Chapter 2

Weak solution methods in variational analysis

2.1 ▪ The Dirichlet problem: Historical presentation

Throughout this book, we adopt the following notation: Ω is an open subset of $\mathbf{R}^N$ ($N \leq 3$ for applications in classical mechanics), and $x = (x_1, x_2, \dots, x_N)$ is a generic point in Ω. The topological boundary of Ω is denoted by $\partial\Omega$.

Let $g : \partial\Omega \longrightarrow \mathbf{R}$ be a given function which is defined on the boundary of Ω. The Dirichlet problem consists in finding a function $u : \overline{\Omega} \longrightarrow \mathbf{R}$ which satisfies

$$\Delta u = 0 \quad \text{on} \quad \Omega, \tag{2.1}$$

$$u = g \quad \text{on} \quad \partial\Omega. \tag{2.2}$$

The operator Δ is the Laplacian

$$\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_N^2};$$

it is equal to the sum of the second partial derivatives of u with respect to each variable $x_1, x_2, \dots, x_N$. Equation (2.1) is the Laplace equation, and a solution of this equation is said to be harmonic on Ω. Clearly, there are many harmonic functions.

The following examples of harmonic functions illustrate how rich this family is:

- $u(x) = \sum_{i=1}^{N} a_i x_i + b$ (affine function) is harmonic on $\mathbf{R}^N$;
- $u(x) = a x_1^2 + b x_2^2 + c x_3^2$ is harmonic on $\mathbf{R}^3$ iff $a + b + c = 0$;
- $u(x) = e^{x_1} \cos x_2$ and $v(x) = e^{x_1} \sin x_2$ are harmonic on $\mathbf{R}^2$ (note that $u(x_1, x_2) = \operatorname{Re} e^{(x_1 + i x_2)}$ and $v(x_1, x_2) = \operatorname{Im} e^{(x_1 + i x_2)}$);
- $u(x) = (x_1^2 + x_2^2 + x_3^2)^{-1/2}$ is harmonic on $\mathbf{R}^3 \setminus \{0\}$ (Newtonian potential).

The study of harmonic functions is a central topic of the so-called potential theory and of harmonic analysis. We will see further, as the above examples suggest, the close connections between this theory, the theory of the complex variable, and the potential theory.

Thus, we can reformulate the Dirichlet problem by saying that we are looking for a harmonic function on Ω which satisfies the boundary data $u = g$ on $\partial\Omega$. It is called a boundary value problem. The condition $u=g$ on $\partial\Omega$, which consists in prescribing the value of the function u on the boundary of Ω, is called the Dirichlet boundary condition and gives rise to the name of the problem.

Let us consider, for illustration, the elementary case $N=1$. Take $\Omega =]a,b[$ an open bounded interval. Given two real numbers g_1 and g_2, the Dirichlet problem reads as follows: find $u:[a,b]\longrightarrow \mathbf{R}$ such that

$$\begin{cases} u''=0 \text{ on }]a,b[, \\ u(a)=g_1, \quad u(b)=g_2. \end{cases}$$

Clearly, this problem has a unique solution, whose graph in $\mathbf{R}^2$ is the line segment joining point (a,g_1) to point (b,g_2).

We will see that for an open bounded subset Ω and under some regularity hypotheses on Ω and g covering most practical situations, one can prove existence and uniqueness of a solution of the Dirichlet problem. Indeed, this is a long story whose important steps are summarized below.

It is in 1782 that the Laplace equation appears for the first time. When studying the orbits of the planets, Laplace discovered that the Newtonian gravitational potential of a distribution of mass of density ρ on a domain $\Omega\subset\mathbf{R}^3$, which is given by the formula

$$u(x)=\frac{1}{4\pi}\int_\Omega \frac{\rho(y)}{|x-y|}dy, \tag{2.3}$$

satisfies the equation

$$\Delta u = 0 \quad \text{on} \quad \mathbf{R}^3\backslash\bar{\Omega}. \tag{2.4}$$

Indeed, it is a good exercise to establish this formula. One first verifies that the Newtonian potential

$$v(x_1,x_2,x_3)=(x_1^2+x_2^2+x_3^2)^{-1/2}$$

satisfies $\Delta v=0$ on $\mathbf{R}^3\backslash\{0\}$. Then a direct derivation under the integral sign yields that u is harmonic on $\mathbf{R}^3\backslash\bar{\Omega}$.

In 1813, Poisson establishes that on Ω the potential u satisfies

$$-\Delta u = \rho \quad \text{on } \Omega, \tag{2.5}$$

which is the so-called Poisson equation.

The central role played by the Laplace and Poisson equations in mathematical physics appeared with more and more evidence, especially because of the work of Gauss. In 1813, Gauss established the following formula (which is often called the Gauss formula or the divergence theorem). Given a vector field $\overrightarrow{V}:\Omega\subset\mathbf{R}^3\longrightarrow\mathbf{R}^3$,

$$\iiint_\Omega \operatorname{div}\overrightarrow{V}(x)\,dx = \iint_{\partial\Omega}\overrightarrow{V}(x)\cdot\overrightarrow{n}(x)\,ds(x), \tag{2.6}$$

which states that the volume integral of the divergence of the vector field $\overrightarrow{V}$ is equal to the global outward flux of $\overrightarrow{V}$ through the boundary of Ω. In the above formula, if we denote

$$\overrightarrow{V}(x)=(v_1(x),v_2(x),v_3(x)),$$

$$\operatorname{div}\overrightarrow{V}(x)=\sum_i \frac{\partial v_i}{\partial x_i}(x)=\frac{\partial v_1}{\partial x_1}(x)+\frac{\partial v_2}{\partial x_2}(x)+\frac{\partial v_3}{\partial x_3}(x)$$

is the divergence of $\overrightarrow{V}$. The vector $\overrightarrow{n}(x)$ is the unit vector which is orthogonal to $\partial\Omega$ at x and which is oriented toward the outside of Ω. The measure ds is the two-dimensional Hausdorff measure on $\partial\Omega$.

Let us briefly explain how the mathematical formulation of conservation laws in physics leads to the Laplace equation. Suppose that the vector field $\overrightarrow{V}$ derives from a potential u, that is,

$$\overrightarrow{V}(x) = Du(x) = \mathrm{grad}u(x) = \nabla u(x) = \left(\frac{\partial u}{\partial x_i}(x)\right)_{i=1,\dots,N}. \tag{2.7}$$

(This is the most commonly used notation.) Suppose, moreover, that the vector field $\overrightarrow{V}(x)$ is such that $\int\int_{\partial G} \overrightarrow{V}\cdot\overrightarrow{n}\, ds = 0$ for all closed surfaces $\partial G \subset \Omega$.

By the Gauss theorem, it follows that

$$\int\int\int_G \mathrm{div}\,\overrightarrow{V}\, dx = 0 \quad \forall\, G \subset \Omega$$

and hence

$$\mathrm{div}\,\overrightarrow{V} = 0 \quad \text{ on } \Omega. \tag{2.8}$$

From (2.7) and (2.8), noticing that

$$\mathrm{div}(\mathrm{grad}u) = \Delta u,$$

we obtain

$$\Delta u = 0,$$

that is, u is harmonic.

The above argument is valid both in the case of the gravitational vector field of Newton and in the case of the electrostatic field of Coulomb in the regions where there is no mass (respectively, charges). With the help of this formula, Gauss was able to prove a number of important properties of harmonic functions, like the mean value property; this was the beginning of the potential theory.

Riemann, who was successively a student of Gauss (1846–1847 in Göttingen) and of Dirichlet (1847–1849 in Berlin), established the foundations of the theory of the complex variable and made the link (when $N = 2$) with the Laplace equation.

Let us recall that for any function $z \in \mathbf{C} \longmapsto f(z)$, which is assumed to be derivable as a function of the complex variable z (f is then said to be holomorphic), its real and imaginary parts P and Q ($f(z) = P(x,y) + iQ(x,y)$, where $z = x + iy$) satisfy the so-called Cauchy–Riemann equations

$$\begin{cases} \dfrac{\partial P}{\partial x} = \dfrac{\partial Q}{\partial y}, \\ \dfrac{\partial P}{\partial y} = -\dfrac{\partial Q}{\partial x}. \end{cases} \tag{2.9}$$

It follows that

$$\Delta P = \frac{\partial^2 Q}{\partial y \partial x} - \frac{\partial^2 Q}{\partial x \partial y} = 0, \qquad \Delta Q = -\frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 P}{\partial y \partial x} = 0.$$

Thus, the real part and the imaginary part of a holomorphic function are harmonic. This approach allows, for example, solution in an elegant way of the Dirichlet problem in a disc. Take $\Omega = D(0,1) = \{z \in \mathbf{C} : |z| < 1\}$, the unit disc centered at the origin in $\mathbf{R}^2$. Given $g : \partial D \longrightarrow \mathbf{R}$ a continuous function, we want to solve the Dirichlet problem

$$\begin{cases} \Delta u = 0 \text{ on } D, \\ u = g \text{ on } \partial D. \end{cases} \tag{2.10}$$

Let us start with the Fourier expansion of the 2π-periodic function

$$g(e^{i\theta}) = \sum_{n\in\mathbf{Z}} c_n(g) e^{in\theta}, \tag{2.11}$$

where $c_n(g) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} g(e^{it})e^{-int}dt$. Note that the above Fourier series converges in an $L^2(-\pi,+\pi)$ norm sense (Dirichlet theorem). Indeed, when starting with the Fourier expansion of the boundary data g, one can give an explicit formula for the solution u of the corresponding Dirichlet problem:

$$u(re^{i\theta}) := \sum_{n\in\mathbf{Z}} c_n(g) r^{|n|} e^{in\theta}. \tag{2.12}$$

Clearly, when taking $r = 1$, one obtains $u = g$ on ∂D. Thus, the only point one has to verify is that u is harmonic on D. Take $r < 1$ and replace $c_n(g)$ by its integral expression in (2.12). By a standard argument based on the uniform convergence of the series, one can exchange the symbols $\sum_n$ and $\int_{-\pi}^{+\pi}$ to obtain

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} g(e^{it}) \sum_{n\in\mathbf{Z}} r^{|n|} e^{in(\theta-t)} dt.$$

Let us introduce

$$P_r(\theta) = \sum_{n\in\mathbf{Z}} r^{|n|} e^{in\theta}, \tag{2.13}$$

the so-called Poisson kernel, and observe that (we denote $z = re^{i\theta}$)

$$P_r(\theta - t) = \mathrm{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right) = \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}. \tag{2.14}$$

Hence,

$$u(re^{i\theta}) = \mathrm{Re}\ \frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{e^{it}+z}{e^{it}-z} g(e^{it}) dt, \tag{2.15}$$

and u appears as the real part of a holomorphic function, which says that u is harmonic on D. Using (2.14) one obtains the Poisson formula

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2} g(e^{it}) dt. \tag{2.16}$$

By similar arguments, one can explicitly solve the Dirichlet problem on a square $]0,a[\times]0,a[$ of $\mathbf{R}^2$. Unfortunately, these methods can be used only in the two-dimensional case.

The modern general treatment of the Dirichlet problem starts with the Dirichlet principle, whose formulation goes back to Gauss (1839), Lord Kelvin, and Dirichlet. It can be formulated as follows.

The solution of the Dirichlet problem is the solution of the following minimization problem:

$$\min\left\{\int_\Omega \sum_{i=1}^N \left(\frac{\partial v}{\partial x_i}\right)^2 dx : v = g \text{ on } \partial\Omega\right\}. \tag{2.17}$$

The functional

$$v \mapsto J(v) := \int_\Omega \sum_{i=1}^N \left(\frac{\partial v}{\partial x_i}\right)^2 dx$$

is called the Dirichlet integral or Dirichlet energy. Thus, the Dirichlet principle states that the solution of the Dirichlet problem minimizes, over all functions v satisfying the boundary data $v = g$ on $\partial\Omega$, the Dirichlet energy. Equivalently,

$$\begin{cases} J(u) \leq J(v)\ \forall v, \quad v = g \text{ on } \partial\Omega, \\ u = g \text{ on } \partial\Omega, \end{cases}$$

that is, u is characterized by a minimization of the energy principle.

One can easily verify, at least heuristically, that the solution u of the minimization problem (2.17) is a solution of the Dirichlet problem (2.1), (2.2). Indeed, the Laplace equation (2.1) can be seen as the optimality condition satisfied by the solution of the minimization problem (2.17). The celebrated Fermat's rule which asserts that the derivative of a function is equal to zero at any point u where f achieves a minimum (respectively, maximum) can be developed in our situation by using the notion of directional derivative. When dealing with problems coming from variational analysis, the so-obtained optimality condition is called the Euler equation.

Thus let us take $v : \Omega \longrightarrow \mathbf{R}$, which satisfies $v = 0$ on $\partial\Omega$. Then, for any $t \in \mathbf{R}$, $u + tv$ still satisfies $u + tv = g$ on $\partial\Omega$ and, since u minimizes J on the set $\{w = g \text{ on } \partial\Omega\}$, we have

$$J(u) \leq J(u + tv).$$

Let us compute for any $t \in \mathbf{R}$, $t \neq 0$

$$\begin{aligned} \frac{1}{t}\big[J(u+tv) - J(u)\big] &= \frac{1}{t}\int_\Omega |Du + tDv|^2 - |Du|^2 \\ &= 2\int_\Omega Du \cdot Dv\, dx + t\int_\Omega |Dv|^2. \end{aligned}$$

For any $t > 0$, this is a nonnegative quantity, and thus by letting $t \longrightarrow 0^+$

$$\int_\Omega Du \cdot Dv\ dx \geq 0.$$

By taking $t < 0$ and letting $t \longrightarrow 0^-$, we obtain the reverse inequality

$$\int_\Omega Du \cdot Dv\, dx \leq 0.$$

Finally

$$\int_\Omega Du \cdot Dv\, dx = 0 \quad \text{for any } v = 0 \text{ on } \partial\Omega.$$

Taking v regular and after integration by parts, we obtain

$$\int_\Omega (\Delta u) v \, dx = 0 \quad \text{for any } v \text{ regular, } v = 0 \text{ on } \partial\Omega,$$

that is, $\Delta u = 0$ on $\partial\Omega$.

Riemann recognized the importance of this principle but he did not discuss its validity. In 1870, Weierstrass, who was a very systematic and rigorous mathematician, discovered when studying some results of his friend Riemann that the Dirichlet principle raises some difficulties. Indeed, Weierstrass proposed the following example (apparently close to the Dirichlet problem!):

$$\min \left\{ \int_{-1}^{+1} x^2 \left(\frac{dv}{dx} \right)^2 dx : v(-1) = a, v(+1) = b \right\}, \tag{2.18}$$

which fails to have a solution. A minimizing sequence for (2.18) can be obtained by considering the viscosity approximation problem

$$\min \left\{ \int_{-1}^{+1} \left(x^2 + \frac{1}{n^2} \right) \left(\frac{dv}{dx} \right)^2 dx : v(-1) = a, v(+1) = b \right\}, \tag{2.19}$$

which now has a unique solution u_n given by

$$u_n(x) = \frac{a+b}{2} - \frac{a-b}{2} \frac{\arctan nx}{\arctan n}, \quad n = 1, 2, \ldots.$$

One can directly verify that u_n satisfies the boundary data and that

$$\int_{-1}^{+1} x^2 \left(\frac{du_n}{dx} \right)^2 dx \longrightarrow 0 \text{ as } n \longrightarrow +\infty.$$

Thus the value of the infimum of (2.18) is zero. But there is no regular function (continuous, piecewise $\mathbf{C}^1$) which satisfies the boundary conditions and such that

$$\int_{-1}^{+1} x^2 \left(\frac{du}{dx} \right)^2 dx = 0.$$

Such a function would satisfy $u =$ constant on $]-1, +1[$ which is incompatible with the boundary data when $a \neq b$.

As we will see, the pathology of the Weierstrass example comes from the coefficient x^2 in front of $(\frac{dv}{dx})^2$ which vanishes (at zero) on the domain $\Omega = (-1, +1)$. As a result, there is a lack of uniform ellipticity or coercivity, which is not the case in the Dirichlet problem. Thus, the Weierstrass example is not a counterexample to the Dirichlet principle; its merit is to underline the shortcomings of this principle. Moreover, it raises a decisive question, which is to understand in which class of functions one has to look for the solution of the Dirichlet principle. Until that date, it was commonly admitted that the functions to consider have to be regular $\mathbf{C}^1$ or $\mathbf{C}^2$ (differentiation being taken in the classical sense), depending on the situation.

It is only in 1900 with a famous conference at the Collège de France in Paris that Hilbert formulates the foundations of the modern variational approach to the Dirichlet principle and hence to the Dirichlet problem. These ideas, which have been worked out in the classical book of mathematical physics of Courant and Hilbert (1937) [181], can be summarized as follows.

The basic idea of Hilbert is to enlarge the class of functions in which one looks for a solution of the Dirichlet principle and simultaneously to generalize the notion of solution. More precisely, Hilbert proposed the following general method for solving the Dirichlet principle:

1. First, construct a minimizing sequence of functions $(u_n)_{n\in\mathbf{N}}$.

2. Then, extract from this minimizing sequence a convergent subsequence, say, $u_{n_k} \longrightarrow \bar{u}$. The so-obtained function $\bar{u}$ is the (generalized) solution of the original problem.

This is what in modern terminology is called a compactness argument. Let us first notice that, even when starting with a sequence $(u_n)_{n\in\mathbf{N}}$ of smooth functions, its limit $\bar{u}$ may be no more differentiable in the classical sense. Such construction of a generalized solution, which corresponds to finding a space obtained by a completion procedure, is very similar to the one which consists of passing from the set of rational numbers to the set of real numbers or from the Riemann integrable functions to Lebesgue integrable functions.

This was a decisive step, and this program was developed throughout the 20th century by a number of mathematicians from different countries.

The functional space in which to find the generalized solution was only in an implicit form in the work of Courant and Hilbert (as a completion of piecewise $\mathbf{C}^1$ functions). The celebrated Sobolev spaces were gradually introduced in the work of Friedrichs (1934) [221] and, for the Soviet mathematical school, Sobolev (1936) [336], and Kondrakov.

The compactness argument, that is, the compact embedding of the Sobolev space $H^1(\Omega)$ into $L^2(\Omega)$ when Ω is bounded, was proved by Rellich (1930).

The modern language of distributions which provides a generalized notion of derivatives for nonsmooth functions (and much more) was systematically developed by L. Schwartz (1950), who was teaching at the École Polytechnique in Paris. This has proved to be a very flexible tool for handling generalized solutions for PDEs.

The ideas of the compactness method introduced by Hilbert to solve the Dirichlet principle were developed in a systematic way by the Italian school. Tonelli (1921) had the intuition to put together the semicontinuity notion of Baire and the Ascoli–Arzelà compactness theorem. So doing, he was able to transfer from real functions to functionals of the calculus of variations (like the Dirichlet integral) the classical compactness argument. He developed the so-called direct methods in the calculus of variations, whose basic topological ingredients are the lower semicontinuity of the functional and the compactness of the lower level sets of the functional. In the line of the Hilbert approach, he founded the basis of the topological method for minimization problems in infinite dimensional spaces.

Thus, modern tools in variational analysis provide a general and quite simple approach to existence results of generalized solutions for a large number of boundary value problems from mathematical physics. The natural question which then arises is to study the regularity of such solutions and to establish under which conditions on the data and the domain we have a classical solution. A large number of contributions have been devoted to this difficult question. Let us say that in the case of the Dirichlet problem, if the domain Ω and g are sufficiently smooth, then there exists a classical solution $u \in \mathbf{C}^2(\bar{\Omega})$. For a detailed bibliography on the regularity problem, see Brezis [137, Chapter IX].

So far, we have considered the Dirichlet problem as it has been introduced historically. Indeed, one can reformulate it in an equivalent form which is more suitable for a variational treatment.

Let us introduce $\tilde{g} : \Omega \longrightarrow \mathbf{R}$, a function defined on the whole of Ω and whose restriction on $\partial\Omega$ is equal to g:

$$\tilde{g}|_{\partial\Omega} = g. \tag{2.20}$$

One usually prescribes $\tilde{g}$ to preserve the regularity properties on g and $\partial\Omega$ (for example, continuity or Lipschitz continuity) and, in most practical situations, this is quite easy to achieve. Take as a new unknown function

$$v := u - \tilde{g}. \tag{2.21}$$

Clearly it is equivalent to find v or u. The boundary value problem satisfied by v is

$$\begin{cases} -\Delta v = f & \text{on } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{2.22}$$

with $f = \Delta\tilde{g}$. The Dirichlet boundary data $v = 0$ on $\partial\Omega$ is then said to be homogeneous, and problem (2.22) is often called the homogeneous Dirichlet problem.

Note that a number of important physical situations lead to (2.22).

For example, when describing the electrostatic potential u in a domain Ω with a density of charge f and whose boundary is connected with the earth, then

$$\begin{cases} -\Delta u = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us consider an elastic membrane in the horizontal plane $x_3 = 0$ occupying a domain Ω in the (x_1, x_2) plane. Suppose that at each point $x \in \Omega$ a vertical force of intensity $f(x)$ is exerted and that the membrane is fixed on its boundary. Let us denote by $u(x)$ the vertical displacement of the point x of the membrane when the equilibrium is attained. Then

$$\begin{cases} -c\,\Delta u = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $c > 0$ is the elasticity coefficient of the membrane.

2.2 ▪ Test functions and distribution theory

2.2.1 ▪ Definition of distributions

The concept of distribution is quite natural if we start from some simple physical observations. Let us first consider a function $f \in L^1_{loc}(\Omega)$, where Ω is an open subset of $\mathbf{R}^N$. One cannot, for an arbitrary $x \in \Omega$, give a meaning to $f(x)$. But, from a physical point of view, it is meaningful to consider the average of f on a small ball with center x and radius $\varepsilon > 0$ and let ε go to zero. Indeed, it follows from the Lebesgue theory that for almost every $x \in \Omega$,

$$\lim_{\varepsilon \longrightarrow 0} \frac{1}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)} f(\xi)\,d\xi$$

exists and the limit is a representative of f. Such points x are called Lebesgue points of f. In the formula above $B(x,\varepsilon)$ denotes the ball with center x and radius ε and $|B(x,\varepsilon)|$ its Lebesgue measure. Let us notice that

$$\frac{1}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)} f(\xi)\,d\xi = \int_{\Omega} f(\xi) v_{x,\varepsilon}(\xi)\,d\xi,$$

where

$$v_{x,\varepsilon}(\xi)=\begin{cases}\frac{1}{|B(x,\varepsilon)|} & \text{if } \xi\in B(x,\varepsilon),\\ 0 & \text{elsewhere.}\end{cases}$$

Thus, it is equivalent to know f as an element of $L^1_{loc}(\Omega)$ or to know the value of the integrals $\int_\Omega f(x)v(x)dx$ for v belonging to a sufficiently large class of functions. This is the starting point of the notion of distribution. Functions v will be called test functions. It is equivalent to know f as a function or as a distribution, the distribution being viewed as the mapping assigning the real number $\int_\Omega f v dx$ to each test function v:

$$f : v \mapsto \int_\Omega f v d\xi .$$

If one is concerned only with L^1_{loc} functions, there are many possibilities for the choice of the class of test functions.

Let us go further and suppose we want to model the concept of a Dirac mass, that is, of a unit mass concentrated at a point. This is an important physical notion which can be viewed as the limiting case of the unit mass concentrated in a ball of radius $\varepsilon > 0$ with $\varepsilon \longrightarrow 0$. For example, consider the Dirac mass at the origin $0\in\Omega$ and the functions

$$f_\varepsilon(x)=\begin{cases}\frac{1}{|B(0,\varepsilon)|} & \text{if } x\in B(0,\varepsilon),\\ 0 & \text{elsewhere.}\end{cases}$$

Then, the distribution attached to f_ε is the mapping

$$v\mapsto \int_\Omega f_\varepsilon(\xi)v(\xi)d\xi = \frac{1}{|B(0,\varepsilon)|}\int_{B(0,\varepsilon)} v(\xi)d\xi .$$

When passing to the limit as $\varepsilon \longrightarrow 0$, we need to take test functions at least continuous (at the origin) in order for the above limit to exist. The limiting distribution is the mapping

$$v\in \mathbf{C}_c(\Omega)\mapsto v(0),$$

where $\mathbf{C}_c(\Omega)$ is the set of continuous real-valued functions, with compact support in Ω. This is the modern way to consider a Dirac mass (at the origin) as the linear mapping which to a regular test function v associates its value at the origin. Note that this distribution is no longer attached to a function.

A similar device can be developed to attach a distribution to more general mathematical objects, such as the derivative of an L^1_{loc} function. Take $f\in L^1_{loc}(\Omega)$ and try to define the distribution attached to $\frac{\partial f}{\partial x_i}$. Let us approximate f by a sequence f_n of smooth functions f_n. Then the distribution of $\frac{\partial f_n}{\partial x_i}$ is the mapping

$$v\mapsto \int_\Omega \frac{\partial f_n}{\partial x_i}(x)v(x)dx.$$

But we cannot pass to the limit on this quantity just by taking v continuous, like in the previous step. So let us assume v to be of class $\mathbf{C}^1$ on Ω with a compact support. Then, let us integrate by parts

$$\int_\Omega \frac{\partial f_n}{\partial x_i} v dx = -\int_\Omega f_n \frac{\partial v}{\partial x_i} dx;$$

we can now pass to the limit on this last expression. Finally, the distribution attached to $\frac{\partial f}{\partial x_i}$ is the mapping

$$v \in \mathbf{C}^1_c(\Omega) \mapsto -\int_\Omega f \frac{\partial v}{\partial x_i} dx,$$

where $\mathbf{C}^1_c(\Omega)$ is the set of real-valued functions of class $\mathbf{C}^1$ with compact support in Ω.

We are now ready to define the concept of distribution. We consider the space of test functions $\mathscr{D}(\Omega)$, which is the vector space of real or complex valued functions on Ω which are indefinitely derivable and with compact support in Ω. (This allows us to cover all the previous situations and much more!) For $v \in \mathscr{D}(\Omega)$, we say that the support of v is contained in a compact subset $K \subset \Omega$, and we write $\operatorname{spt} v \subset K$ if $v = 0$ on $\Omega \setminus K$ (equivalently, $\{v \neq 0\} \subset K$).

We use the following notation. An element $p \in \mathbf{N}^N$, $p = (p_1, p_2, \dots, p_N)$, where N is the dimension of the space ($\Omega \subset \mathbf{R}^N$), is called a multi-index. The integer $|p| = p_1 + p_2 + \cdots + p_N$ is called the length of the multi-index p.

For $v \in \mathscr{D}(\Omega)$, we write

$$D^p v := \frac{\partial^{|p|} v}{\partial x_1^{p_1} \partial x_2^{p_2} \dots \partial x_N^{p_N}}.$$

The operator D^p can be viewed as the composition of elementary partial derivation operators

$$D^p = \left(\frac{\partial}{\partial x_1}\right)^{p_1} \circ \cdots \circ \left(\frac{\partial}{\partial x_N}\right)^{p_N},$$

where $(\frac{\partial}{\partial x_i})^{p_i} = \frac{\partial}{\partial x_i} \circ \frac{\partial}{\partial x_i} \circ \cdots \circ \frac{\partial}{\partial x_i}$, p_i times.

Let us introduce the notion of sequential convergence on $\mathscr{D}(\Omega)$. It is the only topological notion on $\mathscr{D}(\Omega)$ that we use.

Definition 2.2.1. *A sequence $(v_n)_{n\in\mathbf{N}}$ of functions converges in the sense of the space $\mathscr{D}(\Omega)$ to a function $v \in \mathscr{D}(\Omega)$ if the two following conditions are satisfied:*

(i) *There exists a compact subset K in Ω such that $\operatorname{spt} v_n \subset K$ for all $n \in \mathbf{N}$ and $\operatorname{spt} v \subset K$.*

(ii) *For all multi-index $p \in \mathbf{N}^N$, $D^p v_n \longrightarrow D^p v$ uniformly on K.*

One can prove the existence of a locally convex topology on the space $\mathscr{D}(\Omega)$ with respect to which a linear functional F is continuous iff it is sequentially continuous, that is, $F(v_n) \longrightarrow F(v)$ whenever $v_n \longrightarrow v$ in the sense of $\mathscr{D}(\Omega)$. But this topology is not easy to handle (it is not metrizable); we don't really need to use it, so we will use only the notion of convergent sequence in $\mathscr{D}(\Omega)$ as defined above.

Definition 2.2.2. *A distribution T on Ω is a continuous linear form on $\mathscr{D}(\Omega)$. Equivalently, a linear form $T : \mathscr{D}(\Omega) \longrightarrow \mathbf{R}$ is a distribution on Ω if for any sequence $(v_n)_{n\in\mathbf{N}}$ in $\mathscr{D}(\Omega)$, the following implication holds:*

$$v_n \longrightarrow 0 \quad \text{in the sense of} \quad \mathscr{D}(\Omega) \Longrightarrow T(v_n) \longrightarrow 0.$$

The space of distributions on Ω is denoted by $\mathscr{D}'(\Omega)$. It is the topological dual space of $\mathscr{D}(\Omega)$ and we will write $\langle T, v\rangle_{(\mathscr{D}'(\Omega),\mathscr{D}(\Omega))} := T(v)$ the duality pairing between $T \in \mathscr{D}'(\Omega)$ and $v \in \mathscr{D}(\Omega)$.

Let us now give a practical criterion which allows us to verify that a linear form on $\mathscr{D}(\Omega)$ is continuous (and hence is a distribution).

Proposition 2.2.1. *Let T be a linear form on $\mathscr{D}(\Omega)$. Then T is a distribution on Ω iff for all compact K in Ω, there exists $n \in \mathbf{N}$ and $C \geq 0$, possibly depending on K, such that*

$$\forall v \in \mathscr{D}(\Omega) \text{ with } \operatorname{spt} v \subset K, \quad |T(v)| \leq C \sum_{|p| \leq n} ||D^p v||_\infty.$$

PROOF. Clearly, the above condition implies that T is continuous on $\mathscr{D}(\Omega)$. To prove the converse statement, let us argue by contradiction. Thus, given $T \in \mathscr{D}'(\Omega)$, let us assume that there exists a compact K in Ω and a sequence $(v_n)_{n \in \mathbf{N}}$ in $\mathscr{D}(\Omega)$ such that for each $n \in \mathbf{N}$

$$\operatorname{spt} v_n \subset K \text{ and } |T(v_n)| > n \sum_{|p| \leq n} ||D^p v_n||_\infty.$$

Let us define

$$w_n := \frac{1}{n \sum_{|p| \leq n} ||D^p v_n||_\infty} v_n.$$

Then $w_n \in \mathscr{D}(\Omega)$, $\operatorname{spt} w_n \subset K$, and for each $m \in \mathbf{N}$

$$D^m w_n = \frac{1}{n \sum_{|p| \leq n} ||D^p v_n||_\infty} D^m v_n,$$

so that

$$\forall\, n > m \quad ||D^m w_n||_\infty \leq \frac{1}{n},$$

and $w_n \longrightarrow 0$ in $\mathscr{D}(\Omega)$. By linearity of T

$$|T(w_n)| > 1,$$

so that $T(w_n)$ does not tend to zero, a contradiction with the fact that $T \in \mathscr{D}'(\Omega)$. □

Proposition 2.2.1 allows us to naturally introduce the notion of distribution with finite order.

Definition 2.2.3. *A distribution $T \in \mathscr{D}'(\Omega)$ has a finite order if there exists an integer $n \in \mathbf{N}$ such that for each compact subset $K \subset \Omega$, there exists a constant $C(K)$ such that*

$$\forall\, v \in \mathscr{D}(\Omega) \quad \text{with } \operatorname{spt} v \subset K, \ |T(v)| \leq C(K) \sup_{|p| \leq n} ||D^p v||_\infty.$$

If T has a finite order, the order of T is the smallest integer n for which the above inequality holds.

In Proposition 2.2.1, the integer n a priori depends on the compact set K. A distribution has a finite order if n can be taken independent of K.

Let us describe some first examples of distributions.

2.2.2 ▪ Locally integrable functions as distributions: Regularization by convolution and mollifiers

Take $f \in L^1_{loc}(\Omega)$, which means that for each compact subset K of Ω, $\int_K |f(x)|\,dx < +\infty$. One can associate to f the linear mapping

$$T_f : v \in \mathscr{D}(\Omega) \longmapsto \int_\Omega f(x)v(x)\,dx.$$

For any compact subset $K \subset \Omega$, for any $v \in \mathscr{D}(\Omega)$ with $\operatorname{spt} v \subset K$, the following inequality holds:

$$|T_f(v)| \le C(K)\|v\|_\infty$$

with $C(K) = \int_K |f(x)|\,dx < +\infty$. By Proposition 2.2.1 T_f is a distribution of order zero.

Indeed, a function $f \in L^1_{loc}(\Omega)$ is uniquely determined by its corresponding distribution T_f, as stated in the following.

Theorem 2.2.1. *Let $f \in L^1_{loc}(\Omega)$, $g \in L^1_{loc}(\Omega)$ be such that*

$$\forall v \in \mathscr{D}(\Omega) \quad \int_\Omega f(x)v(x)\,dx = \int_\Omega g(x)v(x)\,dx.$$

Then $f = g$ almost everywhere (a.e.) on Ω.

The above result allows us to identify $f \in L^1_{loc}$ with the corresponding distribution T_f, which gives the injection $L^1_{loc}(\Omega) \hookrightarrow \mathscr{D}'(\Omega)$.

The proof of Theorem 2.2.1 is a direct consequence of the density of the space of test functions $\mathscr{D}(\Omega)$ in the space $\mathbf{C}_c(\Omega)$.

Proposition 2.2.2. *$\mathscr{D}(\Omega)$ is dense in $\mathbf{C}_c(\Omega)$ for the topology of the uniform convergence. More precisely, for every $v \in \mathbf{C}_c(\Omega)$, there exists a sequence $(v_n)_{n\in\mathbf{N}}$, $v_n \in \mathscr{D}(\Omega)$, and a compact set $K \subset \Omega$ such that $v_n \to v$ uniformly and* $\operatorname{spt} v_n \subset K$.

PROOF. Take $v \in \mathbf{C}_c(\Omega)$ and extend v by zero outside of Ω. We so obtain an element, which we still denote by v, which is continuous on $\mathbf{R}^N$ and with compact support in Ω. Let us use the *regularization method by convolution* and introduce a smoothing kernel ρ:

$$\begin{cases} \rho \in \mathscr{D}(\mathbf{R}^N),\ \rho \ge 0, \\ \operatorname{spt} \rho \subset B(0,1), \\ \displaystyle\int_{\mathbf{R}^N} \rho(x)dx = 1. \end{cases}$$

Take, for example,

$$\rho(x) = \begin{cases} m\, e^{-1/(1-|x|^2)} & \text{if } |x| \le 1, \\ 0 & \text{elsewhere}, \end{cases}$$

m being chosen in order to have $\int_{\mathbf{R}^N} \rho(x)dx = 1$.

Then let us define for each integer $n = 1, 2, \dots,$

$$\rho_n(x) := n^N \rho(n\,x),$$

which satisfies

$$\begin{cases} \rho_n \in \mathscr{D}(\mathbf{R}^N),\ \rho_n \geq 0, \\ \operatorname{spt} \rho_n \subset B(0, 1/n), \\ \displaystyle\int_{\mathbf{R}^N} \rho_n(x)dx = 1. \end{cases}$$

The sequence $(\rho_n)_{n\in\mathbf{N}}$ is said to be a *mollifier*. Given $v \in \mathbf{C}_c(\mathbf{R}^N)$, let us define $v_n = v \star \rho_n$, that is,

$$v_n(x) = \int_{\mathbf{R}^N} v(y)\rho_n(x-y)dy.$$

We have

$$\begin{aligned} \operatorname{spt} v_n &\subset \operatorname{spt} v + \operatorname{spt} \rho_n \\ &\subset \operatorname{spt} v + B(0, 1/n) \end{aligned}$$

and v_n has a compact support in Ω for n large enough. The classical derivation theorem under the integral sign yields

$$\forall\ \alpha \in \mathbf{N}^N \qquad D^\alpha v_n = v \star D^\alpha \rho_n$$

and v_n belongs to $\mathbf{C}^\infty(\Omega)$. Thus v_n belongs to $\mathscr{D}(\Omega)$.

Let us now prove that v_n converges uniformly to v. To that end, we use the other equivalent formulation of v_n,

$$v_n(x) = \int_{\mathbf{R}^N} v(x-y)\rho_n(y)dy,$$

and the fact that $\int_{\mathbf{R}^N} \rho_n = 1$ to obtain

$$v_n(x) - v(x) = \int_{\mathbf{R}^N} [v(x-y) - v(x)]\rho_n(y)dy.$$

Using that $\operatorname{spt} \rho_n \in B(0,1)$, we have

$$\sup_{x\in\mathbf{R}^N} |v_n(x) - v(x)| \leq \sup_{\substack{x,z\in\mathbf{R}^N \\ \|x-z\|\leq 1/n}} |v(z) - v(x)|.$$

This last quantity goes to zero as $n \to +\infty$; this is a consequence of the uniform continuity of v on $\mathbf{R}^N$. (Recall that v is continuous with compact support.) □

Let us now complete the proof.

PROOF OF THEOREM 2.2.1. Take $h = f - g$, $h \in L^1_{loc}$, which satisfies

$$\forall v \in \mathscr{D}(\Omega) \quad \int_\Omega h(x)v(x)\,dx = 0.$$

By density of $\mathscr{D}(\Omega)$ in $\mathbf{C}_c(\Omega)$ (see Proposition 2.2.2) for any $v \in \mathbf{C}_c(\Omega)$ there exists a sequence $(v_n)_{n\in\mathbf{N}}$ in $\mathscr{D}(\Omega)$ such that

$$\begin{cases} \operatorname{spt} v_n \subset K & \text{for some fixed compact } K \text{ in } \Omega, \\ v_n \longrightarrow v & \text{uniformly on } K. \end{cases}$$

Since $\int_\Omega h(x)v_n(x)\,dx = \int_K h(x)v_n(x)\,dx = 0$, and $h \in L^1(K)$, by passing to the limit as $n \longrightarrow \infty$, we obtain

$$\forall\, v \in \mathbf{C}_c(\Omega) \quad \int_\Omega h(x)v(x)dx = 0.$$

The conclusion $h = 0$ follows by the Riesz–Alexandrov theorem (see Theorem 2.4.7) and the uniqueness of the representation. Let us give a direct independent proof of the fact that $h = 0$. It will mostly rely on the Tietze–Urysohn separation lemma.

One can first reduce to consider the case $h \in L^1(\Omega)$ with $|\Omega| < +\infty$ (write $\Omega = \bigcup_{n\in\mathbf{N}} \Omega_n$ with Ω_n open and $\overline{\Omega}_n$ compact, and take $h|_{\Omega_n}$). By density of $\mathbf{C}_c(\Omega)$ in $L^1(\Omega)$, for each $\varepsilon > 0$ there exists some $h_\varepsilon \in \mathbf{C}_c(\Omega)$ such that $\|h - h_\varepsilon\|_{L^1(\Omega)} < \varepsilon$. Hence

$$\left| \int_\Omega h_\varepsilon(x)v(x)\,dx \right| \le \varepsilon \|v\|_{L^\infty(\Omega)} \quad \forall v \in \mathbf{C}_c(\Omega). \tag{2.23}$$

Consider

$$K_1 = \{x \in \Omega : h_\varepsilon(x) \ge \varepsilon\}, \qquad K_2 = \{x \in \Omega : h_\varepsilon(x) \le -\varepsilon\}.$$

These two sets are disjoint and compact. By the Tietze–Urysohn separation lemma, there exists a function $\varphi \in \mathbf{C}_c(\Omega)$ such that

$$\begin{cases} \varphi(x) = 1 & \text{on } K_1, \\ \varphi(x) = -1 & \text{on } K_2, \\ -1 \le \varphi(x) \le 1 & \forall\, x \in \Omega. \end{cases}$$

Taking $K = K_1 \cup K_2$,

$$\int_\Omega |h_\varepsilon| dx = \int_K |h_\varepsilon| dx + \int_{\Omega\setminus K} |h_\varepsilon|\,dx.$$

On K, we have $|h_\varepsilon| = h_\varepsilon\, \varphi$, so that

$$\int_K |h_\varepsilon|\,dx = \int_K h_\varepsilon \varphi = \int_\Omega h_\varepsilon\, \varphi\,dx - \int_{\Omega\setminus K} h_\varepsilon\, \varphi\,dx.$$

By (2.23), $\int_\Omega h_\varepsilon \varphi\,dx \le \varepsilon\|\varphi\|_{L^\infty} \le \varepsilon$ and, since $|h_\varepsilon| \le \varepsilon$ on $\Omega \setminus K$,

$$\left| \int_{\Omega\setminus K} h_\varepsilon\, \varphi\,dx \right| \le \varepsilon|\Omega|.$$

So,

$$\int_K |h_\varepsilon|\,dx \le \varepsilon(1 + |\Omega|).$$

Finally,

$$\int_\Omega |h_\varepsilon|\,dx \le \varepsilon(1 + |\Omega|) + \varepsilon|\Omega| = \varepsilon + 2\varepsilon|\Omega|$$

and

$$\int_\Omega |h|\,dx \le \int_\Omega |h - h_\varepsilon|\,dx + \int_\Omega |h_\varepsilon|\,dx \le 2\varepsilon(1 + |\Omega|).$$

This being true for any $\varepsilon > 0$, we conclude that $h = 0$. $\square$

Noticing that $L^p(\Omega) \hookrightarrow L^1_{loc}(\Omega)$ for any $1 \le p \le +\infty$, as a direct consequence of Proposition 2.2.2, we obtain the following corollary.

Corollary 2.2.1. *Given $1 \le p \le +\infty$, let us suppose that $f \in L^p(\Omega)$, $g \in L^p(\Omega)$ satisfy*

$$\int_\Omega f(x)v(x)\,dx = \int_\Omega g(x)v(x)\,dx \quad \forall v \in \mathcal{D}(\Omega);$$

then $f = g$ a.e. on Ω.

Let us notice that when taking $1 < p < +\infty$, we can obtain this result in a more direct way, by using the density of $\mathcal{D}(\Omega)$ in $L^q(\Omega)$, $1 \le q < +\infty$. Then take $q = p'$ the Hölder conjugate exponent of p, $\frac{1}{p} + \frac{1}{p'} = 1$. Clearly the density of $\mathcal{D}(\Omega)$ in $L^q(\Omega)$ is a consequence of Proposition 2.2.2. Because of the importance of this result, let us give another proof of it, of independent interest, which relies only on L^p techniques.

Proposition 2.2.3. *Let Ω be an arbitrary open subset of $\mathbf{R}^N$. Then $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$ for $1 \le p < +\infty$.*

This will result from the following.

Proposition 2.2.4. *Let $f \in L^p(\mathbf{R}^N)$ with $1 \le p < +\infty$. Then, for any mollifier $(\rho_n)_{n\in\mathbf{N}}$, the following properties hold:*

(i) $f \star \rho_n \in L^p(\mathbf{R}^N)$,

(ii) $\|f \star \rho_n\|_{L^p(\mathbf{R}^N)} \le \|f\|_{L^p(\mathbf{R}^N)}$,

(iii) $f \star \rho_n \longrightarrow f$ *in* $L^p(\mathbf{R}^N)$ *as* $n \to +\infty$.

PROOF. To prove (i) and (ii) we omit the subscript $n \in \mathbf{N}$. Let us consider the case $1 < p < \infty$ and introduce p' with $\frac{1}{p} + \frac{1}{p'} = 1$:

$$\begin{aligned}|(f \star \rho)(x)| &\le \int_{\mathbf{R}^N} |f(x-y)|\rho(y)dy \\ &\le \int_{\mathbf{R}^N} |f(x-y)|\rho(y)^{1/p}\,\rho(y)^{1/p'}\,dy.\end{aligned}$$

Let us apply the Hölder inequality

$$|(f \star \rho)(x)| \le \left(\int_{\mathbf{R}^N} |f(x-y)|^p \rho(y)dy\right)^{1/p} \left(\int_{\mathbf{R}^N} \rho(y)\,dy\right)^{1/p'}.$$

Since $\int_{\mathbf{R}^N} \rho(y)dy = 1$, we obtain

$$|(f \star \rho)(x)|^p \le \int_{\mathbf{R}^N} |f(x-y)|^p \rho(y)dy.$$

Let us integrate with respect to $x \in \mathbf{R}^N$ and apply the Fubini–Tonelli theorem

$$\begin{aligned}\int_{\mathbf{R}^N} |(f \star \rho)(x)|^p\,dx &\le \int_{\mathbf{R}^N}\left(\int_{\mathbf{R}^N} |f(x-y)|^p \rho(y)\,dy\right)dx \\ &\le \int_{\mathbf{R}^N} \rho(y)\left(\int_{\mathbf{R}^N} |f(x-y)|^p\,dx\right)dy \\ &\le \int_{\mathbf{R}^N} |f(x)|^p\,dx,\end{aligned}$$

where we have used again that $\int \rho\, dx = 1$ and the fact that the Lebesgue measure on $\mathbf{R}^N$ is invariant by translation. Thus, $f \star \rho \in L^p$ and

$$\|f \star \rho\|_{L^p} \le \|f\|_{L^p}.$$

The convergence of $f \star \rho_n$ to f in $L^p(\mathbf{R}^N)$ relies on a quite similar computation. Using that $\int \rho_n\, dx = 1$, we can write

$$f(x) - (f \star \rho_n)(x) = \int_{\mathbf{R}^N} [f(x) - f(x-y)]\rho_n(y)\, dy$$

and

$$|f(x) - (f \star \rho_n)(x)| \le \int_{\mathbf{R}^N} |f(x) - f(x-y)|\rho_n(y)\, dy.$$

Let us rewrite this last inequality as

$$|f(x) - (f \star \rho_n)(x)| \le \int_{\mathbf{R}^N} |f(x) - f(x-y)|\rho_n(y)^{1/p}\rho_n(y)^{1/p'}\, dy$$

and apply the Hölder inequality to obtain

$$|f(x) - (f \star \rho_n)(x)|^p \le \int_{\mathbf{R}^N} |f(x) - f(x-y)|^p \rho_n(y)\, dy.$$

Integrating with respect to x on $\mathbf{R}^N$ and applying the Fubini–Tonelli theorem, we obtain

$$\|f - (f \star \rho_n)\|_{L^p}^p \le \int_{\mathbf{R}^N} \rho_n(y)\, \|f - \tau_y f\|_{L^p}^p\, dy.$$

Let us introduce $\varphi(y) := \|f - \tau_y f\|_{L^p}^p$. Since $f \in L^p(\mathbf{R}^N)$, φ is a continuous function on $\mathbf{R}^N$ such that $\varphi(0) = 0$. We conclude thanks to the following property:

$$\forall \varphi : \mathbf{R}^N \longrightarrow \mathbf{R}^N \text{ continuous with } \varphi(0) = 0, \text{ we have}$$

$$\lim_{n \to +\infty} \int_{\mathbf{R}^N} \varphi(y)\rho_n(y)\, dy = 0.$$

This results from the inequality

$$\begin{aligned}\left| \int_{\mathbf{R}^N} \varphi(y)\rho_n(y)\, dy - \varphi(0) \right| &= \left| \int_{\mathbf{R}^N} (\varphi(y) - \varphi(0))\rho_n(y)\, dy \right| \\ &\le \int_{B(0,1/n)} |\varphi(y) - \varphi(0)|\rho_n(y)\, dy \\ &\le \sup_{|y| \le 1/n} |\varphi(y) - \varphi(0)|,\end{aligned}$$

which tends to zero as $n \to +\infty$. Indeed, we will interpret this last result as a convergence in $\mathscr{D}'(\Omega)$ of the sequence (ρ_n) to δ_0, the Dirac mass at the origin. □

We can now complete the proof.

PROOF OF PROPOSITION 2.2.3. Take $f \in L^p(\Omega)$, $\varepsilon > 0$, and $g \in \mathbf{C}_c(\Omega)$ such that

$$||f - g||_{L^p(\Omega)} < \varepsilon.$$

Then let us extend g outside of Ω by zero to obtain a function that we still denote by g which belongs to $\mathbf{C}_c(\mathbf{R}^N)$. Take $f_n = g \star \rho_n$. Then, $f_n \in \mathscr{D}(\mathbf{R}^N)$, and in fact $f_n \in \mathscr{D}(\Omega)$ for n large enough, because $\operatorname{spt} f_n \subset \operatorname{spt} g + B(0, 1/n)$. Moreover, $f_n \longrightarrow g$ in $L^p(\Omega)$, so that

$$||f - f_n||_{L^p} \leq \varepsilon$$

for n large enough. □

2.2.3 ▪ Radon measures

Let us recall that a Radon measure μ is a linear form on $\mathbf{C}_c(\Omega)$ such that for each compact $K \subset \Omega$, the restriction of μ to $\mathbf{C}_K(\Omega)$ is continuous, that is, for each $K \subset \Omega$, K compact, there exists some $C(K) \geq 0$ such that

$$\forall\, v \in \mathbf{C}_c(\Omega) \quad \text{with } \operatorname{spt} v \subset K, \quad |\mu(v)| \leq C(K)||v||_\infty.$$

To such a Radon measure, one can associate its restriction to $D(\Omega)$,

$$T_\mu : v \in \mathscr{D}(\Omega) \longrightarrow \int_\Omega v(x)\, d\mu(x),$$

which by the definition of μ is a distribution of order zero.

Conversely, μ is completely determined by the corresponding distribution T_μ. This is a consequence of the density of $\mathscr{D}(\Omega)$ in $\mathbf{C}_c(\Omega)$; see Proposition 2.2.2. As a consequence, we can identify any Radon measure with its corresponding distribution and $\mathscr{M} \hookrightarrow \mathscr{D}'(\Omega)$.

As a typical example of a distribution measure which is not in $L^1_{loc}(\Omega)$, if $0 \in \Omega$, take $\mu = \delta_0$ the Dirac mass at the origin, with

$$\langle \mu, v \rangle_{(\mathscr{D}', \mathscr{D})} := v(0).$$

To describe further examples of great importance in applications, we need to introduce further notions, namely, the derivation of distributions and weak limits of distributions.

2.2.4 ▪ Derivation of distributions, introduction to Sobolev spaces

Definition 2.2.4. *Let $T \in \mathscr{D}'(\Omega)$ be a distribution on Ω. Then $\frac{\partial T}{\partial x_i}$ is defined as the linear mapping on $\mathscr{D}(\Omega)$,*

$$\frac{\partial T}{\partial x_i} : v \in \mathscr{D}(\Omega) \longmapsto - \left\langle T, \frac{\partial v}{\partial x_i} \right\rangle_{(\mathscr{D}', \mathscr{D})}.$$

More generally, for any multi-index $p = (p_1, \dots, p_N)$, we define

$$D^p T : v \in \mathscr{D}(\Omega) \longmapsto (-1)^{|p|} \langle T, D^p v \rangle_{(\mathscr{D}', \mathscr{D})}.$$

Proposition 2.2.5. *For any distribution T on Ω, for any multi-index $p \in \mathbf{N}^N$, we have that $D^p T$ is still a distribution on Ω. Therefore, for any $v \in \mathscr{D}(\Omega)$*

$$\langle D^p T, v \rangle_{(\mathscr{D}'(\Omega), \mathscr{D}(\Omega))} = (-1)^{|p|} \langle T, D^p v \rangle_{(\mathscr{D}'(\Omega), \mathscr{D}(\Omega))}.$$

PROOF. One just needs to notice that for $p \in \mathbf{N}^N$ fixed, the mapping $v \longmapsto D^p v$ is continuous from $\mathscr{D}(\Omega)$ into $\mathscr{D}(\Omega)$. This is an immediate consequence of the definition of the sequential convergence in $\mathscr{D}(\Omega)$, which, we recall, involves a compact support condition and the uniform convergence of the derivatives of arbitrary order. These two properties are clearly preserved by the operations D^p. □

Therefore, every distribution in $\mathscr{D}'(\Omega)$ possesses derivatives of arbitrary orders in $\mathscr{D}'(\Omega)$. Indeed, the notion of derivative $D^p T$ of a distribution has been defined so as to extend the classical notion of derivative for a smooth function. Let us recall the identification we make between $f \in L^1_{loc}$ and the corresponding distribution T_f.

Proposition 2.2.6. *Let f be some function in the set $\mathbf{C}^m(\Omega)$ of real-valued functions of class $\mathbf{C}^m$ in Ω. Then, for any $p \in \mathbf{N}^N$ with $|p| \leq m$, the distribution derivative $D^p f$ coincides with the classical derivative $D^p f$ of functions.*

PROOF. It is a direct consequence of the integration by parts formula. If $f \in \mathbf{C}^1(\Omega)$ and $v \in \mathscr{D}(\Omega)$,

$$\int_\Omega \frac{\partial f}{\partial x_i}(x)\, v(x)\, dx = -\int_\Omega f(x) \frac{\partial v}{\partial x_i}(x)\, dx.$$

Similarly, integration by parts $|p|$ times gives the following formula:

$$\int_\Omega (D^p f)(x)\, v(x)\, dx = (-1)^{|p|} \int_\Omega f(x)\, D^p v(x)\, dx;$$

this formula is valid for $f \in \mathbf{C}^{|p|}(\Omega)$ and $v \in \mathscr{D}(\Omega)$. □

The fact that the test function v has a *compact support* is essential in making, in the integration by parts formula, the integral term on $\partial\Omega$ equal to zero. We stress the fact that the notion of derivative $D^p T$ of a distribution $T \in \mathscr{D}'(\Omega)$ takes as a definition the integration by parts formula, the derivation operation being transferred, by this operation, on the test functions. This can be done at an arbitrary order since the test functions have been taken *indefinitely differentiable*. The two previous remarks justify the *choice of test functions $v \in \mathscr{D}(\Omega)$*.

We can now describe a fundamental example of distribution coming from the theory of Sobolev spaces. This theory, which plays a central role in the variational approach to a large number of boundary value problems (like the Dirichlet problem) will be developed in detail in Chapter 5. Here we give some definitions and elementary examples. For any $m \in \mathbf{R}$, $p \in [1, +\infty]$,

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega) : D^j f \in L^p(\Omega) \quad \forall\, j,\ |j| \leq m\}.$$

One of the most important Sobolev spaces is the space

$$W^{1,2}(\Omega) = H^1(\Omega) = \left\{ f \in L^2(\Omega) : \frac{\partial f}{\partial x_i} \in L^2(\Omega),\ i = 1, 2, \ldots, N \right\}.$$

In the above definition, the derivation $\frac{\partial f}{\partial x_i}$ (or more generally $D^j f$) is taken in the distribution sense. We will see that the choice of this notion of derivation is fundamental to obtain the desirable properties for the corresponding spaces.

As an elementary example, let us consider $\Omega=(-1,1)$ and $f(x)=|x|$. Clearly, f is not differentiable in the classical sense at the origin. The function f is continuous on Ω, it belongs to any $L^p(\Omega)$, $1\le p\le+\infty$, and thus it defines a distribution and we can compute its first distribution derivative Df

$$\langle Df,v\rangle_{(\mathscr{D}'(\Omega),\mathscr{D}(\Omega))}:=-\int_{-1}^{1} f(x)v'(x)dx.$$

Let us write

$$\int_{-1}^{1} f(x)v'(x)dx=\int_{-1}^{0} f(x)v'(x)dx+\int_{0}^{1} f(x)v'(x)dx$$

and let us integrate by parts on each interval $(-1,0)$ and $(0,1)$. This is possible since now $f\in\mathbf{C}^1([-1,0])$ and $f\in\mathbf{C}^1([0,1])$. We have

$$\int_{-1}^{0} f(x)v'(x)dx=f(0)v(0)-f(-1)v(-1)-\int_{-1}^{0} f'(x)v(x)dx,$$

$$\int_{0}^{1} f(x)v'(x)dx=f(1)v(1)-f(0)v(0)-\int_{0}^{1} f'(x)v(x)dx.$$

Note that since $v\in\mathscr{D}(-1,1)$, we have $v(-1)=v(1)=0$, but for a general $v\in\mathscr{D}(-1,1)$, $v(0)\neq 0$. By adding the two above equalities, the terms containing $v(0)$ cancel and we obtain

$$\forall v\in\mathscr{D}(-1,1)\quad \int_{-1}^{1} f(x)v'(x)\,dx=-\left[\int_{-1}^{0} -v(x)\,dx+\int_{0}^{1} v(x)\,dx\right],$$

that is,

$$\int_{-1}^{1} f(x)v'(x)dx=-\int_{-1}^{1} g(x)v(x)dx,$$

where

$$g(x)=\begin{cases}-1 & \text{if } -1<x<0,\\ 1 & \text{if } 0<x<1.\end{cases}$$

The above function g is then the distributional derivative of $f(x)=|x|$ on $\Omega=(-1,1)$. It belongs to $L^p(\Omega)$ for any $1\le p\le+\infty$, so that $f\in W^{1,p}(\Omega)$ for any $1\le p\le+\infty$.

The parameters $m\in\mathbf{N}$ and $p\in[1,+\infty]$ yield a scale of spaces which allow us to distinguish, for example, in our situation the different behavior of the functions $f_\alpha(x)=|x|^\alpha$ and of their derivatives at zero. A similar computation as above yields

$$Df_\alpha=\alpha|x|^{\alpha-2}x \quad \text{in } \mathscr{D}'(-1,1).$$

Hence, $f_\alpha\in W^{1,p}(-1,1)$ iff $\int_{-1}^{1}|x|^{p(\alpha-1)}dx<+\infty$, that is, $p<\frac{1}{1-\alpha}$. When $p=2$, we have that $f_\alpha(x)=|x|^\alpha$ belongs to $H^1(-1,1)$ iff $\alpha>\frac{1}{2}$: this expresses that in some sense the derivative of f_α at x does not tend to $+\infty$ too rapidly when x goes to zero.

Let us now examine the other parameter m, which is relative to the order of derivation. Take again $f(x)=|x|$ and compute the second-order derivative of f on $(-1,1)$. This amounts to computing the first-order derivative of $g(x)=\operatorname{sign} x$. Thus

$$\langle D^2 f,v\rangle_{(\mathscr{D}',\mathscr{D})}=\langle Dg,v\rangle_{(\mathscr{D}',\mathscr{D})}=-\int_{-1}^{1} g(x)v'(x)dx.$$

As before, let us split the integral over $(-1,1)$ into two parts,

$$\begin{aligned}\int_{-1}^{1} g(x)v'(x)\,dx &= -\int_{-1}^{0} v'(x)\,dx + \int_{0}^{1} v'(x)\,dx \\ &= -[v(0)-v(-1)] + [v(1)-v(0)] \\ &= -2v(0),\end{aligned}$$

since $v \in \mathscr{D}(-1,1)$ and $v(-1) = v(1) = 0$. Thus

$$\langle Dg, v\rangle_{(\mathscr{D}'(-1,1),\mathscr{D}(-1,1))} = 2v(0)$$

and $Dg = 2\delta_0$, where δ_0 is the Dirac mass at the origin.

But the distribution δ_0 is a measure which is not representable by a function: suppose that there exists a function $h \in L^1_{loc}$ such that

$$\forall v \in \mathscr{D}(-1,1) \quad v(0) = \int_{-1}^{1} h(x)v(x)\,dx.$$

Then, taking successively $v \in \mathscr{D}(-1,0)$ and $v \in \mathscr{D}(0,1)$, we conclude by Theorem 2.2.1 that $h = 0$ a.e. on $(-1,0)$ and on $(0,1)$. Hence $h = 0$ a.e. on $(-1,1)$, which would imply $v(0) = 0$ for every $v \in \mathscr{D}(-1,1)$, a clear contradiction. Therefore $f \in W^{1,2}(-1,1)$ but $f \notin W^{2,1}(-1,1)$.

The above computation of the distributional derivative of a function g which has a discontinuity is very important in a number of applications (phase transitions, plasticity, image segmentation, etc.). This situation will be considered in detail in Chapter 10 and will lead us to the introduction of the functional space $BV(\Omega)$, the space of functions with bounded variation, which can be characterized as the space of integrable functions whose first distributional derivatives are bounded measures.

The next operation on distributions which is very useful for applications is the notion of limit of a sequence of distributions.

2.2.5 ▪ Convergence of sequences of distributions

Definition 2.2.5. *Let $T_n \in \mathscr{D}'(\Omega)$ for all $n \in \mathbf{N}$ and $T \in \mathscr{D}'(\Omega)$. The sequence $(T_n)_{n\in\mathbf{N}}$ is said to converge to T in $\mathscr{D}'(\Omega)$ if*

$$\forall v \in \mathscr{D}(\Omega) \quad \lim_{n\to+\infty} T_n(v) = T(v).$$

We will write $T_n \longrightarrow T$ in $\mathscr{D}'(\Omega)$ or $\lim_{n\to+\infty} T_n = T$ in $\mathscr{D}'(\Omega)$.

In Section 2.4, we will interpret this convergence as a weak* convergence in the dual space $\mathscr{D}'(\Omega)$ of $\mathscr{D}(\Omega)$.

Let us recall that any distribution $T \in \mathscr{D}'(\Omega)$ possesses derivatives or arbitrary orders. The following result expresses that for any multi-index $p \in \mathbf{N}^N$ the mapping $T \longrightarrow D^p T$ is continuous.

Proposition 2.2.7. *Let $p \in \mathbf{N}^N$. The mapping*

$$T \in \mathscr{D}'(\Omega) \longmapsto D^p T \in \mathscr{D}'(\Omega)$$

is continuous, which means that for any sequence $(T_n)_{n\in\mathbf{N}}$, T in $\mathscr{D}'(\Omega)$, the following implication holds:

$$T_n \longrightarrow T \text{ in } \mathscr{D}'(\Omega) \implies D^p T_n \longrightarrow D^p T \text{ in } \mathscr{D}'(\Omega).$$

PROOF. The proof is a direct consequence of the definitions. Let us assume that $T_n \longrightarrow T$ in $\mathscr{D}'(\Omega)$ and take $v \in \mathscr{D}(\Omega)$.

By definition of D^p

$$\langle D^p T_n, v\rangle = (-1)^{|p|}\langle T_n, D^p v\rangle \quad \forall\, v \in \mathscr{D}(\Omega).$$

Since $v \in \mathscr{D}(\Omega)$, $D^p v$ still belongs to $\mathscr{D}(\Omega)$, and the convergence of T_n to T in $\mathscr{D}'(\Omega)$ implies

$$\lim_{n\to+\infty} \langle T_n, D^p v\rangle = \langle T, D^p v\rangle.$$

Thus, again by the definition of D^p,

$$\lim_{n\to+\infty} \langle D^p T_n, v\rangle = (-1)^{|p|}\langle T, D^p v\rangle = \langle D^p T, v\rangle,$$

which expresses that $D^p T_n \longrightarrow D^p T$ in $\mathscr{D}'(\Omega)$. □

The above proposition is one of the reasons for the success of the theory of distributions. It makes this theory a very flexible tool for the study of PDEs. We will often use this type of argument—for example, in the chapter on Sobolev spaces. Suppose $(v_n)_{n\in\mathbf{N}}$ is a sequence in $H^1(\Omega)$ such that

$$\begin{aligned} v_n &\longrightarrow v \text{ in } L^2(\Omega), \\ \frac{\partial v_n}{\partial x_i} &\longrightarrow g_i \text{ in } L^2(\Omega), \qquad i = 1,2,\dots,N. \end{aligned}$$

Then, $v \in H^1(\Omega)$ and $g_i = \frac{\partial v}{\partial x_i}$, $i = 1,2,\dots,N$. This can be justified with the language of distribution as follows. Since $v_n \longrightarrow v$ in $L^2(\Omega)$, $v_n \longrightarrow v$ in $\mathscr{D}'(\Omega)$ and hence $\frac{\partial v_n}{\partial x_i} \longrightarrow \frac{\partial v}{\partial x_i}$ in $\mathscr{D}'(\Omega)$. On the other hand, $\frac{\partial v_n}{\partial x_i} \longrightarrow g_i$ in $L^2(\Omega)$ and hence in $\mathscr{D}'(\Omega)$. The uniqueness of the limit in $\mathscr{D}'(\Omega)$ implies that $\frac{\partial v}{\partial x_i} = g_i$ for all $i = 1,2,\dots,N$ and $v \in H^1(\Omega)$.

Let us give another illustration of the above tools and compute the fundamental solution of the Laplacian in $\mathbf{R}^3$.

Proposition 2.2.8. *Take $N = 3$ and consider the Newtonian potential*

$$f(x) = \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}.$$

Then $-\Delta\left(\frac{1}{4\pi} f\right) = \delta$.

PROOF. Let us denote by $r(x) = \sqrt{x_1^2 + x_2^2 + x_3^2}$ the Euclidean distance of $x = (x_1, x_2, x_3)$ from the origin, and notice that $f(x) = \frac{1}{r(x)}$ belongs to $L^1_{loc}(\mathbf{R}^3)$ and thus defines a distribution on $\mathbf{R}^3$. Let us compute Δf in $\mathscr{D}'(\Omega)$. A standard approach consists in approximating f by a sequence f_ε of smooth functions, then computing Δf_ε in a classical sense by Proposition 2.2.6 and passing to the limit in $\mathscr{D}'(\Omega)$ as $\varepsilon \to 0$. Clearly, the difficulty is at the origin where f has a singularity. Thus, the parameter ε is intended to isolate the origin and regularize f at the origin. At this point, there are two possibilities that lead to different computations and that we examine now.

First, take for $\varepsilon > 0$

$$f_\varepsilon(x) = \begin{cases} 1/\varepsilon & \text{if } r(x) \le \varepsilon, \\ 1/r(x) & \text{if } r(x) \ge \varepsilon. \end{cases}$$

Clearly, f_ε is now a continuous, piecewise smooth function (it is not $\mathbf{C}^1$) on $\mathbf{R}^3$ and a standard computation yields that $\frac{\partial f_\varepsilon}{\partial x_i}$ belongs to $L^2(\mathbf{R}^3)$ with

$$\frac{\partial f_\varepsilon}{\partial x_i} = \begin{cases} 0 & \text{if } r < \varepsilon, \\ -x_i/r^3 & \text{if } r > \varepsilon. \end{cases}$$

Note that $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$ for $i = 1, 2, 3$.

Let us now compute $-\Delta f_\varepsilon$. By definition,

$$\begin{aligned} \langle -\Delta f_\varepsilon, v\rangle_{(\mathcal{D}'(\mathbf{R}^3),\mathcal{D}(\mathbf{R}^3))} &= \langle f_\varepsilon, -\Delta v\rangle_{(\mathcal{D}',\mathcal{D})} \\ &= \sum_{i=1}^{3} \left\langle \frac{\partial f_\varepsilon}{\partial x_i}, \frac{\partial v}{\partial x_i} \right\rangle_{(\mathcal{D}',\mathcal{D})}. \end{aligned}$$

Since $\frac{\partial f_\varepsilon}{\partial x_i}$ belongs to L^1_{loc},

$$\begin{aligned} \langle -\Delta f_\varepsilon, v\rangle &= \sum_{i=1}^{3} \int_{\mathbf{R}^3} \frac{\partial f_\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_i} dx \\ &= -\sum_{i=1}^{3} \int_{r \ge \varepsilon} \frac{x_i}{r^3} \cdot \frac{\partial v}{\partial x_i}\, dx. \end{aligned}$$

Let us now integrate by parts this last expression. Noticing that on $\mathbf{R}^3 \setminus \{0\}$,

$$\begin{aligned} \sum_i \frac{\partial}{\partial x_i}\left(\frac{x_i}{r^3}\right) &= \frac{3}{r^3} + \sum x_i \cdot \frac{-3}{r^4} \cdot \frac{x_i}{r} \\ &= \frac{3}{r^3} - 3\sum \frac{x_i^2}{r^5} = \frac{3}{r^3} - \frac{3}{r^3} = 0 \end{aligned}$$

(which means that $\Delta f = 0$ on $\mathbf{R}^3 \setminus \{0\}$), we obtain

$$\langle -\Delta f_\varepsilon, v\rangle = -\int_{S_\varepsilon} \sum_{i=1}^{3} \frac{x_i}{r^3}\left(-\frac{x_i}{r}\right) v\, dx,$$

where $S_\varepsilon = \{x \in \mathbf{R}^3 : r(x) = \varepsilon\}$ is the sphere of radius ε centered at the origin. Note that the unit normal to S_ε at x which is oriented toward the outside of $\{r \ge \varepsilon\}$ is equal to $-\frac{x}{r}$. Hence

$$\begin{aligned} \langle -\Delta f_\varepsilon, v\rangle_{(\mathcal{D}',\mathcal{D})} &= \int_{S_\varepsilon} \frac{1}{r^2}(x) v(x)\, dx \\ &= \langle \mu_\varepsilon, v\rangle_{(\mathcal{D}',\mathcal{D})}, \end{aligned}$$

where $\mu_\varepsilon = \varepsilon^{-2}\mathscr{H}^2 \lfloor S_\varepsilon$ and $\mathscr{H}^2 \lfloor S_\varepsilon$ is the two-dimensional Hausdorff measure supported by S_ε. An elementary calculus yields that

$$\lim_{\varepsilon\to 0} \frac{1}{4\pi}\mu_\varepsilon = \delta \text{ in } \mathscr{D}'(\mathbf{R}^3).$$

By definition of the convergence in $\mathscr{D}'(\mathbf{R}^3)$ we have

$$-\Delta\left(\frac{1}{4\pi}f_\varepsilon\right) \longrightarrow \delta \text{ in } \mathscr{D}'(\mathbf{R}^3).$$

On the other hand, f_ε converges to f in $L^1(\mathbf{R}^3)$ (for example, by the dominated convergence theorem) and hence in $\mathscr{D}'(\mathbf{R}^3)$. By the continuity of the differential operator Δ in $\mathscr{D}'(\mathbf{R}^3)$ (cf. Proposition 2.2.7) we finally obtain that $-\Delta\left(\frac{1}{4\pi}f\right) = \delta$.

Another regularization consists of building f_ε a $\mathbf{C}^1(\mathbf{R}^3)$ function which approximates f. Take, for example,

$$f_\varepsilon(x) = \begin{cases} a_\varepsilon r^2(x) + b_\varepsilon & \text{if } r(x) \le \varepsilon, \\ 1/r(x) & \text{if } r(x) \ge \varepsilon, \end{cases}$$

a_ε and b_ε being chosen in order to have $f_\varepsilon \in \mathbf{C}^1(\mathbf{R}^3)$. This is equivalent to the system

$$\begin{cases} a_\varepsilon \varepsilon^2 + b_\varepsilon = 1/\varepsilon, \\ \qquad\quad a_\varepsilon = -1/(2\varepsilon^3), \end{cases}$$

which gives $b_\varepsilon = 3/(2\varepsilon)$. Noticing that

$$\begin{aligned} a_\varepsilon r^2 + b_\varepsilon &= -\frac{r^2}{2\varepsilon^3} + \frac{3}{2\varepsilon} \\ &= \frac{1}{2\varepsilon}\left[3 - \frac{r^2}{\varepsilon^2}\right] \\ &\le \frac{3}{2\varepsilon} \quad \text{for } r(x) \le \varepsilon, \end{aligned}$$

we have that

$$0 \le f_\varepsilon(x) \le \frac{3}{2r(x)} = \frac{3}{2}f(x) \text{ on } \mathbf{R}^3.$$

Hence, by the Lebesgue dominated convergence, $f_\varepsilon \longrightarrow f$ in $L^1(\mathbf{R}^3)$ as $\varepsilon \longrightarrow 0$. So $f_\varepsilon \longrightarrow f$ in $\mathscr{D}'(\mathbf{R}^3)$ and

$$\Delta f_\varepsilon \longrightarrow \Delta f \text{ in } \mathscr{D}'(\mathbf{R}^3).$$

An elementary computation yields

$$\begin{aligned} -\Delta f_\varepsilon &= -6a_\varepsilon 1_{B(0,\varepsilon)} \\ &= +\frac{3}{\varepsilon^3}1_{B(0,\varepsilon)}. \end{aligned}$$

Hence

$$-\Delta\left(\frac{1}{4\pi}f_\varepsilon\right) = \frac{1}{\frac{4}{3}\pi\varepsilon^3}1_{B(0,\varepsilon)}.$$

Noticing that $\frac{4}{3}\pi\varepsilon^3$ is precisely the volume of the ball $B(0,\varepsilon)$, we have

$$\lim_{\varepsilon\to 0} \frac{1}{\frac{4}{3}\pi\varepsilon^3}1_{B(0,\varepsilon)} = \delta,$$

which finally implies

$$-\Delta\left(\frac{1}{4\pi}f\right)=\delta$$

and completes the proof. □

Remark 2.2.1. In the case when $N=2$, using a similar calculation (see, for instance, [243, Theorem 3.2]), it can be shown that the fundamental solution of the Laplacian is given by the logarithmic potential. More precisely, consider the function f defined by

$$f(x)=\ln\Big(\frac{1}{\sqrt{x_1^2+x_2^2}}\Big).$$

Then $-\Delta(\frac{1}{2\pi}f)=\delta$.

2.3 ▪ Weak solutions

2.3.1 ▪ Weak formulation of the model examples

The Dirichlet problem. Let Ω be an open subset of $\mathbf{R}^N$ and $f:\Omega\longrightarrow\mathbf{R}$ a given function; take $f\in L^2(\Omega)$, for example. We recall that the Dirichlet problem is to find a function $u:\overline{\Omega}\longrightarrow\mathbf{R}$ which solves

$$-\Delta u=f \quad \text{on } \Omega, \qquad u=0 \quad \text{on } \partial\Omega. \tag{2.24}$$

In Section 2.1, we explained that it is difficult to prove directly the existence of a classical solution to this problem. By classical solution, we mean a function u which is continuous on $\overline{\Omega}$ and of class $\mathbf{C}^2$ on Ω. So, the idea is to allow u to be less regular (at least in a first stage) and to interpret Δu in a weak sense, namely, in a distribution sense.

Taking test functions $v\in\mathscr{D}(\Omega)$, (2.24) is equivalent to

$$\langle -\Delta u, v\rangle_{(\mathscr{D}'(\Omega),\mathscr{D}(\Omega))}=\int_\Omega f v\,dx \quad \forall\, v\in\mathscr{D}(\Omega).$$

The definition of the derivation of distributions (see Definition 2.2.4) is precisely based on the integration by parts formula and allows us to transfer the derivation operation from u onto the test functions $v\in\mathscr{D}(\Omega)$. At this point, we have two possibilities. The two equalities

$$\langle -\Delta u, v\rangle=\sum_{i=1}^N\left\langle\frac{\partial u}{\partial x_i},\frac{\partial v}{\partial x_i}\right\rangle_{(\mathscr{D}'(\Omega),\mathscr{D}(\Omega))}, \tag{2.25}$$

$$\langle -\Delta u, v\rangle=\langle u,-\Delta v\rangle_{(\mathscr{D}'(\Omega),\mathscr{D}(\Omega))} \tag{2.26}$$

correspond, respectively, to a partial transfer and a global transfer of the derivatives on the test functions. They give rise to two distinct weak formulations of the initial problem, which indeed depend on the regularity properties which we expect the solution u to satisfy.

If we expect the solution u to have first distribution derivatives which are integrable, then (2.25) gives rise to

$$\begin{cases}\displaystyle\int_\Omega\sum_{i=1}^N\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx=\int_\Omega f v\,dx \quad \forall\, v\in\mathscr{D}(\Omega),\\ u=0 \quad \text{on } \partial\Omega.\end{cases} \tag{2.27}$$

If we don't expect u to have first derivatives which are integrable and just expect u to be $L^1(\Omega)$ or in $\mathbf{C}(\overline{\Omega})$, then by using (2.26), we obtain the (very) weak formulation

$$\begin{cases} -\int_\Omega u \Delta v \, dx = \int_\Omega f v \, dx \quad \forall v \in \mathscr{D}(\Omega), \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.28}$$

For many reasons, the weak formulation (2.27) is the one which is well adapted to our situation:

(a) As a general rule, we will see that it is preferable to perform the integrations by parts which are necessary and no more. Otherwise, the solution which is obtained satisfies the equation in a very weak sense, we have only poor information on this solution, and the study of its uniqueness and regularity becomes quite involved.

(b) When trying to give a sense to the boundary condition $u = 0$ on $\partial\Omega$, it is useful to have some information on the derivatives of u on Ω. To find a weak solution u for which we know just that u belongs to some $L^p(\Omega)$ space is not sufficient to give meaning to the trace of u on $\partial\Omega$. (Recall that $\partial\Omega$ has a zero Lebesgue measure.)

Since we will be able to find a weak solution of the Dirichlet problem in the space $H^1(\Omega)$ (that is, with first-order distribution derivatives in $L^2(\Omega)$), we will use (2.27) as a variational formulation of the Dirichlet problem. Note, too, that (2.27) has another advantage over (2.28): the left member of the equation, which is the important part and which involves the partial differential operator governing the equation, is symmetric with respect to u and v in (2.27), while it is not in (2.28). This has important consequences on the variational formulation of the problem; see Section 2.3.2 and Chapter 3.

Let us summarize the above comments and give a first definition of the notion of a weak solution for the Dirichlet problem. It will be made precise later and solved in Chapter 6.

Given $f \in L^2(\Omega)$, a weak solution u of the Dirichlet problem (2.24) is a function $u \in H^1(\Omega)$ which satisfies

$$\begin{cases} \int_\Omega \sum_{i=1}^N \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx = \int_\Omega f v \, dx \quad \forall v \in \mathscr{D}(\Omega), \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.29}$$

Indeed, we will justify the choice of the functional space $H^1(\Omega)$ and explain how to interpret the trace of such functions on $\partial\Omega$. We will reformulate (2.29) by introducing the subspace $H_0^1(\Omega)$ of $H^1(\Omega)$:

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}.$$

Indeed, $H_0^1(\Omega)$ is equal to the closure of $\mathscr{D}(\Omega)$ in $H^1(\Omega)$. As a consequence, the equality (2.29) can be extended by a density and continuity argument to $H_0^1(\Omega)$. In this way we obtain the classical weak formulation of the Dirichlet problem.

Definition 2.3.1. *A weak solution of the Dirichlet problem is a solution of the following system:*

$$\begin{cases} \int_\Omega \sum_{i=1}^N \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx = \int_\Omega f v \, dx \quad \forall v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega). \end{cases} \tag{2.30}$$

Note that (2.30) can be written in the following abstract form: find $u \in V$ such that $a(u,v) = L(v)$ for all $v \in V$, where $a : V \times V \longrightarrow \mathbf{R}$ is a bilinear form which is symmetric and positive ($a(v,v) \geq 0$ for every $v \in V$) and L is a linear form on V.

In Chapter 3, an existence result for such an abstract problem will be proved (Lax–Milgram theorem); in Chapter 5 the basic ingredients of the theory of Sobolev spaces will be developed, for example, to treat the case $V = H_0^1(\Omega)$. Thus, in Chapter 6 we will be able to prove the existence of a weak solution to the Dirichlet problem.

The Neumann problem. We recall that the Neumann problem consists of finding a solution u to the boundary problem

$$u - \Delta u = f \text{ on } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{2.31}$$

where $\frac{\partial u}{\partial n} = Du \cdot n$ is the outward normal derivative of u on $\partial\Omega$. A major difference between the Dirichlet and the Neumann problem is that in the Neumann problem, the value of u on the boundary is not prescribed (it is $\frac{\partial u}{\partial n}$ which is prescribed). As a consequence, we have to test u on Ω and on $\partial\Omega$; it is not sufficient to take test functions $v \in \mathscr{D}(\Omega)$.

We will take test functions $v \in \mathbf{C}^1(\overline{\Omega})$. We are no longer in the setting of the distribution theory, but we can follow the lines of this theory. Let us first assume that u is regular and let us multiply (2.31) by $v \in \mathbf{C}^1(\overline{\Omega})$ and integrate by parts. Recall that from the divergence theorem,

$$\int_\Omega \operatorname{div}(v\,Du)\,dx = \int_{\partial\Omega} v\,Du \cdot n\,d\sigma.$$

Thus

$$\int_\Omega (v\,\Delta u + Du \cdot Dv)\,dx = \int_{\partial\Omega} v \frac{\partial u}{\partial n}\,d\sigma. \tag{2.32}$$

By using (2.31) and (2.32) we obtain

$$\int_\Omega (uv + Du \cdot Dv)\,dx = \int_\Omega f\,v\,dx \quad \forall v \in \mathbf{C}^1(\overline{\Omega}). \tag{2.33}$$

Now (2.33) makes sense even for a function u for which we are only able to define first generalized derivatives as functions. So, we will take (2.33) in a first step as a notion of the weak solution. Precisely, given $f \in L^2(\Omega)$, a weak solution of the Neumann problem is a function $u \in H^1(\Omega)$ such that

$$\int_\Omega (uv + Du \cdot Dv)\,dx = \int_\Omega f\,v\,dx \quad \forall v \in \mathbf{C}^1(\overline{\Omega}). \tag{2.34}$$

The striking feature is that in this weak formulation (2.34), the Neumann boundary condition has disappeared! It is important to verify that, so doing, we have not lost any information. In other words, we need to show that, conversely, if $u \in H^1(\Omega)$ verifies (2.34), then u satisfies (2.31). The second condition in (2.31) is the so-called Neumann boundary condition.

First take $v \in \mathscr{D}(\Omega)$. Clearly $\mathscr{D}(\Omega)$ is a subspace of $\mathbf{C}^1(\overline{\Omega})$ and so we obtain

$$u - \Delta u = f \quad \text{in } \mathscr{D}'(\Omega). \tag{2.35}$$

To recover the Neumann boundary condition, we have to perform the integration by parts in a reverse way. To do so, we assume that we have been able to prove that the weak solution u of (2.34) is in fact a regular function. So, by using (2.32) and (2.34),

$$\int_\Omega (u-\Delta u)v\,dx + \int_{\partial\Omega} v\,\frac{\partial u}{\partial n}d\sigma = \int_\Omega f\,v\,dx \quad \forall v\in \mathbf{C}^1(\overline{\Omega}). \tag{2.36}$$

By (2.35), $u-\Delta u = f$, so we can simplify (2.36) to obtain

$$\int_{\partial\Omega} v\,\frac{\partial u}{\partial n}\,d\sigma = 0 \quad \forall v\in \mathbf{C}^1(\overline{\Omega}),$$

which implies $\frac{\partial u}{\partial n}=0$.

Thus, the Neumann boundary condition is implicitly contained in the weak variational formulation (2.34). Indeed, just like for the Dirichlet problem, we will prove a density result, namely, "$\mathbf{C}^1(\overline{\Omega})$ is dense in $H^1(\Omega)$." As a consequence, the equality (2.34) can be extended to all $v\in H^1(\Omega)$ and the final variational formulation of the Neumann problem will be the following.

Definition 2.3.2. *A weak solution of the Neumann problem is a solution u of the following system:*

$$\begin{cases} \displaystyle\int_\Omega (uv + Du\cdot Dv)\,dx = \int_\Omega f\,v\,dx \quad \forall\, v\in H^1(\Omega),\\ u\in H^1(\Omega). \end{cases} \tag{2.37}$$

Note again that the above problem can be written as

$$\begin{cases} \text{find } u\in V = H^1(\Omega) \text{ such that}\\ a(u,v) = L(v) \quad \forall\, v\in V, \end{cases}$$

where $a(u,v)=\int_\Omega (uv+Du\cdot Dv)\,dx$ is a bilinear form, symmetric, and positive and

$$L(v) = \int_\Omega f\,v\,dx \quad \text{is a linear form on } V.$$

The basic difference between the weak variational formulations of the Dirichlet and Neumann problems is in the choice of the space V which reflects the choice of the test functions:

$V = H_0^1(\Omega)$ in the Dirichlet problem;

$V = H^1(\Omega)$ in the Neumann problem.

The Stokes system. Given $\vec{f} = (f_1, f_1, \dots, f_N)\in L^2(\Omega)^N$ and $\mu > 0$, we are looking for the velocity vector field of the fluid $\vec{u} = (u_1, u_2, \dots, u_N)$ and the pressure $p:\Omega\longrightarrow \mathbf{R}$ of the fluid which satisfy

$$-\mu\Delta u_i + \frac{\partial p}{\partial x_i} = f_i \text{ on } \Omega, \quad i=1,\dots,N, \tag{2.38}$$

$$\operatorname{div}\vec{u} = 0 \qquad \text{on } \Omega, \tag{2.39}$$

$$u_i = 0 \qquad \text{on } \partial\Omega, \quad i=1,\dots,N. \tag{2.40}$$

The condition $\operatorname{div}\vec{u} = \sum_{i=1}^N \frac{\partial u_i}{\partial x_i} = 0$ expresses that the fluid is incompressible.

The choice of the test functions is not as immediate as in the two previous situations. A guideline is to choose the test functions smooth enough to perform the integration by parts and which look like the function or vector field we want to test. A clever choice (J. Leray developed this method) is to take test fields $\vec{v} \in \mathscr{V}$, where

$$\mathscr{V} = \{\vec{v} = (v_1, \dots, v_n),\ v_i \in \mathscr{D}(\Omega),\ i = 1, \dots, N, \text{ and } \operatorname{div} \vec{v} = 0\}.$$

One may require the v_i to be $\mathbf{C}^1$ function with compact support as well. The important point is to assume that the divergence of $\vec{v}$ is equal to zero.

Let us interpret (2.38) in the sense of distributions. If we expect to find u_i with first partial derivatives in $L^2(\Omega)$ (i.e., $u_i \in H^1(\Omega)$) and $p \in L^2(\Omega)$, this is equivalent to writing for each $i = 1, 2, \dots, N$

$$\mu \int_\Omega Du_i \cdot Dv_i \, dx - \int_\Omega p \cdot \frac{\partial v_i}{\partial x_i} dx = \int_\Omega f_i v_i \, dx \quad \forall v_i \in \mathscr{D}(\Omega). \tag{2.41}$$

The trick is now to add these equalities ($i = 1, 2, \dots, N$). Since the test functions $v_1, \dots, v_N$, by definition of $\mathscr{V}$, verify $\sum \frac{\partial v_i}{\partial x_i} = 0$, we obtain

$$\mu \sum_{i=1}^N \int_\Omega Du_i \cdot Dv_i \, dx = \sum_{i=1}^N \int_\Omega f_i v_i \, dx \quad \forall \vec{v} \in \mathscr{V}. \tag{2.42}$$

Conversely, it is easy to verify that if u is regular and satisfies (2.42), then

$$\sum_{i=1}^N \int_\Omega (-\mu \Delta u_i - f_i) v_i \, dx = 0 \quad \forall \vec{v} \in \mathscr{V}.$$

In other words, the vector $(\mu \Delta u_i + f_i)_{i=1,\dots,N}$ is orthogonal to $\mathscr{V}$ in $L^2(\Omega)^N$. One can prove—indeed, this is quite an involved result (see Chapter 6)—that this property implies the existence of $p \in L^2(\Omega)$ such that

$$\mu \, \Delta u_i + f_i = \frac{\partial p}{\partial x_i}, \quad i = 1, 2, \dots, N.$$

Indeed, as in the previous examples, the equality (2.42) can be extended by a density and continuity argument to

$$V = \{\vec{v} \in H_0^1(\Omega)^N : \operatorname{div} \vec{v} = 0\}.$$

Finally, the variational formulation of the Stokes system is given below.

Definition 2.3.3. *A weak solution of the Stokes system is a solution $\vec{u} = (u_1, u_2, \dots, u_N)$ of the system*

$$\begin{cases} \mu \displaystyle\int_\Omega \sum_{i=1}^N Du_i \cdot Dv_i \, dx = \sum_{i=1}^N \int_\Omega f_i \, v_i \, dx \quad \forall \vec{v} \in V, \\ \vec{u} \in V, \end{cases} \tag{2.43}$$

where $V = \{\vec{v} \in H_0^1(\Omega)^N : \operatorname{div} \vec{v} = 0\}$.

The choice of the functional space V (which is obtained by a completion of the space $\mathscr{V}$ of test functions) is of fundamental importance. The pressure p has apparently disappeared in this formulation. It is contained implicitly in it, since p can be interpreted as a Lagrange multiplier of the constraint $\operatorname{div} v = 0$.

Notice that, once more, the weak formulation we have obtained can be written in the following form: find $u \in V$ such that

$$a(u,v) = L(v) \quad \forall v \in V,$$

where $a(u,v) = \mu \int_\Omega \sum_{i=1}^N Du_i Dv_i dx$ and $L(v) = \int_\Omega \sum f_i v_i$ are, respectively, a bilinear form and a linear form on V.

2.3.2 ▪ Positive quadratic forms and convex minimization

The weak formulations of the model examples studied in the previous section have very similar structures. Indeed, they can be viewed as particular cases of the following abstract problem.

Given V a linear vector space, $a : V \times V \longrightarrow \mathbf{R}$ a bilinear form, and $L : V \longrightarrow \mathbf{R}$ a linear form, find $u \in V$ such that

$$a(u,v) = L(v) \quad \forall v \in V. \tag{2.44}$$

In Chapter 3, we will study in detail the existence of solutions to such problems. This will require some topological assumptions on the data V, a, L.

For the moment, we will examine algebraic properties of such problems and make the link, when $a(\cdot,\cdot)$ is symmetric and positive, with convex minimization problems.

Let us first make precise these notions concerning bilinear and quadratic forms.

Definition 2.3.4. *Let V be a linear vector space and $a : V \times V \longrightarrow \mathbf{R}$ a bilinear form, i.e.,*

$$\begin{cases} \forall\, u \in V \quad v \longmapsto a(u,v) \text{ is a linear form,} \\ \forall\, v \in V \quad u \longmapsto a(u,v) \text{ is a linear form.} \end{cases}$$

The bilinear form is said to be symmetric if

$$\forall u, v \in V \quad a(u,v) = a(v,u).$$

When a is symmetric, one can associate to $a(\cdot,\cdot)$ the quadratic form $q : V \longrightarrow \mathbf{R}$ which is equal to $q(v) = a(v,v)$.

The bilinear form a is said to be positive (one can say as well that the associated quadratic form q is positive) if

$$\forall\, v \in V \quad a(v,v) \geq 0.$$

We say that $a(\cdot,\cdot)$ (or $q(\cdot)$) is positive definite if

$$\forall\, v \in V \quad a(v,v) \geq 0 \quad \textit{and} \quad a(v,v) = 0 \Longrightarrow v = 0.$$

We can now make the link between problem (2.44) and a minimization problem. All the notions used in the following statement are algebraic.

Proposition 2.3.1. *Let V be a linear vector space, $L : V \longrightarrow \mathbf{R}$ a linear form, and $a : V \times V \longrightarrow \mathbf{R}$ a bilinear, symmetric, positive form. Then the two following statements are equivalent:*

(i) *$u \in V$, $a(u,v) = L(v)\ \forall v \in V$;*

(ii) *$u \in V$, $J(u) \leq J(v)\ \forall\, v \in V$, where*

$$J(v) := \frac{1}{2} a(v,v) - L(v).$$

PROOF. Let us first prove (i) $\Longrightarrow$ (ii). Since V is a linear space, it is equivalent to prove that

$$J(u) \le J(u+v) \quad \forall v \in V.$$

A simple computation gives

$$J(u+v) - J(u) = \left[\frac{1}{2}a(u+v,u+v) - L(u+v)\right] - \left[\frac{1}{2}a(u,u) - L(u)\right].$$

Note that because of the symmetry assumption on the bilinear form $a(\cdot,\cdot)$,

$$a(u+v,u+v) = a(u,u) + 2a(u,v) + a(v,v).$$

Thus

$$\begin{aligned} J(u+v) - J(u) &= \left[\frac{1}{2}a(u,u) + a(u,v) + \frac{1}{2}a(v,v)\right] \\ &\quad - \frac{1}{2}a(u,u) - [L(u) + L(v)] + L(u) \\ &= [a(u,v) - L(v)] + \frac{1}{2}a(v,v). \end{aligned}$$

Since, by assumption, u is a solution of (i), $a(u,v) = L(v)$ and

$$J(u+v) - J(u) = \frac{1}{2}a(v,v),$$

which is nonnegative, since a has been assumed to be positive.

Let us now prove (ii) $\Longrightarrow$ (i). We know that u is a solution of the minimization problem, i.e., u minimizes $J(\cdot)$. One is naturally tempted to write an optimality condition which expresses that some derivative of J at u is equal to zero. Since V was only assumed to be a linear vector space, the only derivation notion we can use is the directional derivative which always makes sense since it relies only on the topological structure of the real line. Since u minimizes J, for any $t \in \mathbf{R}$, for any $v \in V$,

$$J(u+tv) - J(u) \ge 0.$$

Dividing by $t > 0$, we have

$$\frac{1}{t}[J(u+tv) - J(u)] \ge 0.$$

Before letting t go to zero, let us compute this last expression:

$$\begin{aligned} \frac{1}{t}[J(u+tv) - J(u)] &= \frac{1}{t}\left[\frac{1}{2}a(u+tv,u+tv) - L(u+tv) - \frac{1}{2}a(u,u) + L(u)\right] \\ &= \frac{1}{t}\left[ta(u,v) + \frac{t^2}{2}a(v,v) - tL(v)\right] \\ &= a(u,v) + \frac{t}{2}a(v,v) - L(v). \end{aligned}$$

Thus, by letting t go to zero, we obtain

$$\lim_{t\to 0^+} \frac{1}{t}[J(u+tv)-J(u)] = a(u,v)-L(v) \geq 0.$$

Then, one can either make the same argument by using $t<0$ or replace v by $-v$ in the above inequality to obtain the opposite inequality and conclude that

$$a(u,v) = L(v) \quad \forall v \in V. \qquad \square$$

Let us return to the model examples studied in Section 2.3.1 and use their weak formulations together with Proposition 2.3.1 to obtain the results below.

Corollary 2.3.1. *With the notation of Section* 2.3.1, *the following facts hold:*

(a) *The weak solution u of the Dirichlet problem*

$$\begin{cases} -\Delta u = f & on\ \Omega, \\ u = 0 & on\ \partial\Omega \end{cases}$$

is a solution of the minimization problem

$$\begin{cases} J(u) \leq J(v) & \forall\, v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega), \end{cases}$$

where $J(v) := \frac{1}{2}\int_\Omega |Dv|^2 dx - \int_\Omega f v dx$. *This is the Dirichlet variational principle.*

(b) *The weak solution u of the Neumann problem*

$$\begin{cases} u - \Delta u = f & on\ \Omega, \\ \frac{\partial u}{\partial n} = 0 & on\ \partial\Omega \end{cases}$$

is a solution of the minimization problem

$$\begin{cases} J(u) \leq J(v) & \forall\, v \in H^1(\Omega), \\ u \in H^1(\Omega), \end{cases}$$

where $J(v) := \frac{1}{2}\int_\Omega (|Dv|^2 + v^2)dx - \int_\Omega f\, v\, dx$.

(c) *The weak solution u of the Stokes system*

$$\begin{cases} -\mu\Delta u_i + \frac{\partial p}{\partial x_i} = f_i, & i = 1,2,\ldots,N\ on\ \Omega, \\ \operatorname{div} \vec{u} = 0 & on\ \Omega, \\ \vec{u} = \vec{0} & on\ \partial\Omega \end{cases}$$

is a solution of the minimization problem

$$\begin{cases} J(u) \leq J(v) & \forall\, v \in V = \{v \in H_0^1(\Omega)^N : \operatorname{div} v = 0\}, \\ u \in V, \end{cases}$$

where $J(v) = \frac{1}{2}\sum_{i=1}^N \int_\Omega |Dv_i|^2 dx - \sum_{i=1}^N \int_\Omega f_i v_i dx$.

Note that in all these examples, the bilinear form $a(\cdot,\cdot)$ is symmetric and positive. We now come to the question of the nature of the minimization problem and the properties of the functional J given by

$$J(v) = \frac{1}{2}a(v,v) - L(v).$$

Note that J is the sum of a quadratic form and of a linear form.

We are ready to introduce a property of fundamental importance in the study of the minimization problems—convexity.

Recall that a function $J : V \longrightarrow \mathbf{R}$, where V is a linear vector space, is convex if

$$\forall\, u,v \in V, \quad \forall\, \lambda \in [0,1] \quad J(\lambda u + (1-\lambda)v) \le \lambda J(u) + (1-\lambda)J(v).$$

The role of convexity in minimization problems will be examined in detail in Chapters 3, 9, 13, and 15. The class of convex functionals is stable with respect to the sum, it contains the linear forms, and we are going to see that it contains the positive quadratic forms. Thus, functionals of the form $J(v) = \frac{1}{2}a(v,v) - L(v)$ with a and L as above will be convex.

Let us now formulate the convexity property for the positive quadratic forms.

Proposition 2.3.2. *Let V be a linear vector space and $a : V \times V \longrightarrow \mathbf{R}$ a bilinear form which is symmetric and positive. Then, the quadratic form $q : V \longrightarrow \mathbf{R}$ which is associated with a, i.e., $q(v) = a(v,v)$ is a convex function.*

PROOF. This is just an algebraic computation. For any $u, v \in V$ and $\lambda \in [0,1]$,

$$\begin{aligned} q(\lambda u + (1-\lambda)v) &= a(\lambda u + (1-\lambda)v, \lambda u + (1-\lambda)v) \\ &= \lambda^2 a(u,u) + (1-\lambda)^2 a(v,v) + 2\lambda(1-\lambda)a(u,v). \end{aligned}$$

Thus,

$$\begin{aligned} \lambda q(u) + (1-\lambda)q(v) - q(\lambda u + (1-\lambda)v) &= (\lambda - \lambda^2)a(u,u) - 2\lambda(1-\lambda)a(u,v) \\ &\quad + [(1-\lambda) - (1-\lambda)^2]a(v,v) \\ &= \lambda(1-\lambda)[a(u,u) - 2a(u,v) + a(v,v)] \\ &= \lambda(1-\lambda)a(u-v, u-v), \end{aligned}$$

which is nonnegative, because $\lambda \in [0,1]$ and a is positive. $\square$

When examining the question of the uniqueness of the solution of the previous problems, the notion which plays a central role is the strict convexity. Recall that $J : V \longrightarrow \mathbf{R}$ is strictly convex if J is convex and the convexity inequality

$$J(\lambda u + (1-\lambda)v) < \lambda J(u) + (1-\lambda)J(v)$$

is strict whenever $u \neq v$ and $\lambda \in]0,1[$.

The importance of this notion is justified by the following elementary result.

Proposition 2.3.3. *Let V be a linear space and $J : V \longrightarrow \mathbf{R}$ a strictly convex function. Then there exists at most one solution u to the minimization problem*

$$\begin{cases} J(u) \le J(v)\ \forall\, v \in V, \\ u \in V. \end{cases}$$

PROOF. Suppose that we have two distinct solutions u_1 and u_2 to the above minimization problem. Then

$$J\left(\frac{u_1 + u_2}{2}\right) < \frac{1}{2}[J(u_1) + J(u_2)] = \inf J(u),$$

a clear contradiction. Hence $u_1 = u_2$. $\square$

Proposition 2.3.4. *Let V be a linear vector space and $a : V \times V \longrightarrow \mathbf{R}$ a bilinear form which is symmetric and positive definite. Then, the quadratic form $q : V \longrightarrow \mathbf{R}$ which is associated with a, i.e., $q(v) = a(v,v)$ is strictly convex.*

PROOF. The proof is the same computation as in the proof of Proposition 2.3.2: for any $u, v \in V$, for any $\lambda \in [0,1]$,

$$\lambda q(u) + (1-\lambda) q(v) - q(\lambda u + (1-\lambda) v) = \lambda(1-\lambda) a(u-v, u-v).$$

When taking $\lambda \in]0,1[$ we have $\lambda(1-\lambda) > 0$, and when taking $u \neq v$ we have $a(u-v, u-v) > 0$ because a is positive definite. So, for $\lambda \in]0,1[$ and $u \neq v$, $\lambda q(u) + (1-\lambda) q(v) > q(\lambda u + (1-\lambda) v)$ and q is strictly convex. □

Proposition 2.3.5. *The sum of a convex function and a strictly convex function is strictly convex.*

PROOF. The proof is a direct consequence of the fact that when adding an inequality and a strict inequality, one obtains a strict inequality. □

Let us return to the model examples and their variational formulations as minimization problems as given in Corollary 2.3.1. Noticing that in all these situations the quadratic form $q(v) = a(v,v)$ is positive definite, we obtain that the corresponding functional J is strictly convex. So, the weak solution of the problems under consideration, when it exists, is characterized as the unique solution of the associated minimization problems. This makes a natural transition to Chapter 3, where the existence question will be examined.

2.4 ▪ Weak topologies and weak convergences

In recent decades, weak topologies have proved useful as a basic tool in variational analysis in the study of PDEs, and more generally in all fields using tools from functional analysis. Let us explain some of the reasons for the success of weak convergence methods.

(a) Distributions are defined as continuous linear forms on $\mathscr{D}(\Omega)$. In other words, a distribution $T \in \mathscr{D}'(\Omega)$ is viewed via its action on test functions $v \in \mathscr{D}(\Omega)$:

$$T \in \mathscr{D}'(\Omega) : v \in \mathscr{D}(\Omega) \longmapsto \langle T, v \rangle_{(\mathscr{D}'(\Omega), \mathscr{D}(\Omega))}.$$

Given a sequence $T_1, T_2, \ldots, T_n, \ldots$ of distributions (for example, functions, measures), a natural mode of convergence for such sequences is to assume that

$$\forall\, v \in \mathscr{D}(\Omega) \quad \lim_{n \to \infty} \langle T_n, v \rangle_{(\mathscr{D}', \mathscr{D})} = \langle T, v \rangle.$$

This is a typical example of weak convergence.

(b) A celebrated theorem from Riesz asserts that the closed unit ball of a normed linear space is compact iff the space has a finite dimension. Thus, when looking for topologies making bounded sets relatively compact in infinite dimensional spaces, one is naturally led to introduce new topologies which are weaker than the topology of the norm. This is why weak topologies play a decisive role.

(c) Besides the importance of weak topologies from a theoretical point of view, we will see that weak convergences naturally occur when describing concrete situations. For example, weak convergences allow us to describe high oscillations of a sequence of functions, as well as concentration phenomena on zero Lebesgue measure sets.

Before introducing weak topologies on normed linear spaces, let us recall some basic facts from general topology.

2.4.1 ▪ Topologies induced by functions in general topological spaces

First we need to fix the notation. Recall that a topology on a space X is a family θ of subsets of X, called the family of the *open sets* of X, satisfying the axioms of the open sets, namely,

(i) X and $\emptyset$ belong to θ;

(ii) for all $(G_i)_{i\in I}$ $G_i \in \theta$, I arbitrary, $\cup_{i\in I} G_i \in \theta$;

(iii) for all $(G_i)_{i\in I}$ $G_i \in \theta$, I finite, $\cap_{i\in I} G_i \in \theta$.

In other words, the open sets of X for a given topology are a family of subsets of X which is *stable with respect to arbitrary unions and finite intersection*.

We will often denote by τ a topology on a space X and by θ_τ the family of the τ-open sets. A topology can be seen as a subset of $P(X)$, where $P(X)$ is the family of all subsets of X. There is a natural partial ordering on the topologies on a given space X, which is induced by the inclusion ordering on the subsets of $P(X)$: we will say that a topology τ_1 *is coarser or weaker* than a topology τ_2 and we write $\tau_1 < \tau_2$ if $\theta_{\tau_1} \subset \theta_{\tau_2}$, that is, if any element G of θ_{τ_1} also belongs to θ_{τ_2}. Conversely, we will say that τ_2 is *stronger or finer than* τ_1.

Proposition 2.4.1. *The family of the topologies on a set X forms a complete lattice for the relation $\tau_1 < \tau_2$ (τ_1 weaker than τ_2), that is, given an arbitrary collection of topologies $(\tau_i)_{i\in I}$ on X, the following hold:*

(a) *There exists a lower bound, that is a topology which is the largest among all the topologies weaker than the τ_i, $i \in I$. We denote $\tau = \wedge_{i\in I}\tau_i$ the lower bound (or infimum) of the τ_i. Clearly $\theta_\tau = \cap_{i\in I}\theta_{\tau_i}$, that is, $G \in \theta_\tau$ iff G belongs to θ_{τ_i} for all $i \in I$.*

(b) *There exists an upper bound (or supremum) that is a topology which is the smallest among the topologies which are stronger than all the τ_i. We denote $\tau = \vee_{i\in I}\tau_i$ the upper bound of the τ_i. We have that θ_τ is generated by $\cup_{i\in I}\theta_{\tau_i}$ in the sense of Proposition* 2.4.2.

PROOF. (a) Clearly if $(\theta_{\tau_i})_{i\in I}$ is a family of topologies on X, then $\cap_{i\in I}\theta_{\tau_i} = \{G \in P(X) : G \in \theta_{\tau_i}\ \forall i \in I\}$ still satisfies the axioms of the open sets, and it is a topology. The topology τ attached to the family $\theta = \cap_{i\in I}\theta_{\tau_i}$ is weaker than all the topologies τ_i, $i \in I$, and clearly it is the largest among the weaker ones.

(b) In contradiction to the previous case, if a topology τ is stronger than all the topologies τ_i, then θ_τ must contain all the families θ_{τ_i}, that is,

$$\theta_\tau \supset \cup_{i\in I}\theta_{\tau_i}.$$

But now $\mathscr{A} = \cup_{i\in I}\theta_{\tau_i}$ does not satisfy (in general) the axioms of the open sets. So, one is naturally led to address the following question: given a class $\mathscr{A}$ of subsets of an abstract space X, does there exist a smallest topology θ_τ on X which contains $\mathscr{A}$?

Clearly by (a), the answer is yes, one has to take for θ_τ the intersection (or, with an equivalent terminology, the infimum) of all the topologies containing $\mathscr{A} = \cup_{i\in I}\theta_{\tau_i}$. This is made precise in Proposition 2.4.2. $\square$

Proposition 2.4.2. *Let X be an abstract space and let $\mathscr{A}$ be any class of subsets of X. Then there exists a smallest (weakest) topology on X containing $\mathscr{A}$, denoted by $\tau_{\mathscr{A}}$, called the topology generated by $\mathscr{A}$. It is equal to the intersection of all the topologies containing $\mathscr{A}$. It can be obtained via the following two-step procedure:*

1. *First, take the finite intersections of elements of $\mathscr{A}$. One so obtains a family of sets which we call $\mathscr{B}_{\mathscr{A}}$.*
2. *Then $\mathscr{B}_{\mathscr{A}}$ is a base for the topology τ_A generated by $\mathscr{A}$, that is, any member of $\tau_{\mathscr{A}}$ can be obtained as the union of a family of members of $\mathscr{B}_{\mathscr{A}}$.*

We stress that in this construction, one has first to take finite intersections of elements of $\mathscr{A}$, then arbitrary unions of the so-obtained sets. When reversing the two operations, one obtains a family which is not stable by union. For a proof, see one of several books on general topology (for instance, Bourbaki [119]).

We now come to the situation which is of interest when considering weak topologies. Suppose X is an abstract space and $(Y_i,\tau_i)_{i\in I}$ is a family of topological spaces. Suppose that for each $i \in I$, a function $f_i : X \to Y_i$ is given. We want to investigate the topologies on X with respect to which all the functions f_i are continuous, and, among these topologies, examine the question of the existence of a smallest (weakest) one.

Noticing that for each $i \in I$, $f_i^{-1}(\theta_{\tau_i}) := \{f_i^{-1}(G_i) : G_i \in \theta_{\tau_i}\}$ still satisfies the axioms of the open sets, we denote $f_i^{-1}(\tau_i)$ the corresponding topology on X, which is the weakest making f_i (for i fixed) continuous. It follows from Propositions 2.4.1 and 2.4.2 that the answer to the previous question is given by $\tau = \vee_{i\in I} f_i^{-1}(\tau_i)$, whose precise description is given in the following.

Theorem 2.4.1. *Let X be an abstract space and let $(Y_i,\tau_i)_{i\in I}$ be an arbitrary collection of topological spaces with for each $i \in I$, $f_i : X \to Y_i$ a given function. Then, there exists a weakest topology τ on X making all the functions $(f_i)_{i\in I}$ continuous, $f_i : (X,\tau) \longrightarrow (Y_i,\tau_i)$ $i \in I$. This topology τ is equal to $\vee_{i\in I} f_i^{-1}(\tau_i)$, that is, τ is generated by the class*

$$\mathscr{A} = \{f_i^{-1}(G_i), G_i \in \theta_{\tau_i}, i \in I\}.$$

The class

$$\mathscr{B}_\tau = \left\{ \bigcap_{i\in J} f_i^{-1}(G_i),\ G_i \in \theta_{\tau_i}, J \subset I, J \text{ finite} \right\}$$

is a base for this topology, that is, each element of θ_τ can be written as a union of elements of $\mathscr{B}_\tau$.

We say that the topology τ is *induced by the family* $(f_i)_{i\in I}$. The following properties are quite elementary.

Proposition 2.4.3. *Let $f_i : X \longrightarrow (Y_i,\tau_i)$, $i \in I$, be given, and let $\tau = \vee f_i^{-1}(\tau_i)$ be the topology on X induced by the $(f_i)_{i\in I}$. For any sequence $(x_n)_{n\in\mathbf{N}}$ of elements of X, the two conditions are equivalent:*

(i) $x_n \xrightarrow{\tau} x$ *as* $n \to \infty$;

(ii) *for all* $i \in I$ $f_i(x_n) \xrightarrow{\tau_i} f_i(x)$ *as* $n \to \infty$.

PROOF. Since the topology τ makes each f_i continuous, we have clearly (i) $\Longrightarrow$ (ii).

Conversely, let us assume (ii) and prove (i). When considering a neighborhood of x, it is equivalent to take an element of the base $\mathscr{B}_\tau$ which contains x. So let,

$x \in \bigcap_{i\in J} f_i^{-1}(G_i)$, $G_i \in \theta_{\tau_i}$, $J \subset I$, J finite. For each $i \in J$, since $f_i(x_n) \xrightarrow{\tau_i} f_i(x)$ we have that $x_n \in f_i^{-1}(G_i)$ for $n \geq n_i$. Take $N = \max_{i\in J} n_i$, since J is finite, N is a finite integer, and $x_n \in \bigcap_{i\in J} f_i^{-1}(G_i)$ for $n \geq N$, which expresses that $x_n \xrightarrow{\tau} x$. □

A similar type argument yields the following result.

Proposition 2.4.4. *Let $(Z, \mathcal{T})$ be a topological space and let $g : (Z, \mathcal{T}) \longrightarrow (X, \tau)$ be a given function, where τ is the topology induced by the family $f_i : X \longrightarrow (Y_i, \tau_i)$. Then g is continuous iff $f_i \circ g$ is continuous from $(Z, \mathcal{T})$ into (Y_i, τ_i) for each $i \in I$.*

2.4.2 ▪ The weak topology $\sigma(V, V^*)$

We now assume that X is a vector space. To enhance this property, we denote it by V (like vector) and assume that V is a normed linear space, the norm of $v \in V$ being denoted by $||v||_V$ or $||v||$ when no confusion is possible.

We denote by V^* the topological dual of V, which is the set of all linear continuous forms on V. To avoid confusion, generic elements of V and V^* are denoted, respectively, by $v \in V$ and $v^* \in V^*$. We will write $\langle v^*, v\rangle = v^*(v)$ for the canonical pairing between V^* and V, which is just the evaluation of $v^* \in V^*$ at $v \in V$. Recall that V^* is a normed linear space (indeed, it is a Banach space) when equipped with the dual norm

$$||v^*||_{V^*} = \sup\{|v^*(v)| : ||v||_V \leq 1\}.$$

With this definition, we have

$$\forall v \in V, \forall v^* \in V^* \quad |\langle v^*, v\rangle| \leq ||v^*||\,||v||,$$

and $||v^*||$ is precisely the smallest constant for which the above inequality holds.

Definition 2.4.1. *Let $(V, ||\cdot||)$ be a normed linear space with topological dual V^*. The topology $\sigma(V, V^*)$, called the weak topology on V, is the weakest topology on V making continuous all the elements of V^*.*

Let us first comment on this definition. By definition, each element $v^* \in V^*$ is a function from V into $\mathbf{R}$, that is,

$$v^* : V \longrightarrow \mathbf{R}, \qquad v \longmapsto \langle v^*, v\rangle_{(V^*,V)}.$$

The weak topology on V is defined as the weakest topology on V making all these functions $\{v^* : v^* \in V^*\}$ continuous. By Theorem 2.4.1, such a topology exists, and it is weaker than the norm topology (since by definition all the elements v^* of V^* are continuous for the norm topology). We collect below some first results on the topology $\sigma(V, V^*)$ which are direct consequences of its definition.

Proposition 2.4.5. *Let V be a normed space and $\sigma(V, V^*)$ the weak topology on V.*

(i) *A local base of neighborhoods of $v_0 \in V$ for $\sigma(V, V^*)$ consists of all sets of the form*

$$N(v_0) = \{v \in V : |\langle v_i^*, v - v_0\rangle| < \varepsilon \ \forall i \in I\},$$

where I is a finite index set, $v_i^* \in V^*$ for each $i \in I$, and $\varepsilon > 0$.

(ii) $(V, \sigma(V, V^*))$ *is a Hausdorff topological space.*

(iii) *The topology* $\sigma(V, V^*)$ *is coarser than the topology of the norm on* V.

(iv) *When* V *is finite dimensional, the weak topology and the norm topology coincide.*

(v) *When* V *is infinite dimensional, the weak topology* $\sigma(V, V^*)$ *is strictly coarser than the norm topology.*

PROOF. (i) We have

$$N(v_0) = \bigcap_{i \in I} (v_i^*)^{-1}(]\alpha_i - \varepsilon, \alpha_i + \varepsilon[),$$

where $\alpha_i = \langle v_i^*, v_0 \rangle$. By definition of the weak topology $\sigma(V, V^*)$, $N(v_0)$ is an open set for this topology.

Let us prove that such sets form a local base of open neighborhoods of v_0 for $\sigma(V, V^*)$. Take A an open set for $\sigma(V, V^*)$ containing v_0. By Theorem 2.4.1, there exists some open set B for $\sigma(V, V^*)$ such that

$$v_0 \in B \subset A$$

with $B = \bigcap_{i \in I} (v_i^*)^{-1}(G_i)$, $v_i^* \in V^*$, G_i open in $\mathbf{R}$, I finite.

Since $v_i^*(v_0) \in G_i$ and G_i is open in $\mathbf{R}$, there exists some $\varepsilon > 0$ such that $|v_i^*(v) - v_i^*(v_0)| < \varepsilon$ for all $i \in I$ implies $v_i^*(v) \in G_i$ for all $i \in I$. Hence

$$v_0 \in N(v_0) \subset B \subset A$$

with

$$N(v_0) = \bigcap_{i \in I} (v_i^*)^{-1}(]\alpha_i - \varepsilon, \alpha_i + \varepsilon[), \ \alpha_i = \langle v_i^*, v_0 \rangle.$$

(ii) Let us prove that the topology $\sigma(V, V^*)$ is Hausdorff. Take v_1 and v_2 two distinct elements of V and prove that there exist A_1 and A_2 two open sets for $\sigma(V, V^*)$ such that $v_1 \in A_1$, $v_2 \in A_2$, and $A_1 \cap A_2 = \emptyset$. This is a direct consequence of the Hahn–Banach separation theorem. There exists a closed hyperplane which strictly separates v_1 and v_2, that is, there exists some $v^* \in V^*$ and $\alpha \in \mathbf{R}$ such that

$$\langle v^*, v_1 \rangle < \alpha < \langle v^*, v_2 \rangle.$$

Take

$$A_1 = \{v \in V : \langle v^*, v \rangle < \alpha\},$$
$$A_2 = \{v \in V : \langle v^*, v \rangle > \alpha\}.$$

They are open for the topology $\sigma(V, V^*)$ and separate v_1 and v_2.

Assertion (iii) is obvious since all the elements v^* of V^* are continuous for the norm topology.

(iv) Since the weak topology $\sigma(V, V^*)$ is coarser than the norm topology, it has fewer open sets. Let us prove that when V is a finite dimensional space, the opposite inclusion is true, that is, any open set A for the norm topology is also an open set for the weak topology.

Take $v_0 \in B(v_0, \varepsilon) \subset A$, where $B(v_0, \varepsilon)$ is an open ball in $(V, ||\cdot||)$, and prove that there exists some open set U for $\sigma(V, V^*)$ such that

$$v_0 \in U \subset B(v_0, \varepsilon) \subset A.$$

Let us choose a base $e_1, \ldots, e_N$ of V with $\|e_i\| = 1$, $i = 1, \ldots, N$. Each element v of V can be uniquely written as $v = \sum x_i e_i$ and the mappings $v \xrightarrow{e_i^*} x_i$ are linear continuous forms on V, i.e., they are elements of V^*. We have (with $v_0 = \sum x_{oi} e_i$)

$$\begin{aligned} \|v - v_0\| &= \left\| \sum_i (x_i - x_{oi}) e_i \right\| \\ &\leq \sum_i |x_i - x_{oi}| \\ &\leq \sum_i |\langle e_i^*, v - v_o \rangle|. \end{aligned}$$

Therefore, $v \in B(v_0, \varepsilon)$ as soon as $v \in U := \bigcap_{i=1,\ldots,n} (e_i^*)^{-1}(]\alpha_i - \frac{\varepsilon}{N}, \alpha_i + \frac{\varepsilon}{N}[)$, where $\alpha_i = \langle e_i^*, v_0 \rangle = x_{oi}$. Then notice that U is open for $\sigma(V, V^*)$, which concludes the proof of (iv).

(v) There are different ways to prove that in infinite dimensional spaces the weak topology is strictly coarser than the norm topology. One of them consists of proving that the unit sphere $S = \{v \in V : \|v\| = 1\}$ is never closed in infinite dimensional spaces for the topology $\sigma(V, V^*)$. Indeed $\bar{S}^{\sigma(V,V^*)} = \{v \in V : \|v\| \leq 1\}$; see, for instance, [137, Proposition III.6] and related comments for a detailed proof.

Remark 2.4.1. It is quite convenient when formulating topological properties to express them with the help of sequences. This can be done without loss of generality when the topology under consideration is metrizable. But *the weak topology* $\sigma(V, V^*)$ when V is an infinite dimensional normed space is a locally convex topology (the basic operations on V, vectorial sum and multiplication by a scalar, are continuous for the topology $\sigma(V, V^*)$) which is *not metrizable*.

Therefore, it is important to state the properties of this topology with general topological arguments as we have done up to now.

Nevertheless, in most practical situations, one can just use weakly convergent sequences. This will follow from deep results like the Eberlein–Smulian compactness theorem or from simpler observations like the following one: if V^* is separable, the weak topology $\sigma(V, V^*)$ is metrizable on each bounded set of V.

Consequently we now focus on properties of sequences which are $\sigma(V, V^*)$ convergent. Let us start with the following elementary results, which are direct consequences of the definition and of Proposition 2.4.3.

Proposition 2.4.6. *Let V be a normed linear space and $\sigma(V, V^*)$ the weak topology on V. For any sequence $(v_n)_{n \in \mathbf{N}}$ in V the following properties hold:*

(i) $v_n \xrightarrow{\sigma(V,V^*)} v \iff \forall v^* \in V^* \ \langle v^*, v_n \rangle \longrightarrow \langle v^*, v \rangle$;

(ii) $v_n \xrightarrow{\|\cdot\|} v \Longrightarrow v_n \xrightarrow{\sigma(V,V^*)} v$;

(iii) $v_n \xrightarrow{\sigma(V,V^*)} v \Longrightarrow$ *the sequence* $(v_n)_{n \in \mathbf{N}}$ *is bounded and* $\|v\| \leq \liminf_n \|v_n\|$;

(iv) $v_n \xrightarrow{\sigma(V,V^*)} v$ *and* $v_n^* \xrightarrow{\|\cdot\|_*} v^* \Longrightarrow \langle v_n^*, v_n \rangle \longrightarrow \langle v^*, v \rangle$.

PROOF. (iii) The fact that the sequence $(\|v_n\|)_{n \in \mathbf{N}}$ is bounded is a consequence of the Banach–Steinhaus theorem: consider the family of linear operators from the Banach space

V^* into $\mathbf{R}$

$$T_n : V^* \longrightarrow \mathbf{R} \quad n \in \mathbf{N}, \qquad v^* \longmapsto \langle v^*, v_n \rangle.$$

For each $n \in \mathbf{N}$, T_n is a linear continuous operator with norm

$$\|T_n\|_{\mathscr{L}(V^*,\mathbf{R})} = \sup_{\|v^*\|_* \leq 1} |\langle v^*, v_n \rangle| = \|v_n\|.$$

(This last equality is a consequence of the Hahn–Banach theorem.) For each $v^* \in V^*$, the sequence $(T_n(v^*))n \in \mathbf{N}$ is bounded in $\mathbf{R}$. This is a direct consequence of the equality $T_n(v^*) = \langle v^*, v_n \rangle$ and of the weak convergence of the sequence $(v_n)_{n \in \mathbf{N}}$. By the Banach–Steinhaus theorem, $\sup_{n \in \mathbf{N}} \|T_n\|_{\mathscr{L}(V^*,\mathbf{R})} < +\infty$, which is equivalent to $\sup_{n \in \mathbf{N}} \|v_n\| < \infty$.

Let us now prove the inequality $\|v\| \leq \liminf_n \|v_n\|$. By assumption, for each $v^* \in V^*$

$$\langle v^*, v \rangle = \lim_{n \to +\infty} \langle v^*, v_n \rangle.$$

By using the inequality $|\langle v^*, v_n \rangle| \leq \|v^*\|_* \|v_n\|$, we infer

$$\forall v^* \in V^* \qquad |\langle v^*, v \rangle| \leq (\liminf_{n \to +\infty} \|v_n\|) \|v^*\|_*.$$

By using the Hahn–Banach theorem, we obtain

$$\|v\| = \sup_{\|v^*\|_* \leq 1} |\langle v^*, v \rangle| \leq \liminf_n \|v_n\|.$$

(iv) This is just a triangulation argument. Write

$$\langle v_n^*, v_n \rangle - \langle v^*, v \rangle = \langle v_n^* - v^*, v_n \rangle + \langle v^*, v_n - v \rangle.$$

Hence

$$|\langle v_n^*, v_n \rangle - \langle v^*, v \rangle| \leq \|v_n^* - v^*\|_* \|v_n\| + |\langle v^*, v_n - v \rangle|.$$

The previous result (iii) tells us that there exists some constant $C \in \mathbf{R}^+$ such that $\|v_n\| \leq C$ for all $n \in \mathbf{N}$. So,

$$|\langle v_n^*, v_n \rangle - \langle v^*, v \rangle| \leq C \|v_n^* - v^*\| + |\langle v^*, v_n - v \rangle|,$$

which clearly implies the result. □

Remark 2.4.2. We will interpret in Section 3.2.3 the property

$$v_n \overset{\sigma(V,V^*)}{\longrightarrow} v \Longrightarrow \|v\| \leq \liminf_n \|v_n\|$$

as a lower semicontinuity property of the norm $\|\cdot\|_V$ for the topology $\sigma(V, V^*)$. Indeed, more generally, this can be viewed as a consequence of the fact that $\|\cdot\|_V$ is convex and continuous on V (and hence lower semicontinuous for the topology $\sigma(V, V^*)$).

Because of its importance, let us say a few words about the weak convergence in Hilbert spaces (which we denote by H). The Riesz representation theorem tells us that any element of the topological dual space can be represented as

$$H \ni v \longmapsto \langle f, v \rangle,$$

where f is a given element of H and $\langle\cdot,\cdot\rangle$ is the scalar product in H. Let us complete this observation by a few elementary results.

Proposition 2.4.7. *Let H be a Hilbert space. A sequence $(v_n)_{n\in\mathbf{N}}$ is weakly convergent in H iff*

$$\forall z\in H \quad \langle v_n,z\rangle \underset{n\to+\infty}{\longrightarrow} \langle v,z\rangle.$$

Moreover, we have the following implication:

$$v_n \overset{\sigma(H,H)}{\longrightarrow} v \quad \text{and} \quad \|v_n\|\longrightarrow\|v\| \Longrightarrow v_n \overset{\|\cdot\|}{\longrightarrow} v.$$

PROOF. We just need to prove the last statement. We have

$$\|v_n-v\|^2=\|v_n\|^2+\|v\|^2-2\langle v_n,v\rangle.$$

Hence

$$\lim_{n\to+\infty}\|v_n-v\|^2=\|v\|^2+\|v\|^2-2\langle v,v\rangle=0,$$

that is, $v_n \overset{\|\cdot\|}{\longrightarrow} v$. □

In the next section we will prove that this last property, which is quite important in the applications, is valid in a much larger class than the Hilbert spaces, namely, the uniformly convex Banach spaces. For the moment, we pause in these theoretical developments to give some examples of sequences which are $\sigma(V,V^*)$ convergent but not $\|\cdot\|_V$ convergent.

Example 2.4.1. Take $V=l^2$. An element v of V is a sequence of real numbers, $v=(v_k)_{k\in\mathbf{N}}$, such that $\sum_{k\in\mathbf{N}}|v_k|^2<+\infty$. The scalar product $\langle u,v\rangle:=\sum_{k\in\mathbf{N}}u_kv_k$ and the corresponding norm $\|v\|=(\sum|v_k|^2)^{1/2}$ give to V a Hilbert space structure.

Consider the sequence $e_1,e_2,\dots,e_n,\dots$ with

$$e_n=(\delta_{n,k})_{k\in\mathbf{N}},$$

where $\delta_{n,k}$ (the Kronecker symbol) takes the value 1 if $k=n$ and 0 elsewhere. The family $(e_n)_{n\in\mathbf{N}}$ is called the canonical basis of l^2 (it is a Hilbertian basis).

Let us show that $(e_n)_{n\in\mathbf{N}}$ weakly converges to 0 in V, that is,

$$\forall v\in V \quad \langle e_n,v\rangle \underset{n\to+\infty}{\longrightarrow} 0.$$

Observe that $\langle e_n,v\rangle=v_n$, where $v=(v_n)_{n\in\mathbf{N}}$. Since $\Sigma_{n\in\mathbf{N}}|v_n|^2<+\infty$, the general term of this convergent series, that is, v_n, tends to zero as n goes to $+\infty$, which proves the result.

The sequence $(e_n)_{n\in\mathbf{N}}$ is not norm convergent in V; otherwise it would necessarily norm converge to zero. (Recall that the norm convergence implies the weak convergence.) This is impossible since for each $n\in\mathbf{N}$, $\|e_n\|=1$. This can be equivalently obtained when observing that for all $n\neq m$

$$\|e_n-e_m\|=\sqrt{2},$$

the sequence $(e_n)_{n\in\mathbf{N}}$ is not a Cauchy sequence, and hence it is not norm convergent.

Example 2.4.2 (weak convergence in $L^p(\Omega)$, $1\le p<\infty$). Take Ω a bounded open set in $\mathbf{R}^N$, $1\le p<\infty$, and

$$V=L^p(\Omega)=\left\{v:\Omega\longrightarrow\mathbf{R}\text{ Lebesgue measurable: }\int_\Omega|v(x)|^p\,dx<+\infty\right\}.$$

V equipped with the norm $\|v\| = \left(\int_\Omega |v(x)|^p dx\right)^{1/p}$ is a Banach space with dual $V^* = L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$ (with the convention that the conjugate exponent of 1 is $+\infty$, i.e., $L^1(\Omega)^* = L^\infty(\Omega)$).

The weak convergence in $V = L^p(\Omega)$ can be formulated as follows:

$$v_n \overset{\sigma(L^p, L^{p'})}{\longrightarrow} v \iff \forall z \in L^{p'}(\Omega) \quad \int_\Omega v_n(x) z(x) dx \underset{n\to+\infty}{\longrightarrow} \int_\Omega v(x) z(x) dx.$$

The weak convergence in L^p allows us to model two different types of phenomena (they may occur simultaneously):

1. *Oscillations.* We describe the simplest situation of wild oscillations. Take $\Omega = (a, b)$ a bounded open interval of the real line, dx the Lebesgue measure on Ω, and $v_n(x) = \sin(nx)$, $n \in \mathbf{N}$. Clearly v_n oscillates between -1 and $+1$ with period equal to $T_n = 2\pi/n$. When n goes to $+\infty$, $T_n \longrightarrow 0$ and simultaneously its frequency goes to $+\infty$.

Let us prove that $v_n \overset{\sigma(L^p, L^{p'})}{\longrightarrow} 0$ for any $1 \le p < \infty$. Indeed, we can state a slightly more precise result: for any $z \in L^1(\Omega)$

$$\int_a^b z(x) \sin nx dx \longrightarrow 0 \quad \text{as } n \to +\infty.$$

(Note that for all $1 \le p < +\infty$, $L^p(a, b) \subset L^1(a, b)$.)

Indeed, that is exactly the Riemann's theorem which states that the Fourier coefficients of an integrable function tend to zero as $n \to +\infty$. For convenience, we give the proof, which is a nice illustration of a density argument. We also emphasize that this result is a particular case of an ergodic theorem (see Section 12.4.1 or Section 13.2, Proposition 13.2.1, Remark 13.2.5).

For an arbitrary $z \in L^1$, it is difficult to compute or get information on the integral $\int_a^b z(x) \sin nx\, dx$. So, let us first consider the case where z belongs to some dense subspace Z of $L^1(\Omega)$, Z being chosen to make the computation easier.

For example, take $Z = \mathbf{C}_c^1(a, b)$, the subspace of $\mathbf{C}^1$ functions with compact support in (a, b). By integration by parts, for any $z \in \mathbf{C}_c^1(a, b)$,

$$\int_a^b z(x) \sin nx dx = \int_a^b z'(x) \frac{\cos nx}{n} dx,$$

which implies

$$\left| \int_a^b z(x) \sin nx dx \right| \le \frac{1}{n} \int_a^b |z'(x)| dx \underset{n\to+\infty}{\longrightarrow} 0.$$

Another choice would consist of taking for Z the subspace of the step functions on (a, b). In that case a direct computation yields a similar result. So, for any z belonging to a dense subspace Z of $L^1(\Omega)$, we have that

$$\int_a^b z(x) \sin nx dx \underset{n\to+\infty}{\longrightarrow} 0.$$

We complete the proof by a density argument.

Take z an arbitrary element of $L^1(\Omega)$. By the density of Z in $L^1(\Omega)$, for any $\varepsilon > 0$, there exists some element $z_\varepsilon \in Z$ such that $||z - z_\varepsilon||_1 < \varepsilon$. Let us write

$$\int_a^b z(x)\sin nx\,dx = \int_a^b z_\varepsilon(x)\sin nx\,dx + \int_a^b (z(x) - z_\varepsilon(x))\sin nx\,dx.$$

Thus,

$$\begin{aligned}\left|\int_a^b z(x)\sin nx dx\right| &\le \left|\int_a^b z_\varepsilon(x)\sin nx dx\right| + \int_a^b |z(x) - z_\varepsilon(x)| dx\\ &\le \left|\int_a^b z_\varepsilon(x)\sin nx dx\right| + \varepsilon.\end{aligned}$$

Since $z_\varepsilon \in Z$, we obtain

$$\limsup_{n\to+\infty}\left|\int_a^b z(x)\sin nx dx\right| \le \varepsilon \quad \forall\, \varepsilon > 0,$$

which implies $\lim_{n\to+\infty}\int_a^b z(x)\sin nx dx = 0$.

We now observe that the sequence $(v_n)_{n\in\mathbf{N}}$, $v_n(x) = \sin nx$ does not norm converge in $V = L^p(a,b)$. Otherwise, it would be norm convergent to zero, but this is impossible since, for example, with $p = 2$,

$$\begin{aligned}||v_n||_2 = \int_a^b (\sin nx)^2 dx &= \int_a^b \frac{1}{2}(1 - \cos 2nx) dx\\ &= \frac{b-a}{2} - \frac{1}{4n}(\sin 2nb - \sin 2na)\\ &\underset{n\to+\infty}{\longrightarrow} \frac{b-a}{2}, \quad \text{which is different from zero!}\end{aligned}$$ □

2. *Concentration*. Take $\Omega = (0,1)$ and $(v_n)_{n\in\mathbf{N}}$ a sequence of step functions which is described as follows:

let $A_n = \bigcup_{k=1,\dots,n}\left[\frac{k}{n+1} - \frac{1}{2n^2}, \frac{k}{n+1} + \frac{1}{2n^2}\right]$ and take $v_n = \sqrt{n}$ on A_n and $v_n = 0$ elsewhere.

Let us examine the mode of convergence of the sequence $(v_n)_{n\in\mathbf{N}}$ in $V = L^2(0,1)$. One can first observe that

(a) $\int_0^1 v_n^2(x)\,dx = n \cdot \frac{1}{n^2} \cdot n = 1$ for all $n \in \mathbf{N}$;

(b) the sequence $(v_n)_{n\in\mathbf{N}}$ converges to zero in measure, that is,

$$\forall \delta > 0 \quad \text{meas}\{x \in (0,1) : |v_n(x)| > \delta\} \longrightarrow 0.$$

In fact, one just needs to observe that $\{x \in (0,1) : |v_n(x)| > \delta\} = A_n$ and $\text{meas}(A_n) = n \cdot \frac{1}{n^2} = \frac{1}{n} \underset{n\to+\infty}{\longrightarrow} 0$.

Therefore, the sequence $(v_n)_{n\in\mathbf{N}}$ does not norm converge in $L^2(0,1)$; otherwise it would converge to zero (recall that the norm convergence in L^2 implies the convergence in measure), which is impossible since $||v_n||_{L^2} = 1$.

Let us now prove that the sequence $(v_n)_{n\in\mathbf{N}}$ weakly converges to zero in $V = L^2(0,1)$. Indeed, by using the same density argument as in the previous oscillation example, we just need to prove that for any step function $z : (0,1) \longrightarrow \mathbf{R}$,

$$\int_0^1 v_n(x)z(x)dx \underset{n\to+\infty}{\longrightarrow} 0.$$

By linearity of the integral, we just need to compute for any $0 < a < b < 1$ the integral $\int_a^b v_n(x)dx$. Let us now observe that as n goes to $+\infty$,

$$\int_a^b v_n(x)dx \simeq n(b-a)\cdot\frac{1}{n^2}\cdot\sqrt{n} = \frac{b-a}{\sqrt{n}} \longrightarrow 0$$

($\simeq$ stands for equivalent). $\square$

We stress the fact that in the concentration example, the weak convergence occurs simultaneously with the pointwise convergence. What happens in this situation is that the mass of $|v_n|^2$ is concentrated in a set of small Lebesgue measure. This is the concentration phenomenon. Let us notice, too, that the sequence $(v_n)_{n\in\mathbf{N}}$, in the above example norm converges to zero in any $L^p(0,1)$, $1 \le p < 2$! To see this, just compute

$$\int_0^1 |v_n(x)|^p\, dx = n\cdot\frac{1}{n^2}\cdot n^{p/2} = n^{(p/2)-1} \longrightarrow 0$$

as $n \to +\infty$ as soon as $(p/2)-1 < 0$, that is, $p < 2$.

Thus $p = 2$, in this situation, is a *critical exponent*, for which we pass from strong convergence of the sequence $(v_n)_{n\in\mathbf{N}}$ in L^p, $p < 2$, to weak convergence in L^2. As we will see, weak convergences related to concentration effect, often occurs in situations where some critical exponent is involved (like the critical Sobolev exponent).

The two previous examples illustrate the utility of the density arguments when proving weak convergence. Let us state it in an abstract setting.

Proposition 2.4.8. *Let V be a normed linear space and Z a dense subset of V^*. For any bounded sequence $(v_n)_{n\in\mathbf{N}}$ in V, the following assertions are equivalent:*

(i) $v_n \overset{\sigma(V,V^*)}{\longrightarrow} v$.

(ii) *For all $z^* \in Z$, $\langle z^*, v_n\rangle \longrightarrow \langle z^*, v\rangle$ as $n \to +\infty$.*

PROOF. Clearly (i) $\Longrightarrow$ (ii). So let us assume (ii) and prove that for any $v^* \in V^*$, we have

$$\lim_{n\to+\infty}\langle v^*, v_n\rangle = \langle v^*, v\rangle.$$

By the density of Z in V^* for any $\varepsilon > 0$, there exists some element $z^*_\varepsilon \in Z$ such that $\|v^* - z^*_\varepsilon\|_* < \varepsilon$. Let us write

$$\langle v^*, v_n - v\rangle = \langle z^*_\varepsilon, v_n - v\rangle + \langle v^* - z^*_\varepsilon, v_n - v\rangle,$$

which by the triangle inequality and the definition of the dual norm $\|\cdot\|_*$ yields

$$|\langle v^*, v_n - v\rangle| \le |\langle z^*_\varepsilon, v_n - v\rangle + \|v^* - z^*_\varepsilon\|_*\cdot\|v_n - v\|.$$

Using the assumption that the sequence $(v_n)_{n\in\mathbf{N}}$ is bounded in V and that $\|v^* - z^*_\varepsilon\|_* < \varepsilon$, we obtain that for some constant $C \in \mathbf{R}^+$,

$$|\langle v^*, v_n - v\rangle| \le |\langle z^*_\varepsilon, v_n - v\rangle| + C\varepsilon.$$

Now let n tend to $+\infty$, and use assumption (ii) together with $z^*_\varepsilon \in Z$ to get

$$\limsup_{n\to+\infty} |\langle v^*, v_n - v\rangle| \le C\varepsilon.$$

This inequality being true for any $\varepsilon > 0$, we finally infer

$$\forall v^* \in V^* \quad \lim_{n\to+\infty} \langle v^*, v_n - v\rangle = 0,$$

that is, $v = \sigma(V, V^*)\lim_{n\to+\infty} v_n$. □

2.4.3 ▪ Weak convergence and geometry of uniformly convex spaces

In this section we pay attention to a particular class of Banach spaces, namely, the uniformly convex Banach spaces, where we will be able to extend the result of Proposition 2.4.7, that is, weak convergence and convergence of the norms imply the strong convergence.

Definition 2.4.2. *A Banach space $(V, \|\cdot\|)$ is said to be uniformly convex if for any sequences $(u_n)_{n\in\mathbf{N}}$, $(v_n)_{n\in\mathbf{N}}$ in V with $\|u_n\| = \|v_n\| = 1$ for all $n \in \mathbf{N}$, the following implication holds:*

$$\left\|\frac{u_n + v_n}{2}\right\| \underset{n\to+\infty}{\longrightarrow} 1 \Longrightarrow \|u_n - v_n\| \longrightarrow 0.$$

This result reflects a geometrical property of the unit ball which has to be well rotund. Note that this definition is not stable when replacing a norm by an equivalent one. As an elementary example, one can observe that $V = \mathbf{R}^N$ equipped with the norm $\|x\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$ is uniformly convex, whereas the norms $\|x\|_1 = \sum_{i=1}^N |x_i|$ and $\|x\|_\infty = \max_{1\le i\le N} |x_i|$ are not uniformly convex.

The uniform convexity of the norm expresses that if u and v are on the unit sphere, the fact that $\frac{u+v}{2}$ is close to the sphere forces u and v to be close to each other.

Proposition 2.4.9. (a) *The Hilbert spaces are uniformly convex.*

(b) *The L^p spaces, $1 < p < \infty$ are uniformly convex.*

PROOF. (a) The uniform convexity of Hilbert spaces is a direct consequence of the parallelogram equality:

$$\forall u, v \in V \quad \|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

Notice that this property characterizes Hilbert spaces among general Banach spaces. So, let us take u_n, v_n in V such that $\|u_n\| = \|v_n\| = 1$. We have

$$\|u_n - v_n\|^2 = 4\left(1 - \left\|\frac{u_n + v_n}{2}\right\|^2\right).$$

So $\|\frac{u_n+v_n}{2}\| \longrightarrow 1$ forces $\|u_n - v_n\|$ to converge to zero as $n \to +\infty$.

(b) The same type of argument works in L^p spaces, $1 < p < \infty$, when replacing the parallelogram identity by the so-called Clarkson's inequalities; if $\|\cdot\|$ is the L^p norm, one has to distinguish two cases:

$$\left\|\frac{u-v}{2}\right\|^p + \left\|\frac{u+v}{2}\right\|^p \le \frac{1}{2}(\|u\|^p + \|v\|^p) \qquad \text{if } 2 \le p < \infty,$$

$$\left\|\frac{u-v}{2}\right\|^{p'} + \left\|\frac{u+v}{2}\right\|^{p'} \le \left[\frac{1}{2}(\|u\|^{p'} + \|v\|^{p'})\right]^{p-1} \qquad \text{if } 1 < p \le 2$$

with $\frac{1}{p} + \frac{1}{p'} = 1$. □

The following result justifies the introduction of the notion of uniform convexity in this section devoted to the weak convergence.

Proposition 2.4.10. *Let V be a uniformly convex Banach space. Then for any sequence $(v_n)_{n\in\mathbf{N}}$ in V the following implication holds:*

$$v_n \overset{\sigma(V,V^*)}{\longrightarrow} v \textit{ and } \|v_n\| \longrightarrow \|v\| \Longrightarrow v_n \overset{\|\cdot\|}{\longrightarrow} v.$$

PROOF. Let us reduce ourselves to the case $\|v_n\| = \|v\| = 1$. To that end, let us consider $w_n := \frac{v_n}{\|v_n\|}$ and $w := \frac{v}{\|v\|}$. (The case $v = 0$ is obvious, so we can assume $v \neq 0$.)

One can notice that $\|w_n\| = 1$ and $w_n \overset{\sigma(V,V^*)}{\longrightarrow} w$: indeed, for any $v^* \in V^*$,

$$\begin{aligned}\langle v^*, w_n - w\rangle &= \left\langle v^*, \frac{v_n}{\|v_n\|} - \frac{v}{\|v\|}\right\rangle \\ &= \frac{1}{\|v_n\|}\langle v^*, v_n - v\rangle + \left(\frac{1}{\|v_n\|} - \frac{1}{\|v\|}\right)\langle v^*, v\rangle,\end{aligned}$$

which goes to zero as $n \to +\infty$, since $v_n \overset{\sigma(V,V^*)}{\longrightarrow} v$ and $\|v_n\| \longrightarrow \|v\| \neq 0$.

Let us show that $\left\|\frac{w_n+w}{2}\right\| \to 1$ as $n \to +\infty$. Since $\frac{w_n+w}{2} \overset{\sigma(V,V^*)}{\longrightarrow} w$, by the lower semicontinuity of the norm for the weak topology (Proposition 2.4.6(iii)),

$$1 = \|w\| \le \liminf_n \left\|\frac{w_n + w}{2}\right\| \le \limsup_n \left\|\frac{w_n + w}{2}\right\| \le 1.$$

The last inequality follows from the triangle inequality and the fact that $\|w_n\| = \|w\| = 1$. So we have $\|w_n\| = \|w\| = 1$ and $\|\frac{w_n+w}{2}\| \longrightarrow 1$. It follows from the uniform convexity property that $w_n \overset{\|\cdot\|}{\longrightarrow} w$. Using once more that $\|v_n\| \longrightarrow \|v\|$ we derive that $v_n = \|v_n\| w_n$ norm converges to $v = \|v\| w$. □

Remark 2.4.3. It is a quite useful method, when proving that a sequence $(u_n)_{n\in\mathbf{N}}$ is norm converging in a Hilbert space, or more generally in a uniformly convex Banach space, to prove first that the weak convergence holds and then to prove that the norms converge, too. For example, when minimizing the norm over a closed convex bounded subset of a uniformly convex Banach space, one automatically obtains that any minimizing sequence is norm convergent.

The property for a Banach space to verify for any sequence $(v_n)_{n\in\mathbf{N}}$ in V the implication $v_n \overset{\sigma(V,V^*)}{\longrightarrow} v$ and $\|v_n\| \longrightarrow \|v\| \Longrightarrow v_n \overset{\|\cdot\|}{\longrightarrow} v$ is often called the *Kadek property*.

Remark 2.4.4. Let us observe that the Kadek property fails to be true in general Banach spaces. For example, it is false in the space $L^1(\Omega)$, $\Omega \subset \mathbf{R}^N$ equipped with the Lebesgue measure: indeed, take

$$\Omega = (0, \pi), \quad v_n(x) = 1 + \sin nx, \quad n = 1, 2, \ldots.$$

Then $v_n \overset{\sigma(L^1, L^\infty)}{\longrightarrow} v \equiv 1$, $||v_n||_1 = \int_0^\pi v_n(x)dx = \pi + \frac{1}{n}(1 - \cos n\pi)$, so that $||v_n||_1 \underset{n\to\infty}{\longrightarrow} \pi = ||v||_1$. But $||v_n - v||_1 = \int_0^\pi |\sin nx| dx$ does not converge to zero as $n \to +\infty$. Indeed,

$$\int_0^\pi |\sin nx| dx = n \cdot \int_0^{\pi/n} \sin nx dx = 2.$$

2.4.4 ▪ Weak compactness theorems in reflexive Banach spaces

We have already observed that L^p spaces enjoy quite different properties with respect to the weak convergence, depending on the two situations $1 < p < \infty$ and $p = 1$, $p = \infty$. One can distinguish them by introducing the concept of uniform convexity, or local uniform convexity of the space as done in Section 2.4.3. But this is a geometrical concept related to the choice of the norm, and when dealing with topological concepts like compactness, one is naturally led to consider notions which are of topological nature (i.e., invariant by the choice of an equivalent norm). This is where the notion of reflexive Banach space plays a fundamental role. Let us first recall its definition.

Let V be a Banach space, V^* its topological dual, and V^{**} its topological bidual equipped, respectively, with the norms

$$||\cdot||_V = ||\cdot||, \quad ||v^*||_* = \sup_{||v||\le 1} |\langle v^*, v\rangle|, \quad ||v^{**}||_{**} = \sup_{||v^*||_* \le 1} |\langle v^{**}, v^*\rangle|.$$

There exists a canonical embedding of V into V^{**} denoted by $J : V \longrightarrow V^{**}$, which is defined as follows:

$$\forall v \in V,\ \forall v^* \in V^* \quad \langle Jv, v^*\rangle_{(V^{**}, V^*)} = \langle v^*, v\rangle_{(V^*, V)}.$$

Let us comment on this definition. For any $v \in V$, the mapping

$$v^* \in V^* \longmapsto \langle v^*, v\rangle_{(V^*, V)} \in \mathbf{R}$$

is linear and continuous on V^*, so it defines uniquely an element of V^{**} which is denoted by Jv. Let us observe that

$$|\langle v^*, v\rangle| \le ||v^*||_* \cdot ||v|| \quad \forall\ v \in V,$$

so that $||Jv||_{**} \le ||v||$. Indeed, as a consequence of the Hahn–Banach theorem we have

$$||Jv||_{**} = \sup_{||v^*||_* \le 1} |\langle v^*, v\rangle| = ||v||,$$

so that J is a *linear isometry* from V into V^{**}. As a consequence, J is an embedding of V into V^{**}.

Definition 2.4.3. *A Banach space V is said to be reflexive if $J(V) = V^{**}$. When V is reflexive one can identify V and V^{**} with the help of J.*

Remark 2.4.5. J is a linear isometry. Thus it preserves the linear and the normed structures and allows us to identify V and V^{**} when J is onto, that is, in the case of reflexive Banach spaces. One has to pay attention to the following fact: the definition of reflexive Banach spaces says that the map J realizes an isometrical isomorphism between V and V^{**}. It is essential to use J in the definition since one can exhibit a nonreflexive Banach space V such that there exists an isometry from V onto V^{**}!

Proposition 2.4.11. *Let V be a uniformly convex Banach space. Then V is reflexive.*

For a proof of this result, see, for instance, [361], [137]. When considering L^p spaces, this result is in accordance with the results concerning the dual of L^p spaces. When $1 < p < \infty$, L^p is uniformly convex, $(L^p)^* = L^{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$, so that $(L^p)^{**} = (L^{p'})^* = L^p$ (equalities above mean isometric isomorphisms).

Remark 2.4.6. Note that there exist reflexive Banach spaces which do not admit an equivalent norm which makes the space uniformly convex. However, one can always renorm a reflexive Banach space with a norm (equivalent) which is locally uniformly convex both with its dual norm. With this renorming, it will satisfy the Kadek property (as well as its dual); see Section 2.4.3.

The importance of reflexive Banach spaces is justified by the following theorem.

Theorem 2.4.2. (a) *In a reflexive Banach space $(V, \|\cdot\|)$ the closed unit ball*

$$B = \{v \in V : \|v\|_V \leq 1\}$$

is compact for the topology $\sigma(V, V^*)$. As a consequence, the bounded subsets of V are relatively compact for the topology $\sigma(V, V^*)$.

(b) *The above property characterizes the reflexive Banach spaces: a Banach space is reflexive iff the closed unit ball is compact for the topology $\sigma(V, V^*)$.*

PROOF. The proof is a direct consequence of the Banach–Alaoglu–Bourbaki theorem, Theorem 2.4.8. It makes use of the weak* topology on the dual of a Banach space. □

Let us now state a theorem from Eberlein and Smulian which states that from every bounded sequence in a reflexive Banach space one can extract a sequence which converges for the topology $\sigma(V, V^*)$. This is an important and quite surprising result, since the weak topology is not metrizable. One does not expect to have such a sequential compactness result!

Theorem 2.4.3. *Let V be a reflexive Banach space. Then, from each bounded sequence $(u_n)_{n\in\mathbf{N}}$ in V, one can extract a subsequence $(u_{n_k})_{k\in\mathbf{N}}$ which converges for the topology $\sigma(V, V^*)$.*

PROOF. (a) Let us first assume that V^* is separable, that is, there exists a countable set $D = (v_k^*)_{k\in\mathbf{N}}$ which is dense in V^*. The proof relies on a diagonalization argument. First, let us notice that for all $v^* \in V^*$,

$$|\langle v^*, u_n\rangle| \leq \|v^*\|_* \cdot \|u_n\| \leq C\|v^*\|_*,$$

where $C = \sup_{n\in\mathbf{N}} \|u_n\| < +\infty$. So the sequence $\{\langle v^*, u_n\rangle : n \in \mathbf{N}\}$ is bounded in $\mathbf{R}$. For each $v^* \in V^*$, one can extract from the sequence $\{\langle v^*, u_n\rangle : n \in \mathbf{N}\}$ a convergent subsequence. The difficult point is that without any further argument, the so extracted sequence depends on v^*. This is where the separability of V^* and the diagonalization argument take place.

Let us start with $v_1^* \in D$ (D dense countable subset of V^*) and extract a convergent subsequence $\{\langle v_1^*, u_{\sigma_1(n)}\rangle : n \in \mathbf{N}\}$, where $\sigma_1 : \mathbf{N} \to \mathbf{N}$ is a strictly increasing mapping. In a similar way, the sequence $\{\langle v_2^*, u_{\sigma_1(n)}\rangle : n \in \mathbf{N}\}$ is bounded in $\mathbf{R}$, so there exists a convergent subsequence $\{\langle v_2^*, u_{\sigma_1\circ\sigma_2(n)}\rangle : n \in \mathbf{N}\}$. Let us iterate this argument by induction. We can so construct for each $n \in \mathbf{N}$ an increasing mapping $\sigma_n : \mathbf{N} \longrightarrow \mathbf{N}$ such that the sequence

$$\{\langle v_n^*, u_{\sigma_1\circ\sigma_2\circ\cdots\circ\sigma_n(k)}\rangle : k \in \mathbf{N}\}$$

is convergent in $\mathbf{R}$. Note that it is important to have the composition of the mapping σ_i in the precise order $\sigma_1 \circ \cdots \circ \sigma_n$ to have this subsequence extracted from all the previous ones, $\sigma_1, \sigma_1 \circ \sigma_2, \dots, \sigma_1 \circ \cdots \circ \sigma_{n-1}$.

Now the diagonalization argument consists of taking $\tau : \mathbf{N} \longrightarrow \mathbf{N}$ defined by

$$\tau(n) = (\sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n)(n).$$

In other words, $\tau(n)$ is the element of rank n of the subsequence which has been extracted at step n. Clearly τ is strictly increasing.

It is important to notice that for each $n \in \mathbf{N}$, the sequence $(u_{\tau(k)})_{k\geq n}$ is extracted from the sequence $(u_{\sigma_1\circ\cdots\circ\sigma_n(k)})_{k\geq n}$: indeed, for $k \geq n$,

$$u_{\tau(k)} = u_{\sigma_1\circ\sigma_2\circ\cdots\circ\sigma_k(k)} = u_{(\sigma_1\circ\cdots\circ\sigma_n)(p)},$$

where $p = (\sigma_{n+1} \circ \cdots \circ \sigma_k)(k)$. Since $k \geq n$ and σ_i are strictly increasing, we also have $p \geq n$.

It follows that for each $k \in \mathbf{N}$, the sequence $(\langle v_k^*, u_{\tau(n)}\rangle : n \in \mathbf{N})$, as a subsequence of a convergent sequence, is convergent.

By the density of D in V^*, the result is easily extended to an arbitrary element v^* of V^*: for each $\varepsilon > 0$ take $v_{k_\varepsilon}^* \in \mathscr{D}$ such that $\|v^* - v_{k_\varepsilon}^*\| < \varepsilon$. Then for each $n, m \in \mathbf{N}$,

$$\begin{aligned}|\langle v^*, u_{\tau(n)}\rangle - \langle v^*, u_{\tau(m)}\rangle| &\leq |\langle v_{k_\varepsilon}^*, u(\tau_n) - u(\tau_m)\rangle| \\ &\quad + |\langle v^* - v_{k_\varepsilon}^*, u(\tau_n) - u(\tau_m)\rangle| \\ &\leq 2C\varepsilon + |\langle v_{k_\varepsilon}^*, u(\tau_n) - u(\tau_m)\rangle|.\end{aligned}$$

Hence,

$$\limsup_{n,m\to\infty} |\langle v^*, u_{\tau(n)} - u_{\tau(m)}\rangle| \leq 2C\varepsilon.$$

This being true for any $\varepsilon > 0$, we infer that the sequence $\{\langle v^*, u_{\tau(n)}\rangle : n \in \mathbf{N}\}$ satisfies the Cauchy criteria and is thus convergent in $\mathbf{R}$.

Let us denote for all $v^* \in V^*$, $L(v^*) := \lim_{n\to+\infty}\langle v^*, u_{\tau(n)}\rangle$. Clearly, L is a linear continuous form on V^*; note that by passing to the limit in the inequality $|\langle v^*, u_n\rangle| \leq C\|v^*\|_*$, we also have $|L(v^*)| \leq C\|v^*\|_*$. Hence $L \in V^{**}$.

Let us stress the fact that up to this point, we have not used the reflexivity hypothesis. We now use that V is reflexive to assert that $L = J(u)$ for some $u \in V$, where J is the canonical embedding of V into V^{**}. So for all $v^* \in V^*$ we have

$$\lim_{n\to+\infty} \langle v^*, u_{\tau(n)}\rangle = \langle v^*, u\rangle,$$

that is, $u = \sigma(V, V^*)\lim_{n\to\infty} u_{\tau(n)}$.

(b) Now take V a reflexive Banach space and do not make any separability assumptions. Take E the subspace of V generated by the $(u_n)_{n\in\mathbf{N}}$ and define $W=\bar{E}$, the closure of E in $(V,\|\cdot\|)$.

It is easy to verify that W is reflexive and separable. This in turn implies that W^* is reflexive and separable (see [137, Corollary III.24]). We are now in the situation studied in the first part of the proof. One can extract a subsequence $(u_{\tau(n)})_{n\in\mathbf{N}}$ which converges in W for the topology $\sigma(W,W^*)$. Since $V^*\subset W^*$ (by restriction of the linear continuous forms on V to W) we derive that $(u_{\tau(n)})_{n\in\mathbf{N}}$ is convergent for the topology $\sigma(V,V^*)$. □

2.4.5 ▪ The Dunford–Pettis weak compactness theorem in $L^1(\Omega)$

When $1<p<\infty$, the L^p spaces are reflexive Banach spaces, and it follows from Theorem 2.4.3 that the relatively compact subsets of L^p for the topology $\sigma(L^p,L^{p'})$ are exactly the bounded subsets of L^p. The situation in the case $p=1$ is very different (L^1 is not a reflexive Banach space), and the comprehension of the weak convergence properties of bounded sequences in L^1 is a subject of great importance and is quite involved.

Let us first examine the following example. Take $\Omega=(-1,1)$ equipped with the Lebesgue measure and take

$$v_n(x)=\begin{cases} n & \text{if } -\frac{1}{2n}\le x\le +\frac{1}{2n},\\ 0 & \text{elsewhere.}\end{cases}$$

Clearly, the sequence $(v_n)_{n\in\mathbf{N}}$ satisfies

$$\begin{cases} v_n\ge 0, \displaystyle\int_\Omega v_n(x)dx=1,\\ v_n(x)\longrightarrow 0 \quad \text{for a.e. } x\in\Omega,\\ \displaystyle\int_\Omega v_n(x)z(x)dx \underset{n\to+\infty}{\longrightarrow} z(0) \quad \text{for any } z\in\mathbf{C}(\Omega).\end{cases}$$

We can observe that the sequence $(v_n)_{n\in\mathbf{N}}$ is bounded in $L^1(\Omega)$, but one cannot extract a weakly convergent subsequence in the sense $\sigma(L^1,L^\infty)$. By contrast, we will see in the next section that one can interpret the convergence of the sequence $(v_n)_{n\in\mathbf{N}}$ with the help of a weak topology of dual $\sigma(V^*,V)$, for example, $\sigma(\mathcal{M}_b(\Omega),\mathbf{C}_0(\Omega))$.

For the moment, we just retain from this example that to obtain $\sigma(L^1,L^\infty)$ compactness of a sequence of functions, it is not sufficient to assume that the sequence is bounded in L^1. This is where the notion of uniform integrability plays a central role.

Definition 2.4.4. *Let $(\Omega,\mathcal{A},\mu)$ be a measure space with μ a positive and finite measure $(\mu(\Omega)<+\infty)$. Let $\mathbf{K}$ be a subset of $L^1(\Omega,\mathcal{A},\mu)$. We say that $\mathbf{K}$ is uniformly integrable if* (a) *and* (b) *hold:*

(a) *$\mathbf{K}$ is bounded in $L^1(\Omega,\mathcal{A},\mu)$;*

(b) *for every $\varepsilon>0$ there exists some $\delta(\varepsilon)>0$ such that*

$$A\in\mathcal{A},\ \mu(A)<\delta(\varepsilon)\Longrightarrow \sup_{v\in\mathbf{K}}\int_A |v(x)|d\mu(x)<\varepsilon.$$

It is sometimes convenient to consider the following criterion of uniform integrability (also called equi-integrability)

Proposition 2.4.12. *Let $(\Omega, \mathscr{A}, \mu)$ be a measure space satisfying the hypotheses of Definition 2.4.4. Then* **K** *is uniformly integrable iff*

$$\lim_{R\to+\infty} \sup_{v\in\mathbf{K}} \int_{[|v|>R]} |v(x)|\, d\mu(x) = 0. \tag{2.45}$$

PROOF. Assume that (a) and (b) hold and consider $R_0 = \frac{1}{\delta} \sup_{v\in\mathbf{K}} \int_\Omega |v(x)|\, d\mu(x)$, which is finite according to (a). Let $R \geq R_0$; from

$$\mu\big([|v| > R]\big) \leq \frac{1}{R} \int_\Omega |v(x)|\, d\mu(x) \leq \frac{1}{R_0} \int_\Omega |v(x)|\, d\mu(x)$$

we infer that $\mu\big([|v| > R]\big) < \delta$ so that, from (b), $\int_{[|v|>R]} |v(x)|\, d\mu(x) < \varepsilon$.

Conversely, assume that (2.45) holds. For each $A \in \mathscr{A}$ and each $R > 0$ one has

$$\begin{aligned}\int_A |v(x)|\, d\mu(x) &= \int_{A\cap[|v|\leq R]} |v(x)|\, d\mu(x) + \int_{A\cap[|v|>R]} |v(x)|\, d\mu(x)\\ &\leq R\mu(A) + \sup_{v\in\mathbf{K}} \int_{[|v|>R]} |v(x)|\, d\mu(x). \end{aligned} \tag{2.46}$$

Taking $A = \Omega$ in (2.46) gives (a). On the other hand, for all $\varepsilon > 0$, choose R large enough so that $\sup_{v\in\mathbf{K}} \int_{[|v|>R]} |v(x)|\, d\mu(x) < \frac{\varepsilon}{2}$, and take A in $\mathscr{A}$ satisfying $\mu(A) < \frac{\varepsilon}{2R}$. Then (2.46) yields (b). □

A comprehensive characterization of this property is given by the De La Vallée–Poussin theorem.

Theorem 2.4.4. *Let $(\Omega, \mathscr{A}, \mu)$ be a measure space with μ a positive and finite measure and* **K** *a subset of $L^1(\Omega, \mathscr{A}, \mu)$. The following properties are equivalent:*

(i) **K** *is uniformly integrable;*

(ii) *there exists a function $\theta : [0,+\infty[\to [0,+\infty[$ (θ can be taken convex and increasing) such that $\lim_{s\to+\infty} \frac{\theta(s)}{s} = +\infty$ and*

$$\sup_{v\in\mathbf{K}} \int_\Omega \theta(|v(x)|) d\mu(x) < +\infty.$$

PROOF. The implication (ii) $\Longrightarrow$ (i) is important for applications. Let us prove it.

First, one can observe that since θ has a superlinear growth, for each $M \in \mathbf{R}^+$ there exists some $C(M) \in \mathbf{R}^+$ such that

$$\forall s \in \mathbf{R}^+ \quad 0 \leq s \leq \frac{1}{M}\theta(s) + C(M).$$

Let us fix $M_0 > 0$. We have for each $v \in \mathbf{K}$

$$\int_\Omega |v(x)| d\mu(x) \leq \frac{1}{M_0} \int_\Omega \theta(|v(x)|) d\mu(x) + C(M_0)\mu(\Omega)$$

and hence $\sup_{v\in\mathbf{K}} \int |v| d\mu < +\infty$, which proves (a) of Definition 2.4.4.

Let us now prove (b). Fix $\varepsilon > 0$; for any $v \in \mathbf{K}$, $A \in \mathscr{A}$, and $M \in \mathbf{R}^+$,

$$\int_A |v(x)| d\mu(x) \le \frac{1}{M} \int_A \theta(|v(x)|) d\mu(x) + C(M)\mu(A)$$
$$\le \frac{1}{M} \sup_{v\in\mathbf{K}} \int_\Omega \theta(|v|) d\mu + C(M)\mu(A).$$

Take

$$M(\varepsilon) := \frac{2}{\varepsilon} \sup_{v\in\mathbf{K}} \int_\Omega \theta(|v|)\, d\mu, \qquad \delta(\varepsilon) := \frac{\varepsilon}{2C(M(\varepsilon))}.$$

Then if $\mu(A) \le \delta(\varepsilon)$ we have

$$\sup_{v\in\mathbf{K}} \int_A |v(x)| d\mu(x) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which proves (b).

For the proof of the reverse implication (i) $\Longrightarrow$ (ii), see, for instance, [198, Theorem 22] or [153, Theorem 2.12]. □

We are now ready to state the Dunford–Pettis theorem, which gives a characterization of the weak compactness property in L^1.

Theorem 2.4.5 (Dunford–Pettis theorem). *Let $(\Omega, \mathscr{A}, \mu)$ be a measure space with μ a positive and finite measure. Let $\mathbf{K}$ be a subset of $L^1(\Omega, \mathscr{A}, \mu)$. The following properties are equivalent:*

(i) $\mathbf{K}$ *is relatively compact for the weak topology $\sigma(L^1, L^\infty)$.*

(ii) $\mathbf{K}$ *is uniformly integrable.*

(iii) *From each sequence $(v_n)_{n\in\mathbf{N}}$ contained in $\mathbf{K}$, one can extract a subsequence converging for the topology $\sigma(L^1, L^\infty)$.*

PROOF. See [198, Theorem 25], [203, Theorem IV.8.9, Corollary IV.8.11], and [153]. When Ω is an open subset of $\mathbf{R}^N$ and μ the Lebesgue measure on Ω, one can find a proof of implication (ii) $\Longrightarrow$ (iii) in Proposition 4.3.7 by using the notion of Young measures. □

Remark 2.4.7. As an illustration of the Dunford–Pettis theorem, let us consider $(v_n)_{n\in\mathbf{N}}$ a sequence in $L^1(\Omega, \mathscr{A}, \mu)$, $\mu(\Omega) < +\infty$ such that $\sup_{n\in\mathbf{N}} \int_\Omega |v_n| \ln |v_n| d\mu < +\infty$. We claim that the sequence (v_n) is $\sigma(L^1, L^\infty)$ relatively compact. To obtain this result, just use the De La Vallée–Poussin theorem with $\theta(r) = r \ln r$ (which is superlinear) and then use the Dunford–Pettis theorem.

It is immediate that any dominated sequence $(v_n)_{n\in\mathbf{N}}$ in $L^1(\Omega, \mathscr{A}, \mu)$, i.e., satisfying for a.e. $x \in \Omega$, $|v_n(x)| \le g(x)$, where g is some function in $L^1(\Omega, \mathscr{A}, \mu)$, is uniformly integrable. The following theorem extends the Lebesgue dominated convergence theorem for uniformly integrable sequences.

Theorem 2.4.6 (Vitali convergence theorem). *Let $(\Omega, \mathscr{A}, \mu)$ be a measure space with μ a positive and finite measure. Let $(v_n)_{n\in\mathbf{N}}$ be a uniformly integrable sequence of functions in*

$L^1(\Omega, \mathscr{A}, \mu)$ which converges a.e. to some function $v \in L^1(\Omega, \mathscr{A}, \mu)$. Then $(v_n)_{n\in\mathbf{N}}$ strongly converges to v in $L^1(\Omega, \mathscr{A}, \mu)$.

PROOF. For each $R > 0$ one has

$$\begin{aligned}\int_\Omega |v_n - v|\, d\mu &\le \int_{[v_n - v| \le R]} |v_n - v|\, d\mu + \int_{|v_n - v| > R} |v_n - v|\, d\mu \\ &\le \int_{[v_n - v| \le R]} |v_n - v|\, d\mu + \sup_{n\in\mathbf{N}} \int_{|v_n - v| > R} |v_n - v|\, d\mu. \end{aligned} \tag{2.47}$$

Consider the continuous function $\phi_R : [0, +\infty) \to [0, R]$ defined by $\phi_R(t) = \min(t, R)$. From (2.47) we infer

$$\int_\Omega |v_n - v|\, d\mu \le \int_\Omega \phi_R(|v_n - v|)\, d\mu + \sup_{n\in\mathbf{N}} \int_{|v_n - v| > R} |v_n - v|\, d\mu. \tag{2.48}$$

Since $\phi_R(|v_n - v|)$ converges a.e. to 0 and is dominated by R, according to the Lebesgue dominated convergence theorem, we deduce that

$$\lim_{n\to+\infty} \int_\Omega \phi_R(|v_n - v|)\, d\mu = 0.$$

On the other hand the sequence $(v_n - v)_{n\in\mathbf{N}}$ is clearly uniformly integrable; hence, from Proposition 2.4.12,

$$\lim_{R\to+\infty} \sup_{n\in\mathbf{N}} \int_{|v_n - v| > R} |v_n - v|\, d\mu = 0.$$

The conclusion then follows by letting first $n \to +\infty$, then $R \to +\infty$ in (2.48).

2.4.6 ▪ The weak* topology $\sigma(V^*, V)$

Let $(V, \|\cdot\|)$ be a normed linear space with topological dual V^*. On V^* we have already defined two topologies:

- The norm topology associated to the dual norm

  $$\|v^*\|_* = \sup\{|\langle v^*, v\rangle| : v \in V, \|v\| \le 1\},$$

  which makes V^* a Banach space;

- The weak topology $\sigma(V^*, V^{**})$, where V^{**} is the topological bidual of V. But this topology is often difficult to handle because the space V^{**} may have a rather involved structure (think, for example, $V = L^1$, $V^* = L^\infty$, and $V^{**} = (L^\infty)^*$), and moreover it may be too strong to enjoy desirable compactness properties.

The idea is to introduce a topology weaker than $\sigma(V^*, V^{**})$ by considering a weak topology on V^* induced not by all the linear continuous forms on V^* but only by a subfamily. At this point, there is a natural candidate which consists in taking $J(V) \subset V^{**}$, where $J : V \longrightarrow V^{**}$ is the canonical embedding from V into its bidual V^{**}. Recall that (see Section 2.4.4)

$$\forall v \in V, \forall v^* \in V^* \quad \langle J(v), v^*\rangle_{(V^{**}, V^*)} := \langle v^*, v\rangle_{(V^*, V)}.$$

Definition 2.4.5. *Let V be a normed space with topological dual V^*. The weak* topology $\sigma(V^*,V)$ on V^* is the weakest topology on V^* making continuous all the mappings $(J(v))_{v\in V}$, where J is the canonical embedding from V into V^{**}:*

$$\begin{aligned} J(v): V^* &\longrightarrow \mathbf{R}, \\ v^* &\longmapsto \langle J(v), v^*\rangle_{(V^{**},V^*)} = \langle v^*, v\rangle. \end{aligned}$$

Let us collect in the following proposition some first elementary properties of the topology $\sigma(V^*,V)$.

Proposition 2.4.13. *Let V be a normed space and $\sigma(V^*,V)$ the weak* topology on the topological dual V^*. Then*

(i) *A local base of neighborhoods of $v_0^* \in V^*$ for the topology $\sigma(V^*,V)$ consists of all sets of the form*
$$N(v_0^*) = \{v^* \in V^* : |\langle v^* - v_0^*, v_i\rangle| < \varepsilon \;\; \forall i \in I\},$$
where I is a finite index set, $v_i \in V$ for each $i \in I$, and $\varepsilon > 0$.

(ii) *For any sequence $(v_n^*)_{n\in\mathbf{N}}$ in V^*, we have*

(a) $v_n^* \overset{\sigma(V^*,V)}{\longrightarrow} v^* \Longleftrightarrow \forall\, v \in V \quad \langle v_n^*, v\rangle \longrightarrow \langle v^*, v\rangle$;

(b) $v_n^* \overset{\|\cdot\|}{\longrightarrow} v^* \Longrightarrow v_n^* \overset{\sigma(V^*,V)}{\longrightarrow} v^*$.

Assume now that V is a Banach space. Then

(c) $v_n^* \overset{\sigma(V^*,V)}{\longrightarrow} v^* \Longrightarrow$ *the sequence (v_n^*) is bounded and*
$$\|v^*\|_* \leq \liminf_n \|v_n^*\|_*;$$

(d) $v_n^* \overset{\sigma(V^*,V)}{\longrightarrow} v^*$ *and* $v_n \overset{\|\cdot\|}{\longrightarrow} v \Longrightarrow \langle v_n^*, v_n\rangle \longrightarrow \langle v^*, v\rangle$.

PROOF. (i) and (ii)(a) are direct consequences of the general properties of topologies induced by functions; see, respectively, Theorem 2.4.1 and Proposition 2.4.3. Then (ii)(b), (ii)(c), and (ii)(d) are obtained in a similar way as in the proof of Proposition 2.4.6. Just notice that to apply the uniform boundedness theorem, one needs to assume that V is a Banach space. (In Proposition 2.4.6 one works on V^*, which is always a Banach space!) □

An important example: $\mathbf{V = C_0(\Omega)}$, $\mathbf{V^* = \mathscr{M}_b(\Omega)}$. Let Ω be a locally compact topological space which is σ-compact (i.e., Ω can be written as $\Omega = \bigcup_{n\in\mathbf{N}} K_n$ with K_n compact). For example, we may take $\Omega = \mathbf{R}^N$, or Ω an open subset of $\mathbf{R}^N$, or Ω an arbitrary topological compact set.

Take $V = \mathbf{C}_0(\Omega)$ the linear space of real continuous functions on Ω which tend to zero at infinity; more precisely, we say that a continuous function u is in $\mathbf{C}_0(\Omega)$ if for every $\varepsilon > 0$ there exists a compact set K_ε such that $|u(x)| < \varepsilon$ on $\Omega \setminus K_\varepsilon$. Notice that $\mathbf{C}_0(\Omega)$ reduces to $\mathbf{C}(\Omega)$ when Ω is compact. We may endow the space $\mathbf{C}_0(\Omega)$ with the norm

$$\|v\|_V = \sup_{x\in\Omega} |v(x)|.$$

Then V is a Banach space whose topological dual V^* can be described thanks to the celebrated result below.

Theorem 2.4.7 (Riesz–Alexandroff representation theorem). *The topological dual of $\mathbf{C}_0(\Omega)$ can be isometrically identified with the space of bounded Borel measures. More precisely, to each bounded linear functional Φ on $\mathbf{C}_0(\Omega)$ there is a unique Borel measure μ on Ω such that for all $f \in \mathbf{C}_0(\Omega)$,*

$$\Phi(f) = \int_\Omega f(x)d\mu(x).$$

Moreover, $||\phi|| = |\mu|(\Omega)$.

For a proof of this theorem, see, for instance, [331, Theorem 6.19] or [143, Theorem 1.4.22]. Notice that this theorem holds true also when Ω is not supposed to be σ-compact, but in that case one has to consider measures μ which are regular. We recall that a Borel measure $\mu \geq 0$ is said to be regular if

$$\begin{cases} \forall B \in \mathscr{B}(\Omega) & \mu(B) = \inf\{\mu(V) : V \supset B, V \text{ open}\}, \\ \forall B \in \mathscr{B}(\Omega) & \mu(B) = \sup\{\mu(K) : K \subset B, K \text{ compact}\}, \end{cases}$$

and a signed measure μ is said to be regular if its *total variation measure* $|\mu|$ is regular. When Ω is σ-compact, this property is automatically satisfied by Borel measures which are bounded. Note also that since $\mathbf{C}_c(\Omega)$ is dense in $\mathbf{C}_0(\Omega)$, these two spaces have the same topological dual. We prefer to consider $\mathbf{C}_0(\Omega)$ in this construction because it is a Banach space for the sup norm.

Thus a bounded Borel measure μ can be as well considered as a σ-additive set function on the Borel σ-algebra (that's the probabilistic approach) or as a continuous linear form on $\mathbf{C}_0(\Omega)$ or $\mathbf{C}_c(\Omega)$ (that's the functional analysis approach). Given a sequence $(\mu_n)_{n\in\mathbf{N}}$ of bounded Borel measures, we can consider these measures as elements of the topological dual space $V^* = \mathscr{M}_b(\Omega)$ of $V = \mathbf{C}_0(\Omega)$. This leads to the following definition.

Definition 2.4.6.

(i) *A sequence $(\mu_n)_{n\in\mathbf{N}} \subset \mathscr{M}_b(\Omega)$ converges weakly to $\mu \in \mathscr{M}_b(\Omega)$, and we write*

$$\mu_n \longrightarrow \mu \quad \textit{in } \mathscr{M}_b(\Omega)$$

provided

$$\int_\Omega \varphi\, d\mu_n \longrightarrow \int_\Omega \varphi\, d\mu \quad \textit{as } n \to \infty$$

for each $\varphi \in \mathbf{C}_c(\Omega)$.

(ii) *A sequence $(\mu_n) \subset \mathscr{M}_b(\Omega)$ converges $\sigma(\mathscr{M}_b, \mathbf{C}_0)$ to $\mu \in \mathscr{M}_b(\Omega)$, and we write*

$$\mu_n \overset{\sigma(\mathscr{M}_b, \mathbf{C}_0)}{\longrightarrow} \mu,$$

provided

$$\int_\Omega \varphi\, d\mu_n \longrightarrow \int_\Omega \varphi\, d\mu$$

for each $\varphi \in \mathbf{C}_0(\Omega)$.

The relation between these two close concepts is given by the following result.

Proposition 2.4.14. *Given $(\mu_n)_{n\in\mathbf{N}} \subset \mathscr{M}_b(\Omega)$, $\mu \in \mathscr{M}_b(\Omega)$ one has the equivalence*

$$\mu_n \overset{\sigma(\mathscr{M}_b, \mathbf{C}_0)}{\longrightarrow} \mu \Longleftrightarrow \mu_n \longrightarrow \mu \quad \textit{and} \quad \sup_{n\in\mathbf{N}} |\mu_n|(\Omega) < +\infty.$$

PROOF. Since $\mathbf{C}_c(\Omega) \subset \mathbf{C}_0(\Omega)$ the implication $\mu_n \overset{\sigma(\mathscr{M}_b,\mathbf{C}_0)}{\longrightarrow} \mu \Longrightarrow \mu_n \longrightarrow \mu$ is clear. Moreover, since $\mathscr{M}_b(\Omega) = V^*$ with $V = \mathbf{C}_0(\Omega)$ which is a Banach space, the uniform boundedness theorem implies (see Proposition 2.4.13(ii)(c))

$$\mu_n \overset{\sigma(\mathscr{M}_b,\mathbf{C}_0)}{\longrightarrow} \mu \Longrightarrow \sup \|\mu_n\| < +\infty,$$

that is, $\sup_{n\in\mathbf{N}} |\mu_n|(\Omega) < +\infty$. The converse statement follows from a density argument which is similar to the one developed in Proposition 2.4.8. (Note that $\mathbf{C}_c(\Omega)$ is dense in $\mathbf{C}_0(\Omega)$.) □

Corollary 2.4.1. *On any bounded subset of $\mathscr{M}_b(\Omega)$ there is the equivalence*

$$\mu_n \overset{\sigma(\mathscr{M}_b,\mathbf{C}_0)}{\longrightarrow} \mu \Longleftrightarrow \mu_n \longrightarrow \mu.$$

Let us now return to the general abstract properties of the weak* topologies and state the following compactness theorem, which explains the importance of these topologies.

Theorem 2.4.8 (Banach–Alaoglu–Bourbaki). *Let V be a normed linear space. Then the unit ball $B_{V^*} = \{v^* \in V^* : \|v^*\|_* \leq 1\}$ of the topological dual V^* is compact for the topology $\sigma(V^*, V)$.*

PROOF. An element $v^* \in V^*$ is a function from V into $\mathbf{R}$. Let us write briefly $\mathbf{R}^V$ for the set of all functions from V into $\mathbf{R}$ and denote by i the canonical embedding

$$\begin{aligned} i : V^* &\longrightarrow \mathbf{R}^V, \\ v^* &\longmapsto \{\langle v^*, v\rangle\}_{v\in V}. \end{aligned}$$

When $\|v^*\|_* \leq 1$, we have indeed $|\langle v^*, v\rangle| \leq \|v\|$, so

$$\begin{aligned} i : B_{V^*} &\longmapsto \prod_{v\in V} [-\|v\|, +\|v\|] := Y, \\ i(v^*) &= \{\langle v^*, v\rangle\}_{v\in V}. \end{aligned}$$

Let us endow Y with the product topology, which is the weakest topology on Y making all the projections continuous. This topology induces on $i(B_{V^*})$ the weak* topology $\sigma(V^*, V)$; this is exactly the way it has been defined. The topological space Y, which is a product of compact spaces and which is equipped with the product topology, is compact; this is the compactness Tikhonov theorem. So, we just have to verify that $i(B_{V^*})$ is closed in Y. This is clear since for all generalized sequence $(v_\nu^*)_{\nu\in I}$, the convergence of $i(v_\nu^*)$ for the product topology

$$\langle v_\nu^*, v\rangle \longrightarrow \Phi(v) \in Y$$

implies that Φ is still linear and $|\Phi(v)| \leq \|v\|$, so that

$$\Phi(v) = \langle v^*, v\rangle$$

for some $v^* \in B_{V^*}$ and $i(v_\nu^*) \longrightarrow i(v^*)$. □

To obtain a weak* sequential compactness result on B_{V^*} we need to assume a separability condition on V.

Let us recall that a topological space V is said to be separable if there exists a dense countable subset of V. Typically that is the case of spaces $\mathbf{C}_0(\Omega)$, $L^p(\Omega)$ for $1 \le p < +\infty$ but not $L^\infty(\Omega)$.

Theorem 2.4.9. *Let V be a separable normed space. Then the unit ball B_{V^*} of V^* is metrizable for the topology $\sigma(V^*, V)$.*

Before proving Theorem 2.4.9 let us formulate the following important result, which is a direct consequence of Theorems 2.4.8 and 2.4.9.

Corollary 2.4.2. *Let V be a separable normed linear space and $(v_n^*)_{n\in\mathbf{N}}$ a bounded sequence in V^*. Then one can extract a subsequence $(v_{n_k}^*)_{k\in\mathbf{N}}$ which converges for the topology $\sigma(V^*, V)$.*

PROOF OF THEOREM 2.4.9. Let $(v_n)_{n\ge 1}$ be a dense countable subset of the unit ball B_V of V (which exists since V is assumed to be separable; note that separability is a hereditary property).

Then define on the unit ball B_{V^*} of V^* the following distance d:

$$\forall\ u^*, v^* \in B_{V^*} \ \ d(u^*, v^*) = \sum_{n=1}^{\infty} \frac{1}{2^n} |\langle u^* - v^*, v_n \rangle|.$$

Let us verify that the topology associated to the distance d coincides with the weak* topology $\sigma(V^*, V)$ on B_{V^*}. This can be done with the help of neighborhoods or by using generalized sequences (nets): one has to verify that for an arbitrary net $(v_i^*)_{i\in I}$ contained in B_{V^*},

$$v_i^* \xrightarrow{\sigma(V^*,V)} v^* \Longleftrightarrow d(v_i^*, v) \underset{I}{\longrightarrow} 0.$$

It is easy to verify that since $\sup_{i\in I} \|v_i^*\| \le 1$,

$$d(v_i^*, v) \underset{I}{\longrightarrow} 0 \Longleftrightarrow \forall k \in \mathbf{N} \quad \langle v_i^*, v_k \rangle \underset{I}{\longrightarrow} \langle v^*, v_k \rangle.$$

So we have to verify that

$$\forall v \in B_V \langle v_i^*, v \rangle \longrightarrow \langle v^*, v \rangle \Longleftrightarrow \forall k \in \mathbf{N}\ \langle v_i^*, v_k \rangle \longrightarrow \langle v^*, v_k \rangle.$$

This is exactly the same argument as in Proposition 2.4.8, where one just uses nets instead of sequences. $\square$

Back to weak* convergence of measures. Let us denote by Ω a locally compact topological metrizable space which is σ-compact. Recall that $V = \mathbf{C}_0(\Omega)$ is separable: one can first notice that $\mathbf{C}_c(\Omega)$ is dense in $\mathbf{C}_0(\Omega)$, $\mathbf{C}_c(\Omega) = \bigcup_n \mathbf{C}(K_n)$, where K_n are compact, because of the σ-compactness assumption. Then observe that $\mathbf{C}(K)$ is separable; this can be obtained as a consequence of the Stone–Weierstrass theorem. Note that the metrizability of Ω is equivalent to the separability of $\mathbf{C}(K)$ (see [143, Theorem 2.3.29]).

We can now reformulate Corollary 2.4.2 in the case of sequences of measures.

Proposition 2.4.15. *Let Ω be a locally compact, metrizable, σ-compact topological space. Then, from any bounded sequence of Borel measures $(\mu_n)_{n\in\mathbf{N}}$ on Ω, i.e., verifying*

$$\sup_{n\in\mathbf{N}} |\mu_n|(\Omega) < +\infty,$$

one can extract a subsequence $(\mu_{n_k})_{k\in\mathbf{N}}$ which is $\sigma(\mathscr{M}_b,\mathbf{C}_0)$ convergent to some bounded Borel measure μ:

$$\forall\varphi\in\mathbf{C}_0(\Omega)\quad \int_\Omega \varphi\, d\mu_{n_k} \underset{k\to+\infty}{\longrightarrow} \int_\Omega \varphi\, d\mu.$$

As a particular important situation where the result above can be directly applied let us mention the following.

Corollary 2.4.3. *Let Ω be an open subset of $\mathbf{R}^N$ equipped with the Lebesgue measure dx and $(f_n)_{n\in\mathbf{N}}$ a sequence of functions which is bounded in $L^1(\Omega)$, i.e., $\sup_{n\in\mathbf{N}}\int_\Omega |f_n(x)|dx<+\infty$. Then there exists a subsequence $(f_{n_k})_{k\in\mathbf{N}}$ and a bounded Borel measure μ on Ω such that*

$$\forall\varphi\in\mathscr{C}_0(\Omega)\quad \int_\Omega \varphi(x)f_{n_k}(x)dx \underset{k\to+\infty}{\longrightarrow} \int_\Omega \varphi(x)d\mu(x).$$

PROOF. Apply Proposition 2.4.15 to the sequence $\mu_n=f_n\,dx$ and note that $|\mu_n|(\Omega)=\int_\Omega|f_n|\,dx$. □

Remark 2.4.8. (a) The above result, which allows us to extract from any bounded sequence $(f_n)_{n\in\mathbf{N}}$ in $L^1(\Omega)$ a subsequence which is $\sigma(\mathscr{M}_b,\mathbf{C}_0)$ convergent to a bounded Borel measure μ, relies on the embedding of L^1 into $\mathscr{M}_b(\Omega)$ which is a dual, namely, of $\mathbf{C}_0$. This is a general method which consists, when one has some estimations on a sequence $(v_n)_{n\in\mathbf{N}}$ in some space X, to embed $X\hookrightarrow Y^*$, where Y^* is the topological dual of some (separable) normed space Y. Then one can extract a subsequence $v_{n_k}\overset{\sigma(Y^*,Y)}{\longrightarrow} y^*$, the limit y^* belonging to Y^*.

(b) As an example of application of Corollary 2.4.3, take the sequence $(v_n)_{n\in\mathbf{N}}$ defined at the beginning of Section 2.4.5; one has

$$v_n \overset{\sigma(\mathscr{M}_b,\mathbf{C}_0)}{\longrightarrow} \delta_0,$$

where δ_0 is the Dirac measure at the origin.

(c) As a counterpart of its generality, the information given by a weak* convergence $f_n dx \overset{\sigma(\mathscr{M}_b,\mathbf{C}_0)}{\longrightarrow}\mu$ or even $f_n \overset{\sigma(L^1,L^\infty)}{\longrightarrow} f$ is often not sufficient to analyze some situations, for example, occurring in the study of some nonlinear PDEs. To treat such situations, we will introduce the concept of Young measures in Chapter 4.