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Chapter 6

Variational problems: Some classical examples

This chapter shows how the direct variational method can be used to solve some classical boundary value problems.

The following examples have been selected because of their importance in continuum mechanics, physics, biology, and so forth, and because of their relative simplicity. Indeed, in most of these examples, the unknown function is a scalar valued function and the variational problem can be expressed as a convex minimization problem in a reflexive Banach space V; typically V is a Sobolev space $W^{m,p}(\Omega)$ with $1 < p < \infty$. The existence of a (weak) solution can be obtained by application of the convex coercive minimization theorem, Theorem 3.3.4. This contrasts with more involved situations where the unknown function is a vector-valued function, and/or when the functional is no longer convex, and/or the functional space is no longer reflexive, and/or the functional is no longer lower semicontinuous and coercive. Most of these questions will be considered in the next chapters.

When the functional which has to be minimized is the sum of a convex quadratic (positive) form and a linear form, an equivalent approach consists in working with the Euler equation (Proposition 2.3.1) and the corresponding existence theorem, namely, the Lax–Milgram theorem, Theorem 3.1.2. In that case, one can treat the problem by either of the two above equivalent methods.

We stress that it is part of the skill of the mathematician to find a variational formulation (if it exists) of the studied problem. It is not a priori given! In particular, one has to find a functional setting which is well adapted to the problem under consideration. In the examples which are considered in this chapter we use various Sobolev spaces like $H_0^1(\Omega)$, $H^1(\Omega)$, $H^2(\Omega)$, $H^2_{per}(\Omega)$, and $W^{1,p}(\Omega)$.

As a general rule, the variational methods provide only weak solutions. It is an important (and often quite involved) question to study the regularity of the variational solution and hence to decide whether it is a classical solution. We will just give some indications on this question in the case of the Dirichlet problem.

Notation. Ω is a bounded open set in $\mathbf{R}^N$; $\partial\Omega$ is its topological boundary (also denoted by Γ). Ω is said to be a regular open set if $\partial\Omega$ is piecewise of class $\mathbf{C}^1$, and $n(x)$ is the outward unit normal vector to $\partial\Omega$ at x. Given $v : \Omega \to \mathbf{R}$ we write

$$\frac{\partial v}{\partial n}(x) = \nabla v(x) \cdot n(x),$$

the outward normal derivative of v at $x \in \partial\Omega$.

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6.1 ▪ The Dirichlet problem

Given $f : \Omega \to \mathbf{R}$, we are looking for a solution $u : \bar{\Omega} \to \mathbf{R}$ of the following boundary value problem:

$$\begin{cases} -\Delta u = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

The condition "$u = 0$ on $\partial\Omega$" is called the homogeneous Dirichlet boundary condition. Problem (6.1) is called the homogeneous Dirichlet boundary value problem for the Laplace operator. We call it the Dirichlet problem.

The nonhomogeneous Dirichlet problem consists in finding the $u : \bar{\Omega} \to \mathbf{R}$ solution of

$$\begin{cases} -\Delta u = f & \text{on } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where $f : \Omega \to \mathbf{R}$ and $g : \partial\Omega \to \mathbf{R}$ are given functions. The word *homogeneous* refers precisely to the case $g = 0$. We will see at the end of this section that, in general, the nonhomogeneous problem can be reduced to the homogeneous problem.

6.1.1 ▪ The homogeneous Dirichlet problem

Let us recall that when $N = 2$, problem (5.1) can be interpreted, for example, as describing the vertical motion of an elastic membrane under the action of a vertical force of density f. The Dirichlet boundary condition expresses that the membrane is fixed on its boundary.

Theorem 6.1.1. *The variational approach to the Dirichlet problem is described in the following statements:*

(a) *For every $f \in L^2(\Omega)$ there exists a unique $u \in H_0^1(\Omega)$ which satisfies*

$$\begin{cases} \displaystyle\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega). \end{cases} \tag{6.2}$$

(b) *The solution u of* (6.2) *satisfies*

$$\begin{cases} -\Delta u = f & \text{in } \mathscr{D}'(\Omega) & (\text{equality as distributions}), \\ \gamma_0(u) = 0 & \text{on } \partial\Omega & (\gamma_0 \text{ is the trace operator}). \end{cases} \tag{6.3}$$

Indeed, for $u \in H_0^1(\Omega)$ there is equivalence between (6.2) *and* (6.3). *The solution u of* (6.2) *is called the weak solution of the Dirichlet problem* (6.1).

(c) *The solution u of* (6.2) *is the unique solution of the minimization problem*

$$\min\left\{ \frac{1}{2}\int_\Omega |\nabla v|^2 \, dx - \int_\Omega f v \, dx \; : \; v \in H_0^1(\Omega) \right\}. \tag{6.4}$$

This is the Dirichlet variational principle. We also call u the variational solution of the Dirichlet problem.

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PROOF. (a) Let us solve (6.2) by using the Lax–Milgram theorem. To that end, take $V = H_0^1(\Omega)$ equipped with the scalar product

$$\langle u, v\rangle = \int_\Omega (uv + \nabla u \cdot \nabla v)\, dx,$$

which makes V a Hilbert space. Then, set for any $u, v \in V$,

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx,$$

$$l(v) = \int_\Omega f v \, dx.$$

Let us first verify that the bilinear form $a : V \times V \to \mathbf{R}$ is continuous. For arbitrary $u, v \in V$, by using successively the Cauchy–Schwarz inequality in $\mathbf{R}^N$ and $L^2(\Omega)$, we obtain

$$\begin{aligned}|a(u,v)| &\le \int_\Omega |\nabla u|\, |\nabla v|\, dx\\ &\le \left(\int_\Omega |\nabla u|^2\, dx\right)^{1/2} \left(\int_\Omega |\nabla v|^2\, dx\right)^{1/2}\\ &\le \|u\|_V \, \|v\|_V.\end{aligned}$$

Let us now verify that the linear form $l : V \to \mathbf{R}$ is continuous. For arbitrary $v \in V$

$$\begin{aligned}|l(v)| &\le \int_\Omega |f||v|\, dx\\ &\le \left(\int_\Omega |f|^2\, dx\right)^{1/2} \left(\int_\Omega |v|^2\, dx\right)^{1/2}\\ &\le C\|v\| \quad \text{with } C = \|f\|_{L^2}.\end{aligned}$$

The only point which remains to verify is that the bilinear form a is coercive. To that end, we use the Poincaré inequality (Theorem 5.3.1). Since Ω has been assumed to be bounded, there exists some positive constant C such that

$$\forall v \in H_0^1(\Omega) \quad \int_\Omega v(x)^2\, dx \le C \int_\Omega |\nabla v(x)|^2\, dx.$$

By adding $\int_\Omega |\nabla v|^2\, dx$ to each side of the above inequality, we obtain

$$\int_\Omega (v(x)^2 + |\nabla v(x)|^2)\, dx \le (1+C) \int_\Omega |\nabla v(x)|^2\, dx.$$

Equivalently,

$$\forall v \in H_0^1(\Omega) \quad a(v,v) \ge \frac{1}{1+C} \|v\|_V^2,$$

and a is α-coercive (or α-elliptic) with $\alpha = \frac{1}{1+C} > 0$. Thus, all the assumptions of the Lax–Milgram theorem are satisfied. This implies existence and uniqueness of the solution u of problem (6.2).

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(b) Let u be the solution of (6.2). Since $\mathscr{D}(\Omega) \subset H_0^1(\Omega)$, we have

$$\forall v \in \mathscr{D}(\Omega) \quad \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx, \tag{6.5}$$

which, by definition of the derivation in the distribution sense, is equivalent to

$$-\Delta u = f \quad \text{in } \mathscr{D}'(\Omega). \tag{6.6}$$

Moreover, we know by Proposition 5.6.1 that $H_0^1(\Omega) = \ker \gamma_0$, where γ_0 is the trace operator. Hence

$$\gamma_0(u) = 0 \quad \text{in trace sense}$$

and u satisfies (6.3). Conversely, if u satisfies $-\Delta u = f$ in the distribution sense, we have (6.5). Then use the density of $\mathscr{D}(\Omega)$ in $H_0^1(\Omega)$ and the fact that $u \in H_0^1(\Omega)$ and $f \in L^2(\Omega)$ to obtain (6.2).

(c) The equivalence between (6.5) and (6.3) is an immediate consequence of Proposition 2.3.1. To that end, just note that the bilinear form $a(\cdot,\cdot)$ is symmetric and positive.

The corresponding minimization problem is

$$\min\{J(v) \, : \, v \in H_0^1(\Omega)\},$$

where

$$\begin{aligned} J(v) &= \frac{1}{2} a(v,v) - l(v) \\ &= \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx - \int_\Omega f v \, dx. \end{aligned}$$

That's the Dirichlet variational principle. Note that the functional J is convex, continuous, and coercive on $H_0^1(\Omega)$. □

Let us now make the link between the notion of a classical solution of the Dirichlet problem and the notion of a weak solution which was introduced in Theorem 6.1.1. Let us first make precise the notion of classical solution.

Definition 6.1.1. *A function $u : \bar{\Omega} \to \mathbf{R}$ is said to be a classical solution of the Dirichlet problem if $u \in \mathbf{C}^2(\bar{\Omega})$ satisfies $-\Delta u = f$ in the sense of the classical differential calculus, while the restriction of u to $\partial\Omega$ is equal to zero:*

$$\begin{cases} -\Delta u(x) = f(x) & \forall x \in \Omega, \\ u(x) = 0 & \forall x \in \partial\Omega. \end{cases}$$

Proposition 6.1.1. (a) *If $u \in \mathbf{C}^2(\bar{\Omega})$ is a classical solution of the Dirichlet problem* (6.1), *then it is equal to the weak solution of* (6.2). *As a consequence, the classical solution, if it exists, is unique.*

(b) *If the weak solution u of* (6.2) *is regular, that is, $u \in \mathbf{C}^2(\bar{\Omega})$ and Ω is of class $\mathbf{C}^1$, then u is the classical solution of the Dirichlet problem* (6.1).

PROOF. (a) Let $u \in \mathbf{C}^2(\bar{\Omega})$ be a classical solution of the Dirichlet problem. Then u and $\frac{\partial u}{\partial x_i}$ for any $1 \le i \le N$ are continuous functions on the compact set $\bar{\Omega}$ (recall that Ω is

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bounded) and hence bounded on $\bar{\Omega}$. Since $L^\infty(\Omega) \subset L^2(\Omega)$ (we use again that Ω is bounded) we obtain that $u \in H^1(\Omega)$.

Since $u \in H^1(\Omega) \cap \mathbf{C}(\bar{\Omega})$, by using Proposition 5.6.1, we have $u = 0$ on $\partial\Omega$ so that $u \in H_0^1(\Omega)$. Note that this implication holds true without assuming any regularity assumption on $\partial\Omega$.

Take an arbitrary $v \in \mathscr{D}(\Omega)$. We have

$$-\int_\Omega v\Delta u\, dx = \int_\Omega f v\, dx.$$

Let us integrate by parts on Ω. We obtain

$$\forall v \in \mathscr{D}(\Omega) \quad \int_\Omega \nabla u \cdot \nabla v\, dx = \int_\Omega f v\, dx.$$

By density of $\mathscr{D}(\Omega)$ in $H_0^1(\Omega)$, we obtain

$$\forall v \in H_0^1(\Omega) \quad \int_\Omega \nabla u \cdot \nabla v\, dx = \int_\Omega f v\, dx.$$

Hence, u is the weak solution of (6.2).

(b) Let us assume that the weak solution u of (6.2) is regular, that is, $u \in \mathbf{C}^2(\bar{\Omega})$. Using again Proposition 5.6.1 and the fact that Ω is now assumed to be of class $\mathbf{C}^1$, we obtain

$$u \in \mathbf{C}(\bar{\Omega}) \cap H_0^1(\Omega) \Longrightarrow u = 0 \quad \text{on } \partial\Omega \quad \text{(as a restriction to } \partial\Omega).$$

On the other hand, we have

$$-\Delta u = f \quad \text{in } \mathscr{D}'(\Omega).$$

Since $u \in \mathbf{C}^2(\Omega)$, the distributional derivatives of u (up to the second order) coincide with the classical derivatives and

$$-\Delta u(x) = f(x) \quad \forall x \in \Omega$$

in the classical sense. Hence, u is the classical solution of the Dirichlet problem. □

Remark 6.1.1. By Proposition 6.1.1, the question of the existence of a classical solution has been converted into the problem of the regularity of the weak solution. For this quite involved and important question, see [8], [9], [228], and [310].

6.1.2 ▪ The nonhomogeneous Dirichlet problem

Given $g : \partial\Omega \to \mathbf{R}$ and $f : \Omega \to \mathbf{R}$, we look for a solution $u : \bar{\Omega} \to \mathbf{R}$ of the following boundary value problem:

$$\begin{cases} -\Delta u = f & \text{on } \Omega, \\ \quad\; u = g & \text{on } \partial\Omega. \end{cases} \tag{6.7}$$

Let us first give a variational formulation of this problem, as a constrained minimization problem on the set $C = \{v \in H^1(\Omega) \; : \; v = g \;\text{ on } \partial\Omega\}$. Then, we will see how this problem can be reduced to the homogeneous Dirichlet problem.

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Theorem 6.1.2. *Let Ω be a bounded regular connected open set in $\mathbf{R}^N$. Let $g : \partial\Omega \to \mathbf{R}$ be a given function such that $g = \gamma_0(\tilde{g})$ for some $\tilde{g} \in H^1(\Omega)$, i.e., g belongs to $H^{1/2}(\partial\Omega)$, the trace space of $H^1(\Omega)$ on $\partial\Omega$. Let us denote*

$$C = \{v \in H^1(\Omega) \, : \, \gamma_0(v) = g \text{ on } \partial\Omega\}. \tag{6.8}$$

(i) *The set C is a closed convex nonempty subset of $H^1(\Omega)$. Indeed, $C = \tilde{g} + H_0^1(\Omega)$ is an affine subspace in $H^1(\Omega)$ which is parallel to $H_0^1(\Omega)$.*

(ii) *For any $f \in L^2(\Omega)$, there exists a unique solution u of the minimization problem*

$$\min\left\{\frac{1}{2}\int_\Omega |\nabla v(x)|^2\,dx - \int_\Omega f\,v\,dx \, : \, v \in C\right\}. \tag{6.9}$$

(iii) *The solution u of (6.9) is characterized by*

$$\begin{cases} \displaystyle\int_\Omega \nabla u \cdot \nabla v\,dx = \int_\Omega f\,v\,dx \qquad \forall v \in H_0^1(\Omega), \\ u \in C, \end{cases} \tag{6.10}$$

and it is a weak solution of the nonhomogeneous Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{on } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{6.11}$$

where $-\Delta u = f$ is interpreted in the distribution sense and $u = g$ in the trace sense.

PROOF. (i) The structure of C and its topological properties follow immediately from the fact that γ_0 is a linear continuous map from $H^1(\Omega)$ into $L^2(\partial\Omega)$.

(ii) The only point which is not immediate is the fact that the functional

$$J(v) := \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega f\,v\,dx + \delta_C(v) \tag{6.12}$$

is coercive on $H^1(\Omega)$. Given $\lambda \in \mathbf{R}$, let us prove that the set

$$\{J \le \lambda\} = \left\{v \in C \, : \, \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega f\,v\,dx \le \lambda\right\}$$

is bounded in $H^1(\Omega)$. Set $v = w + \tilde{g}$ with $w \in H_0^1(\Omega)$. Equivalently, we have to prove that

$$K := \left\{w \in H_0^1(\Omega) \, : \, \frac{1}{2}\int_\Omega |\nabla w + \nabla\tilde{g}|^2\,dx - \int_\Omega f(w + \tilde{g})\,dx \le \lambda\right\}$$

is bounded in $H_0^1(\Omega)$. Let us observe that

$$K \subset \left\{w \in H_0^1(\Omega) \, : \, \int_\Omega |\nabla w|^2\,dx \le 2\int_\Omega |\nabla w||\nabla\tilde{g}|\,dx + 2\int_\Omega |f||w|\,dx + \gamma\right\}$$

with $\gamma = 2\lambda + 2\|f\|_{L^2}\|\tilde{g}\|_{L^2}$. By using the Poincaré inequality in $H_0^1(\Omega)$ and an elementary computation, one obtains that K is bounded, and hence J is coercive on $H^1(\Omega)$.

The functional J being convex lower semicontinuous (we use that C is closed convex) and coercive, the minimization problem of J on $H^1(\Omega)$, that is, (6.9), admits a solution u.

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The uniqueness of the solution u follows from the strict convexity property of the Dirichlet integral: for any u_1, u_2 in C, we have

$$\int_\Omega \left|\nabla\left(\frac{u_1+u_2}{2}\right)\right|^2 dx = \frac{1}{2}\left[\int_\Omega |\nabla u_1|^2\,dx + \int_\Omega |\nabla u_2|^2\,dx\right] - \frac{1}{4}\int_\Omega |\nabla(u_1-u_2)|^2\,dx. \tag{6.13}$$

If u_1 and u_2 are two distinct solutions of (6.9), then the last term in (6.13) is strictly less than zero, which leads to a contradiction. We have used that $u_1 = u_2 = g$ on $\partial\Omega$ implies $u_1 - u_2 = 0$ on $\partial\Omega$ and that $\int_\Omega |\nabla(u_1-u_2)|^2\,dx = 0$ implies $u_1 - u_2$ is constant on Ω and hence, $u_1 = u_2$.

(iii) The general optimality condition for a minimization problem of the form

$$\min\left\{\frac{1}{2}a(v,v) - l(v) \,:\, v \in C\right\}$$

is, according to Theorem 3.3.5,

$$\begin{cases} a(u, v-u) - l(v-u) \geq 0 & \forall v \in C, \\ u \in C. \end{cases} \tag{6.14}$$

Because of the particular structure of the set $C = \tilde{g} + H_0^1(\Omega)$, we have $v \in C$ iff $v - u \in H_0^1(\Omega)$ which is a subspace of $H^1(\Omega)$. As a consequence, (6.14) is equivalent to

$$\begin{cases} a(u,v) - l(v) = 0 & \forall v \in H_0^1(\Omega), \\ u \in C, \end{cases}$$

that is, (6.10).

Then use that $\mathscr{D}(\Omega)$ is dense in $H_0^1(\Omega)$ to obtain the equivalent formulation (6.11) in terms of distributions. □

Let us now make the link with the homogeneous Dirichlet problem. Indeed, we are going to show that just by taking as a new unknown function $w := u - \tilde{g}$, one can reduce the nonhomogeneous Dirichlet problem to some homogeneous Dirichlet problem. Formally, w satisfies

$$\begin{cases} -\Delta w = f + \Delta\tilde{g} & \text{on } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

which is a homogeneous Dirichlet problem, with a different right-hand side in the partial differential equation on Ω, namely, $f + \Delta\tilde{g}$, which now belongs to the space $H^{-1}(\Omega)$! Let us make this precise in the following statement.

Theorem 6.1.3. *Let us make the same assumptions and use the same notations as in Theorem 6.1.2, and fix some $\tilde{g} \in H^1(\Omega)$ such that $\gamma_0(\tilde{g}) = g$ on $\partial\Omega$.*

(i) *There exists a unique solution $w \in H_0^1(\Omega)$ of the problem*

$$\begin{cases} \displaystyle\int_\Omega \nabla w \cdot \nabla v\,dx = \int_\Omega f v\,dx - \int_\Omega \nabla\tilde{g}\cdot\nabla v\,dx & \forall v \in H_0^1(\Omega), \\ w \in H_0^1(\Omega), \end{cases} \tag{6.15}$$

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and $u = w + \tilde{g}$ is the variational solution of the nonhomogeneous Dirichlet problem (6.9).

(ii) *The variational solution w of* (6.15) *is a weak solution of the boundary value problem*

$$\begin{cases} -\Delta w = f + \Delta \tilde{g} & on\ \Omega, \\ w = 0 & on\ \partial\Omega. \end{cases} \tag{6.16}$$

PROOF. Replacing u by $w + \tilde{g}$ in (6.10) gives exactly (6.15). Hence Theorem 6.1.3 is just an equivalent formulation of Theorem 6.1.2.

We could, as well, treat the nonhomogeneous Dirichlet problem by solving, in an independent way, the variational problem (6.15). Note that to apply the Lax–Milgram theorem, one needs to verify that

$$l(v) = \int_\Omega f v\, dx - \int_\Omega \nabla \tilde{g} \cdot \nabla v\, dx$$

is a linear continuous form on $H_0^1(\Omega)$, which is clear. This corresponds to the fact that in (6.16) w is a solution of the Laplace equation with a right-hand side $f + \Delta\tilde{g}$, which in general is no longer in $L^2(\Omega)$ but belongs to $H^{-1}(\Omega)$! □

Remark 6.1.2. From a practical and especially numerical point of view, it is much easier to work with $w = u - \tilde{g}$ and to solve the corresponding homogeneous Dirichlet problem by means of variational methods in the Sobolev space $H_0^1(\Omega)$. This justifies our special interest in studying the relations between the two approaches.

6.2 ▪ The Neumann problem

We are going to study successively the coercive Neumann problem (both in the homogeneous and the nonhomogeneous case) and then the semicoercive Neumann problem. In this section, Ω is a bounded open connected set in $\mathbf{R}^N$ which is assumed to be regular (piecewise of class $\mathbf{C}^1$). The outward unit normal vector to $\partial\Omega$ at $x \in \partial\Omega$ is denoted by $n(x)$. Generally speaking, the Neumann boundary condition expresses that the normal derivative $\frac{\partial u}{\partial n}$ of the unknown function is prescribed on $\partial\Omega$.

6.2.1 ▪ The coercive homogeneous Neumann problem

Let us give a function $a_0 \in L^\infty(\Omega)$ which satisfies the following assumption: there exists some positive real number $\alpha_0 > 0$ such that

$$a_0(x) \geq \alpha_0 \quad \text{for a.e. } x \in \Omega. \tag{6.17}$$

Given $f \in L^2(\Omega)$, we are looking for a solution u of the following boundary value problem:

$$\begin{cases} -\Delta u + a_0 u = f & \text{on } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.18}$$

The word *homogeneous* refers to the fact that $\frac{\partial u}{\partial n}$ is prescribed as equal to zero on the boundary. The word *coercive* is related to the fact that problem (5.18) is a well-posed problem (existence and uniqueness of a solution) whose variational resolution involves a coercive bilinear form (equivalently, a coercive convex functional) on the space $H^1(\Omega)$. Let us make this precise in the statement below.

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Theorem 6.2.1. *The following facts hold:*

(i) *There exists a unique solution $u \in H^1(\Omega)$ of the problem*

$$\begin{cases} \int_\Omega (\nabla u \cdot \nabla v + a_0 u v)\, dx = \int_\Omega f v\, dx \quad \forall v \in H^1(\Omega), \\ u \in H^1(\Omega). \end{cases} \tag{6.19}$$

(ii) *Equivalently, u is the unique solution of minimization problem*

$$\min \left\{ \frac{1}{2} \int_\Omega (|\nabla v|^2 + a_0 v^2)\, dx - \int_\Omega f v\, dx \, : \, v \in H^1(\Omega) \right\}. \tag{6.20}$$

(iii) *Let us assume that the solution u of* (6.19) *(or* (6.20)*) is regular, i.e., $u \in \mathbf{C}^2(\bar{\Omega})$. Then u is a classical solution of the Neumann problem* (6.18).

(iv) *Conversely, if there exists a classical solution of the Neumann problem* (6.18), *it is equal to the solution of the variational problem* (6.19) *(or* (6.20)*).*

PROOF. (i) The functional space which is well adapted to the Neumann problem is $V = H^1(\Omega)$. Since Ω has been assumed to be regular, we know (see Proposition 5.4.1) that $\mathscr{D}(\bar{\Omega})$ is a dense subspace of $H^1(\Omega)$. Recall that $\mathscr{D}(\bar{\Omega}) = \{v_{|\Omega} \, : \, v \in \mathscr{D}(\mathbf{R}^N)\}$. We will use elements $v \in \mathscr{D}(\bar{\Omega})$ as smooth test functions. Existence and uniqueness of a solution u of (6.19) is an immediate consequence of the Lax–Milgram theorem. Let us briefly give the proof. The bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbf{R}$ which is defined by

$$a(u,v) = \int_\Omega (\nabla u \cdot \nabla v + a_0 u v)\, dx$$

verifies

$$\begin{aligned} |a(u,v)| &\le \max\{1, \|a_0\|_{L^\infty}\} \int_\Omega (|\nabla u||\nabla v| + |u||v|) dx \\ &\le \max\{1, \|a_0\|_{L^\infty}\} \|u\|_{H^1} \|v\|_{H^1} \end{aligned}$$

and a is continuous. On the other hand, for any $v \in H^1(\Omega)$

$$a(v,v) \ge \min\{1, \alpha_0\} \int_\Omega (|\nabla v|^2 + v^2)\, dx = \min\{1, \alpha_0\} \|v\|^2_{H^1}$$

and a is α-coercive on $V = H^1(\Omega)$ with $\alpha = \min\{1, \alpha_0\}$.

The linear form $l(v) = \int_\Omega f v\, dx$ verifies

$$|l(v)| \le \|f\|_{L^2} \|v\|_{L^2} \le \|f\|_{L^2} \|v\|_{H^1}$$

and l is continuous on $H^1(\Omega)$.

Thus, all the conditions of the Lax–Milgram theorem are satisfied and, as a consequence, there exists a solution u, which is unique, of problem (6.19).

(ii) The equivalence between (6.19) and the resolution of the minimization problem (6.20) follows immediately from Proposition 2.3.1 and the fact that a is positive and symmetric.

(iii) We now come to the most interesting question, which is the study of the relationship between the Neumann boundary value problem (6.18) and the variational problem (6.19) (or (6.20)). The striking feature of the variational approach is that the Neumann boundary condition does not appear explicitly in the variational formulation. Indeed, it is implicitly contained in it; this is what we are going to verify now. Suppose that the solution u of (6.19) is regular, i.e., $u \in \mathbf{C}^2(\bar{\Omega})$. Take as a test function $v \in \mathscr{D}(\bar{\Omega}) \subset H^1(\Omega)$ and make an integration by parts on (6.19). (This is possible since u and v are regular up to the boundary.) We obtain

$$\forall v \in \mathscr{D}(\bar{\Omega}) \quad \int_\Omega (-\Delta u + a_0 u - f) v\, dx + \int_{\partial\Omega} v \frac{\partial u}{\partial n}\, d\sigma = 0. \tag{6.21}$$

Thanks to (6.21), we are going to test u successively on Ω and on $\partial\Omega$.

Let us first test u on Ω by taking $v \in \mathscr{D}(\Omega)$ in (6.21). Since $v = 0$ on $\partial\Omega$, the integral term on $\partial\Omega$ in (6.21) is equal to zero and we obtain

$$\forall v \in \mathscr{D}(\Omega) \quad \int_\Omega (-\Delta u + a_0 u - f) v\, dx = 0.$$

This implies that (see Theorem 9.3.1)

$$-\Delta u + a_0 u = f \quad \text{a.e. on } \Omega. \tag{6.22}$$

Let us now test u on $\partial\Omega$. To do so, we return to (6.21) with a general $v \in \mathscr{D}(\bar{\Omega})$ and use the previous information (6.22).

Formula (6.22) just expresses that $-\Delta u + a_0 u - f$, which is an $L^2(\Omega)$ function, is equal to zero almost everywhere on Ω. As a consequence, the first integral term on Ω in (6.21) is equal to zero and we obtain

$$\forall v \in \mathscr{D}(\bar{\Omega}) \quad \int_{\partial\Omega} v \frac{\partial u}{\partial n}\, d\sigma = 0. \tag{6.23}$$

We conclude thanks to the following lemma of independent interest.

Lemma 6.2.1. *Let Ω be a regular bounded open set in $\mathbf{R}^N$. Then, for any function $h \in L^2(\partial\Omega)$, we have*

$$\int_{\partial\Omega} h v\, d\sigma = 0 \quad \forall v \in \mathscr{D}(\bar{\Omega}) \Longrightarrow h = 0 \quad \text{on } \partial\Omega.$$

PROOF. Take $h \in L^2(\partial\Omega)$, which satisfies

$$\int_{\partial\Omega} h v\, d\sigma = 0 \quad \forall v \in \mathscr{D}(\bar{\Omega}). \tag{6.24}$$

By the density of $\mathscr{D}(\bar{\Omega})$ in $H^1(\Omega)$ and the continuity of the trace operator from $H^1(\Omega)$ into $L^2(\partial\Omega)$, property (6.24) can be extended by continuity to $H^1(\Omega)$, i.e.,

$$\int_{\partial\Omega} h \gamma_0(v)\, d\sigma = 0 \quad \forall v \in H^1(\Omega).$$

We know, by Proposition 5.6.3, that the image of $H^1(\Omega)$ by γ_0 is equal to $H^{1/2}(\partial\Omega)$. Hence

$$\int_{\partial\Omega} h v\, d\sigma = 0 \quad \forall v \in H^{1/2}(\partial\Omega).$$

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The conclusion follows now from the density property of $H^s(\partial\Omega)$ in $L^2(\partial\Omega)$ for any $s>0$. This last property can be easily deduced (by using local coordinates) from the fact that for any $s>0$, $\mathscr{D}(\mathbf{R}^N)$ is included in $H^s(\mathbf{R}^N)$, and from the density property of $\mathscr{D}(\mathbf{R}^N)$ in $L^2(\mathbf{R}^N)$. □

PROOF OF THEOREM 6.2.1 CONTINUED. Let us complete the proof of Theorem 6.2.1 and prove the last point (iv). Let us assume that $u\in\mathbf{C}^2(\bar{\Omega})$ is a classical solution of the Neumann boundary value problem (6.18). Take $v\in\mathscr{D}(\bar{\Omega})$ an arbitrary test function, multiply (6.18) by v, and integrate on Ω. We obtain

$$\int_\Omega(-\Delta u+a_0u)v\,dx=\int_\Omega fv\,dx\quad\forall v\in\mathscr{D}(\bar{\Omega}).$$

Integrating by parts, the above expression gives

$$\int_\Omega(\nabla u\cdot\nabla v+a_0uv)dx-\int_{\partial\Omega}v\frac{\partial u}{\partial n}\,d\sigma=\int_\Omega fv\,dx\quad\forall v\in\mathscr{D}(\bar{\Omega}).$$

Since $\frac{\partial u}{\partial n}=0$ on $\partial\Omega$, we get

$$\int_\Omega(\nabla u\cdot\nabla v+a_0uv)\,dx=\int_\Omega fv\,dx\quad\forall v\in\mathscr{D}(\bar{\Omega}).$$

We now use the density of $\mathscr{D}(\bar{\Omega})$ in $H^1(\Omega)$ to extend this equality to any $v\in H^1(\Omega)$. Note also that $u\in\mathbf{C}^2(\bar{\Omega})$ clearly implies that $u\in H^1(\Omega)$. □

Remark 6.2.1. The following facts should be pointed out:

1. The solution u of (6.19) (equivalently, of the minimization problem (6.20)) is called the variational solution, or the weak solution of the Neumann boundary value problem (6.18). As in the case of the Dirichlet problem, the term *weak solution* is justified by the fact that the weak solution always exists, but on the counterpart, the equation on Ω and the Neumann boundary condition are satisfied only in a weak sense (respectively, distribution sense and trace sense). The existence of a classical solution is equivalent to the study of the regularity of the weak solution.

2. According to the variational approach to the Dirichlet problem, the most natural choice for the function space in which to solve the Neumann problem is given by the closure V in $H^1(\Omega)$ of

$$\mathscr{V}=\left\{v\in\mathscr{D}(\bar{\Omega})\ :\ \frac{\partial v}{\partial n}=0\quad\text{on }\partial\Omega\right\}.$$

It is a good exercise to verify that $V=H^1(\Omega)$!

One may be convinced of this fact just by observing the following elementary situation: take $\Omega=(0,1)$ and $v(x)=x$. For each $n\in\mathbf{N}$, take

$$v_n(x)=\begin{cases}\frac{1}{n} & \text{if } 0\le x\le\frac{1}{n},\\ x & \text{if } \frac{1}{n}\le x\le 1-\frac{1}{n},\\ 1-\frac{1}{n} & \text{if } 1-\frac{1}{n}\le x\le 1.\end{cases}$$

Clearly $v_n\in H^1(0,1)$, $\frac{dv_n}{dx}(0)=\frac{dv_n}{dx}(1)=0$, and $v_n\to v$ in $H^1(0,1)$ as $n\to+\infty$:

$$||v_n-v||^2_{H^1(0,1)}=2\int_0^{1/n}\left[\left(x-\frac{1}{n}\right)^2+1\right]dx\le\frac{2}{n}+\frac{2}{n^3}.$$

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Hence the property "$\frac{\partial v}{\partial x} = 0$ on $\partial\Omega$" is not stable for the convergence in the $H^1(\Omega)$ topology! As a conclusion, $H^1(\Omega)$ is the right functional space to solve the Neumann problem by variational methods.

3. As a general rule, homogeneous Neumann boundary conditions naturally occur in problems where the boundary is free, i.e., no constraints, and no external action is exerted on $\partial\Omega$. This explains the importance of Neumann-type boundary value problems in mechanics, physics, and so forth.

6.2.2 ▪ The coercive nonhomogeneous Neumann problem

We keep the same notation and assumptions as in the previous section. In addition, we suppose that there is some function $g \in L^2(\partial\Omega)$ which is given. We consider the (nonhomogeneous) Neumann problem

$$\begin{cases} -\Delta u + a_0 u = f & \text{on } \Omega, \\ \frac{\partial u}{\partial n} = g & \text{on } \partial\Omega. \end{cases} \tag{6.25}$$

The variational approach to (6.25) is similar to that in the proof of the homogeneous case (Theorem 6.2.1). One just needs to change the linear form $l : H^1(\Omega) \to \mathbf{R}$ by introducing the integral term $\int_{\partial\Omega} g v \, d\sigma$. Let us make this precise.

Theorem 6.2.2.

(i) *Given $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$, there exists a unique solution $u \in H^1(\Omega)$ of the problem*

$$\begin{cases} \displaystyle\int_\Omega (\nabla u \cdot \nabla v + a_0 u v)\, dx = \int_\Omega f v \, dx + \int_{\partial\Omega} g v \, d\sigma \quad \forall v \in H^1(\Omega), \\ u \in H^1(\Omega). \end{cases} \tag{6.26}$$

(ii) *Equivalently, u is the unique solution of the minimization problem*

$$\min \left\{ \frac{1}{2}(|\nabla v|^2 + a_0 v^2)\, dx - \int_\Omega f v \, dx - \int_{\partial\Omega} g v \, d\sigma \; : \; v \in H^1(\Omega) \right\}. \tag{6.27}$$

(iii) *Let us assume that the solution u of* (6.26) *(equivalently,* (6.27)*) is regular, i.e., $u \in \mathbf{C}^2(\bar{\Omega})$. Then u is a classical solution of the nonhomogeneous Neumann problem* (6.25).

(iv) *Conversely, if u is a classical solution of* (6.25)*, then it is equal to the solution of the variational problem* (6.26) *(equivalently,* (6.27)*).*

PROOF. (i) We follow the lines of the proof of Theorem 6.2.1. We know, by Theorem 5.6.1, that the trace operator $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$ is continuous. From this, and by using the fact that $g \in L^2(\partial\Omega)$, we deduce that the linear form $v \in H^1(\Omega) \mapsto \int_{\partial\Omega} g v \, d\sigma$ is continuous on $H^1(\Omega)$. This clearly implies that the linear mapping $v \mapsto \int_\Omega f v \, dx + \int_{\partial\Omega} g v \, d\sigma$ is continuous on $H^1(\Omega)$. The assumptions of the Lax–Milgram theorem are satisfied. (The bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbf{R}$ is the same as in the homogeneous case.) As a consequence, problem (6.26) admits a unique solution u.

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(iii) Let us assume that u is a regular solution of (6.26). By taking first $v \in \mathscr{D}(\bar{\Omega})$ and by using the above equality, we infer that

$$\forall v \in \mathscr{D}(\bar{\Omega}) \quad \int_{\partial\Omega} v\left(\frac{\partial u}{\partial n} - g\right) d\sigma = 0.$$

By using Lemma 6.2.1, we conclude that $\frac{\partial u}{\partial n} = g$ on $\partial\Omega$. $\square$

6.2.3 ▪ The semicoercive homogeneous Neumann problem

Let us now consider the following boundary value problem. Given $f \in L^2(\Omega)$, find $u : \bar{\Omega} \to \mathbf{R}$, which satisfies

$$\begin{cases} -\Delta u = f & \text{on } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.28}$$

Note that in (6.28), the term a_0 which was supposed to be positive in the previous section is now equal to zero. This makes a big difference; problem (6.28) is no longer a well-posed problem. To see this, let us assume for a moment that u is a smooth solution of (6.28). Let us integrate (6.28) on Ω and use the divergence theorem. We obtain

$$\begin{aligned} \int_\Omega f(x)\,dx &= -\int_\Omega \Delta u\,dx \\ &= -\int_{\partial\Omega} \frac{\partial u}{\partial n}\,d\sigma \\ &= 0. \end{aligned}$$

Hence, a necessary condition for the existence of a (classical) solution of problem (6.28) is that $\int_\Omega f(x)\,dx = 0$. Moreover, we can observe also that if u is a solution of (6.28), then for any constant C, $u + C$ is also a solution.

Indeed, we are going to prove that the condition $\int_\Omega f(x)\,dx = 0$ is a necessary and sufficient condition for the existence of a solution of problem (6.28) and that any two solutions of this problem differ by a constant.

We are going to present two different variational approaches of independent interest. The first one consists in working with the space

$$V = \left\{ v \in H^1(\Omega) : \int_\Omega v(x)\,dx = 0 \right\},$$

and we will see that, in this framework, the problem becomes coercive. The second approach consists in regularizing problem (6.28) by considering for each $\varepsilon > 0$, the now coercive problem

$$\begin{cases} \varepsilon u_\varepsilon - \Delta u_\varepsilon = f & \text{on } \Omega, \\ \frac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, by passing to the limit as $\varepsilon \to 0$, one obtains that the sequence $(u_\varepsilon)_{\varepsilon\to 0}$ converges to a particular solution of the initial problem (6.28).

Theorem 6.2.3. *Let us give $f \in L^2(\Omega)$ such that $\int_\Omega f(x)\,dx = 0$. We introduce the space $V = \{v \in H^1(\Omega) : \int_\Omega v(x)\,dx = 0\}$ which is equipped with the scalar product and norm of $H^1(\Omega)$, i.e.,*

$$\langle u, v\rangle = \int_\Omega (uv + \nabla u \cdot \nabla v)\,dx,$$

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which makes V a closed subspace of $H^1(\Omega)$ and hence a Hilbert space.

(i) *There exists a unique $u \in V$ which satisfies*

$$\begin{cases} \displaystyle\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in V, \\ u \in V. \end{cases} \tag{6.29}$$

Equivalently, u is the unique solution of the minimization problem

$$\min \left\{ \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx - \int_\Omega f v \, dx \; : \; v \in V \right\}. \tag{6.30}$$

(ii) *Let us assume that the solution u of* (6.29) *is regular. Then u is a classical solution of the problem*

$$\begin{cases} -\Delta u = f & \text{on } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega u(x) \, dx = 0, \end{cases} \tag{6.31}$$

and all the other classical solutions of (6.28) *are obtained by adding a constant to u.*

(iii) *Conversely, if u is a classical solution of* (6.28)*, then $u - \frac{1}{|\Omega|}\int_\Omega u(x)\,dx$ is equal to the solution of problem* (6.29)*.*

PROOF. (i) The central point is to prove that the bilinear form $a : V \times V \to \mathbf{R}$, which is defined by

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx,$$

is coercive on V. Indeed,

$$\forall v \in V \quad a(v,v) = \int_\Omega |\nabla v|^2 \, dx; \tag{6.32}$$

so, we need some Poincaré inequality to conclude that a is coercive. In our setting, it is the Poincaré–Wirtinger inequality (cf. Corollary 5.4.1) which is well adapted. Let us recall that there exists some constant C such that

$$\forall v \in H^1(\Omega) \quad \left\| v - \frac{1}{|\Omega|} \int_\Omega v(x) \, dx \right\|_{L^2(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega)^N}.$$

When $v \in V$, we have $\int_\Omega v(x)\,dx = 0$, and hence

$$\forall v \in V \quad \|v\|_{L^2(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega)^N}. \tag{6.33}$$

From (6.32) and (6.33) we obtain

$$\forall v \in V \quad a(v,v) \geq \frac{1}{1+C^2} \|v\|^2_{H^1(\Omega)}, \tag{6.34}$$

and a is coercive.

(ii) Let us now interpret the variational problem (6.29) in a classical sense when its solution u is regular. To that end, let us notice that

$$\forall v \in \mathscr{D}(\bar{\Omega}) \quad v - \frac{1}{|\Omega|}\int_\Omega v(x)\,dx \in V.$$

By using such test functions in (6.29) and by integration by parts, we obtain

$$\forall v \in \mathscr{D}(\bar{\Omega}) \quad \int_\Omega \nabla u \cdot \nabla v\,dx = \int_\Omega f\Big(v - \frac{1}{|\Omega|}\int_\Omega v(x)\,dx\Big)\,dx. \tag{6.35}$$

It is only now that we use the information $\int_\Omega f(x)\,dx = 0$ to obtain

$$\forall v \in \mathscr{D}(\bar{\Omega}) \quad \int_\Omega \nabla u \cdot \nabla v\,dx = \int_\Omega f v\,dx. \tag{6.36}$$

The end of the proof runs as before and we obtain that u satisfies (6.31).

If u^* is another classical solution of (6.29), then

$$\begin{cases} -\Delta(u-u^*) = 0 & \text{on } \Omega, \\ \frac{\partial}{\partial n}(u-u^*) = 0. \end{cases}$$

Let us multiply the first equation by $u-u^*$ and integrate by parts. We obtain

$$\int_\Omega |\nabla(u-u^*)|^2\,dx = 0$$

and, hence, $u-u^* \equiv C$ for some constant C.

(iii) Conversely, let u be a classical solution of (6.28). Then, $u - \frac{1}{|\Omega|}\int_\Omega u(x)\,dx := u^*$ belongs to V and satisfies

$$\begin{cases} -\Delta u^* = f & \text{on } \Omega, \\ \frac{\partial u^*}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us multiply the first equation by $v \in \mathscr{D}(\bar{\Omega})$ and integrate by parts

$$\forall v \in \mathscr{D}(\bar{\Omega}) \quad \int_\Omega \nabla u^* \cdot \nabla v\,dx = \int_\Omega f v\,dx.$$

By density of $\mathscr{D}(\bar{\Omega})$ in $H^1(\Omega)$ we obtain

$$\forall v \in H^1(\Omega) \quad \int_\Omega \nabla u^* \cdot \nabla v\,dx = \int_\Omega f v\,dx,$$

and since $V \subset H^1(\Omega)$ we obtain (6.29), i.e., u^* is the unique solution of problem (6.29). $\square$

Let us now describe the other approach, which is an illustration of the so-called Tikhonov regularization method.

Theorem 6.2.4. *Let $f \in L^2(\Omega)$ be given such that $\int_\Omega f(x)\,dx = 0$. For any $\varepsilon > 0$, let u_ε be the unique solution of the variational problem*

$$\begin{cases} \displaystyle\int_\Omega (\varepsilon u_\varepsilon v + \nabla u_\varepsilon \cdot \nabla v)\,dx = \int_\Omega f v\,dx \quad \forall v \in H^1(\Omega), \\ u_\varepsilon \in H^1(\Omega). \end{cases} \tag{6.37}$$

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Equivalently, u_ε is the weak solution of the homogeneous Neumann problem

$$\begin{cases} \varepsilon u_\varepsilon - \Delta u_\varepsilon = f & \text{on } \Omega, \\ \frac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then the family $(u_\varepsilon)_{\varepsilon\to 0}$ norm converges in $H^1(\Omega)$, as $\varepsilon \to 0$, to a function $u \in H^1(\Omega)$, which is the solution of problem (6.29)*:*

$$\begin{cases} \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in V, \\ u \in V, \end{cases}$$

where $V = \{v \in H^1(\Omega) : \int_\Omega v(x)\, dx = 0\}$. Equivalently, u is the weak solution of

$$\begin{cases} -\Delta u = f & \text{on } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \\ \int_\Omega u(x)\, dx = 0. \end{cases}$$

PROOF. For $\varepsilon > 0$, take $a_0 \equiv \varepsilon$. We are in the coercive situation described by Theorem 6.2.1. Therefore, problem (6.37) admits a unique solution u_ε. Let us first prove that the family $(u_\varepsilon)_{\varepsilon\to 0}$ remains bounded in $H^1(\Omega)$. By taking $v = u_\varepsilon$ in (6.37), we obtain

$$\varepsilon \int_\Omega u_\varepsilon^2 \, dx + \int_\Omega |\nabla u_\varepsilon|^2 \, dx = \int_\Omega f u_\varepsilon \, dx. \tag{6.38}$$

By taking $v \equiv 1$ in (6.37), and using that $\int_\Omega f(x)\, dx = 0$, we obtain

$$\int_\Omega u_\varepsilon(x)\, dx = 0. \tag{6.39}$$

Let us invoke again the Poincaré–Wirtinger inequality (Corollary 5.4.1), which, in particular, implies the existence of a constant $C > 0$ such that

$$\forall v \in V \quad ||v||_{L^2(\Omega)} \leq C\left(\int_\Omega |\nabla v|^2 \, dx \right)^{1/2}. \tag{6.40}$$

Combining (6.38), (6.39), and (6.40) we obtain

$$\begin{aligned} \int_\Omega u_\varepsilon^2 \, dx &\leq C^2 \int_\Omega |\nabla u_\varepsilon|^2 \, dx \\ &\leq C^2 \left(\int_\Omega f^2 \, dx \right)^{1/2} \left(\int_\Omega u_\varepsilon^2 \, dx \right)^{1/2} \end{aligned}$$

and hence

$$\left(\int_\Omega u_\varepsilon^2 \, dx \right)^{1/2} \leq C^2 \left(\int_\Omega f^2 \, dx \right)^{1/2}.$$

Returning to (6.38), we get

$$\int_\Omega |\nabla u_\varepsilon|^2 \, dx \leq C^2 \int_\Omega f^2 \, dx.$$

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Finally,

$$||u_\varepsilon||_{H^1(\Omega)} \le C(1+C)||f||_{L^2(\Omega)}. \tag{6.41}$$

Let us extract a subsequence (which we still denote by (u_ε)) which weakly converges in $H^1(\Omega)$ to some $u^* \in H^1(\Omega)$. When passing to the limit on (6.37) we immediately obtain

$$\begin{cases} \int_\Omega \nabla u^* \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in H^1(\Omega), \\ u^* \in V. \end{cases} \tag{6.42}$$

The problem (6.42) clearly admits at most a solution: take two solutions u_1^* and u_2^*, make the difference, and take $v = u_1^* - u_2^*$; one obtains $u_1^* - u_2^* =$ constant and $u_1^* - u_2^* \in V$, which implies $u_1^* = u_2^*$.

Hence, the whole sequence $(u_\varepsilon)_{\varepsilon\to 0}$ weakly converges to the unique solution u^* of (6.42), and $u^* = u$, which is the solution that we obtained in Theorem 6.2.3 by a different argument. Let us complete the argument by noticing that by passing to the limit on (6.38)

$$\lim_{\varepsilon\to 0} \int_\Omega |\nabla u_\varepsilon|^2 \, dx = \int_\Omega f u.$$

On the other hand, by taking $v = u$ in (6.29) we have

$$\int_\Omega f u = \int_\Omega |\nabla u|^2 \, dx.$$

Hence $u_\varepsilon \to u$ in $w - H^1(\Omega)$ and $||u_\varepsilon||_{H^1} \to ||u||_{H^1}$. This implies that $u_\varepsilon \to u$ strongly in $H^1(\Omega)$. $\square$

6.2.4 ▪ The semicoercive nonhomogeneous Neumann problem

Given $f : \Omega \to \mathbf{R}$ and $g : \partial\Omega \to \mathbf{R}$, we are looking for a solution $u : \bar{\Omega} \to \mathbf{R}$ of the following boundary value problem:

$$\begin{cases} -\Delta u = f & \text{on } \Omega, \\ \frac{\partial u}{\partial n} = g & \text{on } \partial\Omega. \end{cases}$$

Theorem 6.2.5. *Let $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ be given functions which satisfy the so-called compatibility condition:*

$$\int_\Omega f(x)\, dx + \int_{\partial\Omega} g(x)\, d\sigma(x) = 0. \tag{6.43}$$

Set

$$V = \left\{ v \in H^1(\Omega) : \int_\Omega v(x)\, dx = 0 \right\}.$$

Then there exists a unique solution u of the problem

$$\begin{cases} \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx + \int_{\partial\Omega} g v \, d\sigma \quad \forall v \in V, \\ u \in V. \end{cases} \tag{6.44}$$

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Equivalently, u is the unique solution of the minimization problem

$$\min\left\{\frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega f\,v\,dx - \int_{\partial\Omega} g\,v\,d\sigma \;:\; v\in V\right\}. \tag{6.45}$$

The function u is a weak solution of the following nonhomogeneous Neumann boundary value problem:

$$\begin{cases} -\Delta u = f & \text{on } \Omega,\\ \frac{\partial u}{\partial n} = g & \text{on } \partial\Omega. \end{cases} \tag{6.46}$$

PROOF. Most of the ingredients of the proof were introduced in the previous sections. Therefore, we just briefly sketch the main lines of the proof. The only difference with the semicoercive homogeneous case comes from the linear form $l : V \to \mathbf{R}$,

$$l(v) = \int_\Omega f\,v\,dx + \int_{\partial\Omega} g\,v\,d\sigma.$$

The continuity of l follows from the continuity of the trace operator γ_0 from $H^1(\Omega)$ into $L^2(\partial\Omega)$. The only point which deserves particular attention is the interpretation of (6.44) as a boundary value problem.

Let us assume that the solution u of (6.44) is regular. Given an arbitrary $v \in \mathscr{D}(\bar\Omega)$, let us notice that $v - \frac{1}{|\Omega|}\int_\Omega v(x)\,dx$ belongs to V. Taking such a test function in (6.44), we infer that

$$\int_\Omega \nabla u\cdot\nabla v\,dx = \int_\Omega f\,v\,dx + \int_{\partial\Omega} g\,v\,d\sigma - \mathscr{M}(v)\left[\int_\Omega f(x)\,dx + \int_{\partial\Omega} g(x)\,d\sigma(x)\right],$$

where we set $\mathscr{M}(v) := \frac{1}{|\Omega|}\int_\Omega v(x)\,dx$. By using (6.43) we obtain

$$\forall v\in\mathscr{D}(\bar\Omega) \quad \int_\Omega \nabla u\cdot\nabla v\,dx = \int_\Omega f\,v\,dx + \int_{\partial\Omega} g\,v\,d\sigma$$

and conclude by using arguments similar to those in the proof of Theorem 6.2.2. □

Remark 6.2.2. (1) We have described two different methods to reduce the semicoercive Neumann problem to the coercive one. Each time, the idea is to reduce the problem to a situation where one can apply some Poincaré inequality. As another possibility let us mention (it is a good exercise to develop it) the parallel approach which consists in working with the space W (instead of V) defined by

$$W = \left\{v\in H^1(\Omega) \;:\; \int_{\partial\Omega} v(x)\,d\sigma(x) = 0\right\}.$$

(2) The term *semicoercive* comes from the fact that the lack of coercivity concerns only the lower-order terms (here only the zero-order term, namely, u!).

6.3 ▪ Mixed Dirichlet–Neumann problems

We are going to consider boundary value problems whose boundary conditions contain both u and $\frac{\partial u}{\partial n}$.

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6.3.1 ▪ The Dirichlet–Neumann problem

Let Ω be an open bounded set in $\mathbf{R}^N$ which is assumed to be connected and regular. (Its boundary $\Gamma = \partial\Omega$ is piecewise of class $\mathbf{C}^1$.) Let us suppose that $\Gamma = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$ with $H^{N-1}(\Gamma_0) > 0$, where Γ is the union of the two disjoint sets Γ_0 and Γ_1.

Given $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_1)$ we are looking for a solution u of the following boundary value problem:

$$\begin{cases} -\Delta u = f & \text{on } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} = g & \text{on } \Gamma_1. \end{cases}$$

Note that two different types of boundary conditions are imposed to u: on Γ_0 it is a Dirichlet condition, while on the complementary $\Gamma_1 = \Gamma \setminus \Gamma_0$ it is a Neumann condition. This is a simplified model for an important situation in mechanics, where an elastic material is fixed on a part Γ_0 of its boundary, while on the complementary a surface density of force g is present.

Let us introduce the functional space

$$V := \{v \in H^1(\Omega) \,:\, \gamma_0(v) = 0 \text{ on } \Gamma_0\}. \tag{6.47}$$

The space V, as a subspace of $H^1(\Omega)$, is equipped with the scalar product of $H^1(\Omega)$ and the corresponding norm

$$\|v\|_V = \left(\int_\Omega (v^2 + |\nabla v|^2)\,dx\right)^{1/2}.$$

It follows immediately from the continuity of the trace operator $\gamma_0 : H^1(\Omega) \to L^2(\Gamma)$ that V is a closed subspace of $H^1(\Omega)$. Note that $\gamma_0(v) = 0$ in Γ_0 has to be understood in the following sense: $\gamma_0(v)(x) = 0$ for a.e. x with respect to the measure $H^{N-1}\lfloor\Gamma_0$. Hence V is a Hilbert space. We can now state the variational formulation of the Dirichlet–Neumann problem.

Theorem 6.3.1. (i) *Let $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_1)$ be given functions. Let us assume moreover that $H^{N-1}(\Gamma_0) > 0$. Then, there exists a unique solution u of the problem*

$$\begin{cases} \displaystyle\int_\Omega \nabla u \cdot \nabla v\,dx = \int_\Omega f v\,dx + \int_{\Gamma_1} g v\,d\sigma \quad \forall v \in V, \\ u \in V. \end{cases} \tag{6.48}$$

Equivalently, u is the unique solution of the minimization problem

$$\min\left\{\frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega f v\,dx - \int_{\Gamma_1} g v\,d\sigma \,:\, v \in V\right\}. \tag{6.49}$$

(ii) *Assume that Γ_0 is sufficiently regular to have the property*

$$\left\{v|_{\Gamma_1} \,:\, v \in \mathscr{D}(\bar{\Omega}),\ v = 0 \text{ on } \Gamma_0\right\} \quad \textit{is dense in } L^2(\Gamma_1). \tag{6.50}$$

Then the solution u of the variational problem (6.48) *is a weak solution of the following Dirichlet–Neumann boundary value problem:*

$$\begin{cases} -\Delta u = f & \textit{on } \Omega, \\ u = 0 & \textit{on } \Gamma_0, \\ \frac{\partial u}{\partial n} = g & \textit{on } \Gamma_1. \end{cases} \tag{6.51}$$

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PROOF. (i) Clearly, the bilinear form

$$a(u,v)=\int_{\Omega} \nabla u \cdot \nabla v \, dx$$

is continuous on $V \times V$. The coercivity of a is a consequence of the generalized Poincaré inequality (Theorem 5.4.3): we have seen that V is a closed subspace of $H^1(\Omega)$. On the other hand, the assumption $H^{N-1}(\Gamma_0) > 0$ implies that the only constant function belonging to V is the zero function. Hence, there exists some $C > 0$ such that

$$\forall v \in V \quad \|v\|_{L^2(\Omega)} \leq C \left(\int_{\Omega} |\nabla v|^2 \, dx \right)^{1/2}.$$

This immediately implies that

$$\forall v \in V \quad a(v,v) \geq \frac{1}{1+C^2} \|v\|^2_{H^1(\Omega)}.$$

Let us now consider the linear form $l : V \to \mathbf{R}$ defined by

$$l(v)=\int_{\Omega} f v \, dx + \int_{\Gamma_1} g v \, d\sigma.$$

The continuity of the trace operator $\gamma_0 : H^1(\Omega) \to L^2(\Gamma)$ (Theorem 5.6.1) clearly implies that l is continuous.

(ii) Let us now suppose that the solution u of the variational problem (6.48) is smooth, and let us prove that u is a classical solution of (6.51).

Let us first take $v \in \mathcal{D}(\Omega)$ (note that $\mathcal{D}(\Omega) \subset V$). We obtain

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \quad \forall v \in \mathcal{D}(\Omega),$$

which classically implies

$$-\Delta u = f \text{ on } \Omega. \tag{6.52}$$

Let us now take $v \in \mathcal{D}(\bar{\Omega})$ such that $v = 0$ on Γ_0. Since $v \in V$, we have, by (6.48),

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx + \int_{\Gamma_1} g v \, d\sigma \quad \forall v \in \mathcal{D}(\bar{\Omega}),\ v = 0 \text{ on } \Gamma_0.$$

After integration by parts, and by using (6.52), we obtain

$$\int_{\Gamma_1} \left(\frac{\partial u}{\partial n} - g \right) v \, d\sigma = 0 \qquad \forall v \in \mathcal{D}(\bar{\Omega}), v = 0 \text{ on } \Gamma_0. \tag{6.53}$$

From the regularity assumption (6.50) on Γ_0 and (6.53) we infer that

$$\frac{\partial u}{\partial n} = g \text{ on } \Gamma_1,$$

which completes the proof. □

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Remark 6.3.1. (1) Conversely, when proving that a classical solution of (6.51) is a variational solution of (6.48), one needs to make the following assumption: $\mathscr{V}$ is dense in V for the $H^1(\Omega)$ norm, where

$$\mathscr{V} = \{v \in \mathscr{D}(\bar{\Omega}) \,:\, v = 0 \text{ on } \Gamma_0\}. \tag{6.54}$$

(2) The relationship between the classical and the variational solution for the Dirichlet–Neumann problem is quite delicate. Beside the classical regularity assumptions on Ω and its boundary $\Gamma = \partial\Omega$, it requires extra regularity assumptions on the portion Γ_0 of Γ, where the Dirichlet condition is imposed. Otherwise we could take some Γ_0, with $H^{N-1}(\Gamma_0) > 0$ which is dense in Γ; then, u regular and $u = 0$ on Γ_0 would force u to be equal to zero everywhere on Γ, which makes the above argumentation no longer valid.

6.3.2 ▪ Mixed Dirichlet–Neumann boundary conditions

We still consider an open bounded regular set Ω in $\mathbf{R}^N$. Let us give some positive measurable function $a_0 : \partial\Omega \to \mathbf{R}^+$ such that there exists a positive real number $\alpha > 0$ with

$$a_0(x) \geq \alpha \quad \text{for a.e. } x \in \partial\Omega \quad \text{with respect to } H^{N-1}|\partial\Omega. \tag{6.55}$$

Theorem 6.3.2. *Let $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ be given functions and assume* (6.55).

(i) *Then, there exists a unique solution u of the following system:*

$$\begin{cases} \displaystyle\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} a_0 u v \, d\sigma = \int_\Omega f v \, dx + \int_{\partial\Omega} g v \, d\sigma \quad \forall v \in H^1(\Omega). \\ u \in H^1(\Omega). \end{cases} \tag{6.56}$$

Equivalently, u is the unique solution of the minimization problem

$$\min\left\{\frac{1}{2}\int_\Omega |\nabla v|^2 \, dx + \frac{1}{2}\int_{\partial\Omega} a_0 v^2 \, d\sigma - \int_\Omega f v \, dx - \int_{\partial\Omega} g v \, d\sigma \,:\, v \in H^1(\Omega)\right\}. \tag{6.57}$$

(ii) *The solution u of* (6.56) *is a weak solution of the following boundary value problem:*

$$\begin{cases} -\Delta u = f & \text{on } \Omega, \\ a_0 u + \dfrac{\partial u}{\partial n} = g & \text{on } \partial\Omega. \end{cases} \tag{6.58}$$

PROOF. (i) The only point which is not standard is to verify that the bilinear form

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} a_0 u v \, d\sigma$$

is coercive on $H^1(\Omega)$. By assumption (6.55) we have

$$\forall v \in H^1(\Omega) \quad a(v,v) \geq \int_\Omega |\nabla v|^2 \, dx + \alpha \int_{\partial\Omega} v^2 \, d\sigma. \tag{6.59}$$

Let us apply the generalized Poincaré inequality (Theorem 5.4.3) to the space

$$V = \left\{v \in H^1(\Omega) \,:\, \int_{\partial\Omega} v(x)\, d\sigma(x) = 0\right\}, \tag{6.60}$$

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where, as usual, for simplicity of notation, we write v instead of $\gamma_0(v)$. By using again the continuity of the trace operator $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$, we have that V is a closed subspace of $H^1(\Omega)$. Moreover, if v is a constant function which belongs to V, say, $v \equiv C$, we necessarily have $CH^{N-1}(\partial\Omega) = 0$ which forces C to be equal to zero.

Let us finally observe that

$$\forall v \in H^1(\Omega) \quad v - \mathcal{M}_{\partial\Omega}(v) \in V,$$

where $\mathcal{M}_{\partial\Omega}(v) := \frac{1}{|\partial\Omega|}\int_{\partial\Omega} v(x)\,d\sigma(x)$. Thus, by applying the generalized Poincaré inequality to the space V, we obtain the existence of a positive constant C such that

$$\forall v \in H^1(\Omega) \quad ||v - \mathcal{M}_{\partial\Omega}(v)||_{L^2(\Omega)} \le C\left(\int_\Omega |\nabla v|^2\,dx\right)^{1/2}. \tag{6.61}$$

From (6.61) we easily obtain the inequality

$$\forall v \in H^1(\Omega) \quad \int_\Omega v(x)^2\,dx \le 2C^2\int_\Omega |\nabla v|^2\,dx + 2|\Omega|\mathcal{M}_{\partial\Omega}(v)^2. \tag{6.62}$$

On the other hand, by using the Cauchy–Schwarz inequality in $L^2(\partial\Omega)$, we obtain

$$\mathcal{M}_{\partial\Omega}(v) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} v(x)\,d\sigma(x) \le \frac{1}{|\partial\Omega|^{1/2}}\left(\int_{\partial\Omega} v(x)^2\,d\sigma(x)\right)^{1/2}. \tag{6.63}$$

Combining (6.62) and (6.63), we obtain

$$\forall v \in H^1(\Omega) \quad \int_\Omega v(x)^2\,dx \le 2C^2\int_\Omega |\nabla v|^2\,dx + 2\frac{|\Omega|}{|\partial\Omega|}\int_{\partial\Omega} v(x)^2\,d\sigma(x). \tag{6.64}$$

It is an elementary computation to obtain, by using (6.59) and (6.64), the inequality

$$\forall v \in H^1(\Omega) \quad a(v,v) \ge \gamma ||v||^2_{H^1(\Omega)} \tag{6.65}$$

with

$$\gamma = \frac{\min\{\alpha, 1\}}{1 + 2\max\{C^2, \frac{|\Omega|_N}{|\partial\Omega|_{N-1}}\}}.$$

(ii) Let us assume that the solution u of (6.56) is regular. Then, by taking $v \in \mathcal{D}(\bar{\Omega})$ in (6.56) and integrating by parts, we obtain

$$\int_\Omega (-\Delta u - f)dx + \int_{\partial\Omega}\left(a_0 u + \frac{\partial u}{\partial n} - g\right)v\,d\sigma = 0 \quad \forall v \in \mathcal{D}(\bar{\Omega}).$$

An analysis similar to the one used in the case of the Neumann problem gives that u satisfies (6.58). □

Remark 6.3.2. It is interesting to notice that the boundary condition $a_0 u + \frac{\partial u}{\partial n} = g$ contains, at least formally, all the previous boundary conditions that we have examined.

(i) Take $a_0 = 0$; then one obtains the Neumann boundary condition $\frac{\partial u}{\partial n} = g$. Note that in this case, the coercivity property is lost.

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(ii) Take $a_0 \equiv +\infty$; then formally one obtains the Dirichlet boundary condition, $u = 0$ on $\partial\Omega$.

(iii) Take $a_0 = 0$ on Γ_1 and $a_0 = +\infty$ on $\Gamma_0 = \Gamma \setminus \Gamma_0$; then one obtains the Dirichlet–Neumann boundary condition.

Indeed, it is a good exercise to justify these formal results. For example, take $a_0(x) \equiv n$ in (6.56) and prove that the corresponding sequence (u_n) norm converges to the solution u of the Dirichlet problem.

6.4 ▪ Heterogeneous media: Transmission conditions

Let us consider the following model situation coming from electrostatics. In the open set Ω of $\mathbf{R}^N$ we have two distinct materials with respective conductivity coefficients α and β; take $0 < \alpha < \beta < +\infty$, for example. The subsets of Ω occupied by the two materials are denoted, respectively, by Ω_α and Ω_β. We assume that Ω_α and Ω_β are open sets with a common boundary denoted by Σ which is a $\mathbf{C}^1$ manifold and

$$\Omega = \Omega_\alpha \cup \Omega_\beta \cup \Sigma.$$

In this model, the conductivity coefficient $a : \Omega \to \mathbf{R}$ takes only two values, $a \equiv \alpha$ on Ω_α and $a \equiv \beta$ on Ω_β:

$$a(x) = \begin{cases} \alpha & \text{if } x \in \Omega_\alpha, \\ \beta & \text{if } x \in \Omega_\beta. \end{cases}$$

Note that a is not continuous (it is discontinuous through Σ). It belongs to $L^\infty(\Omega)$. Indeed, in the following developments, $a(\cdot)$ plays a role only as a function defined a.e. with respect to the Lebesgue measure. Since Σ has zero Lebesgue measure, we don't need to define a on Σ.

For each $x \in \Sigma$, $n(x)$ is the unit normal vector to Σ at x which is directed outward with respect to Ω_α (and inward with respect to Ω_β).

For any (electrostatic) potential function $v : \Omega \to \mathbf{R}$ the corresponding stored internal energy is

$$\Phi(v) = \int_\Omega a(x)|\nabla v(x)|^2\,dx.$$

Suppose that the conductor is connected to the earth on its boundary. For a given density of charge $f : \Omega \to \mathbf{R}$ the equilibrium potential function $u : \bar{\Omega} \to \mathbf{R}$ solves the minimization problem

$$\min\left\{\frac{1}{2}\int_\Omega a(x)|\nabla v(x)|^2\,dx - \int_\Omega f(x)v(x)\,dx \;:\; v = 0 \text{ on } \partial\Omega\right\}.$$

The variational approach to the above problem, and the corresponding transmission conditions through the interface Σ satisfied by the solution u, are described in the following statement.

Theorem 6.4.1. (a) *For every $f \in L^2(\Omega)$ there exists a unique $u \in H_0^1(\Omega)$ solution of the following minimization problem:*

$$\min\left\{\frac{1}{2}\int_\Omega a(x)|\nabla v(x)|^2\,dx - \int_\Omega f(x)v(x)\,dx \;:\; v \in H_0^1(\Omega)\right\}. \tag{6.66}$$

(b) *Equivalently, the solution u of* (6.66) *verifies*

$$\begin{cases} \int_\Omega a(x)\nabla u(x)\cdot\nabla v(x)\,dx = \int_\Omega f(x)v(x)\,dx \quad \forall v\in H_0^1(\Omega), \\ u\in H_0^1(\Omega). \end{cases} \tag{6.67}$$

(c) *Equivalently, the solution u of* (6.66) *is a weak solution of the boundary value problem*

$$\begin{cases} -\mathrm{div}(a(x)\nabla u(x)) = f \text{ on } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{6.68}$$

in the following sense: $u\in H_0^1(\Omega)$, the first equation is satisfied in the distribution sense, while $u=0$ is satisfied in the trace sense.

(d) *Let us denote $u_\alpha = u|_{\Omega_\alpha}$ and $u_\beta = u|_{\Omega_\beta}$, and let us assume that the variational (weak) solution of* (6.66) *is regular in the following sense: $u_\alpha \in \mathbf{C}^2(\bar{\Omega}_\alpha)$ and $u_\beta\in\mathbf{C}^2(\bar{\Omega}_\beta)$. Then u is a classical solution of the following transmission problem:*

$$\begin{cases} -\alpha\Delta u_\alpha = f & \text{on } \Omega_\alpha, \\ -\beta\Delta u_\beta = f & \text{on } \Omega_\beta, \\ u_\alpha = u_\beta & \text{on } \Sigma, \\ \alpha\frac{\partial u_\alpha}{\partial n} = \beta\frac{\partial u_\beta}{\partial n} & \text{on } \Sigma, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.69}$$

In particular, u is continuous through Σ, but $\frac{\partial u}{\partial n}$ is discontinuous through Σ.

(e) *Conversely, if u is a classical solution, i.e., $u_\alpha\in\mathbf{C}^2(\bar{\Omega}_\alpha)$, $u_\beta\in\mathbf{C}^2(\bar{\Omega}_\beta)$, and u satisfies* (6.69), *then it is equal to the variational (weak) solution of* (6.66).

PROOF. (a) The argument is quite similar to the one developed in the variational approach to the Dirichlet problem. The functional $J: H_0^1(\Omega)\to\mathbf{R}$ defined by

$$J(v) := \frac{1}{2}\int_\Omega a(x)|\nabla v(x)|^2\,dx - \int_\Omega f v\,dx$$

is convex and continuous on $H_0^1(\Omega)$. The coercivity of J follows from the inequality

$$J(v) \geq \frac{\min(\alpha,\beta)}{2}\int_\Omega |\nabla v|^2\,dx - \|f\|_{L^2}\|v\|_{L^2}$$

and the Poincaré inequality on $H_0^1(\Omega)$.

The conditions of the convex minimization theorem, Theorem 3.3.4, are fulfilled. Hence, there exists a solution u to problem (6.66). It is unique because J is strictly convex. Indeed, this follows from the strict convexity of $\Phi(v) = \int a(x)|\nabla v|^2\,dx$, which is a positive definite quadratic form on $H_0^1(\Omega)$ (see Proposition 2.3.4).

(b) The equivalence between (6.66) and (6.67) is a direct consequence of Proposition 2.3.1 and the fact that the bilinear form

$$b(u,v) := \int_\Omega a(x)\nabla u(x)\cdot\nabla v(x)\,dx$$

is symmetric and positive. Indeed, we could as well solve the variational problem by applying the Lax–Milgram theorem, Theorem 3.1.2, to the formulation (6.67).

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(c) It follows from the density of $\mathscr{D}(\Omega)$ in $H_0^1(\Omega)$ that it is equivalent in the variational formulation (6.67) to take only v belonging to $\mathscr{D}(\Omega)$. One then obtains the equivalent formulation (6.68) just by using the notion of derivation in the distribution sense.

(d) Let us now come to the point which deserves some particular attention, namely, the interpretation of the distribution formula

$$-\operatorname{div}(a(x)\nabla u(x)) = f \text{ on } \Omega.$$

As we just pointed out, all the information we have on u is contained in this distribution formula and the fact that $u \in H_0^1(\Omega)$. Equivalently, we have

$$\begin{cases} \displaystyle\int_\Omega a(x)\nabla u(x)\cdot\nabla v(x)\,dx - \int_\Omega f(x)v(x)\,dx = 0 \quad \forall v \in \mathscr{D}(\Omega), \\ u \in H_0^1(\Omega). \end{cases} \tag{6.70}$$

Let us particularize the test function v to test successively u on Ω_α, Ω_β and then on Σ. We assume that $u_\alpha \in \mathbf{C}^2(\bar{\Omega}_\alpha)$ and $u_\beta \in \mathbf{C}^2(\bar{\Omega}_\beta)$.

(1) Let us first take $v \in \mathscr{D}(\Omega_\alpha)$. From (6.70) and the fact that $a \equiv \alpha$ on Ω_α, we obtain that $u_\alpha = u|_{\Omega_\alpha}$ satisfies

$$\alpha\int_{\Omega_\alpha} \nabla u_\alpha \cdot \nabla v\,dx - \int_{\Omega_\alpha} f v\,dx = 0 \quad \forall v \in \mathscr{D}(\Omega_\alpha).$$

This yields

$$-\alpha\Delta u_\alpha = f \quad \text{on } \Omega_\alpha. \tag{6.71}$$

(2) Similarly, by taking test functions $v \in \mathscr{D}(\Omega_\beta)$ one obtains

$$-\beta\Delta u_\beta = f \quad \text{on } \Omega_\beta. \tag{6.72}$$

(3) Let us now analyze the transmission conditions through Σ which are satisfied by u.

Let us first observe that $u_\alpha = u_\beta$ on Σ. Indeed, since $u \in H_0^1(\Omega)$ there exists some approximating sequence $(u_n)_{u\in\mathbf{N}}$, $u_n \in \mathscr{D}(\Omega)$ for each $n \in \mathbf{N}$ such that $u_n \to u$ in $H^1(\Omega)$ as $n \to +\infty$. Let us identify u_n and its extension by zero outside of Ω. By definition of $\mathscr{D}(\bar{\Omega}_\alpha)$ and $\mathscr{D}(\bar{\Omega}_\beta)$ we have that for every $n \in \mathbf{N}$,

$$u_n|_{\Omega_\alpha} \in \mathscr{D}(\bar{\Omega}_\alpha) \quad \text{and} \quad u_n|_{\Omega_\beta} \in \mathscr{D}(\bar{\Omega}_\beta).$$

Moreover,

$$u_n|_{\Omega_\alpha} \to u_\alpha \quad \text{in } H^1(\Omega_\alpha),$$
$$u_n|_{\Omega_\beta} \to u_\beta \quad \text{in } H^1(\Omega_\beta).$$

By using the continuity of the trace operator $\gamma_{0,\alpha} : H^1(\Omega_\alpha) \to L^2(\Sigma)$ and the fact that for functions in $\mathscr{D}(\bar{\Omega}_\alpha)$ the trace coincides with the restriction, we obtain that

$$u_n|_\Sigma \to \gamma_{0,\alpha}(u_\alpha) \quad \text{in } L^2(\Sigma).$$

Similarly, we have

$$u_n|_\Sigma \to \gamma_{0,\beta}(u_\beta) \quad \text{in } L^2(\Sigma).$$

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Hence $\gamma_{0,\alpha}(u_\alpha)=\gamma_{0,\beta}(u_\beta)$, that is, the traces of u from both sides of Σ are the same. Since u has been assumed to be $\mathbf{C}^2(\bar{\Omega}_\alpha)$ and $\mathbf{C}^2(\bar{\Omega}_\beta)$, these traces coincide with the respective values of u_α and u_β on Σ, that is,

$$u_\alpha = u_\beta \quad \text{on } \Sigma, \tag{6.73}$$

which makes the function u continuous on Ω.

Let us now take a general test function $v \in \mathscr{D}(\Omega)$ and rewrite (6.70) as

$$\alpha \int_{\Omega_\alpha} \nabla u_\alpha \cdot \nabla v \, dx + \beta \int_{\Omega_\beta} \nabla u_\beta \cdot \nabla v \, dx - \int_{\Omega_\alpha} f v \, dx - \int_{\Omega_\beta} f v \, dx = 0. \tag{6.74}$$

Integrating by parts gives

$$\alpha \int_{\Omega_\alpha} \nabla u_\alpha \cdot \nabla v \, dx = -\int_{\Omega_\alpha} (\alpha \Delta u_\alpha) v \, dx + \alpha \int_\Sigma v \frac{\partial u_\alpha}{\partial n} \, d\sigma. \tag{6.75}$$

Similarly (note that the outward normal to Ω_β on Σ is now the opposite vector $-\vec{n}$), we have

$$\beta \int_{\Omega_\beta} \nabla u_\beta \cdot \nabla v \, dx = -\int_{\Omega_\beta} (\beta \Delta u_\beta) v \, dx - \beta \int_\Sigma v \frac{\partial u_\beta}{\partial n} \, d\sigma. \tag{6.76}$$

Combining (6.74), (6.75), and (6.76) we obtain

$$-\int_{\Omega_\alpha} (\alpha \Delta u_\alpha + f) v \, dx - \int_{\Omega_\beta} (\beta \Delta u_\beta + f) v \, dx + \int_\Sigma \left(\alpha \frac{\partial u_\alpha}{\partial n} - \beta \frac{\partial u_\beta}{\partial n} \right) v \, d\sigma = 0.$$

By using (6.71) and (6.72), we finally obtain

$$\forall v \in \mathscr{D}(\Omega) \quad \int_\Sigma \left(\alpha \frac{\partial u_\alpha}{\partial n} - \beta \frac{\partial u_\beta}{\partial n} \right) v(x) \, d\sigma(x) = 0, \tag{6.77}$$

which implies

$$\alpha \frac{\partial u_\alpha}{\partial n} = \beta \frac{\partial u_\beta}{\partial n} \text{ on } \Sigma. \tag{6.78}$$

Let us finally notice that the condition $u = 0$ on $\partial\Omega$ follows from the fact that $u \in H_0^1(\Omega)$ and the regularity assumptions on $\partial\Omega$ and u.

(e) To pass from the boundary value problem (6.69) to (6.67) we proceed in a similar way, just making the integration by parts in the reverse way. The only point which requires some attention is the proof of the property $u \in H^1(\Omega)$. Indeed, this is a consequence of the following lemma of independent interest. $\square$

Lemma 6.4.1. *Let $u_\alpha \in H^1(\Omega_\alpha)$ and $u_\beta \in H^1(\Omega_\beta)$, Ω_α and Ω_β being two disjoint open sets with a common interface Σ. We define*

$$u = \begin{cases} u_\alpha & \text{on } \Omega_\alpha, \\ u_\beta & \text{on } \Omega_\beta. \end{cases}$$

Then the following derivation rule holds:

$$\nabla u = \chi_{\Omega_\alpha} \tilde{\nabla} u_\alpha + \chi_{\Omega_\beta} \tilde{\nabla} u_\beta + [u]_\Sigma \vec{n} \, d\sigma,$$

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where $[u]_\Sigma := \gamma_{0,\beta}(u_\beta) - \gamma_{0,\alpha}(u_\alpha)$ *is the jump of* u *through* Σ, $d\sigma = \mathscr{H}^{N-1}\lfloor\Sigma$, *and* $\tilde{\nabla u_\alpha}$, $\tilde{\nabla u_\beta}$ *denote the extensions by zero of* ∇u_α *and* ∇u_β, *respectively, on* $\Omega \setminus \Omega_\alpha$ *and* $\Omega \setminus \Omega_\beta$.

PROOF. Take $1 \le i \le N$ and compute $\frac{\partial u}{\partial x_i}$ in $\mathscr{D}'(\Omega)$:

$$\left\langle \frac{\partial u}{\partial x_i}, \varphi \right\rangle_{(\mathscr{D}'(\Omega),\mathscr{D}(\Omega))} := -\int_{\Omega_\alpha} u_\alpha \frac{\partial \varphi}{\partial x_i}\, dx - \int_{\Omega_\beta} u_\beta \frac{\partial \varphi}{\partial x_i}\, dx.$$

Let us integrate by parts by using the Green's formula (Proposition 5.6.2):

$$\begin{aligned}\left\langle \frac{\partial u}{\partial x_i}, \varphi \right\rangle_{(\mathscr{D}',\mathscr{D})} &= \int_{\Omega_\alpha} \frac{\partial u_\alpha}{\partial x_i}\varphi\, dx - \int_\Sigma \gamma_{0,\alpha}(u_\alpha)\varphi n_i\, d\sigma + \int_{\Omega_\beta} \frac{\partial u_\beta}{\partial x_i}\varphi\, dx \\ &\quad + \int_\Sigma \gamma_{0,\beta}(u_\beta)\varphi n_i\, d\sigma \\ &= \int_\Omega \left(\widetilde{\frac{\partial u_\alpha}{\partial x_i}} \chi_{\Omega_\alpha} + \widetilde{\frac{\partial u_\beta}{\partial x_i}} \chi_{\Omega_\beta} \right) \varphi\, dx \\ &\quad + \int_\Sigma \big(\gamma_{0,\beta}(u_\beta) - \gamma_{0,\alpha}(u_\alpha)\big) n_i \varphi\, d\sigma.\end{aligned}$$

Together with the equality

$$\frac{\partial u}{\partial x_i} = \widetilde{\frac{\partial u_\alpha}{\partial x_i}} \chi_{\Omega_\alpha} + \widetilde{\frac{\partial u_\beta}{\partial x_i}} \chi_{\Omega_\beta} + [u]_\Sigma n_i H^{N-1} \lfloor \Sigma,$$

this ends the proof. □

Remark 6.4.1. (1) We stress the fact that the transmission law through Σ is contained in the formula

$$-\mathrm{div}(a(x)\nabla u(x)) = f \quad \text{on } \Omega,$$

which holds in the distribution sense. Let us first notice that this formula makes sense. We have $u \in H_0^1(\Omega)$, $a \in L^\infty(\Omega)$. Hence, for each $i \in \mathbf{N}$, $a(x)\frac{\partial u}{\partial x_i} \in L^2(\Omega)$, which defines a distribution on Ω and $\frac{\partial}{\partial x_i}(a(x)\frac{\partial u}{\partial x_i})$, is well defined as a distribution. The point is that we have to treat $a(x)\frac{\partial u}{\partial x_i}$ as a block; we cannot derive this product by using the classical derivation rule, because $a(\cdot) \in L^\infty(\Omega)$ and $\frac{\partial a}{\partial x_i}$ is a measure which is singular with respect to the Lebesgue measure. This fact reveals some of the strengths and weaknesses of the distribution theory. It is a powerful tool which allows us to formulate in a quite simple and unifying way a lot of phenomena. On the counterpart, the physical laws are often implicit and hidden in the distribution formulation.

(2) Taking $\Omega = (0,1)$, $\Omega_\alpha =]0, c[$, $\Omega_\beta =]c, 1[$, and $\Sigma = \{c\}$, the profile of the solution of the corresponding transmission problem corresponds to the classical Descartes law in geometrical optics, with the equality

$$\beta \frac{d^+ u}{dx}(c) = \alpha \frac{d^- u}{dx}(c).$$

6.5 ▪ Linear elliptic operators

Let Ω be an open subset of $\mathbf{R}^N$. Let us give a family $\{a_{ij}(\cdot) \,:\, 1 \leq i,j \leq n\}$ of functions belonging to $L^\infty(\Omega)$ and which satisfies the following condition: there exists some positive real number $\alpha > 0$ such that

$$\forall \xi \in \mathbf{R}^N \quad \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2 \quad \text{a.e. on } \Omega. \tag{6.79}$$

Let us introduce the corresponding bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbf{R}$,

$$\forall u,v \in H^1(\Omega) \quad a(u,v) := \int_\Omega \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx. \tag{6.80}$$

Property (6.79) implies that the bilinear form enjoys a so-called ellipticity condition:

$$\begin{aligned}\forall v \in H^1(\Omega) \qquad a(v,v) &= \int_\Omega \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx \\ &\geq \alpha \int_\Omega |\nabla v(x)|^2\,dx.\end{aligned} \tag{6.81}$$

By using (6.81) one can prove existence and uniqueness results for variational problems involving the bilinear form a. The proof is similar to the ones of the previous sections. Note that the case of the Dirichlet integral and of the corresponding Laplace equation corresponds to the particular situation $a_{ij}(x) = \delta_{ij}$.

Since we are now familiar with a number of boundary value problems (Dirichlet, Neumann, mixed Dirichlet–Neumann), let us give in the present situation a unified approach to these problems by taking as a functional space a closed subspace V of $H^1(\Omega)$ such that $H_0^1(\Omega) \subset V \subset H^1(\Omega)$.

Theorem 6.5.1. *Let us assume that Ω is a bounded open set in $\mathbf{R}^N$ which is regular (piecewise of class $\mathbf{C}^1$) and connected. Let V be a closed subspace of $H^1(\Omega)$ which satisfies conditions*
(a) $H_0^1(\Omega) \subset V \subset H^1(\Omega)$,
(b) $v \in V, v \equiv \textit{constant} \Longrightarrow v \equiv 0$.
On the other hand, let us give a family $(a_{ij})_{i,j=1,\dots,N}$ of $L^\infty(\Omega)$ functions which satisfies the ellipticity condition (6.79).

(i) *Then, for any $f \in L^2(\Omega)$ and any $g \in L^2(\partial\Omega)$, there exists a unique solution $u \in V$ of the problem*

$$\begin{cases} \displaystyle\int_\Omega \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx = \int_\Omega f v\,dx + \int_{\partial\Omega} g v\,d\sigma \quad \forall v \in V, \\ u \in V. \end{cases} \tag{6.82}$$

(ii) *The solution u of* (6.82) *satisfies*

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) = f \quad \textit{on } \Omega$$

in the distribution sense.

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(iii) *When the matrix* $(a_{ij}(x))_{1\le i,j\le N}$ *is symmetric,* (6.82) *is equivalent to saying that* u *is the unique solution of the minimization problem*

$$\min\left\{\frac{1}{2}\int_\Omega \sum_{i,j=1}^N a_{ij}(x)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx - \int_\Omega f v\,dx - \int_{\partial\Omega} g v\,d\sigma \;:\; v\in V\right\}.$$

PROOF. (i) To apply the Lax–Milgram theorem to problem (6.82), we are going to work in the space V which is considered as a subspace of $H^1(\Omega)$ and is equipped with the scalar product of $H^1(\Omega)$:

$$\langle u,v\rangle_V = \int_\Omega (uv + \nabla u\cdot\nabla v)\,dx.$$

Since V is a closed subspace of $H^1(\Omega)$, it is a Hilbert space.

Problem (6.82) can be written

$$\begin{cases} \text{find } u\in V \text{ such that} \\ a(u,v) = l(v) \quad \forall v\in V, \end{cases}$$

where a is given by (6.80) and $l(v) = \int_\Omega f v\,dx + \int_{\partial\Omega} g v\,d\sigma$.

Let us first verify that the bilinear form $a : V\times V \to \mathbf{R}$ is continuous on V. Set $M := \sup_{1\le i,j\le N} \|a_{ij}\|_{L^\infty(\Omega)}$. Then, for any $u,v\in V$,

$$\begin{aligned} |a(u,v)| &\le M\int_\Omega \sum_{i,j=1}^N \left|\frac{\partial u}{\partial x_i}\right|\left|\frac{\partial v}{\partial x_j}\right| dx \\ &\le M\int_\Omega N|\nabla u|\,|\nabla v|\,dx \\ &\le MN\left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2}\left(\int_\Omega |\nabla v|^2\,dx\right)^{1/2} \\ &\le MN\|u\|_V\,\|v\|_V, \end{aligned} \tag{6.83}$$

which implies that a is continuous. The continuity of l follows from the inequality

$$|l(v)| \le \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\|\gamma_0(v)\|_{L^2(\partial\Omega)}$$

and the continuity of the trace operator from $H^1(\Omega)$ into $L^2(\Sigma)$. Let us now verify the crucial point, that is, the coercivity of $a : V\times V\to\mathbf{R}$. By (6.81), since $V\subset H^1(\Omega)$ we have

$$\forall v\in V \quad a(v,v) \ge \alpha\|\nabla v\|^2_{L^2(\Omega)^N}. \tag{6.84}$$

We now use the generalized Poincaré inequality (Theorem 5.4.3): the assumptions (a) and (b) on V imply the existence of a positive constant C such that

$$\forall v\in V \quad \|v\|_{L^2(\Omega)} \le C\|\nabla v\|_{L^2(\Omega)^N}. \tag{6.85}$$

As a consequence of (6.84) and (6.85) we obtain that

$$\forall v\in V \quad a(v,v) \ge \frac{\alpha}{1+C^2}\|v\|_V^2$$

and a is coercive on V.

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Part (ii) is a direct consequence of (6.82) and of the fact that V contains $\mathscr{D}(\Omega)$.

Part (iii) is an equivalent formulation to (6.82) when the matrix $(a_{ij})_{i,j}$ is symmetric. Indeed, in that case the bilinear form $a : V \times V \to \mathbf{R}$ is symmetric, and the equivalence follows from Proposition 2.3.1. $\square$

As a particular case of Theorem 6.5.1, let us consider the following situation: take

$$V = \{v \in H^1(\Omega) \, : \, \gamma_0(v) = 0 \text{ on } \Gamma_0\},$$

where $\Gamma_0 \subset \Gamma = \partial\Omega$ is a measurable subset of the boundary Γ with a strictly positive surface measure: $H^{N-1}(\Gamma_0) > 0$. We set $\Gamma_1 = \Gamma \setminus \Gamma_0$. We know (see Section 6.3) that V satisfies all the assumptions of Theorem 6.5.1. The question we are going to examine is the interpretation of the boundary conditions satisfied by the solution u of (6.82).

Proposition 6.5.1. *When* $V = \{v \in H^1(\Omega) \, : \, \gamma_0(v) = 0 \text{ on } \Gamma_0\}$, *where* $H^{N-1}(\Gamma_0) > 0$, *the unique solution* u *of the problem*

$$\begin{cases} \displaystyle\int_\Omega \sum_{i,j=1}^N a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx = \int_\Omega f v \, dx + \int_{\partial\Omega} g v \, d\sigma \quad \forall v \in V, \\ u \in V, \end{cases} \tag{6.86}$$

is a weak solution of the following boundary value problem:

$$\begin{cases} \displaystyle -\sum_{i,j=1}^N \frac{\partial}{\partial x_j}\left(a_{ij} \frac{\partial u}{\partial x_i}\right) = f \quad \textit{on } \Omega, \\ u = 0 \quad \textit{on } \Gamma_0, \\ \frac{\partial u}{\partial \nu_A} = g \quad \textit{on } \Gamma_1, \end{cases} \tag{6.87}$$

where $\frac{\partial u}{\partial \nu_A} := \sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_i} n_j$ *is called the conormal derivative of* u *associated with the operator* $A : v \mapsto -\Sigma \frac{\partial}{\partial x_j}(a_{ij} \frac{\partial v}{\partial x_i})$.

PROOF. To find the boundary condition satisfied by u on Γ_1 let us make the following regularity assumptions: $u \in H^2(\Omega)$ and $a_{ij} \in \mathbf{C}^1(\bar{\Omega})$ for any $1 \leq i, j \leq N$.

Then, for any $i = 1, 2, \ldots, N$ we have

$$\xi_j := \sum_{i=1}^N a_{ij} \frac{\partial u}{\partial x_i} \in H^1(\Omega).$$

Let us write the Green's formula (Proposition 5.6.2),

$$\int_\Omega \sum_{j=1}^N \xi_j \frac{\partial v}{\partial x_j} \, dx = -\int_\Omega v \sum_{j=1}^N \frac{\partial \xi_j}{\partial x_j} \, dx + \int_{\Gamma_1} v \sum_{j=1}^N \xi_j n_j \, d\sigma. \tag{6.88}$$

From (6.86), (6.87), and (6.88) we infer that

$$\forall v \in V \quad \int_{\Gamma_1} \left(\sum_{j=1}^N \xi_j n_j \right) v \, d\sigma = \int_{\Gamma_1} g v \, d\sigma,$$

which implies

$$\sum_{j=1}^{N} \xi_j n_j = g,$$

that is, $\frac{\partial u}{\partial \nu_A} = g$ on Γ_1. $\square$

Remark 6.5.1. When the above regularity properties on a_{ij} and u are not satisfied, the formula $\frac{\partial u}{\partial \nu_A} = g$ on Γ_1 is just a formal way to express the boundary conditions implicitly contained in (6.86).

6.6 ▪ The linearized elasticity system

In this section, we are concerned with the study of the deformation of an N-dimensional elastic body ($N = 2$ or 3), occupying a domain Ω of $\mathbf{R}^N$, clamped on a part of its boundary, and subjected to a vector field of applied forces. Under the hypothesis that the body undergoes small deformations, we will see that the system of equations that models the equilibrium of the body furnishes a special case of elliptic problem.

We first specify the notation. The physical space is identified with $\mathbf{R}^N$ ($N = 2$ or 3) equipped with the canonical basis $(e_1, \dots, e_N)$ and the standard Euclidean inner product. We denote by $\Omega \subset \mathbf{R}^N$ an open bounded and connected set with smooth boundary Γ in the sense that the theory of traces can apply (for instance, piecewise of class $\mathbf{C}^1$). In the elasticity framework, Ω is referred to as the *interior reference configuration* of the body. We denote by Γ_0 the measurable subset of Γ where the body is clamped and set $\Gamma_1 = \Gamma \setminus \Gamma_0$ (the free boundary).

We denote by $\mathbf{M}^N$ and $\mathbf{M}^N_s$ the spaces of $N \times N$ and $N \times N$ symmetric matrices, respectively, equipped with the Hilbert–Schmidt inner product: for $A = (a_{ij})$ and $B = (b_{ij})$, $A : B := \sum_{i,j=1}^{N} a_{ij} b_{ij}$. The standard Euclidean scalar product of two vectors u and v in $\mathbf{R}^N$ is denoted by $u.v$. We use the same notation $|.|$ for the associated norms in $\mathbf{M}^N$ and $\mathbf{R}^N$. The gradient of a vector field $v = (v_i)_{i=1,\dots,N} : \Omega \to \mathbf{R}^N$ in $Ł^1_{loc}(\Omega)^N$ is the distribution matrix field $\nabla v : \Omega \to \mathbf{M}^N$ whose entries are $\frac{\partial v_i}{\partial x_j}$, where i is the row index. The *linearized strain tensor field* associated with an arbitrary vector field v in $L^1_{loc}(\Omega)^N$ is the distribution matrix field $\mathcal{E}(v) : \Omega \to \mathbf{M}^N_s$ given by $\mathcal{E}_{ij}(v) = \frac{1}{2}\big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\big)$. The divergence of a matrix field $M : \Omega \to \mathbf{M}^N$ in $L^1_{loc}(\Omega, \mathbf{M}^N)$ is the distribution vector field $\mathrm{div}(M) : \Omega \to \mathbf{R}^3$ given by $\mathrm{div}(M) = (\sum_{j=1}^{N} \frac{\partial M_{i,j}}{\partial x_j})_i$.

We are given two vector fields, $f : \Omega \to \mathbf{R}^N$ and $g : \Gamma_1 \to \mathbf{R}^N$, which represent the applied forces densities. For instance, in the case when $N = 3$, if the body of mass density $\rho : \Omega \to \mathbf{R}$ is subjected to gravity and to a constant pressure on Γ_1, then the densities f and g are given by $f(x) = -g_r \rho(x) e_3$ and $g(x) = -\pi \nu(x)$, where g_r is the constant of gravity, π the constant pressure, and ν the unit outer normal to Γ_1.

The configuration occupied by the body when it is subjected to forces with density f and g is defined by means of the deformation, i.e., the mapping $\Phi : \bar{\Omega} \to \mathbf{R}^3$ whose image is the *deformed configuration* of the body. It is convenient to describe the deformed configuration in terms of the *displacement vector field* $u = \Phi - I_{\mathbf{R}^3}$. Under the hypothesis that $|\nabla u(.)| = o(1)$ in Ω, and when the body is made up of an isotropic and homogeneous material, physical and mechanical considerations together with approximating theory lead

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to the (formal) equations of equilibrium in the reference configuration Ω,

$$\begin{cases} -\mathrm{div}(\sigma(u)) = f \quad \text{in } \Omega; \\ u = 0 \quad \text{on } \Gamma_0; \\ \sigma(u)\nu = g \quad \text{on } \Gamma_1; \\ \sigma(u) = \lambda \,\mathrm{trace}\,(\mathscr{E}(u)) I_{\mathbf{R}^3} + 2\mu\, \mathscr{E}(u), \end{cases} \tag{6.89}$$

or, equivalently, by using the components,

$$\begin{cases} -\displaystyle\sum_{j=1}^{N} \frac{\partial}{\partial x_j}\sigma_{ij}(u) = f_i \quad \text{in } \Omega \text{ for } i = 1,\dots,N; \\ u_i = 0 \quad \text{on } \Gamma_0 \text{ for } i = 1,\dots,N; \\ \displaystyle\sum_{j=1}^{N} \sigma_{ij}(u)\nu_j = g_i \quad \text{on } \Gamma_1 \text{ for } i = 1,\dots,N; \\ \sigma_{ij}(u) = \lambda\Big(\displaystyle\sum_{k=1}^{N} \mathscr{E}_{kk}(u)\Big)\delta_{i,j} + 2\mu\mathscr{E}_{ij}(u). \end{cases}$$

For a complete explanation of how to derive the system (6.89) from the general theory of elasticity, we refer the reader to [174]. The matrix field σ is called the *Cauchy stress tensor*. The last equation, called Hooke's law, which approximates the response of the material to external stimuli, is specific to each material. The coefficients $\lambda > 0$ and $\mu > 0$, called *the Lamé constants*, are determined experimentally and are often expressed in terms of the Poisson coefficients ν_P and Young's modulus E_Y through the relations

$$\nu_P = \frac{\lambda}{2(\lambda+\mu)}, \quad E_Y = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}.$$

In the general setting of elasticity, the relation between the stress and the strain tensors, of which Hooke's law is a particular case, is called the *stress-strain constitutive equations of the body*.

The boundary value problem (6.89) is said to be a *pure displacement problem* when $\Gamma_0 = \Gamma$, a *pure traction problem* when $\Gamma_1 = \Gamma$, and a *displacement-traction problem* when $\mathscr{H}^{N-1}(\Gamma_0) > 0$ and $\mathscr{H}^{N-1}(\Gamma_1) > 0$.

We are going to provide a mathematical setting for problem (6.89). For this we first extend the Green's formula (Proposition 5.6.2) to the vectorial setting. In what follows, to shorten the notation, we do not indicate the trace operator in the integrals. According to Proposition 5.6.2, and reasoning with the components, for all σ in $H^1(\Omega, M_s^N)$, and for all v in $H^1(\Omega)^N$, we have

$$\sum_{i,j=1}^{N} \int_\Omega \sigma_{ij} \frac{\partial v_i}{\partial x_j}\, dx = -\sum_{i,j=1}^{N} \int_\Omega \frac{\partial}{\partial x_j}\sigma_{ij} v_i \, dx + \sum_{i,j}^{N} \int_\Gamma \sigma_{ij} v_i \nu_j \, d\mathscr{H}^{N-1} \tag{6.90}$$

and

$$\sum_{i,j=1}^{N} \int_\Omega \sigma_{ij} \frac{\partial v_i}{\partial x_j}\, dx = -\sum_{i,j=1}^{N} \int_\Omega \frac{\partial}{\partial x_i}\sigma_{ij} v_j \, dx + \sum_{i,j}^{N} \int_\Gamma \sigma_{ij} v_j \nu_i \, d\mathscr{H}^{N-1}, \tag{6.91}$$

where $\nu = (\nu_i)_{i=1,\dots,N}$ is the outer unit normal to Γ. According to the fact that σ is a symmetric vector field, we have

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$$\sum_{i,j}^{N}\int_{\Gamma}\sigma_{ij}v_i\nu_j\,d\mathcal{H}^{N-1}=\sum_{i,j}^{N}\int_{\Gamma}\sigma_{ij}v_j\nu_i\,d\mathcal{H}^{N-1}=\int_{\Gamma}\sigma\nu.v\,d\mathcal{H}^{N-1};$$
$$\sum_{i,j=1}^{N}\int_{\Omega}\frac{\partial}{\partial x_j}\sigma_{ij}v_i\,dx=\sum_{i,j=1}^{N}\int_{\Omega}\frac{\partial}{\partial x_i}\sigma_{ij}v_j\,dx=\int_{\Omega}v.\operatorname{div}\sigma\,dx.$$

Hence, summing (6.90) and (6.91), and using $\mathcal{E}_{ij}(v)=\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)$, we obtain the following Green's formula:

$$\int_{\Omega}\sigma:\mathcal{E}(v)\,dx=-\int_{\Omega}v.\operatorname{div}\sigma\,dx+\int_{\Gamma}\sigma\nu.v\,d\mathcal{H}^{N-1}. \tag{6.92}$$

Let us assume that $f\in L^2(\Omega)^N$, $g\in Ł^2(\Gamma_1)^N$, and that $u\in H^2(\Omega)^N$. Let $v\in H^1(\Omega)^N$, satisfying $\gamma_0(v)=0$ on Γ_0. Multiplying the first equation in (6.89) by v with respect to the Euclidean scalar product of $\mathbf{R}^N$, integrating over Ω, and using Green's formula (6.92) with $\sigma=\sigma(u)$ together with the two boundary conditions in (6.89), we deduce

$$\int_{\Omega}\sigma(u):\mathcal{E}(v)\,dx=\int_{\Omega}f.v\,dx+\int_{\Gamma_1}g.v\,d\mathcal{H}^{N-1}. \tag{6.93}$$

Then, taking into account the constitutive equation, and noticing that trace$(\mathcal{E}(.))=$ div(.), we derive the *variational formulation* of (6.89)

$$\lambda\int_{\Omega}\operatorname{div}(u).\operatorname{div}(v)\,dx+2\mu\int_{\Omega}\mathcal{E}(u):\mathcal{E}(v)\,dx=\int_{\Omega}f.v\,dx+\int_{\Gamma_1}g.v\,d\mathcal{H}^{N-1}, \tag{6.94}$$

which makes sense under the weaker condition that $u\in H^1(\Omega)^N$ with $\gamma_0(u)$ on Γ_0, and for all $v\in H^1(\Omega)^N$, with $\gamma_0(u)=\gamma_0(v)=0$ on Γ_0. This leads to the following weak formulation of the displacement-traction problem: assume that $\mathcal{H}^{N-1}(\Gamma_0)>0$; then a vector field u is said to be a *weak solution of* (6.89) if it belongs to $V:=\{v\in H^1(\Omega)^N:\gamma_0(v)=0\text{ on }\Gamma_0\}$ and satisfies the variational problem

$$\lambda\int_{\Omega}\operatorname{div}(u).\operatorname{div}(v)\,dx+2\mu\int_{\Omega}\mathcal{E}(u):\mathcal{E}(v)\,dx=\int_{\Omega}f.v\,dx+\int_{\Gamma_1}g.v\,d\mathcal{H}^{N-1}\quad\forall v\in V.$$

To mimic the proof of Theorem 6.5.1 for establishing the existence of a weak solution, we are led to consider the bilinear form $a:V\times V\to\mathbf{R}$ defined by

$$a(u,v):=\int_{\Omega}\sigma(u):\mathcal{E}(v)\,dx=\lambda\int_{\Omega}\operatorname{div}(u).\operatorname{div}(v)\,dx+2\mu\int_{\Omega}\mathcal{E}(u):\mathcal{E}(v)\,dx \tag{6.95}$$

and to apply the Lax–Milgram theorem. The bilinear form a is clearly continuous when V is equipped with the standard norm of $H^1(\Omega)^N$, but, in contrast to the problems studied in the previous sections, the difficulty is to establish the coercivity of a in the space V. Indeed the form a is clearly coercive when V is equipped with the seminorm $v\mapsto\|\mathcal{E}(v)\|_{L^2(\Omega,\mathbf{M}_s)}$ (actually we will see that it is a norm), but, according to this seminorm, the boundedness of any sequences $(v^n)_{n\in\mathbf{N}}$ a priori does not provide information on all the derivatives $\frac{\partial v_i^n}{\partial x_j}$, but only on $\frac{\partial v_i^n}{\partial x_i}$ and $\frac{1}{2}\left(\frac{\partial v_i^n}{\partial x_j}+\frac{\partial v_j^n}{\partial x_i}\right)$. Therefore a compactness procedure could fail. It is remarkable that $v\mapsto\|\mathcal{E}(v)\|_{L^2(\Omega,\mathbf{M}_s)}$ is a norm equivalent to the standard norm in $H^1(\Omega)^N$ (then all the derivatives of v^n are bounded in $L^2(\Omega)$). This nontrivial result is the consequence of the famous Korn inequalities stated below.

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In what follows, for any $v \in H^1(\Omega)^N$, $||v||_{H^1(\Omega)^N}$ is the Hilbert norm associated with the scalar product

$$\langle u, v\rangle := \int_\Omega u(x).v(x)\,dx + \int_\Omega \nabla u(x) : \nabla v(x)\,dx$$

and

$$||\mathscr{E}(v)||_{L^2(\Omega,\mathbf{M}_s^N)} := \left(\int_\Omega \mathscr{E}(v) : \mathscr{E}(v)\,dx\right)^{1/2}.$$

From $\mathscr{E}(v) = \frac{1}{2}(\nabla v + \nabla v^T)$ we see that $||\mathscr{E}(v)||_{L^2(\Omega,\mathbf{M}_s^N)} \le ||\nabla v||_{L^2(\Omega,\mathbf{M}^N)}$.

Proposition 6.6.1 (Korn's inequalities). *Let Ω be an open bounded and connected set of $\mathbf{R}^N$ which is piecewise of class $\mathbf{C}^1$.*

(i) *Then there exists a constant $C(\Omega) > 0$ such that for all $v \in H^1(\Omega)^N$,*

$$\left(||\mathscr{E}(v)||^2_{L^2(\Omega,\mathbf{M}_s^N)} + ||v||^2_{L^2(\Omega)^N}\right)^{1/2} \ge C(\Omega)||v||_{H^1(\Omega)^N}. \tag{6.96}$$

(ii) *There exists a constant $C'(\Omega) > 0$ such that for all $v \in V$,*

$$||\mathscr{E}(v)||_{L^2(\Omega,\mathbf{M}_s^N)} \ge C'(\Omega)||v||_{H^1(\Omega)^N}. \tag{6.97}$$

In other words, $v \mapsto ||\mathscr{E}(v)||_{L^2(\Omega,\mathbf{M}_s^N)}$ is a norm on V, equivalent to the standard norm $||v||_{H^1(\Omega)^N}$.

(iii) *There exists a constant $C''(\Omega) > 0$ such that for all $v \in H^1(\Omega)^N$,*

$$||\mathscr{E}(v)||_{L^2(\Omega,\mathbf{M}_s^N)} \ge C''(\Omega) \inf_{w\in\ker(\mathscr{E})} ||v + w||_{H^1(\Omega)^N}. \tag{6.98}$$

Remark 6.6.1. The infimum in the second member of inequality (6.98) is nothing but the norm of $\bar{v}$ in the quotient space $H^1(\Omega)/\ker(\mathscr{E})$. Therefore (6.98) can be rewritten as $||\bar{\mathscr{E}}(\bar{v})||_{L^2(\Omega,\mathbf{M}_s^N)/\ker(\mathscr{E})} \ge C"(\Omega)||\bar{v}||_{H^1(\Omega)/\ker(\mathscr{E})}$, where $\bar{\mathscr{E}}$ is well defined by

$$\bar{\mathscr{E}}(\bar{v}) := \mathscr{E}(v)$$

for every $v \in \bar{v}$.

For proving Proposition 6.6.1 we need the two lemmas below, each having its own interest.

Lemma 6.6.1 (the kernel of $\mathscr{E}$: the rigid displacements). *Let Ω be an open bounded and connected set of $\mathbf{R}^N$, $N \in \mathbf{N}^*$. Then the kernel of $\mathscr{E} : \mathscr{D}'(\Omega)^N \to \mathscr{D}'(\Omega)^N$, $v \mapsto \mathscr{E}(v)$, called the space of infinitesimal rigid displacements of the set Ω, is given by*

$$\ker(\mathscr{E}) = \{v \in \mathscr{D}'(\Omega)^N : v(x) = a + Bx : a \in \mathbf{R}^N,\ B \in \mathbf{M}^N \textit{ is antisymmetric}\}.$$

In the specific cases $N = 2, 3$, one has

case $N = 2$: $\ker(\mathscr{E}) = \{v = (v_1, v_2) : \exists (a_1, a_2, b) \in \mathbf{R}^3$ *s.t.* $v_1(x) = a_1 + b x_2,\ v_2(x) = a_2 - b x_1\}$;

case $N = 3$: $\ker(\mathscr{E}) = \{v : \exists a \in \mathbf{R}^3,\ \exists b \in \mathbf{R}^3$ *s.t.* $v(x) = a + b \wedge x\}$.

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Consequently, when $N=2$, if $v \in \ker(\mathscr{E})$ vanishes at two distinct points, then $v=0$, while when $N=3$, if $v \in \ker(\mathscr{E})$ vanishes at three distinct and noncollinear points, then $v=0$.

PROOF. Every function $v : x \mapsto v(x)=a+Bx$, where $a \in \mathbf{R}^N$ and $B \in \mathbf{M}^N$ is antisymmetric, clearly belongs to $\ker(\mathscr{E})$. Conversely, let $v \in \ker(\mathscr{E})$ and consider the antisymmetric part $\mathscr{A}(v)$ of ∇v defined by

$$\mathscr{A}_{ij}(v)=\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}-\frac{\partial v_j}{\partial x_i}\right).$$

For all i,j,k in $\{1,\dots,N\}$ we have

$$\begin{aligned}\frac{\partial}{\partial x_k}\mathscr{A}_{ij}(v)&=\frac{1}{2}\left(\frac{\partial^2 v_i}{\partial x_j\partial x_k}-\frac{\partial^2 v_j}{\partial x_i\partial x_k}\right)\\&=\frac{1}{2}\left(\frac{\partial^2 v_i}{\partial x_j\partial x_k}+\frac{\partial^2 v_k}{\partial x_i\partial x_j}\right)-\frac{1}{2}\left(\frac{\partial^2 v_k}{\partial x_i\partial x_j}+\frac{\partial^2 v_j}{\partial x_i\partial x_k}\right)\\&=\frac{\partial}{\partial x_j}\mathscr{E}_{ik}(v)-\frac{\partial}{\partial x_i}\mathscr{E}_{kj}(v)=0.\end{aligned}\tag{6.99}$$

Hence, since $\frac{\partial v_i}{\partial x_j}=-\frac{\partial v_j}{\partial x_i}$, we have

$$\frac{\partial^2 v_i}{\partial x_j\partial x_k}+\frac{\partial^2 v_j}{\partial x_i\partial x_k}=0.\tag{6.100}$$

Comparing (6.99) and (6.100) gives

$$\frac{\partial^2 v_i}{\partial x_j\partial x_k}=\frac{\partial^2 v_j}{\partial x_i\partial x_k}=0.$$

We infer, since Ω is connected, that for $i=1,\dots,N$, v_i is an affine function. Hence there exist $a \in \mathbf{R}^N$ and $B \in \mathbf{M}^N$ such that $v(x)=a+Bx$. Since $\mathscr{E}(v)=0$ is the symmetric part of B, the matrix B is antisymmetric. The end of the proof is standard. □

Lemma 6.6.2. *Let Ω be an open bounded connected set of $\mathbf{R}^N$ which is piecewise of class $\mathbf{C}^1$, and $v \in \mathscr{D}'(\Omega)$. Then the following equivalence holds:*

$$v \in H^{-1}(\Omega) \text{ and } \frac{\partial v}{\partial x_i} \in H^{-1}(\Omega) \text{ for } i=1,\dots,N \iff v \in L^2(\Omega).$$

Implication $\left(v \in L^2(\Omega) \Longrightarrow v \in H^{-1}(\Omega) \text{ and } \frac{\partial v}{\partial x_i} \in H^{-1}(\Omega)\right)$ is trivial. Indeed, for all $\varphi \in \mathscr{D}(\Omega)$ we have

$$\begin{aligned}|\langle v,\varphi\rangle|&=\left|\int_\Omega v\varphi\,dx\right| \le \|v\|_{L^2(\Omega)}\|\varphi\|_{H_0^1(\Omega)};\\\left|\left\langle \frac{\partial v}{\partial x_i},\varphi\right\rangle\right|&=\left|-\int_\Omega v\frac{\partial \varphi}{\partial x_i}\,dx\right| \le \|v\|_{L^2(\Omega)}\|\varphi\|_{H_0^1(\Omega)}.\end{aligned}$$

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The converse implication is more involved and was first established by J. L. Lions. The proof can be found in [204, Theorem 3.2]. For an extension relative to H^m-Sobolev spaces ($m \in \mathbf{Z}$) and to the case when the boundary of Ω is Lipschitz continuous, see [31, Proposition 2.10].

PROOF OF PROPOSITION 6.6.1. The proof proceeds in five steps.

Step 1. We claim that the vector space $E := \{v \in L^2(\Omega)^N : \mathscr{E}(v) \in L^2(\Omega, \mathbf{M}_s^N)\}$ coincides with the vector space $H^1(\Omega)^N$.

Let v be any element of E. For each $k = 1,\dots,N$, and for all $j = 1,\dots,N$, since $v_k \in L^2(\Omega)$ we have

$$\frac{\partial v_k}{\partial v_j} \in H^{-1}(\Omega). \tag{6.101}$$

On the other hand, since $\mathscr{E}(v)$ belongs to $L^2(\Omega, \mathbf{M}_s^N)$, the elementary identity in $\mathscr{D}'(\Omega)$,

$$\frac{\partial}{\partial x_i}\left(\frac{\partial v_k}{\partial x_j}\right) = \frac{\partial}{\partial x_i}\mathscr{E}_{jk}(v) + \frac{\partial}{\partial x_j}\mathscr{E}_{ik}(v) - \frac{\partial}{\partial x_k}\mathscr{E}_{ij}(v),$$

yields $\frac{\partial}{\partial x_i}\left(\frac{\partial v_k}{\partial x_j}\right) \in H^{-1}(\Omega)$, which, together with (6.101), and according to Lemma 6.6.2, implies that $\frac{\partial v_k}{\partial v_j} \in L^2(\Omega)$. This proves that $E \subset H^1(\Omega)^N$ and completes the claim since the converse inclusion is trivial.

Step 2. We establish (6.96). The spaces $H^1(\Omega)^N$ and E equipped with the norms $||.||_{H^1(\Omega)^N}$ and $||.||_E := \left(||.||^2_{L^2(\Omega)^N} + ||\mathscr{E}(.)||^2\right)^{1/2}$, respectively, are two Hilbert spaces. The mapping $I_d : H^1(\Omega)^N \to E$ defined by $I_d(v) = v$ is then a bijective linear continuous operator between the two Banach spaces $H^1(\Omega)^N$ and E. (The fact that I_d is surjective comes from Step 1, and the continuity of I_d comes from the trivial inequality $||v||_E \le ||v||_{H^1(\Omega)^N}$.) Therefore, according to the Banach open mapping theorem (see [137, 361]), I_d is open: there exists $C > 0$ such that for all $v \in H^1(\Omega)^N$, $||v||_{H^1(\Omega)^N} \le C||v||_E$. The constant $C(\Omega) = C^{-1}$ is suitable.

Step 3. We prove that $\ker \mathscr{E} \cap V = \{0\}$. Let $v \in \ker \mathscr{E} \cap V$. Since $\mathscr{H}^2(\Gamma_0) > 0$, Γ_0 contains at least two distinct points when $N = 2$ and three distinct and noncollinear points when $N = 3$. Since v vanishes on Γ_0, from Lemma 6.6.1 we infer that $v = 0$.

Step 4. We establish (6.97) by proceeding by contradiction. If (6.97) is false, there exists a sequence $(v_n)_{n\in\mathbf{N}}$ in V such that

$$||v_n||_{H^1(\Omega)^N} = 1; \tag{6.102}$$

$$||\mathscr{E}(v_n)||_{L^2(\Omega,\mathbf{M}_s^N)} \to 0. \tag{6.103}$$

From the Rellich–Kondrakov compactness theorem, there exists a subsequence of $(v_n)_{n\in\mathbf{N}}$ (that we do not relabel) which strongly converges to some v in $L^2(\Omega)^N$. We deduce, with (6.103), that $(v_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in $E = H^1(\Omega)^N$ equipped with the norm $||.||_E$; thus, from (6.96) established in Step 2, $(v_n)_{n\in\mathbf{N}}$ is also a Cauchy sequence with the norm $||.||_{H^1(\Omega)^N}$. Since V is a closed subspace of $H^1(\Omega)^N$, $v_n \to v$ strongly in V, so that, from (6.102), we infer that

$$||v||_{H^1(\Omega)^N} = 1. \tag{6.104}$$

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But, from (6.103), $\mathscr{E}(v_n) \to \mathscr{E}(v) = 0$ in $L^2(\Omega, \mathbf{M}_s^N)$, hence $v \in \ker \mathscr{E} \cap V$. From Step 3, we deduce that $v = 0$, which contradicts (6.104).

Step 5. We establish (6.98). We proceed in a similar way by contradiction. If (6.98) is false, there exists a sequence $(v_n)_{n\in\mathbf{N}}$ in V such that

$$\inf_{w\in\ker(\mathscr{E})} \|v_n + w\|_{H^1(\Omega)^N} = 1; \tag{6.105}$$

$$\|\mathscr{E}(v_n)\|_{L^2(\Omega,\mathbf{M}_s^N)} \to 0. \tag{6.106}$$

Recall that from Lemma 6.6.1, $\ker(\mathscr{E})$ is a finite dimensional subspace of $H^1(\Omega)^N$ and thus possesses a closed orthogonal $\ker(\mathscr{E})^\perp$. Consider the decomposition $v_n = u_n + w_n$, where $u_n \in \ker(\mathscr{E})^\perp$ and $w_n \in \ker(\mathscr{E})$. We have

$$\begin{aligned} &v_n - w_n \perp \ker(\mathscr{E});\\ &w_n \in \ker(\mathscr{E}), \end{aligned}$$

so that w_n is the orthogonal projection of v_n onto $\ker(\mathscr{E})$. Hence, from (6.105),

$$\|u_n\|_{H^1(\Omega)^N} = \|v_n - w_n\|_{H^1(\Omega)^N} = \inf_{w\in\ker(\mathscr{E})} \|v_n + w\|_{H^1(\Omega)^N} = 1.$$

With (6.106), we finally obtain

$$\|u_n\|_{H^1(\Omega)^N} = 1; \tag{6.107}$$

$$\|\mathscr{E}(u_n)\|_{L^2(\Omega,\mathbf{M}_s^N)} \to 0. \tag{6.108}$$

Reasoning as in Step 4, from (6.107) and (6.108) we deduce that there exists a subsequence of $(u_n)_{n\in\mathbf{N}}$ (that we do not relabel), and $u \in H^1(\Omega)^N$ such that $u_n \to u$ strongly in $H^1(\Omega)^N$. Thus $u \in \ker(\mathscr{E})^\perp$ and $\|u\|_{H^1(\Omega)^N} = 1$. On the other hand, from (6.108) we infer that $u \in \ker(\mathscr{E})$. Consequently $u = 0$, which contradicts $\|u\|_{H^1(\Omega)^N} = 1$. □

Theorem 6.6.1 (displacement-traction problem). *Let us assume that Ω is a bounded connected open set in $\mathbf{R}^N$ which is piecewise of class $\mathbf{C}^1$, and let $V := \{v \in H^1(\Omega)^N : \gamma_0(v) = 0\}$, where $\mathscr{H}^{N-1}(\Gamma_0) > 0$.*

(i) *Then, for any $f \in L^2(\Omega)^N$ and any $g \in L^2(\partial\Omega)^N$, there exists a unique solution $u \in V$ of the problem*

$$\begin{cases} \displaystyle\int_\Omega \sigma(u) : \mathscr{E}(v)\, dx = \int_\Omega f.v\, dx + \int_{\Gamma_1} g.v\, d\mathscr{H}^{N-1} \quad \forall v \in V;\\ u \in V, \end{cases} \tag{6.109}$$

where $\sigma(u) = \lambda\, trace\, (\mathscr{E}(u)) I_{\mathbf{R}^3} + 2\mu\, \mathscr{E}(u)$.

(ii) *Problem* (6.109) *is equivalent to saying that u is the unique solution of the minimization problem*

$$\min\left\{\frac{1}{2}\int_\Omega \sigma(v) : \mathscr{E}(v)\, dx - \int_\Omega f.v\, dx - \int_{\partial\Omega} g.v\, d\mathscr{H}^{N-1} \ :\ v \in V\right\}, \tag{6.110}$$

where $\sigma(v) = \lambda\, trace\, (\mathscr{E}(v)) I_{\mathbf{R}^3} + 2\mu\, \mathscr{E}(v)$.

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(iii) *The solution* u *of* (6.109) *satisfies*

$$-\mathrm{div}(\sigma(u)) = f \quad \textit{in } \Omega,$$

or equivalently

$$-\sum_{j=1}^{N} \frac{\partial}{\partial x_j} \sigma_{ij}(u) = f_i \quad \textit{in } \Omega \textit{ for } i = 1\dots, N$$

in the distribution sense.

PROOF. Problem (6.109) can be written

$$\begin{cases} \text{find } u \in V \text{ such that} \\ a(u,v) = l(v) \quad \forall v \in V, \end{cases}$$

where a is given by (6.95) and $l(v) = \int_\Omega f.v\, dx + \int_{\partial\Omega} g.v\, d\mathcal{H}^{N-1}$. By reproducing a calculation similar to that of the proof of Theorem 6.5.1, we see that the bilinear form $a : V \times V \to \mathbf{R}$ as well as the linear form l are continuous on V. Finally, from Korn's inequality (6.97), a is coercive on V. By applying the Lax–Milgram theorem to problem (6.109) in V, which is a closed subspace of the Hilbert space $H^1(\Omega)^N$ equipped with the norm $||.||_{H^1(\Omega)^N}$, we infer that (6.109) possesses a unique solution.

Part (ii) follows from Proposition 2.3.1 because $v \mapsto \frac{1}{2}\int_\Omega \sigma(v) : \mathcal{E}(v)\, dx$ is symmetric.

Part (iii) is a direct consequence of (6.109) and the fact that V contains $\mathcal{D}(\Omega)^N$. □

Remark 6.6.2. (a) In the elasticity framework, the space V is called *the space of kinematically admissible displacements*. The variational formulation (6.109) expresses the *principle of virtual work*: if v is a (virtual) admissible displacement, $\lambda \int_\Omega \mathrm{div}(u).\mathrm{div}(v)\, dx + 2\mu \int_\Omega \mathcal{E}(u) : \mathcal{E}(v)\, dx$ represents the deformation work of the elastic solid corresponding to the virtual displacement v, while $\int_\Omega f.v\, dx + \int_{\Gamma_1} g.v\, d\mathcal{H}^2$ represents the work of the external forces (or loading).

(b) The formulation (ii) expresses the principle of least action: among all of the possible displacements (i.e., the virtual displacements) the solution u minimizes the action. The action, i.e., the functional $v \mapsto \frac{1}{2}\int_\Omega \sigma(v) : \mathcal{E}(v)\, dx - \int_\Omega f.v\, dx - \int_{\partial\Omega} g.v\, d\mathcal{H}^{N-1}$, is called the *elastic potential energy*, which is the sum of the *deformation energy* of the body $v \mapsto \frac{1}{2}\int_\Omega \sigma(v) : \mathcal{E}(v)\, dx$ and the *potential energy of the external forces* $v \mapsto \int_\Omega f.v\, dx + \int_{\partial\Omega} g.v\, d\mathcal{H}^{N-1}$.

For any displacement field v, let us denote with $\bar{v}$ its class in the quotient space $H^1(\Omega)/\ker(\mathcal{E})$. The two operators $\bar{\mathrm{div}}$ and $\bar{\mathcal{E}}$ are clearly well defined by the relations $\bar{\mathrm{div}}(\bar{v}) = \mathrm{div}(v)$ and $\bar{\mathcal{E}}(\bar{v}) = \mathcal{E}(v)$. In what follows we still denote them by div and $\mathcal{E}$. The mapping $a : H^1(\Omega)/\ker(\mathcal{E}) \times H^1(\Omega)/\ker(\mathcal{E}) \to \mathbf{R}$ defined by

$$a(\bar{u},\bar{v}) := \int_\Omega \sigma(\bar{u}) : \nabla\bar{v}\, dx = \lambda \int_\Omega \mathrm{div}(\bar{u}).\mathrm{div}(\bar{v})\, dx + 2\mu \int_\Omega \mathcal{E}(\bar{u}) : \mathcal{E}(\bar{v})\, dx \qquad (6.111)$$

is then clearly a continuous bilinear form. Furthermore, according to Korn inequality (6.98), a is coercive in $H^1(\Omega)/\ker(\mathcal{E})$. Then, by using arguments similar to those of the

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previous theorem, we deduce existence and uniqueness for the pure traction problem up to an infinitesimal rigid displacement field. More precisely, we have the next theorem.

Theorem 6.6.2 (pure traction problem). *Let Ω be a bounded connected open set in $\mathbf{R}^N$ which is piecewise of class $\mathbf{C}^1$ with $\Gamma_1 = \Gamma$. Assume furthermore that $f \in L^2(\Omega)^N$ and $g \in L^2(\partial\Omega)^N$ satisfy the condition $\int_\Omega f.v\, dx + \int_\Gamma g.v\, d\mathcal{H}^{N-1} = 0$ for all $v \in \ker(\mathcal{E})$.*

(i) *Then, there exists a unique solution $\bar{u} \in H^1(\Omega)/\ker(\mathcal{E})$ of the problem*

$$\begin{cases} \displaystyle\int_\Omega \sigma(\bar{u}) : \mathcal{E}(\bar{v})\, dx = \int_\Omega f.\bar{v}\, dx + \int_\Gamma g.\bar{v}\, d\mathcal{H}^{N-1} \quad \forall v \in H^1(\Omega)/\ker(\mathcal{E}); \\ \bar{u} \in H^1(\Omega)/\ker(\mathcal{E}), \end{cases} \tag{6.112}$$

where $\sigma(\bar{u}) = \lambda\, trace\, (\mathcal{E}(\bar{u}))I_{\mathbf{R}^3} + 2\mu\, \mathcal{E}(\bar{u})$.

(ii) *Problem* (6.112) *is equivalent to saying that $\bar{u}$ is the unique solution of the minimization problem*

$$\min\left\{\frac{1}{2}\int_\Omega \sigma(\bar{v}) : \mathcal{E}(\bar{v})\, dx - \int_\Omega f.\bar{v}\, dx - \int_{\partial\Omega} g.\bar{v}\, d\mathcal{H}^{N-1} \;:\; \bar{v} \in H^1(\Omega)/\ker(\mathcal{E})\right\},$$

where $\sigma(\bar{v}) = \lambda\, trace\, (\mathcal{E}(\bar{v}))I_{\mathbf{R}^3} + 2\mu\, \mathcal{E}(\bar{v})$.

(iii) *Let $u \in \bar{u}$, where $\bar{u}$ is the solution u of* (6.112)*. Then u satisfies*

$$-\mathrm{div}(\sigma(u)) = f \quad in\ \Omega,$$

or equivalently

$$-\sum_{j=1}^N \frac{\partial}{\partial x_j}\sigma_{ij}(u) = f_i \quad in\ \Omega\ for\ i = 1\ldots, N$$

in the distribution sense.

Remark 6.6.3. *Condition $\int_\Omega f.v\, dx + \int_{\Gamma_1} g.v\, d\mathcal{H}^{N-1} = 0$ for all $v \in \ker(\mathcal{E})$, which is necessary from the mathematical point of view, is natural and says that the external forces do not work on the infinitesimal rigid displacements.*

Assume now that Ω is a connected open set of class $\mathbf{C}^2$. When the exterior loading is regular, and there is no change of boundary condition along a connected portion of Γ, one can establish that the solution of (6.109) or (6.112) is regular. More precisely, if $f \in L^2(\Omega)^N$ and $g \in H^{-1/2}(\Gamma)^N$, then u belongs to $H^2(\Omega)^N$ in the pure displacement or in the pure traction case (see [173, Theorem 6.3-6]). In each of these two cases we can interpret (6.109) and (6.112) in terms of the boundary value problem (6.89) by completing (iii) with the boundary conditions satisfied by the unique solution u and $\bar{u}$, respectively.

Proposition 6.6.2. *Under the conditions of Theorem* 6.6.1 *in the pure displacement case and Theorem* 6.6.2 *in the pure traction case, and if furthermore $g \in H^{-1/2}(\Gamma)^N$, and Ω is a connected open set of class $\mathbf{C}^2$, then the solutions u and $\bar{u}$ of* (6.109) *and* (6.112) *belong to $H^2(\Omega)^N$ and $H^2(\Omega)^N/\ker(\mathcal{E})$, respectively, and satisfy the following:*

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(i) *Pure displacement case:*

$$\begin{cases} -\mathrm{div}(\sigma(u)) = f \quad \textit{a.e. in } \Omega; \\ u = 0 \quad \textit{on } \Gamma \textit{ in the trace sense}; \end{cases}$$

(ii) *Pure traction case: for any $u \in \bar{u}$,*

$$\begin{cases} -\mathrm{div}(\sigma(u)) = f \quad \textit{a.e. in } \Omega; \\ \sigma(u)\nu = g \quad \textit{on } \Gamma \textit{ in the trace sense}. \end{cases}$$

PROOF. Use the Green's formula (6.92) and proceed as in the proof of Proposition 6.5.1. □

Remark 6.6.4. Proposition 6.6.2 can be extended to displacement-traction problems if the closures of Γ_0 and Γ_1 do not intersect, for example, when Ω is the ring $\Omega = \{x \in \mathbf{R}^N : r_1 < |x| < r_2\}$, Γ_0 is the sphere of radius $r_1 > 0$, and Γ_1 is the sphere of radius $r_2 > r_1$.

Regularity is lost at corners along the boundary even if the boundary condition does not change (see [233], [234]). There is a vast literature on the regularity for elliptic systems including the boundary value problems of linearized elasticity as a special case; we mention in particular [215], [226].

6.7 ▪ Introduction to the Signorini problem

We complete the previous section with a short introduction to the Signorini problem. With the notation of Section 6.6, we assume that the body surface Γ is decomposed into three disjoint measurable parts $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_S$ with strictly positive $\mathscr{H}^{N-1}$-measure, and, as previously, that the body is clamped on Γ_0 and subjected to a surface force on the Neumann part Γ_1. The set Γ_S denotes the possible contact boundary with a rigid foundation S disjoint from Ω. By comparison with the mechanical system of displacement-traction, the elastic body is impressed on the rigid support S. Let us denote by ν the unit outer normal to Γ_S and by u_ν the normal component $u.\nu$ of u, and we introduce the normal and tangential component of $\sigma(u)$:

$$\sigma_\nu(u) = \sigma(u)\nu.\nu, \;\; \sigma_T(u) = \sigma(u)\nu - \sigma_\nu(u)\nu.$$

(Note that the normal component $\sigma_\nu(u)$ of $\sigma(u)$ is a scalar field.) We assume that there is pure contact between the body and the rigid support contact, i.e., no friction occurs on Γ_S. Then physical and mechanical considerations, together with approximation theory, lead to the following (formal) equations of equilibrium in the reference configuration Ω, originally introduced by Signorini (see [333], [214], [216], [357]):

$$\begin{cases} -\mathrm{div}(\sigma(u)) = f \quad \text{in } \Omega; \\ u = 0 \quad \text{on } \Gamma_0; \\ \sigma(u)\nu = g \quad \text{on } \Gamma_1; \\ u_\nu \le 0,\ \sigma_\nu(u) \le 0,\ u_\nu \sigma_\nu(u) = 0,\ \sigma_T(u) = 0 \quad \text{on } \Gamma_S; \\ \sigma(u) = \lambda \,\mathrm{trace}\,(\mathscr{E}(u)) I_{\mathbf{R}^3} + 2\mu\, \mathscr{E}(u). \end{cases} \tag{6.113}$$

The set Γ_S is called the *Signorini boundary* or *set of coincidence* or also the *contact boundary*. Compared to the boundary value problem (6.89), the additional four conditions on Γ_S describe the nature of the contact between the body and the support S: the first condition states that no penetration in the normal direction occurs, and the second and the third

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conditions state that only compressive normal stress is allowed and that there must be vanishing contact stress in case of no contact, respectively. The last condition states that the contact is without friction.

As for problem (6.89), we are going to provide a weak formulation for problem (6.113). For this, let us introduce the set

$$K := \{v \in H^1(\Omega)^N : \gamma_0(v) = 0 \text{ on } \Gamma_0,\ \gamma_0(v).\nu \le 0 \text{ a.e. on } \Gamma_S\}.$$

(In what follows, to simplify the notation, we do not indicate the trace operator γ_0.) Assume that $f \in L^2(\Omega)^N$, $g \in Ł^2(\Gamma_1)^N$, and that the solution u of (6.113) belongs to $H^2(\Omega)^N \cap K$. Multiplying with respect to the Euclidean scalar product of $\mathbf{R}^N$ each two members of the first equation in (6.113) by $v-u$, where $v \in K$ is arbitrary, integrating over Ω, and using Green's formula (6.92) with $\sigma = \sigma(u)$, we obtain

$$\begin{aligned}\int_\Omega \sigma(u) : \mathcal{E}(v-u)\,dx &= \int_\Omega f.(v-u)\,dx + \int_\Gamma \sigma(u)\nu.(v-u)\,d\mathcal{H}^{N-1} \\ &= \int_\Omega f.(v-u)\,dx + \int_{\Gamma_1} g.(v-u)\,d\mathcal{H}^{N-1} \\ &\quad + \int_{\Gamma_S} \sigma(u)\nu.(v-u)\,d\mathcal{H}^{N-1}. \end{aligned} \tag{6.114}$$

Let us decompose the stress $\sigma(u)\nu$ on the contact boundary Γ_S with respect to its normal and tangential component. Using the boundary conditions fulfilled by u_ν and $\sigma_\nu(u)$ on Γ_S, and the fact that $v.\nu \le 0$ on Γ_S, we obtain

$$\begin{aligned}\sigma(u)\nu.(v-u) &= \big(\sigma_T(u) + \sigma_\nu(u)\nu\big).(v-u) \\ &= \sigma_\nu(u)\nu.(v-u) \\ &= \sigma_\nu(u)v.\nu \ge 0\end{aligned}$$

$\mathcal{H}^{N-1}$ a.e. in Γ_S. Consequently, (6.114) yields

$$\int_\Omega \sigma(u) : \mathcal{E}(v-u)dx \ge \int_\Omega f.(v-u)\,dx + \int_{\Gamma_1} g.(v-u)d\mathcal{H}^{N-1},$$

which makes sense for all $v \in K$ under the weaker condition that $u \in H^1(\Omega)^N$ and $u \in K$.

Summing up, this leads to the following weak formulation: assume that $\mathcal{H}^{N-1}(\Gamma_0) > 0$, $\mathcal{H}^{N-1}(\Gamma_1) > 0$, and $\mathcal{H}^{N-1}(\Gamma_S) > 0$. Then a vector field u is said to be a *weak solution of* (6.113) if it belongs to

$$K := \{v \in H^1(\Omega)^N : \gamma_0(v) = 0 \text{ on } \Gamma_0,\ \gamma_0(v).\nu \le 0 \text{ a.e. on } \Gamma_S\}$$

and satisfies the variational inequality

$$\int_\Omega \sigma(u) : \mathcal{E}(v-u)\,dx \ge \int_\Omega f.(v-u)\,dx + \int_{\Gamma_1} g.(v-u)\,d\mathcal{H}^{N-1} \quad \forall v \in K,$$

where $\sigma(u) = \lambda\,\mathrm{trace}\,(\mathcal{E}(u))I_{\mathbf{R}^3} + 2\mu\ \mathcal{E}(u)$. It can be seen immediately that the set K of *admissible displacements* is a nonempty closed convex subset of $V := \{v \in H^1(\Omega)^N : \gamma_0(v) = 0 \text{ on } \Gamma_0\}$.

Theorem 6.7.1 (contact without friction problem). *Let us assume that Ω is a bounded connected open set in $\mathbf{R}^N$ which is piecewise of class $\mathbf{C}^1$, and $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_S$ with $\mathcal{H}^{N-1}(\Gamma_0) > 0$, $\mathcal{H}^{N-1}(\Gamma_1) > 0$, and $\mathcal{H}^{N-1}(\Gamma_S) > 0$.*

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(i) *Then, for any $f \in L^2(\Omega)$ and any $g \in L^2(\partial\Omega)$, there exists a unique weak solution $u \in K$ of the Signorini problem*

$$\begin{cases} \displaystyle\int_\Omega \sigma(u) : \mathscr{E}(v-u)\, dx \geq \int_\Omega f.(v-u)\, dx + \int_{\Gamma_1} g.(v-u)\, d\mathscr{H}^2 \quad \forall v \in K; \\ u \in K, \end{cases} \tag{6.115}$$

where $\sigma(u) = \lambda\, trace\,(\mathscr{E}(u)) I_{\mathbf{R}^3} + 2\mu\, \mathscr{E}(u)$.

(ii) *Problem* (6.115) *is equivalent to saying that u is the unique solution of the minimization problem*

$$\min\left\{ \frac{1}{2}\int_\Omega \sigma(v) : \mathscr{E}(v)\, dx - \int_\Omega f.v\, dx - \int_{\partial\Omega} g.v\, d\mathscr{H}^{N-1} \ :\ v \in K \right\}. \tag{6.116}$$

(iii) *The solution u of* (6.115) *satisfies*

$$-div(\sigma(u)) = f \quad in\ \Omega,$$

or equivalently

$$-\sum_{j=1}^{N} \frac{\partial}{\partial x_j}\sigma_{ij}(u) = f \quad in\ \Omega\ for\ i = 1\ldots, N$$

in the distribution sense.

PROOF. The minimization problem in (ii) is equivalent to

$$\min\left\{ \frac{1}{2}\int_\Omega \sigma(v) : \mathscr{E}(v)\, dx - \int_\Omega f.v\, dx - \int_{\partial\Omega} g.v\, d\mathscr{H}^{N-1} + \delta_K(v) :\ v \in V \right\},$$

where δ_K is the indicator function of the set K. Since K is a nonempty closed convex subset of V, the indicator function δ_V is a lower semicontinuous convex and proper function. On the other hand, from the Korn's inequality (6.97), the quadratic form $v \mapsto \frac{1}{2}\int_\Omega \sigma(v) : \mathscr{E}(v)\, dx$ is coercive in V. Therefore (ii) concerns the minimization of the functional

$$\begin{aligned} &F : V \to \mathbf{R} \cup \{+\infty\} \\ &v \mapsto \frac{1}{2}\int_\Omega \sigma(v) : \mathscr{E}(v)\, dx - \left(\int_\Omega f.v\, dx + \int_{\partial\Omega} g.v\, d\mathscr{H}^{N-1} \right) + \delta_K, \end{aligned}$$

which is sum of three lower semicontinuous convex proper functions, one of them being coercive on the Hilbert space V. Thus, it is the minimization of a lower semicontinuous convex coercive function on V. According to Theorem 3.3.4, it admits a solution. The strict convexity of the functional F yields unicity. Passing from formulation (ii) to formulation (i) is obtained by writing the first-order necessary and sufficient condition, as in Theorem 3.3.5, or more generally by applying the subdifferential calculus rules of Chapter 9; see Theorem 9.5.5.

Taking $v = u \pm \varphi$, $\varphi \in \mathscr{D}(\Omega)^N$ as a test function in (6.115), and using Green's formula (6.92), the proof of of (iii) becomes identical to that of Theorem 6.6.1(iii). □

Remark 6.7.1. (a) Formulations (i) and (ii) of Theorem 6.7.1 enter the setting of obstacle problems studied in detail in Section 6.12 in a scalar framework.

(b) If we assume that the weak solution belongs to $H^2(\Omega)^N$, using Green's formula (6.92), and choosing suitable test functions, we can interpret the formulations of (i) or (ii) in terms of the boundary value problem (6.113) (see [256]).

(c) To prove the existence of a weak solution for linear elasticity contact problems in a domain with inclusions, Korn's inequalities established in Proposition 6.6.1 are not suitable. This question, very important for its applications, is treated in [187], where unilateral inequalities of the Korn type are established.

6.8 ▪ The Stokes system

In this section, we are going to make precise the variational approach to the Stokes system, which was introduced in Section 2.3.1. Let us recall that the Stokes system for an incompressible viscous fluid in a domain Ω of $\mathbf{R}^N$ consists in finding functions $u_1, u_2, \dots u_N : \Omega \longrightarrow \mathbf{R}$ and $p : \Omega \longrightarrow \mathbf{R}$ which satisfy

$$\begin{cases} -\mu\Delta u_i + \dfrac{\partial p}{\partial x_i} = f_i \quad \text{on } \Omega,\ i = 1,\dots,N, \\ \displaystyle\sum_{i=1}^{N} \frac{\partial u_i}{\partial x_i} = 0 \quad \text{on } \Omega, \\ u_i = 0 \quad \text{on } \partial\Omega,\ i = 1,\dots,N. \end{cases}$$

The given vector $f = (f_1, f_2, \dots, f_N) \in L^2(\Omega)^N$ represents a volumic density of forces, and $\mu > 0$ is the viscosity coefficient. (It is a positive scalar which is inversely proportional to the Reynolds number.) The vector function $u = (u_1, \dots, u_N) : \Omega \longrightarrow \mathbf{R}^N$ is the velocity vector field of the fluid; it assigns to each point $x \in \Omega$ the velocity vector $u(x) = (u_i(x))_{i=1,\dots,N}$ of the fluid at x. The scalar function $p : \Omega \longrightarrow \mathbf{R}$ is the pressure; for each $x \in \Omega$, $p(x)$ is the pressure of the fluid at x.

The Stokes system can be written in the following form:

$$\begin{cases} -\mu\Delta u + \nabla p = f \quad \text{on } \Omega, \\ \operatorname{div}(u) = 0 \quad \text{on } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

The condition $\operatorname{div}(u) = 0$ expresses that the fluid is incompressible. The Stokes system is a linear system of $(N+1)$ partial differential equations on Ω involving $(N+1)$ unknown functions $(u_1, \dots, u_N, p)$.

The variational formulation of the Stokes system was introduced by Leray around 1934. The idea is to work in the functional space

$$V = \left\{ v \in H_0^1(\Omega)^N : \operatorname{div}(v) = 0 \right\} \tag{6.117}$$

and make the pressure appear as a Lagrange multiplier of the constraint $\operatorname{div}(v) = 0$. Let us assume that Ω is a bounded connected open subset of $\mathbf{R}^N$ whose boundary is piecewise $\mathbf{C}^1$. The space V is equipped with the scalar product of $H_0^1(\Omega)^N$

$$\langle u, v \rangle_{H_0^1(\Omega)^N} = \sum_{i=1}^{N} \langle u_i, v_i \rangle_{H_0^1(\Omega)},$$

where

$$\langle u_i, v_i \rangle_{H_0^1(\Omega)} = \int_\Omega (u_i v_i + \nabla u_i \cdot \nabla v_i)\, dx,$$

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and the corresponding norm

$$||v||_{H_0^1(\Omega)^N} = \left(\sum_{i=1}^{N} \int_\Omega (u_i^2 + |\nabla u_i|^2)\, dx \right)^{1/2}.$$

The space V is equal to the kernel of the divergence operator div,

$$\mathrm{div} : v \in H_0^1(\Omega)^N \longrightarrow \mathrm{div}(v) \in L^2(\Omega),$$

which is a linear continuous operator from $H_0^1(\Omega)^N$ into $L^2(\Omega)$. The continuity of the div operator follows from the following inequality:

$$\begin{aligned} \forall v \in H_0^1(\Omega)^N \quad ||\,\mathrm{div}(v)||^2_{L^2(\Omega)} &= \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial v_i}{\partial x_i} \right|^2 dx \\ &\leq \sum_{i=1}^{N} \int_\Omega |\nabla v_i|^2\, dx \\ &\leq ||v||^2_{H_0^1(\Omega)^N}. \end{aligned}$$

Hence, V is a closed subspace of $H_0^1(\Omega)^N$ and V is a Hilbert space.

We can now state the variational formulation of the Stokes system.

Theorem 6.8.1. (a) *For every $f \in L^2(\Omega)^N$ there exists a unique $u \in V$ which satisfies*

$$\begin{cases} \mu \sum_{i=1}^{N} \int_\Omega \nabla u_i \cdot \nabla v_i \, dx = \sum_{i=1}^{N} \int_\Omega f_i v_i \, dx \qquad \forall v \in V, \\ u \in V. \end{cases} \tag{6.118}$$

(b) *Let u be the solution of* (6.118)*. Then the relation* (6.118) *determines a unique $p \in L^2(\Omega)$ (up to an additive constant) such that the couple $(u,p) \in V \times L^2(\Omega)$ satisfies*

$$\mu \sum_{i=1}^{N} \int_\Omega \nabla u_i \cdot \nabla v_i \, dx - \sum_{i=1}^{N} \int_\Omega f_i v_i \, dx = \int_\Omega p\, \mathrm{div}(v)\, dx \quad \forall v \in H_0^1(\Omega)^N. \tag{6.119}$$

(c) *The couple (u,p) is a weak solution of the Stokes system:*

$$\begin{cases} -\mu \Delta u + \nabla p = f \quad \text{in } \mathscr{D}'(\Omega)^N, \\ \mathrm{div}(u) = 0 \quad \text{in } \mathscr{D}'\Omega), \\ u = 0 \quad \text{on } \partial\Omega \text{ in the trace sense.} \end{cases}$$

The couple (u,p) is called the variational solution of the Stokes system.

PROOF. (a) Let us consider the bilinear form $a : V \times V \longrightarrow \mathbf{R}$

$$a(u,v) = \mu \sum_{i=1}^{N} \int_\Omega \nabla u_i \cdot \nabla v_i \, dx$$

and the linear form $l : V \longrightarrow \mathbf{R}$

$$l(v) = \sum_{i=1}^{N} \int_\Omega f_i v_i \, dx.$$

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The bilinear form a is clearly continuous and its coercivity follows, by standard argument, from the Poincaré inequality in $H_0^1(\Omega)$. The continuity of l is also immediate. Thus, all the assumptions of the Lax–Milgram theorem, Theorem 3.1.2, are satisfied. This implies the existence and uniqueness of the solution u of (6.118).

(b) The difficulty comes from the fact that $\mathscr{D}(\Omega)^N$ is not contained in the space V, and one cannot interpret directly (6.118) in terms of distributions. Moreover, up to now, the pressure p still has not appeared in the above variational formulation. Let us reformulate (6.118) as an orthogonality relation. Let us consider the linear form $L : H_0^1(\Omega)^N \longrightarrow \mathbf{R}$, which is defined by

$$L(v) = \mu \sum_{i=1}^{N} \int_\Omega \nabla u_i \cdot \nabla v_i \, dx - \sum_{i=1}^{N} \int_\Omega f_i v_i \, dx. \tag{6.120}$$

The linear form L is clearly continuous on $H_0^1(\Omega)^N$ and (6.118) precisely tells us that $L(v) = 0$ for all $v \in V$. In other words $L \in \left(H_0^1(\Omega)^N\right)^*$ and $L(v) = 0$ for all $v \in V$, i.e., $L \in V^\perp$ the orthogonal subspace of V (for the pairing between $H_0^1(\Omega)^N$ and its topological dual). The precise description of such elements is provided by the following theorem, obtained by de Rham in 1955; see [335].

Theorem 6.8.2. *Let Ω be a bounded connected set in $\mathbf{R}^N$ whose boundary is piecewise $\mathbf{C}^1$. Let $L \in \left(H_0^1(\Omega)^N\right)^*$, a linear continuous form on $H_0^1(\Omega)^N$. Set $V = \left\{v \in H_0^1(\Omega)^N : \operatorname{div}(v) = 0\right\}$. Then*

$$L(v) = 0 \quad \forall v \in V \iff \exists p \in L^2(\Omega) \textit{ such that } L(v) = \int_\Omega p \operatorname{div}(v) \, dx \quad \forall v \in H_0^1(\Omega)^N.$$

Proof of Theorem 6.8.1 continued. Let us admit the de Rham theorem (the implication $L \in V^\perp \Longrightarrow \exists p \ldots$, which is the interesting part of the theorem, is a nontrivial result) and apply it to the bilinear form L which is defined in (6.120). We thus have the existence of $p \in L^2(\Omega)$ such that

$$\mu \sum_{i=1}^{N} \int_\Omega \nabla u_i \cdot \nabla v_i \, dx - \sum_{i=1}^{N} \int_\Omega f_i v_i \, dx = \int_\Omega p \operatorname{div}(v) \, dx \quad \forall v \in H_0^1(\Omega)^N.$$

This is precisely (6.119).

(c) Since $\mathscr{D}(\Omega)^N$ is dense in $H_0^1(\Omega)^N$, the solution of (6.119) is characterized by $(u, p) \in V \times L^2(\Omega)$ and

$$\mu \sum_{i=1}^{N} \int_\Omega \nabla u_i \cdot \nabla v_i \, dx - \int_\Omega p \operatorname{div}(v) \, dx = \sum_{i=1}^{N} \int_\Omega f_i v_i \, dx \quad \forall v \in \mathscr{D}(\Omega)^N.$$

Taking $v = (0, \ldots, v_i, \ldots 0)$, $i = 1, \ldots, N$, yields

$$\mu \int_\Omega \nabla u_i \cdot \nabla v_i \, dx - \int_\Omega p \frac{\partial v_i}{\partial x_i} \, dx = \int_\Omega f_i v_i \, dx \quad \forall v_i \in \mathscr{D}(\Omega),$$

that is,

$$-\mu \Delta u_i + \frac{\partial p}{\partial x_i} = f_i \text{ in } \mathscr{D}'(\Omega).$$

Moreover, $u \in V$ contains the information

$$\begin{aligned} &\operatorname{div}(u) = 0 \quad \text{in } \mathscr{D}'(\Omega), \\ &u = 0 \quad \text{on } \partial\Omega \text{ in the trace sense.} \end{aligned}$$

Hence, (u, p) is a weak solution of the Stokes system. □

6.9 ▪ Convection-diffusion equations

In the following example, we apply the Lax–Milgram theorem in the nonsymmetric case. Let Ω be a bounded open set in $\mathbf{R}^N$. Let us give N functions $b_1, b_2, \dots, b_N$ which belong to $L^\infty(\Omega)$. We set $\vec{b} = (b_1, b_2, \dots, b_N) \in L^\infty(\Omega)^N$. Given $f \in L^2(\Omega)$, we are looking for a solution of the convection-diffusion boundary value problem

$$\begin{cases} -\Delta u - \operatorname{div}(u\vec{b}) = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.121}$$

The first-order differential operator $\operatorname{div}(u\vec{b}) = \sum_i \frac{\partial}{\partial x_i}(b_i u)$ describes the convection of a physical quantity which is moving with velocity $\vec{b}$ (it is also called advection, of drift term). The Laplacian Δu is a second-order differential operator associated with the diffusion. Note that (6.121) can be written as $-\operatorname{div}(\nabla u + u\vec{b}) = f$, which is the divergence form of a conservation law from physics. There are many situations where the phenomena of diffusion and convection occur simultaneously (heat propagation, dynamic of population, reaction-diffusion equations).

We use the following notation: for any $x \in \Omega$, $|b(x)|_{\mathbf{R}^N} = \left(\sum_i b_i(x)^2\right)^{\frac{1}{2}}$ is the Euclidian norm of the vector $\vec{b}(x) \in \mathbf{R}^N$, and $\|\,|b|\,\|_{L^\infty(\Omega)}$ is the (essential) sup norm of the function $x \mapsto |b(x)|_{\mathbf{R}^N}$.

Since Ω has been assumed to be bounded, by the Poincaré inequality,

$$\forall v \in H_0^1(\Omega) \quad \|v\|_{L^2(\Omega)} \le C_P(\Omega)\|\,|\nabla v|\,\|_{L^2(\Omega)},$$

where the Poincaré constant $C_P(\Omega)$ is the smallest constant for which the above inequality holds.

The variational approach of (6.121) is described in the following statement.

Theorem 6.9.1. *Let us give $f \in L^2(\Omega)$, and suppose that*

$$\|\,|b|\,\|_{L^\infty(\Omega)} < \frac{1}{C_P(\Omega)}, \tag{6.122}$$

where $C_P(\Omega)$ is the Poincaré constant on Ω. Then the following hold:

(a) *There exists a unique $u \in H_0^1(\Omega)$ which satisfies*

$$\begin{cases} \displaystyle\int_\Omega \nabla u \cdot \nabla v \, dx + \sum_i \int_\Omega b_i(x) u(x) \frac{\partial v}{\partial x_i} \, dx = \int_\Omega f v \, dx \quad \forall v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega). \end{cases} \tag{6.123}$$

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(b) *The solution u of* (6.123) *satisfies*

$$\begin{cases} -\Delta u - \operatorname{div}(u\vec{b}) = f & \text{in } \mathscr{D}'(\Omega) \quad (\text{equality as distributions}), \\ \gamma_0(u) = 0 & \text{on } \partial\Omega \quad (\gamma_0 \text{ is the trace operator}). \end{cases} \tag{6.124}$$

Indeed, for $u \in H_0^1(\Omega)$ there is equivalence between (6.123) *and* (6.124)*. The solution u of* (6.123) *is called the weak solution of* (6.124)*.*

PROOF. (a) Let us solve (6.123) by using the Lax–Milgram theorem. To that end, take $V = H_0^1(\Omega)$ equipped with the scalar product

$$\langle u, v\rangle = \int_\Omega (uv + \nabla u \cdot \nabla v)\, dx,$$

which makes V a Hilbert space. Then, set for any $u, v \in V$,

$$a(u,v) = \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx + \sum_i \int_\Omega b_i(x) u(x) \frac{\partial v}{\partial x_i}(x)\, dx,$$

$$l(v) = \int_\Omega f(x) v(x)\, dx.$$

Let us first verify that the bilinear form $a : V \times V \to \mathbf{R}$ is continuous. For arbitrary $u, v \in V$, by using successively the Cauchy–Schwarz inequality in $\mathbf{R}^N$ and $L^2(\Omega)$, we obtain

$$\begin{aligned} |a(u,v)| &\le \int_\Omega |\nabla u(x)|\, |\nabla v(x)|\, dx + \int_\Omega \left(\sum_i |b_i(x)| \left| \frac{\partial v}{\partial x_i}(x) \right| \right) |u(x)|\, dx \\ &\le \int_\Omega |\nabla u(x)|\, |\nabla v(x)|\, dx + \int_\Omega |b(x)|_{\mathbf{R}^N} |\nabla v(x)|_{\mathbf{R}^N} |u(x)|\, dx \\ &\le \int_\Omega |\nabla u(x)|\, |\nabla v(x)|\, dx + \|\, |b|\, \|_{L^\infty(\Omega)} \int_\Omega |\nabla v(x)|_{\mathbf{R}^N} |u(x)|\, dx \\ &\le \|\, |\nabla u|\, \|_{L^2(\Omega)} \|\, |\nabla v|\, \|_{L^2(\Omega)} + \|\, |b|\, \|_{L^\infty(\Omega)} \|\, |\nabla v|\, \|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \\ &\le \left(1 + \|\, |b|\, \|_{L^\infty(\Omega)}\right) \|u\|_V\, \|v\|_V. \end{aligned}$$

Let us now verify that the linear form $l : V \to \mathbf{R}$ is continuous. For arbitrary $v \in V$

$$\begin{aligned} |l(v)| &\le \int_\Omega |f||v|\, dx \\ &\le \left(\int_\Omega |f|^2 dx \right)^{1/2} \left(\int_\Omega |v|^2 dx \right)^{1/2} \\ &\le C\|v\| \quad \text{with } C = \|f\|_{L^2}. \end{aligned}$$

Let us verify that the bilinear form a is coercive. By the definition of a, for any $v \in V = H_0^1(\Omega)$

$$\begin{aligned} a(v,v) &= \int_\Omega |\nabla v(x)|^2 dx + \sum_i \int_\Omega b_i(x) v(x) \frac{\partial v}{\partial x_i}(x)\, dx \\ &\ge \int_\Omega |\nabla v(x)|^2 dx - \left| \sum_i \int_\Omega b_i(x) v(x) \frac{\partial v}{\partial x_i}(x)\, dx \right|. \end{aligned} \tag{6.125}$$

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For a minorization of $a(v,v)$, due to the negative sign in front of the last term of the above formula, we first look for an upper bound of this term.

$$\begin{aligned}\left|\sum_i \int_\Omega b_i(x)v(x)\frac{\partial v}{\partial x_i}(x)\,dx\right| &\le \int_\Omega \left(\sum_i |b_i(x)|\left|\frac{\partial v}{\partial x_i}(x)\right|\right)|v(x)|\,dx\\ &\le \int_\Omega |b(x)|_{\mathbf{R}^N}|\nabla v(x)|_{\mathbf{R}^N}|v(x)|\,dx\\ &\le \|\,|b|\,\|_{L^\infty(\Omega)}\int_\Omega |\nabla v(x)|_{\mathbf{R}^N}|v(x)|\,dx\\ &\le \|\,|b|\,\|_{L^\infty(\Omega)}\|\,|\nabla v|\,\|_{L^2}\|v\|_{L^2}.\end{aligned}$$

Hence, by the Poincaré inequality

$$\left|\sum_i \int_\Omega b_i v\frac{\partial v}{\partial x_i}\,dx\right| \le C_P(\Omega)\|\,|b|\,\|_{L^\infty(\Omega)}\|\,|\nabla v|\,\|^2_{L^2(\Omega)}. \tag{6.126}$$

Combining (6.125) and (6.126) we obtain

$$a(v,v) \ge \left(1 - C_P(\Omega)\|\,|b|\,\|_{L^\infty(\Omega)}\right)\|\,|\nabla v|\,\|^2_{L^2(\Omega)}.$$

By assumption (6.122), $1 - C_P(\Omega)\|\,|b|\,\|_{L^\infty(\Omega)} > 0$. Using again the Poincaré inequality, we obtain

$$a(v,v) \ge \alpha\|v\|_V^2$$

with

$$\alpha = \frac{1 - C_P(\Omega)\|\,|b|\,\|_{L^\infty(\Omega)}}{1 + C_P(\Omega)^2} > 0.$$

Hence, a is coercive, and all the assumptions of the Lax–Milgram theorem are satisfied. This implies existence and uniqueness of the solution u of problem (6.123).

(b) Let u be the solution of (6.123). Since $\mathscr{D}(\Omega) \subset H_0^1(\Omega)$, we have for all $v \in \mathscr{D}(\Omega)$

$$\sum_i \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,dx + \sum_i \int_\Omega b_i(x)u(x)\frac{\partial v}{\partial x_i}\,dx = \int_\Omega f v\,dx. \tag{6.127}$$

Let us interpret (6.127) in the distribution sense. Since $b_i u \in L^1(\Omega)$ is a distribution

$$\sum_i \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i}\right\rangle_{(\mathscr{D}'(\Omega),\mathscr{D}(\Omega))} + \sum_i \left\langle b_i u, \frac{\partial v}{\partial x_i}\right\rangle_{(\mathscr{D}'(\Omega),\mathscr{D}(\Omega))} = \langle f, v\rangle_{(\mathscr{D}'(\Omega),\mathscr{D}(\Omega))}. \tag{6.128}$$

By definition of the derivation in the distribution sense, we obtain

$$-\Delta u - \operatorname{div}(u\vec{b}) = f \ \ in\ \mathscr{D}'(\Omega).$$

Moreover, $H_0^1(\Omega) = \ker \gamma_0$, where γ_0 is the trace operator. Hence

$$\gamma_0(u) = 0 \quad \text{in the trace sense}$$

and u satisfies (6.124). Conversely, if u satisfies $-\Delta u - \operatorname{div}(u\vec{b}) = f$ in the distribution sense, we have (6.128). Then use the density of $\mathscr{D}(\Omega)$ in $H_0^1(\Omega)$ and the fact that $u \in H_0^1(\Omega)$, $b_i u \in L^2(\Omega)$, and $f \in L^2(\Omega)$ to obtain (6.123). □

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Remark 6.9.1. The condition (6.122) expresses the fact that the velocity vector $\vec{b}$ governing the convection is not too large. There is another type of condition on $\vec{b}$, for which the conclusion of Theorem 6.9.1 is still valid, namely,

$$-\mathrm{div}\vec{b} \geq 0 \quad \text{in } \mathscr{D}'(\Omega).$$

Under this condition, we have for all $v \in \mathscr{D}(\Omega)$

$$\begin{aligned}\sum_i \int_\Omega b_i(x) v(x) \frac{\partial v}{\partial x_i}(x)\,dx &= \frac{1}{2}\sum_i \int_\Omega b_i(x) \frac{\partial}{\partial x_i}(v^2)(x)\,dx \\ &= \frac{1}{2}\langle -\mathrm{div}\vec{b}, v^2\rangle_{(\mathscr{D}'(\Omega),\mathscr{D}(\Omega))} \\ &\geq 0.\end{aligned}$$

Then, by density, the result can be extended to an arbitrary $v \in H_0^1(\Omega)$, which gives the coercivity of a. The rest of the proof is unchanged.

6.10 ▪ Semilinear equations

Let Ω be a bounded open set in $\mathbf{R}^N$. Let us give $g : (x,r) \in \Omega \times \mathbf{R} \mapsto g(x,r)$, which is a Carathéodory function, i.e., g is measurable with respect to x and continuous with respect to r. We are looking for a (variational) solution of the semilinear boundary value problem

$$\begin{cases} -\Delta u = g(x,u) & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.129}$$

where $g(x,u)$ is the function $x \in \Omega \mapsto g(x,u(x)) \in \mathbf{R}$ (for short we write $g(u)$). The semilinear terminology comes from the fact that the nonlinear term $g(u)$ depends only on u (and not its partial derivatives). We will reformulate (6.129) as a fixed point problem. Then, depending on the type of (growth) assumption on g, we will apply the fixed point theorem of Banach and Picard or Schauder.

As a basic ingredient of the fixed point approach, we use the operator $T : L^2(\Omega) \to L^2(\Omega)$ which is the inverse of the Laplace–Dirichlet operator. Let us state its precise definition.

Definition 6.10.1. *$T : L^2(\Omega) \longrightarrow L^2(\Omega)$ is defined for every $h \in L^2(\Omega)$ by the following: $Th \in H_0^1(\Omega) \subset L^2(\Omega)$ is the unique solution of the variational problem*

$$\begin{cases} \displaystyle\int_\Omega \nabla(Th)(x) \cdot \nabla v(x)\,dx = \int_\Omega h(x)v(x)\,dx \quad \forall v \in H_0^1(\Omega), \\ Th \in H_0^1(\Omega). \end{cases}$$

Equivalently, Th is the variational solution of the Dirichlet problem

$$\begin{cases} -\Delta(Th) = h \quad \text{on } \Omega, \\ Th = 0 \quad \text{on } \partial\Omega. \end{cases}$$

We have

$$(-\Delta) \circ T = id_H, \qquad H = L^2(\Omega),$$

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i.e., T is the right inverse of $-\Delta$. Let us state the continuity properties of T (see Chapter 8 for the proof).

Proposition 6.10.1. *The operator T satisfies the following properties: $T : L^2(\Omega) \longrightarrow L^2(\Omega)$ is a linear continuous operator, and*

(i) $\forall h \in L^2(\Omega) \quad \|Th\|_{L^2(\Omega)} \le C_P(\Omega)^2 \|h\|_{L^2(\Omega)}$,

(ii) $\forall h \in L^2(\Omega) \quad \|Th\|_{H_0^1(\Omega)} \le C_P(\Omega)\sqrt{1 + C_P(\Omega)^2}\|h\|_{L^2(\Omega)}$,

where $C_P(\Omega)$ is the Poincaré constant on Ω.

6.10.1 ▪ Lipschitz nonlinearity

Let us suppose that there exists some constant $L_g \ge 0$ such that for all $r, s \in \mathbf{R}$

$$|g(x,r) - g(x,s)| \le L_g |r - s| \quad \text{for a.e. } x \in \Omega, \tag{6.130}$$

$$x \mapsto g(x,0) \in L^2(\Omega). \tag{6.131}$$

Assumption (6.130) expresses that $g(x,.) : \mathbf{R} \to \mathbf{R}$ is L_g-Lipschitz continuous. Assumption (6.131) implies that for any $u \in L^2(\Omega)$, $g(u)$ still belongs to $L^2(\Omega)$. Indeed, $x \mapsto g(x,u(x)$ is Lebesgue measurable as the composition of the measurable mappings $x \mapsto (x,u(x))$ and $(x,r) \mapsto g(x,r)$. Moreover

$$|g(x,u(x))| \le |g(x,0)| + L_g |u(x)| \quad \text{for a.e. } x \in \Omega.$$

Noticing that $|g(.,0)| + L_g |u(.)|$ belongs to $L^2(\Omega)$, we deduce that $g(u)$ belongs to $L^2(\Omega)$. As a consequence, we can define the operator

$$\begin{aligned} G \colon L^2(\Omega) &\longrightarrow L^2(\Omega), \\ u &\longmapsto G(u) \quad \text{with } G(u)(x) = g(x,u(x)). \end{aligned}$$

As a straight consequence of assumption (6.130), we obtain that G is L_g-Lipschitz continuous, i.e., for every $u, v \in L^2(\Omega)$

$$\|G(u) - G(v)\|_{L^2(\Omega)} \le L_g \|u - v\|_{L^2(\Omega)}. \tag{6.132}$$

Clearly, u is a solution of the semilinear boundary value problem (6.129) if and only if $T \circ G(u) = u$, i.e., u is a fixed point of the operator $T \circ G$.

Theorem 6.10.1. *Let us suppose that $g : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function that satisfies assumptions* (6.130), (6.131) *with*

$$L_g < \frac{1}{C_P(\Omega)^2}, \tag{6.133}$$

where $C_P(\Omega)$ is the Poincaré constant on Ω. Then, the operator $T \circ G : L^2(\Omega) \to L^2(\Omega)$ is Lipschitz continuous with a Lipschitz constant strictly less than one. Hence, $T \circ G$ admits a unique fixed point u, which is the unique variational solution of the semilinear boundary value problem

$$\begin{cases} -\Delta u = g(x,u) & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

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PROOF. By Proposition ??, the operator $T : L^2(\Omega) \longrightarrow L^2(\Omega)$ is linear continuous and hence Lipschitz continuous with a Lipschitz constant equal to its norm, which is less than or equal to $C_P(\Omega)^2$. By (6.132), the operator $G : L^2(\Omega) \to L^2(\Omega)$ is Lipschitz continuous with Lipschitz constant L_g. Hence, the composition $T \circ G : L^2(\Omega) \to L^2(\Omega)$ of the two operators is Lipschitz continuous with Lipschitz constant $L_g \times C_P(\Omega)^2$. When $L_g C_P(\Omega)^2 < 1$, that is, assumption (6.133), we can apply the Banach–Picard fixed point theorem; see Theorem 3.1.3. (Note that we work in the complete metric space $L^2(\Omega)$, which is indeed an Hilbert space.) Hence, there exists a unique fixed point u of $T \circ G$, which completes the proof. □

Remark 6.10.1. Note that condition (6.133), $L_g < \frac{1}{C_P(\Omega)^2}$, is sharp. Indeed, in chapter 8, we will obtain that $\frac{1}{C_P(\Omega)^2}$ is equal to the first eigenvalue λ_1 of the Laplace–Dirichlet operator. Hence condition (6.133) can be equivalently formulated as

$$L_g < \lambda_1.$$

When $g(x,u) = \lambda u + h(x)$, this condition becomes $\lambda < \lambda_1$. It is sharp because, for $\lambda = \lambda_1$, the equation

$$\begin{cases} -\Delta u = \lambda_1 u + h(x) & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{6.134}$$

does not admit a solution for an arbitrary $h \in L^2(\Omega)$. By the Fredholm alternative, (6.134) has a solution iff h is orthogonal to the eigenspace (of dimension one) associated with λ_1. Then, all the solutions differ by the addition of an element of this eigenspace.

6.10.2 ▪ Bounded nonlinearity

Let us now suppose that $g : (x,r) \in \Omega \times \mathbf{R} \mapsto g(x,r)$, and g is a Carathéodory function which satisfies the following: there exists $h \in L^2(\Omega)$ such that

$$|g(x,r)| \le h(x) \quad \forall\ r \in \mathbf{R} \text{ and a.e. } x \in \Omega. \tag{6.135}$$

Note that (6.135) implies that for almost every $x \in \Omega$, $r \mapsto g(x,r)$ is bounded. Let us show that

$$G : L^2(\Omega) \to L^2(\Omega) \ \text{ is continuous.} \tag{6.136}$$

Let (u_n) be a sequence which converges to u in $L^2(\Omega)$. We can extract a sequence (u_{n_k}) converging almost everywhere to u. By continuity of g with respect to r,

$$g(x, u_{n_k}(x)) \to g(x, u(x)) \quad \text{for a.e. } x \in \Omega. \tag{6.137}$$

Moreover

$$|g(x, u_{n_k}(x)) - g(x, u(x))| \le 2h(x)^2 \quad \text{which belongs to } L^1(\Omega). \tag{6.138}$$

From (6.137), (6.138), and the Lebesgue dominated convergence theorem, we obtain

$$g(u_{n_k}) \longrightarrow g(u) \quad \text{in } L^2(\Omega).$$

This implies that the whole sequence $(g(u_n))_n$ converges to $g(u)$ in $L^2(\Omega)$. Otherwise, there would exist some $\epsilon_0 > 0$ and a subsequence (n_l) such that

$$||g(u_{n_l}) - g(u)||_{L^2(\Omega)} \ge \epsilon_0 > 0.$$

Applying the above argument to this subsequence, we obtain a contradiction.

In order to develop a fixed point argument using only a continuity property, we use the Leray–Schauder fixed point theorem, which we recall below. It is an extension to infinite dimensional spaces of the Brouwer fixed point theorem.

Theorem 6.10.2. *Let V be a Banach space and $K \subset V$ a convex compact nonempty subset of V. Let $S : K \mapsto K$ be a continuous mapping from K into K. Then, there exists at least a fixed point u of S, i.e., $u \in K$ satisfies $S(u) = u$.*

We have all the ingredients to obtain the next theorem.

Theorem 6.10.3. *Let us suppose that $g : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function that satisfies*

$$|g(x,r)| \le h(x) \quad \forall\ r \in \mathbf{R} \text{ and a.e. } x \in \Omega$$

with $h \in L^2(\Omega)$. Then, there exists a variational solution of the semilinear boundary value problem

$$\begin{cases} -\Delta u = g(x,u) & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

PROOF. By Proposition 6.10.1, the operator $T : L^2(\Omega) \longrightarrow L^2(\Omega)$ is linear continuous. By (6.136), the operator $G : L^2(\Omega) \to L^2(\Omega)$ is continuous. Hence, the composition $T \circ G : L^2(\Omega) \to L^2(\Omega)$ of the two operators is continuous. In order to apply the Leray–Schauder fixed point theorem, we just need to find a convex compact nonempty subset K of $V = L^2(\Omega)$ such that $T \circ G$ sends K into K. First, let us notice that (6.135) implies that for any $v \in L^2(\Omega)$,

$$||G(v)||_{L^2(\Omega)} \le ||h||_{L^2(\Omega)}.$$

By the continuity property of T, Proposition 6.10.1, we obtain that the range of $T \circ G$ is contained in the convex set

$$K = \left\{ v \in H_0^1(\Omega) : \ ||v||_{H_0^1(\Omega)} \le C_P(\Omega)\sqrt{1 + C_P(\Omega)^2}\,||h||_{L^2(\Omega)} \right\}.$$

The set K is bounded in $H_0^1(\Omega)$ (it is a ball). By the Rellich–Kondrakov compactness embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ (see Theorem 5.3.3), the set K is relatively compact in $L^2(\Omega)$. Moreover, it is closed in $L^2(\Omega)$. Note that whenever $u_n \to u$ in $L^2(\Omega)$, $u_n \in K$, then u_n is bounded in $H_0^1(\Omega)$ and hence converges weakly to u in $H_0^1(\Omega)$. Since K is a closed convex set in $H_0^1(\Omega)$, it is closed for the weak topology of $H_0^1(\Omega)$, and hence $u \in K$. Thus, K is a convex compact nonempty set in $L^2(\Omega)$, and $T \circ G$ sends K into K. The conclusion follows from the Leray–Schauder fixed point theorem. □

6.10.3 ▪ Critical point methods

As a model example, let us examine the case $g(u) = |u|^{l-1}u$ for some $l > 1$. We are looking for a nontrivial solution (i.e., $u \neq 0$) of the semilinear problem

$$\begin{cases} -\Delta u = |u|^{l-1}u & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.139}$$

The idea is to consider the minimization problem

$$\min_{v \in \Sigma} \int_\Omega |\nabla v(x)|^2 dx, \tag{6.140}$$

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which consists in the minimization of the Dirichlet integral $J(v) = \int_\Omega |\nabla v(x)|^2 dx$ over the manifold

$$\Sigma = \left\{ v \in H_0^1(\Omega) : \frac{1}{p+1} \int_\Omega |v(x)|^{l+1} dx = 1 \right\}. \tag{6.141}$$

Let us observe that (6.140) is a nonconvex minimization problem. (The constraint is a sphere, which is not convex.) Condition (6.141) prevents any possible solution from being equal to 0.

First, let us examine under which condition on l there exists a solution to (6.140). Following the general topological approach (see Chapter 3), we consider a minimizing sequence (u_n) of (6.140). It is bounded in $H_0^1(\Omega)$. Let us examine for which p it is relatively compact in L^{p+1}. This is a crucial property in order to pass to the limit on the constraint $\int_\Omega |u_n|^{l+1} dx = p+1$. By the Sobolev embedding theorem, Theorem 5.7.2, $H_0^1(\Omega) \hookrightarrow L^{2^*}$ with $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}$. By the Rellich–Kondrakov theorem, Theorem 5.3.3, the embedding of $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. As a consequence, the embedding of $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact for all $p < 2^*$. An elementary computation gives

$$l + 1 < 2^* = \frac{2N}{N-2} \Leftrightarrow l < \frac{N+2}{N-2}.$$

Indeed, $l = \frac{N+2}{N-2}$ is a critical exponent for this problem. We can now state the following existence result for (6.139).

Theorem 6.10.4. *Let us suppose that $g(u) = |u|^{l-1} u$ with the exponent l that satisfies*

$$1 < l < \frac{N+2}{N-2}. \tag{6.142}$$

Then, there exists a nontrivial solution (i.e., $u \neq 0$) of the semilinear problem

$$\begin{cases} -\Delta u = |u|^{l-1} u & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.143}$$

PROOF. Let us first complete the proof of the existence of a solution of the minimization problem (6.140). Given (u_n) a minimizing sequence of (6.140), it is bounded in $H_0^1(\Omega)$. By the above argument involving the Sobolev and Rellich–Kondrakov theorems, and condition (6.142), it is relatively compact in L^{p+1}. As a consequence, we can extract a subsequence (u_{n_k}) which converges weakly in $H_0^1(\Omega)$ and strongly in L^{p+1} to some $\bar{u}$. By the lower semicontinuity property of the Dirichlet integral for the weak topology of $H_0^1(\Omega)$ we have

$$\int_\Omega |\nabla \bar{u}(x)|^2 dx \leq \inf_{v \in \Sigma} \int_\Omega |\nabla v(x)|^2 dx.$$

Moreover (u_{n_k}) converges strongly in L^{p+1}, and $\int_\Omega |u_{n_k}|^{l+1} dx = p+1$ implies $\int_\Omega |\bar{u}|^{l+1} dx = p+1$. Hence $\bar{u} \in \Sigma$, and $\bar{u}$ is a solution of (6.140).

Let us write the first-order optimality condition satisfied by $\bar{u}$. There exists a Lagrange multiplier $\lambda \in \mathbf{R}^*$ such that

$$\begin{cases} -\Delta \bar{u} = \lambda |\bar{u}|^{l-1} \bar{u} & \text{on } \Omega, \\ \bar{u} \in H_0^1(\Omega) \cap L^{p+1}(\Omega), \\ \displaystyle\int_\Omega |\bar{u}|^{l+1} dx = p+1. \end{cases} \tag{6.144}$$

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Until now, the above argument works with an arbitrary function g satisfying the appropriate growth condition. We now use the specific form of $g(u) = |u|^{l-1}u$ and the fact that it is homogeneous. Take as a new function

$$u = \frac{1}{c}\bar{u}$$

with c a positive constant, which is to be determined, in order to have $-\Delta u = |u|^{l-1}u$. Replacing in (6.144), we obtain

$$c(-\Delta u) = \lambda c^l |u|^{l-1} u.$$

Taking c such that $c^{l-1} = \frac{1}{\lambda}$ (this is always possible because $1 < l$ and λ is positive), we obtain that u is a solution of (6.143). $\square$

Remark 6.10.2. There is a famous counterexample from Pohozaev which states that if $l \geq \frac{N+2}{N-2}$, then there exists no solution to the equation (except $u = 0$)

$$\begin{cases} -\Delta u = |u|^l & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By the maximum principle, an equivalent statement is that there exists no solution $u \geq 0$ of

$$\begin{cases} -\Delta u = u^l & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

6.10.4 ▪ Monotone nonlinearity

Let us suppose that $g : \mathbf{R} \to \mathbf{R}$ is monotone, i.e.,

$$\forall r, s \in \mathbf{R}, \quad (g(r) - g(s))(r - s) \geq 0.$$

The following result establishes the existence and uniqueness of a solution of the semilinear boundary value problem:

$$\begin{cases} -\Delta u + g(u) = h & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.145}$$

Without seeking the most general results, we assume that g is continuously differentiable and Lipschitz continuous (by contrast with section 6.10.1, the Lipschitz constant may be arbitrary large), which allows us to give a fairly simple proof. In addition, it is the preparatory stage for the general monotone case (see Remark 6.10.3).

Theorem 6.10.5. *Let us suppose that $g : \mathbf{R} \to \mathbf{R}$ is monotone, continuously differentiable, and Lipschitz continuous. Then, for any $h \in L^2(\Omega)$, there exists a unique variational solution $u \in H_0^1(\Omega)$ of the semilinear problem* (6.145). *It satisfies $g(u) \in L^2(\Omega)$, and for all $v \in H_0^1(\Omega)$*

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx + \int_\Omega g(u(x))v(x)\, dx = \int_\Omega h(x)v(x)\, dx.$$

We have $\Delta u \in L^2(\Omega)$. Hence when Ω is regular, $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

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PROOF. (a) First, let us prove uniqueness. Let u_i, $i=1,2$, be two variational solutions of (6.145), i.e., $u_i \in H_0^1(\Omega)$, and for all $v \in H_0^1(\Omega)$

$$\int_\Omega \nabla u_i(x)\cdot \nabla v(x)\,dx + \int_\Omega g(u_i(x))v(x)\,dx = \int_\Omega h(x)v(x)\,dx.$$

Taking the difference between the equations ($i=1,2$), and choosing $v=u_1-u_2$, we obtain

$$\int_\Omega |\nabla(u_1-u_2)(x)|^2\,dx + \int_\Omega (g(u_2(x))-g(u_1(x)))(u_2(x)-u_1(x))\,dx = 0.$$

Since g is monotone $\int_\Omega (g(u_2(x))-g(u_1(x)))(u_2(x)-u_1(x))\,dx \geq 0$. Hence

$$\int_\Omega |\nabla(u_1-u_2)(x)|^2\,dx \leq 0,$$

which readily implies $u_1=u_2$.

(b) Let us prove existence. First, let us remark that by rewriting the equation as $-\Delta u + \tilde{g}(u) = h - g(0)$, with $\tilde{g}(r)=g(r)-g(0)$, we can reduce our study to the case $g(0)=0$. We consider the approximating problems which are obtained by truncating g. For every $n\in\mathbf{N}$, let us define

$$g_n(r) = \begin{cases} +n & \text{if } g(r) > n, \\ g(r) & \text{if } |g(r)| \leq n, \\ -n & \text{if } g(r) \leq -n. \end{cases}$$

Then $g_n : \mathbf{R}\to\mathbf{R}$ is a monotone Lipschitz continuous function which is *bounded*

$$\forall r\in\mathbf{R} \quad |g_n(r)| \leq n.$$

We are in the situation which has been studied in Section 6.10.2. Hence, there exists a (unique) solution u_n of the approximate problem

$$\begin{cases} -\Delta u_n + g_n(u_n) = h & \text{on } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

More precisely $u_n \in H_0^1(\Omega)$, and for all $v\in H_0^1(\Omega)$

$$\int_\Omega \nabla u_n(x)\cdot\nabla v(x)\,dx + \int_\Omega g_n(u_n(x))v(x)\,dx = \int_\Omega hv\,dx. \tag{6.146}$$

We will first establish estimates of the sequence (u_n). Then by using a compactness argument, we will extract a convergent subsequence. The difficult point is to pass to the limit on the nonlinear terms $g_n(u_n)$ and so obtain a solution to our problem.

The first estimation is the standard energy estimate. By taking $v=u_n$ in (6.146), we obtain

$$\int_\Omega |\nabla u_n(x)|^2\,dx + \int_\Omega g_n(u_n(x))u_n(x)\,dx = \int_\Omega hu_n\,dx. \tag{6.147}$$

Since $g_n(r)r\geq 0$ for all $r\in\mathbf{R}$ (a direct consequence of g increasing, and $g(0)=0$), we obtain

$$\int_\Omega |\nabla u_n(x)|^2\,dx \leq \int_\Omega hu_n\,dx. \tag{6.148}$$

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By the Cauchy–Schwarz inequality and the Poincaré inequality, we deduce easily

$$\sup_n \|u_n\|_{H_0^1(\Omega)} < +\infty. \tag{6.149}$$

Now use the fact that the contractions operate on $H_0^1(\Omega)$. Let us notice that g_n can be obtained by composition of the contractions $r \in \mathbf{R} \to r^+ \in \mathbf{R}$, $r \in \mathbf{R} \to r^- \in \mathbf{R}$ with the smooth Lipschitz function g. Moreover $g_n(0) = 0$. Hence, by combining Proposition 5.8.1 and Theorem5.8.2, we obtain that $g_n(u_n)$ belongs to $H_0^1(\Omega)$, and the following equality holds almost everywhere on Ω:

$$\nabla g_n(u_n) = g_n'(u_n)\nabla u_n. \tag{6.150}$$

One can also consult [243, Chapter 10, Example 13]. Taking $v = g_n(u_n)$ in (6.146), and using (6.150), we obtain

$$\int_\Omega |\nabla u_n(x)|^2 g_n'(u_n(x))\,dx + \int_\Omega |g_n(u_n(x))|^2\,dx = \int_\Omega h(x) g_n(u_n(x))\,dx.$$

Since g_n' is nonnegative (recall that g and g_n are monotone), and using again the Cauchy–Schwarz inequality, we obtain

$$\sup_n \int_\Omega |g_n(u_n(x))|^2\,dx < +\infty. \tag{6.151}$$

We now use a topological compactness argument. By (6.149) and the Rellich–Kondrakov theorem, Theorem 5.3.3, we can extract a subsequence u_{n_k} and find some $u \in H_0^1(\Omega)$ such that

$$u_{n_k} \to u \quad \text{in } L^2(\Omega) \text{ and a.e. on } \Omega.$$

Since g is continuous

$$g(u_{n_k}(x)) \to g(u(x)) \quad \text{for a.e. } x \in \Omega.$$

Since $u(x)$ is finite for almost every $x \in \Omega$, so is $g(u(x))$, and

$$g_{n_k}(u_{n_k}(x)) \to g(u(x)) \quad \text{for a.e. } x \in \Omega.$$

By (6.151) the sequence of functions $(g_{n_k}(u_{n_k}))_k$ is bounded in $L^2(\Omega)$ and hence equi-integrable. By the classical Vitali theorem, convergence almost everywhere and equi-integrability imply strong convergence in $L^1(\Omega)$ (cf. Theorem 2.4.6). Hence

$$g_{n_k}(u_{n_k}) \to g(u) \quad \text{strongly in } L^p(\Omega)\ \forall 1 \le p < 2. \tag{6.152}$$

Moreover, by the Fatou lemma and (6.151), we obtain

$$\int_\Omega |g(u(x))|^2\,dx < +\infty.$$

By using (6.152), and the weak convergence in $L^2(\Omega)^N$ of ∇u_{n_k} to ∇u, taking first $v \in \mathscr{D}(\Omega)$, we can pass to the limit on

$$\int_\Omega \nabla u_{n_k}(x)\cdot\nabla v(x)\,dx + \int_\Omega g_{n_k}(u_{n_k}(x))v(x)\,dx = \int_\Omega hv\,dx.$$

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Then, by a classical density argument, $u \in H_0^1(\Omega)$ and $g(u) \in L^2(\Omega)$, we finally obtain, for all $v \in H_0^1(\Omega)$,

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx + \int_\Omega g(u(x))v(x)\, dx = \int_\Omega hv\, dx,$$

which completes the proof. $\square$

Remark 6.10.3. (a) Indeed, one can show that the conclusion of Theorem 6.10.5 holds true for an arbitrary monotone mapping g. Even more general, one can take g equal to a maximal monotone graph in $\mathbf{R} \times \mathbf{R}$; see [134]. The proof uses the Yosida approximation g_λ of the graph g (see Chapter 17). The mapping g_λ is monotone Lipschitz continuous. So doing, we are in the situation studied in Theorem 6.10.5. By using similar arguments, we obtain estimates on the corresponding approximate solutions u_λ. Passing to the limit, as $\lambda \to 0$, relies on a maximal monotonicity argument and provides the solution.

(b) The proof of Theorem 6.10.5 can also be extended in a different direction, by weakening the monotonicity assumption and replacing it by the following: g is a continuous function that satisfies the following sign condition: for all $r \in \mathbf{R}$

$$r g(r) \geq 0.$$

The proof follows the same lines: estimations (6.147)-(6.148)-(6.149) are unchanged. (We don't use monotonicity but only the above sign condition.) But we no longer have the L^2 estimate on the $g_n(u_n)$, which uses the monotonicity. Instead, by (6.147), we have

$$\sup_n \int_\Omega g_n(u_n(x))u_n(x)\, dx < +\infty.$$

From this, we can deduce that the sequence of functions $(g_n(u_n))_n$ is equi-integrable. The rest of the proof is similar. Note that in this situation, we don't have uniqueness, and the solution is taken in a weaker sense: we have only $g(u) \in L^1(\Omega)$.

6.11 ▪ The nonlinear Laplacian Δ_p

Most of the results of the previous sections have a natural extension when replacing the Dirichlet integral $\int_\Omega |\nabla v|^2\, dx$ by $\int_\Omega |\nabla v|^p\, dx$ with $1 < p < +\infty$ and the space $H^1(\Omega)$ by the space $W^{1,p}(\Omega)$. So doing, the Laplace operator Δ is replaced by the nonlinear Laplacian Δ_p, which is defined by

$$\begin{aligned} \Delta_p v &= \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(|\nabla v|^{p-2}\frac{\partial v}{\partial x_i}\right) \\ &= \operatorname{div}(|\nabla v|^{p-2}\nabla v). \end{aligned} \tag{6.153}$$

Note that when $p = 2$ one obtains $\Delta_2 = \Delta$.

The assumption $1 < p < +\infty$ is crucial. When $1 < p < +\infty$, the space $W^{1,p}(\Omega)$ is a reflexive Banach space and most of the variational techniques which have been developed in the space $H^1(\Omega)$ can be generalized to this setting. As a variational tool, we use the convex minimization Theorem 3.3.4 which holds in general reflexive Banach spaces. The cases $p = 1$ and $p = +\infty$, which are important too for applications, are much more involved; a major reason is the lack of reflexivity of these spaces. As an illustration of a boundary value problem for the Δ_p operator, let us first consider the Dirichlet problem.

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Theorem 6.11.1. *Let p be a positive real number $1 < p < +\infty$. Let Ω be a bounded open set in $\mathbf{R}^N$ and let $f \in L^\infty(\Omega)$ be a given function.*

(i) *There exists a unique solution $u \in W_0^{1,p}(\Omega)$ of the following minimization problem:*

$$\min\left\{\frac{1}{p}\int_\Omega |\nabla v(x)|^p\,dx - \int_\Omega f\,v\,dx \;:\; v \in W_0^{1,p}(\Omega)\right\}. \tag{6.154}$$

(ii) *Equivalently, u is solution of the problem*

$$\begin{cases} \displaystyle\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx = \int_\Omega f\,v\,dx \quad \forall v \in W_0^{1,p}(\Omega),\\ u \in W_0^{1,p}(\Omega). \end{cases} \tag{6.155}$$

(iii) *The solution u of* (6.155) *is a weak solution of the boundary value problem*

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f & \text{on } \Omega,\\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.156}$$

where the first equation is satisfied in the sense of distributions and $u = 0$ in the sense of the trace operator $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$.

PROOF. Let us recall that the space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|v\|_{W^{1,p}(\Omega)} = \left(\int_\Omega (|v(x)|^p + |\nabla v(x)|^p)dx\right)^{1/p},$$

where $|\nabla v(x)|$ is the Euclidean norm of the vector $\nabla v(x)$ in $\mathbf{R}^N$.

We know (cf. Theorem 5.1.2) that $W^{1,p}(\Omega)$ is a Banach space, and since $1 < p < +\infty$, it is a reflexive Banach space.

By definition, $W_0^{1,p}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$. Let us equip $W_0^{1,p}(\Omega)$ with the norm of $W^{1,p}(\Omega)$. Hence, $W_0^{1,p}(\Omega)$, which is a closed subspace of a reflexive Banach space, is still a reflexive Banach space. Let us now consider the functional $J : W_0^{1,p}(\Omega) \to \mathbf{R}$ defined by

$$J(v) := \frac{1}{p}\int_\Omega |\nabla v(x)|^p\,dx - \int_\Omega f\,v\,dx \tag{6.157}$$

and verify that the assumptions of the convex minimization theorem, Theorem 3.3.4, are satisfied. Let us first verify that J is convex: For any $v_1, v_2 \in W_0^{1,p}(\Omega)$, for any $0 \le \lambda \le 1$,

$$\begin{aligned} J(\lambda v_1 + (1-\lambda)v_2) &= \frac{1}{p}\int_\Omega |\lambda\nabla v_1(x) + (1-\lambda)\nabla v_2(x)|^p\,dx\\ &\quad -\lambda\int_\Omega f\,v_1\,dx - (1-\lambda)\int_\Omega f\,v_2\,dx\\ &\le \frac{1}{p}\int_\Omega (\lambda|\nabla v_1(x)| + (1-\lambda)|\nabla v_2(x)|)^p\,dx\\ &\quad -\lambda\int_\Omega f\,v_1\,dx - (1-\lambda)\int_\Omega f\,v_2\,dx. \end{aligned}$$

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Let us now use the convexity of the function $\varphi(r) = r^p$ from $\mathbf{R}^+$ into $\mathbf{R}^+$ (note that the second-order derivative $\varphi''(r) = p(p-1)r^{p-2}$ is nonnegative) to obtain

$$\begin{aligned} J(\lambda v_1 + (1-\lambda)v_2) &\le \lambda\left[\frac{1}{p}\int_\Omega |\nabla v_1|^p\,dx - \int_\Omega f v_1\,dx\right] \\ &\quad + (1-\lambda)\left[\frac{1}{p}\int_\Omega |\nabla v_2|^p\,dx - \int_\Omega f v_2\,dx\right] \\ &\le \lambda J(v_1) + (1-\lambda)J(v_2). \end{aligned}$$

Clearly J is continuous on $W_0^{1,p}(\Omega)$: note that

$$\int_\Omega \big||\nabla v(x)| - |\nabla w(x)|\big|^p\,dx \le \int_\Omega |\nabla(v-w)(x)|^p\,dx \le \|v-w\|^p_{W^{1,p}(\Omega)}.$$

The above inequality implies the continuity of the mapping $v \mapsto |\nabla v|$ from $W^{1,p}$ into L^p and hence of the functional $v \mapsto \int_\Omega |\nabla v|^p\,dx$ on $W^{1,p}$. Note that since $f \in L^\infty(\Omega)$,

$$\begin{aligned} \left|\int_\Omega f v\,dx\right| &\le \|f\|_\infty \|v\|_1 \\ &\le \|f\|_\infty |\Omega|^{1/p'} \|v\|_p \quad \left(\text{where } \frac{1}{p} + \frac{1}{p'} = 1\right) \\ &\le \|f\|_\infty |\Omega|^{1/p'} \|v\|_{W^{1,p}} \end{aligned} \tag{6.158}$$

and $v \mapsto \int_\Omega f v$ is a linear continuous form on $W^{1,p}(\Omega)$.

Thus, the only point which remains to verify is the coercivity of J on $W_0^{1,p}(\Omega)$. To that end, we use the classical Poincaré inequality on $W_0^{1,p}(\Omega)$ which does not require any regularity assumptions on Ω, just Ω to be bounded. By Theorem 5.3.1, there exists a positive constant C such that

$$\forall v \in W_0^{1,p}(\Omega) \quad \int_\Omega |v(x)|^p\,dx \le C\int_\Omega |\nabla v(x)|^p\,dx. \tag{6.159}$$

To verify that J is coercive, let us prove that its sublevel sets are bounded. (Indeed these two properties are equivalent; see Proposition 3.2.8.) Let us fix some $\lambda \in \mathbf{R}$ and consider

$$lev_\lambda J = \{v \in W_0^{1,p}(\Omega) : J(v) \le \lambda\}.$$

For $v \in lev_\lambda J$, by using (6.157) we have

$$\begin{aligned} \int_\Omega |\nabla v|^p\,dx &\le p\int_\Omega |f v|\,dx + p\lambda \\ &\le p\|f\|_\infty |\Omega|^{1/p'} \|v\|_p + p\lambda. \end{aligned} \tag{6.160}$$

By using the Poincaré inequality (6.159), we obtain from (6.160)

$$\|v\|_p^p \le pC\|f\|_\infty |\Omega|^{1/p'} \|v\|_p + pC\lambda.$$

Hence

$$\|v\|_p^{p-1} \le pC\|f\|_\infty |\Omega|^{1/p'} + \frac{pC\lambda}{\|v\|_p},$$

which implies

$$\|v\|_p \le \max\left\{1, (pC\|f\|_\infty |\Omega|^{1/p'} + pC\lambda)^{\frac{1}{p-1}}\right\}. \tag{6.161}$$

Returning to (6.160) we obtain that v remains in a bounded subset of $W_0^{1,p}(\Omega)$.

Let us summarize the previous results: $W_0^{1,p}(\Omega)$ is a reflexive Banach space and $J : W_0^{1,p}(\Omega) \to \mathbf{R}$ is a convex, continuous, coercive functional. All the assumptions of Theorem 3.3.4 are satisfied and the minimization problem (6.154) admits a solution. The uniqueness of the solution is a consequence of the strict convexity on $W_0^{1,p}(\Omega)$ of the functional $v \mapsto \int_\Omega |\nabla v|^p\, dx$, which is a consequence of the strict convexity of the function $r \to r^p$ from $\mathbf{R}^+$ into $\mathbf{R}^+$.

(ii) Let us now establish the corresponding Euler equation. To that end, let us write, for any $v \in W_0^{1,p}(\Omega)$, for any $t > 0$,

$$\frac{1}{t}[J(u+tv) - J(u)] \ge 0,$$

and pass to the limit on this inequality as $t \to 0^+$. We have

$$\forall v \in W_0^{1,p}(\Omega) \quad \frac{1}{p}\int_\Omega \frac{|\nabla u + t\nabla v|^p - |\nabla u|^p}{t}\, dx - \int_\Omega f v\, dx \ge 0. \tag{6.162}$$

To pass to the limit in (6.162) we use the Lebesgue dominated convergence theorem:

Set $h(t) = |\nabla u + t\nabla v|^p$. We have $h'(t) = p|\nabla u + t\nabla v|^{p-2}(\nabla u + t\nabla v)\cdot \nabla v$. Hence

$$\begin{aligned}\frac{1}{t}\left[|\nabla u + t\nabla v|^p - |\nabla u|^p\right] &= \frac{1}{t}(h(t) - h(0))\\ &= \frac{1}{t}\int_0^1 p|\nabla u + s\nabla v|^{p-2}(\nabla u + s\nabla v)\cdot\nabla v\, ds.\end{aligned}$$

From this, by taking $0 < t \le 1$, we obtain

$$\begin{aligned}\frac{1}{t}\left||\nabla u + t\nabla v|^p - |\nabla u|^p\right| &\le \frac{p}{t}\int_0^t |\nabla u + s\nabla v|^{p-1}|\nabla v|\, ds\\ &\le p(|\nabla u| + |\nabla v|)^{p-1}|\nabla v|.\end{aligned} \tag{6.163}$$

Let us notice that $|\nabla v| \in L^p(\Omega)$ and that $(|\nabla u| + |\nabla v|)^{p-1}$ belongs to $L^{p'}(\Omega)$ (because of the equality $(p-1)p' = p$). Hence the right-hand side of (6.163) is a function which belongs to $L^1(\Omega)$ and which is independent of $t \in]0,1]$. We can now pass to the limit on (6.162) and obtain

$$\forall v \in W_0^{1,p}(\Omega) \quad \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\, dx - \int_\Omega f v\, dx \ge 0.$$

Let us now replace v by $-v$. We obtain

$$\forall v \in W_0^{1,p}(\Omega) \quad \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\, dx = \int_\Omega f v\, dx.$$

(iii) By taking $v \in \mathscr{D}(\Omega)$, we obtain, by definition of the derivation in distributions, the following equality:

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f \quad \text{in } \mathscr{D}'(\Omega).$$

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On the other hand, when Ω is regular, from $u \in W_0^{1,p}(\Omega)$ we infer that $u = 0$ on $\partial\Omega$ in the trace sense (Proposition 5.6.1). $\square$

Remark 6.11.1. We could as well consider Neumann-type boundary value problems for the Δ_p operator. Let us just notice that, assuming $u \in \mathbf{C}^2(\bar{\Omega})$, we have the following integration by parts formula:

$$\forall v \in \mathscr{D}(\bar{\Omega}) \quad \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx = -\int_\Omega v\Delta_p u \, dx + \int_{\partial\Omega} |\nabla u|^{p-2}\frac{\partial u}{\partial n} v \, d\sigma.$$

It follows that variational problems in $W^{1,p}(\Omega)$ lead to boundary conditions of the following type:

$$|\nabla u|^{p-2}\frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega.$$

(Note that when $p = 2$, one recovers the classical Neumann boundary condition.)

6.12 ▪ The obstacle problem

As a model example, we consider the variational problem with unilateral constraint

$$\min\left\{ \frac{1}{2}\int_\Omega |\nabla v(x)|^2 \, dx - \int_\Omega f(x)v(x)\, dx \; : \; v \in H_0^1(\Omega), \; v \geq g \text{ on } \Omega \right\}, \tag{6.164}$$

where Ω is a bounded open set in $\mathbf{R}^N$, $g : \Omega \to \mathbf{R}$ is the obstacle function, and $f : \Omega \to \mathbf{R}$ is an external force. From a mechanical point of view, taking $\Omega \subset \mathbf{R}^2$, the solution $u : \Omega \to \mathbf{R}$ of (6.164) gives the equilibrium position of an elastic membrane whose boundary is fixed, i.e., $u = 0$ on $\partial\Omega$, and which must lie over an obstacle $g : \Omega \to \mathbf{R}$. The membrane is submitted to the action of a vertical force $f : \Omega \to \mathbf{R}$. The constraint K of the admissible displacements is given by

$$K = \left\{ v \in H_0^1(\Omega) \; : \; v \geq g \text{ on } \Omega \right\}. \tag{6.165}$$

To ensure that K is not empty, we assume that $g \leq 0$ on $\partial\Omega$. This problem plays a central role in potential theory. Taking $f = 0$, and g equal to the characteristic function of an open set $A \subset \Omega$ (i.e., $g = 1$ on A, and $g = 0$ on $\Omega \setminus A$), the infimal value of (6.164) is half of the (harmonic) capacity of the set A; see Definition 5.8.1. This suggests that in the study of (6.164), where g is a general obstacle (not regular, thin, etc.), a central question is to determine in what sense the inequality $v \geq g$ is taken.

Let us first examine the case where

$$K = \left\{ v \in H_0^1(\Omega) \; : \; v(x) \geq g(x) \text{ for a.e. } x \in \Omega \right\},$$

i.e., the inequality constraint in (6.165) is intended in the sense almost everywhere. The existence and uniqueness of the solution of (6.164) are described below. This is a direct consequence of the variational principles of Chapter 5.

Theorem 6.12.1. *Let us suppose that $g : \Omega \to \mathbf{R}$ satisfies $K_g \neq \emptyset$, where*

$$K_g = \left\{ v \in H_0^1(\Omega) \; : \; v(x) \geq g(x) \textit{ for a.e. } x \in \Omega \right\}.$$

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(i) *Then, for any* $f \in L^2(\Omega)$, *there exists a unique solution* $u \in H_0^1(\Omega)$ *of the obstacle problem*

$$\min\left\{\frac{1}{2}\int_\Omega |\nabla v(x)|^2\,dx - \int_\Omega f(x)v(x)\,dx \;:\; v\in H_0^1(\Omega),\ v(x)\geq g(x) \text{ for a.e. } x\in\Omega\right\}. \tag{6.166}$$

(ii) *The solution* u *of* (6.166) *satisfies the variational inequality*

$$\begin{cases} \displaystyle\int_\Omega \nabla u(x)\cdot\nabla(v-u)(x)\,dx - \int_\Omega f(x)(v-u)(x)\,dx \geq 0 \qquad \forall v\in K_g, \\ u\in K_g. \end{cases} \tag{6.167}$$

PROOF. Take $V = H_0^1(\Omega)$ equipped with its classical Hilbertian structure. To show that K_g is closed in V, it suffices to notice that every convergent sequence in $H_0^1(\Omega)$ converges in $L^2(\Omega)$ and hence contains a subsequence that converges pointwise almost everywhere. As a consequence, K_g is a closed convex nonempty subset of V, and its indicator function δ_{K_g} is lower semicontinuous, convex, and proper. On the other hand, the Dirichlet integral $\Phi(v) := \frac{1}{2}\int_\Omega |\nabla v(x)|^2\,dx$ is a lower semicontinuous convex coercive function on V. (The coercivity comes from the Poincaré inequality of Theorem 5.3.1 and from the fact that Ω is bounded.) The linear integral functional $v \mapsto L_f(v) := \int_\Omega f(x)v(x)\,dx$ is continuous on $L^2(\Omega)$ and hence on $H_0^1(\Omega)$. Problem (6.166) is then the minimization of the sum of three lower semicontinuous convex proper functions

$$\min\left\{\Phi(v) + L_f(v) + \delta_{K_g}(v) \;:\; v\in H_0^1(\Omega)\right\},$$

one of them being coercive. Thus, it is the minimization of a lower semicontinuous convex coercive function on the Hilbert space $V = H_0^1(\Omega)$. By Theorem 3.3.4, it admits a solution. This solution is unique because the Dirichlet integral is strongly convex, and hence strictly convex, which makes problem (6.166) strictly convex.

Passing from (6.166) to (6.167) can be readily obtained by writing the first-order necessary and sufficient condition, as in Theorem 3.3.5, or more generally by applying the subdifferential calculus rules of Chapter 9; see Theorem 9.5.5. □

Let us interpret (6.167) as a free boundary problem. Let us define the coincidence set $C\subset\Omega$ as the region where $u=g$. The noncoincidence set $N=\Omega\setminus C$ is the region where u is not equal to g (i.e., $u>g$), and the free boundary ∂C is the interface between the two. Suppose that all these data are regular. On C, the value of u is prescribed $u=g$. Let us show that, formally, on the complementary set $N=\Omega\setminus C$, u is solution of the Poisson problem

$$\begin{cases} -\Delta u = f & \text{on } N=\Omega\setminus C, \\ u = g & \text{on } \partial C, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.168}$$

Since u and g are supposed regular, the set N where $u>g$ is open. On any compact subset of N, the continuous positive function $u-g$ is minorized by a positive constant. Therefore, for any test function $\phi\in\mathcal{D}(N)$, there exists some positive t such that

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$u+t\phi > g$ on N. Since $u+t\phi = u$ on C, we have $u+t\phi \in K_g$. Taking $v = u+t\phi$ in (6.167), after dividing by $t>0$, we obtain

$$\int_\Omega \nabla u(x)\cdot \nabla\phi(x)\,dx - \int_\Omega f(x)\phi(x)\,dx \geq 0 \qquad \forall \phi \in \mathscr{D}(N). \tag{6.169}$$

Since (6.169) holds for ϕ and $-\phi$, we can replace the inequality by an equality in (6.169), which gives

$$-\Delta u = f \qquad \text{on } N = \Omega \setminus C.$$

Note that the boundary of N is contained in $\partial C \cup \partial\Omega$. On these sets, the value of u is required to be respectively equal to g and zero. Hence (6.168) is a well-posed Poisson problem. It is a free boundary problem, because the set N, equivalently, its boundary $\partial C = \partial N$, is not given. It is part of the problem. As soon as the interface (the free boundary) between the two phases $\{u = g\}$ and $\{u > g\}$ is known, the obstacle problem reduces to the classical Poisson problem. Free boundary problems arise in various situations; another well-known example is the Stefan problem, describing the phase transition between ice and water.

The above approach is formal. By contrast with the classical Poisson problem, the solution of the obstacle problem is not smooth in general. In the one-dimensional case, with $f = 0$, the solution is affine in the noncoincidence set. Consequently, whatever the smoothness of the obstacle function g is, the second derivative of u has discontinuities at the points which are at the boundary of the contact set.

In order to consider general obstacles (possibly thin obstacles, which are supported by sets of zero Lebesgue measure) and interpret the inequality constraint $v \geq g$ in the most general sense, we use the elements of potential and capacity theory which were introduced in Section 5.8. Recall that any $v \in H_0^1(\Omega)$ has a quasi-continuous representative $\tilde{v}$ (unique up to the quasi-everywhere equality). A positive finite energy measure is a positive Radon measure which belongs to the dual of $H_0^1(\Omega)$, i.e., $\mu \geq 0$ and $\mu \in H^{-1}(\Omega)$. A positive finite energy measure does not charge the sets which have zero capacity; hence it is a capacitary measure (see Section 5.8.4). Moreover, for any $v \in H_0^1(\Omega)$, the quasi-continuous representative $\tilde{v}$ of v is integrable with respect to μ and

$$\langle v, \mu\rangle_{(H_0^1(\Omega), H^{-1}(\Omega))} = \int_\Omega \tilde{v}\, d\mu. \tag{6.170}$$

Let us introduce the general concept of unilateral constraint; see [51, Definition 3.1]. We denote by $H_0^1(\Omega)^+$ the positive cone of all nonnegative functions of $H_0^1(\Omega)$.

Definition 6.12.1. *A subset K of $H_0^1(\Omega)$ is said to be a unilateral convex set if it satisfies*

(i) *K is a closed convex nonempty subset of $H_0^1(\Omega)$;*

(ii) *$K + H_0^1(\Omega)^+ \subset K$;*

(iii) *$u \wedge v \in K$ for all $u, v \in K$.*

Let us give the functional description of a unilateral convex set; see [51, Theorem 3.2].

Theorem 6.12.2. *Let $K \subset H_0^1(\Omega)$ be a unilateral convex set. Then, there exists a sequence (g_n) of elements of K, and a mapping $g : \Omega \to \bar{\mathbf{R}}$ such that*

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(i) $\tilde{g}_n$ *decreases q.e. to* g*;*

(ii) g *is quasi-upper semicontinuous;*

(iii) $K = \{v \in H_0^1(\Omega) : \tilde{v} \geq g \text{ q.e. on } \Omega\}$.

In the general case, in order to mathematically justify the free boundary formulation of the obstacle problem, the idea is to rewrite it as a complementary problem:

$$\begin{cases} -\Delta u - f \geq 0 & \text{on } \Omega, \\ u - g \geq 0 & \text{on } \Omega, \\ (u-g)(\Delta u + f) = 0 & \text{on } \Omega. \end{cases} \tag{6.171}$$

Then notice that $-\Delta u - f$ is a positive distribution. As a consequence, it is a positive Radon measure; let us set $-\Delta u - f = \mu \geq 0$. The last condition of (6.171) can be rewritten

$$\int_\Omega (u-g)\, d\mu = 0,$$

which makes sense by considering the quasi-continuous representative of u and the above description of K with g quasi-upper semicontinuous. Let us make this precise in the following theorem.

Theorem 6.12.3. *Let* $K \subset H_0^1(\Omega)$ *be a unilateral convex set, i.e.,*

$$K = \{v \in H_0^1(\Omega) : \tilde{v} \geq g \text{ q.e. on } \Omega\}$$

with g *quasi-upper semicontinuous. Then there exists a unique solution* u *of the obstacle problem*

$$\min\left\{ \frac{1}{2}\int_\Omega |\nabla v(x)|^2\, dx - \int_\Omega f(x) v(x)\, dx \ :\ v \in H_0^1(\Omega),\ \tilde{v} \geq g \text{ q.e. on } \Omega \right\}. \tag{6.172}$$

Equivalently, u *is the solution of the following complementary problem:*

(i) $\tilde{u}(x) - g(x) \geq 0$ *q.e. on* Ω*;*

(ii) $\mu := -\Delta u - f \geq 0$ *is a positive finite energy measure;*

(iii) $\int_\Omega (\tilde{u} - g)\, d\mu = 0$.

Proof. The existence and uniqueness of u are obtained in the same manner as in Theorem 6.12.1.

Item (i) follows from $u \in K$. Let us prove that u satisfies (ii) and (iii). The first-order optimality condition for (6.172) gives

$$\begin{cases} \displaystyle\int_\Omega \nabla u(x) \cdot \nabla(v-u)(x)\, dx - \int_\Omega f(x)(v-u)(x)\, dx \geq 0 \qquad \forall v \in K, \\ u \in K. \end{cases} \tag{6.173}$$

Taking $v = u + \phi$, with $\phi \in \mathscr{D}(\Omega)$, $\phi \geq 0$, readily implies that $\mu := -\Delta u - f \geq 0$ is a positive finite energy measure. Since $\tilde{u}(x) - g(x) \geq 0$ q.e., we have $\tilde{u}(x) - g(x) \geq 0$ μ-a.e. Hence

$$\int_\Omega (\tilde{u} - g)\, d\mu \geq 0. \tag{6.174}$$

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On the other hand, by Theorem 6.12.2, there exists a sequence (g_n) of elements of K such that

$$\tilde{g}_n \quad \text{decreases q.e. to } g.$$

Taking $v = g_n$ in (6.173), we obtain

$$0 \leq \langle g_n - u, -\Delta u - f \rangle_{(H_0^1(\Omega), H^{-1}(\Omega))} = \int_\Omega (\tilde{g}_n - \tilde{u})\, d\mu.$$

From the monotone convergence theorem, we deduce that

$$\int_\Omega (g - \tilde{u})\, d\mu \geq 0. \tag{6.175}$$

Combining (6.174) and (6.175) gives

$$\int_\Omega (\tilde{u} - g)\, d\mu = 0.$$

Conversely, let us suppose that $\mu := -\Delta u - f \geq 0$ is a positive finite energy measure such that $\int_\Omega (\tilde{u} - g)\, d\mu = 0$. By (6.170), for any $v \in K$, we have

$$\begin{aligned} \langle v - u, -\Delta u - f \rangle_{(H_0^1(\Omega), H^{-1}(\Omega))} &= \int_\Omega (\tilde{v} - \tilde{u})\, d\mu \\ &= \int_\Omega (\tilde{v} - g)\, d\mu + \int_\Omega (g - \tilde{u})\, d\mu \\ &= \int_\Omega (\tilde{v} - g)\, d\mu \geq 0. \end{aligned}$$

Equivalently

$$\begin{cases} \displaystyle\int_\Omega \nabla u(x) \cdot \nabla (v - u)(x)\, dx - \int_\Omega f(x)(v - u)(x)\, dx \geq 0 \qquad \forall v \in K, \\ u \in K, \end{cases}$$

i.e., u is the solution of the obstacle problem (6.172). $\square$

Remark 6.12.1. (a) A similar analysis can be developed for the obstacle problem

$$\min \left\{ \frac{1}{p} \int_\Omega |\nabla v(x)|^p\, dx - \int_\Omega f(x) v(x)\, dx \;:\; v \in W_0^{1,p}(\Omega),\ v \geq g \text{ on } \Omega \right\},$$

where $1 < p < \infty$. The crucial property is that for $1 < p < \infty$, the Sobolev space $W_0^{1,p}(\Omega)$ is a reflexive Banach space and that the contractions operate on $W_0^{1,p}(\Omega)$; see Theorem 5.8.2 and Corollary 5.8.2. By contrast, the theory cannot be directly extended to the obstacle problem for the bi-Laplacian Δ^2, because the contractions do not operate on the Sobolev space $H^2(\Omega)$. (Truncations induce discontinuities of the first derivatives.)

(b) As an important variant of the above study, the unilateral constraint can be imposed on the boundary of Ω. The variational problem

$$\min \left\{ \frac{1}{2} \int_\Omega |\nabla v(x)|^2\, dx - \int_\Omega f(x) v(x)\, dx \;:\; v \in H^1(\Omega),\ v \geq 0 \text{ a.e. on } \partial\Omega \right\}$$

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gives rise to the following complementary problem (where the normal derivative on $\partial\Omega$ plays the role of the Laplace operator):

$$\begin{cases} -\Delta u - f = 0 & \text{on } \Omega, \\ u \geq 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial n} \geq 0 & \text{on } \partial\Omega, \\ u \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

This is a scalar problem. It can serve as a model for systems in linear elasticity, where unilateral constraint are imposed on the boundary of Ω, like the celebrated Signorini problem introduced in Section 6.7 (see also [204], [216]).