0.1 The continuous and discrete problem

Let $\alpha > 0$, $\Omega \subset \mathbb{R}^n$ bounded polyhedral Lipschitz domain, and $f \in L^2(\Omega)$.

The continuous problem minimizes

$$E(v) := \frac{\alpha}{2}\|v\|^2_{L^2(\Omega)} + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \int_\Omega f\, v \,\mathrm{d}x \tag{0.1}$$

amongst all $v \in V := \mathrm{BV}(\Omega) \cap L^2(\Omega)$ where the BVseminorm $|v|_{\mathrm{BV}(\Omega)}$ is equals to the $W^{1,1}$ seminorm for any $v \in W^{1,1}(\Omega)$.

The nonconforming problem minimizes

$$E_{\mathrm{NC}}(v_{\mathrm{CR}}) := \frac{\alpha}{2}\|v_{\mathrm{CR}}\|^2_{L^2(\Omega)} + |v_{\mathrm{CR}}|_{1,1,\mathrm{NC}} - \int_\Omega f\, v_{\mathrm{CR}} \,\mathrm{d}x \tag{0.2}$$

amongst all $v_{\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T})$ where $|\bullet|_{1,1,\mathrm{NC}} := \|\nabla_{\mathrm{NC}} \bullet\|_{L^1(\Omega)}$.

0.2 Estimator and guaranteed lower energy bound

For some $n \in$ (TODO) (here $n = 2$) and $0 < \beta \leqslant 1$ define the error estimator (TODO there are squares missing, right) $\eta := \sum_{T\in\mathcal{T}} \eta(T)$ with

$$\eta(T) := \underbrace{|T|^{2/n}\|f - \alpha u_{\mathrm{CR}}\|^2_{L^2(T)}}_{=:\eta_{\mathrm{Vol}}(T)} + \underbrace{|T|^{\beta/n} \sum_{F\in\mathcal{F}(T)} \|[u_{\mathrm{CR}}]_F\|_{L^1(F)}}_{=:\eta_{\mathrm{Jumps}}(T)} \tag{0.3}$$

for any $T \in \mathcal{T}$.

For $f \in H^1_0(\Omega)$ and $u \in H^1_0(\Omega)$ $\big(u_{\mathrm{CR}} \in \mathrm{CR}^1_0(\Omega)\big)$ continuous (discrete) minimzer with minimal energy $E(u)$ ($E_{\mathrm{NC}}(u_{\mathrm{CR}})$) it holds

$$E_{\mathrm{NC}}(u_{\mathrm{CR}}) + \frac{\alpha}{2}\|u - u_{\mathrm{CR}}\|^2_{L^2(\Omega)} - \frac{\kappa_{\mathrm{CR}}}{\alpha}\|h_{\mathcal{T}}(f - \alpha u_{\mathrm{CR}})\|_{L^2(\Omega)}|f|_{1,2} \leqslant E(u) \tag{0.4}$$

where $|\bullet|_{1,2} = \|\nabla \bullet\|_{L^2(\Omega)}$.

0.3 Example with exact solution

For $\alpha = \beta = 1$ define f as a function of the radius as

$$f(r) := \begin{cases} \alpha - 12(2 - 9r) & \text{if } 0 \leqslant r \leqslant \frac{1}{6}, \\ \alpha(1 + (6r-1)^\beta) - \frac{1}{r} & \text{if } \frac{1}{6} \leqslant r \leqslant \frac{1}{3}, \\ 2\alpha + 6\pi\sin(\pi(6r-2)) - \frac{1}{r}\cos(\pi(6r-2)) & \text{if } \frac{1}{3} \leqslant r \leqslant \frac{1}{2}, \\ 2\alpha(\frac{5}{2} - 3r)^\beta + \frac{1}{r} & \text{if } \frac{1}{2} \leqslant r \leqslant \frac{5}{6}, \\ -3\pi\sin(\pi(6r-5)) + \frac{1+\cos(\pi(6r-5))}{2r} & \text{if } \frac{5}{6} \leqslant r \leqslant 1, \end{cases} \tag{0.5}$$

with exact solution

$$u(r) := \begin{cases} 1 & \text{if } 0 \leqslant r \leqslant \frac{1}{6}, \\ 1 + (6r-1)^\beta & \text{if } \frac{1}{6} \leqslant r \leqslant \frac{1}{3}, \\ 2 & \text{if } \frac{1}{3} \leqslant r \leqslant \frac{1}{2}, \\ 2(\frac{5}{2} - 3r)^\beta & \text{if } \frac{1}{2} \leqslant r \leqslant \frac{5}{6}, \\ 0 & \text{if } \frac{5}{6} \leqslant r \leqslant 1. \end{cases} \tag{0.6}$$

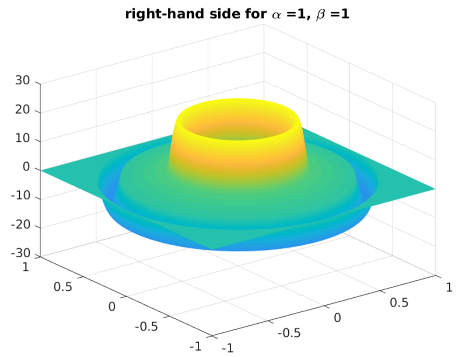

Figure 0.1: right-hand side f

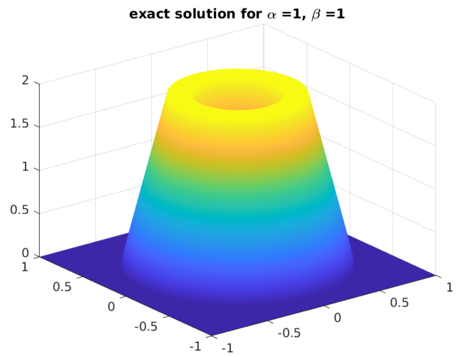

Figure 0.2: exact solution u

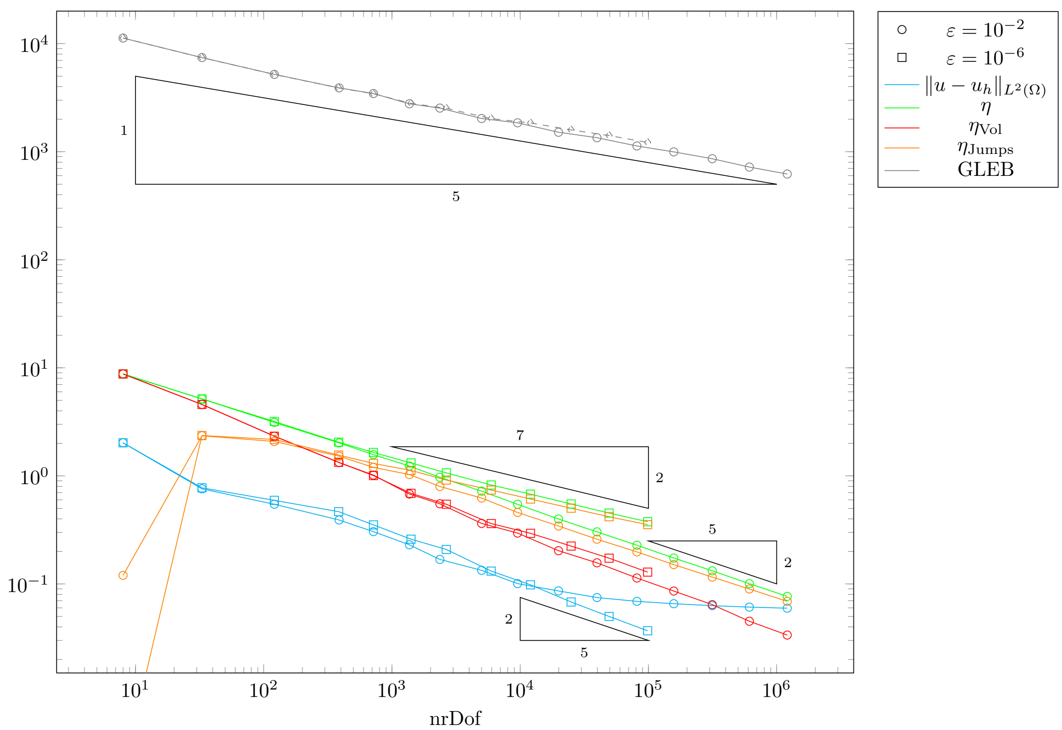

Figure 0.3: convergence history plot for the L^2 error, η, η_{Vol}, η_{Jumps}, and the guaranteed lower energy bound

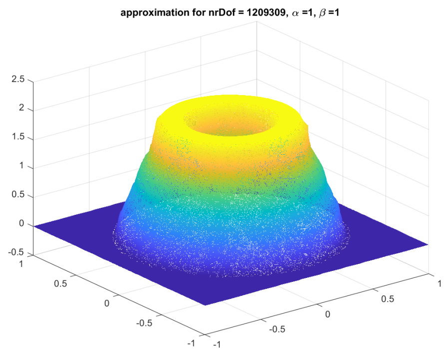

Figure 0.4: last iterate for $\varepsilon = 10^{-2}$

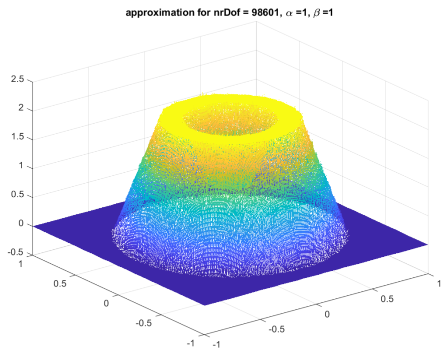

Figure 0.5: last iterate for $\varepsilon = 10^{-6}$

0.4 Application to an image

For $\alpha = 10000$ and $\beta = 1$ let f represent the grayscale of an image in $[0,1]^{256\times 256}$ scaled to the domain $\Omega \in (0,1)^2$ as seen in fig. 0.6.

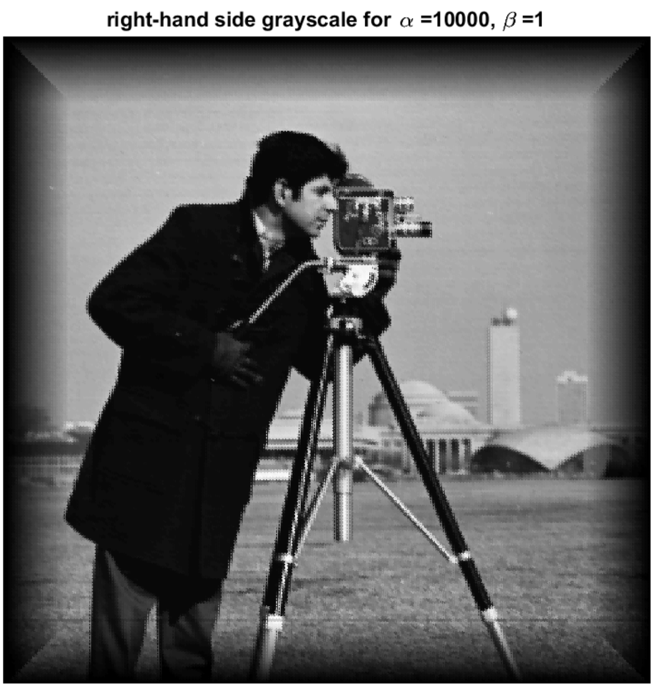

Figure 0.6: grayscale plot of the right-hand side f (view from above onto the x-y plane)

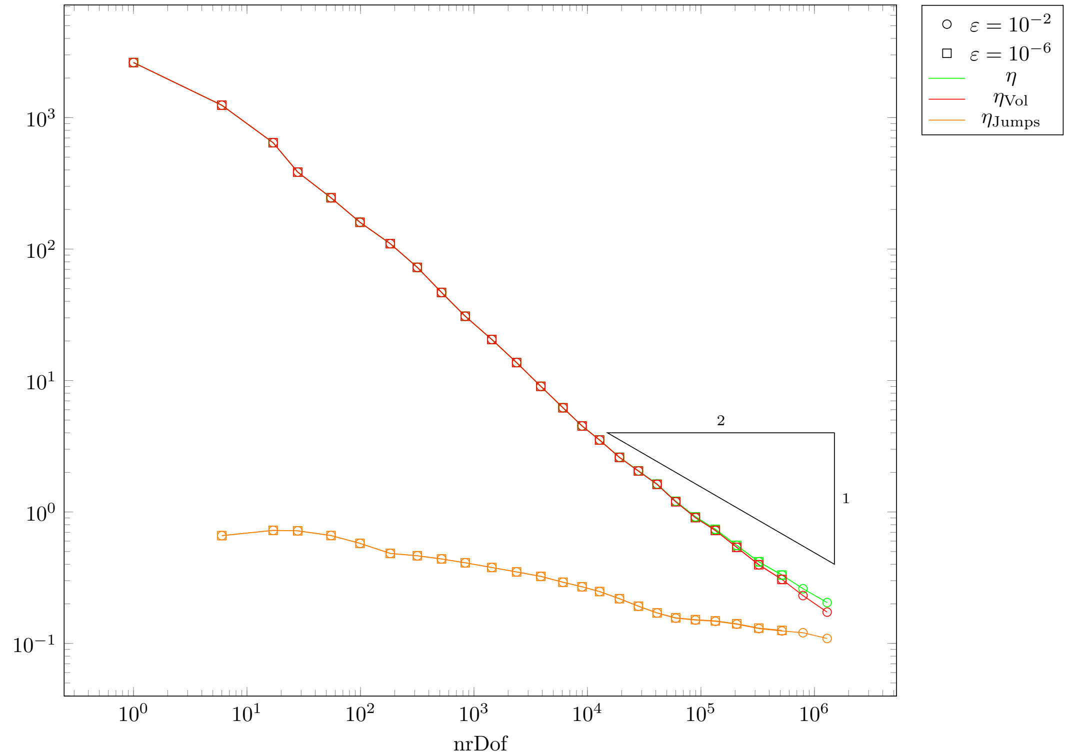

Figure 0.7: convergence history plot for η, η_{Vol}, and η_{Jumps}

Figure 0.8: grayscale plot of last iterate for $\varepsilon = 10^{-2}$

Figure 0.9: grayscale plot of last iterate for $\varepsilon = 10^{-6}$

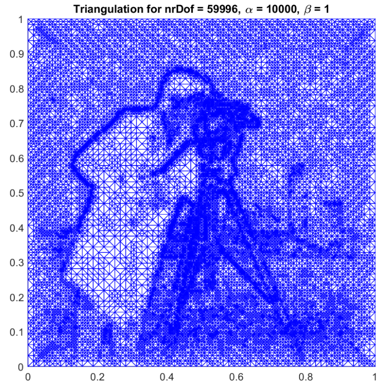

Figure 0.10: adaptive triangulation for 59996 degrees of freedom for $\varepsilon = 10^{-2}$

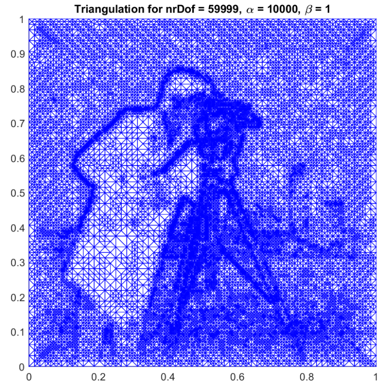

Figure 0.11: adaptive triangulation for 59999 degrees of freedom for $\varepsilon = 10^{-6}$