.8%
Decision Tree	16.6%	57.89%
Extra-Trees	4.89%	34.02%

Table 4.1: Lowest classification errors achieved with LDA, Decision Trees, and Extra-Trees.

4.1 Cross Validation

In an effort to justify our choice of hyperparameters and examine the stability of our classifiers irrespective of variations in the training and test sets, we performed a 5-fold cross validation across the tuning parameters in each algorithm. For decision and extra trees, this consists of `minLeaf`. Future implementations could include more complex parameters. In k -fold cross validation, k classifiers are trained on different subsets of the entire set,

ensuring that each sample of the dataset is used only once for validation. This helps to mitigate the effects of variation in the dataset and illustrate the general performance of the classifier at hand. The k classifiers' error rates are averaged to give a strong estimate of the algorithm's true performance. The figure below illustrates this process for $k = 4$.

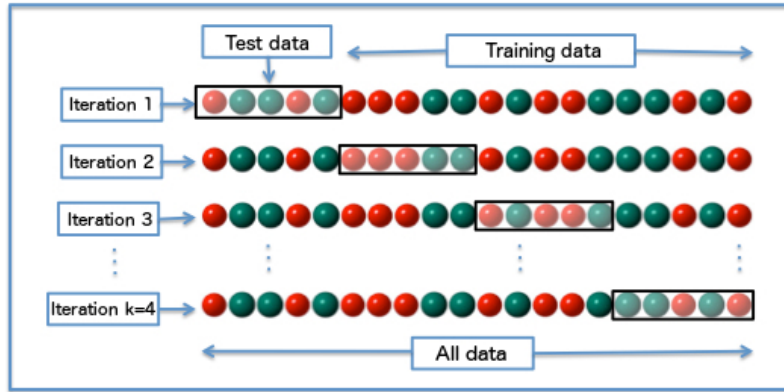


Figure 4.1: 4-fold cross validation on a dataset.

The best results of our cross validation are represented in the table below. The error rates are averaged over the 5 folds. Note that because the 5-fold cross validation yields a larger test set for Yale B than in previous results (§4), the error rate for decision trees on the Yale B dataset was higher. If we used a higher-fold cross validation, the results would have been better.

Algorithm	MNIST	Yale B
LDA	%	%
Decision Tree	17.9% (<code>minLeaf = 1</code>)	74.9% (<code>minLeaf = 1</code>)
Extra-Trees	5.4% (100 extra-trees)	36.3% (100 extra-trees)

Table 4.2: Average classification errors achieved with LDA, Decision Trees, and Extra-Trees after Cross Validation.

4.2 Hyperparameter Optimization

4.2.1 Decision Trees

The figure below represents the accuracy and time performance of our decision tree algorithm on MNIST and Yale B after averaging with cross validation. Using cross validation allows us to generalize the effects of our changing hyperparameter, `minLeaf`, which is a value that defines the minimum number of samples required to make a leaf node in the tree. As soon as the set is split to a level below `minLeaf`, a class label is applied as the *mode* of the samples in the set.

Note that the timing results for MNIST have high variability. This is because we ran our tests on a shared server that was at high occupancy. In this way, training our decision trees on MNIST was impacted by the other users. On our local computers, the decision tree was training in about 4 minutes, with a clear correlation between `minLeaf` and the time it took to complete. This process is evinced in the figure for the Yale B dataset, which was interrupted less on the server.

Our cross-validated results indicate that the highest accuracy is achieved by setting `minLeaf = 1`.

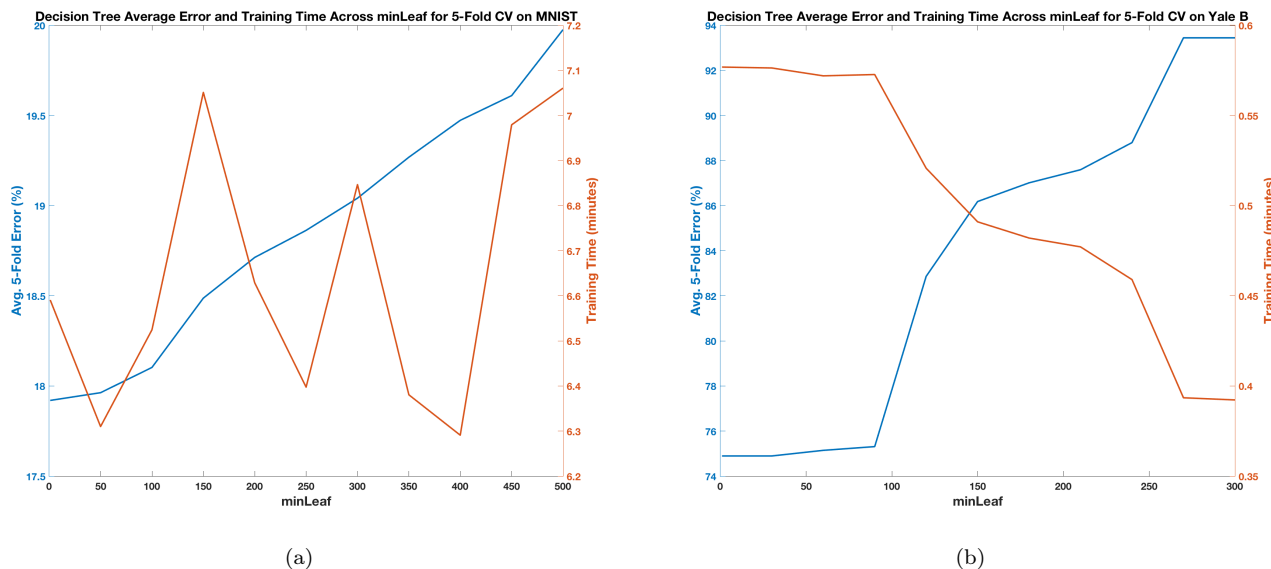


Figure 4.2: Cross Validated Decision Tree Hyperparameter Results on MNIST (a) and Yale B (b).

4.2.2 Extra Trees

The figure below represents the accuracy and time performance of our extra-tree classifier on MNIST and Yale B after averaging with cross validation. Using cross validation allows us to generalize the effects of our changing hyperparameter, `numTrees`, which is a value that defines the number of trees used for the ensemble. In general, more trees to vote on the classification label improves the results. Though a clear case of diminishing returns is observed after `numTrees = 50`.

Note that the time to train the extra-trees increases linearly with the number of trees, as the training of a decision tree, especially in the random case, is of constant time complexity. This does not include the testing of trees, which increases as $O(numTrees * \log n)$. This is derived by considering that `numTrees` trees must be passed through to decide a class label, which takes $\log n$ time.

Our cross-validated results indicate that the highest accuracy is achieved by setting `numTrees > 50`, with `numTrees = 100` being a good compromise between time to execute and accuracy.

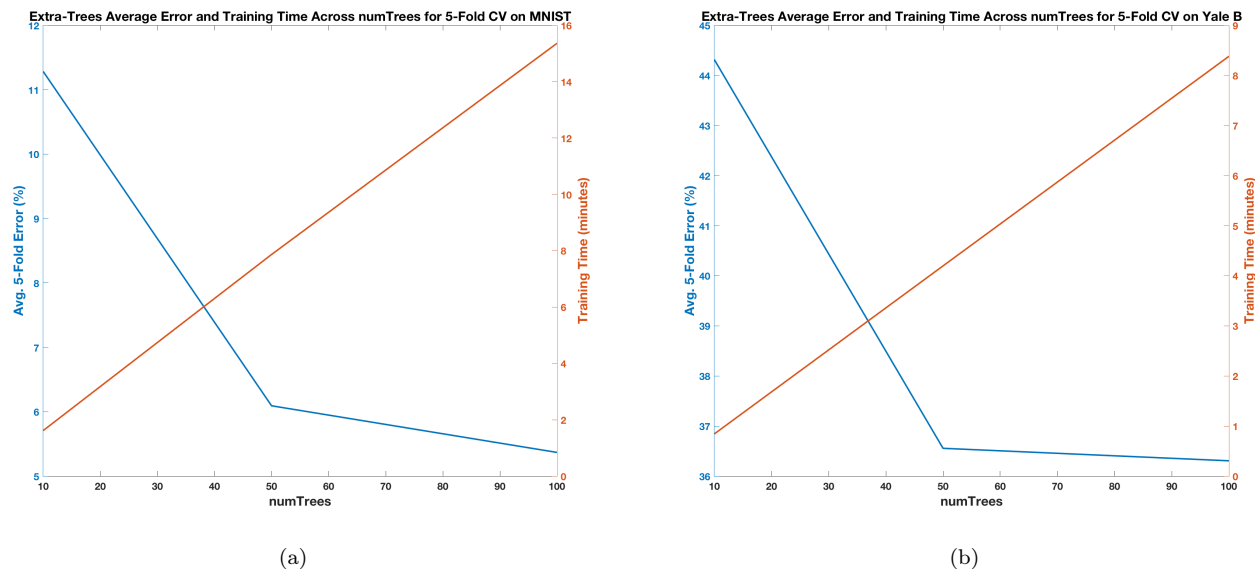


Figure 4.3: Cross Validated Extra Tree Hyperparameter Results on MNIST (a) and Yale B (b).

5 Analysis

5.1 Decision Trees

5.1.1 Advantages of Decision Trees

Some advantages of decision trees include [8].

1. Ability to handle both quantitative and qualitative data.
2. Easily represented in a graph form. The classifier is not a black box and can be fully understood.
3. Well suited to large data sets.

5.1.2 Disadvantages of Decision Trees

Some disadvantages of decision trees include [8].

1. High computational complexity in our implementation.
2. Decision trees can become too complex if limits aren't placed on their growth. This can lead to overfitting.
3. Single decision trees are sensitive to data order and variations in data.
4. Decision trees are susceptible to favoring the class mode of the dataset.

5.1.3 Improving Decision Trees

Much of the drawbacks of using decision trees can be mitigated by ensemble methods such as boosting, bagging, and random forests. These utilize stochastic factors that improve the classification power of a single, thorough decision tree by training many poorer decision trees that then vote on a classification. The majority vote is then chosen for the class label in testing. Randomized decision trees train much faster than a classical decision tree, and our experiments show that they yield an improvement in accuracy, as well.

5.2 Extra Trees

5.2.1 Advantages of Extra Trees

Extra trees have nearly the same advantages as decision trees. Though, due to the ensemble of randomly assigned splitting criteria, extra trees have less “reasonability” with respect to their decision making. The entirety of each tree is available, so they still are not black boxes, but their decision making is without any particular method. Additionally, the computational complexity of extra trees is less than that of classical decision trees. Though, taking the majority vote of all the decision trees introduces a logarithmic complexity on the order of the number of extra-trees employed.

Additionally, extra trees need no feature generation to be trained, they can run on the raw pixel data. Additionally, their training is less dependent on the size of the dataset and number of features it carries; in this way, extra trees scale well with the number of data samples.

5.2.2 Disadvantages of Extra Trees

Like classical decision trees, extra trees can also become too complex if limits aren’t placed on their growth. Likewise, they may favor the class mode of the dataset.

Additionally, compared to classical decision trees, extra-trees require more work to label data, as the test sample must pass through every tree to obtain a vote. Then all of the votes are aggregated into an estimate. This means utilizing the classifier can be costly as the number of trees increases.

5.2.3 Improving Extra Trees

A possible improvement on extra-trees is to utilize a sub-window technique to split each sample into subframes that are associated with the original class. This serves to augment the dataset and may be particularly useful in cases where there are a lack of samples available.

In testing the data, a parallel computation can be utilized as each classifier votes independently of the others. In this way, the computational burden of testing the classifier can be reduced.

5.3 LDA

5.3.1 Advantages of LDA

Some advantages of LDA include [9].

1. The analytical solutions makes the problem simpler and gives more insight.
2. The terms are all linear which eases the computational costs
3. Well suited to large data sets.

5.3.2 Disadvantages of Decision Trees

Some disadvantages of LDA include [10].

1. LDA produces only $(C - 1)$ feature projections.
2. We are assuming Gaussian likelihoods, but if distributions are non-Gaussian then LDA projections might lose some significant information.
3. LDA focuses more on the mean rather in the variance, thus if information more contained in the variance then we lose the distinctive features of classes.

5.3.3 Improving LDA

1. Since the LDA does not capture the rotation and translation well, some deskewing methods might be a good idea to perform.
2. Covariance matrix smoothing

6 Conclusion

In conclusion, we have shown that utilizing Linear Discriminant Analysis, Decision Trees, and Extra-Trees on the Extended Yale Dataset B and MNIST datasets yields a favorable result. We have also shown the results of cross validation performed on our datasets with our algorithms and their hyperparameters. This served to generalize the performance of our classifiers.

For a demonstration of our functions, please navigate to the `code` folder and execute the functions given by `demonstration_dt_MNIST.m` and others, where `dt` stands for decision trees and `et` stands for extra-trees. These demonstrations illustrate the entire process of partitioning our data, training our classifiers, and calculating the results.

7 References

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- [10] Ricardo Gutierrez-Osuna. Linear discriminant analysis, 2018. [Online; accessed 2018-04-20].

8 Code Listings

Below are some of the primary scripts that execute the project. Please see the `code` folder for supporting functions and scripts.

Listing 1: Demonstration Code for Decision Tree on MNIST

```
1  %{
2
3  kudiyar orazymbetov
4  n casale
5
6  ECE 759 Project
7  18/03/16
8
9  this script orchestrates the training and testing of
10 the decision tree classifier
11
12 %}
13
14 function [errorRate] = demonstration_dt_MNIST()
15
16     addpath('utility', 'MNIST', 'MNIST/data', 'MNIST/loadMNIST', 'lda', ...
17           'decisionTree');
18
19     fprintf('begin MNIST decision tree demonstration\n');
20
21     % hyper-parameters
22     N_tr = 35e3; % training samples
23     N_te = 35e3; % test samples
24
25     % for feature selection
26     numFeatures = 10;
27
28     % for decision tree
29     minLeaf = 1; % to prevent overfitting
30
31     % partition data
32     % MNIST contains 70k examples
33     [train, test] = loadMNIST(N_tr);
34
```

```

35     %% dimensionality reduction / feature generation
36     % via linear discriminant analysis (lda)
37     st = cputime;
38
39     [train, test] = lda_features(train, test, 0:numFeatures-1);
40
41     fprintf('Features Generated in %4.2f minutes\n', (cputime - st)/60);
42
43     %% train
44     st = cputime;
45
46     tree = trainDecisionTree({train{2:3}}, minLeaf);
47
48     fprintf('Trained in %4.2f minutes\n', (cputime - st)/60);
49
50     %% test
51     st = cputime;
52
53     test = testDecisionTree(test, tree);
54
55     fprintf('Tested in %4.2f minutes\n', (cputime - st)/60);
56
57     % Classification Error
58     errors = nnz(test{2}(:,1) ~= test{2}(:,2));
59     errorRate = (errors/N_te)*100;
60
61     fprintf('\nnumFeatures: %d, minLeaf: %d, error rate: %2.2f\n', ...
62             numFeatures, minLeaf, errorRate);
63
64 end

```

Listing 2: Demonstration Code for Extra-Trees on Yale B

```

1  %{
2
3  kudiyar orazymbetov
4  n casale
5
6  ECE 759 Project
7  18/03/16
8
9  this script orchestrates the training and testing of
10 the decision tree classifier
11
12 %}
13
14 function [errorRate] = demonstration_et_YaleB()
15
16     addpath('utility', 'YaleB', 'YaleB/data', ...
17             'extraTree');
18
19     fprintf('begin Yale B extra tree demonstration\n');
20
21     % hyper-parameters
22     N = 2414;
23     N_tr = 2000; % training samples
24     N_te = N - N_tr; % test samples

```

```

25
26     % for decision tree
27     minLeaf = 1; % to prevent overfitting
28     numTrees = 50; % ensemble for majority voting
29
30     % partition data
31     [train, test] = loadYaleB(N_tr);
32
33     % features are the raw pixels, so we reorder the cell
34     train = {train{2}, train{1}};
35     test = {test{2}, test{1}};
36
37     %% train
38     st = cputime;
39
40     % create an ensemble of random trees
41     trees = cell(numTrees, 1);
42     for tree = 1:numTrees
43
44         fprintf('tree: %d\n', tree);
45         trees{tree} = trainExtraTree(train, minLeaf);
46
47     end
48
49     fprintf('Trained in %4.2f minutes\n', (cputime - st)/60);
50
51     %% test
52     st = cputime;
53
54     test = testExtraTrees(test, trees);
55
56     fprintf('Tested in %4.2f minutes\n', (cputime - st)/60);
57
58     % Classification Error
59     errors = nnz(test{1}(:,1) ~= test{1}(:,2));
60     errorRate = (errors/N_te)*100;
61
62     fprintf('\nnumTrees: %d, minLeaf: %d, error rate: %2.2f\n', ...
63         numTrees, minLeaf, errorRate);
64
65 end

```

Listing 3: Cross Validation Code for Decision Tree on MNIST

```

1  %{
2
3  kudiyar orazymbetov
4  n casale
5
6  ECE 759 Project
7  18/03/16
8
9  this script orchestrates the cross validation of
10 the decision tree classifier
11 across all hyperparameters,
12 taking performance and time results along the way
13

```

```

14 features are generated using lda, as they yield the best performance
15
16 %}
17
18 clear; close all;
19 addpath('utility', 'MNIST', 'MNIST/data', 'MNIST/loadMNIST', 'lda', ...
20         'decisionTree');
21
22 fprintf('begin Cross Validation on MNIST decision trees\n');
23
24 % hyper-parameters
25 k = 5; % k-fold cross validation
26 % use k to partition data
27 N_te = 70e3/k;
28 N_tr = 70e3 - N_te;
29
30 % for lda
31 numFeatures = 10;
32
33 % for decision tree
34 minLeaves = 0:50:500;
35 minLeaves(1) = 1;
36
37 % partition data
38 % MNIST contains 70k examples
39 [train, test] = loadMNIST(N_tr);
40
41 % merge sets, they're already randomly shuffled
42 all = {[train{1}, test{1}], [train{2}, test{2}]};
43
44 %% k-fold cross validation across minLeaf
45
46 trainTimes = zeros(length(minLeaves), k);
47 errorRates = zeros(length(minLeaves), k);
48
49 for minLeaf = minLeaves
50
51     % random indices
52     %inds = randperm(length(all{2}));
53     %all{1} = all{1}(:, inds);
54     %all{2} = all{2}(inds);
55
56     for fold = 1:k
57
58         % choose fold
59         inds_bool = false(70e3,1);
60         ind_start = (fold-1)*N_te + 1;
61         inds_bool(ind_start:ind_start+N_te-1) = 1;
62
63         test = {all{1}(:, inds_bool), all{2}(inds_bool)};
64         train = {all{1}(:, ~inds_bool), all{2}(~inds_bool)};
65
66         % dimensionality reduction / feature generation
67         [train, test] = lda_features(train, test, 0:numFeatures-1);
68
69         % train

```

```

70         st = cputime;
71         tree = trainDecisionTree({train{2:3}}, minLeaf);
72         thisTrainTime = (cputime - st)/60;
73         fprintf('Trained in %4.2f minutes\n', thisTrainTime);
74         trainTimes(minLeaves == minLeaf, fold) = thisTrainTime;
75
76         % test
77         test = testDecisionTree(test, tree);
78
79         % Classification Error
80         thisError = nnz(test{2}(:,1) ~= test{2}(:,2));
81         thisErrorRate = (thisError/N_te)*100;
82         errorRates(minLeaves == minLeaf, fold) = thisErrorRate;
83
84         fprintf('minLeaf: %d, fold: %d, error rate: %2.2f\n', ...
85                 minLeaf, fold, thisErrorRate);
86
87     end
88
89 end
90
91 %% save, print results
92 filename = 'decisionTree/crossValidation/cv_mnist_dt.mat';
93 save(filename, 'errorRates', 'trainTimes');
94
95 fprintf('end Cross Validation on MNIST decision trees\n');

```

Listing 4: Main Code for LDA on MNIST

```

1  %{
2
3  kudiyar orazymbetov
4  n casale
5
6  ECE 759 Project
7  18/03/16
8
9  this script orchestrates the training and testing of
10 the linear discriminant analysis classifier
11
12 %}
13
14 clear; close all;
15 addpath(' ../utility', ' ../MNIST', ' ../MNIST/data', ' ../MNIST/loadMNIST', ...
16         ' ../lda');
17
18 seed = 152039828;
19 rng(seed); % for reproducibility
20
21 % define parameters
22 N_tr = 35e3; % training samples
23 N_te = 35e3; % test samples
24 k = 10; % number of classes
25 % partition data
26
27 %{
28         1/2 of the dataset should be for training

```

```

29         the other for testing
30
31         MNIST contains 70k examples
32     %}
33
34     [train, test] = loadMNIST(N_tr);
35     %% use PCA to see the results
36     % numFeatures = 20;
37     % [train, U, V] = pca_(train, numFeatures);
38
39     %% Construct scatter matrices and calculate within-class and between class
40     % covariance
41     mu = mean(train{1,1}, 2);
42     num_variables = size(train{1,1},1);
43     % Let's standardize the data;
44     %variance = var(train{1,1}, 0,2);
45     %train{1,1} = (train{1,1}-repmat(mu,1, 60000))./variance;
46
47     %
48     Si = zeros(num_variables); Sb = zeros(num_variables);
49     S_cov = zeros(num_variables);
50     for i = 0:k-1
51         ind = (train{1,2} == i);
52         N_i = sum(ind);
53         x = train{1,1}(:, ind);
54         mu_i = mean(x, 2);
55         S_cov = S_cov + cov(x');
56         Si = Si + (1/N_tr)*(x - (repmat(mu_i,1, N_i)))*(x - (repmat(mu_i,1, N_i)))
57             ' ;
58         Sb = Sb + (N_i/N_tr)*(mu_i - mu)*(mu_i - mu)'; % (1/k)
59     end
60     % We apply singular value decomposition in order to find eigenvalues and
61     % eigenvectors
62     [U D V] = svd(pinv(Si)*Sb); % lets try S_cov/k instead of Si; but it is the
63         same result
64     a = [];
65     for i = 1:(k)
66         a = [a D(i,i)];
67     end
68     % from here we can see that we only have 9 highest values as we expected
69
70     % We transform the training and testing data to a subspace
71     transf_matrix = U(:,1:(k-1));
72     transf_train = train{1,1}'* transf_matrix;
73     transf_test = test{1,1}'*transf_matrix;
74     % We find mean vector and covariance matrix for each class
75     mu_each_class = zeros(k-1, k);
76     cov_each_class = {};
77     sum_cov = zeros(k-1,k-1);
78     for i =0:k-1
79         ind = find(train{1,2} == i);
80         X = transf_train(ind, :);
81         mu_each_class(:, i+1) = mean(X, 1)';
82         cov_each_class{1, i+1} = cov(X);

```

```

83     sum_cov = sum_cov + cov(X);
84 end
85 % since we assume equal covariance in all classes, we take the average of
86 % covariance matrices
87 average_cov = sum_cov/k;
88 cov_equal_each_class = {average_cov average_cov average_cov average_cov
89     average_cov average_cov average_cov average_cov average_cov average_cov};
89 % this part is just a test on how nearest neighbors work
90 % parfor n = 1:13
91 % % we apply Nearest neighbors in order to find which class it belongs
92 %     accuracy(n) = classifyNN(n,transf_test', transf_train', test{1,2}, train
93     {1,2});
93 % end
94 [acc_test acc_train] = classify_comparison_same_cov(k,5,mu_each_class,
95     cov_each_class, average_cov, ...
96     transf_test', test{1,2}, transf_train', train{1,2}); % 0.88 and 0.89 resp using
97     just knn
98 % [acc_test_comp acc_train_comp] = classify_comparison(k,5,mu_each_class,
99     cov_equal_each_class, transf_test', test{1,2}, transf_train', train{1,2});
100 % 0.88 and 0.89 resp using just knn
101 %% This part uses kNN in order to class after transformation
102 [acc_test_5 acc_train_5] = classifyNN(k,5,mu_each_class, cov_each_class,
103     transf_test', test{1,2}, transf_train', train{1,2}); % 0.88 and 0.89 resp
104     using just knn
105 [acc_test_5 acc_train_5] = classifyNN(k,5,mu_each_class, cov_equal_each_class,
106     transf_test', test{1,2}, transf_train', train{1,2}); % 0.87 and 0.88 resp
107     using just knn
108 [acc_test_5_p acc_train_5_p] = classifyNN_pure(5,transf_test', transf_train',
109     test{1,2}, train{1,2});
110
111 % cov each class separately works better in each case
112 % we plot the results to see the best number of nearest neighbors
113 % [acc_test_5 acc_train_5] = classifyNN(5,transf_test', transf_train', test
114     {1,2}, train{1,2}); % 0.86 and 0.85 resp using just knn
115 % this is my first attempt
116
117 f = instantiateFig(1);
118 plot([1:13],accuracy*100, 'r.')
119 prettyPictureFig(f);
120 xlabel('Nearest neighbor number');
121 ylabel('Accuracy of test model');
122
123 print('.../images/NN after LDA', '-dpng');
124 % instead we can use Euclidean distance metric to evaluate the classes by
125 % calculating the distances from each class centroid
126 centroid = zeros(k, k-1);
127 for i = 0:k-1
128     ind = (train{1,2} == i);
129     N_i = sum(ind);
130     centroid(i+1, :) = mean(transf_train(ind,:), 1);
131 end
132
133 accuracy1_train = classify_from_centroid(transf_train', train{1,2}, centroid);
134 accuracy1 = classify_from_centroid(transf_test', test{1,2},centroid);
135 %%

```

```

127 t0 = cputime;
128 N_cross_val = 5;
129 CVO = cvpartition(train{1,2},'k',N_cross_val);
130 err = zeros(CVO.NumTestSets,1);
131 for j = 1:N_cross_val
132     ind = CVO.training(j);
133     train_cv = transf_train(ind,:); train_cv_label = train{1,2}(ind,:);
134     test_cv = transf_train(~ind,:); test_cv_label = train{1,2}(~ind);
135     mu_each_class = zeros(k-1, k);
136     sum_cov = zeros(k-1,k-1);
137     for i = 0:(k-1)
138         ind1 = find(train_cv_label == i);
139         X = train_cv(ind1, :);
140         mu_each_class(:, i+1) = mean(X, 1)';
141         sum_cov = sum_cov + cov(X);
142     end
143     E_cov = sum_cov/k;
144     [acc_cross_valid_test(j) acc_cross_valid_train(j)] =
        classify_comparison_same_cov(k,5,mu_each_class, E_cov, test_cv',
        test_cv_label, train_cv', train_cv_label);
145 end
146 mean_acc_test = mean(acc_cross_valid_test);
147 sd_test = sqrt(var(acc_cross_valid_test));
148 mean_acc_train = mean(acc_cross_valid_train);
149 sd_train = sqrt(var(acc_cross_valid_train));
150 fprintf('Tested in %4.2f minutes\n', (cputime - t0)/60);
151
152
153 for i = 0:k-1
154     ind = find(train{1,2} == i);
155     X = transf_train(ind, :);
156     mu_each_class(:, i+1) = mean(X, 1)';
157     cov_each_class{1, i+1} = cov(X);
158     sum_cov = sum_cov + cov(X);
159 end
160 % since we assume equal covariance in all classes, we take the average of
161 % covariance matrices
162 average_cov = sum_cov/k;
163 %%
164 load('fisheriris');
165 CVO = cvpartition(species,'k',10);
166 err = zeros(CVO.NumTestSets,1);
167 for i = 1:CVO.NumTestSets
168     trIdx = CVO.training(i);
169     teIdx = CVO.test(i);
170     ytest = classify(meas(teIdx,:),meas(trIdx,:),...
171                     species(trIdx,:));
172     err(i) = sum(~strcmp(ytest,species(teIdx)));
173 end
174 cvErr = sum(err)/sum(CVO.TestSize);
175
176
177 %% this is classification through matlab discriminant classification
178 mdl = fitcdiscr(transf_train, train{1,2},'DiscrimType','linear'); % this one
    works since n>m in tansf_train
179 %mdl = fitcdiscr(train{1,1}', train{1,2},'DiscrimType','linear'); % error

```



```
    saying Predictor x1 has zero within-class variance.
180 pred = predict mdl, transf_test);
181 count = 0;
182 for i = 1:size(transf_test,1)
183     if pred(i) == test{1,2}(i);
184         count = count + 1;
185     end
186 end
187 acc = count/size(transf_test,1) % 0.8639
188 % we get a similar result as our methodp
189 % Lets try the same thing in decision tree
190 mdl_tree = fitctree(transf_train, train{1,2});
191 pred = predict(mdl_tree, transf_test);
192 count = 0;
193 for i = 1:size(transf_test,1)
194     if pred(i) == test{1,2}(i);
195         count = count + 1;
196     end
197 end
198 acc_tree = count/size(transf_test,1)
199 % this result is from pca
200 mdl_tree_pca = fitctree(train{3}, train{2});
201 pred = predict(mdl_tree_pca, test{3});
202 count = 0;
203 for i = 1:size(test{2},1)
204     if pred(i) == test{2}(i);
205         count = count + 1;
206     end
207 end
208 acc_tree_pca = count/size(test{2},1)
209 % with pure data
210 mdl_tree_pca = fitctree(train{1}', train{2});
211 pred = predict(mdl_tree_pca, test{1}');
212 count = 0;
213 for i = 1:size(test{2},1)
214     if pred(i) == test{2}(i);
215         count = count + 1;
216     end
217 end
218 acc_tree_pca = count/size(test{2},1)
```