Chapter 1

Introduction

Let us detail the contents of each of the two parts of the book.

Part I: Basic Variational Principles. In Part I, we follow as a guideline the variational treatment of the celebrated Dirichlet problem. We show how the program of D. Hilbert, which was first delineated in his famous lecture at Collège de France in 1900 [241], has been progressively solved throughout the 20th century. We introduce the basic elements of variational analysis which allow one to solve this classical problem and closely related ones, like the Neumann problem and the Stokes problem.

Chapter 2 contains an extensive exposition of weak solution methods in variational analysis and of the accompanying notions: test functions, the distribution theory of L. Schwartz, weak convergences, and topologies.

Chapter 3 provides an exposition of the basic abstract variational principles. We enhance the importance of the direct method for solving minimization problems and put to the fore some of its basic topological ingredients: lower semicontinuity, coercivity, and inf-compactness. We show how weak topologies, reflexivity, and convexity properties come naturally into play. We insist on the modern approach to optimization theory where the concept of epigraph of a function plays a central role; see the monograph of Rockafellar and Wets [330] on variational analysis, where the epigraphical analysis is systematically developed.

Chapter 4 contains some complements on geometric measure theory. We introduce in a self-contained way the notion of Hausdorff measure, which allows us to recover, as special cases, both the Lebesgue measure on an open set of $\mathbf{R}^N$ and surface measures (which play an important role, for example, in the definition of the space trace of Sobolev spaces).

These two basic ingredients, roughly speaking, the generalized differential calculus of distribution theory and the generalized integration theory of Lebesgue, allow us to introduce in Chapter 5 the classical Sobolev spaces which provide the right functional setting for the variational approach to the studied problems. In this new edition, we have completed the section related to capacity by introducing the notions of quasi-continuity, quasi-open sets, and capacitary measures. These play a central role in the analysis of the limiting behavior of variational problems in wildly varying domains (finely perforated domains, cloud of ice, etc.) and shape optimization (Chapter 16).

All the ingredients of the variational approach to the model examples are now available: in Chapter 6 we describe some of them, including Dirichlet, Neumann, and mixed problems. With regard to these examples, we are in the classical favorable situation: we have to minimize a convex coercive lower semicontinuous function on a reflexive Banach space. In such a situation, the direct method of Hilbert and Tonelli does apply, although for the model of linearized elasticity treated in this second edition, coercivity is a delicate point. We also have completed the first edition with three models slightly less classical: the reaction-diffusion equations for which we apply the Lax–Milgram theorem in a nonsymmetric case, the semilinear equations, and the obstacle problem, including the Signorini problem in the framework of linearized elasticity.

Chapters 7 and 8 complement this classic portrait of variational methods by introducing two of the most powerful numerical tools which allow one to compute approximate solutions of variational problems: finite element methods and spectral analysis methods. Each of these two methods corresponds to a very specific type of Galerkin approximation of an infinite dimensional problem by a sequence of finite dimensional ones. Each method has its own advantages; for example, finite element methods allow one to treat engineering problems involving general domains, like the wing of a plane, which explains their great success.

Around 1970, the study of constrained problems and variational inequalities led Stampacchia, Browder, Brezis, Moreau, Rockafellar, et al. to develop the elements of a unilateral variational analysis. In particular, convex variational analysis has known considerable success and has familiarized mathematicians with the idea that sets play a decisive role in analysis. The Fenchel duality, the subdifferential calculus of convex functions, and the extension of the Fermat rule are striking examples of this new approach. The role of the epigraph has progressively emerged as essential in the geometrical understanding of these concepts. Chapter 9 provides a thorough exposition of these elements of convex variational analysis in infinite dimensional spaces. We stress the importance of the Fenchel duality, which allows us to associate to each convex variational problem a dual one, whose solutions have in general a deep physical (or numerical or economical) interpretation as multipliers.

Part II: Advanced Variational Analysis. This second part corresponds to Chapters 10 through 16 and deals with our second objective, which is to present new trends in variational analysis. Indeed, in recent years, variational methods have proved to be very flexible. They have been developed to study a number of advanced problems of modern technology, like composite materials, image processing, and shape optimization. To grasp these phenomena, the classical framework of variational analysis, which was studied in Part I, must be enlarged. Let us describe some of these extensions:

1. The modelization of a large number of problems in physics, image processing, requires the introduction of new functional spaces permitting *discontinuities* of the solution. In phase transitions, image segmentation, plasticity theory, and the study of cracks and fissures, in the study of the wake in fluid dynamics and the shock theory in mechanics, the solution of the problem presents discontinuities along one-codimensional manifolds. Its first distributional derivatives are now measures which may charge zero Lebesgue measure sets, and the solution of these problems cannot be found in classical Sobolev spaces.

   The classical theory of Sobolev spaces, which was developed in Chapter 5, is completed in Chapter 10 by a self-contained and detailed presentation of these spaces, $BV(\Omega)$, $SBV(\Omega)$, $BD(\Omega)$. The space $BV(\Omega)$, for example, is the space of functions with bounded variations, and a function u belongs to $BV(\Omega)$ iff its first distributional derivatives are bounded measures. The $SBV(\Omega)$ space is the subspace of

$BV(\Omega)$ which consists of functions whose first distributional derivatives are bounded measures with no Cantor part.

2. In Chapter 12, we introduce the concept of Γ-convergence, which provides a parametrized version of the direct method in variational analysis.

   Following Stampacchia's work, Mosco [304], [305] and Joly [251] introduced the Mosco-epiconvergence (1970) of sequences of convex functions to study approximation and perturbation schemes in variational analysis and potential theory.

   The general topological concept, without any convexity assumption, has progressively emerged, and De Giorgi in 1975 introduced the notion of Γ-convergence for sequences of functions. It corresponds to the topological set convergence of the epigraphs, whence the equivalent terminology "epi-convergence." This concept has been successfully applied to a large variety of approximation and perturbation problems in calculus of variations and mechanics: homogenization of composite materials, materials with many small holes and porous media, thin structures and reinforcement problems, and so forth.

   We illustrate the concept by describing some recent applications to thin structures, composite material, phase transitions, and image segmentation. We have completed the first edition with an introduction to the ergodic theory of subadditive processes and its application to stochastic homogenization.

3. Chapters 11 and 13 deal with the question of lower semicontinuity and relaxation of functionals of calculus of variations. Indeed, as a general rule, when applying the direct method to a functional F which is not lower semicontinuous, one obtains that minimizing sequences converge to solutions of the relaxed problem, which is the minimization of the lower semicontinuous envelope $\mathrm{cl}\, F$ of F.

   In the vectorial case, that is, when functionals are defined on Sobolev spaces $W^{1,p}(\Omega, \mathbf{R}^m), \Omega \subset \mathbf{R}^N$, relaxation with respect to the weak topology of $W^{1,p}(\Omega, \mathbf{R}^m)$ (or strong topology of $L^p(\Omega, \mathbf{R}^m)$) leads to the important concepts of quasi-convexity (in the sense of Morrey), polyconvexity, and rank-one convexity. We consider as well the case of functionals with linear growth and the corresponding lower semicontinuity and relaxation problems on BV and SBV spaces. All these notions play an important role in the modeling of large deformations in mechanics and plasticity, as described in Chapter 14. Following the microstructure school of Ball and James, in the modeling of the solid/solid phase transformations, the density energy possesses a multiwell structure. An alternative and appropriate procedure consists in relaxing the corresponding free energy functional in the space of Young measures generated by gradients.

   To complete Chapter 11, in this second edition, we have introduced the Kantorovich relaxed formulation of the Monge transport problem: the goal is to find a probability on the product space $\mathbf{R}^N \times \mathbf{R}^N$, which minimizes a suitable relaxed transportation cost, i.e., the pth power of the so-called Wasserstein distance between two probability measures on $\mathbf{R}^N$.

4. Another important aspect of the direct method concerns the coercivity property. In Chapter 15, we examine how the method works when the variational problem has a lack of coercivity. In that case, existence of solutions relies on compatibility conditions, whose general formulation involves recession functions.

5. The next topic considered, in Chapter 16, is shape optimization, which is a good illustration of the powerfulness of direct methods in variational analysis and also of their limitations. This chapter has been completed, in this second edition, by the description of another interesting case of the shape optimization problem, consisting in establishing the existence of optimal potentials for some suitable cost functionals, as, for example, the integral cost functionals newly introduced in Chapter 5.

5. The final chapter is the main contribution of this new edition. Indeed, the previous edition was entirely devoted to the mathematical tools related to variational problems in a static framework. This new edition completes the previous one by treating in some depth the concept of gradient flows. We have chosen to introduce this notion through optimization: when the potential is continuously differentiable, the Cauchy problem governed by the associated gradient vector field is nothing but the classical continuous steepest descent implemented to minimize the potential. The analysis of the generalized steepest descent is valid for arbitrary convex lower semicontinuous potentials on a Hilbert space and also may be generalized to complete metric spaces. This last generalization is not fortuitous and is involved in some cases of cost functionals coming from mass transportation theory. Evolution equations in general describe the changing of a physical (or economic, or social, etc.) system with respect to the time. The Cauchy problem governed by a gradient flow arises, for instance, when one wants to model the heat equation or the Stefan-type problem. Another important aspect of the concept of gradient flows intervenes in the study of some homogenization problems, in which suitable variational convergences for sequences of gradient flows provide a powerful framework to describe the limit Cauchy problems. These problems arise, for example, in the analysis of the diffusion through heterogeneous media, or in first-order evolution problems with small parameters, provided that the potentials involved are convex.