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Chapter 15

Variational problems with a lack of coercivity

As we have seen in Section 3.2, every minimization problem of a coercive lower semicontinuous function admits a solution. On the other hand, without the coercivity assumption, in general we cannot apply the direct method and the existence of a minimizer may fail. This may occur even if the cost function is convex, as it happens, for instance, in the case

$$\min\{e^x \; : \; x \in \mathbf{R}\}.$$

However, some minimum problems, even if not coercive, still admit a solution, as, for instance, the case

$$\min\{x^2 \; : \; (x,y) \in \mathbf{R}^2\}$$

trivially shows.

In this chapter we present some methods which allow us to identify the noncoercive minimum problems which admit a solution. The history of these tools goes back to Stampacchia [338] and Fichera [214], who developed them to treat noncoercive cases in the framework of variational inequalities and of unilateral contact problems in elasticity, respectively. On the other hand, at least in the finite dimensional convex situations, the geometrical tool of *recession function* was introduced by Rockafellar in 1964, and this has been shown to be very useful in a large number of cases. The theory we present in Section 15.1 for the convex cases and in Section 15.2 for the general ones appeared first in the paper by Baiocchi et al. in 1988 [75] and makes it possible to treat in a unified way problems of geometrical type as well as problems coming from continuum mechanics.

15.1 ▪ Convex minimization problems and recession functions

In this section we will treat convex minimum problems, not necessarily coercive, and we will prove the existence of minimizers provided some compatibility conditions are satisfied.

The simplest example of a variational noncoercive minimization problem is the classical Neumann problem presented in Section 6.2,

$$\min\left\{\frac{1}{2}\int_\Omega |Du|^2\,dx - \langle L, u\rangle \; : \; u \in H^1(\Omega)\right\},$$

where Ω is a connected bounded Lipschitz domain of $\mathbf{R}^n$ and L belongs to the dual space $\left(H^1(\Omega)\right)'$. The Euler–Lagrange equation of the minimization problem above can be

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written in the weak form

$$\int_\Omega DuD\phi\,dx = \langle L,\phi\rangle \qquad \forall \phi\in H^1(\Omega). \tag{15.1}$$

Now, if the term L on the right-hand side of (15.1) is of the form

$$\langle L,\phi\rangle = \int_\Omega f\phi\,dx + \int_{\partial\Omega} g\phi\,d\mathcal{H}^{n-1}$$

with $f\in L^2(\Omega)$ and $g\in L^2(\partial\Omega)$, which we write as $L=f+g$, integrating by parts the left-hand side of (15.1) we obtain the PDE problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega,\\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases}$$

It is well known that a solution of the problem above exists iff the compatibility condition

$$\langle L,1\rangle = 0 \tag{15.2}$$

is fulfilled. This can be seen in a simple way by remarking that if a solution of (15.1) exists, condition (15.2) follows straightforwardly by taking as a test function $\phi=1$; on the other hand, if the compatibility condition (15.2) is fulfilled, then by the Poincaré inequality (Theorem 5.3.1), the minimization problem becomes coercive as soon as it is restricted to the class of functions in $H^1(\Omega)$ with zero average. Therefore we obtain the existence of a solution u for the problem written in weak form as

$$\int_\Omega DuD\psi\,dx = \langle L,\psi\rangle \qquad \forall \psi\in H^1(\Omega),\ \int_\Omega \psi\,dx = 0.$$

Now, this implies the existence of a solution for (15.1) by noticing that every function ϕ in $H^1(\Omega)$ can be written as $\phi=\psi+c$, where ψ is a function with zero average and $c\in\mathbf{R}$, so that condition (15.2) yields

$$\int_\Omega DuD\phi\,dx = \int_\Omega DuD\psi\,dx = \langle L,\psi\rangle = \langle L,\phi\rangle.$$

In this section, (V,σ) will denote a real locally convex Hausdorff topological vector space, and $F: V\to\,]-\infty,+\infty]$ will be a proper convex and sequentially σ-lsc mapping. The minimization problem we are interested in is

$$\min\bigl\{F(v)\ :\ v\in V\bigr\}.$$

From the discussions above we know that the existence or nonexistence of a solution depends on some compatibility conditions that we want to identify. To do this we recall the classical definition of recession function introduced by Rockafellar [325] in the finite dimensional case (see also Sections 11.3 and 13.3).

Definition 15.1.1. *Given a proper convex and sequentially σ-lower semicontinuous functional $F: V\to\,]-\infty,+\infty]$, the recession functional F^∞ of F is defined, for every $v\in V$, by*

$$F^\infty(v) = \lim_{t\to+\infty} \frac{F(v_0+tv)}{t}, \tag{15.3}$$

where v_0 is any element of $\operatorname{dom} F = \{v\in V\ :\ F(v)<+\infty\}$.

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The main properties of the recession functional F^∞ are listed in the following proposition.

Proposition 15.1.1. *We have the following:*

(i) *The limit in* (15.3) *exists and is independent of* v_0.

(ii) *The functional* F^∞ *can equivalently be expressed by*

$$F^\infty(v) = \sup\{F(u+v) - F(u) \,:\, u \in \operatorname{dom} F\}$$
$$= \sup\left\{\frac{F(v_0 + tv) - F(v_0)}{t} \,:\, t > 0\right\}$$

for any $v_0 \in \operatorname{dom} F$.

(iii) F^∞ *is proper, convex, sequentially* σ*-lsc, and positively* 1*-homogeneous, that is,*

$$F^\infty(tv) = tF^\infty(v) \qquad \forall t \geq 0, \quad \forall v \in V.$$

(iv) *For every* $F_1, \dots, F_n$ *proper, convex, sequentially* σ*-lsc mappings, with* $(\operatorname{dom} F_1) \cap \cdots \cap (\operatorname{dom} F_n) \neq \emptyset$, *it is*

$$\left(\sum_{i=1}^n F_i\right)^\infty = \sum_{i=1}^n F_i^\infty.$$

(v) $F^\infty(v) + F^\infty(-v) \geq 0$ *for every* $v \in V$.

PROOF. Let us prove that the limit in (15.3) exists. This follows from the fact that for every $v_0 \in \operatorname{dom} F$ and $v \in V$ the function $\phi(t) = F(v_0 + tv)$ is convex on $\mathbf{R}$; hence the mapping $t \mapsto (\phi(t) - \phi(0))/t$ is nondecreasing and so it admits a limit as $t \to +\infty$. Moreover, the fact that the definition of F^∞ does not depend on $v_0 \in \operatorname{dom} F$ follows from property (ii).

Let us prove now that for every $v_0 \in \operatorname{dom} F$ and $v \in V$ it is

$$\lim_{t \to +\infty} \frac{F(v_0 + tv)}{t} = \sup\left\{\frac{F(v_0 + tv) - F(v_0)}{t} \,:\, t > 0\right\}.$$

The inequality $\leq$ is trivial because

$$\lim_{t \to +\infty} \frac{F(v_0 + tv)}{t} = \lim_{t \to +\infty} \frac{F(v_0 + tv) - F(v_0)}{t}$$
$$\leq \sup\left\{\frac{F(v_0 + tv) - F(v_0)}{t} \,:\, t > 0\right\}.$$

To prove the opposite inequality, fix $s > 0$ and $t > s$; by the convexity of F we have

$$F(v_0 + sv) = F\left(\left(1 - \frac{s}{t}\right)v_0 + \frac{s}{t}(v_0 + tv)\right)$$
$$\leq \left(1 - \frac{s}{t}\right)F(v_0) + \frac{s}{t}F(v_0 + tv),$$

so that

$$\frac{F(v_0 + sv) - F(v_0)}{s} \leq \frac{F(v_0 + tv) - F(v_0)}{t}.$$

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By letting first $t \to +\infty$, and then taking the supremum for $s > 0$, we get

$$\sup \left\{ \frac{F(v_0 + sv) - F(v_0)}{s} \ : \ s > 0 \right\} \leq \lim_{t \to +\infty} \frac{F(v_0 + tv)}{t}.$$

To conclude the proof of (ii) it remains to prove that for every $v_0 \in \operatorname{dom} F$ and $v \in V$ the equality

$$\sup_{u \in \operatorname{dom} F} \big(F(u+v) - F(u) \big) = \sup_{t > 0} \frac{F(v_0 + tv) - F(v_0)}{t} \tag{15.4}$$

holds. Let $u, v_0 \in \operatorname{dom} F$ and $v \in V$; by using the convexity and lower semicontinuity of F we obtain

$$\begin{aligned} F(u+v) &\leq \liminf_{t \to +\infty} F\left(\left(1 - \frac{1}{t}\right) u + \frac{1}{t}(v_0 + tv) \right) \\ &\leq \liminf_{t \to +\infty} \left[\left(1 - \frac{1}{t}\right) F(u) + \frac{1}{t} F(v_0 + tv) \right] \\ &= F(u) + \lim_{t \to +\infty} \frac{F(v_0 + tv) - F(v_0)}{t}; \end{aligned}$$

hence, inequality $\leq$ in (15.4) is proved. To prove the opposite inequality, we denote by S the left-hand side of (15.4); it is clear that without loss of generality we may assume $S < +\infty$. Then $u + v \in \operatorname{dom} F$ for every $u \in \operatorname{dom} F$ and so, from $F(u+v) \leq S + F(u)$, we deduce for every integer $k \geq 0$

$$F(u + kv) = F(u) + \sum_{i=1}^{k} \big[F(u + iv) - F(u + (i-1)v) \big] \leq F(u) + kS.$$

Take now two nonnegative integers h, k; by using the convexity of F and the inequality above we obtain

$$\begin{aligned} F\left(u + \frac{h}{k} v \right) &= F\left(\left(1 - \frac{1}{k}\right) u + \frac{u + hv}{k} \right) \\ &\leq \left(1 - \frac{1}{k}\right) F(u) + \frac{1}{k} F(u + hv) \\ &\leq \left(1 - \frac{1}{k}\right) F(u) + \frac{1}{k} (F(u) + hS) = F(u) + \frac{h}{k} S. \end{aligned}$$

Finally, by using the lower semicontinuity of F, we have

$$F(u + tv) \leq F(u) + tS \qquad \forall t \geq 0.$$

Thus, taking $u = v_0$, we obtain

$$\frac{F(v_0 + tv) - F(v_0)}{t} \leq S \qquad \forall t > 0,$$

which concludes the proof of (15.4).

The fact that F^∞ is proper, convex, and sequentially σ-lsc follows from assertion (ii). Indeed, for every $u \in \operatorname{dom} F$ the mapping $v \mapsto F(u+v) - F(u)$ is clearly convex and

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sequentially σ-lsc, and so is F^∞, thanks to the well-known properties of supremum of convex functions. The fact that F^∞ is positively 1-homogeneous follows easily from the definition. Indeed, given $v \in V$ and $s > 0$, we have

$$F^\infty(sv) = \lim_{t\to+\infty} \frac{F(v_0 + tsv)}{t}$$

and, setting $\tau = ts$,

$$F^\infty(sv) = s \lim_{\tau\to+\infty} \frac{F(v_0 + \tau v)}{\tau} = sF^\infty(v).$$

Assertion (iv) follows straightforwardly by the definition of recession function. Finally, to prove (v), we may reduce ourselves to the case when both $F^\infty(v)$ and $F^\infty(-v)$ are finite; otherwise the statement is trivial. Therefore, by using (ii), we deduce that

$$u + v \in \operatorname{dom} F, \quad u - v \in \operatorname{dom} F \qquad \forall u \in \operatorname{dom} F;$$

hence, by using (ii) again, taking $u - v$ instead of u in the supremum associated to $F^\infty(v)$, we have

$$\begin{aligned} F^\infty(v) + F^\infty(-v) \geq & \sup_{u\in\operatorname{dom} F} \big(F(u) - F(u-v)\big) \\ & + \sup_{u\in\operatorname{dom} F} \big(F(u-v) - F(u)\big) \geq 0, \end{aligned}$$

which concludes the proof of Proposition 15.1.1. $\square$

Here are some simple examples in which the recession functional can be explicitly computed.

Example 15.1.1. Let $V = \mathbf{R}$ and let $F(x) = e^x$ for every $x \in \mathbf{R}$. Then an easy calculation shows that in this case

$$F^\infty(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ +\infty & \text{if } x > 0. \end{cases}$$

Analogously, if $V = \mathbf{R}^2$ and $F(x) = x_1^2$, then

$$F^\infty(x) = \begin{cases} 0 & \text{if } x_1 = 0, \\ +\infty & \text{if } x_1 \neq 0. \end{cases}$$

As we will see more precisely later, this example shows how the recession function indicates the *directions of coercivity*.

Example 15.1.2. Let $F : V \to [0, +\infty]$ be a nonnegative convex σ-lsc functional which is positively homogeneous of degree $p > 1$, that is,

$$F(tv) = t^p F(v) \qquad \forall t > 0, \forall v \in V.$$

Then we have

$$F^\infty(v) = \begin{cases} 0 & \text{if } F(v) = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

On the other hand, if F is positively homogeneous of degree 1, then it is clear that $F^\infty = F$.

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Example 15.1.3. Let Ω be an open subset of $\mathbf{R}^n$ with a Lipschitz boundary, and let $V = W^{1,p}(\Omega;\mathbf{R}^m)$ ($p \geq 1$) be the Sobolev space of all $\mathbf{R}^m$-valued functions which are in $L^p(\Omega)$ along with their first derivatives. Consider the functional

$$F(u) = \int_\Omega f(x, Du)\,dx \qquad \forall u \in W^{1,p}(\Omega, \mathbf{R}^m),$$

where

(i) $f : \Omega \times \mathbf{R}^{mn} \to [0, +\infty]$ is a Borel function,

(ii) for a.e. $x \in \Omega$ the function $f(x,\cdot)$ is convex and lower semicontinuous on $\mathbf{R}^{mn}$,

(iii) there exists $u_0 \in W^{1,p}(\Omega, \mathbf{R}^m)$ such that $F(u_0) < +\infty$.

It is well known (see, for instance, Section 13.1) that under the assumptions above, the functional F turns out to be proper, convex, and sequentially lower semicontinuous with respect to the weak topology of $W^{1,p}(\Omega, \mathbf{R}^m)$. Moreover, we have

$$F^\infty(u) = \int_\Omega f^\infty(x, Du)\,dx \qquad \forall u \in W^{1,p}(\Omega, \mathbf{R}^m),$$

where $f^\infty(x,\cdot)$ is the recession function of $f(x,\cdot)$. In fact, because of the convexity of f, for all $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ the function

$$g(x,t) = \frac{f\big(x, Du_0(x) + tDu(x)\big) - f\big(x, Du_0(x)\big)}{t}$$

is nondecreasing with respect to t for a.e. $x \in \Omega$. Therefore the monotone convergence theorem gives

$$\begin{aligned} F^\infty(u) &= \lim_{t \to +\infty} \frac{F(u_0 + tu) - F(u_0)}{t} \\ &= \lim_{t \to +\infty} \int_\Omega g(x,t)\,dx = \int_\Omega f^\infty\big(x, Du(x)\big)\,dx. \end{aligned}$$

As a consequence, if

$$F(u) = \int_\Omega |Du|^p\,dx \qquad \forall u \in W^{1,p}(\Omega, \mathbf{R}^m)$$

with $p > 1$, we get

$$F^\infty(u) = \begin{cases} 0 & \text{if } u \text{ is locally constant in } \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

We are now in a position to give a first result on necessary conditions in convex minimization.

Proposition 15.1.2. *Assume that*

$$\inf\big\{F(v) : v \in V\big\} > -\infty$$

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(which always occurs, for instance, if F admits a minimum point on V). Then

$$F^\infty(v) \geq 0 \qquad \forall v \in V. \tag{15.5}$$

PROOF. Let m be the infimum of F on V, let v_0 be any point in $\operatorname{dom} F$, and let $v \in V$. Since F is proper the infimum m is finite and so, by the definition of F^∞ we obtain

$$F^\infty(v) = \lim_{t\to+\infty} \frac{F(v_0 + tv)}{t} \geq \lim_{t\to+\infty} \frac{m}{t} = 0. \qquad \square$$

Example 15.1.4. Let $P : V \to [0,+\infty]$ be a convex sequentially σ-lower semicontinuous functional which is positively homogeneous of degree $p > 1$ (for instance, the pth power of a seminorm), let $L \in V'$, and let F be the functional defined by

$$F(v) = P(v) - \langle L, v\rangle \qquad \forall v \in V.$$

By Example 15.1.2 we have

$$F^\infty(v) = \begin{cases} -\langle L, v\rangle & \text{if } P(v) = 0, \\ +\infty & \text{otherwise;} \end{cases}$$

hence, by the necessary condition of Proposition 15.1.2 we deduce that if the minimum problem

$$\min\{F(v) \,:\, v \in V\}$$

admits a solution, then the linear functional L has to satisfy the compatibility condition

$$\langle L, v\rangle \leq 0 \qquad \forall v \in V \text{ with } P(v) = 0.$$

Example 15.1.5. The necessary condition of Proposition 15.1.2 is clearly not sufficient to obtain the existence of a minimizer, as the example of the function $F(x) = e^x$ with $V = \mathbf{R}$ shows. In fact, in this case it is $F^\infty \geq 0$ (see Example 15.1.1) but the function F has no minimum points on $\mathbf{R}$.

We give now an existence result for convex minimum problems without coercivity; the existence of a minimizer for the functional F will be obtained by adding to the necessary condition (15.5) some more requirements, namely, semicontinuity, compactness, and compatibility conditions, in the sense specified below. We will use the notation $\ker F^\infty$ for the set $\{v \in V \,:\, F^\infty(v) = 0\}$, which, if condition (15.5) is satisfied, is a sequentially σ-closed convex cone.

Theorem 15.1.1. *Let V be a reflexive and separable Banach space with norm $||\cdot||$, and let $F : V \to]-\infty, +\infty]$ be a proper convex sequentially weakly lower semicontinuous functional. Assume that the following conditions are satisfied:*

(i) compactness: *if $t_h \to +\infty$, $v_h \to v$ weakly, and $F(t_h v_h)$ is bounded from above, then $||v_h - v|| \to 0$;*

(ii) necessary condition: *$F^\infty(v) \geq 0$ for every $v \in V$;*

(iii) compatibility: *$\ker F^\infty$ is a linear subspace of V.*

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Then the minimum problem

$$\min\{F(v)\,:\,v\in V\} \tag{15.6}$$

admits at least a solution.

PROOF. For convenience, we divide the proof into several steps.

Step 1. For every $h\in\mathbf{N}$ consider the minimum problem

$$\min\{F(v)\,:\,v\in B_h\}, \tag{$\wp_h$}$$

where $B_h=\{v\in V\;:\;||v||\le h\}$. Since F is sequentially weakly lower semicontinuous and B_h is sequentially weakly compact, by the direct method of the calculus of variations (see Section 3.2) we obtain that for every $h\in\mathbf{N}$ there exists a solution v_h of problem $(\wp_h)$.

Step 2. If for some $h\in\mathbf{N}$ it is $||v_h||<h$, we claim that the proof of the theorem is achieved because such v_h is a solution of problem (15.6). Indeed, due to the convexity of F, for every $v\in V$ and every $\theta\in]0,1[$

$$F\big(v_h+\theta(v-v_h)\big)\le\theta F(v)+(1-\theta)F(v_h);$$

hence

$$F(v)-F(v_h)\ge\frac{1}{\theta}\Big[F\big(v_h+\theta(v-v_h)\big)-F(v_h)\Big].$$

Due to the definition of v_h, the right-hand side is nonnegative whenever $v_h+\theta(v-v_h)\in B_h$, which always occurs when θ is chosen small enough to have $\theta\big(||v||-||v_h||\big)\le h-||v_h||$. Therefore, to conclude the proof of the theorem, it remains to show that the case

$$||v_h||=h\qquad\forall h\in\mathbf{N} \tag{15.7}$$

leads to existence of a solution of problem (15.6) too. Thus, we assume for the rest of the proof that (15.7) holds.

Step 3. For every $h\in\mathbf{N}$ set $w_h=v_h/h$; the sequence (w_h) is weakly compact in V. Then we may extract a subsequence (which we still denote by (w_h)) weakly converging to some $w\in V$. We have

$$F(hw_h)=F(v_h)\le F(v_0)<+\infty \tag{15.8}$$

for every h large enough, where v_0 is any point in $\operatorname{dom}F$. Therefore, by the compactness assumption (i), we obtain $||w_h-w||\to 0$. Moreover, by using the lower semicontinuity and convexity of F, we have for every $t>0$

$$\begin{aligned}F(tw+v_0)&\le\liminf_{h\to+\infty}F\left(tw_h+\left(1-\frac{t}{h}\right)v_0\right)\\&\le\liminf_{h\to+\infty}\left(\frac{t}{h}F(hw_h)+\left(1-\frac{t}{h}\right)F(v_0)\right)\le F(v_0),\end{aligned}$$

where the last inequality follows from (15.8). Hence

$$F^\infty(w)=\lim_{t\to+\infty}\frac{F(tw+v_0)}{t}\le\lim_{t\to+\infty}\frac{F(v_0)}{t}=0,$$

which, together with the necessary condition (ii), implies

$$w\in\ker F^\infty.$$

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Step 4. By the compatibility condition (iii) we obtain $F^\infty(-w)=0$, and this implies, by Proposition 15.1.1(ii), that

$$F(v-tw)\le F(v) \qquad \forall v\in V,\ \forall t\ge 0. \tag{15.9}$$

In particular, by taking $v=v_h$ and $t=h$, from the fact that $||w_h-w||\to 0$ we have $||v_h-hw||<h$ for h large enough, and then from (15.9) it follows that v_h-hw is a solution of problem $(\wp_h)$ for h large enough, with $||v_h-hw||<h$. Hence we found a solution of problem $(\wp_h)$, for h large enough, with norm strictly less than h, and by repeating the argument used in Step 2, this provides a solution of the minimum problem (15.6)

Remark 15.1.1. In the case of nonreflexive Banach spaces V the result above still holds; it is enough to assume that V is a Banach space with norm $||\cdot||$ and to consider a topology σ on V coarser than the norm topology and such that (V,σ) is a Hausdorff vector space with the closed unit ball of $(V,||\cdot||)$ sequentially σ-compact. This happens, for instance, when V is the dual W' of a separable Banach space W and σ is the weak* topology on V. Then the proof above can be repeated if $F:V\to]-\infty,+\infty]$ is assumed to be proper convex and sequentially σ-lsc and such that (ii), (iii) hold together with the compactness condition:

> if $t_h\to+\infty$, $v_h\to v$ with respect to σ, and $F(t_hv_h)$ is bounded from above, then $||v_h-v||\to 0$.

Remark 15.1.2. Notice that if $\operatorname{dom} F$ is bounded in $(V,||\cdot||)$, then conditions (i), (ii), and (iii) are automatically fulfilled. In fact, in this case the set $\ker F^\infty$ reduces to $\{0\}$, as can be seen immediately.

Remark 15.1.3. Consider the particular case when the so-called condition of Lions–Stampacchia type (see [277])

$$F^\infty(v)>0 \qquad \forall v\ne 0 \tag{15.10}$$

holds. Then we obtain immediately that assumptions (ii) and (iii) are fulfilled. By Theorem 15.1.1 we obtain that, if the compactness assumption (i) also holds, then the set of solutions of problem (15.6) is nonempty. In this case the set of solutions is also bounded. In fact, by contradiction, assume there exist v_h solutions of (15.6) with $||v_h||\to+\infty$; arguing as in the proof of Theorem 15.1.1, it is possible to prove that the sequence $w_h=v_h/||v_h||$ converges in norm to some $w\in\ker F^\infty$. From assumption (15.10) it follows that $w=0$, and this contradicts the fact that $||w_h||=1$ for every h.

We consider now the particular case of quadratic forms on a Hilbert space. More precisely, let V be a separable Hilbert space, let $a:V\times V\to\mathbf{R}$ be a symmetric bilinear continuous form, and let $L\in V'$. Define

$$F(v)=\frac{1}{2}a(v,v)-\langle L,v\rangle \qquad \forall v\in V$$

and consider the minimum problem

$$\min\{F(v)\,:\,v\in V\}, \tag{15.11}$$

which is equivalent to the equation

$$a(v,\cdot)=L \qquad \text{in } V'.$$

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On the bilinear form a we assume that the following conditions are satisfied:

$$a(v,v) \geq 0 \qquad \forall v \in V, \tag{15.12}$$

$$v_h \to 0 \text{ weakly, and } a(v_h,v_h) \to 0 \quad \Rightarrow \quad v_h \to 0 \text{ strongly.} \tag{15.13}$$

In this case, since $\ker(F^\infty - L) = (\ker F^\infty) \cap (\ker L)$, as a corollary of Theorem 15.1.1 we obtain the following result.

Proposition 15.1.3. *The minimum problem* (15.11) *admits a solution iff the compatibility condition*

$$L \textit{ is orthogonal to } \ker a \ (\textit{i.e., } \langle L, v\rangle = 0 \textit{ whenever } a(v,v) = 0) \tag{15.14}$$

is fulfilled. Note that the condition above reads simply $\ker a \subset \ker L$.

PROOF. An easy computation shows that

$$F^\infty(v) = \begin{cases} -\langle L, v\rangle & \text{if } a(v,v) = 0, \\ +\infty & \text{if } a(v,v) \neq 0; \end{cases}$$

then, by Proposition 15.2, if problem (15.11) admits a solution, we must necessarily have $F^\infty \geq 0$, that is, (15.14). Conversely, let us assume (15.14) holds. The weak lower semicontinuity of the functional F is then a consequence of the continuity of the bilinear form a, the compactness assumption (i) follows from property (15.13), and finally the necessary condition (ii) and the compatibility condition (iii) follow from property (15.14). Therefore, by Theorem 15.1.1 we obtain that the minimum problem (15.11) admits a solution. □

Example 15.1.6. Consider the variational formulation of the classical Neumann problem

$$\min\left\{ \frac{1}{2}\int_\Omega |Du|^2\,dx - \langle L, u\rangle \ : \ u \in H^1(\Omega) \right\},$$

where Ω is a bounded regular open subset of $\mathbf{R}^n$ and $L \in \left(H^1(\Omega)\right)'$. If we consider the bilinear form

$$a(u,v) = \int_\Omega Du\,Dv\,dx \qquad \forall u, v \in H^1(\Omega)$$

and apply Proposition 15.1.3, we obtain that a solution exists iff the compatibility condition

$$\langle L, 1\rangle = 0$$

is fulfilled. In terms of partial differential equations, when $L = f + g$ with $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$, this means that the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega \end{cases}$$

admits a solution iff

$$\int_\Omega f(x)\,dx + \int_{\partial\Omega} g(x)\,d\mathcal{H}^{n-1}(x) = 0.$$

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For every subset K of V we denote by χ_K the indicator function of K defined by

$$\chi_K(v) = \begin{cases} 0 & \text{if } v \in K, \\ +\infty & \text{otherwise.} \end{cases}$$

Notice that if K is a nonempty sequentially σ-closed convex subset of V, then the function χ_K turns out to be a proper convex sequentially σ-lsc mapping.

Definition 15.1.2. *For every nonempty sequentially σ-closed convex subset K of V we define the recession cone K^∞ of K by setting*

$$K^\infty = \operatorname{dom}(\chi_K)^\infty. \tag{15.15}$$

For every function $F : V \to]-\infty, +\infty]$ we denote by $\operatorname{epi} F$ the epigraph of F

$$\operatorname{epi} F = \big\{(v, t) \in V \times \mathbf{R} \,:\, t \geq F(v)\big\}. \tag{15.16}$$

We remark that $\operatorname{epi} F$ is a nonempty sequentially closed convex subset of $V \times \mathbf{R}$ whenever F is a proper sequentially σ-lower semicontinuous convex function.

Proposition 15.1.4. *If $F : V \to]-\infty, +\infty]$ is a proper convex sequentially σ-lsc function, we have*

$$\operatorname{epi} F^\infty = (\operatorname{epi} F)^\infty.$$

PROOF. By Proposition 15.1.1(ii) it is

$$(v, t) \in \operatorname{epi} F^\infty \iff t \geq F(u + v) - F(u) \quad \forall u \in \operatorname{dom} F.$$

In other words, if $(s, u) \in \operatorname{epi} F$, we have

$$F(u + v) \leq t + F(u) \leq t + s,$$

that is,

$$(v, t) + \operatorname{epi} F \subset \operatorname{epi} F,$$

or equivalently,

$$\chi_{\operatorname{epi} F}\big((v, t) + (w, \alpha)\big) = 0 \qquad \forall (w, \alpha) \in \operatorname{epi} F.$$

By Proposition 15.1.1 again, we obtain

$$\big(\chi_{\operatorname{epi} F}\big)^\infty (v, t) = 0,$$

that is, $(v, t) \in (\operatorname{epi} F)^\infty$. $\square$

The definition given in (15.15) of recession cone is a very useful tool in convex analysis (see [325], [76], [77], and [119], where it is called “cône asymptote”); let us now state its main properties.

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Proposition 15.1.5. *For any nonempty sequentially σ-closed convex subset K of V the set K^∞ is a convex sequentially σ-closed cone and the following properties hold true:*

$$K^\infty = \bigcap_{t>0} t^{-1}(K - v_0) \quad \forall v_0 \in K, \tag{15.17}$$

$$K \text{ is a cone} \iff K^\infty = K, \tag{15.18}$$

$$0 \in K \iff K^\infty \subset K, \tag{15.19}$$

$$K \text{ is bounded} \Rightarrow K^\infty = \{0\}, \tag{15.20}$$

$$\operatorname{dom} F^\infty \subset (\operatorname{dom} F)^\infty \text{ whenever } F : V \to]-\infty, +\infty] \text{ is convex and lsc.} \tag{15.21}$$

Moreover, a point $w \in V$ belongs to K^∞ iff one of the following conditions is fulfilled:

$$k + w \in K \quad \forall k \in K, \tag{15.22}$$

$$k + tw \in K \quad \forall k \in K \; \forall t \geq 0, \tag{15.23}$$

$$\exists k \in K \, : \, k + tw \in K \quad \forall t \geq 0. \tag{15.24}$$

PROOF. By Proposition 15.1.1(ii) and the definition (15.15) of K^∞ we have that $v \in K^\infty$ iff

$$\chi_K(v_0 + tv) - \chi_K(v_0) \leq 0 \qquad \forall t > 0, \; \forall v_0 \in K,$$

which is equivalent to

$$v + tv_0 \in K \qquad \forall t > 0, \; \forall v_0 \in K.$$

Therefore (15.17) follows.

The proof of (15.18) follows from Proposition 15.1.1(iii). Assertion (15.19) follows easily from (15.17) by taking $v_0 = 0$.

Always from (15.17) we obtain that K^∞ reduces to $\{0\}$ whenever K is bounded, that is, (15.20).

The proof of (15.21) simply follows by remarking that when $F^\infty(v) < +\infty$, then $v_0 + tv \in \operatorname{dom} F$ for every $t > 0$ and every $v_0 \in \operatorname{dom} F$.

Finally, (15.22), (15.23), (15.24) follow from Proposition 15.1.1(ii). ☐

Remark 15.1.4. We remark that the inclusion in (15.21) may be strict: this can be seen by taking, for instance, $F(v) = ||v||^2$. In this case we have

$$\operatorname{dom} F = (\operatorname{dom} F)^\infty = V \qquad \text{whereas} \qquad \operatorname{dom} F^\infty = \{0\}.$$

In the finite dimensional case, implication (15.20) can be reversed, as the following proposition shows.

Proposition 15.1.6. *Let K be a convex closed subset of $\mathbf{R}^n$ such that $K^\infty = \{0\}$. Then K is bounded.*

PROOF. Assume by contradiction that K is unbounded; then for every $h \in \mathbf{N}$ there exists $x_h \in K$ with $|x_h| \geq h$. The sequence $y_h = x_h/|x_h|$ is bounded, so that we may extract a subsequence (which we still denote for simplicity by y_h) converging to some y with $|y| = 1$. We claim that $y \in K^\infty$. In fact, fix $t > 0$ and let x_0 be any point in K; since K is convex, for h large enough we have

$$\frac{t}{|x_h|} x_h + \left(1 - \frac{t}{|x_h|}\right) x_0 \in K,$$

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and, since K is closed, passing to the limit as $h \to +\infty$,

$$ty + x_0 \in K,$$

which implies that y belongs to K^∞. This gives a contradiction because by assumption $K^\infty = \{0\}$ and $|y| = 1$. $\square$

In general infinite dimensional topological vector spaces, Proposition 15.1.6 is false, as shown by the following example.

Example 15.1.7. Let V be a separable (infinite dimensional) Hilbert space, and let $(e_n)_{n\in\mathbf{N}}$ be a complete orthonormal system in V. Consider the set

$$K = \big\{v \in V \,:\, |(v, e_n)| \le n \text{ for every } n \in \mathbf{N}\big\},$$

where $(\cdot,\cdot)$ denotes the scalar product in V. It is easy to see that K is a nonempty convex weakly closed subset of V; moreover, we have $K^\infty = \{0\}$. In fact, if $v \in K^\infty$, by using the fact that $0 \in K$, we obtain from (15.17)

$$tv \in K \qquad \text{for every } t > 0,$$

that is,

$$|(v, e_n)| \le \frac{n}{t} \quad \text{for every } t > 0 \text{ and } n \in \mathbf{N}.$$

Therefore, as $t \to +\infty$, we get

$$(v, e_n) = 0 \quad \forall n \in \mathbf{N},$$

and so $v = 0$. Nevertheless, K is unbounded, as it contains the points $v_n = ne_n$ for every $n \in \mathbf{N}$.

We will specialize now our existence results on noncoercive minimum problems to the case when the functional F can be written in the form

$$F(v) = J(v) - \langle L, v\rangle + \chi_K(v).$$

This situation arises, for instance, in many problems of mathematical physics, where J represents the stored energy functional depending on the nature of the body, L describes the action of the applied forces, and K is the set of admissible configurations which takes into account the physical constraints of the problems. In this case, the minimization problem we are dealing with takes the form

$$\min\big\{J(v) - \langle L, v\rangle \,:\, v \in K\big\}.$$

Theorem 15.1.2. *Assume that*

$$J : V \to [0, +\infty] \text{ is a proper convex sequentially } \sigma\text{-lsc functional}, \tag{15.25}$$

$$L : V \to \mathbf{R} \text{ is a linear } \sigma\text{-continuous functional}, \tag{15.26}$$

$$K \subset V \text{ is a nonempty convex sequentially } \sigma\text{-closed set}, \tag{15.27}$$

and consider the minimum problem

$$\min\big\{J(v) - \langle L, v\rangle \,:\, v \in K\big\}. \tag{15.28}$$

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Then a necessary condition for the existence of at least a minimizer is

$$J^\infty(v) \geq \langle L, v\rangle \qquad \forall v \in K^\infty. \tag{15.29}$$

On the other hand, the minimum problem (15.28) *admits at least a solution provided the necessary condition* (15.29) *holds and the following compactness and compatibility conditions are fulfilled: if*

$$t_h \to +\infty, \quad v_h \in K, \quad v_h \to v$$

weakly, and $J(t_h v_h) - t_h\langle L, v_h\rangle$ *is bounded from above, then*

$$\|v_h - v\| \to 0, \tag{15.30}$$

$$K^\infty \cap \ker(J^\infty - L) \textit{ is a linear subspace of } V. \tag{15.31}$$

PROOF. Noticing that by Proposition 15.1.1(iv) the equality

$$(J - L + \chi_K)^\infty = J^\infty - L + \chi_{K^\infty}$$

holds, and taking Proposition 15.1.2 into account, we obtain that condition (15.29) is necessary for the existence of at least a solution of the minimum problem (15.28).

Analogously, compactness and compatibility conditions (i) and (iii) become in this case (15.30) and (15.31), so that the conclusion follows by Theorem 15.1.1. □

Remark 15.1.5. When J is a quadratic form, or more generally when J^∞ takes only the values 0 and $+\infty$ (which, for instance, occurs if J is positively p-homogeneous with $p > 1$; see Example 15.1.2, condition (15.31)), it can be written in the simpler form

$$K^\infty \cap \ker J^\infty \cap \ker L \text{ is a linear subspace of } V, \tag{15.32}$$

which will be used in the following.

Let us discuss the structural assumptions of the existence result of Theorem 15.1.1 above.

Example 15.1.8. The compactness assumption (i) in the existence theorem, Theorem 15.1.1, cannot be dropped, as the following example shows. Let V be an infinite dimensional separable Hilbert space, and let $(e_n)_{n\in\mathbf{N}}$ be a complete orthonormal system in V. We denote by $(\cdot,\cdot)$ the scalar product in V, and we define a functional $F : V \to \mathbf{R}$ by setting

$$F(v) = \sum_{n\in\mathbf{N}} 2^{-n}|(v, e_n) - 1|^2 \qquad \forall v \in V.$$

It is easy to see that the functional F is finite-valued, convex, and weakly lower semicontinuous. Moreover, for every $v \in V$

$$\begin{aligned} F^\infty(v) &= \lim_{t\to+\infty} \frac{F(tv)}{t} \\ &= \lim_{t\to+\infty} \sum_{n\in\mathbf{N}} 2^{-n}\left[t|(v, e_n)|^2 - 2(v, e_n) + \frac{1}{t}\right]. \end{aligned}$$

Hence,

$$F^\infty(v) = \begin{cases} 0 & \text{if } v = 0, \\ +\infty & \text{if } v \neq 0, \end{cases}$$

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so that the necessary condition (ii) and the compatibility condition (iii) of Theorem 15.1 are fulfilled. Nevertheless, the functional F does not admit any minimum point in V. In fact, taking for every $k \in \mathbf{N}$

$$v_k = \sum_{i=1}^{k} e_i$$

we get

$$\inf_{v \in V} F(v) \leq F(v_k) = \sum_{n=k+1}^{\infty} 2^{-n}$$

so that

$$\inf_{v \in V} F(v) = 0.$$

But there are no points $v \in V$ such that $F(v) = 0$. Indeed, $F(v) = 0$ would imply $(v, e_n) = 1$ for every $n \in \mathbf{N}$, which contradicts the equality

$$\|v\|^2 = \sum_{n \in \mathbf{N}} |(v, e_n)|^2.$$

Example 15.1.9. The compatibility condition (iii) in Theorem 15.1.1 cannot be dropped, as the following example shows. Take $V = \mathbf{R}$ and define $F : \mathbf{R} \to]-\infty, +\infty]$ by

$$F(x) = \begin{cases} -\log x & \text{if } x > 0, \\ +\infty & \text{if } x \leq 0. \end{cases}$$

The function F is proper, convex, and lower semicontinuous, and the compactness condition (i) is fulfilled since the dimension of V is finite. Moreover, a simple calculation yields

$$F^{\infty}(x) = \begin{cases} 0 & \text{if } x \geq 0, \\ +\infty & \text{if } x < 0, \end{cases}$$

and so the necessary condition (ii) is fulfilled too, but

$$\inf\{F(x) \,:\, x \in \mathbf{R}\} = -\infty.$$

We conclude this section by showing how Theorem 15.1.1 can be used to determine whether the algebraic difference of two closed convex sets in a Banach space is closed. Here we consider a reflexive Banach space V and two nonempty closed convex subsets A, B of V; the algebraic difference $A - B$ is defined by

$$A - B = \{a - b \,:\, a \in A,\ b \in B\}.$$

Example 15.1.10. We emphasize that even when dealing with cones in a finite dimensional space, the convex set $A - B$ may be not closed: take, for instance, $V = \mathbf{R}^3$ and

$$\begin{aligned} A &= \left\{x \in \mathbf{R}^3 \,:\, x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_1 x_3 \geq x_2^2\right\}, \\ B &= \left\{x \in \mathbf{R}^3 \,:\, x_2 = x_3 = 0\right\}. \end{aligned}$$

The sets A and B are closed convex cones, but a simple calculation gives

$$A - B = \left\{x \in \mathbf{R}^3 \,:\, x_2 > 0,\ x_3 > 0\right\} \cup \left\{x \in \mathbf{R}^3 \,:\, x_2 = 0,\ x_3 \geq 0\right\},$$

which is not closed. Indeed, for every $n \in \mathbf{N}$ we have that the point $(0, 1, 1/n)$ belongs to $A - B$, whereas their limit $(0, 1, 0)$ is not in $A - B$.

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In the previous example the convex sets A and B are both unbounded. On the other hand, if A and B are two closed convex subsets of a reflexive Banach space and at least one of them is bounded, then $A-B$ is closed (weak and strong closedness coincide, due to convexity). Indeed, if A is bounded and $x_h = a_h - b_h$ tends weakly to x with $a_h \in A$ and $b_h \in B$, then up to subsequences we have $a_h \to a \in A$ weakly, hence $b_h = x_h - a_h \to x - a \in B$ weakly, so that $x \in A - B$.

The following lemma characterizes the closed convex subsets of V.

Lemma 15.1.1. *Let K be a nonempty convex subset of a reflexive Banach space V. Then the following conditions are equivalent:*

(i) *K is closed.*

(ii) *For every $u \in V$ the function $v \mapsto ||u-v||$ has a minimum on K.*

PROOF. Assume K is closed and let $u \in V$; set

$$M = \inf\{||u-v|| \,:\, v \in K\},$$

and for every $h \in \mathbf{N}$ let $v_h \in K$ be such that

$$||u-v_h|| \le M + \frac{1}{h}.$$

The sequence (v_h) is bounded; since V is reflexive, possibly passing to subsequences, we may assume that v_h converges weakly to some v which belongs to K, because K is weakly closed (being strongly closed and convex). By the weak lower semicontinuity of the norm, we obtain

$$||u-v|| \le \liminf_{h\to+\infty} ||u-v_h|| \le M,$$

which proves (ii).

Conversely, assume (ii) holds, and let (v_h) be a sequence in K strongly convergent to some $v \in V$. By (ii) there exists $\overline{w} \in K$ such that

$$||\overline{w}-v|| \le ||w-v|| \qquad \forall w \in K.$$

In particular,

$$||\overline{w}-v|| \le ||v_h-v|| \qquad \forall h \in \mathbf{N};$$

hence, as $h \to +\infty$, we obtain $\overline{w} = v$, and this proves that $v \in K$. ☐

We are now in a position to prove a closure result for the difference of two closed sets.

Theorem 15.1.3. *Let V be a reflexive Banach space, and let A and B be two nonempty closed convex subsets of V. Assume that A is locally compact for the strong topology and that*

$$A^\infty \cap B^\infty \text{ is a linear subspace.} \tag{15.33}$$

Then the convex set $A-B$ is closed.

PROOF. By Lemma 15.1.1 it is enough to prove that for every $u \in V$ the function $v \mapsto ||u-v||$ has a minimum on $A-B$, or equivalently that the problem

$$\min\{F(a,b) \,:\, (a,b) \in V \times V\} \tag{15.34}$$

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has at least a solution, where

$$F(a,b) = ||u - a + b|| + \chi_A(a) + \chi_B(b).$$

We apply to the minimum problem (15.34) the existence theorem, Theorem 15.1.1. Since F is a nonnegative functional, the necessary condition $F^\infty \geq 0$ is immediately fulfilled. To prove the compactness condition (i) take $a_h \to a$ weakly in V, $b_h \to b$ weakly in V, $t_h \to +\infty$ such that $F(t_h a_h, t_h b_h) \leq C$; then by the definition of F we have

$$t_h a_h \in A, \qquad t_h b_h \in B, \qquad ||u - t_h a_h + t_h b_h|| \leq C.$$

Since the convex set A is assumed to be locally compact for the strong topology of V and $t_h a_h \in A$, the convergence $a_h \to a$ is actually strong; moreover, by $||u - t_h a_h + t_h b_h|| \leq C$ we obtain that $||b_h - a_h|| \to 0$ and so also $b_h \to a$ strongly.

Let us finally prove the compatibility condition (iii). By the definition of F we find

$$F^\infty(a,b) = \lim_{t \to +\infty} \frac{F(a_0 + ta, b_0 + tb)}{t},$$

where $a_0 \in A$ and $b_0 \in B$. Therefore

$$\begin{aligned} F^\infty(a,b) &= \lim_{t \to +\infty} ||a - b|| + \chi_A(a_0 + ta) + \chi_B(b_0 + tb) \\ &= ||a - b|| + \chi_{A^\infty}(a) + \chi_{B^\infty}(b) \end{aligned}$$

so that

$$\begin{aligned} \ker F^\infty &= \{(a,b) \,:\, a \in A^\infty,\ b \in B^\infty,\ a = b\} \\ &= \{(a,a) \,:\, a \in A^\infty \cap B^\infty\}. \end{aligned}$$

Therefore the compatibility condition (iii) follows from assumption (15.33). □

Remark 15.1.6. The requirement that A is locally compact for the strong topology is clearly fulfilled when A is finite dimensional. On the other hand, there are closed convex subsets A which are locally compact for the strong topology but not finite dimensional. For instance, it is enough to take in a separable Hilbert space V the set

$$A = \{v \in V \,:\, |(v, e_n)| \leq 1/n \ \forall n \in \mathbf{N}\},$$

where $(e_n)_{n\in\mathbf{N}}$ is a complete orthonormal system in V.

The results of this section allow us to study the problem of lower semicontinuity for the inf-convolution of two convex functions. We recover some results of Section 9.2. If V is a reflexive Banach space and $f, g : V \to]-\infty, +\infty]$ are two proper convex functions, we recall that the inf-convolution $f \#_e g$ is defined by

$$(f \#_e g)(w) = \inf\{f(u) + g(v) \,:\, u, v \in V,\ u + v = w\}. \tag{15.35}$$

For instance, if $f = \chi_A$ and $g = \chi_B$, with A, B convex subsets of V, we have $f \#_e g = \chi_{A+B}$. Moreover, it is easy to see that in terms of epigraphs the inf-convolution operation turns out to simply reduce to the algebraic sum, that is,

$$\mathrm{epi}_{f \#_e g} = \mathrm{epi}_f + \mathrm{epi}_g.$$

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As we saw in Example 15.1.10, it may happen that f and g are both lower semicontinuous but the lower semicontinuity does not occur for the inf-convolution $f\#_e g$. To see when $f\#_e g$ is lower semicontinuous, we investigate equivalently on the closedness of $\mathrm{epi}_{f\#_e g}$ and we obtain the following result.

Proposition 15.1.7. *Let $f, g : V \to]-\infty, +\infty]$ be two proper convex lower semicontinuous functions. Assume the following:*

(i) compactness: *if $t_h \to +\infty$, u_h and v_h converge weakly, $u_h + v_h \to 0$ strongly, and $f(t_h u_h) + g(t_h v_h)$ is bounded from above, then u_h and v_h converge strongly;*

(ii) compatibility: *if $f^\infty(v) + g^\infty(-v) \le 0$, then $f^\infty(-v) + g^\infty(v) \le 0$.*

Then the inf-convolution $f\#_e g$ defined in (15.35) *is lower semicontinuous.*

PROOF. By Lemma 15.1.1 it is enough to show that for every $M \in \mathbf{R}$ and every $u \in V$ the function $(t, v) \mapsto |t - M| + \|v - u\|$ admits a minimum on $\mathrm{epi}_{f\#_e g}$. In other words, we have to show the existence of a solution for the minimum problem

$$\min\left\{|t - M| + \|v - u\| \,:\, t \ge (f\#_e g)(v)\right\}.$$

It is easy to see that for a fixed v, the optimal t in the minimum problem above is given by $t = M \vee (f\#_e g)(v)$, so that we have to show the existence of a solution for the problem

$$\min\left\{M \vee (f\#_e g)(v) + \|v - u\| \,:\, v \in V\right\}.$$

By the definition of inf-convolution this fact turns out to be equivalent to the existence of a solution for

$$\min\left\{M \vee \big(f(x) + g(y)\big) + \|x + y - u\| \,:\, x, y \in V\right\}.$$

Setting $\Phi(x, y) = M \vee \big(f(x) + g(y)\big) + \|x + y - u\|$ we apply to Φ the existence theorem, Theorem 15.1.1. The convexity and the weak lower semicontinuity of Φ follow straightforwardly. To prove the compactness assumption (i) of Theorem 15.1.1, take $x_h \to x$ and $y_h \to y$ weakly in V, and $t_h \to +\infty$ such that $\Phi(t_h x_h, t_h y_h)$ is bounded from above. Dividing by t_h we obtain

$$\limsup_{h\to+\infty}\left(\frac{f(t_h x_h)}{t_h} + \frac{g(t_h y_h)}{t_h}\right)^+ + \|x_h + y_h\| \le 0,$$

which implies that $x_h + y_h \to 0$ strongly in V. By the compactness assumption (i), the convergence of x_h and of y_h is actually strong in V.

Since $\Phi^\infty(x, y) = \big(f^\infty(x) + g^\infty(y)\big)^+ \|x + y\|$ the necessary condition (ii) of Theorem 15.1.1 is fulfilled. It remains to prove the compatibility condition (iii). Since

$$\ker \Phi^\infty = \left\{(x, y) \in V \times V \,:\, x + y = 0,\ f^\infty(x) + g^\infty(y) \le 0\right\},$$

the fact that $\ker \Phi^\infty$ is a subspace follows immediately from assumption (ii). ☐

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Example 15.1.11. Let Ω be a bounded connected open subset of $\mathbf{R}^n$ and let X be the Sobolev space $H^1(\Omega)$. We denote by (x,y) the points of Ω, where x represents some k coordinates and y the remaining $n-k$. Consider the functionals

$$F(u)=\int_\Omega |D_x u|^2\,dx\,dy-\langle f,u\rangle,$$

$$G(u)=\int_\Omega |D_y u|^2\,dx\,dy-\langle g,u\rangle,$$

where f and g are in the dual space of X. The functionals F and G are convex and lower semicontinuous on $H^1(\Omega)$; we want to see if their inf-convolution $F\#_e G$ is still lower semicontinuous. By Proposition 15.1.7 above it is enough to verify the compactness assumption (i) and the compatibility assumption (ii).

If $t_h\to+\infty$, u_h and v_h converge weakly, $u_h+v_h\to 0$ strongly, and $F(t_h u_h)+G(t_h v_h)$ is bounded from above, then dividing by t_h^2 we have that

$$D_x u_h \to 0 \text{ in } L^2(\Omega), \qquad D_y v_h \to 0 \text{ in } L^2(\Omega),$$

which, together with the fact that $u_h+v_h\to 0$ strongly in $H^1(\Omega)$, implies that u_h and v_h converge actually strongly in $H^1(\Omega)$.

Finally, an easy calculation gives the expressions of the recession functions of F and G:

$$F^\infty(u)=\begin{cases}+\infty & \text{if } D_x u\neq 0,\\ -\langle f,u\rangle & \text{if } D_x u\equiv 0,\end{cases}\qquad G^\infty(v)=\begin{cases}+\infty & \text{if } D_y v\neq 0,\\ -\langle g,v\rangle & \text{if } D_y v\equiv 0.\end{cases}$$

Therefore, to prove the compatibility assumption, and hence the lower semicontinuity of $F\#_e G$, it is enough to assume that $\langle f-g,1\rangle=0$.

15.2 ▪ Nonconvex minimization problems and topological recession

In this section we will consider general minimum problems of the form

$$\min\{F(v)\,:\,v\in V\}, \tag{15.36}$$

where F is a possibly nonconvex functional noncoercive as well. Problems of this kind arise, for instance, in nonlinear elasticity (see Section 11.2), and due to the lack of convexity the results of previous sections cannot be applied. We will introduce a new kind of recession functional for general nonconvex functions and we will prove an abstract existence result under lower semicontinuity, compactness, and compatibility conditions, which will be expressed by means of this new tool.

As in Section 15.1, (V,σ) will denote a real locally convex Hausdorff topological vector space, and $F:V\to\,]-\infty,+\infty]$ will be a proper (not necessarily convex) mapping.

Definition 15.2.1. *The topological recession functional F_∞ of F is defined for every $v\in V$ by*

$$F_\infty(v)=\liminf_{\substack{t\to+\infty\\ w\to v}}\frac{F(tw)}{t}. \tag{15.37}$$

The main properties of the functional F_∞ are listed in the following proposition.

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Proposition 15.2.1. *We have the following:*

(i) F_∞ *is* σ*-lsc and positively homogeneous of degree* 1.

(ii) $F_\infty = F$ *whenever* F *is* σ*-lsc and positively homogeneous of degree* 1.

(iii) $F_\infty = F^\infty$ *whenever* F *is proper, convex, and* σ*-lsc.*

(iv) $(F+G)_\infty(v) \geq F_\infty(v) + G_\infty(v)$ *for every mapping* $G : V \to]-\infty, +\infty]$ *and for every* $v \in V$ *such that the sum at the right-hand side is defined.*

(v) *The equality* $(F+G)_\infty = F_\infty + G_\infty$ *holds in the following cases:*

- (v_a) F *and* G *are proper, convex,* σ*-lsc, and* $\operatorname{dom} F \cap \operatorname{dom} G \neq \emptyset$*;*
- (v_b) G *is positively homogeneous of degree* 1*, finite, and* σ*-continuous;*
- (v_c) F *is convex and* σ*-lower semicontinous,* $F(0) < +\infty$*,* G *is* σ*-lsc and positively homogeneous of degree* 1.

PROOF. The proof of properties (i), (ii), (iv), (v_b) can be obtained immediately from Definition 15.2.1; property (v_a) follows from property (iii) and from Proposition 15.1.1(iv); property (v_c) follows from property (iv) and from the fact that using properties (ii), (iii), and Proposition 15.1.1(i), we get for every $v \in V$

$$\begin{aligned}(F+G)_\infty(v) &\leq \liminf_{t\to+\infty} \frac{F(tv)+G(tv)}{t}\\ &= \left(\liminf_{t\to+\infty}\frac{F(tv)}{t}\right) + G(v)\\ &= F^\infty(v) + G(v) = F_\infty(v) + G_\infty(v).\end{aligned}$$

It remains to prove property (iii). Let $v \in V$ and let $v_0 \in \operatorname{dom} F$ be fixed; taking $w_t = v + v_0/t$ we get

$$F_\infty(v) \leq \liminf_{t\to+\infty}\frac{F(tw_t)}{t} = \liminf_{t\to+\infty}\frac{F(v_0+tv)}{t} = F^\infty(v).$$

On the other hand, by using the convexity and the lower semicontinuity of F, for every $s > 0$ we have

$$\begin{aligned}F(v_0+sv) &\leq \liminf_{\substack{t\to+\infty\\ w\to v}} F\left(\left(1-\frac{s}{t}\right)v_0 + \frac{s}{t}tw\right)\\ &\leq \liminf_{\substack{t\to+\infty\\ w\to v}}\left[\left(1-\frac{s}{t}\right)F(v_0) + \frac{s}{t}F(tw)\right]\\ &= F(v_0) + s\liminf_{\substack{t\to+\infty\\ w\to v}}\frac{F(tw)}{t} = F(v_0) + sF_\infty(v).\end{aligned}$$

Therefore,

$$\frac{F(v_0+sv)-F(v_0)}{s} \leq F_\infty(v)$$

for every $s > 0$, and passing to the limit as $s \to +\infty$, we obtain $F^\infty(v) \leq F_\infty(v)$. ☐

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Analogously to what we made in Section 15.2 in the convex case, for every nonempty subset K of V we may define the topological recession cone K_∞ of K.

Definition 15.2.2. *Let K be a nonempty subset of V. The topological recession cone K_∞ of K is defined by*

$$K_\infty = \mathrm{dom}(\chi_K)_\infty. \tag{15.38}$$

Remark 15.2.1. By Definition 15.2.1 it follows that a point u belongs to K_∞ iff

$$\forall s > 0 \quad \forall U \in \mathfrak{I}(u) \quad \exists t > s \; : \; U \cap \frac{1}{t}K \neq \emptyset,$$

where $\mathfrak{I}(u)$ denotes the family of all σ-neighborhoods of u. In other words,

$$K_\infty = \bigcap_{s>0} \mathrm{cl}_\sigma \left(\bigcup_{t>s} \frac{1}{t}K \right), \tag{15.39}$$

where cl_σ denotes the closure with respect to σ.

The following proposition contains a list of properties of the topological recession cones.

Proposition 15.2.2. *We have the following:*

(i) *K_∞ is a σ-closed cone (possibly nonconvex);*

(ii) *$K_\infty = K$ whenever K is a σ-closed cone;*

(iii) *$K_\infty = K^\infty$ whenever K is a nonempty convex σ-closed set;*

(iv) *$K_\infty = \{0\}$ whenever K is bounded (the converse is false even in the convex case, as shown in Example* 15.1.7*);*

(v) *if V is finite dimensional, then $K_\infty = \{0\}$ implies that K is bounded;*

(vi) *$K_\infty = (K \cup H)_\infty = (K + H)_\infty$ for every bounded subset H of V;*

(vii) *$(\mathrm{epi}\, F)_\infty = \mathrm{epi}\, F_\infty$ for every proper function $F : V \to]-\infty, +\infty]$.*

PROOF. Properties (i), (ii), (iii) follow from the definition (15.38) of K_∞ and from Proposition 15.2.1(i), (ii), (iii), respectively.

To prove property (iv), let U be a closed neighborhood of 0; since K is bounded, there exists $s > 0$ such that $K \subset tU$ for every $t > s$. By the characterization of K_∞ given by (15.39) this implies that $K_\infty \subset U$, and since U is arbitrary, we get $K_\infty = \{0\}$.

Let us prove property (v). Assume by contradiction that K is unbounded; then for every $h \in \mathbf{N}$ there exists $x_h \in K$ with $|x_h| \geq h$. Since the sequence $y_h = x_h/|x_h|$ is bounded, we may extract a subsequence (still denoted by y_h) converging to some $y \in V$ with $|y| = 1$. Therefore

$$\begin{aligned}(\chi_K)_\infty(y) &\leq \liminf_{h\to+\infty} \chi_K(|x_h|y_h) \\ &= \liminf_{h\to+\infty} \chi_K(x_h) = 0,\end{aligned}$$

so that $y \in K_\infty$, and this is impossible because $y \neq 0$.

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Let us prove property (vi). Since the inclusion $K_\infty \subset (K \cup H)_\infty$ is obvious, it is enough to prove the opposite inclusion. Let $x \in (K \cup H)_\infty$; if $x = 0$ we have $x \in K_\infty$ because, by (i), K_∞ is a closed cone. If $x \neq 0$ let U_0, U be two disjoint neighborhoods of 0 and x, respectively. Since H is bounded, there exists $s_0 > 0$ such that $H \subset t\,U_0$ for every $t > s_0$; moreover, since $x \in (K \cup H)_\infty$, by (15.39) we have

$$\forall s > 0 \quad \forall W_x \quad \exists t > s \,:\, W_x \cap \frac{1}{t}(K \cup H) \neq \emptyset, \tag{15.40}$$

where we denoted by W_x a generic neighborhood of x. When $s \geq s_0$ and $W_x \subset U$ we have

$$W_x \cap \frac{1}{t} H \subset U \cap U_0 = \emptyset \qquad \forall t > s,$$

so that, by (15.40),

$$W_x \cap \frac{1}{t} K \neq \emptyset.$$

By (15.39) this proves that $x \in K_\infty$. The equality $K_\infty = (K \cup H)_\infty$ can be proved in a similar way.

Let us now prove property (vii). If $(u,\xi) \in (\operatorname{epi} F)_\infty$, denoting by U and I generic neighborhoods of u and ξ, respectively, by (15.39) we have

$$\forall s > 0 \quad \forall U \quad \forall I \quad \exists t > s \quad \exists v \in U \quad \exists \eta \in I \,:\, F(tv) \leq t\eta.$$

Therefore $F_\infty(u) \leq \xi$, so that $(u,\xi) \in \operatorname{epi} F_\infty$. On the other hand, if $(u,\xi) \in \operatorname{epi} F_\infty$, we have $F_\infty(u) \leq \xi$; hence by the definition of F_∞ we obtain

$$\forall s > 0 \quad \forall U \quad \forall \eta > \xi \quad \exists t > s \quad \exists v \in U \,:\, \frac{F(tv)}{t} < \eta.$$

Therefore $(u,\xi) \in \operatorname{epi} F_\infty$. ☐

Remark 15.2.2. It can be useful to give a characterization of the topological recession functional F_∞ in terms of converging nets: actually, it is easy to show that for every $v \in V$

$$F_\infty(v) = \inf\left\{ \liminf_{\lambda \in \Lambda} \frac{F(t_\lambda v \lambda)}{t_\lambda} \,:\, t_\lambda \to +\infty,\ v_\lambda \to v \right\}, \tag{15.41}$$

where Λ is an arbitrary directed set and $(t_\lambda),(v_\lambda)$ are nets indexed by Λ. Recall that a directed set is a set together with a relation $\geq$ which is both transitive and reflexive such that for any two elements a, b there exists another element c with $c \geq a$ and $c \geq b$.

Definition (15.37) or the equivalent one (15.41) is very general and provides a good extension of the notion of convex recession functional introduced in the last section. However, in many situations, it is not easy to deal with neighborhoods or nets; for this reason, we introduce now a new definition of recession functional, in which only the behavior of converging sequences is involved. More precisely, for every $v \in V$ we set

$$F_\infty^{seq}(v) = \inf\left\{ \liminf_{h \to +\infty} \frac{F(t_h v_h)}{t_h} \,:\, t_h \to +\infty,\ v_h \to v \right\}, \tag{15.42}$$

where (t_h) and (v_h) are sequences. It is clear from (15.41) and (15.42) that $F_\infty \leq F_\infty^{seq}$, and that $F_\infty = F_\infty^{seq}$ whenever the space (V,σ) is metrizable. Moreover, by a proof similar to

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the one of Proposition 15.2.1, it is possible to show that the following properties for F_∞^{seq} hold.

Proposition 15.2.3. *We have the following:*

(i) F_∞^{seq} *is positively homogeneous of degree* 1.

(ii) $F_\infty^{seq} = F$ *whenever* F *is sequentially* σ*-lsc and positively homogeneous of degree* 1.

(iii) $F_\infty^{seq} = F$ *whenever* F *is proper, convex, and sequentially* σ*-lsc.*

(iv) $(F+G)_\infty^{seq}(v) \geq F_\infty^{seq}(v) + G_\infty^{seq}(v)$ *for every mapping* $G : V \to]-\infty, +\infty]$ *and for every* $v \in V$ *such that the sum at the right-hand side is defined.*

(v) *The equality* $(F+G)_\infty^{seq}(v) = F_\infty^{seq}(v) + G_\infty^{seq}(v)$ *holds in the following cases:*

(v_a) F *and* G *are proper, convex, sequentially* σ*-lsc, and* $\mathrm{dom}\, F \cap \mathrm{dom}\, G \neq \emptyset$*;*

(v_b) G *is positively homogeneous of degree* 1 *and sequentially* σ*-continuous;*

(v_c) F *is convex and sequentially* σ*-lsc,* $F(0) < +\infty$*,* G *is sequentially* σ*-lsc and positively homogeneous of degree* 1.

Remark 15.2.3. We point out that in general the functional F_∞^{seq} is neither σ-lsc nor sequentially σ-lsc. However, this simpler definition will be sufficient to obtain an existence theorem for minimizers (Theorem 15.2.1) which will be used in many applications. We stress the fact that the explicit computation of F_∞^{seq} may be difficult; nevertheless, to apply the existence result of Theorem 15.2.1, in many cases it will be enough to show qualitative properties of F_∞^{seq} which are easy to obtain thanks to Proposition 15.2.3.

In an analogous way, for every nonempty subset K of V we may introduce the sequential recession cone K_∞^{seq} by setting

$$K_\infty^{seq} = \mathrm{dom}(\chi_K)_\infty^{seq}. \tag{15.43}$$

In other words, it is

$$x \in K_\infty^{seq} \quad \Longleftrightarrow \quad \exists t_h \to +\infty \quad \exists x_h \to x \quad \forall h \in \mathbf{N} \quad t_h x_h \in K.$$

The set K_∞^{seq} turns out to be a cone (possibly not sequentially σ-closed) with vertex 0 and, by a proof similar to the one of Proposition 15.2.2, we obtain the following properties.

Proposition 15.2.4. *We have the following:*

(i) $K_\infty^{seq} = K$ *if* K *is a sequentially* σ*-closed cone.*

(ii) $K_\infty^{seq} = K^\infty$ *if* K *is a nonempty sequentially* σ*-closed set.*

(iii) $K_\infty^{seq} = \{0\}$ *if* K *is bounded.*

(iv) $K_\infty^{seq} = \{0\}$ *implies* K *bounded if* V *is finite dimensional.*

(v) $K_\infty^{seq} = (K \cup H)_\infty^{seq} = (K + H)_\infty^{seq}$ *if* $H \subset V$ *is bounded.*

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The following result provides a general necessary condition for the existence of minimizers.

Proposition 15.2.5. *Assume that*

$$\inf\{F(v) : v \in V\} > -\infty.$$

Then we have

$$F_\infty(v) \geq 0 \ \ \forall v \in V \ \ \textit{(hence } F_\infty^{seq} \geq 0, \textit{ because } F_\infty \leq F_\infty^{seq}\textit{).}$$

PROOF. Let m be the infimum of F on V, and let $v \in V$. By the definition of F_∞ we get

$$F_\infty(v) = \liminf_{\substack{t \to +\infty \\ w \to v}} \frac{F(tw)}{t} \geq \liminf_{t \to +\infty} \frac{m}{t} = 0. \qquad \square$$

We recall that even in the convex case, the condition $F^\infty \geq 0$ is not sufficient for the existence of a solution of problem

$$\min\{F(v) : v \in V\} \tag{15.44}$$

(see, for instance, Example 15.1.9). To show an existence result for problem (15.44), analogously to what was done in Section 15.1 we set

$$\ker F_\infty^{seq} = \{v \in V : F_\infty^{seq}(v) = 0\}.$$

Theorem 15.2.1. *Assume V is a Banach space with norm $||\cdot||$, let σ be a topology on V coarser than the norm topology and such that (V,σ) is a Hausdorff vector space with the closed unit ball of $(V, ||\cdot||)$ sequentially σ-compact, and let $F : V \to]-\infty, +\infty]$ be a sequentially σ-lower semicontinuous functional. Assume also that the following conditions are satisfied:*

(i) compactness: *if $t_h \to +\infty$, $v_h \to v$ with respect to σ, and $F(t_h v_h)$ is bounded from above, then $||v_h - v|| \to 0$;*

(ii) necessary condition: $F_\infty^{seq}(v) \geq 0$ *for every $v \in V$;*

(iii) compatibility: *for every $u \in \ker F_\infty^{seq}$ there exists $t > 0$ such that $F(v - tu) \leq F(v)$ for all $v \in V$.*

Then the minimum problem (15.44) *has at least a solution.*

PROOF. The proof is similar to the one of Theorem 15.1, and we will follow it step by step.

Step 1. For every $h \in \mathbf{N}$ let v_h be a solution of the minimum problem

$$\min\{F(v) : ||v|| \leq h\}. \tag{$\wp_h$}$$

Since F is sequentially σ-lsc and $\{v \in V : ||v|| \leq h\}$ is sequentially σ-compact, by the direct method of the calculus of variations (see Corollary 3.2.3) we obtain that there exists a solution v_h of problem $(\wp_h)$. Moreover, by the lower semicontinuity of F and by the assumptions on σ, we may choose v_h such that

$$||v_h|| = \min\{||w|| : w \text{ solves } (\wp_h)\}. \tag{15.45}$$

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Step 2. If the sequence (v_h) is bounded in norm, then by the lower semicontinuity of F and by the σ-compactness of bounded sets, the existence of a solution of problem (15.44) follows from the direct method of the calculus of variations. Indeed, if (v_{h_k}) is a subsequence of (v_h) which is σ-convergent to some $\overline{v} \in V$, we have

$$F(\overline{v}) \leq \liminf_{k\to+\infty} F(v_{h_k}) = \inf\{F(v) \,:\, v \in V\}.$$

Step 3. It remains to show that the case (v_h) unbounded cannot occur. By contradiction, assume that a subsequence of $||v_h||$ (which we still index by h) tends to $+\infty$. Since the normalized vectors $w_h = v_h/||v_h||$ are bounded, there exists a subsequence of (w_h) (which we still index by h) which σ-converges to some $w \in V$. We have

$$F(v_{h+1}) \leq F(v_h) \quad \forall h \in \mathbf{N},$$

so that $F(v_h)$ is bounded from above. Hence

$$F^{seq}_\infty(w) \leq \liminf_{h\to+\infty} \frac{F(||v_h||w_h)}{||v_h||} = \liminf_{h\to+\infty} \frac{F(v_h)}{||v_h||} \leq 0,$$

which, together with the necessary condition (ii), gives

$$w \in \ker F^{seq}_\infty. \tag{15.46}$$

Step 4. We have $||v_h|| \to +\infty$, $w_h \to w$ with respect to σ, and $F(||v_h||w_h) = F(v_h)$ bounded from above. By the compactness assumption (i) we obtain $||w_h - w|| \to 0$, and this prevents w from being zero, because $||w_h|| = 1$ for all $h \in \mathbf{N}$. From (15.46) and from the compatibility condition (iii) we get that there exists $t > 0$ such that

$$F(v_h - tw) \leq F(v_h) \quad \forall h \in \mathbf{N}. \tag{15.47}$$

Finally,

$$\begin{aligned}
||v_h - tw|| &= \left\| \left(1 - \frac{t}{||v_h||}\right) v_h + t(w_h - w) \right\| \\
&\leq \left(1 - \frac{t}{||v_h||}\right) ||v_h|| + t||w_h - w|| \\
&= ||v_h|| + t\big(||w_h - w|| - 1\big).
\end{aligned}$$

The right-hand side of the last equality is strictly less than $||v_h||$ for h large enough, and this is in contradiction with (15.47) and (15.45). $\square$

Remark 15.2.4. Looking at the proof of Theorem 15.2.1 we see that the compactness condition (i) can be imposed only on sequences (v_h) which σ-converge to an element $v \in \ker F^{seq}_\infty$.

Remark 15.2.5. In Theorem 15.2.1 a weaker form of the compatibility condition (iii) can be used, namely,

> for every $u \in \ker F^{seq}_\infty$ there exists $R > 0$ such that for every $v \in V$ with $||v|| \geq R$ there exists $t > 0$ such that $F(v - tu) \leq F(v)$.

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Moreover, an inspection of the proof of Theorem 15.2.1 shows that in the case of a product space $V_1 \times \cdots \times V_n$ the compatibility condition (iii) can be replaced by the following weaker one:

for every $(u_1,\dots,u_n) \in \ker F_\infty^{seq}$ there exist positive numbers $t_1,\dots,t_n$ such that

$$F(v_1 - t_1 u_1,\dots,v_n - t_n u_n) \le F(v_1,\dots,v_n) \qquad \forall\, (v_1,\dots,v_n) \in V_1 \times \cdots \times V_n.$$

Remark 15.2.6. Theorem 15.2.1 includes in a certain sense the classical direct method of the calculus of variations, which gives the existence of a solution of problem (15.44) under the following assumptions:

(i) F is sequentially σ-lsc.

(ii) There exist $\alpha > 0$ and $b \in \mathbf{R}$ such that

$$F(v) \ge \alpha\|v\| + b \quad \forall v \in V.$$

Indeed, by (ii) we obtain

$$t_h \to +\infty,\ v_h \to v,\ F(t_h v_h) \le c \quad \Rightarrow \quad \|v_h\| \to 0 \text{ and } v = 0,$$

so that the compactness hypothesis (i) is satisfied. Moreover, by (iii) again we get

$$F_\infty^{seq}(v) \ge \alpha\|v\| \quad \forall v \in V,$$

so that $\ker F_\infty^{seq}$ reduces to $\{0\}$. Hence, hypotheses (ii) and (iii) of Theorem 15.2.1 are also satisfied, and problem (15.44) admits at least a solution. In particular (ii) holds if $\operatorname{dom} F$ is bounded.

As in Section 15.1, we now specialize our results to the case when the functional F is of the form

$$F(v) = J(v) - \langle L, v\rangle + \chi_K(v),$$

where

$J : V \to [0,+\infty]$ is a sequentially σ-lsc functional;
$L : V \to \mathbf{R}$ is a linear σ-continuous functional;
$K \subset V$ is a sequentially σ-closed set.

Taking into account Propositions 15.39 and 15.40, we have that a necessary condition for the existence of a solution of the minimum problem

$$\min\left\{J(v) - \langle L, v\rangle \;:\; v \in K\right\}$$

is given by

$$J_\infty^{seq}(v) \ge \langle L, v\rangle \quad \forall v \in K, \tag{15.48}$$

whereas the compatibility condition (iii) can be written as follows:

(iii′) for every $u \in K_\infty^{seq}$ with $J_\infty^{seq}(u) = \langle L, u\rangle$, there exists $t > 0$ such that for every $v \in K$

$$\begin{cases} v - t u \in K, \\ J(v - t u) + t\langle L, u\rangle \le J(v). \end{cases}$$

In many situations the functional J has a “superlinear growth” so that the functional J_∞^{seq} reduces to

$$J_\infty^{seq}(v)=\begin{cases}0 & \text{if } v\in\ker J_\infty^{seq},\\ +\infty & \text{otherwise;}\end{cases}$$

in this case the necessary condition (ii) becomes

(ii′) $\langle L,v\rangle\le 0$ for all $v\in K_\infty^{seq}\cap\ker J_\infty^{seq}$,

whereas the compatibility condition (iii) is given by (iii′) with $u\in K_\infty^{seq}\cap\ker J_\infty^{seq}\cap\ker L$.

15.3 ▪ Some examples

In this section we show some examples that can be treated with the theory of noncoercive minimum problems developed in the previous sections of the chapter. Some other cases with applications to problems from mechanics were presented in Chapter 14.

Consider the following minimization problem:

$$\min\left\{\int_\Omega\left(\frac{1}{2}|Dv|^2+B(v)\right)dx-\langle L,v\rangle:\quad u\in H^1(\Omega)\right\},\tag{15.49}$$

where Ω is a bounded connected open subset of $\mathbf{R}^n$, L is in the dual space of $H^1(\Omega)$, and $B:\mathbf{R}\to]-\infty,+\infty]$ is a convex lower semicontinuous function.

Proposition 15.3.1. *For the existence of a solution of problem* (15.49) *the assumption*

$$-B^\infty(-1)\le\frac{\langle L,1\rangle}{\text{meas}(\Omega)}\le B^\infty(1)\tag{15.50}$$

is necessary, while

$$-B^\infty(-1)<\frac{\langle L,1\rangle}{\text{meas}(\Omega)}<B^\infty(1)\tag{15.51}$$

is sufficient. The case $-B^\infty(-1)=B^\infty(1)$ *corresponds to an affine function B for which the condition* $B^\infty(1)\,\text{meas}(\Omega)=\langle L,1\rangle$ *is necessary and sufficient for the existence.*

PROOF. It is immediate to see that the functional

$$F(u)=\int_\Omega\left(\frac{1}{2}|Dv|^2+B(v)\right)dx-\langle L,v\rangle$$

is sequentially weakly lower semicontinuous on $H^1(\Omega)$. To verify the compactness property (i) of Theorem 15.1.1 we notice that, the function B being convex and lower semicontinuous, we have for suitable $a,b\in\mathbf{R}$

$$B(s)\ge as+b\qquad\forall s\in\mathbf{R}.\tag{15.52}$$

Therefore, if $t_h\to+\infty$, $v_h\to v$ weakly, and $F(t_hv_h)$ is bounded, we obtain from (15.52)

$$\int_\Omega t_h^2|Dv_h|^2\,dx-ct_h||v_h||_{H^1}\le c$$

for a suitable positive constant c. Therefore, dividing by t_h^2, we obtain that Dv_h strongly converges to zero in $L^2(\Omega)$ and hence v_h strongly converges in $H^1(\Omega)$ to a constant function.

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The computation of the recession functional F^∞ gives for every $v \in H^1(\Omega)$

$$F^\infty(v) = \begin{cases} \int_\Omega B^\infty(v) - \langle L, v\rangle & \text{if } v \text{ is constant,} \\ +\infty & \text{otherwise.} \end{cases}$$

Therefore, the necessity of condition (15.50) follows from Proposition 15.1.2.

To apply the existence Theorem 15.1.1 it remains to prove the compatibility condition (iii). By the above expression of F^∞, we have that $v \in \ker F^\infty$ iff v is constant and

$$\operatorname{meas}(\Omega) B^\infty(v) = v\langle L, 1\rangle.$$

Thus (iii) is a consequence of assumption (15.51). □

Remark 15.3.1. Consider the partial differential equation

$$\begin{cases} -\Delta u + b(u) = h(x) & \text{in } \Omega, \\ \partial u/\partial \nu = g & \text{on } \partial\Omega, \end{cases}$$

where b is the maximal monotone graph $b = \partial B$. It is well known that this problem is equivalent to the minimization problem (15.49) via the Euler–Lagrange equation, where $L = h + g$. Then, it is easy to see that condition (15.50) can be expressed by saying that $\frac{1}{\operatorname{meas}(\Omega)}\langle L, 1\rangle$ belongs to the closure of the range of b, while condition (15.51) can be expressed by saying that $\frac{1}{\operatorname{meas}(\Omega)}\langle L, 1\rangle$ belongs to the algebraic interior of the range of b.

Example 15.3.1. We particularize the discussion above to the case of the differential equation

$$\begin{cases} -\Delta u + u^+ = h & \text{in } \Omega, \\ \partial u/\partial \nu = g & \text{on } \partial\Omega, \end{cases} \tag{15.53}$$

where $h \in H^{-1}(\Omega)$ and $g \in H^{-1/2}(\partial\Omega)$. A simple calculation shows that the functional whose Euler–Lagrange equation is (15.53) is

$$F(v) = \int_\Omega \left(\frac{1}{2}|Dv|^2 + |v^+|^2\right) dx - \langle L, v\rangle, \qquad v \in H^1(\Omega), \tag{15.54}$$

where $L = h + g$, and, due to convexity of the functional F, solving (15.53) is equivalent to minimizing (15.54). As a consequence of Proposition 15.3.1 we obtain immediately that

(i) if $\langle L, 1\rangle < 0$, then no solution of (15.53) exists;

(ii) if $\langle L, 1\rangle > 0$, then (15.53) has a solution.

It remains to consider the case

$$\langle L, 1\rangle = 0. \tag{15.55}$$

If h is uniformly continuous and $g = 0$, a proof of existence can be found in [139]. Note that in this case such regularity assumptions imply that the solution w of the associated linearized problem

$$\begin{cases} -\Delta w = h & \text{in } \Omega, \\ \partial w/\partial \nu = g & \text{on } \partial\Omega \end{cases} \tag{15.56}$$

is essentially bounded. More generally, the following proposition holds.

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Proposition 15.3.2. *Let $h \in H^{-1}(\Omega)$ and $g \in H^{-1/2}(\partial\Omega)$ satisfy* (15.55). *Then the following conditions are equivalent:*

(i) *there exists a solution of problem* (15.53)*;*

(ii) *the linearized problem* (15.56) *admits a negative solution.*

Moreover, any solution u of (15.53) *solves* (15.56) *and $u^+ = 0$.*

PROOF. Assume (i) holds. If u solves (15.53) and $u^+ \neq 0$, for every constant $c > 0$ we have

$$F(u-c) < F(u),$$

which contradicts the fact that u is a minimum point of the functional F. Hence $u^+ = 0$ and u solves (15.56), that is, u is a negative solution of (15.56). Assume now (ii). Let w be a negative solution of (15.56); then, w obviously solves (15.53). □

Remark 15.3.2. For instance, by well-known regularity results for solutions of elliptic partial differential equations, condition (ii) of Proposition 15.3.2 is satisfied if $\partial\Omega$ is smooth, $h \in L^p(\Omega)$, and $g \in W^{1-1/p,p}(\partial\Omega)$ with $p > n/2$.

Remark 15.3.3. The argument used in the proof of Proposition 15.3.2 applies also to problems of the form

$$\min\left\{ \int_\Omega \left(\frac{1}{2}|Dv|^2 + B(v^+) \right) dx - \langle L, v\rangle : \quad v \in H^1(\Omega) \right\} \tag{15.57}$$

for any convex and strictly increasing $B : \mathbf{R}^+ \to \mathbf{R}^+$. In this case the associated Euler–Lagrange equation reads as

$$\begin{cases} -\Delta u + \partial B(u^+) \ni h & \text{in } \Omega, \\ \partial u/\partial \nu = g & \text{on } \partial\Omega, \end{cases} \tag{15.58}$$

where $L = h + g$ and ∂B is the subdifferential of the convex function B, whereas the associated linearized equation coincides with (15.56).

Remark 15.3.4. Consider the obstacle (from below) problem (see also [223])

$$\min\left\{ \int_\Omega \frac{1}{2}|Dv|^2\, dx - \langle L, v\rangle : \quad v \in H^1(\Omega),\ v \le 0 \right\}. \tag{15.59}$$

By introducing the function

$$B(s) = \begin{cases} 0 & \text{if } t \le 0, \\ +\infty & \text{if } t > 0, \end{cases}$$

problem (15.59) can be written in the form (15.58) and the argument above works in the same way. The Euler–Lagrange equation is usually written in this case as a *variational inequality:*

$$u \in H^1(\Omega),\ u \le 0 \qquad \int_\Omega DuD(v-u)\, dx - \langle L, v-u\rangle \ge 0 \quad \forall v \in H^1(\Omega),\ v \le 0.$$

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Consider now, in a Hilbert space V, a linear continuous functional $L : V \to \mathbf{R}$, a closed convex subset $K \subset V$, and a symmetric continuous bilinear form $a : V \times V \to \mathbf{R}$ which we assume to be nonnegative, that is,

$$a(v,v) \geq 0 \qquad \forall v \in V.$$

If F denotes the functional

$$F(v) = \frac{1}{2}a(v,v) - (L,v) \qquad \forall v \in V,$$

we may consider the minimum problem

$$\min\{F(v) \,:\, v \in K\}. \tag{15.60}$$

Proposition 15.3.3. *The following conditions are equivalent:*

(i) *There exists $u \in K$ such that $F(u) \leq F(v)$ for every $v \in K$.*

(ii) *There exists $u \in K$ such that $a(u,u-v) \leq (L,u-v)$ for every $v \in K$.*

PROOF. Assume (i). Then, since K is convex, for every $v \in V$ we have $tv + (1-t)u \in K$ and the map

$$g_v(t) = F\big(tv + (1-t)u\big), \qquad t \in [0,1],$$

achieves its minimum at $t = 0$, hence

$$0 \leq g_v'(0) = -a(u,u) + a(u,v) - (L,v) + (L,u),$$

that is, (ii).

Assume now (ii). We have to prove that the map g_v above achieves its minimum at $t = 0$. Since

$$g_v''(t) = a(v,v) + a(u,u) - 2a(u,v) \geq 0,$$

the function $g_v(t)$ is convex on $[0,1]$. Therefore it is enough to show that $g_v'(0) \geq 0$, which holds true because $g_v'(0) = -a(u,u) + a(u,v) - (L,v) + (L,u)$. ☐

Remark 15.3.5. Since the map $v \mapsto a(v,v)$ is quadratic, the recession functional associated to F is given by

$$F^\infty(v) = \begin{cases} -(L,v) & \text{if } a(v,v) = 0, \\ +\infty & \text{otherwise;} \end{cases}$$

hence (see Theorem 15.1.2), so that a solution of (15.60) exists, a necessary condition is

$$(L,v) \leq 0 \qquad \forall v \in K^\infty \cap \ker a,$$

whereas, if a satisfies the compactness condition

$$v_h \to 0 \text{ weakly},\ a(v_h,v_h) \to 0 \quad \to \quad v_h \to 0 \text{ strongly},$$

a sufficient one is

$$K^\infty \cap \ker a \cap \ker L \quad \text{is a subspace.}$$

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Problem (15.60) written in the form (ii) above is called a variational inequality. For instance, the obstacle problems are of this type, where the Hilbert space is $H^1(\Omega)$ and the convex set K is

$$K = \{v \in H^1(\Omega) \,:\, v \geq \psi \text{ q.e. in } \overline{\Omega}\},$$

where q.e. is intended in the sense of capacity (see Section 5.8). Let us consider more particularly this situation in the case the bilinear form a is given by means of an elliptic operator,

$$a(u,v) = \int_\Omega \sum_{i,j=1}^n a_{ij}(x) D_i u D_j u \, dx,$$

whose coefficient a_{ij} are symmetric and measurable and satisfy the ellipticity condition

$$c_1|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq c_2|\xi|^2 \qquad \forall \xi \in \mathbf{R}^n.$$

We denote by A the corresponding elliptic operator

$$Au = -\sum_{i,j=1}^n D_i\big(a_{ij}(x) D_j u\big).$$

Proposition 15.3.4. *The obstacle problem* (15.60), *equivalent to the variational inequality* (ii) *of Proposition* 15.3.3, *is also equivalent to the complementary problem*

$$\begin{cases} u - \psi \geq 0, \\ Au - L \geq 0, \\ \langle Au - L, u - \psi \rangle = 0. \end{cases} \tag{15.61}$$

PROOF. Let u be a solution of the obstacle problem (15.60); by formulation (ii) of Proposition 15.3.3, for every nonnegative function $\phi \in H^1(\Omega)$, taking $v = u + \phi$, we obtain

$$\langle Au, \phi \rangle \geq \langle L, \phi \rangle,$$

that is, $Au - L \geq 0$. On the other hand, taking $v = \psi$ we have

$$0 \leq \langle Au - L, u - \psi \rangle = \langle Au, u - \psi \rangle - \langle L, u - \psi \rangle \leq 0,$$

that is, $\langle Au - L, u - \psi \rangle = 0$.

On the other hand, if u verifies (15.61), writing a generic $v \geq \psi$ in the form $\psi + \phi$ with $\phi \geq 0$, we have

$$\langle Au - L, u - v \rangle = \langle Au - L, u - \psi \rangle - \langle Au - L, \phi \rangle \leq 0,$$

so that by Proposition 15.3.3, u solves the obstacle problem. □

Remark 15.3.6. Since $Au - L$ is a nonnegative distribution, by the Riesz–Schwartz theorem there exists a nonnegative Borel measure μ on Ω such that $Au - L = \mu$. Moreover, by (15.61) the measure μ is concentrated on the *coincidence set* $\{u = \psi\}$. For further details on the obstacle problems see, for instance, the book by Kinderlehrer and Stampacchia [258]. Here we want only to remark that the obstacle problem is a mathematical model for determining the shape of a thin elastic membrane subject to a vertical load L and where

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the unknown u represents the vertical displacement of the membrane. The measure μ in this framework has a natural interpretation as the upward force due to the constraint reaction of the rigid obstacle.

As a further example we consider now the case of minimum problems of the form

$$\min\left\{\int_0^T\left[\frac{1}{2}|u'|^2+V(t,u)\right]dt\ :\ u\in H^1(0,T;\mathbf{R}^n),\ u(0)=u(T)\right\},\tag{15.62}$$

where V is a Borel function, with $V(t,\cdot)$ lower semicontinuous on $\mathbf{R}^n$, and such that

$$V(t,s)\geq -a(t)-b(t)|s|^q\qquad\text{for a.e. } t\in(0,T)\text{ and for every } s\in\mathbf{R}^n\tag{15.63}$$

for suitable $q<2$ and $a(t)$, $b(t)$ in $L^1(0,T)$. If $V(t,\cdot)$ is smooth the minimum problem above has the Euler–Lagrange equation

$$\begin{cases}-u''+\nabla V(t,u)=0,\\ u(0)=u(T),\ u'(0)=u'(T).\end{cases}\tag{15.64}$$

We are therefore looking for solutions of the differential equations $-u''+\nabla V(t,u)=0$ which are periodic on the interval $[0,T]$. It is convenient to denote by H^1_T the space $\{u\in H^1(0,T;\mathbf{R}^n)\ :\ u(0)=u(T)\}$ and by F the functional

$$F(u)=\int_0^T\left[\frac{1}{2}|u'|^2+V(t,u)\right]dt.$$

Notice that in general the functional F is not convex. Nevertheless the sequential weak lower semicontinuity of F is straightforward. Let us prove now the compactness property (i) of Theorem 15.2.1. If $t_h\to+\infty$ and $u_h\to u$ weakly in H^1_T, with $F(t_hu_h)$ bounded from above, then we deduce by (15.63) that $\int_0^T|u_h'|^2\,dt\to 0$, and this implies immediately the strong convergence of the sequence $\{u_h\}$ to a constant.

To prove the necessary condition $F_\infty(u)\geq 0$ for every $u\in H^1_T$, we notice that it is trivially fulfilled when u is a nonconstant function, because in this case we have $F_\infty(u)=+\infty$. Indeed, if $t_h\to+\infty$ and $u_h\to u$ weakly in H^1_T with u nonconstant, we have

$$\liminf_{h\to+\infty}\int_0^T|u_h'|^2\,dt\geq\int_0^T|u'|^2\,dt>0.$$

Therefore, by (15.63) and using the fact that $q<2$, we obtain

$$\liminf_{h\to+\infty}\frac{F(t_hu_h)}{t_h}\geq\liminf_{h\to+\infty}\int_0^T\left[\frac{t_h}{2}|u_h'|^2-b(t)t_h^{q-1}|u_h|^q\right]dt=+\infty.$$

The necessary condition $F_\infty(u)\geq 0$ is then reduced to

$$F_\infty(c)\geq 0\qquad\text{for every } c\in\mathbf{R}^n.\tag{15.65}$$

Here we assume (15.65) is satisfied and we will particularize some special cases when additional assumptions on the potential V are made.

When $V(t,\cdot)$ is convex on $\mathbf{R}^n$, then the functional F turns out to be convex, and we are in the framework of Section 13.1. Assume there exists a function $u_0\in H^1_T$ such that

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$V\big(t,u_0(t)\big)$ is integrable and introduce for every $c\in\mathbf{R}^n$ the convex function $\Phi_c:\mathbf{R}\to\,]-\infty,+\infty]$ given by

$$\Phi_c(r)=\int_0^T V\big(t,u_0(t)+cr\big)\,dt.$$

Then by applying Theorem 15.1.1 it is easy to see that the existence of at least a solution to problem (15.62) occurs provided for every $c\in\mathbf{R}^n$ the function Φ_c is either constant or such that

$$\lim_{|r|\to+\infty}\Phi_c(r)=+\infty. \tag{15.66}$$

For instance, if

$$V(t,s)=V(s)-f(t)s$$

we have that condition (15.66) is fulfilled for every $f\in L^1(0,T;\mathbf{R}^n)$ whenever the function $V(s)$ has a superlinear growth, that is,

$$\lim_{|s|\to+\infty}\frac{V(s)}{|s|}=+\infty.$$

Another case in which the existence of minimizers for problem (15.62) can be easily obtained is when the potential V satisfies a Lipschitz condition of the form

$$|V(t,s_1)-V(t,s_2)|\le k(t)|s_1-s_2| \qquad \text{for a.e. } t\in(0,T) \text{ and for every } s_1.s_2\in\mathbf{R}^n. \tag{15.67}$$

Here we assume that $k\in L^1(0,T)$, and that the potential V satisfies the coercivity condition

$$\lim_{|s|\to+\infty}\int_0^T V(t,s)\,dt=+\infty. \tag{15.68}$$

In this case problem (15.62) is actually coercive, and the existence result then follows immediately from the direct methods of the calculus of variations of Section 3.2. Indeed, by using the Lipschitz condition (15.67) we obtain for every $u\in H_T^1$

$$F(u)\ge\int_0^T\left[\frac{1}{2}|u'|^2+V\big(t,u(0)\big)\right]dt-\int_0^T k(t)|u(t)-u(0)|\,dt.$$

Moreover, since

$$|u(t)-u(0)|=\left|\int_0^t u'(s)\,ds\right|\le\left(T\int_0^T|u'|^2\,dt\right)^{1/2},$$

we deduce that

$$F(u)\ge\int_0^T\left[\frac{1}{2}|u'|^2+V\big(t,u(0)\big)\right]dt-C\left(\int_0^T|u'|^2\,dt\right)^{1/2}$$

for a suitable positive constant C. Now, by using the coercivity assumption (15.68), and the fact that on H_T^1 the norm is equivalent to

$$\left(\int_0^T|u'|^2\,dt+|u(0)|^2\right)^{1/2},$$

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we obtain that $\|u\|_{H^1_T} \to +\infty$ yields $F(u) \to +\infty$, hence the coercivity of F. The lower semicontinuity of F is a straightforward consequence of the results of Section 13.1.

Consider now the case when the function $V(t,\cdot)$ is periodic. More precisely, we assume that the function V is nonnegative, or more generally bounded from below by an L^1 function, and that for suitable independent vectors $\tau_i \in \mathbf{R}^n$ we have for a.e. $t \in (0,T)$ and for every $s \in \mathbf{R}^n$

$$V(t, s+\tau_i) = V(t,s), \qquad i = 1,\dots,n.$$

Note that in this case the functional F is not convex in general and that, by the nonnegativity of V, the necessary condition (15.65) is clearly fulfilled. Thus, to apply Theorem 15.2.1 and obtain the existence of a minimizer for F, it is enough to show that the compatibility condition (iii′) of Remark 15.2.5 holds. In other words we have to find for every $c \in \mathbf{R}^n$ a vector $\mu \in \mathbf{R}^n$ whose components μ_i are positive, such that

$$F(u_1 - \mu_1 c_1, \dots, u_n - \mu_n c_n) \le F(u_1,\dots,u_n) \qquad \forall u \in H^1_T.$$

This can be achieved if we choose, for instance, $\mu_i = |\tau_i|/|c_i|$ for all $i = 1,\dots,n$ with $\mu_i = 1$ if $c_i = 0$.

15.4 ▪ Limit analysis problems

We consider the so-called limit analysis problems which consist in minimizing functionals of the form

$$F(x) - \gamma L(x), \quad x \in X,$$

where X is a normed space, $F : X \to]-\infty, +\infty]$ is a (possibly nonconvex) functional, and $L \in X'$. We are interested in characterizing the values of γ for which the minimum is attained. As an application we consider nonconvex minimum problems defined on the space of measures and on the BV space.

Let X be a normed space; we consider on X the norm topology τ and another (linear Hausdorff) topology σ weaker than τ and such that the unit ball $\{x \in X \ : \ \|x\| \le 1\}$ is σ-compact. Let $F : X \to]-\infty, +\infty]$ be a functional proper and σ-lsc (on all τ-bounded sets), and let $L : X \to \mathbf{R}$ be a linear σ-continuous functional. The limit analysis problem associated to F and L consists in finding the values $\gamma \in \mathbf{R}$ for which the minimum problem

$$\min\{F(x) - \gamma L(x) : x \in X\} \tag{15.69}$$

admits at least a solution. The problem when F is convex was studied by Bouchitté and Suquet in [118], where the following result is proved.

Theorem 15.4.1. *Assume that F is proper, convex, σ-lsc, and σ-coercive in the sense that*

$$\lim_{\|x\| \to +\infty} F(x) = +\infty.$$

Then, setting

$$\gamma^* = \min\{F^\infty(x) \ : \ x \in X,\ L(x) = 1\},$$
$$\gamma_* = -\min\{F^\infty(x) \ : \ x \in X,\ L(x) = -1\},$$

the following statements are equivalent:

(i) $F-\gamma L$ *is* σ*-coercive.*

(ii) $\gamma_* < \gamma < \gamma^*$.

PROOF. Since F is proper, there exists $x_0 \in X$ such that $F(x_0) < +\infty$; then, by considering the functional $F(x+x_0)-F(x_0)$ it is easy to see that we may reduce ourselves to assume without any loss of generality that $F(0)=0$.

Assume now (i). By the properties of recession functions we have

$$(F-\gamma L)^\infty(x) \geq \frac{F(tx)-\gamma L(tx)}{t} \qquad \forall t>0,\ \forall x \in X;$$

if for some $x \neq 0$ we had $(F-\gamma L)^\infty(x) \leq 0$, this would be in contradiction to the coerciveness of $F-\gamma L$. Then, taking as x^* the solution of

$$\gamma^* = \min\left\{F^\infty(x) \,:\, L(x)=1\right\},$$

we obtain

$$0 < (F-\gamma L)^\infty(x^*) = F^\infty(x^*) - \gamma L(x^*) = \gamma^* - \gamma.$$

Analogously, if x_* is the solution of

$$-\gamma_* = \min\left\{F^\infty(x) \,:\, L(x)=-1\right\},$$

we get

$$0 < (F-\gamma L)^\infty(x_*) = F^\infty(x_*) - \gamma L(x_*) = -\gamma^* + \gamma,$$

that is, (ii).

Assume now (ii) and let $0<\gamma<\gamma^*$. By contradiction, if $F-\gamma L$ were not coercive, we could find a sequence (x_h) such that $||x_h|| \to +\infty$ and

$$(F-\gamma L)(x_h) \leq M \tag{15.70}$$

for a suitable $M \in \mathbf{R}$. Note that $L(x_h) \to +\infty$ because otherwise (15.70) and the coerciveness of F would prove that (x_h) is bounded. Setting $y_h = x_h/||x_h||$ and $t_h = ||x_h||$ we get that (y_h) is σ-relatively compact, hence (up to subsequences) converging to a suitable $y \in X$. By using the lower semicontinuity and convexity of F, and the fact that $F(0)=0$, we have for every $t>0$

$$\begin{aligned}\frac{F(ty)}{t} &\leq \liminf_{h\to+\infty} \frac{F(ty_h)}{t} \leq \liminf_{h\to+\infty} \frac{F(t_h y_h)}{t_h} \\ &\leq \liminf_{h\to+\infty} \gamma \frac{L(x_h)}{||x_h||} = \gamma L(y)\end{aligned}$$

so that, as $t \to +\infty$,

$$F^\infty(y) \leq \gamma L(y). \tag{15.71}$$

Being F lower semicontinuous and σ-coercive, by Proposition 15.1.3 we have $F^\infty \geq 0$, hence $L(y) \geq 0$ by (15.71). The case $L(y)>0$ must be excluded because, taking $z = y/L(y)$ we get $L(z)=1$ and

$$\gamma^* \leq F^\infty(z) = \frac{F^\infty(y)}{L(y)} \leq \gamma,$$

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which contradicts (ii). Therefore $L(y)=0$ and, by (15.71), $F^\infty(y)=0$. Arguing as in the first part of the proof we get that by the coerciveness of F, this implies $y=0$, so that using (15.70) and recalling that $L(x_h)\to+\infty$,

$$F\left(\frac{x_h}{L(x_h)}\right)\le\frac{F(x_h)}{L(x_h)}\le\frac{M}{L(x_h)}+\gamma.$$

Hence $x_h/L(x_h)$ is bounded in X (by the coerciveness of F), and so $y_h/L(y_h)$ is bounded too. But this contradicts the fact that $\|y_h\|=1$ whereas $L(y_h)\to 0$. An analogous argument can be used in the case $\gamma_*<\gamma<0$. □

When F is not necessarily convex we have the following result.

Theorem 15.4.2. *Assume that*

(i) *F is proper and sequentially σ-lsc;*

(ii) *for all $z\in\ker F_\infty$ there exists $\eta=\eta(z)>0$ such that*

$$F(x-\eta z)\le F(x)\qquad\forall x\in X;$$

(iii) *there exist a seminorm $P:X\to[0,+\infty[$ satisfying the compactness condition* (2.3.11) *and a number $C\ge 0$ such that*

$$F(x)\ge P(x)-C\qquad\forall x\in X.$$

Then, setting

$$\gamma^*=\inf\{F_\infty(x)\ :\ L(x)=1\},$$
$$\gamma_*=-\inf\{F_\infty(x)\ :\ L(x)=-1\},$$

we have the following:

(a) *if the minimum problem* (15.69) *admits a solution, then $\gamma_*\le\gamma\le\gamma^*$;*

(b) *if $\gamma_*<\gamma<\gamma^*$, then the minimum problem* (15.69) *admits a solution.*

PROOF. To prove statement (a) we apply Proposition 15.2.5. The necessary condition for the existence of a solution to problem (15.69) is

$$(F-\gamma L)_\infty\ge 0\qquad\text{on } X,$$

which, thanks to Proposition 15.2.1, becomes

$$F_\infty(x)\ge\gamma L(x)\qquad\forall x\in X. \tag{15.72}$$

Taking $L(x)=-1$ in (15.72) gives

$$\gamma\ge\sup\left\{-F_\infty(x)\ :\ L(x)=-1\right\}=\gamma_*;$$

similarly, taking $L(x)=1$ in (15.72) gives

$$\gamma\le\inf\left\{F_\infty(x)\ :\ L(x)=1\right\}=\gamma^*.$$

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To prove statement (b) we are going to verify all the hypotheses of Theorem 15.2.1 for the functional $G = F - \gamma L$ which is clearly sequentially σ-lsc. As seen in the proof of statement (a), the necessary condition (ii) of Theorem 15.2.1 is equivalent to inequalities

$$\gamma_* \le \gamma \le \gamma^*.$$

To verify the compactness hypothesis (i) of Theorem 15.2.1, let $t_h \to +\infty$ and let $x_h \to x$ be a sequence σ-converging in X such that

$$F(t_h x_h) - \gamma L(t_h x_h) \le C. \tag{15.73}$$

Dividing by t_h we get

$$\frac{F(t_h x_h)}{t_h} - \gamma L(x_h) \le \frac{C}{t_h}$$

so that, by definition of the topological recession functional,

$$F_\infty(x) \le \gamma L(x).$$

If $L(x) > 0$ we would obtain

$$F_\infty\left(\frac{x}{L(x)}\right) \le \gamma < \gamma^* = \inf\{F_\infty(y) \,:\, L(y) = 1\},$$

which gives a contradiction. In an analogous way we can exclude the case $L(x) < 0$. Therefore we have

$$L(x) = 0$$

and so also $F_\infty(x) = 0$. Now, if $L(t_h x_h)$ is bounded from above, by (15.73) we would get

$$P(t_h x_h) \le C,$$

and so, since P satisfies the compactness assumption (ii) of Theorem 15.2.1, we would obtain $x_h \to x$ strongly in X. Otherwise, if $L(t_h x_h)$ (or a subsequence of it) tends to $+\infty$, dividing by $L(t_h x_h)$ in (15.73) we get

$$P\left(\frac{x_h}{L(x_h)}\right) \le C.$$

Since $L(x_h) \to L(x) = 0$, by the compactness assumption of P we obtain again $x_h \to x$. Therefore the compactness hypothesis (2.3.11) is satisfied.

Finally we verify the compatibility conditions (iii) of Theorem 15.2.1. Let $z \in \ker(F_\infty - \gamma L)$, i.e.,

$$F_\infty(z) = \gamma L(z).$$

As before, using the strict inequalities $\gamma_* < \gamma < \gamma^*$ we obtain $F_\infty(z) = L(z) = 0$, and so, by assumption (ii)

$$F(x - \eta z) - \gamma L(x - \eta z) = F(x - \eta z) - \gamma L(x) \le F(x) - \gamma L(x)$$

for all $x \in X$, that is, the compatibility condition (iii) is satisfied. □

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Remark 15.4.1. If the infimum

$$\inf\{F_\infty(x)\ :\ L(x)=1\} \qquad \big(\text{respectively}, \inf\{F_\infty(x)\ :\ L(x)=-1\}\big)$$

is not attained, then we can also accept in Theorem 15.4.2 (b) $\gamma=\gamma^*$ (respectively, $\gamma=\gamma_*$).

As an application of the limit analysis theorems above we consider the case of functionals defined on measures. We refer to Section 13.3 for the theory of convex functionals on measures and to Bouchitté and Buttazzo [110], [111], [112] for further details on nonconvex functionals defined on measures.

Consider a measure space $(\Omega,\mathscr{B},\mu)$, where Ω is a separable locally compact metric space, $\mathscr{B}$ is the σ-algebra of all Borel subsets of Ω, and $\mu:\mathscr{B}\to[0,+\infty[$ is a positive, finite, nonatomic measure. Consider a functional defined on $\mathbf{M}(\Omega;\mathbf{R}^n)$ of the form first considered by Bouchitté and Buttazzo [110],

$$F(\lambda)=\int_\Omega f\left(\frac{d\lambda}{d\mu}\right)d\mu+\int_{\Omega\setminus A_\lambda} f^\infty(\lambda^s)+\int_{A_\lambda} g(\lambda(x))\,d\#. \tag{15.74}$$

Here

$f:\mathbf{R}^n\to[0,+\infty]$ is a proper, convex, lower semicontinuous function with $f(0)=0$;

f^∞ is its recession function;

$g:\mathbf{R}^n\to[0,+\infty]$ is a lower semicontinuous function with $g(0)=0$ satisfying the subadditivity condition

$$g(s_1+s_2)\le g(s_1)+g(s_2) \qquad \forall s_1,s_2\in\mathbf{R}^n;$$

$\lambda=\frac{d\lambda}{d\mu}\mu+\lambda^s$ is the Lebesgue–Nikodým decomposition of λ into absolutely continuous and singular parts with respect to μ;

A_λ is the set of all atoms of λ;

$\lambda(x)$ is the value $\lambda(\{x\})$;

is the counting measure.

As already recalled, Bouchitté and Buttazzo [110] proved that if the condition

$$f^\infty(s)=\lim_{t\to 0^+}\frac{g(ts)}{t} \qquad \forall s\in\mathbf{R}^n$$

is fulfilled, then the functional (15.74) is sequentially weakly*-lsc on $\mathbf{M}(\Omega;\mathbf{R}^n)$.

For our purposes, it is convenient to introduce for every function $g:\mathbf{R}^n\to[0,+\infty]$ the functions

$$g^\infty(s)=\liminf_{t\to+\infty}\frac{g(ts)}{t},$$
$$g^0(s)=\limsup_{t\to 0^+}\frac{g(ts)}{t}.$$

The following proposition holds (see [110]).

Proposition 15.4.1. *Let $g:\mathbf{R}^n\to[0,+\infty[$ be a lower semicontinuous and subadditive function with $g(0)=0$. Then, we have the following:*

(i) *the functions* g^0 *and* g^∞ *are convex, lower semicontinuous, and positively* 1*-homogeneous;*

(ii) $g^0(s) = \sup_{t>0} \frac{g(ts)}{t} = \lim_{t\to 0^+} \frac{g(ts)}{t}$ *for every* $s \in \mathbf{R}^n$*;*

(iii) $g^\infty(s) = \inf_{t>0} \frac{g(ts)}{t} = \lim_{t\to +\infty} \frac{g(ts)}{t}$ *for every* $s \in \mathbf{R}^n$.

Remark 15.4.2. From Proposition 15.72 it follows easily that

$$g^\infty(s) \le g(s) \le g^0(s) \qquad \text{for every } s \in \mathbf{R}^n.$$

The following theorem gives sufficient conditions on f and g to apply Theorem 15.4.2 to functionals of the form (15.74).

Theorem 15.4.3. *Let* $f : \mathbf{R}^n \to [0, +\infty]$*, and* $g : \mathbf{R}^n \to [0, +\infty[$ *be given functions. Assume that*

(i) f *is convex, lower semicontinuous, and proper on* $\mathbf{R}^n$*, and* $f(0) = 0$*;*

(ii) *there exist* $C_1 > 0$ *and* $D \in \mathbf{R}$ *such that*

$$f(x) \ge C_1|x| - D \qquad \forall x \in \mathbf{R}^n;$$

(iii) g *is lower semicontinuous and subadditive on* $\mathbf{R}^n$*, and* $g(0) = 0$*;*

(iv) *there exists* $C_2 > 0$ *such that*

$$g(x) \ge C_2|x| \qquad \forall x \in \mathbf{R}^n;$$

(v) $g^0 = f^\infty$ *in* $\mathbf{R}^n$*;*

(vi) $H \in \mathbf{C}_0(\Omega; \mathbf{R}^n)$.

Let $F : \mathbf{M}(\Omega; \mathbf{R}^n) \to [0, +\infty]$ *be the functional defined in* (15.74)*. Then, setting*

$$\gamma^* = \inf\{F_\infty(\lambda) \ : \ \langle H, \lambda\rangle = 1\},$$
$$\gamma_* = -\inf\{F_\infty(\lambda) \ : \ \langle H, \lambda\rangle = -1\},$$

we have the following:

(a) *if the functional* $F - \gamma\langle H, \cdot\rangle$ *admits a minimum on* $\mathbf{M}(\Omega; \mathbf{R}^n)$*, then* $\gamma_* \le \gamma \le \gamma^*$*;*

(b) *the functional* $F - \gamma\langle H, \cdot\rangle$ *admits a minimum on* $\mathbf{M}(\Omega; \mathbf{R}^n)$*, for every* γ *such that* $\gamma_* < \gamma < \gamma^*$.

PROOF. By the assumptions made on f and g the functional F is sequentially weakly*-lsc on $\mathbf{M}(\Omega; \mathbf{R}^n)$. Moreover, by assumptions (ii) and (iv), we have

$$F(\lambda) \ge C_1 \int_\Omega \left|\frac{d\lambda}{d\mu}\right| d\mu + C_1 \int_{\Omega\setminus A_\lambda} |\lambda^s| + C_2 \int_{A_\lambda} |\lambda^s(x)|\, d\# - D\mu(\Omega)$$

so that

$$F(\lambda) \ge C\|\lambda\| - b \tag{15.75}$$

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for suitable $C > 0$ and $b \in \mathbf{R}$. Finally, from (15.75) we get

$$\ker F_\infty = \{0\}$$

so that hypothesis (ii) of Theorem 15.4.2 is satisfied too, and hence the conclusions follow from Theorem 15.4.2. □

We give now an explicit formula for the bounds γ^* and γ_*. To obtain this result we need first an explicit representation for the topological recession function F_∞. We use a representation theorem for the relaxed functional associated to integrals of the form (15.74). More precisely, given a functional $F : \mathbf{M}(\Omega; \mathbf{R}^n) \to [0, +\infty]$ of the form

$$F(\lambda) = \begin{cases} \displaystyle\int_\Omega f\Big(\frac{d\lambda}{d\mu}\Big) d\mu + \int_{A_\lambda} g\big(\lambda(x)\big)\# & \text{if } \lambda^s = 0 \text{ on } \Omega \setminus A_\lambda, \\ +\infty & \text{otherwise,} \end{cases}$$

we consider its relaxed functional $\overline{F}$ defined by

$$\overline{F} = \sup\big\{G \;:\; G \le F,\ G \text{ sequentially weakly}^*\text{-lsc on } \mathbf{M}(\Omega; \mathbf{R}^n)\big\}.$$

Bouchitté and Buttazzo [111] proved that if $f, g : \mathbf{R}^n \to [0, +\infty]$ satisfy the assumptions

- f is convex and lower semicontinuous on $\mathbf{R}^n$, and $f(0) = 0$,
- there exist $\alpha > 0$ and $\beta \ge 0$ such that
$$f(s) \ge \alpha|s| - \beta \qquad \forall s \in \mathbf{R}^n,$$
- g is subadditive and lower semicontinuous on $\mathbf{R}^n$, and $g(0) = 0$,
- $g^0(s) \ge \alpha|s|$ for every $s \in \mathbf{R}^n$,

then the following integral representation holds for $\overline{F}$:

$$\overline{F}(\lambda) = \int_\Omega \overline{f}\Big(\frac{d\lambda}{d\mu}\Big) d\mu + \int_{\Omega\setminus A_\lambda} (\overline{f})^\infty(\lambda^s) + \int_{A_\lambda} \overline{g}(\lambda(x))\, d\#, \tag{15.76}$$

where

$$\overline{f} = f \#_e g^0, \qquad \overline{g} = f^\infty \#_e g.$$

To characterize the topological recession function for functionals of the form (15.74) we introduce the functional

$$G^\infty(\lambda) = \int_\Omega g^\infty(\lambda) = \int_\Omega g^\infty\Big(\frac{d\lambda}{d\mu}\Big) d\mu + \int_\Omega g^\infty(\lambda^s).$$

Theorem 15.4.4. *Under the assumptions of Theorem* 15.4.3, *we have*

$$F_\infty(\lambda) = G^\infty(\lambda) \qquad \forall \lambda \in \mathbf{M}(\Omega; \mathbf{R}^n).$$

PROOF. We prove first that $F_\infty(\lambda) \le G^\infty(\lambda)$ for every $\lambda \in \mathbf{M}(\Omega;\mathbf{R}^n)$. Let $\lambda \in \mathbf{M}(\Omega;\mathbf{R}^n)$ be a measure with a finite number of atoms. Then

$$\begin{aligned} F_\infty(\lambda) &\le \liminf_{t\to+\infty}\left[\int_\Omega \frac{1}{t} f\left(t\frac{d\lambda}{d\mu}\right)d\mu + \int_{\Omega\setminus A_\lambda} f^\infty(\lambda^s) + \int_{A_\lambda} \frac{g(t\lambda(x))}{t}\,d\#\right] \\ &\le \int_\Omega f^\infty\left(\frac{d\lambda}{d\mu}\right)d\mu + \int_{\Omega\setminus A_\lambda} f^\infty(\lambda^s) + \int_{A_\lambda} g^\infty\big(\lambda(x)\big)\,d\#. \end{aligned} \tag{15.77}$$

Now, let λ be any measure in $\mathbf{M}(\Omega;\mathbf{R}^n)$. Setting $A_\lambda^h = \{x \in A_\lambda \;:\; |\lambda|(x) < 1/h\}$ and $\lambda_h = \lambda \cdot 1_{\Omega\setminus A_\lambda^h}$, we get $\lambda_h \to \lambda$; moreover, since $A_{\lambda_h} = \{x \in A_\lambda \;:\; |\lambda|(x) \ge 1/h\}$, we have that λ_h has a finite number of atoms. From the weak*-lower semicontinuity of F_∞, taking into account (15.77), we have

$$\begin{aligned} F_\infty(\lambda) &\le \liminf_{h\to+\infty} F_\infty(\lambda_h) \le \liminf_{h\to+\infty}\left[\int_\Omega f^\infty\left(\frac{d\lambda}{d\mu}\right)d\mu + \int_{\Omega\setminus A_\lambda} f^\infty(\lambda^s) + \int_{A_\lambda\setminus A_\lambda^h} g^\infty\big(\lambda^s(x)\big)\,d\#\right] \\ &\le \int_\Omega f^\infty\left(\frac{d\lambda}{d\mu}\right)d\mu + \int_{\Omega\setminus A_\lambda} \chi_{\{0\}}(\lambda^s) + \int_{A_\lambda} g^\infty\big(\lambda^s(x)\big)\,d\#. \end{aligned} \tag{15.78}$$

By computing the relaxation of the first and the last terms of (15.78), we get by (15.76)

$$F_\infty(\lambda) \le \int_\Omega (f^\infty \#_e g^\infty)\left(\frac{d\lambda}{d\mu}\right)d\mu + \int_{\Omega\setminus A_\lambda} (f^\infty \#_e g^\infty)(\lambda^s) + \int_{A_\lambda} (f^\infty \#_e g^\infty)\big(\lambda^s(x)\big)\,d\#$$

for every $\lambda \in \mathbf{M}(\Omega;\mathbf{R}^n)$. From (v) of Theorem 15.4.3 we deduce that $f^\infty \#_e g^\infty = g^\infty$ so that

$$F_\infty(\lambda) \le G^\infty(\lambda) \qquad \forall \lambda \in \mathbf{M}(\Omega;\mathbf{R}^n).$$

We prove now the opposite inequality. We claim that for every $\varepsilon > 0$ there exists a $k_\varepsilon > 0$ such that

$$g^\infty(s) \le f(s) + \varepsilon|s| + k_\varepsilon \qquad \forall s \in \mathbf{R}^n. \tag{15.79}$$

By contradiction, assume there exists an $\varepsilon_0 > 0$ such that for every $k \in \mathbf{N}$ there exists a $s_k \in \mathbf{R}^n$ with

$$g^\infty(s_k) > f(s_k) + \varepsilon_0|s_k| + k.$$

Setting $v_k = s_k/|s_k|$, and $t_k = |s_k|$ we have

$$g^\infty(v_k) > \frac{f(t_k v_k)}{t_k} + \varepsilon_0 + \frac{k}{t_k}. \tag{15.80}$$

Since $|v_k| = 1$, it is not restrictive to assume $v_k \to v$ for some $v \in \mathbf{R}^n$. If (t_k) is bounded we get, by using Proposition 15.4.1(i),

$$g(v) \ge g^\infty(v) = +\infty,$$

which is impossible since g is finite. Therefore, we can assume $t_k \to +\infty$. Passing to the limit in (15.80) yields, taking into account Proposition 15.4.1(i) again,

$$g^\infty(v) \ge f^\infty(v) + \varepsilon_0 = g^0(v) + \varepsilon_0 > g^\infty(v),$$

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which is a contradiction. Then (15.79) holds, and from the weak*-lower semicontinuity and 1-homogeneity of G^∞ we get

$$\begin{aligned} G^\infty(\lambda) &= \inf\left\{ \liminf_{h\to+\infty} \frac{G^\infty(t_h\lambda_h)}{t_h} \right\} \\ &\le \inf\left\{ \liminf_{h\to+\infty} \frac{1}{t_h}\left[\int_\Omega \left(f\left(t_h \frac{d\lambda_h}{d\mu}\right) + \varepsilon \left| t_h \frac{d\lambda_h}{d\mu}\right| \right) d\mu \right.\right. \\ &\qquad \left.\left. + \int_{\Omega\setminus A_{\lambda_h}} f^\infty(t_h\lambda_h^s) + \int_{A_{\lambda_h}} g\big(t_h\lambda_h^s(x)\big)\, d\# \right] \right\} \\ &\le \inf\left\{ \liminf_{h\to+\infty} \left[\frac{F(t_h\lambda_h)}{t_h} + \varepsilon\|\lambda_h\| \right] \right\}, \end{aligned}$$

where the infimum is taken over all $t_h \to +\infty$ and all $\lambda_h \to \lambda$. Taking (t_h) and (λ_h) such that

$$F_\infty(\lambda) = \liminf_{h\to+\infty} \frac{F(t_h\lambda_h)}{t_h},$$

we have

$$G^\infty(\lambda) \le F_\infty(\lambda) + \varepsilon \limsup_{h\to+\infty} \|\lambda_h\| \le F_\infty(\lambda) + \varepsilon C.$$

Letting $\varepsilon \to 0$, we get

$$G^\infty(\lambda) \le F_\infty(\lambda),$$

and the proof is achieved. □

By virtue of Theorem 15.4.4 we can write

$$\begin{aligned} \gamma^* &= \inf\{G^\infty(\lambda) \,:\, \langle H, \lambda\rangle = 1\}, \\ \gamma_* &= -\inf\{G^\infty(\lambda) \,:\, \langle H, \lambda\rangle = -1\} \end{aligned}$$

or equivalently

$$\gamma^* = \frac{1}{\sup\{\langle H,\lambda\rangle \,:\, G^\infty(\lambda) = 1\}},$$

$$\gamma_* = \frac{1}{\inf\{\langle H,\lambda\rangle \,:\, G^\infty(\lambda) = 1\}}.$$

The last expressions allow us to compute explicitly γ^* and γ_* in terms of g^∞ and H only. Indeed, by using the definition of the Fenchel transform, it is easy to see that

$$\frac{1}{\sup\{\langle H,\lambda\rangle \,:\, G^\infty(\lambda) = 1\}} = \sup\{t \,:\, (G^\infty)^*(tH) = 0\}.$$

Therefore, since

$$(G^\infty)^*(w) = \begin{cases} 0 & \text{if } (g^\infty)^*(w) \equiv 0, \\ +\infty & \text{otherwise,} \end{cases}$$

we obtain

$$\gamma^* = \sup\{t \,:\, (g^\infty)^*(tH) \equiv 0\} = \left[\sup_{x,s} \frac{H(x)s}{g^\infty(s)} \right]^{-1}.$$

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Analogously, it is

$$\gamma_* = \inf\{t \ : \ (g^\infty)^*(tH) \equiv 0\} = \left[\inf_{x,s} \frac{H(x)s}{g^\infty(s)}\right]^{-1}.$$

For instance, if $g^\infty(s) = c|s|$, we get

$$\gamma^* = \frac{c}{||H||_{C_0(\Omega;\mathbf{R}^n)}}, \qquad \gamma_* = -\frac{c}{||H||_{C_0(\Omega;\mathbf{R}^n)}}.$$

The result we obtained allows us to study the limit analysis problem for a class of nonconvex functionals defined on BV. More precisely, let $\Omega =]a,b[$ be an open interval of $\mathbf{R}$, and assume that f and g satisfy all the hypotheses of Theorem 15.4.3 Consider the nonconvex functional $F : BV(\Omega;\mathbf{R}^n) \to [0,+\infty]$ defined by

$$F(u) = \int_\Omega f(\nabla u)\,dx + \int_{\Omega\setminus S_u} f^\infty(D^s u) + \int_{S_u} g(D^s u(x))\,d\#(x), \tag{15.81}$$

where ∇u and $D^s u$, respectively, denote the absolutely continuous and the singular parts of Du with respect to the Lebesgue measure, and S_u is the set of jumps of u, that is, the set of all points $x \in \Omega$ such that the left and right traces $u^+(x)$ and $u^-(x)$ do not coincide. Setting $\lambda = Du$, the functionals of type (15.81) can be interpreted in terms of functionals of type (15.74) on $\mathbf{M}(\Omega;\mathbf{R}^n)$.

The Neumann problem. We deal with functionals G defined on $BV(\Omega;\mathbf{R}^n)$ by

$$G(u) = \int_\Omega f(\nabla u)\,dx + \int_{\Omega\setminus S_u} f^\infty(D^s u) + \int_{S_u} g(D^s u(x))\,d\#(x) - \gamma\langle L, u\rangle, \tag{15.82}$$

where

$$\langle L, u\rangle = \int_\Omega h u\,dx + \int_\Omega \phi Du$$

with $h \in L^1(\Omega;\mathbf{R}^n)$ and $\phi \in C_0(\Omega;\mathbf{R}^n)$. It is easily verified that $\langle L, 1\rangle = 0$ is a necessary condition in order to get a minimum for the functional (15.82). Therefore, setting

$$H(x) = \int_a^x h(y)\,dy,$$

we have that $H \in C_0(\Omega;\mathbf{R}^n)$, and integrating by parts,

$$\langle L, u\rangle = \langle \phi - H, Du\rangle, \qquad \text{with } \phi - H \in C_0(\Omega;\mathbf{R}^n).$$

Hence, by the limit analysis result above, a necessary condition for the existence of a minimizer of the functional G in (15.82) is $\gamma_* \le \gamma \le \gamma^*$, whereas a sufficient condition is $\gamma_* < \gamma < \gamma^*$, with

$$\gamma_* = \left[\inf_{x,s} \frac{(\phi(x) - H(x))\,s}{g^\infty(s)}\right]^{-1},$$

$$\gamma^* = \left[\sup_{x,s} \frac{(\phi(x) - H(x))\,s}{g^\infty(s)}\right]^{-1}.$$

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The Dirichlet problem. To deal with the Dirichlet problem associated to functionals of the form (15.82), it is convenient to consider an open interval Ω_0 containing Ω and the space

$$BV_0 = \left\{u \in BV(\Omega_0;\mathbf{R}^n) \;:\; u = 0 \text{ on } \Omega_0 \setminus \Omega\right\}.$$

Therefore, given $h \in L^1(\Omega;\mathbf{R}^n)$ and $\phi \in \mathbf{C}(\overline{\Omega};\mathbf{R}^n)$, and denoting by $\tilde{h} \in L^1(\Omega_0;\mathbf{R}^n)$ and $\tilde{\phi} \in C_0(\Omega_0;\mathbf{R}^n)$ some extensions of h and ϕ to Ω_0, we may set for every $u \in BV_0$

$$\langle \tilde{L}, u\rangle = \int_{\Omega_0} \tilde{h}\, u\, dx + \int_{\Omega_0} \tilde{\phi} Du = \int_{\Omega} h\, u\, dx + \int_{\overline{\Omega}} \phi Du$$

and consider the problem

$$\min\left\{ \int_{\Omega_0} f(\nabla u)\, dx + \int_{\Omega_0\setminus S_u} f^{\infty}(D^s u) + \int_{S_u} g(D^s u(x))\, d\#(x) - \gamma\langle \tilde{L}, u\rangle \;:\; u \in BV_0 \right\}, \tag{15.83}$$

where S_u denotes now the set of jumps of u on Ω_0. If $H \in \mathbf{C}_0(\Omega_0;\mathbf{R}^n)$ is such that $H' = h$ a.e. on Ω, we have

$$\langle \tilde{L}, u\rangle = \int_{\Omega} H' u\, dx + \int_{\Omega_0} \tilde{\phi} Du = \int_{\overline{\Omega}} (\phi - H) Du$$

and problem (15.83) can be written as

$$\min\left\{ \int_{\overline{\Omega}} f\left(\frac{d\lambda}{dx}\right) dx + \int_{\overline{\Omega}\setminus A_\lambda} f^{\infty}(\lambda^s) + \int_{A_\lambda} g(\lambda(x))\, d\# - \gamma\langle \phi - H, \lambda\rangle : \lambda \in \mathbf{M}(\overline{\Omega};\mathbf{R}^n),\ \lambda(\overline{\Omega}) = 0 \right\}.$$

Therefore, arguing as in the previous case, we obtain that a necessary (respectively, sufficient) condition for existence in the Dirichlet problem (15.83) is $\gamma_* \le \gamma \le \gamma^*$ (respectively, $\gamma_* < \gamma < \gamma^*$), with

$$\gamma_* = \left[\inf\left\{ \frac{\langle \phi - H, \lambda\rangle}{\int_{\overline{\Omega}} g^{\infty}(\lambda)} \;:\; \lambda(\overline{\Omega}) = 0 \right\}\right]^{-1},$$

$$\gamma^* = \left[\sup\left\{ \frac{\langle \phi - H, \lambda\rangle}{\int_{\overline{\Omega}} g^{\infty}(\lambda)} \;:\; \lambda(\overline{\Omega}) = 0 \right\}\right]^{-1}.$$

Remark 15.4.3. The Neumann and Dirichlet problems can be considered in the more general and interesting case of functions u defined on a subset Ω of $\mathbf{R}^n$. The functional $F : BV(\Omega;\mathbf{R}^m) \to [0,+\infty]$ is then of the form (see Sections 10.3 and 10.4)

$$F(u) = \int_{\Omega} f(\nabla u)\, dx + \int_{\Omega\setminus S_u} f^{\infty}(D^s u) + \int_{S_u} g([u], \nu_u)\, d\mathcal{H}^{n-1},$$

where $\mathcal{H}^{n-1}$ is the Hausdorff $(n-1)$-dimensional measure, $[u]$ is the jump of u along S_u, and ν_u is the normal versor to S_u. In this case the associated limit analysis problems have not been studied, and even the study of general conditions on f and g which imply the lower semicontinuity of F leaves some open questions.