Chapter 4

Complements on measure theory

4.1 ▪ Hausdorff measures and Hausdorff dimension

This section aims to define intrinsically the intuitive notion of length, area, and volume. More precisely, we would like to provide a nonnegative measure for any subset of $\mathbf{R}^N$, which agrees with the well-known k-dimensional measure for regular k-dimensional surfaces when k is an integer. Hausdorff's construction, as shown below, is particularly well suited to the geometry of sets and does not require any local parametrization on these sets; therefore, no assumption of regularity is needed. For instance, the Hausdorff measure offers the possibility of measuring fractal sets, as well as defining a new notion of dimension for any set, thereby extending the classical topological dimension. Note that the process described in the first subsection is the Carathéodory general approach to construct a measure from a σ-subadditive set function (or outer measure).

4.1.1 ▪ Outer Hausdorff measures and Hausdorff measures

We denote the collection of all subsets of $\mathbf{R}^N$ by $\mathscr{P}(\mathbf{R}^N)$ and, for any nonempty set E of $\mathscr{P}(\mathbf{R}^N)$, we set $\operatorname{diam}(E) = \sup\{d(x,y) : (x,y) \in E\}$, the diameter of E, where d is the Euclidean distance in $\mathbf{R}^N$. When s is a positive integer we denote the volume of the unit ball of $\mathbf{R}^s$ by ω_s; in the general case $s \geq 0$, we set

$$\omega_s = \frac{\pi^{s/2}}{\Gamma(1+s/2)},$$

where Γ is the well-known Euler function

$$\Gamma(t) = \int_0^{+\infty} x^{t-1} e^{-x}\, dx.$$

We also set $c_s = 2^{-s}\omega_s$. For instance, we have

$$c_0 = 1, \quad c_1 = 1, \quad c_2 = \pi/4, \quad c_3 = \pi/6.$$

Let E be any set of $\mathscr{P}(\mathbf{R}^N)$ and $\delta > 0$. A finite or countable family $(A_i)_{i\in\mathbf{N}}$ of sets in $\mathscr{P}(\mathbf{R}^N)$ satisfying $0 < \operatorname{diam}(A_i) \leq \delta$ and $E \subset \bigcup_{i\in\mathbf{N}} A_i$ will be called a δ-covering of E.

Definition 4.1.1. *For each $s \geq 0$, $\delta > 0$, and $E \subset \mathbf{R}^N$, let us set*

$$\mathcal{H}^s_\delta(E) := \inf\left\{ c_s \sum_{i\in\mathbf{N}} \operatorname{diam}(A_i)^s : (A_i)_{i\in\mathbf{N}}\ \delta\text{-covering of } E \right\}.$$

The s-dimensional outer Hausdorff measure is the set mapping $\mathcal{H}^s$ taking its values in $[0,+\infty]$ defined by

$$\begin{aligned}\mathcal{H}^s(E) &:= \sup_{\delta>0} \mathcal{H}^s_\delta(E)\\ &= \lim_{\delta\to 0} \mathcal{H}^s_\delta(E).\end{aligned}$$

Remark 4.1.1. The constant c_s is a normalization parameter so that when $s \in \mathbf{N}$ and E is an s-dimensional regular hypersurface of $\mathbf{R}^N$, $\mathcal{H}^s(E)$ is the s-dimensional area of E (see Proposition 4.1.6 or Theorem 4.1.1). The positive number δ, intended to tend to zero, forces the sets A_i of the δ-covering to follow the geometry of E.

Proposition 4.1.1. *The set function $\mathcal{H}^s : \mathcal{P}(\mathbf{R}^N) \longrightarrow [0,+\infty]$ is an outer measure, i.e., satisfies*

(i) *$\mathcal{H}^s(\emptyset) = 0$;*

(ii) *(σ-subadditivity) for all sequences $(E_i)_{i\in\mathbf{N}}$ of subsets of $\mathbf{R}^N$ such that $E \subset \cup_{i\in\mathbf{N}} E_i$,*

$$\mathcal{H}^s(E) \leq \sum_{i\in\mathbf{N}} \mathcal{H}^s(E_i);$$

(iii) *$\mathcal{H}^s$ is a nondecreasing set function, that is, $\mathcal{H}^s(A) \leq \mathcal{H}^s(B)$ whenever $A \subset B$.*

PROOF. For all A in $\mathcal{P}(\mathbf{R}^N)$ such that $\operatorname{diam}(A) \leq \delta$, one has $\mathcal{H}^s_\delta(\emptyset) \leq \operatorname{diam}(A)^s \leq \delta^s$ and (i) follows. The monotonicity property (iii) follows straightforwardly from the definition of $\mathcal{H}^s$.

Let now $\varepsilon > 0$ and $(A_{i,j})_{j\in\mathbf{N}}$ be a δ-covering of E_i satisfying

$$c_s \sum_{j\in\mathbf{N}} \operatorname{diam}(A_{i,j})^s \leq \frac{\varepsilon}{2^i} + \mathcal{H}^s_\delta(E_i).$$

Obviously, $(A_{i,j})_{(i,j)\in\mathbf{N}^2}$ is a δ-covering of $\bigcup_{i\in\mathbf{N}} E_i$ so that

$$\mathcal{H}^s_\delta\left(\bigcup_{i\in\mathbf{N}} E_i\right) \leq \sum_{i\in\mathbf{N}} \mathcal{H}^s_\delta(E_i) + 2\varepsilon.$$

Letting $\delta \to 0$, conclusion (ii) follows from the fact that $\mathcal{H}^s$ is a nondecreasing set function and ε is arbitrary. $\square$

Following the classical construction of a measure from an outer measure, we define the subset $\mathcal{M}_s$ of $\mathcal{H}^s$-measurable sets in the sense of Carathéodory:

$$A \in \mathcal{M}_s \Longleftrightarrow \forall X \in \mathcal{P}(\mathbf{R}^N),\ \mathcal{H}^s(X) = \mathcal{H}^s(X \cap A) + \mathcal{H}^s(X \setminus A).$$

Note that $\emptyset$ and $\mathbf{R}^N$ belong to $\mathcal{M}_s$.

Proposition 4.1.2. *The set $\mathcal{M}_s$ is a σ-algebra and $\mathcal{H}^s$ is σ-additive on $\mathcal{M}_s$.*

PROOF. Obviously $\mathbf{R}^N$ belongs to $\mathcal{M}_s$. The proof of stability of $\mathcal{M}_s$ with respect to the complementary operation is easily established from a straightforward calculation. The proof of stability for countable union and the σ-additivity of $\mathcal{H}^s$ is divided into three steps.

First step. Stability for finite unions and finite intersections. Let A_1, A_2 be two sets in $\mathcal{M}_s$ and X some set in $\mathcal{P}(\mathbf{R}^N)$. Since A_1 and A_2 are two measurable sets, an elementary calculation on sets gives

$$\begin{aligned}\mathcal{H}^s(X) &= \mathcal{H}^s(A_1 \cap X) + \mathcal{H}^s(X \setminus A_1)\\ &= \mathcal{H}^s(A_1 \cap X) + \mathcal{H}^s((X \setminus A_1) \cap A_2) + \mathcal{H}^s((X \setminus A_1) \setminus A_2)\\ &= \mathcal{H}^s(A_1 \cap X) + \mathcal{H}^s(X \cap A_2 \setminus A_1) + \mathcal{H}^s(X \setminus (A_1 \cup A_2)).\end{aligned}$$

According to the identities

$$X \cap A_1 = X \cap (A_1 \cup A_2) \cap A_1 \text{ and } X \cap A_2 \setminus A_1 = X \cap (A_1 \cup A_2) \setminus A_1,$$

we derive

$$\mathcal{H}^s(X) = \mathcal{H}^s(X \cap (A_1 \cup A_2)) + \mathcal{H}^s(X \setminus (A_1 \cup A_2))$$

so that $A_1 \cup A_2 \in \mathcal{M}_s$. Stability for finite intersection is then obtained thanks to the stability with respect to the complementary operation.

Note that substituting X by $X \cap (A_1 \cup A_2)$ in $\mathcal{H}^s(X) = \mathcal{H}^s(A_1 \cap X) + \mathcal{H}^s(X \setminus A_1)$ we obtain the following important equality used in the step below:

$$\mathcal{H}^s(X \cap (A_1 \cup A_2)) = \mathcal{H}^s(X \cap A_1) + \mathcal{H}^s(X \cap A_2) \tag{4.1}$$

whenever A_1 and A_2 are disjoint sets in $\mathcal{M}_s$.

Second step. Stability for disjoint countable unions and σ-additivity of $\mathcal{H}^s$. Let $(A_i)_{i \in \mathbf{N}}$ be a family of pairwise disjoint sets in $\mathcal{M}_s$ and A their union. According to the subadditivity of $\mathcal{H}^s$ for all X in $\mathcal{P}(\mathbf{R}^N)$ we have

$$\begin{aligned}\mathcal{H}^s(X) &\leq \mathcal{H}^s(X \setminus A) + \mathcal{H}^s(X \cap A)\\ &\leq \mathcal{H}^s(X \setminus A) + \sum_{i=0}^{\infty} \mathcal{H}^s(X \cap A_i)\\ &= \mathcal{H}^s(X \setminus A) + \lim_{n \to +\infty} \mathcal{H}^s\left(\bigcup_{i=0}^{n} X \cap A_i\right)\\ &\leq \liminf_{n \to +\infty}\left(\mathcal{H}^s\left(X \setminus \bigcup_{i=0}^{n} A_i\right) + \mathcal{H}^s\left(\bigcup_{i=0}^{n} X \cap A_i\right)\right)\\ &= \mathcal{H}^s(X),\end{aligned}$$

where we have used (4.1) in the first equality, the nondecreasing of $\mathcal{H}^s$ in the inequality, and the stability for finite union in the last equality. This proves that A belongs to $\mathcal{M}_s$. The σ-additivity of $\mathcal{H}^s$ is obtained by taking $X = A$.

Last step. Stability for countable union. Let $(A_i)_{i \in \mathbf{N}}$ be a family of sets in $\mathcal{M}_s$. Set

$$B_0 = A_0 \text{ and for } n \geq 1, \quad B_n = A_n \setminus \bigcup_{i=0}^{n-1} A_i.$$

According to the first step, the family $(B_i)_{i \in \mathbf{N}}$ is made up of pairwise disjoint sets of $\mathcal{M}_s$ so that, from step 2, $\bigcup_{i \in \mathbf{N}} B_i = \bigcup_{i \in \mathbf{N}} A_i$ belongs to $\mathcal{M}_s$. $\square$

Definition 4.1.2. *The restriction to $\mathscr{M}_s$ of the set function $\mathscr{H}^s$ is called the s-dimensional Hausdorff measure.*

The s-dimensional Hausdorff measure $\mathscr{H}^s$ is a $[0,+\infty]$-valued Borel measure in the following sense.

Proposition 4.1.3. *The σ-algebra $\mathscr{M}_s$ contains the σ-algebra of all the Borel sets of $\mathbf{R}^N$.*

PROOF. The proof is based on the following Carathéodory criterion. An outer measure satisfying this criterion is sometimes called an outer metric measure.

Lemma 4.1.1. *Let μ be an outer measure on a metric space E equipped with its Borel σ-algebra $\mathscr{B}(E)$ and $\mathscr{M}_\mu$ be the σ-algebra of the measurable sets in the Carathéodory sense. Then $\mathscr{B}(E) \subset \mathscr{M}_\mu$ iff μ is additive on any pair $\{A,B\}$ of sets of E satisfying $d(A,B) > 0$.*

PROOF OF LEMMA 4.1.1. Let us assume $\mathscr{B}(E) \subset \mathscr{M}_\mu$ and let A, B be two subsets of E such that $d(A,B) > 0$. Since $\overline{A} \in \mathscr{M}_\mu$ the two decompositions

$$A = (A \cup B) \cap \overline{A},$$
$$B = (A \cup B) \setminus \overline{A}$$

imply $\mu(A \cup B) = \mu(A) + \mu(B)$.

Conversely, let A be an open subset of E and X any fixed subset of E such that $\mu(X) < +\infty$. According to subadditivity, it is enough to establish the inequality

$$\mu(X) \geq \mu(X \cap A) + \mu(X \setminus A).$$

One may assume $\mu(X) < +\infty$. For all $k \in \mathbf{N}^*$, let us define the sets

$$A_k := \left\{ x \in A : d(x, E \setminus A) > \frac{1}{k} \right\},$$
$$B_k := A_{k+1} \setminus A_k.$$

Noticing that for $k - l \geq 2$, $d(B_k, B_l) \geq \frac{1}{l+1} - \frac{1}{k} > 0$, we have for all $n \in \mathbf{N}$

$$\sum_{i=1}^{n} \mu(X \cap B_{2i}) = \mu\left(\bigcup_{i=1}^{n} X \cap B_{2i} \right) \leq \mu(X),$$

hence

$$\sum_{k \text{ even}} \mu(X \cap B_k) \leq \mu(X).$$

The same calculation gives

$$\sum_{k \text{ odd}} \mu(X \cap B_k) \leq \mu(X)$$

so that

$$\sum_{k \in \mathbf{N}} \mu(X \cap B_k) \leq 2\mu(X)$$

and

$$\lim_{k \to +\infty} \sum_{i \geq k} \mu(X \cap B_i) = 0.$$

We infer

$$\lim_{k\to+\infty}\mu(X\cap(A\setminus A_k))=\lim_{k\to+\infty}\mu\big(X\cap(\cup_{i\geq k}B_i)\big)$$
$$\leq \lim_{k\to+\infty}\sum_{i\geq k}\mu(X\cap B_i)=0,$$

and, by subadditivity, we deduce

$$\mu(X\cap A)\leq\mu(X\cap A_k)+\mu(X\cap(A\setminus A_k))$$
$$\leq\liminf_{k\to+\infty}\mu(X\cap A_k).$$

Since obviously $\limsup_{n\to+\infty}\mu(X\cap A_k)\leq\mu(X\cap A)$, one has

$$\mu(X\cap A)=\lim_{k\to+\infty}\mu(X\cap A_k).$$

From $d(A_k,X\setminus A)>\frac{1}{k}>0$ we now obtain

$$\mu(X)\geq\mu((X\cap A_k)\cup(X\setminus A))=\mu(X\cap A_k)+\mu(X\setminus A),$$

and we complete the proof of Lemma 4.1.1 by letting $k\to+\infty$. □

PROOF OF PROPOSITION 4.1.3 CONTINUED. According to Lemma 4.1.1, it is enough to prove that $\mathscr{H}^s$ is an outer metric measure. Let A and B be two subsets of $\mathbf{R}^N$ such that $d(A,B)>0$ and let $\mathscr{C}=(C_i)_{i\in\mathbf{N}}$ be a covering of $A\cup B$ with $\operatorname{diam}(C_i)\leq\delta<d(A,B)$. We can write this covering as the union of two disjoint δ-coverings: take $\mathscr{A}=\{C\in\mathscr{C}: C\cap A\neq\emptyset\}$ and $\mathscr{B}=\{C\in\mathscr{C}:C\cap B\neq\emptyset\}$. Therefore

$$\sum_{i=1}^{\infty}c_s\operatorname{diam}(C_i)^s=\sum_{C\in\mathscr{A}}c_s\operatorname{diam}(C)^s+\sum_{C\in\mathscr{B}}c_s\operatorname{diam}(C)^s$$

and $\mathscr{H}^s_\delta(A\cup B)\geq\mathscr{H}^s_\delta(A)+\mathscr{H}^s_\delta(B)$. We end the proof by letting $\delta\to0$. □

Remark 4.1.2. The following process is often used to construct an outer measure and thus a measure on Cantor-type sets C in $\mathbf{R}^N$. Let $\mathscr{E}_0=\{C\}$ and for $n\in\mathbf{N}^*$ let $\mathscr{E}_n$ be a finite collection of disjoint subsets E of C, such that each $E\in\mathscr{E}_n$ is contained in one of the sets of $\mathscr{E}_{n-1}$. We assume moreover that $\lim_{n\to+\infty}\{\operatorname{diam}(E):E\in\mathscr{E}_n\}=0$. We set now $\mu(C)=a$, where a is an arbitrary number satisfying $0<a<+\infty$ and, for all the sets E_i, $i=1,\dots,m_1$, of $\mathscr{E}_1$, we define the masses $\mu(E_i)$ such that

$$\sum_{i=1}^{m_1}\mu(E_i)=\mu(C).$$

Similarly, we assign masses $\mu(E_i)$ to the sets of $\mathscr{E}_n$ such that if $E\in\mathscr{E}_{n-1}$, $E=\cup_{i=1}^{m_n}E_i$, $E_i\in\mathscr{E}_n$,

$$\sum_{i=1}^{m_n}\mu(E_i)=\mu(E).$$

We finally set

$$\mu\left(\mathbf{R}^N\setminus\bigcup_{E\in\mathscr{E}_n}E\right)=0.$$

Let us denote the collection of all the complementary sets of $\mathscr{E}_n$ by $\overline{\mathscr{E}}_n$ and set $\mathscr{E} = \cup_{n\in\mathbf{N}}(\mathscr{E}_n \cup \overline{\mathscr{E}}_n)$. For each set $A \in \mathscr{P}(\mathbf{R}^N)$, we now extend μ by setting

$$\mu(A) = \inf\left\{ \sum_{i\in\mathbf{N}} \mu(E_i) : A \subset \bigcup_{i\in\mathbf{N}} E_i,\ E_i \in \mathscr{E} \right\},$$

which defines an outer measure. One can prove that μ is a measure whose σ-algebra of measurable sets contains the σ-algebra of Borel sets of $\mathbf{R}^N$. Moreover, the support of μ, that is, the smallest closed set X of $\mathbf{R}^N$ such that $\mu(\mathbf{R}^N \setminus X) = 0$, is contained in $\cap_{n\in\mathbf{N}} \overline{\cup_{E\in\mathscr{E}_n} E}$.

Remark 4.1.3. The construction of Hausdorff measures $\mathscr{H}^s$ can be made in a general metric space X and indeed most of the properties continue to hold in this larger framework. For a detailed presentation of Hausdorff measures in metric spaces, see Ambrosio and Tilli [29].

Theorem 4.1.1. *For all Lebesgue measurable sets E in $\mathbf{R}^N$, we have $\mathscr{H}^N(E) = \mathscr{L}^N(E)$, where $\mathscr{L}^N$ denotes the Lebesgue measure on $\mathbf{R}^N$. Moreover, $\mathscr{H}^s(E) = 0$ for $s > N$, whereas $\mathscr{H}^0$ is the counting measure.*

PROOF. For all Lebesgue measurable sets E in $\mathbf{R}^N$, let us recall the so-called isodiametric inequality, which asserts that the Lebesgue measure of E is less than the Lebesgue measure of any ball having the same diameter:

$$\mathscr{L}^N(E) \le c_N(\operatorname{diam}(E))^N.$$

For a proof see, for instance, Evans and Gariepy [211]. For all covering $(A_i)_{i\in\mathbf{N}}$ of E we then have

$$\sum_{i\in\mathbf{N}} c_N(\operatorname{diam}(A_i))^N \ge \sum_{i\in\mathbf{N}} \mathscr{L}^N(A_i) \ge \mathscr{L}^N(E),$$

hence $\mathscr{H}^N_\delta(E) \ge \mathscr{L}^N(E)$. Letting $\delta \to 0$ gives $\mathscr{H}^N(E) \ge \mathscr{L}^N(E)$.

The converse inequality is more involved. We will use the so-called Vitali's covering lemma. Let us first define the notion of fine covering.

Definition 4.1.3. *A family $\mathscr{F}$ of closed balls in $\mathbf{R}^N$ is said to cover a set $E \subset \mathbf{R}^N$ finely if, for each $x \in E$ and each $\varepsilon > 0$, there exists $\overline{B}_r(x) \in \mathscr{F}$ with $r < \varepsilon$, where $B_r(x)$ denotes the open ball with radius r and centered at x.*

Lemma 4.1.2 (Vitali's covering theorem). *Let $E \subset \mathbf{R}^N$ with $\mathscr{L}^N(E) < +\infty$ and finely covered by a family of closed balls $\mathscr{F}$. Then there exists a countable subfamily $\mathscr{G}$ of pairwise disjoint elements of $\mathscr{F}$ such that*

$$\mathscr{L}^N\left(E \setminus \bigcup_{B\in\mathscr{G}} B\right) = 0.$$

PROOF. For the proof see, for instance, Ziemer [366]. □

Remark 4.1.4. In the definition of a family finely covering a subset E of $\mathbf{R}^N$ one may replace the family of closed balls by a regular family of closed subsets of $\mathbf{R}^N$ (see [366]).

Vitali's covering theorem is also valid for nonnegative Radon measures μ on $\mathbf{R}^N$ (i.e., locally bounded nonnegative Borel measures; see Section 4.2). For a proof consult [366].

PROOF OF THEOREM 4.1.1 CONTINUED. Let A be a set of finite Lebesgue measure in $\mathbf{R}^N$ and, for each $\eta > 0$, let U be an open subset of $\mathbf{R}^N$ such that $A \subset U$ and $\mathscr{L}^N(U \setminus A) < \eta$. There exists a family $\mathscr{F}$ of closed balls with diameter less than δ, included in U, and covering finely U. Indeed, for all $x \in U$, there exists a closed ball $\overline{B}_{r(x)}(x)$ included in U and $\mathscr{F} = (\overline{B}_r(x))_{x\in U,\ r\le r(x)\wedge\delta/2}$ is such a suitable family. According to Lemma 4.1.2 one can extract a subfamily $(B_i)_{i=1,\dots,\infty}$ of pairwise disjoint elements with diameter less than δ, satisfying

$$\mathscr{L}^N\left(U \setminus \bigcup_{i=1}^{\infty} B_i\right) = 0,\ \bigcup_{i=1}^{\infty} B_i \subset U.$$

Set $A^* := \bigcup_{i=1}^{\infty}(B_i \cap A)$. We have $\mathscr{L}^N(A \setminus A^*) = 0$ and

$$\begin{aligned}
\mathscr{H}^N_\delta(A^*) &\le \sum_{i=1}^{\infty} c_N (\mathrm{diam} B_i)^N \\
&= \sum_{i=1}^{\infty} \mathscr{L}^N(B_i) \\
&= \mathscr{L}^N(U) \le \mathscr{L}^N(A) + \eta.
\end{aligned}$$

Letting $\delta \to 0$ and $\eta \to 0$, we obtain $\mathscr{H}^N(A^*) \le \mathscr{L}^N(A)$. It remains to establish $\mathscr{L}^N(A \setminus A^*) = 0 \Longrightarrow \mathscr{H}^N(A \setminus A^*) = 0$ and more generally that if E is a Borel set satisfying $\mathscr{L}^N(E) = 0$, then $\mathscr{H}^N(E) = 0$. This property is a corollary of the first assertion of Lemma 4.1.3 below, whose proof may be found in [366].

Lemma 4.1.3. *Let $\mathscr{F}$ be a family of closed balls with* $\sup\{\mathrm{diam} B : B \in \mathscr{F}\} < +\infty$. *Then there exists a countable subfamily $\mathscr{G}$ of pairwise disjoint elements of $\mathscr{F}$ such that*

$$\bigcup_{B\in\mathscr{F}} B \subset \bigcup_{B\in\mathscr{G}} B^*,$$

where B^* denotes the closed ball concentric with B with radius five times as big as that of B. Let E be a subset of $\mathbf{R}^N$, finely covered by $\mathscr{F}$; then, for all finite family $\mathscr{G}^* \subset \mathscr{G}$,

$$E \subset \Big(\bigcup_{B\in\mathscr{G}^*} B\Big) \bigcup \Big(\bigcup_{B\in\mathscr{G}\setminus\mathscr{G}^*} B^*\Big).$$

Indeed, for $\eta > 0$, consider an open subset U of $\mathbf{R}^N$ such that $E \subset U$ and $\mathscr{L}^N(U) \le \eta$. The open set U being a union of closed balls included in U with diameter less than δ, according to Lemma 4.1.3, there exists a family $\mathscr{G}$ of pairwise disjoint closed balls included in U such that $U \subset \bigcup_{B\in\mathscr{G}} B^*$ and

$$\begin{aligned}
\mathscr{H}^N_\delta(E) &\le \sum_{B\in\mathscr{G}} H^N_\delta(B^*) \\
&\le \sum_{B\in\mathscr{G}} c_N 5^N (\mathrm{diam}(B))^N \\
&= 5^N \sum_{B\in\mathscr{G}} \mathscr{L}^N(B) = 5^N \mathscr{L}^N\left(\bigcup_{B\in\mathscr{G}} B\right) < 5^N \eta.
\end{aligned}$$

The conclusion follows by letting $\delta \to 0$ and $\eta \to 0$.

We establish now that for all Borel subsets E of $\mathbf{R}^N$, $\mathcal{H}^s(E)=0$ when $s>N$. One can write $E=\cup_{n\in\mathbf{N}}E_n$, where $\mathcal{H}^N(E_n)<+\infty$. (Take $E_n=B_n\cap E$, where B_n denotes the ball of $\mathbf{R}^N$ with radius n, centered at 0.) Let $\delta>0$ and $(A_i)_{i\in\mathbf{N}}$ a δ-covering of E_n. We have

$$c_s\sum_{i\in\mathbf{N}}\mathrm{diam}(A_i)^s\leq\frac{c_s}{c_N}\delta^{s-N}c_N\sum_{i\in\mathbf{N}}\mathrm{diam}(A_i)^N$$

which yields

$$\mathcal{H}^s_\delta(E_n)\leq\frac{c_s}{c_N}\delta^{s-N}\mathcal{H}^N(E_n),$$

and finally, letting $\delta\to 0$, one obtains $\mathcal{H}^s(E_n)=0$. Therefore, by subadditivity,

$$\mathcal{H}^s(E)\leq\sum_{n=0}^{\infty}\mathcal{H}^s(E_n)=0.$$

The fact that $\mathcal{H}^0$ is the counting measure is easy to establish and left to the reader. □

4.1.2 ▪ Hausdorff measures: Scaling properties and Lipschitz transformations

The scaling properties of length, area, or volume are well known. The two propositions below summarize and generalize these properties.

Proposition 4.1.4. *Let A be any subset of $\mathbf{R}^N$ and $\lambda>0$. Then*

$$\mathcal{H}^s(\lambda A)=\lambda^s\mathcal{H}^s(A).$$

PROOF. If $(A_i)_i$ is a δ-covering of A, then $(\lambda A_i)_i$ is a $\lambda\delta$-covering of λA so that

$$\mathcal{H}^s_{\lambda\delta}(\lambda A)\leq\sum_{i=1}^{\infty}c_s(\lambda\mathrm{diam}(A_i))^s=\lambda^s\sum_{i=1}^{\infty}c_s\mathrm{diam}(A_i)^s.$$

Therefore $\mathcal{H}^s_{\lambda\delta}(\lambda A)\leq\lambda^s\mathcal{H}^s_\delta(A)$. Letting $\delta\to 0$, we obtain $\mathcal{H}^s(\lambda A)\leq\lambda^s\mathcal{H}^s(A)$. Replacing λ with $1/\lambda$ and A with λA gives the converse inequality. □

Proposition 4.1.5. *Let A be any subset of $\mathbf{R}^N$ and $f:A\longrightarrow\mathbf{R}^m$ satisfying for all x and y in A*

$$|f(x)-f(y)|\leq L|x-y|^\alpha,$$

where $L>0$ and $\alpha>0$ are two given constants. Then

$$\mathcal{H}^{s/\alpha}(f(A))\leq\frac{c_{s/\alpha}}{c_s}L^{s/\alpha}\mathcal{H}^s(A).$$

Consequently, if f is a Lipschitz function with modulus L, then

$$\mathcal{H}^s(f(A))\leq L^s\,\mathcal{H}^s(A),$$

and if f is an isometry, then

$$\mathcal{H}^s(f(A))=\mathcal{H}^s(A).$$

PROOF. If $(A_i)_i$ is a δ-covering of A, then $(f(A\cap A_i))_i$ is an $L\delta^\alpha$-covering of $f(A)$, and

$$\mathscr{H}^{s/\alpha}_{L\delta^\alpha}(f(A)) \le \frac{c_{s/\alpha}}{c_s}\sum_{i=1}^\infty c_s \operatorname{diam}(f(A\cap A_i))^{s/\alpha} \le \frac{c_{s/\alpha}}{c_s} L^{s/\alpha}\sum_{i=1}^\infty c_s \operatorname{diam}(A_i)^s.$$

We deduce that $\mathscr{H}^{s/\alpha}_{L\delta^\alpha}(f(A)) \le \frac{c_{s/\alpha}}{c_s} L^{s/\alpha}\mathscr{H}^s_\delta(A)$, and the conclusion follows after letting $\delta \to 0$. □

The Hausdorff measure of an N-dimensional regular hypersurface of $\mathbf{R}^m$ is nothing but its classical area when the hypersurface is defined by means of a parametrization.

Proposition 4.1.6. *Let $m \in \mathbf{N}$ with $m \ge N$, $f : \mathbf{R}^N \longrightarrow \mathbf{R}^m$ be a one-to-one Lipschitz map and E a Borel subset of $\mathbf{R}^N$. Then*

$$\mathscr{H}^N(f(E)) = \int_E \left(\sum_{i=1}^{C_m^N} |J_i|^2\right)^{1/2} d\mathscr{L}^N,$$

where J_i, $i = 1,\dots,C_m^N$, are the $N\times N$-minors of the Jacobian matrix of f.

PROOF. To illustrate how the definition of the one-dimensional Hausdorff measure is well adapted to the local geometry of arcs, we only give the proof in the case $N = 1$, where $f : [0,1] \longrightarrow \mathbf{R}^m$ is denoted by $t \mapsto (x_i(t))_{i=1,\dots,m}$. For a complete proof, consult Ambrosio [19] or Evans and Gariepy [211]. We must establish

$$\begin{aligned}\mathscr{H}^1\big(f([0,1])\big) &= \int_{[0,1]} \left(\sum_{i=1}^m |x_i'(t)|^2\right)^{1/2} dt\\ &= \int_{[0,1]} |f'(t)|\, dt,\end{aligned}$$

which is the length of the parametrized arc $\Gamma = f([0,1])$. We use the well-known classical result straightforwardly obtained from Taylor's formula and Riemann integration theory: if f belongs to $\mathbf{C}^1([\alpha,\beta],\mathbf{R}^m)$, we have

$$\int_{[\alpha,\beta]} |f'(t)|\, dt = \lim_{\eta\to 0}\sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)| \ge |f(\alpha) - f(\beta)|,$$

where $t_0 = \alpha < t_1 < \cdots t_i < t_{i+1} < \cdots t_n = \beta$ is some finite subdivision of $[\alpha,\beta]$ with $\eta = \max_{i=0,\dots,n-1}(t_{i+1} - t_i)$.

Step 1. The function f is assumed to belong to $\mathbf{C}^1([0,1],\mathbf{R}^m)$ and we prove $\int_{[0,1]} |f'(t)| dt \ge \mathscr{H}^1(\Gamma)$. For all $\delta > 0$, let $\eta > 0$ be such that

$$|t - t'| < \eta \Longrightarrow |f(t) - f(t')| < \delta.$$

Let us consider a finite subdivision $t_0 = 0 < t_1 < \cdots t_i < t_{i+1} < \cdots t_n = 1$ of $[0,1]$ satisfying $\eta > \max_{i=0,\dots,n-1}(t_{i+1} - t_i)$ and set $\Gamma_i = f([t_i, t_{i+1}])$ for $i = 0,\dots,n-1$. Consider now a_i, b_i in $[t_i, t_{i+1}]$ such that $\operatorname{diam}(\Gamma_i) = |f(a_i) - f(b_i)|$. We have

$$\Gamma \setminus \{f(1)\} = \bigcup_{i=0}^{n-1} \Gamma_i$$

with $\operatorname{diam}(\Gamma_i) < \delta$ and

$$\begin{aligned}\int_{[0,1]} |f'(t)|\,dt &= \sum_{i=0}^{n-1} \int_{[t_i,t_{i+1}]} |f'(t)|\,dt \\ &\geq \sum_{i=0}^{n-1} \int_{[a_i,b_i]} |f'(t)|\,dt \\ &\geq \sum_{i=0}^{n-1} |f(a_i) - f(b_i)| \\ &= \sum_{i=0}^{n-1} \operatorname{diam}(\Gamma_i) \geq \mathcal{H}^1_\delta(\Gamma).\end{aligned}$$

We end this step by letting $\delta \to 0$.

Step 2. The function f is again assumed to belong to $\mathbf{C}^1([0,1],\mathbf{R}^m)$ and we prove the converse inequality $\int_{[0,1]} |f'(t)|\,dt \leq \mathcal{H}^1(\Gamma)$. Let $n \in \mathbf{N}$, $h = 1/n$, $t_i = hi$, $i = 0,\dots,n-1$, and consider the covering

$$\Gamma = \bigcup_{i=0}^{n-1} f([t_i,t_{i+1})) \cup \{f(1)\}$$

by pairwise disjoint Borel subsets $f([t_i,t_{i+1}))$ of Γ (recall that f is one-to-one). We then have

$$\mathcal{H}^1(\Gamma) = \sum_{i=0}^{n-1} \mathcal{H}^1(f([t_i,t_{i+1}))).$$

For each $i = 1,\dots,n-1$, consider the orthogonal projection π of $f([t_i,t_{i+1}))$ onto the line $(f(t_i), f(t_{i+1}))$. As π does not increases distances, from Proposition 4.1.5 we have

$$\mathcal{H}^1\big(f([t_i,t_{i+1}))\big) \geq \mathcal{H}^1\big(\pi(f([t_i,t_{i+1})))\big),$$

which is greater than $\mathcal{H}^1([f(t_i), f(t_{i+1})))$. Indeed, a convexity argument yields

$$[f(t_i), f(t_{i+1})) \subset \pi(f([t_i,t_{i+1}))).$$

By using the definition, it is easy to prove for the segments $[a,b]$ in $\mathbf{R}^m$ that $\mathcal{H}^1([a,b]) = |b-a|$. Therefore $\mathcal{H}^1([f(t_i), f(t_{i+1})]) = |f(t_i) - f(t_{i+1})|$ and

$$\mathcal{H}^1(\Gamma) \geq \sum_{i=0}^{n-1} |f(t_i) - f(t_{i+1})|.$$

The conclusion follows after letting $n \to +\infty$.

Step 3. The function f is assumed to be Lipschitz continuous. The conclusion is a straightforward consequence of the two previous steps and the following approximating result: every Lipschitz map $f : E \to \mathbf{R}^m$ can be approximated in the Lusin sense by functions of class $\mathbf{C}^1$. More precisely, there exists a nondecreasing family $(K_i)_{i\in\mathbf{N}}$ of compact sets included in E such that

$$\mathcal{H}^N\left(E \setminus \bigcup_{i=0}^{\infty} K_i\right) = 0$$

and such that $f\lfloor K_i$ is the restriction of a function of class $\mathbf{C}^1$. For a proof of this property, consult [211]. $\square$

Remark 4.1.5. The measure $\left(\sum_{i=1}^{C_m^N} |J_i|^2\right)^{1/2} \cdot \mathscr{L}^N$ is sometimes called the element of area of the N-dimensional surface $f(E)$ and, for all Borel subset B of $f(E)$,

$$B \mapsto \mathscr{H}^N(B) = \int_{f^{-1}(B)} \left(\sum_{i=1}^{C_m^N} |J_i|^2 \right)^{1/2} d\mathscr{L}^N$$

is its corresponding surface measure.

When f is not one-to-one, the generalization of the formula in Proposition 4.1.6 is the so-called area formula:

$$\int_{f(E)} \mathscr{H}^0(E \cap f^{-1}(x))\, \mathscr{H}^N(dx) = \int_E \left(\sum_{i=1}^{m-N+1} |J_i|^2 \right)^{1/2} d\mathscr{L}^N,$$

where $x \mapsto \mathscr{H}^0(E \cap f^{-1}(x))$ is nothing but the multiplicity function as illustrated in the example of the parametrization of the unit circle: $t \mapsto (\cos(2n\pi t), \sin(2n\pi t))$ from $E = [0,1[$ into $\mathbf{R}^2$.

One can also generalize this formula as follows: let $h : E \to [0,+\infty]$ be a Borel function; then

$$\int_{f(E)} \sum_{x \in f^{-1}(y)} h(x)\, \mathscr{H}^N(dy) = \int_E h(x) \left(\sum_{i=1}^{C_m^N} |J_i|^2 \right)^{1/2} \mathscr{L}^N(dx).$$

For these generalizations, see [19], [211], or Federer [213].

4.1.3 ▪ Hausdorff dimension

Among the wide variety of definitions of dimension, the Hausdorff dimension introduced in Theorem 4.1.2 has the advantage of being defined for any set. For alternative definitions of dimensions, consult, for instance, Falconer [212].

Lemma 4.1.4. *For all fixed set A of $\mathbf{R}^N$, the map $s \mapsto \mathscr{H}^s(A)$ is nonincreasing. More precisely, for all $\delta > 0$ and for $t > s$, we have*

$$\mathscr{H}_\delta^t(A) \le \delta^{t-s} \frac{c_t}{c_s} \mathscr{H}_\delta^s(A). \tag{4.2}$$

PROOF. Let $(A_i)_{i\in\mathbf{N}}$ be a δ-covering of A. One has

$$\begin{aligned}\mathscr{H}_\delta^t(A) &\le \sum_{i\in\mathbf{N}} c_t \operatorname{diam}(A_i)^t \\ &\le \frac{c_t}{c_s} \sum_{i\in\mathbf{N}} c_s \operatorname{diam}(A_i)^s \delta^{t-s}.\end{aligned}$$

The conclusion then follows by taking the infimum on all the δ-covering of A. $\square$

Theorem 4.1.2 (definition of the Hausdorff dimension). *Let A be a set of $\mathbf{R}^N$ and set*

$$s_0 := \inf\{t \geq 0 : \mathscr{H}^t(A) = 0\}.$$

Then s_0 satisfies

$$\mathscr{H}^s(A) = \begin{cases} +\infty & \text{if } s < s_0, \\ 0 & \text{if } s > s_0. \end{cases}$$

The real number s_0 is called the Hausdorff dimension of the set A and is denoted by $\dim_H(A)$. At the critical value s_0, $\mathscr{H}^{s_0}(A)$ may be zero or infinite or may satisfy $0 < \mathscr{H}^{s_0}(A) < +\infty$. In this last case, A is called an s_0-set.

PROOF. Letting $\delta \to 0$ in inequality (4.2), we obtain

$$\mathscr{H}^s(A) < +\infty \Longrightarrow \forall t > s,\ \mathscr{H}^t(A) = 0.$$

Set $s_0 := \inf\{t \geq 0 : \mathscr{H}^t(A) = 0\}$ and take $s < s_0$. Assume that $\mathscr{H}^s(A) < +\infty$. For t satisfying $s < t < s_0$ we have $\mathscr{H}^t(A) = 0$, which contradicts the definition of s_0. Consequently, for $s < s_0$, $\mathscr{H}^s(A) = +\infty$. Since the map $s \mapsto \mathscr{H}^s(A)$ is nonincreasing, for $s > s_0$, we have $\mathscr{H}^s(A) = 0$.

Obviously $\dim_H(\mathbf{R}) = 1$ and $\mathscr{H}^1(\mathbf{R}) = +\infty$. On the other hand, Proposition 4.1.9 provides a nontrivial set A in $\mathbf{R}$ with $\dim_H(A) = 1$ and satisfying $\mathscr{H}^1(A) = 0$. □

Remark 4.1.6. Taking, as δ-covering, the class of balls of $\mathbf{R}^N$, one may define

$$\tilde{\mathscr{H}}^s_\delta(E) := \inf\left\{ c_s \sum_{i=1}^{\infty} \operatorname{diam}(B_i)^s : E \subset \bigcup_{i=1}^{\infty} B_i,\ 0 < \operatorname{diam}(B_i) \leq \delta,\ B_i \text{ ball of } \mathbf{R}^N \right\},$$

and the set mapping $\tilde{\mathscr{H}}^s$, by

$$\begin{aligned} \tilde{\mathscr{H}}^s(E) &:= \sup_{\delta > 0} \tilde{\mathscr{H}}^s_\delta(E) \\ &= \lim_{\delta \to 0} \tilde{\mathscr{H}}^s_\delta(E). \end{aligned}$$

It is easy to establish the following bounds:

$$\mathscr{H}^s(E) \leq \tilde{\mathscr{H}}^s(E) \leq 2^s \mathscr{H}^s(E).$$

Thanks to this estimate, the Hausdorff dimensions defined from the two mappings $\mathscr{H}^s$ and $\tilde{\mathscr{H}}^s$ are equal.

The following are useful for the Hausdorff dimension.

Proposition 4.1.7. *Let A be any set in $\mathbf{R}^N$.*

(i) *The following implications hold true:*

$$\begin{aligned} \mathscr{H}^s(A) < +\infty &\Longrightarrow \dim_H(A) \leq s, \\ \mathscr{H}^s(A) > 0 &\Longrightarrow \dim_H(A) \geq s. \end{aligned}$$

(ii) *Let $f : A \longrightarrow \mathbf{R}^m$ satisfying for all x and y in A,*

$$|f(x) - f(y)| \leq L|x - y|^\alpha,$$

where $L > 0$ and $\alpha > 0$ are two given positive constants. Then $\dim_H(f(A)) \leq \frac{1}{\alpha}\dim_H(A)$.

PROOF. The proof of assertion (i) is a straightforward consequence of the definition. Let us prove (ii). Indeed $s > \dim_H(A) \Longrightarrow \mathcal{H}^s(A) = 0$ and inequality $\mathcal{H}^{s/\alpha}(f(A)) \leq \frac{c_{s/\alpha}}{c_s} L^{s/\alpha} \mathcal{H}^s(A)$ established in Proposition 4.1.5 yields $\mathcal{H}^{s/\alpha}(f(A)) = 0$ so that

$$\frac{s}{\alpha} \geq \dim_H(f(A)).$$

The conclusion follows by letting $s \to \dim_H(A)$. $\square$

Example 4.1.1. When U is an open subset of $\mathbf{R}^N$, then $\dim_H(U) = N$. Indeed, U contains an open ball B and $\mathcal{H}^N(U) \geq \mathcal{H}^N(B) > 0$, and thus $\dim_H(U) \geq N$.

Example 4.1.2. Every countable subset A of $\mathbf{R}^N$ is a set of zero Hausdorff dimension. Indeed, by σ-additivity, one has

$$\mathcal{H}^s(A) = \sum_{a \in A} \mathcal{H}^s(\{a\}).$$

Since $\mathcal{H}^0(\{a\}) = 1$, for all $s > 0$ one has $\mathcal{H}^s(\{a\}) = 0$, and thus $\mathcal{H}^s(A) = 0$. This proves that $\dim_H(A) = 0$.

Example 4.1.3. Let us consider for $N \leq m$ a one-to-one map $f : \mathbf{R}^N \longrightarrow \mathbf{R}^m$ of class $\mathbf{C}^1$. Let E be a compact subset of $\mathbf{R}^N$. We have $\dim_H(f(E)) = N$. Indeed, from Proposition 4.1.6, we have $0 < \mathcal{H}^N(f(E)) \leq C \mathcal{L}^N(E) < +\infty$, where C is a positive constant.

Example 4.1.4. Let us consider now the middle third Cantor set C of the interval $[0, 1]$. We have $\dim_H(C) = \frac{\ln 2}{\ln 3}$. Indeed,

$$C = \left(C \cap \left[0, \frac{1}{3}\right]\right) \cup \left(C \cap \left[\frac{2}{3}, 1\right]\right).$$

Set $C_1 := C \cap [0, \frac{1}{3}]$ and $C_2 := C \cap [\frac{2}{3}, 1]$. These two sets are geometrically similar to C by a ratio $\frac{1}{3}$. According to the properties of the Hausdorff measure established in Proposition 4.1.5, we derive

$$\mathcal{H}^s(C) = \frac{2}{3^s} \mathcal{H}^s(C).$$

The conclusion follows if we assume that C is an s-set, that is, $0 < \mathcal{H}^s(C) < +\infty$. For a proof of this property, consult [212], where various interesting methods are given for the calculation of Hausdorff dimensions of fractal sets.

More generally, let $S_1, \ldots, S_m : \mathbf{R}^N \to \mathbf{R}^N$ be m similarities, i.e., satisfying

$$|S_i(x) - S_i(y)| = c_i |x - y|$$

for all x, y in $\mathbf{R}^N$, where $0 < c_i < 1$ (the ratio of S_i). We assume that the S_i satisfy the *open set condition*, that is, there exists a nonempty bounded open set U such that

$$\bigcup_{i=1}^{m} S_i(U) \subset U.$$

Consider now a so-called self-similar set F satisfying $F = \bigcup_{i=1}^{m} S_i(F)$. Then $\dim_H F = s$, where s is the solution of

$$\sum_{i=1}^{m} c_i^s = 1$$

and $0 < \mathcal{H}^s(F) < +\infty$. For a proof, consult [212].

Note that the middle third Cantor set is such a self-similar set by taking $S_1(x) = \frac{1}{3}x$ and $S_2(x) = \frac{1}{3}x + \frac{2}{3}$ and the *open set condition* holds for $U =]0,1[$.

The Hausdorff dimension of a set gives us some information about its topological structure, as stated in the proposition below.

Proposition 4.1.8. *A subset E of $\mathbf{R}^N$ with $\dim_H(E) < 1$ is totally disconnected: no two of its points lie in the same connected component.*

PROOF. Let x and y be distinct elements of E and consider the distance function d_x to x in $\mathbf{R}^N$, that is, $d_x(z) = |z - x|$. As d_x does not increase distances, from Proposition 4.1.7 we deduce that $\dim_H(d_x(E)) \le \dim_H(E) < 1$. Thus $d_x(E)$ is a subset of $\mathbf{R}$ of $\mathcal{H}^1$ measure (or Lebesgue measure) zero. Consequently, there exists $r > 0$ such that $r < d_x(y)$ and $r \notin d_x(E)$ (otherwise $(0, d_x(y)) \subset d_x(E)$). The set E is then the union of the two disjoint open sets

$$E = \{z \in E : d_x(z) < r\} \cup \{z \in E : d_x(z) > r\}$$

where x is in one set and y is in the other, so that x and y lie in different connected components. □

We end this section by giving a nontrivial set in $\mathbf{R}$, with null Lebesgue measure but with Hausdorff dimension equal to one.

Proposition 4.1.9. *There exists a compact set E of $[0,1]$ such that $\mathcal{H}^1(E) = 0$ and* $\dim_H(E) = 1$.

PROOF. We make use of a Cantor-type set construction but reduce the proportion of intervals removed at each stage. More precisely, we define a family $(K_n)_{n\in\mathbf{N}}$ of closed sets as follows: $K_0 = [0,1]$, $K_1 = K_0 \setminus I_1^1$, where I_1^1 is an open interval centered at 1/2 with length l_0, $0 < l_0 < 1$; for all $n > 1$, K_n is the union of 2^n closed disjoint intervals I_i^n with the same length l_n and K_{n+1} is obtained by removing from each I_i^n an open interval of length $\frac{l_n}{n+1}$ and centered at the center of I_i^n. Let us show that the compact set $E = \cap_{n\in\mathbf{N}} K_n$ answers the question.

A straightforward calculation gives $l_n = 2^{-n}(1 - l_0)/n$ so that

$$\mathcal{H}^1(E) = \lim_{n\to+\infty} \mathcal{H}^1(K_n) = 0.$$

Now fix $0 < r < 1$ and define the integer (depending on r),

$$n_0 = \sup\{n \in \mathbf{N} : \exists i \in \{1, \dots, 2^n\},\ B_r(x) \cap E \subset I_i^n\}.$$

It is easily seen that

$$r \ge \frac{l_{n_0}}{n_0 + 1} = \frac{2^{-n_0}(1 - l_0)}{n_0(n_0 + 1)}. \tag{4.3}$$

On the other hand, for each $0 < \alpha < 1$, there exists a positive constant (i.e., independent of n_0) $C(\alpha, l_0)$ such that

$$2^{-n_0} \le C(\alpha, l_0) \frac{2^{-\alpha n_0}(1 - l_0)^\alpha}{n_0^\alpha (n_0 + 1)^\alpha}; \tag{4.4}$$

indeed, it is enough to take

$$C(\alpha, l_0) = (1 - l_0)^{-\alpha} \max\left\{x^\alpha (1+x)^\alpha 2^{-(1-\alpha)x} \ : \ x \geq 1\right\}.$$

Consider now the outer measure μ in $\mathbf{R}$ satisfying $\mu(I_i^n) = 2^{-n}$ for all n and all $i = 1, \dots, 2^n$, defined following the general construction of Remark 4.1.2. Collecting (4.3) and (4.4), we obtain for all $x \in E$ and all $0 < r < 1$,

$$\mu(B_r(x) \cap E) \leq C(\alpha, l_0) r^\alpha.$$

This estimate yields $\dim_H(E) \geq \alpha$. Indeed, if $(B_i)_{i \in \mathbf{N}}$ is a δ-covering of E made up of balls B_i centered in E with $\delta < 1$,

$$\sum_{i \in \mathbf{N}} c_\alpha \operatorname{diam}(B_i)^\alpha \geq \frac{\omega_\alpha}{C(\alpha, l_0)} \sum_{i \in \mathbf{N}} \mu(B_i \cap E) \geq \frac{\omega_\alpha}{C(\alpha, l_0)} \mu(E) = \frac{\omega_\alpha}{C(\alpha, l_0)},$$

so that $\tilde{\mathscr{H}}^\alpha(E) > 0$, then $\mathscr{H}^\alpha(E) > 0$ and $\dim_H(E) \geq \alpha$.

Since $\alpha < 1$ is arbitrary, we have $\dim_H(E) \geq 1$. The opposite inequality is trivial; then $\dim_H(E) = 1$. □

4.2 ▪ Set functions and duality approach to Borel measures

4.2.1 ▪ Borel measures as set functions

Let Ω be a separable locally compact topological space (for example, $\mathbf{R}^N$ or an open subset of $\mathbf{R}^N$) and $\mathscr{B}(\Omega)$ its Borel field. We denote the set of all $\mathbf{R}^m$-valued Borel measures by $\mathbf{M}(\Omega, \mathbf{R}^m)$. Let us recall that $\mathbf{M}(\Omega, \mathbf{R}^m)$ is the vectorial space of all the set functions $\mu : \mathscr{B}(\Omega) \to \mathbf{R}^m$ satisfying $\mu(\emptyset) = 0$ and the σ-additivity condition:

$$\mu\left(\bigcup_{n \in \mathbf{N}} B_n\right) = \sum_{n \in \mathbf{N}} \mu(B_n) \ \text{ for all pairwise disjoint families } (B_n)_{n \in \mathbf{N}} \text{ in } \mathscr{B}(\Omega).$$

In the case when $m = 1$, we will use the notation $\mathbf{M}(\Omega)$ (or $\mathbf{M}_b(\Omega)$) and the elements of $\mathbf{M}(\Omega)$ are called signed Borel measures. The subset of its nonnegative elements is denoted by $\mathbf{M}^+(\Omega)$.

If A is a fixed Borel subset of Ω, the restriction to A of a Borel measure $\mu \in \mathbf{M}(\Omega, \mathbf{R}^m)$ is the Borel measure $\mu \lfloor A$ of $\mathbf{M}(\Omega, \mathbf{R}^m)$ defined for all Borel sets E of Ω, by $\mu \lfloor A(E) = \mu(E \cap A)$. It is worth noticing that Section 4.1 provides many concrete Borel measures on $\mathbf{R}^N$. Indeed, if A is a Borel subset in $\mathbf{R}^N$ satisfying $\mathscr{H}^s(A) < +\infty$, then $\mu = \mathscr{H}^s \lfloor A$ belongs to $\mathbf{M}^+(\mathbf{R}^N)$.

The support of a measure $\mu \in \mathbf{M}(\Omega, \mathbf{R}^m)$ is the smallest closed set E of Ω, denoted by $\operatorname{spt}(\mu)$, such that $|\mu|(\Omega \setminus E) = 0$. As a straightforward consequence of the definition, we also have

$$\operatorname{spt}(\mu) = \{x \in \Omega \ : \ \forall r > 0, \ |\mu|(B_r(x)) > 0\}.$$

Let us recall that if $\mu \in \mathbf{M}^+(\Omega)$, the measure $\mu(B)$ of all Borel subsets B of Ω can be approximated by the measures of the open or compact subsets of Ω. More precisely, the following holds.

Proposition 4.2.1. *The Borel measures μ in $\mathbf{M}^+(\Omega)$ are regular, i.e., for all Borel sets B of Ω, one has*

$$\begin{aligned} \mu(B) &= \sup\{\mu(K) : K \subset B, \ K \text{ compact set of } \Omega\}, \\ &= \inf\{\mu(U) : B \subset U, \ U \text{ open set of } \Omega\}. \end{aligned}$$

The total variation of a measure $\mu \in \mathbf{M}(\Omega, \mathbf{R}^m)$ is the real-valued set function $|\mu|$, defined for all Borel sets B of Ω by

$$|\mu|(B) = \sup \left\{ \sum_{i=0}^{\infty} |\mu(B_i)| \ : \ \bigcup_{i=0}^{\infty} B_i = B \right\},$$

where the supremum is taken over all the partitions of B in $\mathscr{B}(\Omega)$. We point out that for all μ in $\mathbf{M}(\Omega, \mathbf{R}^m)$ we automatically have $|\mu|(\Omega) < +\infty$ (μ is bounded) and $|\mu|$ is σ-additive, so that $|\mu|$ is a Borel measure in $\mathbf{M}^+(\Omega)$. Actually, it is easily seen that $|\mu|$ is the smallest nonnegative scalar Borel measure ν such that $|\mu(B)| \leq \nu(B)$ for all Borel sets B and that the mapping $\mu \mapsto |\mu|(\Omega)$ is a norm for which $\mathbf{M}(\Omega, \mathbf{R}^m)$ is a Banach space.

In the scalar case, for all $\mu \in \mathbf{M}(\Omega)$ we define in $\mathbf{M}^+(\Omega)$ the positive part μ^+ and the negative parts μ^- of μ by

$$\mu^+ = \frac{|\mu| + \mu}{2}, \qquad \mu^- = \frac{|\mu| - \mu}{2},$$

so that $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$. We define now the nonnegative Radon measures as the locally finite nonnegative Borel measures.

Definition 4.2.1. *A set function $\mu : \mathscr{B}(\Omega) \to [0, +\infty]$ such that for all $\Omega' \subset\subset \Omega$ its restrictions to $\mathscr{B}(\Omega')$ is a Borel measure on Ω' is called a nonnegative Radon measure.*

Remark 4.2.1. It is easily seen that the nonnegative Radon measures are regular.

Given a measure λ in $\mathbf{M}^+(\Omega)$, we denote the set of all Borel functions $f : \Omega \to \mathbf{R}^m$ such that

$$\int_\Omega |f|\, d\lambda < +\infty$$

by $L^1_\lambda(\Omega, \mathbf{R}^m)$ or by $L^1_\lambda(\Omega)$ when $m = 1$. Given now two measures $\mu \in \mathbf{M}(\Omega, \mathbf{R}^m)$ and $\lambda \in \mathbf{M}^+(\Omega)$, we say that the measure μ is absolutely continuous with respect to the measure λ and we write $\mu \ll \lambda$ iff

$$\forall B \in \mathscr{B}(\Omega),\ \lambda(B) = 0 \Longrightarrow \mu(B) = 0.$$

We say that the measure μ is singular with respect to the measure λ and we write $\mu \perp \lambda$ iff there exists B in $\mathscr{B}(\Omega)$ such that $\lambda(B) = 0$ and μ is concentrated on B, i.e., $\mu(C) = 0$ for all Borel set C such that $B \cap C = \emptyset$.

One may establish that

$$\mu \ll \lambda \iff \exists f \in L^1_\lambda(\Omega, \mathbf{R}^m) \quad \text{s.t. } \mu = f\lambda.$$

The following theorem extends this result.

Theorem 4.2.1 (Radon–Nikodým). *Let μ and λ be two Borel measures, respectively, in $\mathbf{M}(\Omega, \mathbf{R}^m)$ and $\mathbf{M}^+(\Omega)$. Then there exist a function f in $L^1_\lambda(\Omega, \mathbf{R}^m)$ and a measure μ^s in $\mathbf{M}(\Omega, \mathbf{R}^m)$ such that*

$$\mu = f\lambda + \mu^s, \quad \mu^s \perp \lambda.$$

Moreover, f is given by

$$f(x)=\lim_{\rho\to 0}\frac{\mu(B_\rho(x))}{\lambda(B_\rho(x))}\quad \textit{for } \mu \textit{ a.e. } x\in\Omega,$$

where $B_\rho(x)$ denotes the open ball in Ω centered at $x\in\Omega$ with radius ρ.

For details about these elementary notions and various proofs of these properties, we refer the reader to the books by Buchwalter [143], Marle [287], and Rudin [331].

We now give three useful lemmas concerning Borel measures in $\mathbf{M}^+(\Omega)$. The first states that one can reduce families of pairwise disjoint Borel sets to a countable subfamily as long as one considers their measures. The second is a localization lemma in $\mathbf{M}^+(\Omega)$. The last lemma is a basic result concerning the s-density of a nonnegative Radon measure and is an essential tool in the study of the structure of sets with finite perimeter (see Chapter 10).

Lemma 4.2.1. *Let μ be a Borel measure in $\mathbf{M}^+(\Omega)$ and $(B_i)_{i\in I}$ a family of pairwise disjoint Borel subsets of Ω. Then the subset of indices $i\in I$ such that $\mu(B_i)\neq 0$ is at most countable.*

PROOF. Since $\{i\in I:\mu(B_i)\neq 0\}=\bigcup_{n\in\mathbf{N}^*}\{i\in I:\mu(B_i)>\frac{1}{n}\}$, it is enough to prove that each set $I_n=\{i\in I:\mu(B_i)>\frac{1}{n}\}$ is finite. Assume that I_n contains an infinite sequence of indices in I, that is, $\{i_0,\dots,i_k,\dots\}\subset I$. We then have

$$+\infty>\mu\left(\bigcup_{k\in\mathbf{N}}B_{i_k}\right)=\sum_{k\in\mathbf{N}}\mu(B_{i_k})=+\infty,$$

a contradiction. □

Example 4.2.1. Let $\mu\in\mathbf{M}(\Omega,\mathbf{R}^m)$ and $B_r(x_0)$ be the open ball in Ω centered at x_0 with radius r. Then for all but countably many r in $\mathbf{R}^+$, one has

$$\int_{\partial B_r(x_0)}|\mu|=0.$$

Lemma 4.2.2. *Let $\mu\in\mathbf{M}^+(\Omega)$, $(f_i)_{i\in\mathbf{N}}$ be a family of nonnegative functions in $L^1_\mu(\Omega)$ and set $f=\sup_i f_i$. Then*

$$\int_\Omega f\,d\mu=\sup\left\{\sum_{i\in I}\int_{A_i}f_i\,d\mu\right\},$$

where the supremum is taken over all finite families $(A_i)_{i\in I}$ of pairwise disjoint open subsets of Ω.

PROOF. Let n be a fixed element of $\mathbf{N}$ and consider the μ-measurable sets

$$E_i:=\left\{x\in\Omega:\ \sup_{0\le k\le n}f_k(x)=f_i(x)\right\}.$$

We now construct the following family $(\Omega_i)_{i=0,\dots,n}$ of pairwise disjoint μ-measurable sets:

$$\Omega_0=E_0,\qquad \Omega_{i+1}=\left(\Omega\setminus\bigcup_{k=1}^{i}E_k\right)\cap E_{i+1},\quad i=0,\dots,n-1.$$

It is easy to check that $\Omega=\bigcup_{i=0}^n E_i=\bigcup_{i=0}^n \Omega_i$. Moreover,

$$\begin{aligned}\int_\Omega \sup_{0\le k\le n} f_k \, d\mu &= \int_\Omega 1_{\bigcup_{i=0}^n \Omega_i} \sup_{0\le k\le n} f_k \, d\mu \\ &= \sum_{i=0}^n \int_{\Omega_i} \sup_{0\le k\le n} f_k \, d\mu \\ &= \sum_{i=0}^n \int_{\Omega_i} f_i \, d\mu.\end{aligned}$$

Let $\mu_i=f_i\cdot\mu$ be the Borel measure in $\mathbf{M}^+(\Omega)$ whose density with respect to μ is f_i. Since each measure μ_i is regular, one has $\mu_i(\Omega_i)=\sup\{\mu_i(K){:}K \text{ compact subset of } \Omega_i\}$, hence

$$\int_\Omega \sup_{0\le i\le n} f_i \, d\mu = \sup\left\{\sum_{i=0}^n \int_{K_i} f_i \, d\mu : (K_i)_{i=1,\dots,n} \text{ pairwise disjoint compact sets}\right\}.$$

From the regularity property of μ_i again, and by compactness, for all $\varepsilon>0$, there exists a family $(\mathscr{O}_i)_{i=0,\dots,n}$ of open sets, that one may assume pairwise disjoint, each $\mathscr{O}_i$ containing K_i, such that $\mu_i(\mathscr{O}_i\setminus K_i)<\varepsilon/n$. Therefore

$$\int_\Omega \sup_{0\le i\le n} f_i \, d\mu = \sup\left\{\sum_{i=0}^n \int_{A_i} f_i \, d\mu : (A_i)_{i=1,\dots,n} \text{ pairwise disjoint open sets}\right\}.$$

Taking the supremum on n and by the monotone convergence theorem, we obtain

$$\int_\Omega f \, d\mu \le \sup\left\{\sum_{i\in I} \int_{A_i} f_i \, d\mu\right\}.$$

Since the converse inequality is obvious, the proof is complete. □

Example 4.2.2. Let Ω be a bounded open set of $\mathbf{R}^N$ and let us consider the following measure associated with a given function u in the space $SBV(\Omega)$ defined in Section 10.5: S_u is a hypersurface in Ω, which is the union of a negligible subset of Ω for the $N-1$-Hausdorff measure and of countable many C^1-hypersurfaces with Hausdorff dimension $N-1$ (S_u is the jump set of u). For H^{N-1} almost every $x\in S_u$, $\nu_u(x)$ is a normal unit vector at x and $\mu:=\mathscr{L}^N\lfloor\Omega+\nu_u\,\mathscr{H}^{N-1}\lfloor S_u$, where $\mathscr{L}^N$ denotes the Lebesgue measure in $\mathbf{R}^N$. On the other hand, let us consider the functions $f_\nu:=|\nabla u\cdot\nu|^p+|\nu_u\cdot\nu|1_{S_u}$ indexed by a countable dense subset D of ν in S^{N-1}. We have

$$\begin{aligned}\int_\Omega \sup_{\nu\in D} f_\nu \, d\mu &= \int_{\Omega\setminus S_u} \sup_{\nu\in D} f_\nu \, d\mu + \int_{S_u} \sup_{\nu\in D} f_\nu \, d\mu \\ &= \int_\Omega |\nabla u|^p \, dx + \mathscr{H}^{N-1}(S_u),\end{aligned}$$

which is the Mumford–Shah energy functional introduced in image segmentation and studied in Sections 12.5 and 14.3.

By applying Lemma 4.2.2 now, we then obtain the following formula, which is a central point to extend to arbitrary dimension, the approximation of the Mumford–Shah energy functional established in one dimension:

$$\int_\Omega |\nabla u|^p \, dx + \mathscr{H}^{N-1}(S_u) = \sup\left\{\sum_{i\in I}\left(\int_{A_i} |\nabla u\cdot\nu_i|^p \, dx + \int_{A_i} |\nu_\nu\cdot\nu_i| \, d\mathscr{H}^{N-1}\lfloor S_u\right)\right\}.$$

The supremum is taken over all finite families $(A_i)_{i\in I}$ of pairwise disjoint open subsets of Ω. Note that for $p=1$, we obtain the expression of the total variation of the measure $Du := \nabla u\ \mathscr{L}^N\lfloor\Omega + [u]\nu_u\ \mathscr{H}^{N-1}\lfloor S_u$. For more details on the spaces $BV(\Omega)$ and $SBV(\Omega)$, see Chapter 10, and for the definition and properties concerning the Mumford–Shah energy functional, consult Sections 12.5 and 14.3. For another application of the localization Lemma 4.2.2, see Section 13.3.

Lemma 4.2.3. *Let Ω be an open bounded subset of $\mathbf{R}^N$, μ a nonnegative Radon measure on Ω, and $0 < s \le N$, $t > 0$. For each Borel subset E of Ω, the following implication holds:*

$$\forall x \in E \quad \limsup_{\rho\to 0} \frac{\mu(B_\rho(x))}{\rho^s} > t \Longrightarrow \mu \ge C t \mathscr{H}^s\lfloor E,$$

where C is a positive constant depending only on s. When s is an integer, one may take $C=\omega_s$, the volume of the unit ball of $\mathbf{R}^s$.

PROOF. One may assume $\mu(E) < +\infty$. Since $\mu(E) = \inf\{\mu(U) : E \subset U,\ U \text{ open set of } \Omega\}$, one may choose an arbitrary open subset U of Ω such that $E \subset U$ and $\mu(U) < +\infty$. Let now $\delta > 0$ and consider the family of closed balls

$$\mathscr{F} := \left\{ \overline{B}_\rho(x) \subset U : x \in E,\ \rho \le \frac{\delta}{2},\ \frac{\mu(B_\rho(x))}{\rho^s} \ge t \right\}.$$

From the hypothesis, it is easily seen that this family finely covers E (Definition 4.1.3). Therefore, according to Lemma 4.1.3, there exists a countable subfamily $\mathscr{G}$ of pairwise disjoint elements of $\mathscr{F}$ such that

$$\bigcup_{B\in\mathscr{F}} B \subset \bigcup_{B\in\mathscr{G}} B^*,$$

where B^* denotes the closed ball concentric with B, with radius five times as big as that of B. Moreover, for each finite family $\mathscr{G}^* \subset \mathscr{G}$,

$$E \subset \left(\bigcup_{B\in\mathscr{G}^*} B \right) \cup \left(\bigcup_{B\in\mathscr{G}\setminus\mathscr{G}^*} B^* \right).$$

Thus,

$$\begin{aligned}
\mathscr{H}^s_{5\delta}(E) &\le c_s \sum_{B\in\mathscr{G}^*} (\operatorname{diam}(B))^s + c_s 5^s \sum_{B\in\mathscr{G}\setminus\mathscr{G}^*} (\operatorname{diam}(B))^s \\
&\le 2^s c_s t^{-1} \left(\sum_{B\in\mathscr{G}^*} \mu(B) \right) + 2^s c_s 5^s t^{-1} \left(\sum_{B\in\mathscr{G}\setminus\mathscr{G}^*} \mu(B) \right).
\end{aligned}$$

Since the second sum of the last inequality can be taken less than δ for an appropriate choice of $\mathscr{G}^*$, we obtain

$$\begin{aligned}
\mathscr{H}^s_{5\delta}(E) &\le \omega_s t^{-1} \left(\sum_{B\in\mathscr{G}^*} \mu(B) \right) + \delta \\
&\le \omega_s t^{-1} \mu(U) + \delta.
\end{aligned}$$

We end the proof by letting $\delta \to 0$ and taking the infimum on U. $\square$

4.2.2 ▪ Duality approach

We recall now some definitions and important results concerning the Riesz functional analysis approach. We set $\mathbf{C}_0(\Omega, \mathbf{R}^m)$ to denote the space of all continuous functions which tend to zero at infinity, i.e.,

$$\forall \varepsilon > 0 \quad \text{there exists a compact set } K_\varepsilon \subset \Omega \text{ such that } \sup_{x \in \Omega \setminus K_\varepsilon} |\varphi(x)| \leq \varepsilon.$$

Recall that equipped with the norm

$$||\varphi||_\infty = \sup_{x \in \Omega} |\varphi(x)|,$$

$\mathbf{C}_0(\Omega, \mathbf{R}^m)$ is a Banach space. We denote its subspace made up of all continuous functions with compact support in Ω by $\mathbf{C}_c(\Omega, \mathbf{R}^m)$. When $m = 1$, the two above spaces will be denoted, respectively, by $\mathbf{C}_0(\Omega)$ and $\mathbf{C}_c(\Omega)$.

According to the vectorial version of the Riesz–Alexandroff representation Theorem 2.4.7, the dual of $\mathbf{C}_0(\Omega, \mathbf{R}^m)$ (and then of $\mathbf{C}_c(\Omega, \mathbf{R}^m)$) can be isometrically identified with $\mathbf{M}(\Omega, \mathbf{R}^m)$. Then, any Borel measure is a continuous linear form on $\mathbf{C}_0(\Omega, \mathbf{R}^m)$ or $\mathbf{C}_c(\Omega, \mathbf{R}^m)$ and the two dual norms $||\cdot||_{\mathbf{C}'_c(\Omega,\mathbf{R}^m)}$ and $||\cdot||_{\mathbf{C}'_0(\Omega,\mathbf{R}^m)}$ are equal to the total mass $|\cdot|(\Omega)$:

$$\begin{aligned}|\mu|(\Omega) &= \sup\{\langle \mu, \varphi \rangle : \varphi \in \mathbf{C}_0(\Omega, \mathbf{R}^m), ||\varphi||_\infty \leq 1\} \\ &= \sup\{\langle \mu, \varphi \rangle : \varphi \in \mathbf{C}_c(\Omega, \mathbf{R}^m), ||\varphi||_\infty \leq 1\},\end{aligned}$$

where $\langle \mu, \varphi \rangle = \mu(\varphi)$. In what follows, $\langle \mu, \varphi \rangle$ will also be denoted by $\int_\Omega \varphi \, d\mu$. Note that $\mathbf{M}(\Omega, \mathbf{R}^m)$ is isomorphic to the product space $\mathbf{M}(\Omega)^m$ and that according to this isomorphism

$$\mu \in M(\Omega, \mathbf{R}^m) \iff \mu = (\mu_1, \dots, \mu_m) \text{ and } \mu_i \in \mathbf{C}_0(\Omega)', \ i = 1, \dots, m.$$

The following proposition is an easy generalization, for vectorial measures, of Proposition 2.4.14 and Corollary 2.4.1. To shorten notation, $\sigma(\mathbf{C}'_0, \mathbf{C}_0)$ and $\sigma(\mathbf{C}'_c, \mathbf{C}_c)$ denote, respectively, the two weak topologies $\sigma(\mathbf{C}'_0(\Omega, \mathbf{R}^m), \mathbf{C}_0(\Omega, \mathbf{R}^m))$ and $\sigma(\mathbf{C}'_c(\Omega, \mathbf{R}^m), \mathbf{C}_c(\Omega, \mathbf{R}^m))$.

Proposition 4.2.2. *The weak topologies $\sigma(\mathbf{C}'_0, \mathbf{C}_0)$ and $\sigma(\mathbf{C}'_c, \mathbf{C}_c)$ induce the same topology on bounded subsets of $\mathbf{M}(\Omega, \mathbf{R}^m)$. Moreover, from any bounded sequence of Borel measures $(\mu_n)_{n \in \mathbf{N}}$ in $\mathbf{M}(\Omega, \mathbf{R}^m)$, one can extract a subsequence $\sigma(\mathbf{C}'_0, \mathbf{C}_0)$-converging (thus $\sigma(\mathbf{C}'_c, \mathbf{C}_c)$-converging) to some Borel measure μ in $\mathbf{M}(\Omega, \mathbf{R}^m)$.*

Note that for unbounded sequences of $\mathbf{M}(\Omega, \mathbf{R}^m)$, the two weak topologies do not agree, as illustrated in the following example: take $\Omega = (0, +\infty)$ and $\mu_n = n\delta_n$. Then

$$n\delta_n \rightharpoonup 0 \text{ in the topology } \sigma(\mathbf{C}'_c(\Omega), \mathbf{C}_c(\Omega))$$

but $\langle n\delta_n, \varphi \rangle \to 1$ as $n \to +\infty$ for any function of $\mathbf{C}_0(\Omega)$ satisfying $\varphi \sim \frac{1}{x}$ in the neighborhood of $+\infty$.

According to Proposition 4.2.2, from now on, for bounded sequences in $\mathbf{M}(\Omega, \mathbf{R}^m)$, we do not distinguish the convergences associated with these two weak topologies and we will refer to these convergences as the weak convergence in $\mathbf{M}(\Omega, \mathbf{R}^m)$. In terms of probabilistic approach, we have the following properties.

Proposition 4.2.3 (Alexandrov). *Let μ, μ_n in $\mathbf{M}^+(\Omega)$ such that μ_n weakly converges to μ; then*

$$\text{for all open subsets } U \text{ of } \Omega, \quad \mu(U) \leq \liminf_{n\to+\infty} \mu_n(U),$$
$$\text{for all compact subsets } K \text{ of } \Omega, \quad \mu(K) \geq \limsup_{n\to+\infty} \mu_n(K).$$

Consequently, for all relatively compact Borel subset B of Ω such that $\mu(\partial B) = 0$, we have

$$\mu(B) = \lim_{n\to+\infty} \mu_n(B).$$

PROOF. Let U be an open subset of Ω and 1_U its characteristic function. Classically, there exists a nondecreasing sequence $(\varphi_p)_{p\in\mathbf{N}}$ in $\mathbf{C}_c(\Omega)$ such that $1_U = \sup_p \varphi_p$. Therefore

$$\begin{aligned}\mu(U) &= \lim_{p\to+\infty} \int_\Omega \varphi_p \, d\mu \\ &= \lim_{p\to+\infty} \lim_{n\to+\infty} \int_\Omega \varphi_p \, d\mu_n \\ &\leq \liminf_{n\to+\infty} \int_U d\mu_n = \liminf_{n\to+\infty} \mu_n(U).\end{aligned}$$

For the other assertion, it suffices to notice that if K is a compact subset of Ω, there exists a nonincreasing sequence $(\varphi_p)_{p\in\mathbf{N}}$ in $\mathbf{C}_c(\Omega)$ such that $1_K = \inf_p \varphi_p$ and to argue similarly.

Let us prove the last assertion. Since μ_n is a nondecreasing set function, according to the first assertions, we have

$$\begin{aligned}\mu(\overset{\circ}{B}) &\leq \liminf_{n\to+\infty} \mu_n(\overset{\circ}{B}) \leq \liminf_{n\to+\infty} \mu_n(B) \\ &\leq \limsup_{n\to+\infty} \mu_n(B) \leq \limsup_{n\to+\infty} \mu_n(\overline{B}) \leq \mu(\overline{B}).\end{aligned}$$

The conclusion follows from $\mu(\overline{B}) = \mu(\overset{\circ}{B})$. □

The following corollary clarifies the relation between the weak convergence of sequences in $\mathbf{M}(\Omega, \mathbf{R}^m)$ and the convergence of the corresponding measures of suitable Borel sets.

Corollary 4.2.1. *Let μ, μ_n in $\mathbf{M}(\Omega, \mathbf{R}^m)$ be such that μ_n weakly converges to μ and $|\mu_n|$ weakly converges to some σ in $\mathbf{M}^+(\Omega)$. Then $|\mu| \leq \sigma$ and, for all relatively compact Borel subsets B of Ω such that $\sigma(\partial B) = 0$, we have $\mu(B) = \lim_{n\to+\infty} \mu_n(B)$.*

PROOF. Let U be an open subset of Ω and φ any function in $\mathbf{C}_c(U, \mathbf{R}^m)$ with $||\varphi||_\infty \leq 1$. Letting $n \to +\infty$ in

$$\left| \int_U \varphi \, d\mu_n \right| \leq \int_U |\varphi| \, d|\mu_n|,$$

we obtain

$$\left| \int_U \varphi \, d\mu \right| \leq \int_U |\varphi| \, d\sigma.$$

Taking the supremum on φ gives $|\mu|(U) \leq \sigma(U)$. The conclusion $|\mu| \leq \sigma$ follows from Proposition 4.2.1.

For proving the last assertion, let us denote the m components of μ_n and μ by μ_n^i and μ^i, $i=1,\dots,m$, respectively, and the weak limits (for a subsequence not relabeled) of the positive and negative parts of μ_n^i by $\nu^{i,+}$ and $\nu^{i,-}$, respectively. Going to the limit when $n \to +\infty$ in $\mu_n^i = \mu_n^{i,+} - \mu_n^{i,-}$ and $\mu_n^{i,\pm} \le |\mu_n|$, we obtain $\mu^i = \nu^{i,+} - \nu^{i,-}$ and $\nu^{i,\pm} \le \sigma$. The conclusion then follows by applying Proposition 4.2.3 to the m components $\mu_n^{i\pm}$. □

We now restrict ourselves to the case when $\Omega = \mathbf{R}^N$ and we study the approximation of a Borel measure in $\mathbf{M}(\mathbf{R}^N, \mathbf{R}^m)$ by a regular function in the sense of the weak convergence of measures. To this end, we define the regularization (or the mollification) of a measure by means of a regularizing kernel. Let us recall that a regularizer ρ_ε is a function in $\mathbf{C}_c^\infty(\mathbf{R}^N)$ defined by $\rho_\varepsilon(x) = \varepsilon^{-N}\rho(x/\varepsilon)$, where ρ is some nonnegative real-valued function in $\mathbf{C}_c^\infty(\mathbf{R}^N)$ satisfying

$$\int_{\mathbf{R}^N} \rho(x)\,dx = 1, \qquad \operatorname{spt}\rho \subset \overline{B}_1(0).$$

Note that the support $\operatorname{spt}\rho_\varepsilon$ of ρ_ε is included in $\overline{B}_\varepsilon(0)$. For any measure μ in $\mathbf{M}(\mathbf{R}^N, \mathbf{R}^m)$, we define the function $\rho_\varepsilon * \mu$ defined on $\mathbf{R}^N$ by

$$\rho_\varepsilon * \mu(x) = \int_{\mathbf{R}^N} \rho_\varepsilon(x-y)\; \mu(dy)$$

and we aim to show that $\rho_\varepsilon * \mu$ is a suitable approximation of μ.

Theorem 4.2.2. *The functions $\rho_\varepsilon * \mu$ belong to $\mathbf{C}^\infty(\mathbf{R}^N, \mathbf{R}^m)$ and for any $\alpha \in \mathbf{N}^N$, $D^\alpha(\rho_\varepsilon * \mu) = D^\alpha \rho_\varepsilon * \mu$. Moreover, when ε goes to zero,*

(i) $\rho_\varepsilon * \mu \rightharpoonup \mu$, *weakly in* $\mathbf{M}(\mathbf{R}^N, \mathbf{R}^m)$;

(ii) $\displaystyle\int_{\mathbf{R}^N} |\rho_\varepsilon * \mu| \le \int_{\mathbf{R}^N} |\mu|$;

(iii) $\displaystyle\int_{\mathbf{R}^N} |\rho_\varepsilon * \mu| \to \int_{\mathbf{R}^N} |\mu|$.

PROOF. The classical derivation theorem under the integral sign yields $D^\alpha(\rho_\varepsilon * \mu) = D^\alpha \rho_\varepsilon * \mu$. Let us establish (i). Let $\varphi \in \mathbf{C}_c(\mathbf{R}^N, \mathbf{R}^m)$. According to Fubini's theorem,

$$\begin{aligned}
&\left| \int_{\mathbf{R}^N} \varphi(x)\rho_\varepsilon * \mu(x)\,dx - \int_{\mathbf{R}^N} \varphi(y)\,\mu(dy) \right| \\
&= \left| \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \varphi(x)\rho_\varepsilon(x-y)\; \mu(dy)\; dx - \int_{\mathbf{R}^N} \varphi(y)\; \mu(dy) \right| \\
&= \left| \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \varphi(x)\rho_\varepsilon(x-y)\,\mu(dy)\,dx - \int_{\mathbf{R}^N} \left(\int_{\mathbf{R}^N} \rho_\varepsilon(x-y)\,dx \right) \varphi(y)\,\mu(dy) \right| \\
&\le \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} |\varphi(x) - \varphi(y)|\rho_\varepsilon(x-y)\,dx\,|\mu|(dy) \\
&\le \sup_{\{(x,y)\in\mathbf{R}^{2N} : |x-y|\le\varepsilon\}} |\varphi(x)-\varphi(y)| \int_{\mathbf{R}^N} |\mu|,
\end{aligned}$$

which, thanks to the uniform continuity of φ, tends to zero when $\varepsilon \to 0$.

We establish now (ii). By Fubini's theorem and a change of scale, we have

$$\begin{aligned}\int_{\mathbf{R}^N} |\rho_\varepsilon * \mu|(x)\,dx &= \varepsilon^{-N}\int_{\mathbf{R}^N}\left|\int_{\mathbf{R}^N}\rho\left(\frac{x-y}{\varepsilon}\right)\mu(dy)\right|dx\\ &\le \varepsilon^{-N}\int_{\mathbf{R}^N}\int_{\mathbf{R}^N}\rho\left(\frac{x-y}{\varepsilon}\right)|\mu|(dy)\,dx\\ &\le \int_{\mathbf{R}^N}\left(\int_{\mathbf{R}^N}\varepsilon^{-N}\rho\left(\frac{x-y}{\varepsilon}\right)dx\right)|\mu|(dy) = \int_{\mathbf{R}^N}|\mu|.\end{aligned}$$

Assertion (iii) is a straightforward consequence of (i), the weak lower semicontinuity of the map $\mu \mapsto \int_{\mathbf{R}^N}|\mu|$ and (ii). □

Let $\mathbf{C}_b(\Omega,\mathbf{R}^m)$ be the set of all bounded continuous functions from Ω into $\mathbf{R}^m$. We introduce now a stronger notion of convergence induced by the weak topology

$$\sigma(\mathbf{C}'_b(\Omega,\mathbf{R}^m),\mathbf{C}_b(\Omega,\mathbf{R}^m)).$$

Definition 4.2.2. *A sequence $(\mu_n)_n$ in $M(\Omega,\mathbf{R}^m)$ narrowly converges to μ in $M(\Omega,\mathbf{R}^m)$ iff*

$$\int \varphi\, d\mu_n \to \int \varphi\, d\mu$$

for all φ in $\mathbf{C}_b(\Omega,\mathbf{R}^m)$.

This convergence is strictly stronger than the weak convergence of measures. Indeed, let $\Omega = (0,1)$ and $\mu_n = \delta_{1/n}$. Then μ_n weakly converges to 0 but $\int_\Omega \mu_n = 1$. Moreover, taking $\varphi \sim \sin(1/x)$ at 0^+, $\int \varphi\mu_n$ does not converge for any subsequence. This example shows that the unit ball of $\mathbf{M}(\Omega)$ is not weakly sequentially compact for this topology. Nevertheless, the Prokhorov theorem below asserts that the bounded sets of $\mathbf{M}^+(\Omega)$ are sequentially compact for the narrow topology as long as a uniform control is assumed outside a compact set whose measure is close to that of Ω.

Theorem 4.2.3 (Prokhorov). *Let $\mathscr{H}$ be a bounded subset of $\mathbf{M}^+(\Omega)$ satisfying*

$$\forall\varepsilon, \exists K_\varepsilon \text{ compact subset of } \Omega \text{ such that } \sup\{\mu(\Omega\setminus K_\varepsilon) : \mu\in\mathscr{H}\} \le \varepsilon.$$

Then $\mathscr{H}$ is sequentially compact for the narrow topology.

For a proof, consult, for instance, Delacherie and Meyer [198]. Any subset of $\mathbf{M}^+(\Omega)$ satisfying the previous uniform bound is said to be tight. Note that in the previous example, $\{\delta_{1/n} : n \in \mathbf{N}^*\}$ is not tight. In the bounded subsets of $\mathbf{M}^+(\Omega)$, the weak and the narrow topology agree when there is no loss of mass, i.e., when $\mu(\Omega) = \lim_{n\to+\infty}\mu_n(\Omega)$. More precisely, we have the following.

Proposition 4.2.4. *Let μ_n, μ in $\mathbf{M}^+(\Omega)$. Then the following assertions are equivalent:*

(i) *$\mu_n \rightharpoonup \mu$ narrowly;*

(ii) *$\mu_n \rightharpoonup \mu$ weakly and $\mu_n(\Omega) \to \mu(\Omega)$.*

PROOF. We prove (ii) $\Longrightarrow$ (i). The converse is obvious. Let $f \in \mathbf{C}_b(\Omega)$, $\varepsilon > 0$ and K a compact subset of Ω such that $\mu(\Omega \setminus K) \leq \varepsilon$. Let moreover $\varphi \in \mathbf{C}_c(\Omega)$ satisfying $0 \leq \varphi \leq 1$, $\varphi = 1$ in K. We have

$$\begin{aligned}\left|\int f\,d\mu_n - \int f\,d\mu\right| &\leq \left|\int f\,d\mu_n - \int f\varphi\,d\mu_n\right| + \left|\int f\varphi\,d\mu_n - \int f\varphi\,d\mu\right| \\ &\quad + \left|\int f\varphi\,d\mu - \int f\,d\mu\right| \\ &\leq \|f\|_\infty \int (1-\varphi)\,d\mu_n + \left|\int f\varphi\,d\mu_n - \int f\varphi\,d\mu\right| \\ &\quad + \|f\|_\infty \int (1-\varphi)\,d\mu\end{aligned}$$

so that

$$\limsup_{n\to+\infty} \left|\int f\,d\mu_n - \int f\,d\mu\right| \leq 2\varepsilon\|f\|_\infty$$

and the conclusion follows after letting $\varepsilon \to 0$. □

In the probabilistic approach we have a similar statement.

Proposition 4.2.5. *Let μ_n, μ in $\mathbf{M}^+(\Omega)$. Then the following assertions are equivalent:*

(i) *μ_n narrowly converges to μ;*

(ii) *$\mu_n(\Omega) \to \mu(\Omega)$ and for all open subset U, $\mu(U) \leq \liminf_{n\to+\infty} \mu_n(U)$;*

(iii) *$\mu_n(\Omega) \to \mu(\Omega)$ and for all closed subset F, $\mu(F) \geq \limsup_{n\to+\infty} \mu_n(F)$;*

(iv) *for all Borel subset B such that $\mu(\partial B) = 0$, $\mu_n(B) \to \mu(B)$.*

PROOF. The only implication we have to establish is (iv) $\Longrightarrow$ (i). Indeed, each of the others is an easy consequence of Propositions 4.2.4 and 4.2.3. According to Proposition 4.2.4, it is enough to establish the weak convergence of μ_n to μ. For this, let $\varphi \in \mathbf{C}_c(\Omega)$, $\|\varphi\|_{L^\infty(\Omega)} = M$, with compact support K, and consider the subdivision

$$\begin{cases} -M = a_0 < a_1 \cdots < a_i < a_{i+1} < \cdots < a_m = M, \\ a_{i+1} - a_i \leq \varepsilon, \\ \mu([\varphi = a_i]) = 0. \end{cases}$$

Such a subdivision exists. Indeed, the last property is a consequence of Lemma 4.2.1. Consider now the Borel subsets $U_i = [\varphi < a_i] \cap K$ and the function

$$\varphi_\varepsilon = \sum_{i=1}^m a_i \chi_{U_i \setminus U_{i-1}}.$$

Since $\mu(\partial U_i) = 0$, from assertion (iv) we have $\mu_n(U_i) \to \mu(U_i)$ so that for all $\varepsilon > 0$

$$\int \varphi_\varepsilon\,d\mu_n \to \int \varphi_\varepsilon\,d\mu.$$

From the equiboundedness of the measures μ_n and the estimate $\|\varphi_\varepsilon - \varphi\|_\infty \le \varepsilon$, we easily deduce

$$\int \varphi d\mu_n \to \int \varphi d\mu,$$

which ends the proof. $\square$

The following proposition is an extension of property (iv). For a proof, consult Marle [287, Proposition 9.9.4].

Proposition 4.2.6. *Let μ_n, μ be Borel measures in $\mathbf{M}^+(\Omega)$ such that μ_n narrowly converges to μ and let f be a μ_n-measurable (for every n) and bounded function from Ω into $\mathbf{R}$ such that the set of its discontinuity points has a null μ-measure. Then f is μ-measurable and*

$$\lim_{n\to+\infty} \int f\, d\mu_n = \int f\, d\mu.$$

We end this subsection by stating two theorems extending in some sense the classical Fubini's theorem. We prove only the first one. For the second, see, for instance, [211].

Let μ in $\mathbf{M}^+(\Omega \times \mathbf{R}^m)$. We denote the projection of μ on Ω by σ. Let us recall that σ is the measure of $\mathbf{M}^+(\Omega)$ defined for all Borel set E of Ω by $\sigma(E) = \mu(E \times \mathbf{R}^m)$. The following slicing decomposition holds.

Theorem 4.2.4. *There exists a family $(\mu_x)_{x\in\Omega}$ of probability measures on $\mathbf{R}^m$, unique up to equality σ-a.e., such that for all f in $\mathbf{C}_0(\Omega \times \mathbf{R}^m)$*

(i) $x \mapsto \displaystyle\int_{\mathbf{R}^m} f(x,y)\, d\mu_x(y)$ *is σ-measurable;*

(ii) $\displaystyle\int_{\Omega\times\mathbf{R}^m} f(x,y)\, d\mu(x,y) = \int_\Omega \Big(\int_{\mathbf{R}^m} f(x,y)\, d\mu_x(y)\Big)\, d\sigma(x).$

We will write $\mu = (\mu_x)_{x\in\Omega} \otimes \sigma$.

PROOF. *First step.* We establish the result for all f of the form $f = g \otimes h$, where $g(x) = 1_B(x)$, B is any Borel subset of Ω, and h belongs to $\mathbf{C}_0(\mathbf{R}^m)$.

Let us first assume that h belongs to a dense countable subset D of $\mathbf{C}_0(\mathbf{R}^m)$ and define $\gamma_h \in \mathbf{M}^+(\Omega)$ by

$$\gamma_h(B) = \int_{B\times\mathbf{R}^m} h(y)\, d\mu(x,y) \quad \forall\, B \in \mathscr{B}(\Omega).$$

Since $\mu(B \times \mathbf{R}^m) = \sigma(B) = 0 \Longrightarrow \gamma_h(B) = 0$, according to the Radon–Nikodým theorem, Theorem 4.2.1, the measure γ_h has a density $a_h \in L^1_\sigma(\Omega)$ with respect to σ, i.e.,

$$\gamma_h = a_h \cdot \sigma.$$

We then obtain

$$\int_{B\times\mathbf{R}^m} h(y)\, d\mu(x,y) = \int_B a_h(x)\, d\sigma(x). \tag{4.5}$$

Moreover, there exists a sequence $(N_h)_{h\in D}$ of σ-null sets such that a_h is given for all $x_0 \in \Omega' := \Omega \setminus (\cup_{h\in D} N_h)$ by

$$a_h(x_0) = \lim_{\rho\to 0} \frac{\gamma_h(B_\rho(x_0))}{\sigma(B_\rho(x_0))} = \lim_{\rho\to 0} \frac{\int_{(B_\rho(x_0))\times\mathbf{R}^m} h(y)\,d\mu(x,y)}{\mu(B_\rho(x_0)\times\mathbf{R}^m)}. \tag{4.6}$$

For all fixed x_0 in Ω', let us now consider the linear map $\Gamma_{x_0} : D \to \mathbf{R}$ defined by

$$\Gamma_{x_0}(h) = a_h(x_0).$$

From (4.6), we easily check that $|\Gamma_{x_0}(h)| \leq ||h||_\infty$. Therefore Γ_{x_0} may be extended by a measure μ_{x_0} in $\mathbf{M}^+(\mathbf{R}^m)$. Since the map $x \mapsto 1_B(x)\mu_x(h) = 1_B(x)a_h(x)$ is σ-measurable for all $h \in D$, it is also σ-measurable for all h in $\mathbf{C}_0(\mathbf{R}^m)$. Now, from (4.5), one can write

$$\int_{B\times\mathbf{R}^m} h(y)\,d\mu(x,y) = \int_B \left(\int_{\mathbf{R}^m} h(y)\,d\mu_x(y)\right) d\sigma(x)$$

with $||\mu_x|| \leq 1$.

Second step. We establish the result for all f of the form $f = g\otimes h$, where $g \in L^1_\sigma(\Omega)$ and $h \in \mathbf{C}_0(\mathbf{R}^m)$.

For all $\varepsilon > 0$, let us consider the step function $g_\varepsilon = \sum_{i\in I} \alpha_i 1_{B_i}$, $B_i \in \mathscr{B}(\Omega)$, I finite, such that

$$\int_\Omega |g - g_\varepsilon|\,d\sigma < \varepsilon. \tag{4.7}$$

We have

$$\begin{aligned}
&\left|\int_{\Omega\times\mathbf{R}^m} g\otimes h\,d\mu - \int_\Omega g(x)\left(\int_{\mathbf{R}^m} h(y)\,d\mu_x(y)\right) d\sigma(x)\right| \\
&\leq \left|\int_{\Omega\times\mathbf{R}^m} g\otimes h\,d\mu - \int_{\Omega\times\mathbf{R}^m} g_\varepsilon\otimes h\,d\mu\right| \\
&\quad + \left|\int_{\Omega\times\mathbf{R}^m} g_\varepsilon\otimes h\,d\mu - \int_\Omega g_\varepsilon(x)\left(\int_{\mathbf{R}^m} h(y)\,d\mu_x(y)\right) d\sigma(x)\right| \\
&\quad + \left|\int_\Omega g_\varepsilon(x)\left(\int_{\mathbf{R}^m} h(y)\,d\mu_x(y)\right) d\sigma(x) - \int_\Omega g(x)\left(\int_{\mathbf{R}^m} h(y)\,d\mu_x(y)\right) d\sigma(x)\right|.
\end{aligned}$$

According to the first step, the second term of the right-hand side is equal to zero. According to (4.7), each of the two other terms is less than $\varepsilon||h||_\infty$. Since ε is arbitrary, we obtain

$$\int_{\Omega\times\mathbf{R}^m} g\otimes h\,d\mu = \int_\Omega g(x)\left(\int_{\mathbf{R}^m} h(y)\,d\mu_x(y)\right) d\sigma(x). \tag{4.8}$$

We are going to prove that μ_x is a probability measure. Let $(h_n)_{n\in\mathbf{N}}$ be a nondecreasing sequence of functions in $\mathbf{C}_0(\mathbf{R}^m)$ pointwise converging to $1_{\mathbf{R}^m}$. From (4.8) we deduce, for all Borel set B in Ω,

$$\int_{B\times\mathbf{R}^m} h_n(y)\,d\mu(x,y) = \int_B \left(\int_{\mathbf{R}^m} h_n(y)\,d\mu_x(y)\right) d\sigma(x),$$

and, by letting $n \to +\infty$,

$$\sigma(B) = \int_B \mu_x(\mathbf{R}^m)\, d\sigma(x).$$

Since for σ a.e. x in Ω, $\mu_x(\mathbf{R}^m) = \|\mu_x\| \le 1$ we infer that $\mu_x(\mathbf{R}^m) = 1$ for σ a.e. x in Ω.

Third step. We assume that f belongs to $\mathbf{C}_0(\Omega \times \mathbf{R}^m)$. The result is now an easy consequence of the density of

$$\left\{ \sum_{i \in I} g_i \otimes h_i \ : \ g_i \in \mathbf{C}_c(\Omega),\ h_i \in \mathbf{C}_c(\mathbf{R}^m),\ I \in PF(\mathbf{N}) \right\}$$

in $\mathbf{C}_0(\Omega \times \mathbf{R}^m)$ for the uniform norm. ($PF(\mathbf{N})$ denotes the family of all finite subsets of $\mathbf{N}$.)

Last step. It remains to establish the uniqueness of the family $(\mu_x)_x$, up to equality σ-a.e. Take $f = 1_{B_\rho(x_0)} \times h$, where h is any function in D and $x_0 \in \Omega$ is such that the limit

$$\lim_{\rho \to 0} \frac{\int_{B_\rho(x_0)} \left(\int_{\mathbf{R}^m} h(y)\, d\mu_x(y) \right) d\sigma(x)}{\sigma(B_\rho(x_0))}$$

exists. According to the theory of differentiation of measures (Radon theorem), we know that there exists a Borel set N_h with $\sigma(N_h) = 0$ such that the above limit exists for $x_0 \in \Omega \setminus N_h$. Now, this limit exists for $x_0 \in \Omega' = \Omega \setminus \cup_{h \in D} N_h$ and for all $h \in D$.

From

$$\int_{\Omega \times \mathbf{R}^m} 1_{B_\rho(x_0)} h(y)\, d\mu(x,y) = \int_{B_\rho(x_0)} \left(\int_{\mathbf{R}^m} h(y)\, d\mu_x(y) \right) d\sigma(x)$$

we deduce that for $x_0 \in \Omega'$

$$\int_{\mathbf{R}^m} h(y)\, d\mu_{x_0}(y) = \lim_{\rho \to 0} \frac{\int_{(B_\rho(x_0)) \times \mathbf{R}^m} h(y)\, d\mu(x,y)}{\sigma(B_\rho(x_0))} = \Gamma_{x_0}(h).$$

Therefore $\mu_x = \Gamma_x$ for all $x \in \Omega'$ and all $h \in D$. This gives the required uniqueness. □

Theorem 4.2.5 (classical coarea formula). *For all Lipschitz functions $f : \mathbf{R}^N \longrightarrow \mathbf{R}$ and for all functions $g : \mathbf{R}^N \longrightarrow \mathbf{R}$ in $L^1(\mathbf{R}^N)$ we have*

$$\int_{\mathbf{R}^N} g(x)|Df|\, dx = \int_{-\infty}^{+\infty} \left(\int_{[f=t]} g(x)\, d\mathcal{H}^{N-1}(x) \right) dt.$$

As a corollary of Theorem 4.2.5 we obtain the so-called curvilinear Fubini theorem.

Corollary 4.2.2. *Let Ω be a bounded open subset of $\mathbf{R}^N$ and Γ_t the set $\{x \in \Omega : d(x, \mathbf{R}^N \setminus \overline{\Omega}) = t\}$. Then*

$$\int_\Omega g(x)\, dx = \int_{-\infty}^{+\infty} \left(\int_{\Gamma_t} g(x)\, d\mathcal{H}^{N-1}(x) \right) dt.$$

PROOF. Take $f = d(\cdot, \mathbf{R}^N \setminus \overline{\Omega})$. □

Taking now $g = 1$ and f the truncation of $d(\cdot, S)$ between s and s', with $s < s'$, we obtain the next corollary.

Corollary 4.2.3. *Let S be a subset of $\mathbf{R}^N$. Then*

$$\mathscr{L}^N\big([s < d(\cdot, S) < s']\big) = \int_s^{s'} \mathscr{H}^{N-1}\big([d(\cdot, S) = t]\big)\,dt$$

and the distributional derivative of $\mathscr{L}^N\big([d(\cdot, S) < t]\big)$ is given by

$$\frac{d}{dt}\mathscr{L}^N\big([d(\cdot, S) < t]\big) = \mathscr{H}^{N-1}\big([d(\cdot, S) = t]\big).$$

4.3 ▪ Introduction to Young measures

We deal now with the notion of Young measure, a measure theoretical tool, well suited to the analysis of oscillations of minimizing sequences (see, for instance, [170]). We will give in Chapter 11 an important application in the scope of relaxation in nonlinear elasticity. For an application to phase transitions for crystals, consult [172]. For a general exposition of the theory, see [78], [80], [162], [344], [354], [355], and the references therein.

4.3.1 ▪ Definition

In this section, Ω is an open bounded subset of $\mathbf{R}^N$ and $E = \mathbf{R}^d$. In Chapter 11, Section 11.4, we will consider the case when $d = mN$ so that E will be isomorphic to the space $\mathbf{M}^{m\times N}$ of $m \times N$ matrices. To shorten notation, we denote the N-dimensional Lebesgue measure restricted to Ω by $\mathscr{L}$.

Definition 4.3.1. *We call a Young measure on $\Omega \times E$ any positive measure $\mu \in \mathbf{M}^+(\Omega \times E)$ whose image $\pi_\Omega\#\mu$ by the projection π_Ω on Ω is the Lebesgue measure $\mathscr{L}$ on Ω, i.e., for all Borel subset B of Ω,*

$$\pi_\Omega\#\mu(B) := \mu(B \times E) = \mathscr{L}(B).$$

We denote the set of all Young measures on $\Omega \times E$ by $\mathscr{Y}(\Omega; E)$.

We now consider the space $\mathbf{C}_b(\Omega; E)$ of Carathéodory integrands, namely, the space of all functions $\psi : \Omega \times E \to \mathbf{R}$, $\mathscr{B}(\Omega) \otimes \mathscr{B}(E)$ measurable, and satisfying

(i) $\psi(x, \cdot)$ is bounded continuous on E for all $x \in \Omega$;

(ii) $x \mapsto \|\psi(x, \cdot)\|$ is Lebesgue integrable.

We equip $\mathscr{Y}(\Omega; E)$ with the narrow topology, i.e., the weakest topology which makes the maps

$$\mu \mapsto \int_{\Omega\times E} \psi\, d\mu$$

continuous, when ψ runs through $\mathbf{C}_b(\Omega; E)$. This topology induces the narrow convergence of Young measures defined as follows: let $(\mu_n)_{n\in\mathbf{N}}$ be a sequence of measures in $\mathscr{Y}(\Omega; E)$ and $\mu \in \mathscr{Y}(\Omega; E)$; then

$$\mu_n \overset{nar}{\rightharpoonup} \mu \Longleftrightarrow \lim_{n\to+\infty}\int_{\Omega\times E} \psi(x, \lambda)d\mu_n(x, \lambda) = \int_{\Omega\times E} \psi(x, \lambda)d\mu(x, \lambda)\ \forall \psi \in \mathbf{C}_b(\Omega; E).$$

Remark 4.3.1. Let $(\mu_n)_{n\in\mathbf{N}}$ be a sequence in $\mathscr{Y}(\Omega;E)$ and $\mu \in \mathscr{Y}(\Omega;E)$. It is easily seen (see Valadier [354], [355]) that

$$\mu_n \overset{nar}{\rightharpoonup} \mu \Longleftrightarrow \lim_{n\to+\infty}\int_{\Omega\times E} 1_B(x)\varphi(\lambda)d\mu_n = \int_{\Omega\times E} 1_B(x)\varphi(\lambda)d\mu \ \forall (B,\varphi)\in \mathscr{B}(\Omega)\times \mathbf{C}_b(E).$$

The space $\mathscr{Y}(\Omega;E)$ is closed in $\mathbf{M}(\Omega\times E)$ equipped with the narrow convergence; more precisely, we have the next proposition.

Proposition 4.3.1. *Let $(\mu_n)_{n\in\mathbf{N}}$ be a sequence in $\mathscr{Y}(\Omega;E)$ narrowly converging to some μ in $\mathbf{M}(\Omega\times E)$. Then μ belongs to $\mathscr{Y}(\Omega;E)$.*

PROOF. From Remark 4.3.1, taking the test function $(x,\lambda)\mapsto \varphi(x,\lambda) = 1_B(x)1_E(\lambda)$, we obtain

$$\mathscr{L}(B) = \lim_{n\to+\infty} \mu_n(B\times E) = \mu(B\times E)$$

for all Borel subsets B of Ω. □

4.3.2 ▪ Slicing Young measures

According to Theorem 4.2.4, for each Young measure μ corresponds a unique family $(\mu_x)_{x\in\Omega}$ (up to equality a.e.) of probability measures on E such that $\mu = (\mu_x)_{x\in\Omega}\otimes\mathscr{L}$. Moreover, the map $x\mapsto\mu_x$ is measurable in the following sense:

$$\forall h\in\mathbf{C}_0(E),\ x\mapsto \int_E h\,d\mu_x \text{ is measurable.}$$

A Young measure μ is then also called a *parametrized measure* and Ω is the set of the parameters.

Let $L_w(\Omega,\mathbf{M}(E))$ be the space of all families $(\mu_x)_{x\in\Omega}$ of measures $\mu_x\in\mathbf{M}(E)$ (not necessarily probability measures) such that $x\mapsto\mu_x$ is measurable in the previous sense. By identifying $\mu = (\mu_x)_{x\in\Omega}\otimes\mathscr{L}$ with $(\mu_x)_{x\in\Omega}$, we have $\mathscr{Y}(\Omega;E)\subset L_w(\Omega,\mathbf{M}(E))$. We equip $L_w(\Omega,\mathbf{M}(E))$ with the following weak convergence:

$$(\mu_x^n)_{x\in\Omega} \overset{L_w}{\rightharpoonup} (\mu_x)_{x\in\Omega} \Longleftrightarrow \forall\ h\in\mathbf{C}_0(E),\ \int_E h\,d\mu_x^n \rightharpoonup \int_E h\,d\mu_x \text{ in } L^\infty(\Omega) \text{ weak star,}$$

i.e.,

$$\int_\Omega g(x)\left(\int_E h\,d\mu_x^n\right)dx \to \int_\Omega g(x)\left(\int_E h\,d\mu_x\right)dx\ \forall\ h\in\mathbf{C}_0(E) \text{ and } \forall g\in L^1(\Omega).$$

Remark 4.3.2. The set $\mathscr{Y}(\Omega;E)$ is not closed in $L_w(\Omega,\mathbf{M}(E))$ equipped with this convergence. Indeed, take $\Omega=(0,1)$, $E=\mathbf{R}$, and $\mu^n = \delta_n\otimes\mathscr{L}$. Then $(\mu_x^n)_{x\in\Omega}$ is the constant family δ_n and $(\mu_x^n)_{x\in\Omega}\overset{L_w}{\rightharpoonup} 0$ which is not a Young measure.

Let us define the tightness notion for Young measures.

Definition 4.3.2. *A subset $\mathscr{H}$ of $\mathscr{Y}(\Omega;E)$ is said to be tight if*

$$\forall\varepsilon>0,\ \exists\mathscr{K}_\varepsilon \textit{ compact subset of } E \textit{ such that } \sup_{\mu\in\mathscr{H}} \mu(\Omega\times(E\setminus\mathscr{K}_\varepsilon)) < \varepsilon.$$

Tight subsets of $\mathscr{Y}(\Omega;E)$ are closed in $L_w(\Omega,\mathbf{M}(E))$. More precisely, we have the next proposition.

Proposition 4.3.2. *Let $(\mu^n)_{n\in\mathbf{N}}$ be a tight sequence in $\mathscr{Y}(\Omega;E)$ with $\mu^n=(\mu^n_x)_{x\in\Omega}\otimes\mathscr{L}$ and assume that $(\mu^n_x)_{x\in\Omega}\overset{L_w}{\rightharpoonup}(\mu_x)_{x\in\Omega}$ in $L_w(\Omega,\mathbf{M}(E))$. Then for a.e. x in Ω, μ_x is a probability measure on E so that $(\mu_x)_{x\in\Omega}\otimes\mathscr{L}$ belongs to $\mathscr{Y}(\Omega;E)$.*

PROOF. Since $\sup_{n\in\mathbf{N}}\mu^n(\Omega\times E)=\mathscr{L}(\Omega)<+\infty$, there exists a subsequence (not relabeled) and some $\mu\in\mathbf{M}^+(\Omega\times E)$ such that

$$\mu^n\rightharpoonup\mu \text{ weakly in the sense of measures in } \mathbf{M}(\Omega\times E).$$

We claim that it suffices to prove that $\mu\in\mathscr{Y}(\Omega;E)$. Indeed, assuming $\mu\in\mathscr{Y}(\Omega;E)$ and denoting by $(\nu_x)_{x\in\Omega}$ the family of probability measures associated with μ, by using a density argument, we easily obtain

$$(\mu^n_x)_{x\in\Omega}\overset{L_w}{\rightharpoonup}(\nu_x)_{x\in\Omega}\quad\text{in } L_w(\Omega,\mathbf{M}(E)).$$

Therefore, by unicity of the weak limit in $L_w(\Omega,\mathbf{M}(E))$, up to a Lebesgue negligible subset of Ω, we will obtain $(\nu_x)_{x\in\Omega}=(\mu_x)_{x\in\Omega}$ so that $\mu=(\mu_x)_{x\in\Omega}\otimes\mathscr{L}\in\mathscr{Y}(\Omega;E)$.

We are going to establish that $\mu\in\mathscr{Y}(\Omega;E)$. According to Alexandrov's theorem (Proposition 4.2.3), for all open subsets U of Ω, one has

$$\pi_\Omega\#\mu(U)=\mu(U\times E)\leq\liminf_{n\to+\infty}\mu^n(U\times E)=\mathscr{L}(U).$$

Let now K be any compact subset of Ω and for all $\varepsilon>0$ let $\mathscr{K}_\varepsilon$ be a compact subset of E given by the tightness hypothesis. According to Alexandrov's theorem again, one has

$$\begin{aligned}\pi_\Omega\#\mu(K)=\mu(K\times E)&\geq\mu(K\times\mathscr{K}_\varepsilon)\\&\geq\limsup_{n\to+\infty}\mu^n(K\times\mathscr{K}_\varepsilon)\\&\geq\limsup_{n\to+\infty}\mu^n(K\times E)-\varepsilon\\&=\mathscr{L}(K)-\varepsilon\end{aligned}$$

so that, since ε is arbitrary, $\pi_\Omega\#\mu(K)\geq\mathscr{L}(K)$. Since the measure $\pi_\Omega\#\mu$ is regular, we deduce that $\pi_\Omega\#\mu(B)=\mathscr{L}(B)$ for all Borel subsets B of Ω. ☐

On $\mathscr{Y}(\Omega;E)$ the narrow convergence and the weak convergence of families of corresponding probability measures are equivalent. More precisely, we have the following theorem.

Theorem 4.3.1. *Let $(\mu^n)_{n\in\mathbf{N}}$ be a sequence in $\mathscr{Y}(\Omega;E)$ and $\mu\in\mathscr{Y}(\Omega;E)$ with $\mu^n=(\mu^n_x)_{x\in\Omega}\otimes\mathscr{L}$ and $\mu=(\mu_x)_{x\in\Omega}\otimes\mathscr{L}$. Then*

$$\mu^n\overset{nar}{\rightharpoonup}\mu\iff(\mu^n_x)_{x\in\Omega}\overset{L_w}{\rightharpoonup}(\mu_x)_{x\in\Omega}\ \textit{in } L_w(\Omega,\mathbf{M}(E)).$$

PROOF. Implication $\Longrightarrow$ is straightforward. We now prove the converse implication.

First step. We establish the tightness of $(\mu^n)_{n\in\mathbf{N}}$. Averaging each family, we define the two probability measures on E,

$$\nu_n := \frac{1}{\mathscr{L}(\Omega)}\int_\Omega \mu_x^n\,dx, \quad \nu := \frac{1}{\mathscr{L}(\Omega)}\int_\Omega \mu_x\,dx,$$

which act on all $\varphi \in \mathbf{C}_0(E)$ as follows:

$$\langle \nu_n, \varphi\rangle := \frac{1}{\mathscr{L}(\Omega)}\int_\Omega \left(\int_E \varphi(\lambda)\,d\mu_x^n(\lambda)\right)dx, \quad \langle \nu, \varphi\rangle := \frac{1}{\mathscr{L}(\Omega)}\int_\Omega \left(\int_E \varphi(\lambda)\,d\mu_x(\lambda)\right)dx.$$

Thus, the weak convergence of $(\mu_x^n)_{x\in\Omega}$ toward $(\mu_x)_{x\in\Omega}$ in $L_w(\Omega,\mathbf{M}(E))$ yields the weak convergence of ν_n toward ν in $\mathbf{M}(E)$. According to the regularity property satisfied by ν, for arbitrary $\varepsilon > 0$, there exists a compact subset $\mathscr{K}_\varepsilon$ of E such that $\nu(E\setminus\mathscr{K}_\varepsilon) < \varepsilon$. From Lemma 4.2.1, one may assume $\nu(\partial\mathscr{K}_\varepsilon) = 0$ so that, according to Alexandrov's theorem, Proposition 4.2.3,

$$\nu_n(\mathscr{K}_\varepsilon) \to \nu(\mathscr{K}_\varepsilon),$$

and, since ν_n and ν are probability measures,

$$\nu_n(E\setminus\mathscr{K}_\varepsilon) \to \nu(E\setminus\mathscr{K}_\varepsilon).$$

We then deduce $\sup_{n\ge N_\varepsilon} \nu_n(E\setminus\mathscr{K}_\varepsilon) < 2\varepsilon$ for a certain N_ε in $\mathbf{N}$. Our claim then follows from $\mu^n(\Omega\times(E\setminus\mathscr{K}_\varepsilon)) = \mathscr{L}(\Omega)\nu_n(E\setminus\mathscr{K}_\varepsilon)$.

Second step. We establish $\mu^n \overset{nar}{\rightharpoonup} \mu$. According to Remark 4.3.1, it suffices to prove

$$\lim_{n\to+\infty}\int_{\Omega\times E} 1_B(x)\varphi(\lambda)\,d\mu^n(x,\lambda) = \int_{\Omega\times E} 1_B(x)\varphi(\lambda)\,d\mu(x,\lambda) \quad \forall B\in\mathscr{B}(\Omega),\ \forall\varphi\in\mathbf{C}_b(E).$$

For $\varepsilon > 0$ let $\mathscr{K}_\varepsilon$ be the compact subset of E given by the tightness of $(\mu^n,\mu)_{n\in\mathbf{N}}$ and consider $\phi_\varepsilon \in \mathbf{C}_c(E)$ satisfying $0\le\phi_\varepsilon\le 1$ and $\phi_\varepsilon = 1$ on $\mathscr{K}_\varepsilon$. We now write

$$\begin{aligned}
&\left|\int_{\Omega\times E} 1_B(x)\varphi(\lambda)\,d\mu^n(x,\lambda) - \int_{\Omega\times E} 1_B(x)\varphi(\lambda)\,d\mu(x,\lambda)\right|\\
&\le \left|\int_{\Omega\times E} 1_B(x)\varphi(\lambda)\,d\mu^n(x,\lambda) - \int_{\Omega\times E} 1_B(x)\phi_\varepsilon(\lambda)\varphi(\lambda)\,d\mu^n(x,\lambda)\right|\\
&+ \left|\int_{\Omega\times E} 1_B(x)\phi_\varepsilon(\lambda)\varphi(\lambda)\,d\mu^n(x,\lambda) - \int_{\Omega\times E} 1_B(x)\phi_\varepsilon(\lambda)\varphi(\lambda)\,d\mu(x,\lambda)\right|\\
&+ \left|\int_{\Omega\times E} 1_B(x)\phi_\varepsilon(\lambda)\varphi(\lambda)\,d\mu(x,\lambda) - \int_{\Omega\times E} 1_B(x)\varphi(\lambda)\,d\mu(x,\lambda)\right|.
\end{aligned} \tag{4.9}$$

According to the tightness of $(\mu^n,\mu)_{n\in\mathbf{N}}$, the first and the last term in the right-hand side of (4.9) are less than $\varepsilon\|\varphi\|_\infty$. On the other hand, since $\phi_\varepsilon\varphi \in \mathbf{C}_0(E)$, by hypothesis, the second term tends to zero when n goes to $+\infty$. Therefore, since ε is arbitrary, we end the proof by letting $n\to+\infty$ in (4.9). $\square$

4.3.3 ▪ Prokhorov's compactness theorem

The theorem below may be considered as a parametrized version of the classical Prokhorov compactness Theorem 4.2.3.

Theorem 4.3.2 (Prokhorov's compactness theorem for Young measures). *Let $(\mu^n)_{n\in\mathbf{N}}$ be a tight sequence in $\mathscr{Y}(\Omega;E)$. Then there exists a subsequence $(\mu^{n_k})_{k\in\mathbf{N}}$ of $(\mu^n)_{n\in\mathbf{N}}$ and μ in $\mathscr{Y}(\Omega;E)$ such that*

$$\mu^{n_k} \overset{nar}{\rightharpoonup} \mu \text{ in } \mathscr{Y}(\Omega;E).$$

PROOF. Since $\sup_{n\in\mathbf{N}} \mu^n(\Omega\times E) = \mathscr{L}(\Omega) < +\infty$, there exists a subsequence (not relabeled) and $\mu \in \mathbf{M}^+(\Omega\times E)$ such that $\mu^n \rightharpoonup \mu$ weakly in the sense of measures in $\mathbf{M}(\Omega\times E)$. Since $(\mu^n)_{n\in\mathbf{N}}$ is tight, arguing as in the proof of Proposition 4.3.2, one may assert that μ belongs to $\mathscr{Y}(\Omega;E)$. It remains to establish the narrow convergence of μ^n toward μ, or, equivalently, according to Theorem 4.3.1, the weak convergence of $(\mu^n_x)_{x\in\Omega}$ toward $(\mu_x)_{x\in\Omega}$ in $L_w(\Omega,\mathbf{M}(E))$. Let $\Phi\in L^1(\Omega)$, $\varphi\in\mathbf{C}_0(E)$, and $\Phi_\varepsilon\in\mathbf{C}_c(\Omega)$ satisfying

$$\int_\Omega |\Phi-\Phi_\varepsilon|\,dx < \varepsilon. \tag{4.10}$$

Since μ^n weakly converges to μ in $\mathbf{M}(\Omega\times E)$ and $(x,\lambda)\mapsto\Phi_\varepsilon(x)\varphi(\lambda)$ belongs to $\mathbf{C}_0(\Omega\times E)$, according to the slicing Theorem 4.2.4, one has

$$\lim_{n\to+\infty}\left|\int_\Omega \Phi_\varepsilon(x)\left(\int_E \varphi(\lambda)\,d\mu^n_x\right)dx - \int_\Omega \Phi_\varepsilon(x)\left(\int_E \varphi(\lambda)\,d\mu_x\right)dx\right| = 0. \tag{4.11}$$

Let us write

$$\begin{aligned}
&\left|\int_\Omega \Phi(x)\left(\int_E \varphi(\lambda)\,d\mu^n_x\right)dx - \int_\Omega \Phi(x)\left(\int_E \varphi(\lambda)\,d\mu_x\right)dx\right|\\
&\le\left|\int_\Omega \Phi(x)\left(\int_E \varphi(\lambda)\,d\mu^n_x\right)dx - \int_\Omega \Phi_\varepsilon(x)\left(\int_E \varphi(\lambda)\,d\mu^n_x\right)dx\right|\\
&+\left|\int_\Omega \Phi_\varepsilon(x)\left(\int_E \varphi(\lambda)\,d\mu^n_x\right)dx - \int_\Omega \Phi_\varepsilon(x)\left(\int_E \varphi(\lambda)\,d\mu_x\right)dx\right|\\
&+\left|\int_\Omega \Phi_\varepsilon(x)\left(\int_E \varphi(\lambda)\,d\mu_x\right)dx - \int_\Omega \Phi(x)\left(\int_E \varphi(\lambda)\,d\mu_x\right)dx\right|.
\end{aligned} \tag{4.12}$$

From (4.10), the first and the last term of the right-hand side of (4.12) are less than $\varepsilon\|\varphi\|_\infty$. Since ε is arbitrary, the claim follows from (4.11) by letting $n\to+\infty$ in (4.12). □

4.3.4 ▪ Young measures associated with functions and generated by functions

Let $u:\Omega\to E$ be a given Borel function and consider the image $\mu = G\#\mathscr{L}$ of the measure $\mathscr{L}$ by the graph function $G:\Omega\to\Omega\times E,\ x\mapsto(x,u(x))$.

Since the image of the measure μ by the projection π_Ω on Ω is the Lebesgue measure $\mathscr{L}$, μ belongs to $\mathscr{Y}(\Omega;E)$. This measure, concentrated on the graph of u, is called the *Young measure associated with the function u*. By definition of the image of a measure,

μ "acts" on $\mathbf{C}_b(\Omega;E)$ as follows:

$$\forall \varphi \in \mathbf{C}_b(\Omega;E), \quad \int_{\Omega\times E} \varphi(x,\lambda)\, d\mu(x,\lambda) = \int_\Omega \varphi(x,u(x))\, dx.$$

This shows that the probability family $(\mu_x)_{x\in\Omega}$ associated with μ is $(\delta_{u(x)})_{x\in\Omega}$.

Let $(u_n)_{n\in\mathbf{N}}$ be a sequence of Borel functions $u_n : \Omega \to E$ and consider the sequence of their associated Young measures $(\mu_n)_{n\in\mathbf{N}}$, $\mu_n = (\delta_{u_n(x)})_{x\in\Omega} \otimes \mathscr{L}$. If $\mu_n \overset{nar}{\rightharpoonup} \mu$ in $\mathscr{Y}(\Omega;E)$, the Young measure μ is said to be *generated by* the sequence $(u_n)_{n\in\mathbf{N}}$. In general (see Examples 4.3.1 and 4.3.2), μ is not associated with a function.

Let us rephrase the tightness of a sequence $(\mu_n)_{n\in\mathbf{N}}$ in terms of the associated sequence $(u_n)_{n\in\mathbf{N}}$. We easily obtain the following equivalence: the sequence $(\mu_n)_{n\in\mathbf{N}}$ is tight iff

$$\forall \varepsilon > 0,\ \exists \mathscr{K}_\varepsilon, \text{compact subset of } E, \text{ such that } \sup_{n\in\mathbf{N}} \mathscr{L}\big\{x\in\Omega \,:\, u_n(x) \in E\setminus \mathscr{K}_\varepsilon\big\} < \varepsilon.$$

Remark 4.3.3. It is worth noticing that a sequence $(\mu_n)_{n\in\mathbf{N}}$ of Young measures associated with a bounded sequence $(u_n)_{n\in\mathbf{N}}$ in $L^1(\Omega,E)$ is tight. Indeed according to the Markov inequality, one has

$$\begin{aligned}\mathscr{L}\big\{x\in\Omega \,:\, |u_n(x)| > M\big\} &\le \frac{1}{M}\int_\Omega |u_n|\, dx \\ &\le \frac{1}{M}\sup_{n\in\mathbf{N}}\int_\Omega |u_n|\, dx,\end{aligned}$$

which tends to zero when $M \to +\infty$. Therefore, according to Prokhorov's theorem, Theorem 4.3.2, for each bounded sequence $(u_n)_{n\in\mathbf{N}}$ in $L^1(\Omega,E)$, one can extract a subsequence generating a Young measure μ, i.e.,

$$(\delta_{u_n(x)})_{x\in\Omega} \otimes \mathscr{L} \overset{nar}{\rightharpoonup} \mu.$$

4.3.5 ▪ Semicontinuity and continuity properties

Here is a first semicontinuity result related to extended real-valued nonnegative functions.

Proposition 4.3.3. *Let $\varphi : \Omega\times E \to [0,+\infty]$ be a $\mathscr{B}(\Omega)\otimes\mathscr{B}(E)$ measurable function such that $\lambda \mapsto \varphi(x,\lambda)$ is lsc for a.e. x in Ω. Moreover, let $(\mu_n)_{n\in\mathbf{N}}$ be a sequence of Young measures in $\mathscr{Y}(\Omega;E)$, narrowly converging to some Young measure μ in $\mathscr{Y}(\Omega;E)$. Then*

$$\int_{\Omega\times E} \varphi(x,\lambda)\, d\mu(x,\lambda) \le \liminf_{n\to+\infty} \int_{\Omega\times E} \varphi(x,\lambda)\, d\mu_n(x,\lambda).$$

PROOF. Let us consider the Lipschitz regularization of φ

$$\varphi_p(x,\lambda) := \inf_{\xi\in E}\big\{\varphi(x,\xi) + p|\lambda-\xi|\big\}$$

for $p \in \mathbf{N}$ intended to go to $+\infty$, and set $\psi_p = \varphi_p \wedge p$. It is easily seen that ψ belongs to $\mathbf{C}_b(\Omega;E)$ (see Theorem 9.2.1) and that $(\psi_p)_{p\in\mathbf{N}}$ is a nondecreasing sequence which pointwise converges to φ. Consequently,

$$\int_{\Omega\times E} \psi_p(x,\lambda)\, d\mu(x,\lambda) = \lim_{n\to+\infty} \int_{\Omega\times E} \psi_p(x,\lambda)\, d\mu_n(x,\lambda)$$
$$\leq \liminf_{n\to+\infty} \int_{\Omega\times E} \varphi(x,\lambda)\, d\mu_n(x,\lambda),$$

and we complete the proof thanks to the monotone convergence theorem by letting $p \to +\infty$ in the left-hand side. □

We would like to improve Proposition 4.3.3 for functions φ taking negative values or more generally for functions φ which are not necessarily bounded from below. We restrict ourselves to sequences of Young measures associated with functions. Let us first recall the notion of uniform integrability: a sequence $(f_n)_{n\in\mathbf{N}}$ of functions $f_n : \Omega \to \mathbf{R}$ in $L^1(\Omega)$ is said to be *uniformly integrable* if

$$\lim_{R\to+\infty} \sup_{n\in\mathbf{N}} \int_{[|f_n|>R]} |f_n| = 0.$$

From Proposition 2.4.12 this definition is equivalent to Definition 2.4.4 (see also Delacherie and Meyer [198]).

Proposition 4.3.4. *Let $(\mu_n)_{n\in\mathbf{N}}$ be a sequence of Young measures associated with a sequence of functions $(u_n)_{n\in\mathbf{N}}$, narrowly converging to some Young measure μ. On the other hand, let $\varphi : \Omega \times E \to \mathbf{R}$ be a $\mathscr{B}(\Omega)\otimes\mathscr{B}(E)$ measurable function such that $\lambda \mapsto \varphi(x,\lambda)$ is lower semicontinuous for a.e. x in Ω. Assume moreover that the negative part $x \mapsto \varphi(x,u_n(x))^-$ is uniformly integrable. Then*

$$\int_{\Omega\times E} \varphi(x,\lambda)\, d\mu(x,\lambda) \leq \liminf_{n\to+\infty} \int_\Omega \varphi(x,u_n(x))\, dx.$$

PROOF. Let $R>0$ intended to tend to $+\infty$ and set $\varphi_R = \sup(-R,\varphi)+R$. Since $\varphi_R \geq 0$ and $\lambda \mapsto \varphi_R(x,\lambda)$ is lower semicontinuous, one may apply Proposition 4.3.3 so that, removing the term $R\mathscr{L}(\Omega)$, one obtains

$$\int_{\Omega\times E} \varphi(x,\lambda)\, d\mu(x,\lambda) \leq \int_{\Omega\times E} \sup(-R,\varphi(x,\lambda))\, d\mu(x,\lambda)$$
$$\leq \liminf_{n\to+\infty} \int_{\Omega\times E} \sup(-R,\varphi(x,\lambda))\, d\mu_n(x,\lambda)$$
$$= \liminf_{n\to+\infty} \int_\Omega \sup(-R,\varphi(x,u_n(x)))\, dx. \tag{4.13}$$

On the other hand,

$$\int_\Omega \sup(-R,\varphi(x,u_n(x)))\, dx = \int_{[\varphi(.,u_n(.))\geq -R]} \varphi(x,u_n(x))\, dx + \int_{[\varphi(.,u_n(.))<-R]} (-R)\, dx$$
$$\leq \int_\Omega \varphi(x,u_n(x))dx - \int_{[\varphi(.,u_n(.))<-R]} \varphi(x,u_n(x))dx. \tag{4.14}$$

But

$$\begin{aligned}
-\int_{[\varphi(.,u_n(.))<-R]} \varphi(x,u_n(x))\,dx &= \int_{[\varphi(.,u_n(.))<-R]} \varphi(x,u_n(x))^-\,dx \\
&= \int_{[\varphi(.,u_n(.))^->R]} \varphi(x,u_n(x))^-\,dx \\
&\le \sup_{n\in\mathbf{N}} \int_{[\varphi(.,u_n(.))^->R]} \varphi(x,u_n(x))^-\,dx, \qquad (4.15)
\end{aligned}$$

which, by hypothesis, tends to 0 when $R \to +\infty$. We end the proof by collecting (4.13), (4.14), and (4.15) and letting $R \to +\infty$. □

As a straightforward consequence of Proposition 4.3.4, we obtain the following useful theorem, a key tool in relaxation theory (see Section 11.4.2).

Theorem 4.3.3. *Let $(\mu_n)_{n\in\mathbf{N}}$ be a sequence of Young measures associated with a sequence of functions $(u_n)_{n\in\mathbf{N}}$, narrowly converging to some Young measure μ. On the other hand, let $\varphi : \Omega \times E \to \mathbf{R}$ be a $\mathscr{B}(\Omega)\otimes\mathscr{B}(E)$ measurable function such that $\lambda \mapsto \varphi(x,\lambda)$ is continuous for a.e. x in Ω. Assume moreover that $x \mapsto \varphi(x,u_n(x))$ is uniformly integrable. Then*

$$\int_{\Omega\times E} \varphi(x,\lambda)\,d\mu(x,\lambda) = \lim_{n\to+\infty} \int_\Omega \varphi(x,u_n(x))\,dx.$$

PROOF. Since $x \mapsto \varphi(x,u_n(x))$ is uniformly integrable, $x \mapsto \varphi(x,u_n(x))^-$ is uniformly integrable so that, according to Proposition 4.3.4,

$$\int_{\Omega\times E} \varphi(x,\lambda)\,d\mu(x,\lambda) \le \liminf_{n\to+\infty} \int_\Omega \varphi(x,u_n(x))\,dx.$$

Let us set $\tilde{\varphi} := -\varphi$. Since $x \mapsto \tilde{\varphi}(x,u_n(x))^-$ is equal to $x \mapsto \varphi(x,u_n(x))^+$, which is also uniformly integrable, we infer from Proposition 4.3.4

$$\int_{\Omega\times E} \tilde{\varphi}(x,\lambda)\,d\mu(x,\lambda) \le \liminf_{n\to+\infty} \int_\Omega \tilde{\varphi}(x,u_n(x))\,dx,$$

that is,

$$\int_{\Omega\times E} \varphi(x,\lambda)\,d\mu(x,\lambda) \ge \limsup_{n\to+\infty} \int_\Omega \varphi(x,u_n(x))\,dx,$$

which completes the proof. □

Remark 4.3.4. Theorem 4.3.3 is often applied when $(\psi(.,u_n(.)))_{n\in\mathbf{N}}$ is a sequence of real-valued functions weakly converging in $L^1(\Omega)$. Indeed, according to the Dunford–Pettis theorem, Theorem 2.4.5, there is equivalence between the weak relative compactness of the sequence $(\psi(.,u_n(.)))_{n\in\mathbf{N}}$ in $L^1(\Omega)$ and its uniform integrability.

4.3.6 ▪ Young measures capture oscillations

Let $(\mu_n)_{n\in\mathbf{N}}$ be a sequence of Young measures associated with a sequence $(u_n)_{n\in\mathbf{N}}$ of Borel functions $u_n : \Omega \to E$ and assume that $\mu_n \overset{nar}{\rightharpoonup} \mu$ in $\mathscr{Y}(\Omega;E)$. Let us show that, roughly speaking, for a.e. x_0 in Ω, each μ_{x_0} gives the limit probability distribution of the values of u_n when x are taken randomly, with the uniform probability law around x_0. Indeed, it is easily seen that the slicing Theorem 4.2.4 holds for functions $(x,\lambda)\mapsto f(x,\lambda)$ of the form $(x,\lambda)\mapsto 1_{B_\rho(x_0)\times A}(x,\lambda)$, where $B_\rho(x_0)$ is the open ball with radius $\rho>0$ centered at $x_0\in\Omega$, and A is any open set of E. Then we have

$$\mu(B_\rho(x_0)\times A)=\int_{\Omega\times E} 1_{B_\rho(x_0)\times A}(x,\lambda)=\int_{B_\rho(x_0)}\mu_x(A)\,dx.$$

Therefore, according to Lebesgue's differentiation theorem, for a.e. $x_0\in\Omega$,

$$\mu_{x_0}(A)=\lim_{\rho\to 0}\frac{1}{\mathscr{L}(B_\rho(x_0))}\mu(B_\rho(x_0)\times A). \tag{4.16}$$

Note that the negligible set of all the points x_0 for which (4.16) does not hold depends on the choice of A in E. But reasoning with a given countable family of sets A, there exists a negligible set $\mathscr{N}$ in Ω such that for all $x_0\in\Omega\setminus\mathscr{N}$, (4.16) holds for all set A of this family. Let then $x_0\in\Omega\setminus\mathscr{N}$ and A be chosen so that (4.16) holds. Choose moreover $\rho>0$ such that $\mu(\partial(B_\rho(x_0)\times A))=0$. Such a choice is possible thanks to Lemma 4.2.1. Since $\mu_n\overset{nar}{\rightharpoonup}\mu$, in particular μ_n weakly converges in the sense of measures in $\mathbf{M}(\Omega\times E)$ so that, according to Theorem 4.2.3,

$$\begin{aligned}\mu(B_\rho(x_0)\times A)&=\lim_{n\to+\infty}\mu_n(B_\rho(x_0)\times A)\\&=\lim_{n\to+\infty}\mathscr{L}\big(\{x\in B_\rho(x_0):u_n(x)\in A\}\big).\end{aligned}\tag{4.17}$$

Collecting (4.16) and (4.17), we finally obtain

$$\mu_{x_0}(A)=\lim_{\rho\to 0}\lim_{n\to+\infty}\frac{\mathscr{L}\big(\{x\in B_\rho(x_0):u_n(x)\in A\}\big)}{\mathscr{L}(B_\rho(x_0))}, \tag{4.18}$$

which proves the thesis.

For a given sequence $(u_n)_{n\in\mathbf{N}}$ the first mode of behavior which can cause a defect of strong convergence is the presence of rapid oscillations in the functions u_n. Estimate (4.18) shows that Young measures capture some information on such oscillations. We will illustrate this property with a few examples.

Let $u:Y=(0,1)^N\to E$ be a given function in $L^p(Y,E)$, $p\geq 1$, extended by Y-periodicity to $\mathbf{R}^N$ and define the sequence $(u_n)_{n\in\mathbf{N}}$ by setting $u_n(x)=u(nx)$ for all $x\in\mathbf{R}^N$. Classically one has

$$u_n\rightharpoonup\overline{u}\text{ in }L^p(\Omega,E),$$

where $\overline{u}$ is the mean value of u defined by $\overline{u}=\int_Y u(y)\,dy$. Obviously, if u is not a constant function, we have neither strong convergence in $L^p(\Omega,\mathbf{R}^N)$ nor a.e. pointwise convergence on Ω of the sequence $(u_n)_{n\in\mathbf{N}}$ toward $\overline{u}$. Let $(\mu_n)_{n\in\mathbf{N}}$ denote the sequence of Young measures associated with the sequence $(u_n)_{n\in\mathbf{N}}$, i.e., $\mu_n=(\delta_{u_n(x)})_{x\in\Omega}\otimes\mathscr{L}$. Then we have the next proposition.

Proposition 4.3.5. *The sequence $(\mu_n)_{n\in\mathbf{N}}$ narrowly converges to $\mu=(\mu_x)_{x\in\Omega}\otimes\mathscr{L}$ in $\mathscr{Y}(\Omega;E)$, where, for a.e. x in Ω, the probability measure μ_x is the image $u\#\mathscr{L}\lfloor Y$ of the*

Lebesgue measure $\mathscr{L}\lfloor Y$ by the function u. In other words, μ acts on all $\varphi \in \mathbf{C}_b(\Omega; E)$ as follows:

$$\int_{\Omega \times E} \varphi(x,\lambda)\, d\mu(x,\lambda) = \int_\Omega \left(\int_Y \varphi(x, u(y))\, dy \right) dx.$$

PROOF. It is enough to establish

$$\lim_{n\to+\infty} \int_{\Omega\times E} \varphi(x,\lambda)\, d\mu^n(x,\lambda) = \int_\Omega \left(\int_Y \varphi(x,u(y))\, dy \right) dx$$

when φ is of the form $\varphi(x,\lambda) = 1_B(x)\phi(\lambda)$, where B belongs to $\mathscr{B}(\Omega)$ and ϕ is a bounded continuous function on E (see Remark 4.3.1).

Since, classically, $x \mapsto \phi(u(nx))$ weakly converges to $\int_Y \phi(u(y))\, dy$ in $L^\infty(\Omega)$ weak star, we have

$$\begin{aligned} \lim_{n\to+\infty} \int_{\Omega\times E} \varphi(x,\lambda)\, d\mu^n(x,\lambda) &= \lim_{n\to+\infty} \int_\Omega 1_B(x)\phi(u(nx))\, dx \\ &= \int_\Omega 1_B(x) \left(\int_Y \phi(u(y))\, dy \right) dx \\ &= \int_\Omega \left(\int_Y \varphi(x,u(y))\, dy \right) dx \end{aligned}$$

and the thesis is proved. □

Example 4.3.1. Take $u : (0,1) \to \mathbf{R}$ the function defined by

$$u(x) = \begin{cases} -1 & \text{if } x \in (0, \frac{1}{2}), \\ +1 & \text{if } x \in (\frac{1}{2}, 1), \end{cases}$$

and consider the sequence $(u_n)_{n\in\mathbf{N}}$ defined as previously with, for instance, $\Omega = (0,1)$. Using Proposition 4.3.5 (or (4.18)), it is easily seen that $\mu_x = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}$ so that μ is the measure on $(0,1)\times\mathbf{R}$, concentrated on the union of the two segments $(0,1)\times\{-1\}\cup(0,1)\times\{1\}$, with the mass 1/2 on each two segments. Therefore, the measure μ, encoding the oscillations, is not associated with a function.

Example 4.3.2. Take $u : (0,1) \to \mathbf{R}$ the function defined by $u(x) = \sin(2\pi x)$ and consider the sequence $(u_n)_{n\in\mathbf{N}}$ defined as previously with, for instance, $\Omega = (0,1)$. An elementary calculation gives $\mu_x = \frac{1}{\pi\sqrt{1-y^2}} \mathscr{L}\lfloor(-1,1)$ so that μ is not associated with a function but concentrated on all the rectangle $(0,1)\times(-1,1)$.

The next proposition shows that Young measures encode oscillations on the weak limits.

Proposition 4.3.6. *Let $(u_n)_{n\in\mathbf{N}}$ be a given sequence of functions in $L^p(\Omega,E)$, weakly converging to some u in $L^p(\Omega,E)$, $p \geq 1$, and assume that the sequence $(\mu_n)_{n\in\mathbf{N}}$ of their associated Young measures narrowly converges to some Young measure μ. Then, for a.e. x in Ω, $u(x)$ is the barycenter (or the expectation) of the probability measure μ_x:*

$$u(x) = \int_E \lambda d\mu_x(\lambda).$$

PROOF. Reasoning with each component of u_n, one may assume, without restrictions, that u_n is a real-valued function and that $E = \mathbf{R}$. Let us first apply Theorem 4.3.3 with φ defined by $\varphi(x,\lambda) = \phi(x)\lambda$, where $\phi \in \mathbf{C}_c(\Omega)$. Since $(u_n)_{n\in\mathbf{N}}$ weakly converges in $L^p(\Omega)$, the sequence $\varphi(.,u_n(.))_{n\in\mathbf{N}}$ is uniformly integrable (see Remark 4.3.4) so that, according to Theorem 4.3.3,

$$\begin{aligned}\int_{\Omega\times E} \phi(x)\lambda \, d\mu(x,\lambda) &= \lim_{n\to+\infty} \int_\Omega \phi(x)u_n(x)\, dx \\ &= \int_\Omega \phi(x)u(x)\, dx.\end{aligned}$$

According now to Theorem 4.2.4, we infer

$$\int_\Omega \phi(x)\left(\int_{\mathbf{R}} \lambda \, d\mu_x(\lambda)\right) dx = \int_\Omega \phi(x)u(x)\, dx.$$

Since ϕ is arbitrary, one obtains $u(x) = \int_{\mathbf{R}} \lambda \, d\mu_x(\lambda)$ for a.e. x in Ω. □

One can now establish the Dunford–Pettis theorem, Theorem 2.4.5.

Proposition 4.3.7. *Let $(u_n)_{n\in\mathbf{N}}$ be a given sequence of uniformly integrable functions in $L^1(\Omega,E)$. Then there exists a subsequence $(u_{n_k})_{k\in\mathbf{N}}$ and u in $L^1(\Omega,E)$ such that*

$$u_{n_k} \rightharpoonup u \qquad \sigma(L^1,L^\infty).$$

PROOF. Since $(u_n)_{n\in\mathbf{N}}$ is uniformly integrable, one can easily establish that

$$\sup_{n\in\mathbf{N}} \int_\Omega |u_n|\, dx < +\infty$$

so that (see Remark 4.3.3) the sequence of Young measures $\mu_n = (\delta_{u_n(x)})_{x\in\Omega} \otimes \mathscr{L}$ is tight. According to Prokhorov compactness Theorem 4.3.2, there exists a subsequence of $(\mu_n)_{n\in\mathbf{N}}$ (not relabeled) and μ in $\mathscr{Y}(\Omega;E)$ satisfying $\mu_n \overset{nar}{\rightharpoonup} \mu$.

Consider $g \in L^\infty(\Omega,E)$ and set $\varphi(x,\lambda) := g(x).\lambda$. The sequence $(\varphi(x,u_n(x)))_{n\in\mathbf{N}}$ obviously satisfies hypotheses of Theorem 4.3.3 so that

$$\lim_{n\to+\infty} \int_\Omega g(x).u_n(x)\, dx = \int_\Omega g(x).\left(\int_E \lambda \, d\mu_x\right) dx.$$

The barycenter

$$u : x \mapsto \int_E \lambda \, d\mu_x$$

then satisfies $u_n \rightharpoonup u \ \ \sigma(L^1,L^\infty)$. □

4.3.7 ▪ Young measures do not capture concentrations

Another mode of behavior which causes a defect of strong convergence for a sequence $(u_n)_{n\in\mathbf{N}}$ weakly converging to some u in $L^p(\Omega,E)$ is the *concentration effect*. Such concentration effects appear when $u_n - u$ converges to zero in measure and when the total mass $\int_\Omega |u_n - u|^p\, dx$ is concentrated at the limit to a set of zero Lebesgue measure. Note that in the examples of Section 4.3.6 no concentration effects occurred because the sequences $(u_n)_{n\in\mathbf{N}}$ did not converge in measure. Let us illustrate the concentration phenomenon with the following elementary example.

Let $\Omega=(-1,1)$, $p=2$ and consider the real-valued function u_n, $n\in\mathbf{N}^*$, defined by

$$u_n(x)=\begin{cases}\sqrt{n} & \text{if } x\in(-\frac{1}{n},\frac{1}{n}),\\ 0 & \text{otherwise.}\end{cases}$$

It is easily seen that u_n converges to 0 a.e. in $(-1,1)$, in measure, and weakly in $L^2(-1,1)$. On the other hand, the total mass is $\int_{(-1,1)}|u_n|^2\,dx=2$, while the measure $|u_n|^2\mathscr{L}$ weakly converges to $2\delta_0$ in the sense of measures in $\mathbf{M}(-1,1)$. Therefore the total mass is concentrated at the point $x=0$.

Proposition 4.3.8 shows that Young measures generated by sequences converging in measure are trivial and therefore do not capture concentration effects. For the analysis of both oscillations and concentration effects, see [346] and [218].

Proposition 4.3.8. *A sequence $(u_n)_{n\in\mathbf{N}}$ of Borel functions converges to u in measure iff the associated sequence of Young measures $(\mu_n)_{n\in\mathbf{N}}$ narrowly converges to the Young measure associated with u, i.e., $\mu=(\delta_{u(x)})_{x\in\Omega}\otimes\mathscr{L}$.*

On the other hand, let $(\mu_n)_{n\in\mathbf{N}}$ be a sequence of Young measures associated with a sequence of Borel functions $(u_n)_{n\in\mathbf{N}}$, narrowly converging to some Young measure μ. Moreover let $(v_n)_{n\in\mathbf{N}}$ be another sequence of Borel functions $v_n:\Omega\to E$ such that v_n-u_n converges to 0 in measure. Then the sequence $(v_n)_{n\in\mathbf{N}}$ generates the same Young measure μ. In other words, $v_n=(v_n-u_n)+u_n$ generates the Young measure μ generated by u_n so that the perturbation by v_n-u_n, for which a concentration phenomenon may occur, has no effect on μ.

PROOF. *First step.* We first claim

$$u_n \text{ converges in measure toward } u \Longrightarrow (\delta_{u_n(x)})_{x\in\Omega}\xrightarrow{L_w}(\delta_{u(x)})_{x\in\Omega},$$

which, according to Theorem 4.3.1, is equivalent to the narrow convergence of corresponding Young measures. Now, after using an easy density argument, it is enough to test the convergence with $\varphi(x,\lambda)=1_B(x)\phi(\lambda)$, where $B\in\mathscr{B}(\Omega)$ and $\phi\in\mathbf{C}_0(E)$. Let $\varepsilon>0$ given arbitrary. From uniform continuity of ϕ, there exists $\eta>0$ such that

$$|\lambda-\lambda'|<\eta\Longrightarrow|\phi(\lambda)-\phi(\lambda')|<\varepsilon.$$

Let us write

$$\begin{aligned}&\left|\int_\Omega\varphi(x,u_n(x))\,dx-\int_\Omega\varphi(x,u(x))\,dx\right|\le\int_\Omega|\phi(u_n(x))-\phi(u(x))|\,dx\\ &=\int_{[|u_n-u|>\eta]}|\phi(u_n(x))-\phi(u(x))|\,dx+\int_{[|u_n-u|\le\eta]}|\phi(u_n(x))-\phi(u(x))|\,dx\\ &\le 2\|\phi\|_\infty\mathscr{L}([|u_n-u|>\eta])+\varepsilon\mathscr{L}(\Omega).\end{aligned}\tag{4.19}$$

Now, by hypothesis, $\lim_{n\to+\infty}\mathscr{L}([|u_n-u|>\eta])=0$ and, since ε is arbitrary, the claim follows after letting $n\to+\infty$ in (4.19).

Second step. We establish the converse implication. Let us consider $\varphi\in\mathbf{C}_b(\Omega;E)$ defined by $\varphi(x,\lambda)=|\lambda-u(x)|\wedge C$, where C is any positive constant. Since $\mu_n\xrightarrow{nar}\mu$, one has

$$\lim_{n\to+\infty}\int_{\Omega\times E}\varphi(x,\lambda)\,d\mu_n(x,\lambda)=\int_{\Omega\times E}\varphi(x,\lambda)\,d\mu(x,\lambda),$$

that is,

$$\lim_{n\to+\infty}\int_\Omega |u_n(x)-u(x)|\wedge C\, dx = \int_\Omega |u(x)-u(x)|\wedge C\, dx = 0. \tag{4.20}$$

On the other hand, for any $\eta > 0$,

$$\mathscr{L}([|u_n-u|>\eta]) \le \frac{1}{\min(\eta, C)}\int_\Omega |u_n(x)-u(x)|\wedge C\, dx.$$

Consequently, (4.20) yields $\lim_{n\to+\infty}\mathscr{L}([|u_n-u|>\eta])=0$.

Last step. We establish the second assertion. As previously, according to Theorem 4.3.1, it suffices to establish

$$\lim_{n\to+\infty}\int_B \phi(v_n)\,dx = \int_{\Omega\times E} 1_B(x)\phi(\lambda)\,d\mu(x,\lambda)$$

for all Borel subsets B of Ω and all $\phi\in\mathbf{C}_c(E)$.

Let $\varepsilon > 0$. Since ϕ is uniformly continuous on E, there exists $\eta > 0$ such that $|\phi(v_n(x))-\phi(u_n(x))| < \varepsilon$ for all x in the set $[|v_n-u_n|<\eta]$. On the other hand, since v_n-u_n tends to 0 in measure, $\lim_{n\to+\infty}\mathscr{L}([|v_n-u_n|\ge\eta])=0$. Therefore

$$\begin{aligned}\left|\int_B \phi(v_n)\,dx-\int_B\phi(u_n)\,dx\right| &\le \int_\Omega |\phi(v_n)-\phi(u_n)|\,dx\\ &= \int_{[|v_n-u_n|\ge\eta]}|\phi(v_n)-\phi(u_n)|\,dx\\ &\quad+\int_{[|v_n-u_n|<\eta]}|\phi(v_n)-\phi(u_n)|\,dx\\ &\le 2\|\phi\|_\infty\mathscr{L}([|v_n-u_n|\ge\eta])+\varepsilon\mathscr{L}(\Omega)\end{aligned}$$

and, since ε is arbitrary, we conclude by letting $n\to+\infty$. □