Chapter 12

Γ-convergence and applications

12.1 ▪ Γ-convergence in abstract metrizable spaces

Given a metrizable space, or more generally a first countable topological space, we would like to define a convergence notion on the space of extended real-valued functions $F : X \longrightarrow \mathbf{R} \cup \{+\infty\}$ so that the maps

$$F \mapsto \inf_X F, \quad F \mapsto \arg\min_X F$$

are sequentially continuous. More precisely, given $F_n, F : X \to \mathbf{R} \cup \{+\infty\}$, under some compactness hypotheses, we wish that the following implications hold true when $n \to +\infty$:

$F_n \to F \Longrightarrow \inf_X F_n \to \min_X F$;

$F_n \to F,\ u_n \in \arg\min_X F_n \Longrightarrow u_n \to u \in \arg\min_X F$ at least for a subsequence.

It is worth noticing that such convergence theory contains in some sense the theory of relaxation of Chapter 11. Indeed, according to the relaxation theorem, Theorem 11.1.2, one has

$$F_n \equiv F \to \mathrm{cl}(F) \Longrightarrow \inf_X = \min_X \mathrm{cl}(F)$$

and every relatively compact minimizing sequence possesses a subsequence converging to $u \in \arg\min_X F$. Therefore, constant sequences $F_n \equiv F$ must converge to $\mathrm{cl}(F)$ in the sense described above (see Remark 12.1.1).

Such an issue is of central importance in the calculus of variations. Indeed, many problems arising from physics, mechanics, economics, and approximation methods in numerical analysis are modeled by means of minimization of functionals depending on some parameter, here formally denoted by n. For instance, we write F_n for F_ε, where ε is a small parameter associated to a thickness, a stiffness in mechanics, or a size of small discontinuities. Then, if the model associated with F_n possesses a variational formulation, the problem of finding a functional F asymptotically equivalent to F_n, formally written $F \sim F_n$, must be posed in terms of variational analysis: $F \sim F_n$ means that when n tends to infinity (or ε tends to zero),

$$\inf_X F \sim \min_X F_n;$$

$$\overline{x} \in \arg\min_X F \sim x_n \in \varepsilon_n\text{-}\arg\min_X F_n,\ \varepsilon_n \to 0,$$

in the sense of some suitable topology on X.

The notion of Γ-convergence, introduced by De Giorgi and Franzoni [197] and studied in this section, corresponds to that issue.

Definition 12.1.1. *Let (X,d) be a metrizable space, or more generally a first countable topological space, $(F_n)_{n\in\mathbf{N}}$ a sequence of extended real-valued functions $F_n : X \longrightarrow \mathbf{R}\cup\{+\infty\}$, and $F : X \longrightarrow \mathbf{R}\cup\{+\infty\}$. The sequence $(F_n)_{n\in\mathbf{N}}$ (sequentially) Γ-converges to F at $x \in X$ iff both the following assertions hold:*

(i) *for all sequences $(x_n)_{n\in\mathbf{N}}$ converging to x in X, one has*

$$F(x) \le \liminf_{n\to+\infty} F_n(x_n);$$

(ii) *there exists a sequence $(y_n)_{n\in\mathbf{N}}$ converging to x in X such that*

$$F(x) \ge \limsup_{n\to+\infty} F_n(y_n).$$

When (i) *and* (ii) *hold for all x in X, we say that $(F_n)_{n\in\mathbf{N}}$ Γ-converges to F in (X,d) and we write $F = \Gamma - \lim_{n\to+\infty} F_n$.*

Note that trivially the system of assertions (i) and (ii) is equivalent to (i) and (ii)′:

(i) for all sequences $(x_n)_{n\in\mathbf{N}}$ converging to x in X, one has

$$F(x) \le \liminf_{n\to+\infty} F_n(x_n);$$

(ii)′ there exists a sequence $(y_n)_{n\in\mathbf{N}}$ converging to x in X such that

$$F(x) = \lim_{n\to+\infty} F_n(y_n).$$

Remark 12.1.1. Let us consider the constant sequence $(F_n)_{n\in\mathbf{N}}$, where $F_n = F : X \longrightarrow \mathbf{R}\cup\{+\infty\}$ is a given function. From Chapter 11, this sequence does not Γ-converge to F but converges to the lsc envelope $\mathrm{cl}(F)$ of F. Consequently, the Γ-convergence is not in general associated with a topology on the family of all functions $F : X \longrightarrow \mathbf{R}\cup\{+\infty\}$. For a detailed analysis of subfamilies of functions on which the Γ-convergence is endowed by a topology, we refer the interested reader to [183].

Remark 12.1.2. Let us recall the definition of the set convergence. Let $(C_n)_{n\in\mathbf{N}}$ be a sequence of subsets of a metric space (X,d) or more generally of any topological space. The lower limit of the sequence $(C_n)_{n\in\mathbf{N}}$ is the subset of X denoted by $\liminf_{n\to+\infty} C_n$ and defined by

$$\liminf_{n\to+\infty} C_n = \left\{x \in X : \exists x_n \to x,\ x_n \in C_n\ \forall n \in \mathbf{N}\right\}.$$

The upper limit of the sequence $(C_n)_{n\in\mathbf{N}}$ is the subset of X denoted by $\limsup_{n\to+\infty} C_n$ and defined by

$$\limsup_{n\to+\infty} C_n = \left\{x \in X : \exists (n_k)_{k\in\mathbf{N}},\ \exists (x_k)_{k\in\mathbf{N}},\ \forall k, x_k \in C_{n_k},\ x_k \to x\right\}.$$

The sequence $(C_n)_{n\in\mathbf{N}}$ is said to be convergent if the following equality holds:

$$\liminf_{n\to+\infty} C_n = \limsup_{n\to+\infty} C_n.$$

The common value is called the limit of $(C_n)_{n\in\mathbf{N}}$ in the Painlevé–Kuratowski sense and denoted by $\lim_{n\to+\infty} C_n$. Therefore, by definition $x \in C := \lim_{n\to+\infty} C_n$ iff the two following assertions hold:

$$\forall x \in C,\ \exists (x_n)_{n\in\mathbf{N}} \text{ such that } \forall n \in \mathbf{N}, x_n \in C_n, \text{ and } x_n \to x;$$

$$\forall (n_k)_{k\in\mathbf{N}},\ \forall (x_k)_{k\in\mathbf{N}} \text{ such that } \forall k \in \mathbf{N},\ x_k \in C_{n_k},\ x_k \to x \Longrightarrow x \in C.$$

One can prove that the Γ-convergence of a sequence $(F_n)_{n\in\mathbf{N}}$ to a function F is equivalent to the convergence of the sequence of epigraphs of the functions F_n to the epigraph of F when the class of subsets of $X \times \mathbf{R}$ is equipped with the set convergence previously defined (see Attouch [37]). This is why Γ-convergence is sometimes called epiconvergence.

We define the extended real-valued functions $\Gamma-\limsup_{n\to+\infty} F_n$ and $\Gamma-\liminf_{n\to+\infty} F_n$ by setting for all $x \in X$

$$\begin{aligned} \Gamma-\limsup_{n\to+\infty} F_n(x) &:= \sup_{m\in\mathbf{N}^*} \limsup_{n\to+\infty} \inf\left\{F_n(y) : d(x,y) < \frac{1}{m}\right\}, \\ \Gamma-\liminf_{n\to+\infty} F_n(x) &:= \sup_{m\in\mathbf{N}^*} \liminf_{n\to+\infty} \inf\left\{F_n(y) : d(x,y) < \frac{1}{m}\right\}. \end{aligned} \tag{12.1}$$

Since clearly $\limsup_{n\to+\infty} \inf\{F_n(y) : d(x,y) < \frac{1}{m}\}$ and $\liminf_{n\to+\infty} \inf\{F_n(y) : d(x,y) < \frac{1}{m}\}$ are nondecreasing with respect to m, each supremum above is a limit when m goes to $+\infty$. The following proposition is a straightforward consequence of the definitions above. For a proof see Attouch [37], Braides [123], and Dal Maso [183].

Proposition 12.1.1. *Let (X,d) be a metrizable space, or more generally a first countable topological space, $(F_n)_{n\in\mathbf{N}}$ a sequence of functions $F_n : X \longrightarrow \mathbf{R}\cup\{+\infty\}$, and $F : X \longrightarrow \mathbf{R}\cup\{+\infty\}$. Then*

(i) *for all $x \in X$,*

$$\begin{aligned} \Gamma-\limsup_{n\to+\infty} F_n(x) &:= \min\left\{\limsup_{n\to+\infty} F_n(x_n) : x_n \to x\right\}, \\ \Gamma-\liminf_{n\to+\infty} F_n(x) &:= \min\left\{\liminf_{n\to+\infty} F_n(x_n) : x_n \to x\right\}; \end{aligned}$$

(ii) *the sequence $(F_n)_{n\in\mathbf{N}}$ (sequentially) Γ-converges to F iff*

$$\Gamma-\limsup_{n\to+\infty} F_n \le F \le \Gamma-\liminf_{n\to+\infty} F_n;$$

(iii) *the functions $\Gamma-\limsup_{n\to+\infty} F_n$ and $\Gamma-\liminf_{n\to+\infty} F_n$ are lsc;*

(iv) *assuming that there exist $\alpha > 0$ and x_0 in X such that for all $n \in \mathbf{N}$ and for all x in X $F_n(x) \ge -\alpha(1+d(x,x_0))$, then*

$$\begin{aligned} \Gamma-\limsup_{n\to+\infty} F_n(x) &:= \sup_{\lambda\ge 0} \limsup_{n\to+\infty} \inf_{y\in X} \{F_n(y) + \lambda d(x,y)\}, \\ \Gamma-\liminf_{n\to+\infty} F_n(x) &:= \sup_{\lambda\ge 0} \liminf_{n\to+\infty} \inf_{y\in X} \{F_n(y) + \lambda d(x,y)\}. \end{aligned}$$

Remark 12.1.3. Note that assertion (iv) expresses the functionals $\Gamma - \limsup_{n\to+\infty} F_n$ and $\Gamma - \liminf_{n\to+\infty} F_n$ in terms of unconstrained problems by penalizing the distance from the point x in (12.1). Note also that $F_n^\lambda(x) := \inf_{y\in X}\{F_n(y)+\lambda d(x,y)\}$ is nothing but the Baire approximation of the functional F_n at the point x introduced in Theorem 9.2.1 in the context of Lipschitz regularization via epi-sum in normed spaces. Recall that $F_n^\lambda(x)$ increases to $F_n(x)$ when $\lambda \to +\infty$ and that F_n^λ is Lipschitz continuous with constant λ. We could replace this penalization by the Moreau–Yoshida approximation $\inf_{y\in X}\left\{F_n(y)+2\lambda\lambda d(x,y)^2\right\}$.

The main interest of the concept of Γ-convergence is its variational nature made precise in item (i) below. For more precise details about epiconvergence or Γ-convergence, see Attouch [37], Braides [123], and Dal Maso [183].

Theorem 12.1.1. *Let $(F_n)_{n\in\mathbf{N}}$ be a sequence of functions $F_n : X \longrightarrow \mathbf{R}\cup\{+\infty\}$ which Γ-converges to some function $F : X \longrightarrow \mathbf{R}\cup\{+\infty\}$. Then the following assertions hold:*

(i) *Let $x_n \in X$ be such that $F_n(x_n) \le \inf\{\, F_n(x) : x \in X \,\} + \varepsilon_n$, where $\varepsilon_n > 0$, $\varepsilon_n \to 0$ when $n \to +\infty$. Assume that $\{x_n,\ n \in \mathbf{N}\}$ is relatively compact; then every cluster point $\overline{x}$ of $\{x_n : n \in \mathbf{N}\}$ is a minimizer of F and*

$$\lim_{n\to+\infty} \inf\{F_n(x) : x \in X\} = F(\overline{x}).$$

(ii) *If $G : X \to \mathbf{R}$ is continuous, then $(F_n + G)_{n\in\mathbf{N}}$ Γ-converges to F+G.*

Let $(F_n)_{n\in\mathbf{N}}$ be a sequence of functions $F_n : X \longrightarrow \mathbf{R}\cup\{+\infty\}$. If there exists a function $F : X \longrightarrow \mathbf{R}\cup\{+\infty\}$ such that each subsequence of $(F_n)_{n\in\mathbf{N}}$ possesses a subsequence which Γ-converges to F, then all the sequence Γ-converges to F.

PROOF. Assertion (ii) is easy to establish and is left to the reader. For a proof of the last assertion, consult Attouch [37, Proposition 2.72]. The proof of assertion (i) is very close to that of Theorem 11.1.2. Let $\overline{x}$ be a cluster point of $\{x_n : n \in \mathbf{N}\}$, let $(x_{\sigma(n)})_{n\in\mathbf{N}}$ be a subsequence of $\{x_n : n\in\mathbf{N}\}$ converging to $\overline{x}$, and set

$$\tilde{x}_m = \begin{cases} x_{\sigma(n)} & \text{if there exists } n \text{ such that } m = \sigma(n),\\ \bar{x} & \text{otherwise.}\end{cases}$$

Then $\tilde{x}_m \to \bar{x}$ when $m \to +\infty$ and, according to (i) of Definition 12.1.1, we have

$$F(\overline{x}) \le \liminf_{n\to+\infty} F_n(\tilde{x}_n) \le \liminf_{n\to+\infty} F_{\sigma(n)}(x_{\sigma(n)}) = \liminf_{n\to+\infty} \inf_X F_{\sigma(n)}. \tag{12.2}$$

Let now x be any element of X. According to (ii) of Definition 12.1.1, there exists a sequence $(y_n)_{n\in\mathbf{N}}$ converging to x and satisfying

$$F(x) \ge \limsup_{n\to+\infty} F_n(y_n) \ge \limsup_{n\to+\infty} F_{\sigma(n)}(y_{\sigma(n)}). \tag{12.3}$$

Combining (12.2) and (12.3), we obtain

$$F(\overline{x}) \le \liminf_{n\to+\infty} \inf_X F_{\sigma(n)} \le \limsup_{n\to+\infty} \inf_X F_{\sigma(n)} \le \limsup_{n\to+\infty} F_{\sigma(n)}(y_{\sigma(n)}) \le F(x). \tag{12.4}$$

This proves that $F(\overline{x}) = \min_X F$.

Taking $x = \overline{x}$ in (12.4) we also obtain $\lim_{n\to+\infty} \inf_X F_{\sigma(n)} = \min_X F$. Since all subsequence of $\inf_X F_n$ possesses a subsequence converging to $\min_X F$, one has $\lim_{n\to+\infty} \inf_X F_n = \min_X F$ as required. □

In the following sections, we give three applications of the Γ-convergence. In Section 14.2, we also show how the Γ-convergence allows us to justify some one-dimensional models in the framework of fracture mechanics.

12.2 ▪ Application to the nonlinear membrane model

Let ω be an open bounded subset of $\mathbf{R}^2$ with boundary γ and consider $\Omega_\varepsilon = \omega \times (0,\varepsilon)$, the reference configuration filled up by some elastic material. This three-dimensional thin structure is clamped on a part $\Gamma_{0,\varepsilon} = \gamma_0 \times (0,\varepsilon)$ of the boundary $\partial\Omega_\varepsilon$ of Ω_ε (see Figure 12.1).

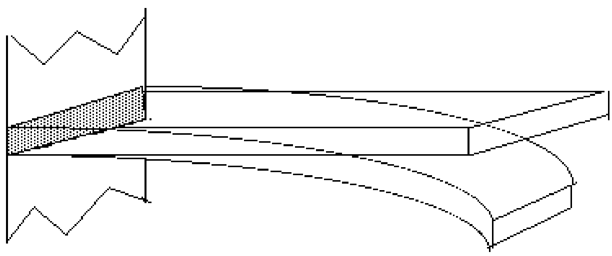

Figure 12.1. *The deformation of a thin layer Ω_ε of size ε.*

To take into account large purely elastic deformation, the constitutive law of the deformable body is associated with a nonconvex elastic density f satisfying a growth condition of order $p > 1$. The stored strain energy associated with a displacement field $u : \Omega_\varepsilon \to \mathbf{R}^3$ is given by the integral functional $F_\varepsilon : L^p(\Omega_\varepsilon, \mathbf{R}^3) \longrightarrow \mathbf{R}^+ \cup \{+\infty\}$ defined by

$$F_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_{\Omega_\varepsilon} f(\nabla u)\,dx & \text{if } u \in W^{1,p}_{\Gamma_{0,\varepsilon}}(\Omega_\varepsilon, \mathbf{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

where the density f satisfies conditions (11.5) and (11.6), namely, there exist three positive constants α, β, L such that

$$\forall a \in \mathbf{M}^{3\times 3} \quad \alpha|a|^p \le f(a) \le \beta(1+|a|^p), \tag{12.5}$$

$$\forall a,\ b \in \mathbf{M}^{3\times 3} \quad |f(a)-f(b)| \le L|b-a|(1+|a|^{p-1}+|b|^{p-1}). \tag{12.6}$$

The scaling parameter ε^{-1} accounts for the stiffness of the material. In the linearized elasticity framework, it corresponds to Lamé coefficients of order ε^{-1}.

The structure is subjected to applied body forces $g_\varepsilon : \Omega_\varepsilon \longrightarrow \mathbf{R}^3$ for which we make the following assumption: there exists a vector valued function $g : \Omega = \omega \times (0,1) \longrightarrow \mathbf{R}^3$, $g \in L^q(\Omega, \mathbf{R}^3)$ $(1/p + 1/q = 1)$, such that $\varepsilon g_\varepsilon(\hat{x}, \varepsilon x_3) = g(x)$, $x = (\hat{x}, x_3)$. The exterior loading is

$$L_\varepsilon(u) = \int_{\Omega_\varepsilon} g_\varepsilon.u\,dx$$

and the equilibrium configuration is given by displacement vector fields $\overline{u}_\varepsilon$, solutions of the problem

$$\inf\left\{F_\varepsilon(u) - L_\varepsilon(u) : u \in L^p(\Omega_\varepsilon, \mathbf{R}^3)\right\}.$$

Due to the very small thickness ε of the layer Ω_ε, for computing an approximate equilibrium displacement field, it is unrealistic to make a direct use of the finite element method described in Chapter 7. The variational property of Γ-convergence (Theorem 12.1.1) provides a new procedure: by letting ε go to zero, we aim at finding the elastic energy of a (fictitious) material occupying the two-dimensional membrane ω. We will finally compute an approximate equilibrium displacement field for the corresponding minimization problem by means of a two-dimensional finite element method.

To work in the fixed space $L^p(\Omega, \mathbf{R}^3)$, $\Omega = \omega \times (0,1)$, the change of scale $(\hat{x}, x_3) = (\hat{x}, \varepsilon x_3')$ transforming $(\hat{x}, x_3) \in \Omega_\varepsilon$ into $(\hat{x}, x_3') \in \Omega$ leads to the following equivalent optimization problem: find $\tilde{u}_\varepsilon$ solution of

$$\inf\left\{\tilde{F}_\varepsilon(u) - \int_\Omega g.u\, dx : u \in L^p(\Omega, \mathbf{R}^3)\right\},$$

where

$$\tilde{F}_\varepsilon(v) = \begin{cases} \displaystyle\int_\Omega f\left(\widehat{\nabla} v, \frac{1}{\varepsilon}\frac{\partial v}{\partial x_3}\right) dx & \text{if } v \in W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

$\Gamma_0 = \gamma_0 \times (0,1)$, and $\widehat{\nabla} v$ denotes the tangential gradient of v (i.e., $\widehat{\nabla} v = (\frac{\partial v_i}{\partial x_j})_{i=1,2,3,\ j=1,2}$).

As suggested above, we first establish the existence of the Γ-limit of the sequence $(\tilde{F}_\varepsilon)_{\varepsilon>0}$ when ε goes to zero. To make precise its domain, we establish the following compactness result.

Proposition 12.2.1 (compactness). *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^p(\Omega, \mathbf{R}^3)$ satisfying*

$$\sup_{\varepsilon>0} \tilde{F}_\varepsilon(u_\varepsilon) < +\infty.$$

Then, there exist a nonrelabeled subsequence and u in $V = \{v \in W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^m) : \frac{\partial v}{\partial x_3} = 0\}$, such that u_ε converges to u, weakly in $W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3)$ and strongly in $L^p(\Omega, \mathbf{R}^3)$.

PROOF. Since $\sup_{\varepsilon>0} \tilde{F}_\varepsilon(u_\varepsilon) < +\infty$, by using the lower bound in (12.5), we obtain

$$\tilde{F}_\varepsilon(u_\varepsilon) = \int_\Omega f\left(\widehat{\nabla} u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial u_\varepsilon}{\partial x_3}\right) dx;$$

$(\nabla u_\varepsilon)_{\varepsilon>0}$ is bounded in $L^p(\Omega, \mathbf{M}^{3\times 3})$;

$\left(\frac{1}{\varepsilon}\frac{\partial u_\varepsilon}{\partial x_3}\right)_{\varepsilon>0}$ is bounded in $L^p(\Omega, \mathbf{R}^3)$.

Consequently, according to the Rellich–Kondrakov theorem, Theorem 5.4.2, and the Poincaré inequality, Theorem 5.3.1, there exist some $u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3)$ and a nonrelabeled subsequence of $(u_\varepsilon)_{\varepsilon>0}$ such that

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^3);$$

$$u_\varepsilon \to u \quad \text{strongly in } L^p(\Omega, \mathbf{R}^3);$$

$$\frac{\partial u_\varepsilon}{\partial x_3} \to 0 \quad \text{strongly in } L^p(\Omega, \mathbf{R}^3).$$

Therefore u belongs to V. □

Note that V is canonically isomorphic to $W^{1,p}_{\gamma_0}(\omega,\mathbf{R}^3)$. In what follows, we will use the same notation for $v \in V$ and its canonical representant in $W^{1,p}_{\gamma_0}(\omega,\mathbf{R}^3)$. The following theorem was established by Le Dret and Raoult [271]. For more general and recent variational models related to thin elastic plates, see [33], [35], [114], [284], and [285]. For a variational model taking into account oscillation-concentration effects, see [272].

Theorem 12.2.1 (Le Dret and Raoult [271]). *Let us equip $L^p(\Omega,\mathbf{R}^3)$ with its strong topology. The sequence of integral functionals $(\tilde{F}_\varepsilon)_{\varepsilon>0}$ Γ-converges to the integral functional F defined in $L^p(\Omega,\mathbf{R}^3)$ by*

$$F(u)=\begin{cases}\displaystyle\int_\omega Qf_0(\widehat{\nabla} u)\, d\hat{x} & \text{if } u \in V,\\ +\infty & \text{otherwise.}\end{cases}$$

The energy density $f_0 : \mathbf{M}^{3\times 2} \longrightarrow \mathbf{R}$ is defined for all m in $\mathbf{M}^{3\times 2}$ by

$$f_0(m)=\inf\left\{f((m|\xi)) : \xi \in \mathbf{R}^3\right\}$$

and Qf_0 denotes the quasi-convex envelope of f_0. We write (m/ξ) to denote the matrix $\mathbf{M}^{3\times 3}$ obtained by completing the matrix m with the column ξ.

Since the map $u \mapsto \int_\Omega g.u\, dx$ is continuous on $L^p(\Omega,\mathbf{R}^3)$, from Theorem 12.1.1 we deduce the following corollary.

Corollary 12.2.1. *The sequence of optimization problems*

$$\inf\left\{\tilde{F}_\varepsilon(u)-\int_{\Omega_\varepsilon} g.u\, dx : u \in L^p(\Omega_\varepsilon,\mathbf{R}^3)\right\} \qquad (\mathscr{P}_\varepsilon)$$

converges to the limit problem

$$\min\left\{F(u)-\int_\omega \overline{g}.u\, d\hat{x} : u \in V\right\}, \qquad (\mathscr{P})$$

where $\overline{g}$ is defined for all $\hat{x} \in \omega$, by $\overline{g}(\hat{x})=\int_0^1 g(\hat{x},s)\, ds$.

Moreover, if $\overline{u}_\varepsilon$ is a solution or an ε-minimizer of $(\mathscr{P}_\varepsilon)$, then $\tilde{u}_\varepsilon$ defined by $\tilde{u}_\varepsilon(x)=\overline{u}_\varepsilon(\hat{x},\varepsilon x_3)$ strongly converges in $L^p(\Omega,\mathbf{R}^3)$ to a solution $\overline{u}$ of the limit problem $(\mathscr{P})$.

Roughly speaking, for very small thickness of the layer Ω_ε, an equilibrium configuration $\overline{u}_\varepsilon$ of $(\mathscr{P}_\varepsilon)$, living in $W^{1,p}_{\Gamma_{0,\varepsilon}}(\Omega_\varepsilon,\mathbf{R}^3)$, is close to $\overline{u}$, living in $W^{1,p}_{\gamma_0}(\omega,\mathbf{R}^3)$, and the layer Ω_ε may be considered as a two-dimensional membrane ω, reference configuration filled up by some elastic material whose strain energy density is the function Qf_0.

PROOF OF THEOREM 12.2.1. For all bounded Borel sets A of $\mathbf{R}^N$, we will sometimes denote its N-dimensional Lebesgue measure by $|A|$ rather than $\mathscr{L}^N(A)$. The proof proceeds in two steps, corresponding to each inequality in the definition of the Γ-convergence. Let us first notice that f_0 satisfies

$$\forall m \in \mathbf{M}^{3\times 2} \quad \alpha|m|^p \le f_0(m) \le \beta(1+|m|^p), \tag{12.7}$$

$$\forall m,\ m' \in \mathbf{M}^{3\times 2} \quad |f_0(m)-f_0(m')| \le L'|m-m'|(1+|m|^{p-1}+|m'|^{p-1}), \tag{12.8}$$

where L' is a positive constant depending only on p and L. These estimates are obtained by easy calculations from (12.5) and (12.6) and are left to the reader. Consequently f_0 fulfills all conditions of Proposition 11.2.2.

First step. Let u_ε strongly converge to u in $L^p(\Omega,\mathbf{R}^3)$. We are going to establish

$$F(u) \le \liminf_{\varepsilon\to 0} \tilde{F}_\varepsilon(u_\varepsilon).$$

Obviously, one may assume $\liminf_{\varepsilon\to 0}\tilde{F}_\varepsilon(u_\varepsilon) < +\infty$ so that for a nonrelabeled subsequence,

$$\tilde{F}_\varepsilon(u_\varepsilon) = \int_\Omega f\left(\widehat{\nabla} u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial u_\varepsilon}{\partial x_3}\right) dx,$$

and from Proposition 12.2.1, u belongs to V. Trivially we have

$$\begin{aligned}\int_\Omega f\left(\widehat{\nabla} u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial u_\varepsilon}{\partial x_3}\right) dx &\ge \int_\Omega f_0(\widehat{\nabla} u_\varepsilon)\, dx \\ &\ge \int_\Omega Qf_0(\widehat{\nabla} u_\varepsilon)\, dx. \end{aligned} \tag{12.9}$$

Let us now consider the function $h : \mathbf{M}^{3\times 3} \longrightarrow \mathbf{R}$ defined by $h(a) = Qf_0(a_1|a_2)$, where a_1 and a_2 denote the two first columns of the matrix $a = (a_1,a_2,a_3)$. We claim that

$$\forall a \in \mathbf{M}^{3\times 3} \quad 0 \le h(a) \le \beta(1+|a|^p), \tag{12.10}$$
$$\forall a,\ b \in \mathbf{M}^{3\times 3} \quad |h(a)-h(b)| \le L''|b-a|(1+|a|^{p-1}+|b|^{p-1}), \tag{12.11}$$

where L'' is some positive constant depending only on p and L'. We also claim that $h = Qh$, where, D denoting any bounded open subset of $\mathbf{R}^3$ with $|\partial D| = 0$, the function Qh is defined by

$$\forall a \in \mathbf{M}^{3\times 3} \quad Qh(a) = \inf\left\{\frac{1}{|D|}\int_D h(a+\nabla\phi)\, dx : \phi \in W_0^{1,p}(D,\mathbf{R}^3)\right\}$$

(see Proposition 11.2.2). Inequalities (12.10) and (12.11) are straightforward consequences of inequalities (12.7) and (12.8). Let us show that $h = Qh$. Indeed, let $Y = \hat{Y}\times(0,1)$, where $\hat{Y} = (0,1)^2$, $\phi \in W_0^{1,p}(Y,\mathbf{R}^3)$, and set $\phi_y(\hat{x}) = \phi(\hat{x},y)$ for a.e. y in $(0,1)$. Clearly ϕ_y belongs to $W_0^{1,p}(\hat{Y},\mathbf{R}^3)$ and we have

$$\begin{aligned}\int_Y h(a+\nabla\phi)\, dx &= \int_Y Qf_0((a_1|a_2)+\widehat{\nabla}\phi)\, dx \\ &= \int_0^1 \left(\int_{\hat{Y}} Qf_0((a_1|a_2)+\nabla\phi_y(\hat{x}))\, d\hat{x}\right) dy \\ &\ge \int_0^1 Qf_0((a_1|a_2))dy = h(a),\end{aligned}$$

where we have used the quasi-convexity inequality satisfied by Qf_0 in the last inequality (see Proposition 11.2.2). Consequently,

$$h(a) = \inf\left\{\int_Y h(a+\nabla\phi)\, dx : \phi \in W_0^{1,p}(Y,\mathbf{R}^3)\right\}$$

and, according to Proposition 11.2.2, the equality $h = Qh$ follows from

$$\inf\left\{\frac{1}{|D|}\int_D h(a+\nabla\phi)\,dx : \phi \in W_0^{1,p}(D,\mathbf{R}^3)\right\}$$
$$= \inf\left\{\int_Y h(a+\nabla\phi)\,dx : \phi \in W_0^{1,p}(Y,\mathbf{R}^3)\right\}.$$

Letting $\varepsilon \to 0$ in (12.9) and according to Remark 11.2.1 and Theorem 13.2.1, we obtain

$$\begin{aligned}\liminf_{\varepsilon\to 0}\int_\Omega f\left(\widehat{\nabla}u_\varepsilon, \frac{1}{\varepsilon}\frac{\partial u_\varepsilon}{\partial x_3}\right)dx &\geq \liminf_{\varepsilon\to 0}\int_\Omega Qf_0(\widehat{\nabla}u_\varepsilon)\,dx\\ &= \liminf_{\varepsilon\to 0}\int_\Omega h(\nabla u_\varepsilon)\,dx\\ &\geq \int_\Omega h(\nabla u)\,dx = \int_\Omega Qf_0(\widehat{\nabla}u)\,dx.\end{aligned}$$

Since u belongs to V, with our convention the last integral is also equal to $\int_\omega Qf_0(\nabla u)\,d\hat{x}$ and the proof of the first step is complete.

Second step. We are going to establish $(\Gamma - \limsup_{\varepsilon\to 0}\tilde{F}_\varepsilon) \leq F$. Let us assume $F(u) < +\infty$ so that $u \in W^{1,p}_{\gamma_0}(\omega,\mathbf{R}^3)$ and

$$F(u) = \int_\omega Qf_0(\nabla u(\hat{x}))\,d\hat{x}.$$

Following the proof of Lemma 11.2.2 about interchange between infimum and integral, one may easily establish

$$\int_\omega f_0(\nabla u)\,d\hat{x} = \inf_{\xi\in\mathscr{D}(\omega,\mathbf{R}^3)}\int_\omega f(\nabla u(\hat{x}),\xi(\hat{x}))\,d\hat{x}. \tag{12.12}$$

Let now ξ be some fixed element in $\mathscr{D}(\omega,\mathbf{R}^3)$ and define in $W^{1,p}_{\Gamma_0}(\Omega,\mathbf{R}^3)$ the following function:

$$w_\varepsilon(\hat{x},x_3) = u(\hat{x}) + \varepsilon x_3\xi(\hat{x}).$$

It is easy to see that w_ε strongly converges to u in $L^p(\Omega,\mathbf{R}^3)$. On the other hand, from (12.6), an easy computation gives

$$\lim_{\varepsilon\to 0}\tilde{F}_\varepsilon(w_\varepsilon) = \int_\omega f(\nabla u(\hat{x}),\xi(\hat{x}))\,d\hat{x}. \tag{12.13}$$

Now, from (12.13), and taking the infimum over $\xi \in \mathscr{D}(\omega,\mathbf{R}^3)$, (12.12), gives

$$\inf\left\{\limsup_{\varepsilon\to 0}\tilde{F}_\varepsilon(v_\varepsilon) : v_\varepsilon \to u \text{ strongly in } L^p(\Omega,\mathbf{R}^3)\right\} \leq \int_\omega f_0(\nabla u)\,d\hat{x},$$

that is to say,

$$\left(\Gamma - \limsup_{\varepsilon\to 0}\tilde{F}_\varepsilon\right)(u) \leq \tilde{F}(u), \tag{12.14}$$

where $\tilde{F}$ is the functional defined in $L^p(\Omega,\mathbf{R}^3)$ by

$$\tilde{F}(v) = \begin{cases}\displaystyle\int_\omega f_0(\nabla v(\hat{x}))\,d\hat{x} & \text{if } v \in V,\\ +\infty & \text{otherwise.}\end{cases}$$

Obviously, (12.14) also holds for any function $u \in L^p(\Omega, \mathbf{R}^3)$. Taking the lower semicontinuous envelope of each of the two members when $L^p(\Omega, \mathbf{R}^3)$ is equipped with its strong topology, and according to Proposition 12.1.1(iii) and to an easy adaptation of Corollary 11.2.1, we obtain

$$\left(\Gamma - \limsup_{\varepsilon \to 0} \tilde{F}_\varepsilon\right)(u) \leq F(u)$$

for all $u \in L^p(\Omega, \mathbf{R}^3)$.

Last step. Collecting the two previous steps gives $\Gamma\text{—}\limsup_{\varepsilon\to 0} \tilde{F}_\varepsilon \leq F \leq \Gamma\text{—}\liminf_{\varepsilon \to 0} \tilde{F}_\varepsilon$ so that $\Gamma - \lim_{\varepsilon \to 0} \tilde{F}_\varepsilon = F$. □

12.3 ▪ Application to homogenization of composite media

12.3.1 ▪ The quadratic case in one dimension

Before giving the general result concerning homogenization of composite media in Subsection 12.3.2, we establish a complete description of Γ-limits of integral functionals with quadratic density in the one-dimensional case. More precisely, given $a_\varepsilon : \mathbf{R} \to \mathbf{R}$ satisfying that there exist $\alpha > 0$ and $\beta > 0$ such that, for all $x \in \mathbf{R}$,

$$\alpha \leq a_\varepsilon \leq \beta, \tag{12.15}$$

we would like to establish the existence of a Γ-limit for the sequence of integral functionals $F_\varepsilon : L^2(0,1) \to \mathbf{R}^+ \cup \{+\infty\}$ defined by

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_{(0,1)} a_\varepsilon(x)\, u'^2(x)\, dx & \text{if } u \in H^1(0,1), \\ +\infty & \text{otherwise,} \end{cases}$$

when $L^2(0,1)$ is equipped with its strong topology.

Theorem 12.3.1. *Assume that a_ε fulfills condition* (12.15)*. Then the following assertions hold:*

(i) *If $\frac{1}{a_\varepsilon} \rightharpoonup \frac{1}{a}$ for the $\sigma(L^\infty, L^1)$ topology, then $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges to the integral functional F defined on $L^2(0,1)$ by*

$$F(u) = \begin{cases} \displaystyle\int_{(0,1)} a(x)\, u'^2(x)\, dx & \textit{if } u \in H^1(0,1), \\ +\infty & \textit{otherwise.} \end{cases}$$

(ii) *Conversely, if $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges to some functional F, then $(\frac{1}{a_\varepsilon})_{\varepsilon>0}$ $\sigma(L^\infty, L^1)$-converges to some b with $a = \frac{1}{b}$ satisfying* (12.15)*, and F has the integral representation*

$$F(u) = \begin{cases} \displaystyle\int_{(0,1)} a(x)\, u'^2(x)\, dx & \textit{if } u \in H^1(0,1), \\ +\infty & \textit{otherwise.} \end{cases}$$

PROOF OF (i). Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence strongly converging to some u in $L^2(0,1)$. We want to establish

$$F(u)\leq \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon). \tag{12.16}$$

Obviously, one may assume $\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon)<+\infty$, so that, for a nonrelabeled subsequence, u_ε belongs to $H^1(0,1)$. From the equiboundedness of u'_ε in $L^2(0,1)$, we deduce that u_ε is bounded in $H^1(0,1)$ so that u_ε weakly converges to u in $H^1(0,1)$. We now take into account the quadratic expression of F_ε and write $F_\varepsilon(u_\varepsilon)$ as follows:

$$\begin{aligned}F_\varepsilon(u_\varepsilon)=\int_{(0,1)} a_\varepsilon\, u_\varepsilon'^2\,dx&=\int_{(0,1)} a_\varepsilon(u'_\varepsilon-au'/a_\varepsilon)^2\,dx\\&\quad+2\int_{(0,1)} u'_\varepsilon u' a\,dx-\int_{(0,1)} u'^2a^2/a_\varepsilon\,dx\\&\geq 2\int_{(0,1)} u'_\varepsilon u' a\,dx-\int_{(0,1)} u'^2a^2/a_\varepsilon\,dx.\end{aligned}$$

Letting $\varepsilon\to 0$ gives (12.16).

Given $u\in L^2(0,1)$, we now must construct a sequence $(v_\varepsilon)_{\varepsilon>0}$ strongly converging to u and satisfying

$$F(u)\geq \limsup_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon). \tag{12.17}$$

One may assume $u\in H^1(0,1)$. Let us set

$$v_\varepsilon(x)=u(0)+\int_0^x a(t)u'(t)/a_\varepsilon(t)\,dt.$$

We recall that u belongs to $\mathbf{C}([0,1])$ so that the previous expression is well defined. Then $v'_\varepsilon=au'/a_\varepsilon$ weakly converges to u' in $L^2(0,1)$. Since v_ε is bounded in $L^\infty(0,1)$, we deduce that v_ε is bounded in $H^1(0,1)$; thus, from the Rellich–Kondrakov theorem, Theorem 5.4.2, it strongly converges to some θ in $L^2(0,1)$ with $\theta'=u'$, thus $\theta=u+c$. According to the continuity of the trace, one has $\theta(0)=u(0)$, so that $\theta=u$ and $v_\varepsilon\to u$ strongly in $L^2(0,1)$. On the other hand,

$$F_\varepsilon(v_\varepsilon)=\int_{(0,1)} a^2u'^2/a_\varepsilon\,dx,$$

and letting $\varepsilon\to 0$ yields

$$\lim_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon)=F(u),$$

hence (12.17).

PROOF OF (ii). From (12.15), $1/a_\varepsilon$ is bounded in $L^\infty(0,1)$. Therefore, for a nonrelabeled subsequence, it $\sigma(L^\infty,L^1)$-converges to some b with $a=\frac{1}{b}$ satisfying (12.15). According to (i), the corresponding subsequence of $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges to the functional G defined by

$$G(u)=\begin{cases}\displaystyle\int_{(0,1)} a(x)\,u'^2(x)\,dx & \text{if } u\in H^1(0,1),\\ +\infty & \text{otherwise.}\end{cases}$$

Consequently $F=G$ and a is uniquely defined by F. Since all subsequence of $(1/a_\varepsilon)_{\varepsilon>0}$ possesses a subsequence which $\sigma(L^\infty,L^1)$-converges to the same limit $1/a$, all the sequence $(1/a_\varepsilon)_{\varepsilon>0}$ $\sigma(L^\infty,L^1)$-converges to $1/a$. □

Example 12.3.1. Let us consider a_ε defined by $a_\varepsilon(x)=a(x/\varepsilon)$, where a is a $(0,1)$-periodic function taking two positive values α and β on $(0,1/2)$ and $(1/2,1)$, respectively. Then $1/a_\varepsilon$ weakly converges for the $\sigma(L^\infty,L^1)$ topology to $(\alpha^{-1}+\beta^{-1})/2$. For a proof, see Example 2.4.2 or Proposition 13.2.1 and the proof of Theorem 13.2.1. Therefore

$$F(u)=\begin{cases} \displaystyle\int_{(0,1)} \left(\frac{\alpha^{-1}+\beta^{-1}}{2}\right)^{-1} u'^2(x)\,dx & \text{if } u\in H^1(0,1),\\ +\infty & \text{otherwise.}\end{cases}$$

The functional F_ε may be interpreted, for example, as the elastic energy of a system of two kinds of small periodically distributed springs with size ε. Theorem 12.3.1 shows that the mechanical behavior of such a system is equivalent to a homogeneous string in the sense of Γ-convergence. The equivalent density is associated with the strain tensor $\left(\frac{\alpha^{-1}+\beta^{-1}}{2}\right)^{-1}$ strictly smaller than the mean value $\frac{\alpha+\beta}{2}$ of the tensors associated with the two kinds of material.

Remark 12.3.1. The same problem in the two-dimensional case, describing, for example, a system of two kinds of small elastic pieces in $\Omega=(0,1)^2$ in a chessboard structure, may be treated as above by using the concept of Γ-convergence. One can show that the equivalent density is quadratic and associated with the strain tensor $\sqrt{\alpha\beta}$. In the three-dimensional case, there is no explicit formula for the strain tensor limit. More generally, when working with general quadratic densities of the form $f_\varepsilon(\xi)=\langle A_\varepsilon\xi,\xi\rangle$, $A_\varepsilon\in\mathscr{M}^{3\times 3}$ satisfying, for all $\xi\in\mathbf{R}^3$,

$$\alpha|\xi|^2\le\langle A_\varepsilon\xi,\xi\rangle\le\beta|\xi|^2,$$

one can show that the Γ-limit of the associated integral functional possesses a density of the form $\langle A\xi,\xi\rangle$. The strategy is then to derive optimal bounds for the constant limit matrix A (see Murat and Tartar [309]).

12.3.2 ▪ Periodic homogenization in the general case

Let Ω be an open bounded subset of $\mathbf{R}^3$ which represents the interior of the reference configuration filled up by some elastic ($p>1$) or pseudoplastic ($p=1$) material which is clamped on a part Γ_0 of the boundary $\partial\Omega$ of Ω. We assume that this material is heterogeneous with a periodic distribution of small heterogeneities of size of order $\varepsilon>0$, so that the stored strain energy density is of the form

$$(x,a)\mapsto f\left(\frac{x}{\varepsilon},a\right),$$

where $f(.,a)$ is Y-periodic, $Y=(0,1)^3$. We assume that f satisfies conditions (12.5) and (12.6) of the previous section and, to take into account large purely elastic deformations, f is not assumed to be convex but possibly quasi-convex. With the notation of Sections 11.2 and 11.3, the stored strain energy associated with a displacement field $u:\Omega\to\mathbf{R}^3$ is given by the integral functional $F_\varepsilon: L^p(\Omega,\mathbf{R}^3)\longrightarrow\mathbf{R}^+\cup\{+\infty\}$ defined by

$$F_\varepsilon(u)=\begin{cases} \displaystyle\int_\Omega f\left(\frac{x}{\varepsilon},\nabla u\right)dx & \text{if } u\in W^{1,p}_{\Gamma_0}(\Omega,\mathbf{R}^3),\\ +\infty & \text{otherwise.}\end{cases}$$

The structure is subjected to applied body forces $g : \Omega \longrightarrow \mathbf{R}^3$, $g \in L^q(\Omega, \mathbf{R}^3)$, $(1/p + 1/q = 1$ if $p > 1$; $q = +\infty$ if $p = 1)$ and the exterior loading is defined by

$$L(u) = \int_\Omega g.u \, dx.$$

The equilibrium configuration is then given by the displacement field $\overline{u}_\varepsilon$ solution of the problem

$$\inf\left\{ F_\varepsilon(u) - \int_\Omega L(u) : u \in L^p(\Omega, \mathbf{R}^3) \right\}.$$

Due to the very small size ε of heterogeneity, for computing an approximate equilibrium displacement field, it is illusory to make direct use of the finite element method. The variational property of Γ-convergence (Theorem 12.1.1) would again provide a new procedure: to find a fictitious material occupying Ω, which appears to be homogeneous when ε goes to zero, and to compute the approximate equilibrium displacement field by means of a finite element method related to a discretization of the new model.

To treat more general situations, we deal with functionals $F_\varepsilon : L^p(\Omega, \mathbf{R}^m) \longrightarrow \mathbf{R}^+ \cup \{+\infty\}$ defined by

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega f\left(\frac{x}{\varepsilon}, \nabla u\right) dx & \text{if } u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

where Ω is an open bounded subset of $\mathbf{R}^N$, m is any positive integer, f satisfies the growth conditions (12.5) and (12.6), and, for all a in the set $\mathbf{M}^{m\times N}$ of $m \times N$ matrices, the Borel function $f(.,a)$ is assumed to be Y-periodic, $Y = (0,1)^N$.

Following the strategy of the previous subsection, we are going to establish the Γ-convergence of the sequence $(F_\varepsilon)_{\varepsilon>0}$ when $L^p(\Omega, \mathbf{R}^m)$ is equipped with its strong topology. In Theorem 12.3.2, we will establish that the Γ-limit of F_ε possesses an integral representation. In the following proposition, we characterize its density ($p > 1$) or its regular part ($p = 1$). It is worth noticing the similarity between this proposition and Proposition 11.2.2, where we defined the relaxed density of the integral functional on Sobolev or BV spaces.

Proposition 12.3.1. *For all open bounded convex set A in $\mathbf{R}^N$ the following limit exists:*

$$f^{hom}(a) = \lim_{\varepsilon\to 0}\left[\inf\left\{ \frac{1}{|A/\varepsilon|} \int_{A/\varepsilon} f(x, a + \nabla u(x))\, dx : u \in W^{1,p}_0(A/\varepsilon, \mathbf{R}^m) \right\} \right].$$

Moreover, this limit does not depend on the choice of the open bounded convex set A and is equal to

$$\inf_{n\in\mathbf{N}^*} \inf\left\{ \frac{1}{n^N} \int_{nY} f(y, a + \nabla u(y))\, dy : u \in W^{1,p}_0(Y, \mathbf{R}^m) \right\}.$$

The proof is based on a convergence result related to subadditive processes. To go further, we first give some notation and definitions. Let us denote the family of all the bounded Borel sets of $\mathbf{R}^N$ by $\mathscr{B}_b(\mathbf{R}^N)$.

A sequence $(B_n)_{n\in\mathbf{N}}$ of sets of $\mathscr{B}_b(\mathbf{R}^N)$ is said to be regular if there exists an increasing sequence of half intervals I_n of the type $[a,b)$ with vertices in $\mathbf{Z}^N$ and a positive constant C independent of n such that $B_n \subset I_n$ and $|I_n| \le C|B_n|$ for all $n \in \mathbf{N}$.

A subadditive $\mathbf{Z}^N$-invariant set function indexed by $\mathscr{B}_b(\mathbf{R}^N)$ is a map $\mathscr{S} : \mathscr{B}_b(\mathbf{R}^N) \longrightarrow \mathbf{R}$, $A \mapsto \mathscr{S}_A$, such that

(i) for all A, $B \in \mathscr{B}_b(\mathbf{R}^N)$ with $A \cap B = \emptyset$, $\mathscr{S}_{A\cup B} \le \mathscr{S}_A + \mathscr{S}_B$;

(ii) for all $A \in \mathscr{B}_b(\mathbf{R}^N)$ and all $z \in \mathbf{Z}^N$, $\mathscr{S}_{z+A} = \mathscr{S}_A$.

Finally, for all A in $\mathscr{B}_b(\mathbf{R}^N)$, we define the positive number $\rho(A) := \sup\{r \ge 0 : \exists \bar{B}_r(x) \subset A\}$, where $\bar{B}_r(x)$ is the closed ball with radius $r > 0$ centered at x. The following lemma generalizes the classical limit theorem related to subadditive processes indexed by cubes. For a proof, we refer the reader to Ackoglu and Krengel [4] or to Licht and Michaille [274].

Lemma 12.3.1. *Let $\mathscr{S}$ be a subadditive $\mathbf{Z}^N$-invariant set function such that*

$$\gamma(\mathscr{S}) := \inf\left\{ \frac{\mathscr{S}_I}{|I|} : I = [a,b[,\ a = (a_i)_{i=1,\dots,N},\ b = (b_i)_{i=1,\dots,N} \in \mathbf{Z}^N, \right.$$
$$\left. \forall i = 1,\dots,N,\ a_i < b_i \right\} > -\infty$$

and which satisfies the following domination property: there exists a positive constant $C(\mathscr{S}) < +\infty$ such that $|\mathscr{S}_A| \le C(\mathscr{S})$ for all Borel sets A included in $[0,1[^N$. Let $(A_n)_{n\in\mathbf{N}}$ be a regular sequence of Borel convex sets of $\mathscr{B}_b(\mathbf{R}^N)$ satisfying $\lim_{n\to+\infty} \rho(A_n) = +\infty$. Then

$$\lim_{n\to+\infty} \frac{\mathscr{S}_{A_n}}{|A_n|} = \inf_{m\in\mathbf{N}^*} \left\{ \frac{\mathscr{S}_{[0,m[^N}}{m^N} \right\} = \gamma(\mathscr{S}).$$

PROOF OF PROPOSITION 12.3.1. Let us notice that in the definition of subadditivity, assertion (i) may be replaced by the following: for all A, $B \in \mathscr{B}_b(\mathbf{R}^N)$ with $A \cap B = \emptyset$ and $|\partial A| = |\partial B| = 0$, $\mathscr{S}_{A\cup B} \le \mathscr{S}_A + \mathscr{S}_B$ (see [274, Remark of Theorem 2.1] or [185]). Then we claim that

$$\mathscr{S} : A \mapsto \inf\left\{ \int_{\overset{\circ}{A}} f(x, a + \nabla u(x))\, dx : u \in W_0^{1,p}(\overset{\circ}{A}, \mathbf{R}^m) \right\}$$

is a subadditive $\mathbf{Z}^N$-invariant process. Indeed, Y-periodicity of $f(\cdot,a)$ yields $\mathscr{S}_{A+z} = \mathscr{S}_A$ for all $A \in \mathscr{B}_b(\mathbf{R}^N)$ and all $z \in \mathbf{Z}^N$. Let now A, B in $\mathscr{B}_b(\mathbf{R}^N)$ such that $A \cap B = \emptyset$ and $|\partial A| = |\partial B| = 0$. For arbitrary fixed $\eta > 0$, consider $\varphi_A \in \mathscr{D}(\overset{\circ}{A}, \mathbf{R}^m)$ and $\varphi_B \in \mathscr{D}(\overset{\circ}{B}, \mathbf{R}^m)$ satisfying

$$\int_{\overset{\circ}{A}} f(x, a + \nabla\varphi_A)\, dx \le \mathscr{S}_A + \eta,$$

$$\int_{\overset{\circ}{B}} f(x, a + \nabla\varphi_B)\, dx \le \mathscr{S}_B + \eta.$$

Extending φ_A and φ_B by zero, respectively, on $\mathbf{R}^N \setminus \overset{\circ}{A}$ and $\mathbf{R}^N \setminus \overset{\circ}{B}$, the function φ which coincides with φ_A on $\overset{\circ}{A}$ and φ_B in $\overset{\circ}{B}$ belongs to $W_0^{1,p}(\overset{\circ}{A \cup B}, \mathbf{R}^m)$. Since $|\partial A| = |\partial B| = 0$,

$|\overset{\circ}{A \cup B} \setminus \overset{\circ}{A} \cup \overset{\circ}{B}| = 0$, thus

$$\begin{aligned}\int_{\overset{\circ}{A \cup B}} f(x, a + \nabla\varphi)\, dx &= \int_{\overset{\circ}{A}} f(x, a + \nabla\varphi_A)\, dx + \int_{\overset{\circ}{B}} f(x, a + \nabla\varphi_B)\, dx \\ &\quad + \int_{\overset{\circ}{A \cup B} \setminus \overset{\circ}{A} \cup \overset{\circ}{B}} f(x, a)\, dx \\ &= \int_{\overset{\circ}{A}} f(x, a + \nabla\varphi_A)\, dx + \int_{\overset{\circ}{B}} f(x, a + \nabla\varphi_B)\, dx \\ &\leq \mathscr{S}_A + \mathscr{S}_B + 2\eta.\end{aligned}$$

Consequently, $\mathscr{S}_{A\cup B} \leq \mathscr{S}_A + \mathscr{S}_B + 2\eta$ and the subadditivity of $\mathscr{S}$ follows by letting $\eta \to 0$. We conclude the proof by applying Lemma 12.3.1 to this process. ☐

Remark 12.3.2. (1) When $a \mapsto f(x, a)$ is a convex function, the limit which defines f^{hom} in Proposition 12.3.1 can be expressed in a reduced form (see Proposition 12.3.4 below).

(2) Lemma 12.3.1 may be generalized for ergodic subadditive processes, i.e., for subadditive processes with value in $L^1(\Sigma, \mathscr{T}, \mathbf{P})$ and whose probability law is, roughly speaking, invariant under a group $(T_z)_{z \in \mathbf{Z}^N}$ of measure-preserving transformations on the probability space $(\Sigma, \mathscr{T}, \mathbf{P})$. This probabilistic version allows us to treat stochastic homogenization in Section 12.4 and a variational model in fracture mechanics in Section 14.2.

We can now establish the main convergence result, a generalization of Theorems 11.2.1 and 11.3.1. In what follows, ε actually denotes a sequence $(\varepsilon_n)_{n\in\mathbf{N}}$ of positive numbers ε_n going to zero when $n \to +\infty$ and we adopt the notation of Section 11.2. For more general problems involving multiple small parameters, see [13], [14]. For problems concerned with nonlocal effects, see [90], [91].

Theorem 12.3.2. *Let f satisfying* (12.5) *and* (12.6) *with $p \geq 1$ and assume that the Borel function $f(.,a)$ is Y-periodic for all a in $\mathbf{M}^{m\times N}$. Let us consider the integral functional F_ε defined in $L^p(\Omega, \mathbf{R}^m)$ by*

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega f\left(\frac{x}{\varepsilon}, \nabla u\right) dx & \text{if } u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

where $L^p(\Omega, \mathbf{R}^m)$ is equipped with its strong topology. Then $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges to the integral functional F^{hom} defined by

(i) *case $p > 1$,*

$$F^{hom}(u) = \begin{cases} \displaystyle\int_\Omega f^{hom}(\nabla u)\, dx & \text{if } u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbf{R}^m), \\ +\infty & \text{otherwise;} \end{cases}$$

(ii) *case $p = 1$,*

$$F^{hom}(u) = \begin{cases} \displaystyle\int_\Omega f^{hom}(\nabla u)\, dx + \int_\Omega (f)^{hom,\infty}\left(\frac{D^s u}{|D^s u|}\right) |D^s u| \\ \displaystyle\qquad\qquad + \int_{\Gamma_0} (f)^{hom,\infty}(\gamma_0(u) \otimes \nu)\, d\mathscr{H}^{N-1} & \text{if } u \in BV(\Omega, \mathbf{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

where ν denotes the outer unit normal to Γ_0, γ_0 the trace operator, and $(f)^{hom,\infty}$ the recession function of f^{hom} defined by

$$(f)^{hom,\infty}(a) = \limsup_{t\to+\infty} \frac{(f)^{hom}(ta)}{t}.$$

The proof of Theorem 12.3.2 is the consequence of Propositions 12.3.2 and 12.3.3. To shorten the proofs we do not take into account the boundary condition, i.e., the domain of F_ε is $W^{1,p}(\Omega, \mathbf{R}^m)$. For treating the general case, it suffices to reproduce exactly the proofs of Corollary 11.2.1 when $p > 1$ and Corollary 11.3.1 when $p = 1$.

Proposition 12.3.2. *For all u in $L^p(\Omega, \mathbf{R}^m)$ and all sequence $(u_n)_{n\in\mathbf{N}}$ strongly converging to u in $L^p(\Omega, \mathbf{R}^m)$, one has*

$$F^{hom}(u) \le \liminf_{n\to+\infty} F_{\varepsilon_n}(u_n). \tag{12.18}$$

PROOF. Our strategy is exactly the one of Proposition 11.2.3 or 11.3.3. Obviously, one may assume $\liminf_{n\to+\infty} F_{\varepsilon_n}(u_n) < +\infty$. For a nonrelabeled subsequence, consider the nonnegative Borel measure $\mu_n := f(\frac{\cdot}{\varepsilon_n}, \nabla u_n(.))\mathscr{L}\lfloor\Omega$; we have

$$\sup_{n\in\mathbf{N}} \mu_n(\Omega) < +\infty.$$

Consequently, there exists a further subsequence (not relabeled) and a nonnegative Borel measure $\mu \in \mathbf{M}(\Omega)$ such that

$$\mu_n \rightharpoonup \mu \quad \text{weakly in } \mathbf{M}(\Omega).$$

Let $\mu = g\mathscr{L}^N\lfloor\Omega + \mu^s$ be the Lebesgue–Nikodým decomposition of μ, where μ^s is a nonnegative Borel measure, singular with respect to the N-dimensional Lebesgue measure $\mathscr{L}\lfloor\Omega$ restricted to Ω. For establishing (12.18) it is enough to prove that

$$g(x) \ge f^{hom}(\nabla u(x)) \quad x \text{ a.e.}, \tag{12.19}$$

$$\mu^s \ge f^{hom,\infty}\left(\frac{D^s u}{|D^s u|}\right) |D^s u| \quad \text{when } p = 1. \tag{12.20}$$

(a) *Proof of* (12.19). Let $\rho > 0$ intended to tend to 0 and let $B_\rho(x_0)$ be the open ball of radius ρ centered at x_0. According to the theory of differentiation of measures, for a.e. $x_0 \in \Omega$

$$g(x_0) = \lim_{\rho\to 0} \frac{\mu(B_\rho(x_0))}{|B_\rho(x_0)|}.$$

Applying Lemma 4.2.1, one may assume $\mu(\partial B_\rho(x_0)) = 0$ for all but countably many $\rho > 0$, so that, from Alexandrov's theorem, Proposition 4.2.3, we have $\mu(B_\rho(x_0)) = \lim_{n\to+\infty} \mu_n(B_\rho(x_0))$ and we finally must establish

$$\lim_{\rho\to 0} \lim_{n\to+\infty} \frac{\mu_n(B_\rho(x_0))}{|B_\rho(x_0)|} \ge f^{hom}(\nabla u(x_0)) \quad \text{for a.e. } x_0 \in \Omega. \tag{12.21}$$

Let us assume for the moment that the trace of u_n on $\partial B_\rho(x_0)$ coincides with the affine function u_0 defined by $u_0(x) := u(x_0) + \langle \nabla u(x_0), x - x_0 \rangle$. It follows from Proposition 12.3.1 that

$$\begin{aligned}
&\lim_{n\to+\infty} \frac{\mu_n(B_\rho(x_0))}{|B_\rho(x_0)|}\\
&= \lim_{n\to+\infty} \frac{1}{|B_\rho(x_0)|} \int_{B_\rho(x_0)} f\left(\frac{x}{\varepsilon_n}, \nabla u(x_0) + \nabla(u_n - u_0)\right) dx\\
&\geq \limsup_{n\to+\infty} \inf \left\{ \frac{1}{|B_\rho(x_0)|} \int_{B_\rho(x_0)} f\left(\frac{x}{\varepsilon_n}, \nabla u(x_0) + \nabla \phi\right) dx : \phi \in W_0^{1,p}(B_\rho(x_0), \mathbf{R}^m) \right\}\\
&= \lim_{n\to+\infty} \inf \left\{ \frac{1}{|\frac{1}{\varepsilon_n} B_\rho(x_0)|} \int_{\frac{1}{\varepsilon_n} B_\rho(x_0)} f(x, \nabla u(x_0) + \nabla \phi)\, dx : \phi \in W_0^{1,p}\left(\frac{1}{\varepsilon_n} B_\rho(x_0), \mathbf{R}^m\right) \right\}\\
&= f^{hom}(\nabla u(x_0)),
\end{aligned}$$

and the proof would be complete. The idea now consists in modifying u_n into a function of $W^{1,p}(B_\rho(x_0), \mathbf{R}^m)$ which coincides with u_0 on $\partial B_\rho(x_0)$ in the trace sense, to follow the previous procedure and to control the additional terms, when ρ goes to zero, thanks to the estimate (see Lemma 11.2.1 and Proposition 10.4.1): for a.e. $x \in \Omega$,

$$\left[\frac{1}{|B_\rho(x_0)|} \int_{B_\rho(x_0)} |u(x) - \big(u(x_0) + \nabla u(x_0)(x - x_0)\big)|^p \, dx \right]^{1/p} = o(\rho).$$

The suitable modification of u_n is exactly the one of Proposition 11.2.3 because of the conditions (12.5) and (12.6) satisfied by f. The proof of (12.21) is then complete.

(b) *Proof of* (12.20). It suffices to reproduce the proof of inequality

$$\mu^s \geq (Qf)^\infty \left(\frac{D^s u}{|D^s u|} \right) |D^s u|$$

obtained in the proof of Proposition 11.3.3 after substituting f by $f(\frac{x}{\varepsilon_n}, \cdot)$ and, according to Proposition 12.3.1, after substituting Qf by f^{hom}. □

Proposition 12.3.3. *For all u in $L^p(\Omega, \mathbf{R}^m)$, $p \geq 1$, there exists a sequence $(u_n)_{n\in\mathbf{N}}$ strongly converging to u in $L^p(\Omega, \mathbf{R}^m)$ such that*

$$F^{hom}(u) \geq \limsup_{n\to+\infty} F_{\varepsilon_n}(u_n).$$

PROOF. The proof will be obtained in two steps.

First step. We assume $u \in W^{1,p}(\Omega, \mathbf{R}^m)$. We reproduce, with minor modifications, the outline of the proof of Proposition 11.2.4. According to Proposition 12.3.1 and to the Lebesgue dominated convergence theorem,

$$F^{hom}(u) = \int_\Omega f^{hom}(\nabla u)\, dx = \lim_{k\to+\infty} \int_\Omega f_k^{hom}(\nabla u)\, dx, \tag{12.22}$$

where

$$f_k^{hom}(a) = \inf\left\{ \frac{1}{|kY|} \int_{kY} f(y, a + \nabla v)\, dy : v \in W_0^{1,p}(kY, \mathbf{R}^m) \right\}.$$

Let us fix $k \in \mathbf{N}^*$. Applying the interchange lemma, Lemma 11.2.2, we have for all $\eta > 0$ (of the form $1/h$ with h integer) and for some $\phi_{k,\eta}$ in $\mathbf{C}_c(\Omega, \mathcal{D}(kY, \mathbf{R}^m))$,

$$\begin{aligned} &\frac{1}{|kY|} \int_{\Omega \times kY} f(y, \nabla u(x) + \nabla_y \phi_{k,\eta}(x,y))\, dx dy \geq \int_\Omega f_k^{hom}(\nabla u)\, dx \\ &> \frac{1}{|kY|} \int_{\Omega \times kY} f(y, \nabla u(x) + \nabla_y \phi_{k,\eta}(x,y))\, dx dy - \eta. \end{aligned} \tag{12.23}$$

Let us extend $y \mapsto \phi_{k,\eta}(x,y)$ by kY-periodicity on $\mathbf{R}^N$ and consider the function $u_{k,\eta,n}$ defined by

$$u_{k,\eta,n}(x) = u(x) + \varepsilon_n \phi_{k,\eta}\left(x, \frac{x}{\varepsilon_n}\right).$$

Note that $\phi_{k,\eta}$ is a Carathéodory function so that $x \mapsto \phi_{k,\eta}(x, \frac{x}{\varepsilon_n})$ is measurable. Clearly $u_{k,\eta,n}$ belongs to $W^{1,p}(\Omega, \mathbf{R}^m)$ and

$$u_{k,\eta,n} \to u \quad \text{strongly in } L^p(\Omega, \mathbf{R}^m)$$

when n goes to ∞. On the other hand, according to the continuity assumption (12.6) on f and to Lemma 11.2.3,

$$\begin{aligned} &\lim_{n\to+\infty} \int_\Omega f\left(\frac{x}{\varepsilon_n}, \nabla u_{k,\eta,n}\right) dx \\ &= \lim_{n\to+\infty} \int_\Omega f\left(\frac{x}{\varepsilon_n}, \nabla u(x) + \nabla_y \phi_{k,\eta}\left(x, \frac{x}{\varepsilon_n}\right) + \varepsilon_n \nabla \phi_{k,\eta}\left(x, \frac{x}{\varepsilon_n}\right)\right) dx \\ &= \lim_{n\to+\infty} \int_\Omega f\left(\frac{x}{\varepsilon_n}, \nabla u(x) + (\nabla_y \phi_{k,\eta})\left(x, \frac{x}{\varepsilon_n}\right)\right) dx \\ &= \frac{1}{|kY|} \int_{\Omega \times kY} f(y, \nabla u(x) + \nabla_y \phi_{k,\eta}(x,y))\, dx dy. \end{aligned}$$

Consequently, from (12.23)

$$\lim_{n\to+\infty} F_{\varepsilon_n}(u_{k,\eta,n}) = \frac{1}{|kY|} \int_{\Omega \times kY} f(y, \nabla u(x) + \nabla_y \phi_{k,\eta}(x,y))\, dx dy \leq \int_\Omega f_k^{hom}(\nabla u)\, dx + \eta.$$

The inequality above and (12.22), letting $\eta \to 0$ (i.e., $h \to +\infty$) and $k \to +\infty$, yield

$$\lim_{k\to+\infty} \lim_{\eta\to+0} \lim_{n\to+\infty} F_{\varepsilon_n}(u_{k,\eta,n}) = F^{hom}(u).$$

Let us now apply the diagonalization Lemma 11.1.1 to the sequence $\left(F_{\varepsilon_n}(u_{k,\eta,n}), u_{k,\eta,n}\right)_{k,\eta,n}$ in the metric space $\mathbf{R} \times L^p(\Omega, \mathbf{R}^m)$: there exists a map $n \mapsto (k,\eta)(n)$ such that

$$\begin{aligned} &\lim_{n\to+\infty} F_{\varepsilon_n}(u_{(k,\eta)(n),n}) = F^{hom}(u), \\ &\lim_{n\to+\infty} u_{(k,\eta)(n),n} = u \quad \text{strongly in } L^p(\Omega, \mathbf{R}^m). \end{aligned}$$

We have proved that $F^{hom}(u) \geq \limsup_{n\to+\infty} F_{\varepsilon_n}(u_n)$ for $u_n = u_{(k,\eta)(n),n}$ converging to u in $W^{1,p}(\Omega,\mathbf{R}^m)$ equipped with the strong convergence of $L^p(\Omega,\mathbf{R}^m)$. If $p > 1$, the proof is complete because the domain of F^{hom} is $W^{1,p}(\Omega,\mathbf{R}^m)$.

Second step ($p = 1$). Let us consider the functional G defined on $L^1(\Omega,\mathbf{R}^m)$ by

$$G(u) = \begin{cases} \int_\Omega f^{hom}(\nabla u)\,dx & \text{if } u \in W^{1,1}(\Omega,\mathbf{R}^m), \\ +\infty & \text{otherwise.} \end{cases}$$

According to Theorem 11.3.1, F^{hom} is nothing but the lower semicontinuous envelope $\mathrm{cl}(G)$ of G. Therefore, for all $u \in L^1(\Omega,\mathbf{R}^m)$, there exists a sequence $(u_l)_{l\in\mathbf{N}}$ in $L^1(\Omega,\mathbf{R}^m)$, strongly converging to u in $L^1(\Omega,\mathbf{R}^m)$ such that

$$F^{hom}(u) = \lim_{l\to+\infty} G(u_l).$$

One may assume $F^{hom}(u) < +\infty$ so that according to the first step, there exists a sequence $(u_{l,n})_{n\in\mathbf{N}}$ strongly converging to u_l in $L^1(\Omega,\mathbf{R}^m)$ when $n \to +\infty$, such that

$$G(u_l) = \lim_{n\to+\infty} F_{\varepsilon_n}(u_{l,n}).$$

Combining these two equalities, we obtain

$$F^{hom}(u) = \limsup_{l\to+\infty} \limsup_{n\to+\infty} F_{\varepsilon_n}(u_{l,n})$$

and

$$\lim_{l\to+\infty} \lim_{l\to+\infty} u_{l,n} = u \quad \text{strongly in } L^1(\Omega,\mathbf{R}^m).$$

We end the proof by applying the diagonalization Lemma 11.1.1. □

Proposition 12.3.4 (convex case). *Assume that the function $a \mapsto f(x,a)$ is convex. Then for all $a \in \mathbf{M}^{m\times N}$, $f^{hom}(a)$ reduces to*

$$f^{hom}(a) = \inf\left\{ \int_Y f(y, a+\nabla u(y))\,dy : u \in W^{1,p}_\#(Y,\mathbf{R}^m) \right\},$$

where $W^{1,p}_\#(Y,\mathbf{R}^m) := \{u \in W^{1,p}(Y,\mathbf{R}^m) : u \text{ is } Y\text{-periodic}\}$.

PROOF. For all $v \in W^{1,p}_\#(Y,\mathbf{R}^m)$ and all $a \in \mathbf{M}^{m\times N}$, set $u_\varepsilon(x) := a.x + \varepsilon u(\frac{x}{\varepsilon})$. Clearly $u_\varepsilon \to l_a$ in $L^p(\Omega,\mathbf{R}^m)$ so that, according to Proposition 12.3.2 with $\Omega = Y$,

$$f^{hom}(a) \leq \liminf_{\varepsilon\to 0} \int_Y f\left(\frac{x}{\varepsilon}, a + \nabla u\left(\frac{x}{\varepsilon}\right)\right) dx. \tag{12.24}$$

But

$$\lim_{\varepsilon\to 0} \int_Y f\left(\frac{x}{\varepsilon}, a + \nabla u\left(\frac{x}{\varepsilon}\right)\right) dx = \int_Y f(y, a+\nabla u(y))\,dy. \tag{12.25}$$

Combining (12.24) and (12.25), we infer that

$$f^{hom}(a) \leq \inf\left\{ \int_Y f(y, a+\nabla u(y))\,dy : u \in W^{1,p}_\#(Y,\mathbf{R}^m) \right\}.$$

We are going to establish the converse inequality. From the subdifferential inequality, for all $u \in W_0^{1,p}(nY,\mathbf{R}^m)$ and all $w \in W_\#^{1,p}(Y,\mathbf{R}^m)$ extended by periodicity in $\mathbf{R}^N$, we have

$$f(y,a+\nabla u(y)) \geq f(y,a+\nabla w(y)) + \langle \partial f(y,a+\nabla w(y)), \nabla u(y) - \nabla w(y)\rangle,$$

where, to shorten the notation, we write $\partial f(y,a+\nabla w(y))$ for some $\xi(y) \in \partial f(y,a+\nabla w(y))$. Thus

$$\begin{aligned}\frac{1}{n^N}\int_{nY} f(y,a+\nabla u(y))\,dy \geq{}& \int_Y f(y,a+\nabla w(y))\,dy \\ &+ \frac{1}{n^N}\int_{nY}\langle \partial f(y,a+\nabla w(y)), \nabla u(y) - \nabla w(y)\rangle\,dy.\end{aligned} \tag{12.26}$$

Take now for w the minimizer of $\inf\{\int_Y f(y,a+\nabla u(y))\,dy : u \in W_\#^{1,p}(Y,\mathbf{R}^m)\}$ which then satisfies

$$\operatorname{div} \partial f(y,a+\nabla w(y)) = 0 \quad \text{a.e. in } nY;$$

$$\partial f(y,a+\nabla w(y)).\nu \quad \text{antiperiodic on } \partial nY,$$

and integrate by parts the last term of (12.26). We obtain

$$\frac{1}{n^N}\int_{nY} f(y,a+\nabla u(y))\,dy \geq \inf\left\{\int_Y f(y,a+\nabla u(y))\,dy : u \in W_\#^{1,p}(Y,\mathbf{R}^m)\right\}$$

for all $u \in W_0^{1,p}(nY,\mathbf{R}^m)$. We conclude the proof by taking the infimum of the left-hand side with respect to all the functions of $W_0^{1,p}(nY,\mathbf{R}^m)$. □

12.4 ▪ Stochastic homogenization

We return to the previous study, but we no longer assume that the density $x \mapsto f(x,a)$ is periodic. From the standpoint of the modeling, the heterogeneities of the medium studied are not assumed to be regularly distributed. However, a too general distribution of heterogeneities does not allow us to perform a mathematical analysis. This is why, although the medium is imperfectly known, we assume that it is statistically homogeneous in the sense that the probability distribution of heterogeneities is invariant under the spatial translations. We provide a precise mathematical definition which clearly extends the periodic framework. Finally, we give two standard examples of statistically homogeneous materials (cf. Examples 12.4.1 and 12.4.2).

The strategy consisting in performing the statistical averages obtained from a large number of samples of the random functional energy has been implemented in [59] by means of the epigraphical sum of realizations. In this section, we choose to continue the strategy of the previous section, initiated by [185], by showing that the sequence of the functional energies Γ-converges almost surely and by identifying the Γ-limit. For this, we only have to mimic the proof of Theorem 12.3.2 by substituting a subadditive theorem for Lemma 12.3.1. In Chapter 17, we deal with stochastic homogenization in a dynamical case and determine the limit Cauchy problem corresponding to the diffusion in random media.

We first establish the mathematical tools coming from ergodic theory and establish the almost sure pointwise convergence of subadditive processes in general. The energy density of the homogenized problem is the almost sure limit of a suitable subadditive process defined from the energy functional of the initial problem. Therefore, it possibly depends on a physical or geometrical parameter (temperature, inclusion shape). This is why it may be interesting to investigate the variational property of the pointwise almost sure convergence through the parameter. Precisely, when the process depends on a parameter in a separable metric space, under a lower semicontinuity dependence, we show that the almost sure convergence of the opposite superadditive process is actually a Γ-convergence. This last result generalizes the epigraphical law of large numbers established in [59].

12.4.1 ▪ The subadditive ergodic theorem

In what follows, a dynamical system $(\Sigma, \mathscr{A}, \mathbf{P}, (T_z)_{z\in\mathbf{Z}^N})$ is a probability space $(\Sigma, \mathscr{A}, \mathbf{P})$ endowed with a group $(T_z)_{z\in\mathbf{Z}^N}$ of $\mathbf{P}$-preserving transformations on $(\Sigma, \mathscr{A})$, i.e., a family of $\big(\mathscr{A}, \mathscr{A}\big)$-measurable maps $T_z : \Sigma \to \Sigma$ satisfying

$$\begin{aligned}&T_z \circ T_{z'} = T_{z+z'},\ T_{-z} = T_z^{-1} \quad \forall (z,z') \in \mathbf{Z}^N \times \mathbf{Z}^N;\\ &T_z^{\#}\mathbf{P} = \mathbf{P} \quad \forall z \in \mathbf{Z}^N.\end{aligned}$$

We use the standard notation $T_z^{\#}\mathbf{P}$ to denote the image measure (or push forward) of $\mathbf{P}$ by T_z. The term *dynamical system* refers to the "evolution" of the elements (or alea) of Σ according to the group $(T_z)_{z\in\mathbf{Z}^N}$. More specific dynamical systems, namely, differential dynamical systems, are introduced in Chapter 17, where the semigroup $(S_t)_{t\geq 0}$ generated by a gradient or a subdifferential of convex potential plays the role of the discrete group $(T_z)_{z\in\mathbf{Z}^N}$.

The dynamical system $(\Sigma, \mathscr{A}, \mathbf{P}, (T_z)_{z\in\mathbf{Z}^N})$ is said to be ergodic if for all E of $\mathscr{A}$ we have

$$T_z E = E \quad \forall z \in \mathbf{Z}^N \Longrightarrow \mathbf{P}(E) = 0 \text{ or } \mathbf{P}(E) = 1.$$

A sufficient condition to ensure ergodicity is the so-called mixing condition which expresses an asymptotic independence: for all sets E and F of $\mathscr{A}$

$$\lim_{|z|\to+\infty} \mathbf{P}(T_z E \cap F) = \mathbf{P}(E)\mathbf{P}(F). \tag{12.27}$$

Ergodicity is obtained from (12.27) by taking $E = F$. The defect of ergodicity is captured by the σ-algebra $\mathscr{F}$ of invariant sets of $\mathscr{A}$ under the group $(T_z)_{z\in\mathbf{Z}^N}$, i.e., $E \in \mathscr{F}$ iff $T_z E = E$ for all $z \in \mathbf{Z}^N$. We denote by $L^1_{\mathbf{P}}(\Sigma)$ the space of $\mathbf{P}$-integrable numerical functions. For $X \in L^1_{\mathbf{P}}(\Sigma)$, $\mathbf{E}^{\mathscr{F}} X$ denotes the conditional expectation of X given $\mathscr{F}$, that is, the unique $\mathscr{F}$-measurable function in $L^1_{\mathbf{P}}(\Sigma)$ satisfying

$$\int_E \mathbf{E}^{\mathscr{F}} X(\omega)\, d\mathbf{P}(\omega) = \int_E X(\omega)\, d\mathbf{P}(\omega) \quad \forall\, E \in \mathscr{F}.$$

It is easy to establish that the function $\mathbf{E}^{\mathscr{F}} X$ is $(T_z)_{z\in\mathbf{Z}^N}$-invariant, i.e.,

$$\mathbf{E}^{\mathscr{F}} X \circ T_z = \mathbf{E}^{\mathscr{F}} X \quad \forall z \in \mathbf{Z}^N.$$

Moreover, if the dynamical system $(\Sigma, \mathscr{A}, \mathbf{P}, (T_z)_{z\in\mathbf{Z}^N})$ is ergodic, then $E^{\mathscr{F}} X$ is constant equal to the expectation value $\mathbf{E}X := \int_\Sigma X\, d\mathbf{P}$ of the function X.

We denote by $\mathscr{B}_b(\mathbf{R}^N)$ the family of bounded Borel subsets of $\mathbf{R}^N$.

Definition 12.4.1 (additive process). *Let $(\Sigma, \mathscr{A}, \mathbf{P}, (T_z)_{z\in\mathbf{Z}^N})$ be a dynamical system. An additive process indexed by $\mathscr{B}_b(\mathbf{R}^N)$, and covariant with respect to the group $(T_z)_{z\in\mathbf{Z}^N}$, is a mapping $\mathbb{A} : \mathscr{B}_b(\mathbf{R}^N) \longrightarrow L^1_{\mathbf{P}}(\Sigma)$, $A \mapsto \mathbb{A}_A$, satisfying the three following conditions:*

(i) *For all* $(A,B) \in \mathscr{B}_b(\mathbf{R}^N) \times \mathscr{B}_b(\mathbf{R}^N)$ *with* $A \cap B = \emptyset$, $\mathbb{A}_{A\cup B} = \mathbb{A}_A + \mathbb{A}_B$.

(ii) *For all* $A \in \mathscr{B}_b(\mathbf{R}^N)$ *and all* $z \in \mathbf{Z}^N$, $\mathbb{A}_{z+A} = \mathbb{A}_A \circ T_z$.

(iii) *There exists a nonnegative function* h *in* $L^1_{\mathbf{P}}(\Sigma)$ *such that* $|\mathbb{A}_A| \le h$ *for all Borel sets* A *included in* $[0,1[^N$.

Condition (ii) is referred to as the covariance property. It expresses the fact that for the map $A \mapsto \mathbb{A}_A$, the spatial translations are transferred to the dynamic. Condition (iii) is referred to as the domination property.

In what follows, $\mathscr{I}$ denotes the family of the half open intervals $[a,b[$ with a and b in $\mathbf{Z}^N$. Let us recall the notion of regularity of sequences in $\mathscr{B}_b(\mathbf{R}^N)$, introduced in Lemma 12.3.1: a sequence $(B_n)_{n\in\mathbf{N}}$ of sets of $\mathscr{B}_b(\mathbf{R}^N)$ is said to be regular if there exists a nondecreasing sequence $(I_n)_{n\in\mathbf{N}}$ of $\mathscr{I}$ and a constant $C_{reg} > 0$ such that $B_n \subset I_n$ and $\sup_{n\in\mathbf{N}} |I_n|/|B_n| \le C_{reg}$.

For every $A \in \mathscr{B}_b(\mathbf{R}^N)$, we set $\rho(A) := \sup\{r \ge 0 \, : \, \exists \bar{B}_r(x) \subset A\}$, where $\bar{B}_r(x)$ is the closed ball with radius $r > 0$ centered at x. The following theorem generalizes the Birkhoff ergodic theorem. For a proof see [311, Corollary 4.20].

Theorem 12.4.1. *Let* $\mathbb{A}$ *be an additive process covariant with respect to* $(T_z)_{z\in\mathbf{Z}^N}$, *and let* $(A_n)_{n\in\mathbf{N}}$ *be a regular sequence of convex sets of* $\mathscr{B}_b(\mathbf{R}^N)$ *satisfying* $\lim_{n\to+\infty} \rho(A_n) = +\infty$. *Then, for* $\mathbf{P}$ *almost every* $\omega \in \Sigma$,

$$\lim_{n\to+\infty} \frac{\mathbb{A}_{A_n}}{|A_n|}(\omega) = \mathbf{E}^{\mathscr{F}} \mathbb{A}_{[0,1[^N}(\omega).$$

If moreover the dynamical system $(\Sigma, \mathscr{A}, \mathbf{P}, (T_z)_{z\in\mathbf{Z}^N})$ *is ergodic, then*

$$\lim_{n\to+\infty} \frac{\mathbb{A}_{A_n}}{|A_n|}(\omega) = \mathbf{E}\mathbb{A}_{[0,1[^N}.$$

Remark 12.4.1. If the process is covariant with respect to the group $(T_z)_{z\in m\mathbf{Z}^N}$, where m is a given integer in $\mathbf{N}^*$, one can establish the analogous pointwise convergence theorem provided that, in the domination condition (iii), we replace $[0,1[^N$ by $[0,m[^N$. Furthermore, in the definition of regularity, we must replace $\mathscr{I}$ by the family $\mathscr{I}_m$ of all the half open intervals $[a,b)$ with a and b in $m\mathbf{Z}^N$. Let us denote by $\mathscr{F}_m$ the σ-algebra of invariant sets of $\mathscr{A}$ under the group $(T_z)_{z\in m\mathbf{Z}^N}$; then $\lim_{n\to+\infty} \frac{\mathbb{A}_{A_n}}{|A_n|}(\omega) = \mathbf{E}^{\mathscr{F}_m} \frac{1}{m^N} \mathbb{A}_{[0,m[^N}(.)$, or, in the case when $(\Sigma, \mathscr{A}, \mathbf{P}, (T_z)_{z\in m\mathbf{Z}^N})$ is ergodic, $\lim_{n\to+\infty} \frac{\mathbb{A}_{A_n}}{|A_n|}(\omega) = \mathbf{E}\frac{1}{m^N} \mathbb{A}_{[0,m[^N}$.

It is sometimes sufficient to consider the restriction of $\mathbb{A}$ to $\mathscr{I}$. More precisely, we have the following.

Definition 12.4.2. *A discrete additive process covariant with respect to* $(T_z)_{z\in\mathbf{Z}^N}$ *is a set function* $\mathbb{A} : \mathscr{I} \longrightarrow L^1_{\mathbf{P}}(\Sigma)$ *satisfying*

(i) *for every* $I \in \mathscr{I}$ *such that there exists a finite family* $(I_j)_{j\in J}$ *of disjoint intervals in* $\mathscr{I}$ *satisfying* $I = \bigcup_{j\in J} I_j$, *then*

$$\mathbb{A}_I(\cdot) = \sum_{j\in J} \mathbb{A}_{I_j}(\cdot);$$

(ii) *for all* $I \in \mathscr{I}$ *and all* $z \in \mathbf{Z}^N$, $\mathbb{A}_I \circ \tau_z = \mathscr{A}_{z+I}$.

Then the same conclusion holds. More precisely, we have the following.

Theorem 12.4.2. *Let $\mathbb{A} : \mathscr{I} \longrightarrow L^1_{\mathbf{P}}(\Sigma)$ be a discrete additive process, and let $(I_n)_{n\in\mathbf{N}}$ be a regular sequence of $\mathscr{I}$ satisfying $\lim_{n\to+\infty} \rho(I_n) = +\infty$. Then, for $\mathbf{P}$ almost every $\omega \in \Sigma$,*

$$\lim_{n\to+\infty} \frac{\mathbb{A}_{I_n}}{|I_n|}(\omega) = \mathbf{E}^{\mathscr{F}} \mathbb{A}_{[0,1[^N}(\omega).$$

If moreover the dynamical system $(\Sigma, \mathscr{A}, \mathbf{P}, (T_z)_{z\in\mathbf{Z}^N})$ is ergodic, then

$$\lim_{n\to+\infty} \frac{\mathbb{A}_{I_n}}{|I_n|}(\omega) = \mathbf{E}\mathbb{A}_{[0,1[^N}.$$

Given a dynamical system $(\Sigma, \mathscr{A}, \mathbf{P}, (T_z)_{z\in\mathbf{Z}})$ $(N = 1)$ and a function Φ in $L^1_{\mathbf{P}}(\Sigma)$ for all a, b in $\mathbf{Z}$, $a < b$, set $\mathbb{A}_{[a,b[} := \sum_{i=a}^{b-1} \Phi \circ T_i$. It is easy to check that $\mathbb{A} : \mathscr{I} \to L^1_{\mathbf{P}}(\Sigma)$ is a discrete additive process, covariant with respect to $(T_z)_{z\in\mathbf{Z}}$. From Theorem 12.4.2 we deduce the standard discrete time Birkhoff ergodic theorem.

Corollary 12.4.1 (Birkhoff's ergodic theorem). *For $\mathbf{P}$-a.e. $\omega \in \Sigma$*

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=1}^{n} \Phi \circ T_i(\omega) = \mathbf{E}^{\mathscr{F}} \Phi(\omega).$$

If the dynamical system $(\Sigma, \mathscr{A}, \mathbf{P}, (T_z)_{z\in\mathbf{Z}})$ is ergodic, then for $\mathbf{P}$-a.e. $\omega \in \Sigma$,

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=1}^{n} \Phi \circ T_i(\omega) = \mathbf{E}\Phi.$$

Let A be a bounded open subset of $\mathbf{R}^N$ and $A^{\mathbf{Z}}$ the set of sequences in A endowed with the σ algebra $\mathscr{T}_{A^{\mathbf{Z}}}$ which is the infinite product of the Borel σ-algebra on A. Let us equip $(A^{\mathbf{Z}}, \mathscr{T}_{A^{\mathbf{Z}}})$ with the probability measure μ, infinite product of the normalized Lebesgue measure $\frac{1}{|A|}\mathscr{L}\lfloor A$ on A. From the Birkhoff ergodic theorem we deduce the following result, which is at the root of the Monte Carlo method for computing the integrals, and which is a useful tool for questions of measurability (see Lemma 12.4.4 below).

Corollary 12.4.2. *Let $g : \mathbf{R}^N \times \mathbf{M}^{m\times N} \to \mathbf{R}$ be a $\mathscr{B}(\mathbf{R}^N) \otimes \mathscr{B}(\mathbf{M}^{m\times N})$-measurable function satisfying $0 \le g(x,a) \le \beta(1 + |a|^p)$ for all $(x,a) \in \mathbf{R}^N \times \mathbf{M}^{m\times N}$, where β is a given nonnegative constant, and let u be a given function in $W^{1,p}(A, \mathbf{R}^m)$. Then*

$$\frac{1}{|A|} \int_A g(x, \nabla u(x))\, dx = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} g(s_i, \nabla u(s_i))$$

for μ almost every sequence $s = (s_i)_{i\in\mathbf{Z}}$ in A.

PROOF. Consider the projection $\pi : A^{\mathbf{Z}} \to A$ defined by $\pi((s)_{i\in\mathbf{Z}}) = s_0$, the shift group $(T_z)_{z\in\mathbf{Z}}$ defined by $T_z((s_i)_{i\in\mathbf{Z}}) = (s_{i+z})_{i\in\mathbf{Z}}$, and set $\Phi(s) = g(\pi(s), \nabla u(\pi(s)))$ for every s in $A^{\mathbf{Z}}$. The dynamical system $(A^{\mathbf{Z}}, \mathscr{T}_{A^{\mathbf{Z}}}, \mu, (T_z)_{z\in\mathbf{Z}})$ is ergodic. Indeed the measure μ restricted to $\mathscr{F}$ is uniquely determined by its values on the cylinders of $\mathscr{F}$, and μ restricted to the cylinders satisfies (12.27). Thus, according to the Birkhoff ergodic theorem, for μ almost every $s \in A^{\mathbf{Z}}$, we infer

$$\begin{aligned}\frac{1}{|A|}\int_A g(x,\nabla u(x))\,dx &= \int_{A^{\mathbf{Z}}} \Phi(s)\,d\mu(s)\\ &= \lim_{n\to+\infty}\frac{1}{n}\sum_{i=1}^{n}\Phi\circ T_i(s)\\ &= \lim_{n\to+\infty}\frac{1}{n}\sum_{i=1}^{n} g(s_i,\nabla u(s_i)),\end{aligned}$$

which completes the proof. □

Additive processes and the pointwise convergence result stated in Theorem 12.4.1 can be generalized to subadditive processes. We address this notion and give detailed proofs.

Definition 12.4.3. *Let $(\Sigma,\mathscr{A},\mathbf{P},(T_z)_{z\in\mathbf{Z}^N})$ be a dynamical system. A subadditive process indexed by $\mathscr{B}_b(\mathbf{R}^N)$, and covariant with respect to $(T_z)_{z\in\mathbf{Z}^N}$, is a mapping $\mathbb{S}:\mathscr{B}_b(\mathbf{R}^N)\longrightarrow L^1_{\mathbf{P}}(\Sigma)$, $A\mapsto \mathbb{S}_A$, satisfying the four following conditions:*

(i) *For all $(A,B)\in\mathscr{B}_b(\mathbf{R}^N)\times\mathscr{B}_b(\mathbf{R}^N)$ with $A\cap B=\emptyset$, $\mathbb{S}_{A\cup B}\le \mathbb{S}_A+\mathbb{S}_B$.*

(ii) *For all $A\in\mathscr{B}_b(\mathbf{R}^N)$ and all $z\in\mathbf{Z}^N$, $\mathbb{S}_{z+A}=\mathbb{S}_A\circ T_z$.*

(iii) *There exists a nonnegative function h in $L^1_{\mathbf{P}}(\Omega)$ such that $|\mathbb{S}_A|\le h$ for all Borel sets A included in $[0,1[^N$.*

(iv) $\gamma(\mathbb{S}):=\inf\left\{\int_\Sigma \frac{\mathbb{S}_I}{|I|}\,d\mathbf{P} : I\in\mathscr{I}\right\}>-\infty$.

The constant $\gamma(\mathbb{S})$ in (iv) is referred to as the spatial constant of the process. It is sometimes sufficient to consider the restriction of $\mathbb{S}$ to $\mathscr{I}$. More precisely, we have the following.

Definition 12.4.4. *A discrete subadditive process, covariant with respect to $(T_z)_{z\in\mathbf{Z}^N}$, is a set function $\mathbb{S}:\ \mathscr{I}\longrightarrow L^1_{\mathbf{P}}(\Sigma)$ satisfying*

(i) *for every $I\in\mathscr{I}$ such that there exists a finite family $(I_j)_{j\in J}$ of disjoint intervals in $\mathscr{I}$ with $I=\bigcup_{j\in J} I_j$, then*
$$\mathbb{S}_I(\cdot)\le\sum_{j\in J}\mathbb{S}_{I_j}(\cdot);$$

(ii) *for all $I\in\mathscr{I}$ and all $z\in\mathbf{Z}^N$, $\mathbb{S}_I\circ\tau_z=\mathscr{S}_{z+I}$;*

(iii) $\gamma(\mathbb{S}):=\inf\left\{\int_\Sigma\frac{\mathbb{S}_I}{|I|}\,d\mathbf{P}:I\in\mathscr{I}\right\}>-\infty$.

We are going to establish a pointwise convergence result (Theorem 12.4.3 below) which generalizes Lemma 12.3.1 to a stochastic situation, and Theorem 12.4.1 for subadditive processes. Its proof is based on a so-called *maximal inequality* (maximal for $-\mathbb{S}$, thus minimal for $\mathbb{S}$), which itself is derived from the Wiener covering lemma below. For any bounded interval I of $\mathbf{R}^N$, we denote by I^* its 3-dilated associated interval, namely, $I^*=\bigcup_{u\in\mathbf{R}^N:(u+I)\cap I\neq\emptyset}(u+I)$. Note that $|I^*|=3^N|I|$.

Lemma 12.4.1. *Let $I_1 \subset \cdots \subset I_n$ be finitely many nested half open bounded intervals of $\mathbf{R}^N$ with $|I_1| > 0$, and let x_0 be a fixed element of I_1. Let A be a finite subset of $\mathbf{R}^N$ and a map $\nu : A \to \{1,\dots,n\}$. Consider the covering*

$$A \subset \bigcup_{x\in A} \left(x - x_0 + I_{\nu(x)}\right);$$

then there exists $A' \subset A$ such that the family $\left(x - x_0 + I_{\nu(x)}\right)_{x\in A'}$ is made up of pairwise disjoint intervals and

$$A \subset \bigcup_{z\in A'} \left(x - x_0 + I_{\nu(x)}\right)^*.$$

PROOF. We adapt the proof of the standard Wiener covering lemma by open Euclidean balls of $\mathbf{R}^N$. (For a proof of the standard Wiener covering lemma, see [264, Lemma 3.5.1].) Note that in the standard Wiener covering lemma $I_i = B_{r_i}(0)$, where $0 < r_1 \le \cdots \le r_n$, $x_0 = 0$, $x - x_0 + I_k = B_{r_k}(x)$, $\left(x - x_0 + I_k\right)^* = B_{3r_k}(x)$, and the crucial argument of the proof is to notice that

$$r_i \le r_j \text{ and } B_{r_i}(x) \cap B_{r_j}(y) \ne \emptyset \Longrightarrow B_{r_i}(x) \subset B_{3r_j}(y).$$

To shorten the notation, we set $K_{\nu(x)} := x - x_0 + I_{\nu(x)}$, and we use the following similar remark throughout the proof: $\nu(x) \le \nu(y)$ and $K_{\nu(x)} \cap K_{\nu(y)} \ne \emptyset \Longrightarrow K_{\nu(x)} \subset K^*_{\nu(y)}$.

We construct A' following a finite iterative procedure. If $A = \emptyset$, then the conclusion of the lemma is trivial. Otherwise choose $x_1 \in A$ such that $\nu(x_1) = \max\{\nu(x) : x \in A\}$ and set $A_1 = A \setminus K^*_{\nu(x_1)}$. If $A_1 = \emptyset$, then $A' = \{x_1\}$ is suitable. Otherwise, if $x_1,\dots,x_k$ are chosen with $\left(K_{\nu(x_i)}\right)_{i=1,\dots,k}$ pairwise disjoint, let $A_k = A \setminus \bigcup_{i=1}^k K^*_{\nu(x_i)}$. If $A_k = \emptyset$, then $A' = \{u_1,\dots,u_k\}$ is suitable. Otherwise, choose x_{k+1} such that $\nu(x_{k+1}) = \max\{\nu(x) : x \in A_k\}$, and set $A_{k+1} = A \setminus \bigcup_{i=1}^{k+1} K^*_{\nu(x_i)}$. The set $K_{\nu(x_{k+1})}$ does not intercept each set $K_{\nu(x_i)}$ for $i = 1,\dots,k$; otherwise $K_{\nu(x_{k+1})} \subset \bigcup_{i=1}^k K^*_{\nu(x_i)}$, in contradiction with $x_{k+1} \in A_k = A \setminus \bigcup_{i=1}^k K^*_{\nu(x_i)}$. This construction must end after finite many steps, i.e., $A \subset \bigcup_{i=1}^l K^*_{\nu(x_i)}$ for some integer l. The set $A' = \{x_1,\dots,x_l\}$ is suitable. □

Lemma 12.4.2 (minimal inequality). *Let $\mathbb{S} : \mathscr{B}_b(\mathbf{R}^N) \longrightarrow L^1_{\mathbf{P}}(\Sigma)$ be a discrete nonpositive subadditive process, covariant with respect to $(T_z)_{z\in\mathbf{Z}^N}$, and $(I_n)_{n\in\mathbf{N}}$ a regular sequence of intervals of $\mathscr{I}$ with a constant of regularity C_{reg}. Then, for every $r > 0$, the probability of the set*

$$E_r := \left\{ \omega \in \Sigma : \inf_n \frac{\mathbb{S}_{I_n}(\omega)}{|I_n|} \le -r \right\}$$

satisfies

$$\mathbf{P}(E_r) \le -\frac{3^N C_{reg}\, \gamma(\mathbb{S})}{r}.$$

PROOF. Since $(I_n)_{n\in\mathbf{N}}$ is a regular family of $\mathscr{I}$, there exists a nondecreasing family $(I'_n)_{n\in\mathbf{N}}$ of $\mathscr{I}$ such that for all $n \in \mathbf{N}$, $I_n \subset I'_n$ and $|I'_n| \le C_{reg}|I_n|$. In what follows, we fix n intended to go to $+\infty$, consider $I'_1 \subset \cdots \subset I'_n$, and fix $z_0 \in I'_1 \cap \mathbf{Z}^N$. Let $k(n) \in \mathbf{N}$ be large enough so that

$$-z_0 + I'_i \subset [-k(n), k(n)[^N \quad \forall i = 1,\dots,n, \tag{12.28}$$

and take an integer $k \geq k(n)$. We set

$$E_n := \left\{ \omega \in \Sigma : \inf_{1\leq i\leq n} \frac{\mathbb{S}_{I_i}}{|I_i|}[\omega) \leq -r \right\},$$

and

$$\forall \omega \in \Sigma,\ A(\omega) := \{z \in [-k+k(n), k-k(n)[^N : T_{z-z_0}\omega \in E_n\}.$$

According to the definition of the set E_n, and by covariance, for each $z \in A(\omega)$ there exists an integer $\nu(z)$ in $\{1,\dots,n\}$ (possibly depending on ω) such that

$$\mathbb{S}_{z-z_0+I_{\nu(z)}}(\omega) \leq -r|I_{\nu(z)}|. \tag{12.29}$$

We consider now the following covering of $A(\omega)$:

$$A(\omega) \subset \bigcup_{z\in A(\omega)} z - z_0 + I'_{\nu(z)}.$$

Then, applying Lemma 12.4.1, one can extract a pairwise disjoint family $\left(z_i - z_0 + I'_{\nu(z_i)}\right)_{i=1,\dots l}$ such that

$$A(\omega) \subset \bigcup_{i=1}^{l} \left(z_i - z_0 + I'_{\nu(z_i)}\right)^*. \tag{12.30}$$

From (12.30) we deduce that

$$\sum_{i=1}^{l} |I_{\nu(x_i}| \geq \frac{\#A(\omega)}{3^N C_{reg}}, \tag{12.31}$$

where $\#A(\omega)$ denotes the cardinal of the set $A(\omega)$. We have used the fact that $\#(I \cap \mathbf{Z}^N) = |I|$ for all $I \in \mathscr{I}$. On the other hand, from (12.28), we have $-z_0 + I_i \subset [-k(n), k(n)[^N$ for all $i = 1,\dots,n$. Thus, for $i = 1,\dots,l$, since $z_i \in A(\omega) \subset [-k+k(n), k-k(n)[^N$, we infer that $z_i - z_0 + I_{\nu(z_i)} \subset [-k,k[^N$, and thus

$$\cup_{i=1}^{l}\left(z_i - z_0 + I_{\nu(z_i)}\right) \subset [-k,k[^N. \tag{12.32}$$

Note that $\left(z_i - z_0 + I_{\nu(z_i)}\right)_{i=1,\dots l}$ are pairwise disjoint. Since $\mathbb{S}$ is a nonpositive subadditive process, it is subadditive and nonincreasing. Therefore, from (12.32), (12.29), and (12.31) we infer that for all $\omega \in \Sigma$

$$\begin{aligned} \mathbb{S}_{[-k,k[^N}(\omega) &\leq \sum_{i=1}^{l} \mathbb{S}_{z_i - z_0 + I_{\nu(z_i)}}(\omega) \\ &\leq -r \sum_{i=1}^{l} |I_{\nu(z_i)}| \\ &\leq -\frac{r}{3^N C_{reg}} \#A(\omega). \end{aligned} \tag{12.33}$$

Integrating (12.33) over Σ, we obtain

$$\gamma(\mathbb{S}) \leq -\frac{r}{3^N C_{reg}(2k)^N} \int_\Sigma \#A(\omega)\, d\mathbf{P}(\omega). \tag{12.34}$$

Noticing that

$$\#A(.) = \sum_{z\in[-k+k(n),k-k(n)[^N} \mathbf{1}_{\{\omega : T_{z-z_0}\omega\in E_n\}},$$

and using the fact that $\mathbf{P}$ is invariant under the group $(T_z)_{z\in\mathbf{Z}^N}$, we have

$$\int_\Sigma \#A(\omega)\, d\mathbf{P}(\omega) = \sum_{z\in[-k+k(n),k-k(n)[^N} \mathbf{P}(E_n) = \big(2(k-k(n))\big)^N \mathbf{P}(E_n)$$

so that (12.34) yields

$$\mathbf{P}(E_n) \le -\frac{3^N C_{reg}\gamma(\mathbb{S})}{r}\,\frac{k^N}{(k-k(n))^N}.$$

The proof is completed first by letting $k\to+\infty$ and then $n\to+\infty$. $\square$

Let $(A_n)_{n\in\mathbf{N}}$ be a regular sequence of convex sets of $\mathscr{B}_b(\mathbf{R}^N)$ such that $\lim_{n\to+\infty}\rho(A_n) = +\infty$. For every $m\in\mathbf{N}^*$, $m<n$, we set

$$\underline{A}_{n,m} = \bigcup_{\{z\in m\mathbf{Z}^N \,:\, z+[0,m[^N\subset A_n\}} \left(z+[0,m[^N\right);$$

$$\overline{A}_{n,m} = \bigcup\nolimits_{\{z\in m\mathbf{Z}^N \,:\, (z+[0,m[^N)\cap A_n\neq\emptyset\}} \left(z+[0,m[^N\right)$$

and denote by $\mathscr{F}_m$ the σ-algebra of invariant sets of $\mathscr{A}$ under the group $(T_z)_{z\in m\mathbf{Z}}$. We will also need the following technical lemma.

Lemma 12.4.3. *Let X and h be two functions in $L^1_{\mathbf{P}}(\Sigma)$ with $h\ge 0$. Then the following assertions hold:*

(i) $\lim_{n\to+\infty}\frac{|\overline{A}_{n,m}\setminus\underline{A}_{n,m}|}{|A_n|} = 0$;

(ii) $\lim_{n\to+\infty}\frac{1}{|A_n|}\sum_{z\in m\mathbf{Z}^N\cap\underline{A}_{n,m}} X\circ T_z = \frac{\mathbf{E}^{\mathscr{F}_m}X}{m^N}$ *almost surely;*

(iii) $\lim_{n\to+\infty}\frac{1}{|A_n|}\sum_{z\in\overline{A}_{n,m}\setminus\underline{A}_{n,m}} h\circ T_z = 0$ *almost surely.*

PROOF. The proof of assertion (i) is standard and is left to the reader. Note that we easily deduce from (i) that $\lim_{n\to+\infty}\frac{|\overline{A}_{n,m}|}{|A_n|} = \lim_{n\to+\infty}\frac{|\underline{A}_{n,m}|}{|A_n|} = 1$. For each convex set A_n and $\delta>0$ set

$$A_n^\delta := \{x\in\mathbf{R}^N : d(x,\partial A_n)\le\delta\}.$$

It is worth noticing that $(A_n\setminus A_n^\delta)_{n\in\mathbf{N}}$ and $(A_n\cup A_n^\delta)_{n\in\mathbf{N}}$ are two families of convex regular sets with $\lim_{n\to+\infty}\rho(A_n\setminus A_n^\delta) = \lim_{n\to+\infty}\rho(A_n\cup A_n^\delta) = +\infty$. For fixed m, we can find some $\delta>0$ such that

$$\begin{aligned}\frac{1}{|A_n|}\sum_{z\in(A_n\setminus A_n^\delta)\cap m\mathbf{Z}^N} h\circ T_z &\le \frac{1}{|A_n|}\sum_{z\in\underline{A}_{n,m}\cap m\mathbf{Z}^N} h\circ T_z\\ &\le \frac{1}{|A_n|}\sum_{z\in\overline{A}_{n,m}\cap m\mathbf{Z}^N} h\circ T_z\\ &\le \frac{1}{|A_n|}\sum_{z\in(A_n\cup A_n^\delta)\cap m\mathbf{Z}^N} h\circ T_z. \qquad (12.35)\end{aligned}$$

Applying Theorem 12.4.1 and Remark 12.4.1 to the additive process $\mathbb{A}$, covariant with respect to $(T_z)_{z\in m\mathbf{Z}^N}$, defined by

$$\mathbb{A}_A := \sum_{z\in m\mathbf{Z}^N\cap A} h\circ T_z,$$

we infer that the left-hand side and the right-hand side of (12.35) converge to the same limit. We then complete the proof of (iii) by going to the limit on (12.35) as $n\to+\infty$. On account of

$$\left|\frac{1}{|A_n|}\sum_{z\in m\mathbf{Z}^N\cap A_n} X\circ T_z - \frac{1}{|A_n|}\sum_{z\in m\mathbf{Z}^N\cap \underline{A}_n^m} X\circ T_z\right| \le \frac{1}{|A_n|}\sum_{z\in m\mathbf{Z}^N\cap(\overline{A}_n^m\setminus\underline{A}_n^m)} |X|\circ T_z,$$

assertion (ii) follows from (iii) by taking $h=|X|$ and from Theorem 12.4.1 applied to the additive process $\mathbb{A}$, covariant with respect to $(T_z)_{z\in m\mathbf{Z}^N}$, defined by

$$\mathbb{A}_A := \sum_{z\in m\mathbf{Z}^N\cap A} X\circ T_z.$$

The proof of Lemma 12.4.3 is complete. $\square$

We are in a position to establish the proof of the so-called subadditive ergodic theorem.

Theorem 12.4.3. *Let $\mathbb{S}:\mathscr{B}_b(\mathbf{R}^N)\longrightarrow L^1_{\mathbf{P}}(\Sigma)$ be a subadditive process covariant with respect to $(T_z)_{z\in\mathbf{Z}^N}$, and let $(A_n)_{n\in\mathbf{N}}$ be a regular sequence of convex sets of $\mathscr{B}_b(\mathbf{R}^N)$ satisfying $\lim_{n\to+\infty}\rho(A_n)=+\infty$. Then, for $\mathbf{P}$-a.e. $\omega\in\Sigma$,*

$$\lim_{n\to+\infty}\frac{\mathbb{S}_{A_n}}{|A_n|}(\omega)=\inf_{m\in\mathbf{N}^*}\mathbf{E}^{\mathscr{F}}\frac{\mathbb{S}_{[0,m[^N}}{m^N}(\omega).$$

If moreover the dynamical system $(\Sigma,\mathscr{A},\mathbf{P},(T_z)_{z\in\mathbf{Z}^N})$ is ergodic, then for $\mathbf{P}$-a.e. $\omega\in\Sigma$,

$$\lim_{n\to+\infty}\frac{\mathbb{S}_{A_n}}{|A_n|}(\omega)=\inf_{m\in\mathbf{N}^*}\mathbf{E}\frac{\mathbb{S}_{[0,m[^N}}{m^N}=\gamma(\mathbb{S}).$$

PROOF. Since $(A_n)_{n\in\mathbf{N}}$ is a regular sequence of convex sets of $\mathscr{B}_b(\mathbf{R}^N)$, there exists an increasing sequence $(I_n)_{n\in\mathbf{N}}$ of $\mathscr{I}$ and a positive constant C_{reg}, which does not depend on n, such that $A_n\subset I_n$ and $|I_n|\le C_{reg}|B_n|$ for all $n\in\mathbf{N}$. For $m\in\mathbf{N}^*$ we set

$$\overline{l}_m := \limsup_{n\to+\infty}\frac{\mathbb{S}_{\underline{A}_{n,m}}}{|\underline{A}_{n,m}|},\quad \underline{l}_m=\liminf_{n\to+\infty}\frac{\mathbb{S}_{\overline{A}_{n,m}}}{|\overline{A}_{n,m}|};$$

$$\overline{l} := \limsup_{n\to+\infty}\frac{\mathbb{S}_{A_n}}{|A_n|},\quad \underline{l}=\liminf_{n\to+\infty}\frac{\mathbb{S}_{A_n}}{|A_n|}.$$

Note that the sets $\underline{I}_{n,m}$ and $\overline{I}_{n,m}$ belong to $\mathscr{I}_m$. (Recall that $\mathscr{I}_m$ denotes the family of the half open intervals $[a,b[$ with a and b in $m\mathbf{Z}^N$.) Finally, for every discrete subadditive process Ψ defined on $\mathscr{I}_m$ and covariant with respect to $(T_z)_{z\in m\mathbf{Z}^N}$, we set

$$\gamma^m(\Psi):=\inf\left\{\int_\Sigma\frac{\Psi_I}{|I|}d\mathbf{P}: I\in\mathscr{I}_m\right\}.$$

First step. We prove that $\underline{l} = \overline{l}$ almost surely. We will denote by l this common value.

The inclusion $\underline{A}_{n,m} \subset A_n$, together with the subadditivity and the domination conditions, yields

$$\begin{aligned}\frac{\mathbb{S}_{A_n}}{|A_n|} &\leq \frac{\mathbb{S}_{\underline{A}_{n,m}}}{|\underline{A}_{n,m}|}\frac{|\underline{A}_{n,m}|}{|A_n|} + \frac{\mathbb{S}_{A_n \setminus \underline{A}_{n,m}}}{|A_n|} \\ &\leq \frac{\mathbb{S}_{\underline{A}_{n,m}}}{|\underline{A}_{n,m}|}\frac{|\underline{A}_{n,m}|}{|A_n|} + \frac{1}{|A_n|}\sum_{z\in\mathbf{Z}^N\cap(\overline{A}_{n,m}\setminus\underline{A}_{n,m})} h\circ T_z \end{aligned} \tag{12.36}$$

with h defined in Definition 12.4.3(iii). On account of (iii) of Lemma 12.4.3, we have the following almost sure limit:

$$\lim_{n\to+\infty}\frac{1}{|A_n|}\sum_{z\in\mathbf{Z}^N\cap(\overline{A}_{n,m}\setminus\underline{A}_{n,m})} h\circ T_z = 0.$$

Hence, letting $n\to+\infty$ in (12.36) we deduce that almost surely

$$\overline{l} \leq \overline{l}_m. \tag{12.37}$$

Similarly from $A_n \subset \overline{A}_{n,m}$ we infer that almost surely

$$\underline{l}_m \leq \underline{l}. \tag{12.38}$$

Fix $r>0$ and set $E_{m,r} := \{\omega : \overline{l}_m(\omega) - \underline{l}_m(\omega) \geq r\}$. From (12.37), (12.38) we infer that $\{\omega : \overline{l}(\omega) - \underline{l}(\omega) \geq r\} \subset E_{m,r}$. Thus, to conclude it suffices to show that for every $\varepsilon>0$, and for m large enough, the inequality

$$\mathbf{P}(E_{m,r}) \leq \frac{2^N C_{reg}\ \varepsilon}{r}$$

holds, provided that we have established that for m large enough, and for $\mathbf{P}$-almost all $\omega\in\Sigma$,

$$-\infty < \underline{l}_m(\omega) \text{ and } \overline{l}_m(\omega) < +\infty.$$

The end of the step consists in establishing these two claims.

According to [274, Theorem 2.1] applied to the $\mathbf{Z}^N$-invariant subadditive set function $A\mapsto \int_\Sigma \mathbb{S}_A d\mathbf{P}$, we have $\lim_{m\to+\infty}\int_\Sigma \frac{\mathbb{S}_{[0,m[^N}}{m^N} d\mathbf{P} = \gamma(\mathbb{S})$. Hence, given $\varepsilon>0$, there exists $m(\varepsilon)\in\mathbf{N}^*$ such that, for $m\geq m(\varepsilon)$,

$$\int_\Sigma \frac{\mathbb{S}_{[0,m[^N}}{m^N} d\mathbf{P} - \gamma(\mathbb{S}) \leq \varepsilon. \tag{12.39}$$

Consider the discrete additive process $\mathbb{A}^m$ covariant with respect to the group $(T_z)_{z\in m\mathbf{Z}^N}$, defined for all $I\in\mathscr{I}_m$ by

$$\mathbb{A}^m_I := \sum_{z\in I\cap m\mathbf{Z}^N} \mathbb{S}_{[0,m[^N}\circ T_z.$$

Subtracting this process from the restriction of $\mathbb{S}$ to $\mathscr{I}_m$, we obtain a nonpositive discrete subadditive process $\mathbb{S}^m$, indexed by $\mathscr{I}_m$, and covariant with respect to $(T_z)_{z\in m\mathbf{Z}^N}$:

$$\mathbb{S}^m := \mathbb{S} - \mathbb{A}^m \leq 0. \tag{12.40}$$

On the other hand, by additivity and covariance, for every I of $\mathscr{I}_m$

$$\int_\Sigma \frac{\mathbb{A}_I^m}{|I|}\, d\mathbf{P} = \int_\Sigma \frac{\mathbb{S}_{[0,m[^N}}{m^N} d\mathbf{P},$$

so that (12.39) yields that for $m \geq m(\varepsilon)$, the spatial constant $\gamma(\mathbb{S}^m)$ of $\mathbb{S}^m$ satisfies

$$\gamma^m(\mathbb{S}^m) \geq -\varepsilon. \tag{12.41}$$

Moreover, according to Lemma 12.4.3(i) and (ii), for $\mathbf{P}$-a.e. ω, we have

$$\begin{aligned} L_m(\omega) &:= \lim_{n\to+\infty} \frac{\mathbb{A}^m_{\overline{A}_{n,m}}(\omega)}{|\overline{A}_{n,m}|} \\ &= \lim_{n\to+\infty} \frac{\mathbb{A}^m_{\underline{A}_{n,m}}(\omega)}{|\underline{A}_{n,m}|}. \\ &= \mathbf{E}^{\mathscr{F}_m} \frac{\mathbb{S}_{[0,m[^N}}{m^N}. \end{aligned}$$

By applying $\mathscr{S}^m$ to $(\underline{A}_{n,m})_{n\in\mathbf{N}}$ and letting $n \to +\infty$, (12.40) yields that for $\mathbf{P}$-a.e. ω

$$\overline{l}_m(\omega) - L_m(\omega) \leq 0, \tag{12.42}$$

from which we deduce $\overline{l}_m < +\infty$.

By applying $\mathscr{S}^m$ to $(\overline{A}_{n,m})_{n\in\mathbf{N}}$ we infer

$$\frac{\mathscr{S}^m_{\overline{A}_{n,m}}}{|\overline{A}_{n,m}|} = \frac{\mathscr{S}_{\overline{A}_{n,m}}}{|\overline{A}_{n,m}|} - \frac{\mathscr{A}^m_{\overline{A}_{n,m}}}{|\overline{A}_{n,m}|},$$

from which we deduce, since $\mathscr{S}^m$ is nonpositive, thus nonincreasing,

$$\frac{|\overline{I}_{n,m}|}{|\overline{A}_{n,m}|} \frac{\mathscr{S}^m_{\overline{I}_{n,m}}}{|\overline{I}_{n,m}|} \leq \frac{\mathscr{S}_{\overline{A}_{n,m}}}{|\overline{A}_{n,m}|} - \frac{\mathscr{A}^m_{\overline{A}_{n,m}}}{|\overline{A}_{n,m}|}.$$

From (i) of Lemma 12.4.3 it is easy to establish $\limsup_{n\to+\infty} \frac{|\overline{I}_{n,m}|}{|\overline{A}_{n,m}|} \leq C_{reg}$. Consequently, by letting $n \to +\infty$ in the above inequality, for $\mathbf{P}$-a.e. ω we obtain

$$\underline{l}_m(\omega) - L_m(\omega) \geq C_{reg} \inf_n \frac{\mathbb{S}^m_{\overline{I}_{n,m}}(\omega)}{|\overline{I}_{n,m}|}. \tag{12.43}$$

From (12.43) we infer

$$\{\omega : \underline{l}_m - L_m \leq -r\} \subset E_r := \left\{ \omega \in \Sigma : \inf_n \frac{\mathbb{S}^m_{\overline{I}_{n,m}}(\omega)}{|\overline{I}_{n,m}|} \leq -\frac{r}{C_{reg}} \right\}.$$

For fixed $m \geq m(\varepsilon)$, from (12.41) and Lemma 12.4.2 applied to the process $\mathbb{S}^m$, covariant with respect to the group $(T_z)_{z\in m\mathbf{Z}^N}$ (note that $(\overline{I}_{n,m})_{n\in\mathbf{N}}$ is nondecreasing and thus is a regular sequence of $\mathscr{I}_m$ with constant 1), we obtain

$$\mathbf{P}(\{\omega : \underline{l}_m - L_m \leq -r\}) \leq \frac{3^N \varepsilon C_{reg}}{r}. \tag{12.44}$$

The almost sure inequality $-\infty < \underline{l}_m$ follows by letting $r \to +\infty$.

Combining (12.43) and (12.42) we deduce that

$$\overline{l}_m(\omega) - \underline{l}_m(\omega) \leq -C_{reg} \inf_n \frac{\mathbb{S}^m_{\overline{I}_{n,m}}(\omega)}{|\overline{I}_{n,m}|},$$

so that $E_{m,r} \subset E_r$. Therefore, by using again Lemma 12.4.2 together with (12.41), for $m \geq m(\varepsilon)$,

$$\begin{aligned}\mathbf{P}(\{\omega \in \Sigma : \ \overline{l}(\omega) - \underline{l}(\omega) \geq r\}) &\leq \mathbf{P}(\{\omega \in \Sigma : \ \overline{l}_m(\omega) - \underline{l}_m(\omega) \geq r\}) \\ &\leq \frac{2^N \varepsilon C_{reg}}{r}.\end{aligned}$$

The step is then completed by letting first $\varepsilon \to 0$ and then $r \to 0$. Since for all $m \in \mathbf{N}^*$, $\underline{l} \leq \underline{l}_m \leq \overline{l}_m \leq \overline{l}$, we have also proved that for all $m \in \mathbf{N}$, we have $\overline{l}_m = \underline{l}_m = l$ almost surely.

Second step. We prove that l is almost surely invariant by $(T_z)_{z\in\mathbf{Z}}$, i.e., for all $z \in \mathbf{Z}^N$, $l(\omega) = l(T_z\omega)$ a.s. Consider $I_n = [0, n[^N$. Fix $z = (z_i)_{i=1,\dots,N}$ and set $|z|_\infty := \max_{i=1,\dots,N} |z_i|$. From $I_n + z \subset [-|z|_\infty, n + |z|_\infty[^N$, and the fact that $\mathscr{S}^1 = \mathscr{S} - \mathscr{A}^1$ is nonpositive (see (12.40)), thus nonincreasing, we infer that

$$\frac{\mathscr{S}^1_{I_n}}{n^N} \circ T_z = \frac{\mathscr{S}^1_{I_n+z}}{n^N} \geq \frac{\mathscr{S}^1_{[-|z|_\infty, n+|z|_\infty[^N}}{(n+2|z|_\infty)^N} \frac{(n+2|z|_\infty)^N}{n^N}.$$

Letting $n \to +\infty$, from the first step and the invariance of L_1 with respect to $(T_z)_{z\in\mathbf{Z}^N}$, we deduce that $l(T_z\omega) \geq l(\omega)$ for $\mathbf{P}$-a.e. ω. Hence we also infer that $l(T_{-z}\omega) \geq l(\omega)$ for $\mathbf{P}$-a.e. ω. Applying T_z we finally obtain $l(\omega) \geq l(T_z\omega)$, thus $l(T_z\omega) = l(\omega)$ for $\mathbf{P}$-a.e. ω.

Last step. This step is devoted to the identification of l. Let us set for all $m \in \mathbf{N}^*$, $f_m(\omega) := \mathbf{E}^{\mathscr{F}}(\mathbb{S}_{[0,m[^N}/m^N)$. We first prove that $l \leq \inf_{m\in\mathbf{N}^*} f_m$. Indeed from (12.42) for every $m \in \mathbf{N}^*$, $l \leq L_m = \mathbf{E}^{\mathscr{F}_m}\mathbb{S}_{[0,m[^N}/m^N$ so that, by invariance of l and from the fact that $\mathscr{F} \subset \mathscr{F}_m$, we infer

$$\begin{aligned} l = \mathbf{E}^{\mathscr{F}} l &\leq \mathbf{E}^{\mathscr{F}} L_m \\ &= \mathbf{E}^{\mathscr{F}}\left(\mathbf{E}^{\mathscr{F}_m} \frac{\mathbb{S}_{[0,m[^N}}{m^d}\right) \\ &= \mathbf{E}^{\mathscr{F}} \frac{\mathbb{S}_{[0,m[^N}}{m^N}, \end{aligned}$$

which proves the claim.

On the other hand, with the notation of the first step, noticing that $(\overline{I}_{n,m})_{n\in\mathbf{N}}$ is regular, we have for $\mathbf{P}$-a.e. ω,

$$\lim_{n\to+\infty} \frac{\mathscr{S}^m_{\overline{I}_{n,m}}}{|\overline{I}_{n,m}|} = l - L_m \leq 0.$$

From Fatou's lemma and (12.41), we deduce that for every E in $\mathscr{F}$, and for $m \geq m(\varepsilon)$,

$$\begin{aligned}
\int_E (L_m - l)\, d\mathbf{P} &= \int_E \lim_{n\to+\infty} -\frac{\mathscr{S}^m_{I_{n,m}}}{|I_{n,m}|}\, d\mathbf{P} \\
&\leq \liminf_{n\to+\infty} \int_E -\frac{\mathscr{S}^m_{I_{n,m}}}{|I_{n,m}|}\, d\mathbf{P} \\
&\leq \liminf_{n\to+\infty} \int_\Sigma -\frac{\mathscr{S}^m_{I_{n,m}}}{|I_{n,m}|}\, d\mathbf{P} \\
&\leq \sup_{J\in\mathscr{I}_m} \int_\Sigma -\frac{\mathscr{S}^m_J}{|J|}\, d\mathbf{P} \\
&= -\gamma(\mathscr{S}^m) \leq \varepsilon.
\end{aligned}$$

According to the definition of the conditional expectation with respect to $\mathscr{F}$, for all $E \in \mathscr{F}$, and for $m \geq m(\varepsilon)$ we infer that

$$\begin{aligned}
\int_E l\, d\mathbf{P} &\geq \int_E L_m\, d\mathbf{P} - \varepsilon \\
&= \int_E \mathbf{E}^{\mathscr{F}_m} \frac{\mathbb{S}_{[0,m[^N}}{m^N}\, d\mathbf{P} - \varepsilon \\
&= \int_E \mathbf{E}^{\mathscr{F}}\left(\mathbf{E}^{\mathscr{F}_m} \frac{\mathbb{S}_{[0,m[^N}}{m^N}\right) d\mathbf{P} - \varepsilon \\
&= \int_E \mathbf{E}^{\mathscr{F}} \frac{\mathbb{S}_{[0,m[^N}}{m^N}\, d\mathbf{P} - \varepsilon \\
&\geq \int_E \inf_{n\in\mathbf{N}^*} f_n\, d\mathbf{P} - \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$, and since $l \leq \inf_{n\in\mathbf{N}^*} f_n$ a.s., we deduce that

$$\forall E \in \mathscr{F} \qquad \int_E l\, d\mathbf{P} = \int_E \inf_{n\in\mathbf{N}^*} f_n\, d\mathbf{P}. \tag{12.45}$$

Since, by definition, f_n is $\mathscr{F}$-measurable, so is $\inf_{n\in\mathbf{N}^*} f_n$. Equality (12.45) being true for every $E \in \mathscr{F}$, we obtain $l = \inf_{n\in\mathbf{N}^*} f_n$, which completes the proof. □

The same conclusion holds for discrete subadditive process. More precisely, we have the following.

Theorem 12.4.4. *Let* $\mathbb{S} : \mathscr{I} \longrightarrow L^1_{\mathbf{P}}(\Sigma)$ *be a discrete subadditive process, and* $(I_n)_{n\in\mathbf{N}}$ *a regular sequence of* $\mathscr{I}$ *satisfying* $\lim_{n\to+\infty} \rho(I_n) = +\infty$. *Then, for* $\mathbf{P}$*-a.e.* $\omega \in \Sigma$,

$$\lim_{n\to+\infty} \frac{\mathbb{S}_{I_n}}{|I_n|}(\omega) = \inf_{m\in\mathbf{N}^*} \mathbf{E}^{\mathscr{F}} \frac{\mathbb{S}_{[0,m[^N}}{m^N}(\omega).$$

If moreover the dynamical system $(\Sigma, \mathscr{A}, \mathbf{P}, (T_z)_{z\in\mathbf{Z}^N})$ *is ergodic, then*

$$\lim_{n\to+\infty} \frac{\mathbb{S}_{I_n}}{|I_n|}(\omega) = \inf_{m\in\mathbf{N}^*} \mathbf{E}\frac{\mathbb{S}_{[0,m[^N}}{m^N} = \gamma(\mathbb{S}).$$

For a proof, we refer the reader to [4]. Theorem 12.4.4 will be used in Section 14.2.

Remark 12.4.2. As for the deterministic case (see Lemma 12.3.1), condition (i) can be weakened by restricting the subadditivity to the sets of $\mathscr{B}_b(\mathbf{R}^N)$ whose boundary is Lebesgue negligible, more precisely: for all A and all $B \in \mathscr{B}_b(\mathbf{R}^N)$ with $A \cap B = \emptyset$ and $|\partial A| = |\partial B| = 0$, $\mathbb{S}_{A\cup B} \le \mathbb{S}_A + \mathbb{S}_B$. Indeed the boundary of the sets considered in the proof is Lebesgue negligible.

12.4.2 ▪ Parametrized subadditive processes

In this section, we assume that the dynamical system $(\Sigma, \mathscr{A}, \mathbf{P}, (T_z)_{z\in\mathbf{Z}^N})$ is ergodic. We are concerned with the variational property of the almost sure convergence stated in Theorem 12.4.3, when the subadditive process depends on a parameter which belongs to a separable metric space. For convenience, in order to use usual concepts of the calculus of variations, the process $\mathbb{S}$ will be assumed superadditive, that is, $-\mathbb{S}$ subadditive. More precisely, given a separable metric space (X,d), we consider a mapping

$$\mathbb{S} : \mathscr{B}_b(\mathbf{R}^N) \times X \to L^1_{\mathbf{P}}(\Sigma),\ (A,x) \mapsto \mathbb{S}_A(x,.)$$

fulfilling the following conditions:

(i) for all $x \in X$, $A \mapsto -\mathbb{S}_A(x,.)$ is a subadditive process, covariant with respect to $(T_z)_{z\in\mathbf{Z}^N}$;

(ii) for all $A \in \mathscr{B}_b(\mathbf{R}^N)$, $(x,\omega) \mapsto \mathbb{S}_A(x,\omega)$ is $\mathscr{B}(X)\otimes\mathscr{A}$-measurable;

(iii) for all $A \in \mathscr{B}_b(\mathbf{R}^N)$ and all $\omega \in \Sigma$, the map $x \mapsto \mathbb{S}_A(x,\omega)$ is lower semicontinuous in X;

(iv) $\exists \alpha > 0$, $\exists x_0 \in X$ such that for all $A \in \mathscr{B}_b(\mathbf{R}^N)$ and all $x \in X$, $\mathbb{S}_A(\omega,x) \ge -\alpha\big(1 + d(x,x_0)\big)$.

Such a mapping $\mathbb{S}$ will be referred to as a parametrized superadditive process covariant with respect to $(T_z)_{z\in\mathbf{Z}^N}$. Under these conditions, Theorem 12.4.5 below generalizes the epigraphical law of large numbers established in [59] (see also [240]).

Theorem 12.4.5 (almost sure Γ-convergence). *Let* $(\Sigma, \mathscr{A}, \mathbf{P}, (T_z)_{z\in\mathbf{Z}^N})$ *be an ergodic dynamical system,* $\mathbb{S}$ *a parametrized superadditive process covariant with respect to* $(T_z)_{z\in\mathbf{Z}^N}$*, and* $(A_n)_{n\in\mathbf{N}}$ *a regular sequence of convex sets of* $\mathscr{B}_b(\mathbf{R}^N)$ *satisfying* $\lim_{n\to+\infty}\rho(A_n) = +\infty$*. Then, for* $\mathbf{P}$*-a.e.* $\omega \in \Sigma$*, we have*

$$\Gamma - \lim_{n\to+\infty} \frac{\mathbb{S}_{A_n}(.,\omega)}{|A_n|} = \sup_{m\in\mathbf{N}^*} \mathbf{E}\frac{\mathbb{S}_{[0,m[^N}}{m^N} = \sup\left\{\int_\Sigma \frac{\mathbb{S}_I(.,\omega)}{|I|}\, d\mathbf{P}(\omega) : I \in \mathscr{I}\right\}.$$

PROOF. Since the map $x \mapsto \alpha\big(1+d(x,x_0)\big)$ is a continuous perturbation of $x \mapsto \frac{\mathbb{S}_{A_n}}{|A_n|}(x,\omega)$, according to (ii) of Theorem 12.1.1, it is enough to establish the Γ-convergence for the nonnegative superadditive process $A \mapsto \mathbb{S}_A(\omega,.) + \alpha\big(1+d(x,x_0)\big)|A|$. We still denote by $\mathbb{S}$ this new process.

First step. We establish the existence of Σ' in $\mathscr{A}$ satisfying $\mathbf{P}(\Sigma')=1$ and such that for all $\omega \in \Sigma'$,

$$\Gamma - \liminf_{n\to+\infty} \frac{S_{A_n}}{|A_n|}(.,\omega) \geq \sup_{m\in\mathbf{N}^*} \left\{ \int_\Sigma \frac{S_{[0,m[^N}}{m^N}(.,\omega)\, d\mathbf{P}(\omega) \right\}.$$

The crucial idea is to notice that the process defined for fixed x and fixed k (intended to go to $+\infty$) by $A \mapsto -\inf_{y\in X}\{\mathscr{S}_A(y,.) + kd(x,y)|A|\}$ is subadditive and satisfies all of the hypothesis of Theorem 12.4.3. (The measurability comes from the measurability of $\omega \mapsto \operatorname{epi} \mathscr{S}_A(.,\omega)$; see [59, 240].)

Let $D \subset X$ be a dense countable subset of X. From the consideration above, there exists $\Sigma' \in \mathscr{A}$ with $\mathbf{P}(\Sigma')=1$ such that for all $(\omega,x)\in \Sigma' \times D$

$$\begin{aligned}\lim_{n\to+\infty} \left(\frac{S_{A_n}}{|A_n|}(.,\omega) \right)^k (x) &= \sup_{m\in\mathbf{N}^*} \left\{ \int_\Sigma \left(\frac{S_{[0,m[^N}}{m^N}(.,\omega) \right)^k (x)\, d\mathbf{P}(\omega) \right\} \\ &\geq \int_\Sigma \left(\frac{S_{[0,m[^N}}{m^d}(.,\omega) \right)^k (x)\, d\mathbf{P}(\omega) \quad \forall m \in \mathbf{N}^*, \qquad (12.46)\end{aligned}$$

where, for all $A \in \mathscr{B}_b(\mathbf{R}^N)$, $(\frac{S_A}{|A|}(.,\omega))^k$ is the Baire approximation of $x \mapsto \frac{S_A}{|A|}x,\omega$ defined by

$$\left(\frac{S_A}{|A|}(.,\omega) \right)^k (x) = \inf_{y\in X} \left\{ \frac{S_A}{|A|}(y,\omega) + kd(x,y) \right\}.$$

Since the Baire approximation is Lipschitz continuous with constant k (see Theorem 9.2.1 and Remark 12.1.3), inequality (12.46) holds for all $(\omega,x)\in\Sigma'\times X$. Noticing that $(\frac{S_{[0,m[^N}}{m^d}(.,\omega))^k$ increases to $\frac{S_{[0,m[^N}}{m^d}(.,\omega)$ when $k\to+\infty$, from Proposition 12.1.1(iv), and the monotone convergence theorem, (12.46) yields

$$\Gamma - \liminf_{n\to+\infty} \frac{S_{A_n}}{|A_n|}(.,\omega) \geq \int_\Sigma \frac{S_{[0,m[^N}}{m^N}(.,\omega)\, d\mathbf{P}(\omega)$$

for all $m \in \mathbf{N}^*$. Hence

$$\Gamma - \liminf_{n\to+\infty} \frac{S_{A_n}}{|A_n|}(.,\omega) \geq \sup_{m\in\mathbf{N}^*} \left\{ \int_\Sigma \frac{S_{[0,m[^N}}{m^N}(.,\omega)\, d\mathbf{P}(\omega) \right\}.$$

Second step. We establish the existence of $\Sigma" \in \mathscr{A}$ satisfying $\mathbf{P}(\Sigma")=1$ and such that for all $\omega\in\Sigma''$

$$\Gamma - \limsup_{n\to+\infty} \frac{S_{A_n}}{|A_n|}(.,\omega) \leq \sup_{m\in\mathbf{N}^*} \left\{ \int_\Sigma \frac{S_{[0,m[^N}}{m^N}(.,\omega)\, d\mathbf{P}(\omega) \right\}.$$

For fixed $\varepsilon > 0$ and $x \in X$, letting $n \to +\infty$ in

$$\inf_{y\in B(x,\varepsilon)} \frac{S_{A_n}}{|A_n|}(\omega,y) \leq \frac{S_{A_n}}{|A_n|}(\omega,x),$$

and according to Theorem 12.4.3, we deduce that there exists $\Sigma_x \in \mathscr{A}$ satisfying $\mathbf{P}(\Sigma_x)=1$, and such that for all $\omega \in \Sigma_x$

$$\limsup_{n\to+\infty} \inf_{y\in B(x,\varepsilon)} \frac{S_{A_n}}{|A_n|}(\omega,y) \le \sup_{m\in\mathbf{N}^*} \left\{ \int_\Sigma \frac{S_{[0,m[^N}}{m^N}(\omega,x)\, d\mathbf{P}(\omega) \right\}.$$

Letting $\varepsilon \to 0$, from (12.1), we deduce that for all $x \in X$ and all $\omega \in \Sigma_x$

$$\Gamma - \limsup_{n\to+\infty} \frac{\mathbb{S}_{A_n}}{|A_n|}(x,\omega) \le \sup_{m\in\mathbf{N}^*} \left\{ \int_\Sigma \frac{\mathbb{S}_{[0,m[^N}}{m^N}(x,\omega)\, d\mathbf{P}(\omega) \right\}. \tag{12.47}$$

Let D_{epi} be a dense countable subset of the epigraph of the map

$$\Phi : x \mapsto \sup_{m\in\mathbf{N}^*} \left\{ \int_\Sigma \frac{\mathbb{S}_{[0,m[^d}}{m^d}(x,\omega)\, d\mathbf{P}(\omega) \right\},$$

$\Pi_X D_{epi}$ its projection on X, and set $\Sigma'' := \cap_{x\in\Pi_X D_{epi}} \Sigma_x$. We have $\mathbf{P}(\Sigma'')=1$. From (12.47) we infer that for all $\omega \in \Sigma''$

$$\{(x,r)\in\mathscr{D} \;:\; \Phi(x)\le r\} \subset \text{epigraph}\left(\Gamma - \limsup_{n\to+\infty} \frac{S_{A_n}}{|A_n|}(.,\omega)\right). \tag{12.48}$$

Noticing that Φ and $\Gamma - \limsup_{n\to+\infty} \frac{S_{A_n}}{|A_n|}(.,\omega)$ are lower semicontinuous, taking the closure of each two sets above, we deduce that for all $\omega \in \Sigma''$

$$\text{epigraph}(\Phi) \subset \text{epigraph}\left(\Gamma - \limsup_{n\to+\infty} \frac{S_{A_n}}{|A_n|}(.,\omega)\right).$$

Hence $\Gamma - \limsup_{n\to+\infty} \frac{\mathbb{S}_{A_n}}{|A_n|}(.,\omega) \le \Phi$ for all $\omega \in \Sigma''$.

Set $\Sigma''' = \Sigma' \cap \Sigma''$. We have $\mathbf{P}(\Sigma''')=1$ and, from the two steps above, the Γ-convergence of the process is obtained for all $\omega \in \Sigma'''$, which completes the proof. □

12.4.3 ▪ Random integrands

We denote by $\mathscr{J}_{\alpha,\beta,L}$ the subset of $\mathbf{R}^{\mathbf{R}^N\times\mathbf{M}^{m\times N}}$ made up of functions $g:\mathbf{R}^N\times\mathbf{M}^{m\times N}\to\mathbf{R}$ measurable in x and satisfying conditions (12.5), (12.6) for some given $\alpha>0$, $\beta>0$, $L>0$, and $p\in[1,+\infty[$. We equip $\mathscr{J}_{\alpha,\beta,L}$ with the σ-algebra $\mathscr{T}_{\mathscr{J}_{\alpha,\beta,L}}$, trace on $\mathscr{J}_{\alpha,\beta,L}$, of the product σ-algebra of $\mathbf{R}^{\mathbf{R}^N\times\mathbf{M}^{m\times N}}$, i.e., the smallest σ-algebra on $\mathscr{J}_{\alpha,\beta,L}$ such that all the evaluation maps

$$e_{(x,\xi)} : g \mapsto g(x,\xi),\ (x,\xi)\in\mathbf{R}^N\times\mathbf{M}^{m\times N}$$

are measurable when $\mathbf{R}$ is endowed with its Borel σ-algebra.

Let us consider a probability space $(\Sigma,\mathscr{A},\mathbf{P})$ and, for any topological space X, denote by $\mathscr{B}(X)$ its Borel σ-algebra.

Definition 12.4.5. *A function $f:\Sigma\times\mathbf{R}^N\times\mathbf{M}^{m\times N}\to\mathbf{R}$ is said to be a random integrand if it is $\big(\mathscr{A}\otimes\mathscr{B}(\mathbf{R}^N)\otimes\mathscr{B}(\mathbf{M}^{m\times N}),\mathscr{B}(\mathbf{R})\big)$ measurable and if $f(\omega,.,.)$ belongs to the class $\mathscr{J}_{\alpha,\beta,L}$ for every $\omega\in\Sigma$.*

In the literature, in the general definition of a random integrand, the class $\mathscr{J}_{\alpha,\beta,L}$ is replace by the larger class of function g measurable in x and lower semicontinuous in ξ. In what follows, we restrict ourselves to the above definition.

Given a random integrand f, the map $\tilde{f}:\Sigma\to\mathscr{J}_{\alpha,\beta,L}$ defined by $\tilde{f}(\omega)=f(\omega,.,.)$ is clearly $\left(\mathscr{A},\mathscr{T}_{\mathscr{J}_{\alpha,\beta,L}}\right)$ measurable. Denote by $\mathscr{O}$ the family of all open bounded subsets of $\mathbf{R}^N$ and consider the class $\mathscr{F}_{\alpha,\beta,L}$ of $\mathbf{R}^{W_{loc}(\mathbf{R}^N,\mathbf{R}^m)\times\mathscr{O}}$ defined by

$$\mathscr{F}_{\alpha,\beta,L}:=\{G=J(g):g\in\mathscr{J}_{\alpha,\beta,L}\},$$

where

$$J(g)(u,A):=\int_A g(x,\nabla u)\,dx,\ (u,A)\in W^{1,p}_{loc}(\mathbf{R}^N,\mathbf{R}^m)\times\mathscr{O}.$$

$\mathscr{F}_{\alpha,\beta,L}$ is endowed with the σ-algebra $\mathscr{T}_{\mathscr{F}_{\alpha,\beta,L}}$, trace of the product σ-algebra of $\mathbf{R}^{W^{1,p}_{loc}(\mathbf{R}^N,\mathbf{R}^m)\times\mathscr{O}}$, i.e., the smallest σ-algebra on $\mathscr{F}_{\alpha,\beta,L}$, such that all the evaluation maps

$$\mathscr{E}_{(u,A)}:G\mapsto G(u,A),\ (u,A)\in W^{1,p}_{loc}(\mathbf{R}^N,\mathbf{R}^m)\times\mathscr{O}$$

are measurable. It is worth noting that the map

$$J:g\to J(g)$$

from $\mathscr{J}_{\alpha,\beta,L}$ into $\mathscr{F}_{\alpha,\beta,L}$ is not $\left(\mathscr{T}_{\mathscr{J}_{\alpha,\beta,L}},\mathscr{T}_{\mathscr{F}_{\alpha,\beta,L}}\right)$ measurable in general so that we cannot deduce the measurability of $\omega\mapsto J\circ\tilde{f}(\omega)$ from the $\left(\mathscr{A},\mathscr{T}_{\mathscr{J}_{\alpha,\beta,L}}\right)$ measurability of $\omega\mapsto\tilde{f}(\omega)$. To overcome this difficulty, from now on, we equip $\mathscr{J}_{\alpha,\beta,L}$ with the smallest σ-algebra $\tilde{\mathscr{A}}$ containing $\mathscr{T}_{\mathscr{J}_{\alpha,\beta,L}}$ such that the map

$$J:\mathscr{J}_{\alpha,\beta,L}\to\mathscr{F}_{\alpha,\beta,L},\ g\mapsto J(g)$$

is $\left(\tilde{\mathscr{A}},\mathscr{T}_{\mathscr{F}_{\alpha,\beta,L}}\right)$ measurable. The explicit knowledge of $\tilde{\mathscr{A}}$ is not really necessary. Indeed we have the next proposition.

Proposition 12.4.1. *Let $f:\Omega\times\mathbf{R}^N\times\mathbf{M}^{m\times N}\to\mathbf{R}$ be a random integrand. Then $\tilde{f}:\Sigma\to\mathscr{J}_{\alpha;,\beta,L}$ is $\left(\mathscr{A},\tilde{\mathscr{A}}\right)$ measurable and the map $\omega\mapsto J\circ\tilde{f}(\omega)$ is $\left(\mathscr{A},\mathscr{T}_{\mathscr{F}_{\alpha,\beta,L}}\right)$ measurable.*

PROOF. As said above, $\tilde{f}$ is $\left(\mathscr{A},\mathscr{T}_{\mathscr{J}_{\alpha,\beta,L}}\right)$ measurable, then, according to the definition of the σ-algebra $\tilde{\mathscr{A}}$, it suffices to establish that the map $\omega\mapsto J\circ\tilde{f}(\omega)$ is $\left(\mathscr{A},\mathscr{T}_{\mathscr{F}_{\alpha,\beta,L}}\right)$ measurable, that is, from the definition of the σ-algebra $\mathscr{T}_{\mathscr{F}_{\alpha,\beta,L}}$, the maps $\omega\mapsto J(\tilde{f}(\omega))(u,A)$ are $\left(\mathscr{A},\mathscr{B}(\mathbf{R})\right)$ measurable for all $u\in W^{1,p}_{loc}(\mathbf{R}^N,\mathbf{R}^m)$ and all $A\in\mathscr{O}$. The thesis follows straightforwardly from the definition of a random integrand and the standard result on the measurability of integrals depending on a parameter. □

12.4.4 ▪ The dynamical system associated with a random integrand

Thanks to $\tilde{f}$, the "phenomenal" probability space $(\Sigma,\mathscr{A},\mathbf{P})$ is transferred into the probability space $(\mathscr{J}_{\alpha,\beta,L},\tilde{\mathscr{A}},f\#\mathbf{P})$, where $f\#\mathbf{P}$ is the image probability measure of $\mathbf{P}$, defined by $\tilde{f}\#\mathbf{P}(E) = \mathbf{P}(\tilde{f}^{-1}(E))$ for every E in $\tilde{\mathscr{A}}$. Since in what follows f is the only random integrand under consideration, in order to shorten the notation we will denote by $(\tilde{\Sigma}, \tilde{\mathscr{A}}, \tilde{\mathbf{P}})$ the transported probability space $(\mathscr{I}_{\alpha,\beta,L}, \tilde{\mathscr{A}}, \tilde{f}\#\mathbf{P})$. We recall below the well-known change of variable theorem (or transfer theorem).

Theorem 12.4.6 (transfer). *Let $X : \tilde{\Sigma} \to \mathbf{R}$ be an $\left(\tilde{\mathscr{A}}, \mathscr{B}(\mathbf{R})\right)$ measurable function. Then*

$$X \in L_{\tilde{\mathbf{P}}}(\tilde{\Sigma}) \iff X \circ \tilde{f} \in L_{\mathbf{P}}(\Sigma)$$

and, in this case,

$$\int_{\tilde{\Sigma}} X \, d\tilde{\mathbf{P}} = \int_{\Sigma} X \circ \tilde{f} \, d\mathbf{P}.$$

Let $(T_z)_{z\in\mathbf{Z}^N}$ be the group of measurable maps $T_z : \tilde{\Sigma} \to \tilde{\Sigma}$ defined by $T_z g(x,.) = g(x+z,.)$ for every $g \in \tilde{\Sigma}$ and all $x \in \mathbf{R}^N$. Then for all $z \in \mathbf{Z}^N$, T_z is $\left(\tilde{\mathscr{A}}, \tilde{\mathscr{A}}\right)$ measurable. Indeed the thesis is a straightforward consequence of the definition of $\tilde{\mathscr{A}}$, and the relation $J(T_z g)(u,A) = J(g)(u(.-z), A+z)$ for every $(u,A) \in W^{1,p}_{loc}(\mathbf{R}^N,\mathbf{R}^m) \times \mathscr{O}$.

Definition 12.4.6. *The dynamical system $(\tilde{\Sigma}, \tilde{\mathscr{A}}, \tilde{\mathbf{P}}, (T_z)_{z\in\mathbf{Z}^N})$ is referred to as the dynamical system associated with the random integrand f. Assume that $(T_z)_{z\in\mathbf{Z}^N}$ is a $\tilde{\mathbf{P}}$-preserving transformation on the measurable space $(\tilde{\Sigma}, \tilde{\mathscr{A}})$, that is, $T_z^{\#}\tilde{\mathbf{P}} = \tilde{\mathbf{P}}$ for all $z \in \mathbf{Z}^N$. Then f is said to be periodic in law, or equivalently the dynamical system $(\tilde{\Sigma}, \tilde{\mathscr{A}}, \tilde{\mathbf{P}}, (T_z)_{z\in\mathbf{Z}^N})$ is said to be stationary. If $\tilde{P}(E) \in \{0,1\}$ for every subset $E \in \tilde{\mathscr{A}}$ such that for every $z \in \mathbf{Z}^N$, $T_z(E) = E$, then f or $(\tilde{\Sigma}, \tilde{\mathscr{A}}, \tilde{\mathbf{P}}, (T_z)_{z\in\mathbf{Z}^N})$ is said to be ergodic.*

The proposition below states that the explicit knowledge of $\tilde{\mathscr{A}}$ is not essential to characterize random integrands periodic in law or ergodic.

Proposition 12.4.2. *Let f be a random integrand. Then we have*

(i) *f is periodic in law iff the laws of the random vectors*

$$\left(f(.,x_i,\xi_i)\right)_{i\in I}, \qquad \left(f(.,x_i+z,\xi_i)\right)_{i\in I}$$

are equal for every $z \in \mathbf{Z}^N$ and every finite family $(x_i,\xi_i)_{i\in I}$ in $\mathbf{R}^N \times \mathbf{M}^{m\times N}$;

(ii) *f is ergodic iff $\tilde{\mathbf{P}}(E) \in \{0,1\}$ for every subset $E \in \mathscr{T}_{\mathscr{I}_{\alpha,\beta,L}}$ such that $T_z(E) = E$ for every $z \in \mathbf{Z}^N$;*

(iii) *if f satisfies the mixing condition*

$$\begin{aligned}&\lim_{|z|\to+\infty} \mathbf{P}\left(\{\omega \in \Sigma : f(\omega,x_i,\xi_i) > s_i \ \forall i \in I,\ f(\omega,y_j+z,\zeta_j) > t_j \ \forall j \in J\}\right)\\ &= \mathbf{P}\left(\{\omega \in \Sigma : f(\omega,x_i,\xi_i) > s_i \ \forall i \in I\}\right)\mathbf{P}\left(\{f(\omega,y_j,\zeta_j) > t_j \ \forall j \in J\}\right)\end{aligned}$$

for every pair of finite families $(x_i,\xi_i,s_i)_{i\in I}$ and $(y_j,\zeta_j,t_j)_{j\in J}$ in $\mathbf{R}^N \times \mathbf{M}^{m\times N} \times \mathbf{R}$, then f is ergodic.

PROOF. From the definition of the σ-algebra $\mathscr{T}_{\mathscr{I}_{\alpha,\beta,L}}$, the condition in (i) is necessary and sufficient to ensure equality of the two probability measures $\tilde{\mathbf{P}}$ and $T_z^{\#}\tilde{\mathbf{P}}$ restricted to $\mathscr{T}_{\mathscr{I}_{\alpha,\beta,L}}$. Therefore to conclude it suffices to establish

$$\tilde{\mathbf{P}} = T_z^{\#}\tilde{\mathbf{P}} \text{ on } \mathscr{T}_{\mathscr{I}_{\alpha,\beta,L}} \Longrightarrow \tilde{\mathbf{P}} = T_z^{\#}\tilde{\mathbf{P}} \text{ on } \tilde{A},$$

that is,

$$\tilde{f}\#\mathbf{P} = T_z \circ \tilde{f}\#\mathbf{P} \text{ on } \mathscr{T}_{\mathscr{I}_{\alpha,\beta,L}} \Longrightarrow \tilde{f}\#\mathbf{P} = T_z \circ \tilde{f}\#\mathbf{P} \text{ on } \tilde{A}.$$

This last implication is a straightforward consequence of the lemma below.

Lemma 12.4.4. *Let $(g_i)_{i\in\mathbf{N}}$ be a countable family of random integrands; then for every $E \in \tilde{A}$ there exists E' in $\mathscr{T}_{\mathscr{I}_{\alpha,\beta,L}}$ such that $\tilde{g}_i\#\mathbf{P}(E\Delta E') = 0$ for every $i \in \mathbf{N}$.*

Indeed Lemma 12.4.4 yields $\tilde{g}_i\#\mathbf{P}(E) = \tilde{g}_i\#\mathbf{P}(E')$ for every $i \in \mathbf{N}$. Assertion (i) then follows by applying this equality to $\tilde{f}$ and $T_z \circ \tilde{f}$. The proof of (ii) is obtained by applying Lemma 12.4.4 to the countable family of random integrands $(T_z f)_{z\in\mathbf{Z}}$, and (iii) follows from (ii), the mixing condition (12.27), and the definition of the σ-algebra $\mathscr{T}_{\mathscr{I}_{\alpha,\beta,L}}$.

We are going to prove Lemma 12.4.4. Let $\mathscr{T}$ be the subfamily of $\tilde{\mathscr{A}}$ made up of the sets for which the thesis holds. The proof consists in establishing that $\mathscr{T} = \tilde{A}$. Clearly $\mathscr{T}$ is a σ-algebra which contains $\mathscr{T}_{\mathscr{I}_{\alpha,\beta,L}}$; therefore it is enough to prove that the map J is $(\mathscr{T}, \tilde{\mathscr{A}})$ measurable, i.e., from the definition of $\tilde{\mathscr{A}}$, that for any A in $\mathscr{O}$, any $u \in W^{1,p}_{loc}(\mathbf{R}^N,\mathbf{R}^m)$ and any $r \in \mathbf{R}$, the set

$$E := \{\varphi \in \mathscr{I}_{\alpha,\beta,L} : J(g)(u,A) > r\}$$

belongs to $\mathscr{T}$. According to Corollary 12.4.2, for every $\varphi \in \mathscr{J}_{\alpha,\beta,L}$, there exists S_φ in $\mathscr{T}_{A^{\mathbf{Z}}}$ with $\mu(S_\varphi) = 1$, such that for every sequence $s = (s_I)_{i\in\mathbf{Z}}$ of S_φ,

$$\frac{1}{|A|}\int \varphi(x,\nabla u(x))\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\varphi(s_k, Du(s_k)). \tag{12.49}$$

Let us consider the subset U of $\Sigma \times A^{\mathbf{Z}}$ made up of all the (ω, s) for which

$$\frac{1}{|A|}\int g_i(\omega,x,\nabla u(x))\,dx = \limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} g_i(\omega,s_k,Du(s_k)) \quad \forall i \in \mathbf{N}.$$

Clearly U belongs to $\mathscr{A} \otimes \mathscr{T}_{A^{\mathbf{Z}}}$ and from (12.49), $\mu\big(\{t \in A^{\mathbf{Z}} : (\omega,t) \in U\}\big) = 1$ for every $\omega \in \Sigma$. By Fubini's theorem we have

$$\begin{aligned}\int_\Sigma\left(\int_{A^{\mathbf{Z}}}\mathbf{1}_{\{t:(\omega,t)\in U\}}(s)\,d\mu(s)\right)d\mathbf{P}(\omega) = 1\\ &= \int_{A^{\mathbf{Z}}}\left(\int_\Sigma \mathbf{1}_{\{\theta:(\theta,s)\in U\}}(\omega)\,d\mathbf{P}(\omega)\right)d\mu(s)\\ &= \int_{A^{\mathbf{Z}}}\mathbf{P}\big(\{\omega\in\Sigma : (\omega,s)\in U\}\big)\,d\mu(s)\end{aligned}$$

so that $\mathbf{P}\big(\{\omega \in \Sigma : (\omega, s) \in U\}\big) = 1$ for μ for almost every $s \in A^{\mathbf{Z}}$. Consider $\bar{s} \in A^{\mathbf{Z}}$ such that $\mathbf{P}\big(\{\omega \in \Sigma : (\omega, s) \in U\}\big) = 1$. We claim that the set

$$E' := \left\{ \varphi \in \mathscr{J}_{\alpha,\beta,L} = \limsup_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(\bar{s}_k, \nabla u(\bar{s}_k)) \geq \frac{r}{|A|} \right\}$$

satisfies $\tilde{g}_i \# \mathbf{P}(E \Delta E') = 0$ for every $i \in \mathbf{N}$. Indeed E' clearly belongs to $\mathscr{T}_{\mathscr{J}_{\alpha,\beta,L}}$, and

$$\begin{aligned}
&\tilde{g}_i \# \mathbf{P}(E \Delta E') \\
&\quad \leq \mathbf{P}\left(\left\{ \omega \in \Sigma : \frac{1}{|A|} \int_A g_i(\omega, x, \nabla u(x))\, dx \neq \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} g_i(\omega, s_k, Du(s_k)) \right\} \right) \\
&\quad \leq \mathbf{P}\big(\{\omega \in \Sigma : (\omega, s) \notin U\}\big) = 0,
\end{aligned}$$

which completes the proof of Lemma 12.4.4. $\square$

Example 12.4.1 (random checkerboard-like materials). Let g_- and g_+ be two homogeneous functions in $\mathscr{J}_{\alpha,\beta,L}$, $(a,b) \in (0,1)^2$ satisfying $a + b = 1$ and consider the product $\Sigma = \{g_-, g_+\}^{\mathbf{Z}^N}$ equipped with the σ-algebra product of the trivial σ- algebra in $\{g_-, g_+\}$, and with the product probability measure $\mathbf{P} = \otimes_{z\in\mathbf{Z}^N} \mu_z$, where $\mu_z = a\delta_{g_-} + b\delta_{g_+}$ for all $z \in \mathbf{Z}^n$. By construction $\mathbf{P}$ is invariant under the shift group $(\tau_z)_{z\in\mathbf{Z}^N}$ defined by $\tau_z(\omega_t)_{t\in\mathbf{Z}^N} = (\omega_{t+z})_{t\in\mathbf{Z}^N}$, i.e., $\tau_z^{\#}\mathbf{P} = \mathbf{P}$ for all $z \in \mathbf{Z}^N$.

We define $f : \Sigma \times \mathbf{R}^N \times \mathbf{M}^{m\times N} \to \mathbf{R}$ as follows: $f(\omega, x, \xi) := \omega_z(\xi)$ whenever $x \in Y + z$, where Y is the unit cell $(0,1)^N$. According to this definition it is straightforward to show that f is a random integrand and that for all $\omega \in \Sigma$, all $x \in \mathbf{R}^N$, and all $\xi \in \mathbf{M}^{m\times N}$,

$$T_z f(\omega, x, \xi) = f(\omega, x + z, .) = f(\tau_z \omega, x, \xi)$$

so that $T_z \tilde{f}(\omega) = \tilde{f}(\tau_z \omega)$. Consequently f is periodic in law. Indeed the thesis follows from the following calculation: for every $E \in \tilde{\mathscr{A}}$

$$\begin{aligned}
T_z^{\#}\tilde{\mathbf{P}}(E) &= T_z^{\#}(\tilde{f}\#\mathbf{P})(E) \\
&= \mathbf{P}(\tilde{f}^{-1}(T_{-z}E)) \\
&= \mathbf{P}\Big(\{\omega \in \Sigma : T_z \tilde{f}(\omega) \in E\}\Big) \\
&= \mathbf{P}\Big(\{\omega \in \Sigma : \tilde{f}(\tau_z \omega) \in E\}\Big) \\
&= \tau_z^{\#}\mathbf{P}(\tilde{f}^{-1}(E)) \\
&= \mathbf{P}(\tilde{f}^{-1}(E)) := \tilde{\mathbf{P}}(E).
\end{aligned}$$

Finally it is easily seen that f satisfies the mixing condition (iii) of Proposition 12.4.2.

In the case when $N = 3$ and in the framework of nonlinear elasticity, the random integrand f may be thought of as an elastic density of a random checkerboard-like material, i.e., whose density takes two values g_- and g_+ at random on the lattice spanned by the unit cell $Y = (0,1)^3$. The probability presence of g_- is a, and that of g_+ is b.

Example 12.4.2 (heterogeneities distributed following a Poisson point process). Let Σ be the set of locally finite sequences $(\omega_i)_{i\in\mathbf{N}}$ in $\mathbf{R}^N$ and let $\mathscr{M}$ be the set of countable and locally finite sums of Dirac measures, equipped with their standard σ-algebra. Given $\lambda > 0$, we consider the Poisson point process $\omega \mapsto \mathscr{N}(\omega, .)$ with intensity $\lambda\,\mathscr{L}_N$ from the probability space $(\Sigma, \mathscr{A}, \mathbf{P})$ into $\mathbf{N}^{\mathscr{B}(\mathbf{R}^N)}$ equipped with the standard product σ-algebra, which is characterized as follows:

(i) for every $A \in \mathscr{B}_b(\mathbf{R}^N)$

$$\mathscr{N}(\omega, A) = \sum_{i \in \mathbf{N}} \delta_{\omega_i}(A);$$

(ii) for every finite and pairwise disjoint family $(A_i)_{i\in I}$ of $\mathscr{B}_b(\mathbf{R}^N)$, $(\mathscr{N}(., A_i))_{i\in I}$ are independent random variables;

(iii) for every bounded Borel set A and every $k \in \mathbf{N}$

$$P([\mathscr{N}(., A) = k]) = \lambda^k \mathscr{L}_N(A)^k \, \frac{\exp(-\lambda \mathscr{L}_N(A))}{k!}.$$

Note that for all $A \in \mathscr{B}_b(\mathbf{R}^N)$, $\mathscr{N}(\omega, A) = \#(A \cap \Omega)$, and that $\mathbf{E}(\mathscr{N}(., A)) = \lambda\,\mathscr{L}_N(A)$.

Given g_- and g_+ two homogeneous functions in $\mathscr{J}_{\alpha,\beta,L}$ and $r > 0$, we define a random integrand f by setting

$$f(\omega, x, \xi) := g_+(\xi) + \big(g_-(\xi) - g_+(\xi)\big)\min(1, \mathscr{N}(\omega, B(x, r))).$$

More explicitly we clearly have

$$f(\omega, x, \xi) = \begin{cases} g_-(\xi) & \text{if } x \in \bigcup_{i\in\mathbf{N}} B(\omega_i, r), \\ g_+(\xi) & \text{otherwise.} \end{cases}$$

According to Proposition 12.4.2(i) and (ii), it is easy to show that f is periodic in law and ergodic.

In the case when $N = 3$ and in the framework of nonlinear elasticity, the random integrand f may be thought of as an elastic density of a material with spherical heterogeneities of size r whose elastic density is g_- and whose centers are randomly distributed with a frequency λ per unit of volume. The elastic density of the matrix is g_+. It should be noted that the spheres can interpenetrate.

12.4.5 • The process $\{F_\varepsilon, F^{hom} : \varepsilon \to 0\}$

In what follows, ε denotes a sequence $(\varepsilon_n)_{n\in\mathbf{N}}$ of positive numbers ε_n going to zero when $n \to +\infty$, and we often briefly write $\varepsilon \to 0$ instead of $\lim_{n\to+\infty} \varepsilon_n = 0$. As in section 12.3.2, Ω is an open bounded subset of $\mathbf{R}^3$ which represents the interior of the reference configuration filled up by some elastic ($p > 1$) or pseudoplastic ($p = 1$) material which is clamped on a part Γ_0 of the boundary $\partial\Omega$ of Ω. But to treat more general situations, Ω is actually an open bounded subset of $\mathbf{R}^N$ with $N \in \mathbf{N}^*$. Given a probability space $(\Sigma, \mathscr{A}, \mathbf{P})$, $m \in \mathbf{N}^*$, and a random integrand, periodic in law,

$$f : \Sigma \times \mathbf{R}^N \times \mathbf{M}^{m\times N} \to \mathbf{R}$$

(recall that $f(\omega, ., .)$ belongs to the class $\mathscr{J}_{\alpha,\beta,L}$), we define the random functional integral

$$F_\varepsilon : \Omega \times L^p(\Omega, \mathbf{R}^m) \longrightarrow \mathbf{R}^+ \cup \{+\infty\}$$

by

$$F_\varepsilon(\omega,u)=\begin{cases}\displaystyle\int_\Omega f\Big(\omega,\frac{x}{\varepsilon},\nabla u\Big)\,dx & \text{if } u\in W^{1,p}_{\Gamma_0}(\Omega,\mathbf{R}^m),\\ +\infty \quad \text{otherwise.}\end{cases}$$

When $N=m=3$, $F_\varepsilon(\omega,.)$ is the random stored strain energy associated with a displacement field $u:\Omega\to\mathbf{R}^3$, and, with the notation of Section 12.3.2, the equilibrium configuration is given by the displacement field $\overline{u}_\varepsilon$ solution of the random problem

$$\inf\left\{F_\varepsilon(\omega,u)-\int_\Omega L(u): u\in L^p(\Omega,\mathbf{R}^3)\right\}.$$

The small parameter ε accounts for the size of the small and randomly distributed heterogeneities.

Following the strategy of Section 12.3.2, we are going to establish the almost sure Γ-convergence of the sequence $(F_\varepsilon)_{\varepsilon>0}$ when $L^p(\Omega,\mathbf{R}^m)$ is equipped with its strong topology. In the following proposition we characterize the density of the Γ-limit, or its singular part when $p=1$.

Proposition 12.4.3. *There exists a set Σ' in $\mathscr{A}$ with $\mathbf{P}(\Sigma')=1$ such that for all $(\omega,a)\in\Sigma'\times\mathbf{M}^{m\times N}$, and for all open bounded convex set A in $\mathbf{R}^N$ the following limit exists:*

$$f^{hom}(\omega,a)=\lim_{\varepsilon\to 0}\left[\inf\left\{\frac{1}{|A/\varepsilon|}\int_{A/\varepsilon} f(\omega,x,a+\nabla u(x))\,dx : u\in W^{1,p}_0(A/\varepsilon,\mathbf{R}^m)\right\}\right].$$

This limit does not depend on the choice of the open bounded convex set A and is given by

$$f^{hom}(\omega,a)=\inf_{n\in\mathbf{N}^*}\mathbf{E}^{\mathscr{F}}\inf\left\{\frac{1}{n^N}\int_{nY} f(\omega,y,a+\nabla u(y))\,dy : u\in W^{1,p}_0(Y,\mathbf{R}^m)\right\},$$

where $\mathscr{F}=\{\tilde f^{-1}(\tilde E):\tilde E\in\tilde{\mathscr{A}},\ T_z\tilde E=\tilde E \text{ for all } z\in\mathbf{Z}^N\}$. If f is ergodic, then

$$f^{hom}(a)=\inf_{n\in\mathbf{N}^*}\mathbf{E}\inf\left\{\frac{1}{n^N}\int_{nY} f(\omega,y,a+\nabla u(y))\,dy : u\in W^{1,p}_0(Y,\mathbf{R}^m)\right\}.$$

Moreover for all $\omega\in\Sigma'$, $f^{hom}(\omega,.)$ satisfies (12.5) *and* (12.6) *with a constant L' depending only on L, p, α, and β.*

PROOF. We begin by reason in the dynamical system $(\tilde\Sigma,\tilde{\mathscr{A}},\tilde{\mathbf{P}},(T_z)_{z\in\mathbf{Z}^N})$ associated with the random integrand f. Fix a matrix a in the subset $\mathbf{M}^{m\times N}_{\mathbf{Q}}$ of $\mathbf{M}^{m\times N}$ made up of the $m\times N$ matrices with rational entries. We claim that $\mathbb{S}:\mathscr{B}_b(\mathbf{R}^N)\to L_{\mathbf{P}}(\tilde\Sigma)$ defined for every A in $\mathscr{B}_b(\mathbf{R}^N)$ and every $\tilde\omega\in\tilde\Sigma$ by

$$\mathbb{S}:A\mapsto\left(\tilde\omega\mapsto\inf\left\{\int_{\mathring{A}}\tilde\omega(x,a+\nabla u(x))\,dx : u\in W^{1,p}_0(\mathring{A},\mathbf{R}^m)\right\}\right)$$

is a subadditive process. The fact that $\tilde\omega\mapsto\mathbb{S}_A(\tilde\omega)$ belongs to $L_{\tilde{\mathbf{P}}}(\tilde\Sigma)$ is a direct consequence of the uniform growth condition satisfied by all the elements of $\tilde\Sigma$, and the measurability may be established by standard arguments. Indeed, from (12.5) and taking $u = 0$ as an admissible function in the definition of $\mathbb{S}_A$ we deduce that

$$\mathbb{S}_A(\tilde{\omega}) \leq \beta |A|(1+|a|^p). \tag{12.50}$$

Note that from the lower growth inequality in (12.5), and according to Jensen's inequality, we infer

$$\mathbb{S}_A(\tilde{\omega}) \geq \beta |A||a|^p. \tag{12.51}$$

The proof of condition (i) in definition 12.4.3 follows point by point the proof of Proposition 12.3.1. Condition (ii) is a straightforward consequence of the definition of the group $(T_z)_{z\in\mathbf{Z}^N}$ and a change of variable. Condition (iii) is obvious. Consequently, according to Theorem 12.4.3, there exists a set $\tilde{\Sigma}_a$ in $\tilde{\mathscr{A}}$ with $\tilde{\mathbf{P}}(\tilde{\Sigma}_a) = 1$ such that for all $\tilde{\omega} \in \tilde{\Sigma}_a$ the limit

$$g^{hom}(\tilde{\omega}, a) := \lim_{\varepsilon\to 0}\left[\inf\left\{ \frac{1}{|A/\varepsilon|}\int_{A/\varepsilon} \tilde{\omega}(x, a+\nabla u(x))\,dx : u \in W_0^{1,p}(A/\varepsilon, \mathbf{R}^m)\right\}\right]$$

exists and is equal to

$$g^{hom}(\tilde{\omega}, a) = \inf_{n\in\mathbf{N}^*} \mathbf{E}^{\tilde{\mathscr{F}}} \inf\left\{ \frac{1}{n^N}\int_{nY} \tilde{\omega}(y, a+\nabla u(y))\,dy : u \in W_0^{1,p}(Y, \mathbf{R}^m)\right\},$$

where $\tilde{\mathscr{F}}$ is the σ-algebra made up of the invariant sets of $\tilde{\mathscr{A}}$ under $(T_z)_{z\in\mathbf{Z}^N}$. Let us set $\tilde{\Sigma}' = \bigcap_{a\in\mathbf{M}_{\mathbf{Q}}^{m\times N}} \tilde{\Sigma}_a$. Clearly $\tilde{\mathbf{P}}(\tilde{\Sigma}') = 1$. From condition (12.6) it is easy to show that $\frac{\mathbb{S}_A(\tilde{\omega},.)}{|A|}$ satisfies the local Lipschitz condition

$$\left| \frac{\mathbb{S}_A(\tilde{\omega}, a)}{|A|} - \frac{\mathbb{S}_A(\tilde{\omega}, b')}{|A|} \right| \leq L'|a-b'|(1+|a|^{p-1}+|b|^{p-1}) \tag{12.52}$$

for all $(a,b) \in \mathbf{M}^{m\times N} \times \mathbf{M}^{m\times N}$, where L' depends only on L, p, α, and β (for a complete proof, see [291]). Therefore $a \mapsto g^{hom}(\tilde{\omega}, a)$ satisfies (12.52) for all $(\tilde{\omega}, a)$ in $\tilde{\Sigma}' \times \mathbf{M}_{\mathbf{Q}}^{m\times N}$. From (12.52) we can extend g^{hom} on $\tilde{\Sigma}' \times \mathbf{M}^{m\times N}$: still denoting by g^{hom} this extension, for every $(\tilde{\omega}, a) \in \tilde{\Sigma}' \times \mathbf{M}^{m\times N}$, $g^{hom}(\tilde{\omega}, a) = \lim_{n\to+\infty} g^{hom}(\tilde{\omega}, a_n)$, where $(a_n)_{n\in\mathbf{N}}$ is any sequence in $\mathbf{M}_{\mathbf{Q}}^{m\times N}$ (note that from (12.52), this limit does not depend on the choice of the sequence $(a_n)_{n\in\mathbf{N}}$). By using the uniform estimate (12.52) with respect to ε, and letting $a_n \to a$, then $\varepsilon \to 0$ in the estimate

$$\begin{aligned}\left| g^{hom}(\tilde{\omega}, a) - \frac{\mathbb{S}_{A/\varepsilon}(\tilde{\omega}, a)}{|A/\varepsilon|}\right| &\leq |g^{hom}(\tilde{\omega}, a) - g^{hom}(\tilde{\omega}, a_n)| \\ &\quad + \left| g^{hom}(\tilde{\omega}, a_n) - \frac{\mathbb{S}_{A/\varepsilon}(\tilde{\omega}, a_n)}{|A/\varepsilon|}\right| \\ &\quad + \left| \frac{\mathbb{S}_{A/\varepsilon}(\tilde{\omega}, a_n)}{|A/\varepsilon|} - \frac{\mathbb{S}_{A/\varepsilon}(\tilde{\omega}, a)}{|A/\varepsilon|}\right|,\end{aligned}$$

we obtain

$$g^{hom}(\tilde{\omega}, a) := \lim_{\varepsilon\to 0}\left[\inf\left\{ \frac{1}{|A/\varepsilon|}\int_{A/\varepsilon} \tilde{\omega}(x, a+\nabla u(x))\,dx : u \in W_0^{1,p}(A/\varepsilon, \mathbf{R}^m)\right\}\right]$$

for all $(\tilde{\omega},a)$ in $\tilde{\Sigma}' \times \mathbf{M}^{m\times N}$. Set $\Sigma' = \tilde{f}^{-1}(\tilde{\Sigma}')$, and, for all $(\omega,a) \in \Sigma' \times \mathbf{M}^{m\times N}$, $f^{hom}(\omega,a) := g^{hom}(\tilde{f}(\omega),a)$. The conclusion follows from the definition of the conditional expectation, Theorem 12.4.6, and (12.50), (12.51), (12.52). □

We can now establish the main convergence result, a generalization of Theorem 12.3.2.

Theorem 12.4.7. *Assume that Ω is piecewise of class $\mathbf{C}^1$. Let f be a random integrand, periodic in law. Then, for $\mathbf{P}$ almost all ω in Σ, $(F_\varepsilon(\omega,.))_{\varepsilon>0}$ Γ-converges to the random integral functional F^{hom} defined on $\Sigma \times L^p(\Omega,\mathbf{R}^m)$ by*

(i) *case $p > 1$,*

$$F^{hom}(\omega,u) = \begin{cases} \displaystyle\int_\Omega f^{hom}(\omega,\nabla u)\,dx & \text{if } u \in W^{1,p}_{\Gamma_0}(\Omega,\mathbf{R}^m), \\ +\infty & \text{otherwise;} \end{cases}$$

(ii) *case $p = 1$,*

$$F^{hom}(\omega,u) = \begin{cases} \displaystyle\int_\Omega f^{hom}(\omega,\nabla u)\,dx + \int_\Omega (f)^{hom,\infty}\left(\omega,\frac{D^s u}{|D^s u|}\right)|D^s u| & \\ \displaystyle\qquad + \int_{\Gamma_0} (f)^{hom,\infty}(\omega,\gamma_0(u)\otimes\nu)\,d\mathscr{H}^{N-1} & \text{if } u \in BV(\Omega,\mathbf{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

where ν denotes the outer unit normal to Γ_0, γ_0 the trace operator, and $(f)^{hom,\infty}(\omega,.)$ the recession function of $f^{hom}(\omega,.)$ defined for every $a \in \mathbf{M}^{m\times N}$ by

$$(f)^{hom,\infty}(\omega,a) = \limsup_{t\to+\infty} \frac{(f)^{hom}(\omega,ta)}{t}.$$

If, moreover, f is ergodic, then the same result holds with a deterministic limit F^{hom} defined by replacing $f^{hom}(\omega,.)$ by $f^{hom}(.)$ in the expression of F^{hom} above.

The proof of Theorem 12.4.7 is the consequence of Propositions 12.4.4 and 12.4.5 below. To shorten the proofs, we do not take into account the boundary condition, i.e., the domain of F_ε is $W^{1,p}(\Omega,\mathbf{R}^m)$. For treating the general case, it suffices to reproduce exactly the proofs of Corollary 11.2.1 when $p > 1$ and Corollary 11.3.1 when $p = 1$, established in the periodic case.

Proposition 12.4.4. *For all ω in the subset Σ' of Proposition* 12.4.3, *for all u in $L^p(\Omega,\mathbf{R}^m)$, and for all sequences $(u_n)_{n\in\mathbf{N}}$ strongly converging to u in $L^p(\Omega,\mathbf{R}^m)$, we have*

$$F^{hom}(\omega,u) \le \liminf_{n\to+\infty} F_{\varepsilon_n}(\omega,u_n). \tag{12.53}$$

PROOF. In what follows ω is fixed in Σ'. Our strategy is exactly that of Proposition 12.3.2. Obviously, one may assume $\liminf_{n\to+\infty} F_{\varepsilon_n}(\omega,u_n) < +\infty$. Then, considering the nonnegative Borel measure $\mu_n(\omega) := f(\omega,\frac{\cdot}{\varepsilon_n},\nabla u_n(.))\mathscr{L}\lfloor\Omega$, for a nonrelabeled subsequence we have

$$\sup_{n\in\mathbf{N}} \mu_n(\omega)(\Omega) < +\infty.$$

Consequently, there exists a further nonrelabeled subsequence and a nonnegative Borel measure $\mu(\omega) \in \mathbf{M}(\Omega)$ such that

$$\mu_n(\omega) \rightharpoonup \mu(\omega) \quad \text{weakly in } \mathbf{M}(\Omega).$$

Let $\mu(\omega) = g(\omega)\mathscr{L}^N \lfloor \Omega + \mu^s(\omega)$ be the Lebesgue–Nikodým decomposition of $\mu(\omega)$, where $\mu^s(\omega)$ is a nonnegative Borel measure, singular with respect to the N-dimensional Lebesgue measure $\mathscr{L}\lfloor\Omega$ restricted to Ω. For establishing (12.53) it is enough to prove that

$$g(\omega)(x) \geq f^{hom}(\omega, \nabla u(x)) \quad x \text{ a.e.}, \tag{12.54}$$

$$\mu^s(\omega) \geq f^{hom,\infty}\left(\omega, \frac{D^s u}{|D^s u|}\right) |D^s u| \quad \text{when } p = 1. \tag{12.55}$$

(a) *Proof of* (12.54). Let $\rho > 0$ intended to tend to 0. With the notation of the proof of Proposition 12.3.2, for a.e. $x_0 \in \Omega$, we have

$$g(\omega, x_0) = \lim_{\rho \to 0} \frac{\mu(\omega)(B_\rho(x_0))}{|B_\rho(x_0)|}.$$

One may assume $\mu(\omega)(\partial B_\rho(x_0)) = 0$ for all but countably many $\rho > 0$, so that, from Alexandrov's theorem, Proposition 4.2.3, we have $\mu(\omega)(B_\rho(x_0)) = \lim_{n\to+\infty} \mu_n(\omega)(B_\rho(x_0))$. We finally are led to establish

$$\lim_{\rho\to 0} \lim_{n\to+\infty} \frac{\mu_n(\omega)(B_\rho(x_0))}{|B_\rho(x_0)|} \geq f^{hom}(\omega, \nabla u(x_0)) \quad \text{for a.e. } x_0 \in \Omega. \tag{12.56}$$

Let us assume for the moment that the trace of u_n on $\partial B_\rho(x_0)$ coincides with the affine function u_0 defined by $u_0(x) := u(x_0) + \langle \nabla u(x_0), x - x_0 \rangle$. It follows from Proposition 12.4.3 that

$$\begin{aligned}
&\lim_{n\to+\infty} \frac{\mu_n(\omega)(B_\rho(x_0))}{|B_\rho(x_0)|} \\
&= \lim_{n\to+\infty} \frac{1}{|B_\rho(x_0)|} \int_{B_\rho(x_0)} f\left(\omega, \frac{x}{\varepsilon_n}, \nabla u(x_0) + \nabla(u_n - u_0)\right) dx \\
&\geq \limsup_{n\to+\infty} \inf \left\{ \frac{1}{|B_\rho(x_0)|} \int_{B_\rho(x_0)} f\left(\omega, \frac{x}{\varepsilon_n}, \nabla u(x_0) + \nabla\phi\right) dx : \phi \in W_0^{1,p}(B_\rho(x_0), \mathbf{R}^m) \right\} \\
&= \lim_{n\to+\infty} \inf \left\{ \frac{1}{|\frac{1}{\varepsilon_n} B_\rho(x_0)|} \int_{\frac{1}{\varepsilon_n} B_\rho(x_0)} f(\omega, x, \nabla u(x_0) + \nabla\phi)\, dx : \phi \in W_0^{1,p}\left(\frac{1}{\varepsilon_n} B_\rho(x_0), \mathbf{R}^m\right) \right\} \\
&= f^{hom}(\omega, \nabla u(x_0)),
\end{aligned}$$

and the proof would be complete. In the general case, following point by point the proof of Proposition 12.3.2, we suitably modify u_n as in Proposition 11.2.3 into a function of $W^{1,p}(B_\rho(x_0), \mathbf{R}^m)$, which coincides with u_0 on $\partial B_\rho(x_0)$ in the trace sense, and follow the previous procedure. Recall that the additional term induced by this modification goes to zero with ρ thanks to the estimate (see Lemma 11.2.1 and Proposition 10.4.1): for a.e. $x \in \Omega$,

$$\left[\frac{1}{|B_\rho(x_0)|}\int_{B_\rho(x_0)} |u(x)-\big(u(x_0)+\nabla u(x_0)(x-x_0)\big)|^p \, dx\right]^{1/p} = o(\rho).$$

The proof of (12.56) is then complete.

(b) *Proof of* (12.55). It suffices to reproduce the proof of inequality

$$\mu^s \geq (Qf)^\infty\left(\frac{D^s u}{|D^s u|}\right)|D^s u|$$

obtained in the proof of Proposition 11.3.3 after substituting $f(\omega, \frac{x}{\varepsilon_n}, .)$ for f and, according to Proposition 12.4.3, after substituting $f^{hom}(\omega, .)$ for Qf. □

Proposition 12.4.5. *For all $\omega \in \Sigma'$ and for all u in $L^p(\Omega, \mathbf{R}^m)$, $p \geq 1$, there exists a sequence $(u_n(\omega, .))_{n \in \mathbf{N}}$ strongly converging to u in $L^p(\Omega, \mathbf{R}^m)$ such that*

$$F^{hom}(\omega, u) \geq \limsup_{n \to +\infty} F_{\varepsilon_n}(\omega, u_n).$$

PROOF. The proof will be obtained in four steps.

First step. We assume that $u = l_a$. For $\eta > 0$, let $(Q_{i,\eta})_{i \in J_\eta}$ be a finite family of open cubes $Q_{i,\eta}$ of the lattice spanned by $]0, \eta[^N$ and a finite subset I_η of J_η such that

$$\bigcup_{i \in I_\eta} Q_{i,\eta} \subset \Omega \subset \bigcup_{i \in J_\eta} Q_{i,\eta},$$
$$\left|\bigcup_{i \in J_\eta \setminus I_\eta} Q_{i,\eta}\right| < \delta(\eta), \ \lim_{\eta \to 0} \delta(\eta) = 0.$$

For each $i \in J_\eta$, consider $u_{i,\eta,n}(\omega, .) \in W_0^{1,p}(Q_{i,\eta}, \mathbf{R}^m)$ such that

$$\frac{1}{\frac{1}{\varepsilon_n}|Q_{i,\eta}|}\int_{\frac{1}{\varepsilon_n}Q_{i,\eta}} f(\omega, x, a + \nabla u_{i,\eta,n}(\omega, x))\, dx - \eta \leq \frac{\mathcal{S}_{\frac{1}{\varepsilon_n}Q_{i,\eta}}(\omega, a)}{\frac{1}{\varepsilon_n}|Q_{i,\eta}|}.$$

(When $p > 1$ one can take for $u_{i,\eta,n}$ an exact minimizer.) Note that

$$\frac{1}{|Q_{i,\eta}|}\int_{Q_{i,\eta}} f\left(\omega, \frac{x}{\varepsilon_n}, a + (\nabla u_{i,\eta,n})\left(\omega, \frac{x}{\varepsilon_n}\right)\right) dx \leq \frac{\mathcal{S}_{\frac{1}{\varepsilon_n}Q_{i,\eta}}(\omega, a)}{\frac{1}{\varepsilon_n}|Q_{i,\eta}|} + \eta. \tag{12.57}$$

Set

$$u_{\eta,n}(\omega, .) = l_a + \sum_{i \in I_\eta} \varepsilon_n u_{i,\eta,n}\left(\omega, \frac{\cdot}{\varepsilon_n}\right)\mathbf{1}_{Q_{i,\eta}},$$

extended by l_a on $\Omega \setminus \bigcup_{i \in I_\eta} \bar{Q}_{i,\eta}$. Clearly $u_{\eta,n}$ belongs to $W^{1,p}(\Omega, \mathbf{R}^m)$. From (12.57), (12.5), (12.50) and by using Poincaré's inequality in $Q_{i,\eta}$, one easily obtains

$$|u_{\eta,n}(\omega, .) - l_a|_{L^p(\Omega, \mathbf{R}^m)} \leq C\eta^p, \tag{12.58}$$

where $C>0$ depends only on α, β, p, and $|\Omega|$. According to Proposition 12.4.3, from (12.57) and (12.50), we infer

$$\begin{aligned}
&F^{hom}(\omega,u)\\
&\geq \sum_{i\in I_\eta} |Q_{i,\eta}| f^{hom}(a)\\
&\geq \sum_{i\in I_\eta} \lim_{n\to+\infty} |Q_{i,\eta}| \frac{\mathbb{S}_{\frac{1}{\varepsilon_n}Q_{i,\eta}}(\omega,a)}{\frac{1}{\varepsilon_n}|Q_{i,\eta}|}\\
&\geq \limsup_{n\to+\infty} \int_{\bigcup_{i\in I_\eta} Q_{i,\eta}} f\left(\omega,\frac{x}{\varepsilon_n},\nabla u_{\eta,n}(\omega,x)\right)dx-\eta|\Omega|\\
&=\limsup_{n\to\infty}\left(\int_\Omega f\left(\omega,\frac{x}{\varepsilon_n},\nabla u_{\eta,n}(\omega,x)\right)dx-\int_{\bigcup_{i\in J_\eta\setminus I_\eta} Q_{i,\eta}} f\left(\omega,\frac{x}{\varepsilon_n},\nabla u_{\eta,n}(\omega,x)\right)dx\right)-\eta|\Omega|\\
&\geq \limsup_{n\to\infty}\int_\Omega f\left(\omega,\frac{x}{\varepsilon_n},\nabla u_{\eta,n}(\omega,x)\right)dx-\delta(\eta)\beta(1+|a|^p)-\eta|\Omega|.
\end{aligned}$$

Letting $\eta\to 0$ yields

$$F^{hom}(\omega,u)\geq \limsup_{\eta\to 0}\limsup_{n\to\infty}\int_\Omega f\left(\omega,\frac{x}{\varepsilon_n},\nabla u_{\eta,n}(\omega,x)\right)dx,$$

so that, by using a diagonalization argument (see [37, Corollary 1.16]), there exists a map $n\mapsto\eta(n)$ (possibly depending on ω) satisfying $\lim_{n\to+\infty}\eta(n)=0$ and such that

$$F^{hom}(\omega,u)\geq \limsup_{n\to+\infty} F_n(\omega,u_{\eta(n),n}(\omega,.)).$$

On the other hand, from (12.58), $\lim_{n\to+\infty} u_{\eta(n),n}(\omega,.)=l_a$ strongly in $L^p(\Omega,\mathbf{R}^m)$. The function $u_n(\omega,.):=u_{\eta(n),n}(\omega,.)$ satisfies the assertion of Proposition 12.4.5.

Second step. We assume that u belongs to the space $\mathrm{Aff}(\Omega,\mathbf{R}^m)$ made up of continuous piecewise affine functions on Ω. Then there exists a finite family $(\Omega_i)_{i\in I}$ of pairwise disjoint open subsets of Ω, piecewise of class $\mathbf{C}^1$ such that $|\Omega\setminus\bigcup_{i\in I}\Omega_i|=0$ and $(a_i,b_i)\in\mathbf{M}^{m\times N}\times\mathbf{R}^m$ such that $u\lfloor\Omega_i=l_{a_i}+b_i$ for all i in I. According to the first step, there exists $u_{i,n}(\omega,.)$ in $W^{1,p}(\Omega_i,\mathbf{R}^m)$ strongly converging to u in $L^p(\Omega_i,\mathbf{R}^m)$ such that for every $i\in I$

$$\int_{\Omega_i} f^{hom}(\omega,\nabla u)\,dx(\omega,u)\geq \limsup_{n\to+\infty}\int_{\Omega_i} f\left(\omega,\frac{x}{\varepsilon_n},\nabla u_{i,n}(\omega,x)\right)dx.$$

We can modify $u_{i,n}(\omega,.)$ in a neighborhood of the boundary of each Ω_i into a function of $W^{1,p}(\Omega_i,\mathbf{R}^m)$, still denoted by $u_{n,i}(\omega,.)$, such that

$$\begin{aligned}
&u_{i,n}(\omega,.)=u \text{ on } \partial\Omega_i,\quad u_{i,n}(\omega,.)\to u \text{ strongly in } L^p(\Omega_i,\mathbf{R}^m)\\
&\int_{\Omega_i} f^{hom}(\omega,\nabla u)\,dx\geq \limsup_{n\to+\infty}\int_{\Omega_i} f\left(\omega,\frac{x}{\varepsilon_n},\nabla u_{i,n}(\omega,x)\right)dx.
\end{aligned}$$

(See the proof of Corollary 11.2.1.) Set $u_n(\omega,.) := \sum_{i\in I} u_{n,i}(\omega,.)\mathbf{1}_{\Omega_i}$. From the above we infer that $u_n(\omega,.) \in W^{1,p}(\Omega,\mathbf{R}^m)$, $u_n(\omega,.) \to u$ strongly in $L^p(\Omega,\mathbf{R}^m)$ and

$$\begin{aligned} F^{hom}(\omega,u) &\geq \sum_{i\in I} \limsup_{n\to+\infty} \int_{\Omega_i} f\left(\omega,\frac{x}{\varepsilon_n},\nabla u_{i,n}(\omega,x)\right) dx \\ &\geq \limsup_{n\to+\infty} \sum_{i\in I} \int_{\Omega_i} f\left(\omega,\frac{x}{\varepsilon_n},\nabla u_{i,n}(\omega,x)\right) dx \\ &= \limsup_{n\to+\infty} \int_{\Omega} f\left(\omega,\frac{x}{\varepsilon_n},\nabla u_n(\omega,x)\right) dx. \end{aligned}$$

The function $u_n(\omega,.)$ satisfies the assertion of Proposition 12.4.5.

Third step. We assume that u belongs to $W^{1,p}(\Omega,\mathbf{R}^m)$. We conclude from the previous step, by the density of $\mathrm{Aff}(\Omega,\mathbf{R}^m)$ in $W^{1,p}(\Omega,\mathbf{R}^m)$. Indeed from (12.5) and (12.6), $u \mapsto F^{hom}(\omega,u)$ is strongly continuous in $W^{1,p}(\Omega,\mathbf{R}^m)$. Therefore there exists a sequence $(u_m)_{m\in\mathbf{N}}$ in $\mathrm{Aff}(\Omega,\mathbf{R}^m)$ strongly converging to u in $W^{1,p}(\Omega,\mathbf{R}^m)$ such that

$$F^{hom}(\omega,u) = \lim_{m\to+\infty} F^{hom}(\omega,u_m). \tag{12.59}$$

On the other hand, from the second step, and Proposition 12.4.4, for each u_m there exists a sequence $(u_{m,n}(\omega,.))_{n\in\mathbf{N}}$ in $W^{1,p}(\Omega,\mathbf{R}^m)$ strongly converging to u_m in $L^p(\Omega,\mathbf{R}^m)$ such that

$$F^{hom}(\omega,u_m) = \lim_{n\to+\infty} \int_{\Omega} f\left(\omega,\frac{x}{\varepsilon_n},\nabla u_{m,n}(\omega,x)\right) dx. \tag{12.60}$$

We conclude from (12.59) and (12.60) by applying the diagonalization Lemma 11.1.1 to the sequence

$$\left(F_{\varepsilon_n}(\omega,u_{m,n}(\omega,.)),u_{m,n}(\omega,.)\right)_{m,n}$$

in the metric space $\mathbf{R}\times L^p(\Omega,\mathbf{R}^m)$. If $p>1$, the proof of Proposition 12.4.5 is complete.

Last step ($p=1$). The conclusion follows from the previous step and the second step of the proof of Proposition 12.3.3. □

12.5 ▪ Application to image segmentation and phase transitions

12.5.1 ▪ The Mumford–Shah model

Let Ω be a bounded open subset of $\mathbf{R}^N$ and g a given function in $L^\infty(\Omega)$. Denoting by $\mathscr{F}$ the class of the closed sets of Ω, for all K in $\mathscr{F}$ and all u in $\mathbf{C}^1(\Omega\setminus K)$ we deal with the functional

$$E(u,K) := \int_\Omega |u-g|^2\,dx + \int_{\Omega\setminus K} |\nabla u|^2\,dx + \mathscr{H}^{N-1}(K)$$

and the associated optimization problem

$$\inf\{E(u,K) : (u,K)\in \mathbf{C}^1(\Omega\setminus K)\times\mathscr{F}\}. \tag{12.61}$$

When Ω is a rectangle in $\mathbf{R}^2$ and $g(x)$ is the light signal striking Ω at a point x, (12.61) is the Mumford–Shah model of image segmentation: K may be considered as the outline of the given light image in computer vision. If it exists, a solution (u^*,K^*) of (12.61) fulfills the three following properties:

(i) the first term in $E(u,K)$ asks that u^* approximates the light signal g in $L^2(\Omega)$;

(ii) in $\Omega\setminus K^*$, u^* does not vary very much (because of the term $\int_{\Omega\setminus K}|\nabla u^*|^2\,dx$);

(iii) the third term asks that the boundaries K^* be as short as possible.

Let us remark that dropping one of the three terms makes the problem trivial, i.e.,

$$\inf\{E(u,K)\,:(u,K)\in \mathbf{C}^1(\Omega\setminus K)\times\mathscr{F}\}=0.$$

Indeed, when $E(u,K)=\int_{\Omega\setminus K}|\nabla u|^2\,dx+\mathscr{H}^{N-1}(K)$, take $u^*=0$ and $K^*=\emptyset$.

When $E(u,K):=\int_\Omega|u-g|^2\,dx+\mathscr{H}^{N-1}(K)$, take $u^*=g$ and $K^*=\emptyset$.

When $E(u,K):=\int_\Omega|u-g|^2\,dx+\int_{\Omega\setminus K}|\nabla u|^2\,dx$, let us decompose Ω by a finite union of open cubes $Q_{i,\eta}$ with diameter η and boundary $K_{i,\eta}$,

$$\mathscr{L}^N\Bigg(\Omega\setminus\bigcup_{i\in I(\eta)}Q_{i,\eta}\Bigg)=0,\quad K_\eta=\bigcup_{i\in I(\eta)}K_{i,\eta},$$

and set

$$u_{i,\eta}:=\frac{1}{|Q_{i,\eta}|}\int_{Q_{i,\eta}}g(x)\,dx,\quad u_\eta:=\sum_{i\in I(\eta)}u_{i,\eta}1_{Q_{i,\eta}}.$$

Then $E(u_\eta,K_\eta)$ tends to zero when η goes to zero. In this last case (14.14) obviously has no solution if g is not constant.

To fit several applications to computer vision problems, one can adjust the functional E by suitable positive constants α, β, and γ and consider

$$E(u,K):=\alpha\int_\Omega|u-g|^2\,dx+\beta\int_{\Omega\setminus K}|\nabla u|^2\,dx+\gamma\mathscr{H}^{N-1}(K).$$

In what follows, to shorten notation, we set $\alpha=\beta=\gamma=1$.

The existence of a solution for the optimization problem (12.61) was conjectured in [308] and has been established in [196] by using the semicontinuity and the compactness results of Ambrosio [16] related to functionals defined in SBV spaces (see Chapter 13). They defined a weak formulation of (12.61) as follows. If (12.61) has a solution (u^*,K^*), the closed set K^* must contain the jump set of u^*. Then, it is natural to solve the problem in $SBV(\Omega)$ and to consider K^* as the closure of the set S_{u*}. That leads us to consider the following weak formulation of the problem (12.61):

$$\inf\left\{\int_\Omega|\nabla u|^2\,dx+\mathscr{H}^{N-1}(S_u)+\int_\Omega|u-g|^2\,dx : u\in SBV(\Omega)\right\},\tag{12.62}$$

where ∇u denotes the density of the Lebesgue part of Du and S_u the jump set of u (see Section 10). The functional E defined on $SBV(\Omega)$ by $E(u)=\int_\Omega|\nabla u|^2\,dx+\mathscr{H}^{N-2}(S_u)+\int_\Omega|u-g|^2\,dx$ will be referred to as the Mumford–Shah energy functional. In Section 14.3 we will establish the following existence result.

Theorem 12.5.1. *There exists at least a solution of the weak problem* (12.62).

12.5.2 ▪ Variational approximation of a more elementary problem: A phase transitions model

To describe a numerical processing of the weak formulation (problem (12.62)), a natural way consists in approximating in a variational sense the Mumford–Shah energy functional by classical integral functionals. Before treating the Mumford–Shah energy, we begin by showing how the Van Der Waals–Cahn–Hilliard thermodynamical model of phase transitions allows us to define a good approximation of the term $\mathscr{H}^{N-1}(S_u)$. For another and more direct method in one or two dimensions (i.e., $N = 1, 2$), we refer the reader to Chambolle [168]. For nonlocal variational approximations of the Mumford–Shah functional we refer the reader to Braides and Dal Maso [126], Gobbino [231], Cortesani and Toader [179], and references therein.

Let Ω be an open bounded subset of $\mathbf{R}^N$, $m > 0$, $0 < \alpha < \beta$ be such that α meas$(\Omega) \leq m \leq \beta$ meas(Ω), and $SBV(\Omega : \{\alpha, \beta\})$ be the subspace of all functions of $SBV(\Omega)$ taking only the two values α or β. We consider the following problem:

$$\inf\left\{ \mathscr{H}^{N-1}(S_u) : u \in SBV(\Omega : \alpha, \beta),\ \int_\Omega u\, dx = m \right\}.$$

As said above, the thermodynamical model of phase transition provides an analogous estimate of this problem. Indeed, consider the functional F_ε defined in $L^1(\Omega)$ by

$$F_\varepsilon(u) = \begin{cases} c_0 \displaystyle\int_\Omega \left(\sqrt{\varepsilon}\, |Du|^2 + \frac{1}{\sqrt{\varepsilon}} W(u) \right) dx & \text{if } u \in H^1(\Omega),\ u \geq 0,\ \displaystyle\int_\Omega u\, dx = m, \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$c_0 = \left(2 \int_\alpha^\beta \sqrt{W(t)}\, dt \right)^{-1},$$

ε is a positive parameter intended to go to zero, and $W : [0, +\infty[\to \mathbf{R}$ is a nonnegative, continuous function with exactly two zeros α, β $(0 < \alpha < \beta)$. The integral functional $c_0^{-1} F_\varepsilon$ is the rescaled Van Der Waals–Cahn–Hilliard energy functional by the ratio $1/\sqrt{\varepsilon}$ and W is a thermodynamical potential of a liquid with constant mass m confined to a bounded container Ω under isothermal conditions and whose density distribution u presents two phases α and β. More precisely, the Van Der Waals–Cahn–Hilliard energy is the functional

$$u \mapsto \int_\Omega W(u)\, dx + \mathrm{Ind}_{\{\int_\Omega v(x)\, dx = m,\ v \geq 0\}}(u) + \varepsilon \int_\Omega |Du|^2 dx,$$

where

$$\mathrm{Ind}_{\{\int_\Omega v(x)\, dx = m,\ v \geq 0\}}(u) = \begin{cases} 0 & \text{if } \displaystyle\int_\Omega u(x)\, dx = m,\ u \geq 0, \\ +\infty & \text{otherwise.} \end{cases}$$

The thickness L of the transition between the two phases is given by

$$L = \sqrt{\varepsilon} \frac{\beta - \alpha}{2 \int_\alpha^\beta \sqrt{W(\tau)}\, d\tau}.$$

Since L is very small, so is ε and the Van Der Waals–Cahn–Hilliard free energy is nothing but a perturbation of the Gibbs free energy functional

$$G : u \mapsto \int_\Omega W(u)\, dx + \mathrm{Ind}_{\{\int_\Omega v(x)\, dx = m,\ v \geq 0\}}(u)$$

by the functional $H : u \mapsto \varepsilon \int_\Omega |Du|^2\, dx$.

The first Gibbs model, which consists in minimizing G, is unsatisfactory. It is indeed easily seen that the set arg min (G) is made by infinitely many piecewise constant functions u taking the value α in an arbitrary subset A of Ω with measure $(\beta \mathrm{meas}(\Omega) - m)/(\beta - \alpha)$, and the value β in $\Omega \setminus A$, with no restriction on the shape of the interface between $[u = \alpha]$ and $[u = \beta]$. In particular, there is no way to recover the physical criterion: the interface has minimal area. This criterion may be recovered by the new model, consisting in minimizing the functional F_ε. We point out that because of $\mathrm{arg\,min}(G) \cap \mathrm{dom}\,(H) = \emptyset$, this last model is a (viscosity) singular perturbation of the first one. For a general study of viscosity perturbations consult Attouch [39].

Modica proved in [293] the following result, previously established in the special case $N = 1$ by Gurtin [235].

Theorem 12.5.2. *The sequence $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges to the functional F defined by*

$$F(u) = \begin{cases} \mathcal{H}^{N-1}(S_u) & \text{if } u \in SBV(\Omega : \alpha, \beta), \text{ and } \displaystyle\int_\Omega u\, dx = m, \\ +\infty & \text{otherwise} \end{cases}$$

in $L^1(\Omega)$ equipped with its strong topology.

Assume moreover that W satisfies the following polynomial behavior at infinity: there exist $t_0 > 0, c_1 > 0,\ c_2 > 0,\ k \geq 2$ such that for all $t \geq t_0$

$$c_1 t^k \leq W(t) \leq c_2 t^k.$$

Then the set $\{u_\varepsilon : \varepsilon \to 0\}$ of minimum points of F_ε has a compact closure in $L^1(\Omega)$, and any cluster point u is a minimum point of F.

PROOF. We only give the proof of the lower bound in the definition of Γ-convergence and establish the compactness result. For a complete proof, consult [293] or the proof of Proposition 12.5.2. We begin by substituting $\sqrt{\varepsilon}$ by ε and we omit the constant c_0 in the definition of F_ε. The expected Γ-limit must be

$$F(u) = \begin{cases} c_0^{-1} \mathcal{H}^{N-1}(S_u) & \text{if } u \in SBV(\Omega : \alpha, \beta), \text{ and } \displaystyle\int_\Omega u\, dx = m, \\ +\infty & \text{otherwise,} \end{cases}$$

which is actually the asymptotic model of Van Der Waals–Cahn–Hilliard.

First step. We begin by proving that for all v in $L^1(\Omega)$ and all sequence $(v_\varepsilon)_{\varepsilon>0}$ strongly converging to v in $L^1(\Omega)$, one has

$$\liminf_{\varepsilon \to 0} F_\varepsilon(v_\varepsilon) \geq F(v).$$

The proof given here is based on a general method described in [351]. One may assume, for a subsequence not relabeled, that $\liminf_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon) = \lim_{\varepsilon \to 0} F_\varepsilon(v_\varepsilon) = C < +\infty$, where

C is a nonnegative constant which does not depend on ε. We then deduce

$$\begin{cases} v \geq 0, \ \int_\Omega v\,dx = m, \\ \int_\Omega W(v_\varepsilon)\,dx \leq C\varepsilon. \end{cases}$$

According to the continuity of W and Fatou's lemma, the last inequality yields

$$\int_\Omega W(v)\,dx \leq \liminf_{\varepsilon\to 0} \int_\Omega W(v_\varepsilon)\,dx \leq 0,$$

so that $W(v(x)) = 0$ a.e. and v takes only the two values α and β. Note that since truncations operate on $H^1(\Omega)$, v is also the strong limit of the truncated functions $\tilde{v}_\varepsilon = \alpha \vee v_\varepsilon \wedge \beta$. Moreover, from the definition of W which achieves its infimum at α and β, $F_\varepsilon(v_\varepsilon) \geq F_\varepsilon(\tilde{v}_\varepsilon)$. According to these remarks, keeping the same notation, we will replace v_ε by $\tilde{v}_\varepsilon$. The elementary Young inequality yields

$$F_\varepsilon(v_\varepsilon) \geq 2\left(\int_\Omega W(v_\varepsilon)\,dx\right)^{1/2}\left(\int_\Omega |Dv_\varepsilon|^2\,dx\right)^{1/2}.$$

This estimate is optimal and may be recovered by studying the map $\varepsilon \mapsto F_\varepsilon(u)$ for a fixed u in $H^1(\Omega)$ whose minimum point is

$$\varepsilon = \left(\frac{\int_\Omega W(u)\,dx}{\int_\Omega |Du|^2\,dx}\right)^{1/2}$$

and for which the minimal value is

$$2\left(\int_\Omega W(u)\,dx\right)^{1/2}\left(\int_\Omega |Du|^2\,dx\right)^{1/2}.$$

By the Cauchy–Schwarz inequality we obtain

$$\begin{aligned} \liminf_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon) &\geq 2\liminf_{\varepsilon\to 0} \int_\Omega \sqrt{W(v_\varepsilon)}|Dv_\varepsilon|\,dx \\ &= 2\liminf_{\varepsilon\to 0} \int_\Omega |D\big(\psi(v_\varepsilon)\big)|\,dx, \end{aligned}$$

where $\psi(t) = \int_\alpha^t \sqrt{W(s)}\,ds$. Since $v_\varepsilon \to v$ strongly in $L^1(\Omega)$, $\alpha \leq v_\varepsilon \leq \beta$, and ψ is continuous, we deduce that $\psi(v_\varepsilon)$ strongly converges to $\psi(v)$ in $L^1(\Omega)$. According to Proposition 10.1.1, we finally deduce that $\psi(v)$ belong to $BV(\Omega)$ and

$$\liminf_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon) \geq 2\int_\Omega |D\psi(v)|. \tag{12.63}$$

Let us compute this last integral. Since

$$\psi(v) = \begin{cases} 0 \quad \text{on } [v = \alpha], \\ \int_\alpha^\beta \sqrt{W(s)}\,ds \quad \text{on } [v = \beta], \end{cases}$$

the function $\psi(v)$ is a simple function of $BV(\Omega)$ and

$$\int_\Omega |D\psi(v)| = \left(\int_\alpha^\beta \sqrt{W(s)}\,ds\right)\mathcal{H}^{N-1}(S_v).$$

Inequality (12.63) finally gives

$$\liminf_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon) \geq \left(2\int_\alpha^\beta \sqrt{W(s)}\,ds\right)\mathcal{H}^{N-1}(S_v), = F(v),$$

concluding the first step.

Second step. Let us now establish the relative compactness in $L^1(\Omega)$ of the set $\{u_\varepsilon : \varepsilon \to 0\}$ of minimum points of F_ε. The letter C will denote various positive constants. Consider $v_\varepsilon = \psi \circ u_\varepsilon$, where ψ is the primitive of the function $W^{1/2}$ defined above. Let us first prove the relative compactness of the set $\{v_\varepsilon : \varepsilon \to 0\}$. From the polynomial behavior of W and the fact that $k/2+1 \leq k$, we have for all $t \geq t_0$,

$$\begin{aligned}\psi(t) &\leq \int_\alpha^{t_0} W^{1/2}(s)\,ds + \int_{t_0}^t W^{1/2}(s)\,ds\\ &\leq C(1+W(t)),\end{aligned}$$

which yields

$$\int_\Omega v_\varepsilon\,dx \leq C(1+\sqrt{\varepsilon}F_\varepsilon(u_\varepsilon))$$

and gives the boundedness in $L^1(\Omega)$ of v_ε. On the other hand, from the proof of the lower bound above

$$\int_\Omega |Dv_\varepsilon|\,dx \leq \frac{1}{2}F_\varepsilon(v_\varepsilon),$$

which finally gives the boundedness of v_ε in $BV(\Omega)$. The relative compactness of $\{v_\varepsilon : \varepsilon \to 0\}$ is a consequence of the compactness of the embedding $BV(\Omega) \hookrightarrow L^1(\Omega)$.

Let us now go back to the functions u_ε. Let v be a strong limit in $L^1(\Omega)$ of a nonrelabeled subsequence of v_ε, consider the inverse function ψ^{-1} of ψ, and set $u = \psi^{-1} \circ v$. We establish the strong convergence of u_ε to u in $L^1(\Omega)$. We proceed as follows: we prove the equi-integrability of u_ε and the convergence in measure of u_ε to u (see, for instance, Marle [287]). From the polynomial behavior of W, we have

$$\begin{aligned}\int_\Omega |u_\varepsilon|^k\,dx &\leq t_0^k \operatorname{meas}(\Omega) + C\int_\Omega W(u_\varepsilon)\,dx\\ &\leq C(1+\sqrt{\varepsilon}F_\varepsilon(u_\varepsilon)) \leq C\end{aligned}$$

and equi-integrability follows from $k \geq 2$. On the other hand, since $\psi'(t) \geq \sqrt{c}_1 t_0^{k/2}$ for all $t > t_0$, ψ^{-1} is a Lipschitz function on $[\psi(t_0), +\infty)$ and hence uniformly continuous on $\mathbf{R}^+$. Therefore, u_ε converges in measure to u. □

For a numerical approach, it suffices now to establish the Γ-convergence of the discretization $F_{\varepsilon,h(\varepsilon)}$ of the functional F_ε by finite elements, to the functional F, with a suitable choice of the size h of discretization. For this study, consult Bellettini [89].

12.5.3 ▪ Variational approximation of the Mumford–Shah functional energy

When neglecting the functional $u \mapsto \int_\Omega |u-g|^2\,dx$ in the expression of the Mumford–Shah functional, to control the jumps of admissible functions, a natural domain is the space $GSBV(\Omega)$ of generalized special functions of bounded variation defined by

$$GSBV(\Omega) := \left\{ u : \Omega \to \mathbf{R} : u \text{ Borel function}, k \wedge u \vee (-k) \in SBV(\Omega)\ \forall k \in \mathbf{N} \right\}.$$

It can be shown (see Ambrosio and Tortorelli [30]) that to each function $u \in GSBV(\Omega)$ there corresponds a Borel function $\nabla u : \Omega \to \mathbf{R}^N$ and $S_u \subset \Omega$ such that $\nabla u = \nabla(k \wedge u \vee (-k))$ a.e. on $[|u| \le k]$ for all $k \in \mathbf{N}$ and $\mathcal{H}^{N-1}(S_{k\wedge u \vee(-k)}) \to \mathcal{H}^{N-1}(S_u)$ when $k \to +\infty$.

Following the strategy of the previous subsection, as in Ambrosio and Tortorelli [30], we establish that the functional F defined in $X := L^1(\Omega) \times L^1(\Omega, [0,1])$ equipped with its strong topology by

$$F(u,s) = \begin{cases} \displaystyle\int_\Omega |\nabla u|^2\,dx + \mathcal{H}^{N-1}(S_u) & \text{if } u \in GSBV(\Omega) \text{ and } s = 1, \\ +\infty & \text{otherwise} \end{cases}$$

can be approximated, in the sense of Γ-convergence, by the functionals defined in X by

$$F_\varepsilon(u,s) = \begin{cases} \displaystyle\int_\Omega (s^2+\varepsilon^2)|\nabla u|^2\,dx + M_\varepsilon(s,\Omega) & \text{if } (u,s) \in \mathbf{C}^1(\Omega) \times \mathbf{C}^1(\Omega,[0,1]) \cap X, \\ +\infty & \text{otherwise.} \end{cases}$$

For all open subsets A of Ω and all s in $\mathbf{C}^1(\Omega,[0,1])$, $M_\varepsilon(.,A)$ denotes the integral functional

$$M_\varepsilon(s,A) := \int_A \left(\varepsilon |\nabla s|^2 + \frac{1}{4\varepsilon}(1-s)^2 \right) dx.$$

The second argument s is, as we will see in the proof, a control parameter on the gradient. The approximation of the Mumford–Shah energy will be the functional G_ε defined by $G_\varepsilon(u,s) = F_\varepsilon(u,s) + \int_\Omega |u-g|^2\,dx$. Indeed, $u \mapsto \int_\Omega |u-g|^2\,dx$ is a continuous perturbation of F_ε and the conclusion will follow from Theorem 12.1.1(ii).

We assume that Ω satisfies the following "reflection condition" $(\mathcal{R})$ on $\partial\Omega$: there exists an open neighborhood U of $\partial\Omega$ in $\mathbf{R}^N$ and a one-to-one Lipschitz function $\varphi : U \cap \Omega \longrightarrow U \setminus \overline{\Omega}$ such that φ^{-1} is Lipschitz.

Theorem 12.5.3. *Assume that Ω satisfies condition $(\mathcal{R})$. Then the sequence of functionals $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges to the functional F.*

The proof proceeds with Propositions 12.5.1 and 12.5.2. We denote the strong topology of $L^1(\Omega) \times L^1(\Omega,[0,1])$ by τ, and the letter C will denote various positive constants which do not depend on ε. We point out that condition $(\mathcal{R})$ is not necessary for obtaining the lower bound in Proposition 12.5.1.

Proposition 12.5.1. *For all $(u,s) \in X$ and all sequences $((u_\varepsilon, s_\varepsilon))_\varepsilon$ τ-converging to (u,s), we have $F(u,s) \le \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, s_\varepsilon)$, or equivalently, $F \le \Gamma - \liminf_{\varepsilon \to 0} F_\varepsilon$.*

PROOF. Obviously, one may assume $\Gamma - \liminf_{\varepsilon\to 0} F_\varepsilon(u,s) < +\infty$ and $s = 1$.

First step. We assume $u \in L^\infty(\Omega)$ and establish the proposition in the one-dimensional case $N = 1$ when Ω is a bounded interval I in $\mathbf{R}$. When I is not an interval, it suffices to argue on each connected component of I and to conclude thanks to the superadditivity of $\Gamma-\liminf_{\varepsilon\to 0} F_\varepsilon$. When working on a bounded open subset A of $\mathbf{R}$, we will denote F and $\Gamma-\liminf_{\varepsilon\to 0} F_\varepsilon$ by $F(.,A)$ and $\Gamma-\liminf_{\varepsilon\to 0} F_\varepsilon(.,A)$, respectively. The key point of the proof is the following lemma.

Lemma 12.5.1. *Assume u in $L^\infty(I)$ and fix $x_0\in I$.*

(i) *If there exists $\eta>0$ such that for all $\rho<\eta$, $u\notin W^{1,2}(B_\rho(x_0))$, then for all $\rho<\eta$*

$$\Gamma-\liminf_{\varepsilon\to 0} F_\varepsilon((u,1),B_\rho(x_0))\geq 1.$$

(ii) *If there exists $\rho>0$ such that $u\in W^{1,2}(B_\rho(x_0))$, then*

$$\Gamma-\liminf_{\varepsilon\to 0} F_\varepsilon((u,1),B_\rho(x_0))\geq \int_{B_\rho(x_0)} |\nabla u|^2\,dx.$$

Assume for the moment that the proof of Lemma 12.5.1 is established. We claim that the set $E:=\{x\in I:\exists\eta>0\ \forall\rho<\eta,\ u\notin W^{1,2}(B_\rho(x_0))\}$ is finite. Indeed, otherwise E would contain an infinite countable subset $D=\{x_i, i\in\mathbf{N}\}$. For all n in $\mathbf{N}$ and ρ small enough such that $B_\rho(x_i)$, $i=0,\ldots,n$, are pairwise disjoint sets, we would have from (i), superadditivity, and nondecreasing properties of $A\mapsto\Gamma-\liminf_{\varepsilon\to 0} F_\varepsilon(.,A)$,

$$+\infty>\Gamma-\liminf_{\varepsilon\to 0} F_\varepsilon((u,1),I)\geq\sum_{i=0}^{n}\Gamma-\liminf_{\varepsilon\to 0}\ F_\varepsilon((u,1),B_\rho(x_i))\geq n.$$

This being true for all $n\in\mathbf{N}$, we obtain a contradiction.

The set E is then made up of a finite number of points $x_0,\ldots,x_n$ and it is easily seen that $u\in W^{1,2}(I\setminus E)$. From $\mathscr{H}^0(E)<+\infty$, we deduce $u\in SBV(I)$ and $E=S_u$. For ρ small enough as previously, according to (ii) of Lemma 12.5.1, we have

$$\begin{aligned}\Gamma-\liminf_{\varepsilon\to 0} F_\varepsilon((u,1),I)&\geq\Gamma-\liminf_{\varepsilon\to 0} F_\varepsilon\left((u,1),I\setminus\overline{\bigcup_{x\in S_u}B_\rho(x_0)}\right)\\&\quad+\Gamma-\liminf_{\varepsilon\to 0} F_\varepsilon\left((u,1),\bigcup_{x\in S_u}B_\rho(x_0)\right)\\&\geq\int_{I\setminus\overline{\bigcup_{x\in S_u}B_\rho(x_0)}}|\nabla u|^2\,dx+\mathscr{H}^0(S_u).\end{aligned}$$

We conclude the step by letting $\rho\to 0$ in the above inequality.

It remains to establish assertions (i) and (ii) of Lemma 12.5.1. Let $(u_\varepsilon,s_\varepsilon)\in X\cap\mathbf{C}^1(I)\times\mathbf{C}^1(I,[0,1])$ τ-converging to $(u,1)$ and satisfying

$$\liminf_{\varepsilon\to 0} F_\varepsilon((u_\varepsilon,s_\varepsilon),B_\rho(x_0))<+\infty.$$

For proving (i), we establish the existence of x_ε, x'_ε, x''_ε in $B_\rho(x_0)$ such that $x'_\varepsilon<x_\varepsilon<x''_\varepsilon$ and satisfying $\lim_{\varepsilon\to 0}s_\varepsilon(x_\varepsilon)=0$, $\lim_{\varepsilon\to 0}s_\varepsilon(x'_\varepsilon)=\lim_{\varepsilon\to 0}s_\varepsilon(x''_\varepsilon)=1$, for a nonrelabeled subsequence. Let us assume this result for the moment. We conclude as follows: by convexity inequality (precisely $a^2+b^2\geq 2ab$),

$$
\begin{aligned}
F_\varepsilon(u_\varepsilon,s_\varepsilon) &\geq M_\varepsilon(s_\varepsilon,B_\rho(x_0))\\
&\geq \int_{B_\rho(x_0)} (1-s_\varepsilon)|\nabla s_\varepsilon|\,dx\\
&\geq \int_{x'_\varepsilon}^{x_\varepsilon} (1-s_\varepsilon)|\nabla s_\varepsilon|\,dx + \int_{x_\varepsilon}^{x''_\varepsilon} (1-s_\varepsilon)|\nabla s_\varepsilon|\,dx\\
&\geq \left|\int_{x'_\varepsilon}^{x_\varepsilon} (1-s_\varepsilon)\nabla s_\varepsilon\,dx\right| + \left|\int_{x_\varepsilon}^{x''_\varepsilon} (1-s_\varepsilon)\nabla s_\varepsilon\,dx\right|\\
&= \left|\left[-\frac{(1-s_\varepsilon)^2}{2}\right]_{x'_\varepsilon}^{x_\varepsilon}\right| + \left|\left[-\frac{(1-s_\varepsilon)^2}{2}\right]_{x_\varepsilon}^{x''_\varepsilon}\right|,
\end{aligned}
$$

which tends to 1 when $\varepsilon \to 0$.

We are going to establish the existence of x_ε, x'_ε, and x''_ε. In what follows, we argue with various nonrelabeled subsequences and C denotes various positive constants independent of ε. Let $\sigma < \rho$ and set $m_\varepsilon := \inf_{B_\sigma(x_0)} s_\varepsilon$. From $F_\varepsilon((u_\varepsilon,s_\varepsilon),B_\rho(x_0)) \leq C$, we derive

$$
m_\varepsilon^2 \int_{B_\sigma(x_0)} |\nabla u_\varepsilon|^2\,dx \leq C.
$$

Up to a subsequence, m_ε converges to some l, $0 \leq l \leq 1$. We claim that $l = 0$. Otherwise,

$$
\lim_{\varepsilon\to 0} \int_{B_\sigma(x_0)} |\nabla u_\varepsilon|^2\,dx \leq \frac{C}{l^2}
$$

and u_ε would weakly converge to u in $W^{1,2}(B_\sigma(x_0))$, which is in contradiction with $u \notin W^{1,2}(B_\rho(x_0))$ for all $\rho < \eta$. Consequently, there exists $x_\varepsilon \in \overline{B_\sigma(x)}$, satisfying $\lim_{\varepsilon\to 0} s_\varepsilon(x_\varepsilon) = \lim_{\varepsilon\to 0} m_\varepsilon = 0$. On the other hand, estimates

$$
\int_{x_0-\rho}^{x_0-\sigma} \frac{(1-s_\varepsilon)^2}{4\varepsilon}\,dx \leq C, \quad \int_{x_0+\sigma}^{x_0+\rho} \frac{(1-s_\varepsilon)^2}{4\varepsilon}\,dx \leq C
$$

and the mean value theorem yield, for a subsequence, the existence of x'_ε and x''_ε satisfying the required assertions.

Let us show (ii). According to

$$
\int_{B_\rho(x_0)} s_\varepsilon^2 |\nabla u_\varepsilon|^2\,dx \leq F_\varepsilon(u_\varepsilon,s_\varepsilon),
$$

it is enough to establish the inequality

$$
\int_{B_\rho(x_0)} |\nabla u|^2\,dx \leq \liminf_{\varepsilon\to 0} \int_{B_\rho(x_0)} s_\varepsilon^2 |\nabla u_\varepsilon|^2\,dx
$$

when $F_\varepsilon(u_\varepsilon,s_\varepsilon)$ is equibounded. Set $v_\varepsilon = (1-s_\varepsilon)^2$. The equiboundedness of ∇v_ε in $L^1(I)$ will provide the following uniform control on v_ε: for all $\delta > 0$, there exists a finite part J_δ of I such that

$$
\text{for all compact subsets } K,\ K \subset I \setminus J_\delta, \text{ one has } \limsup_{\varepsilon\to 0} \left(\sup_K v_\varepsilon\right) < \delta. \tag{12.64}
$$

We will then deduce $\liminf_{\varepsilon\to 0}\left(\inf_K s_\varepsilon\right) > 1 - \delta^{1/2}$.

Let us assume for the moment estimate (12.64). We have for all $\delta > 0$ and all compact subset K with $K \subset I \setminus J_\delta$,

$$C \geq \int_{B_\rho(x_0)} s_\varepsilon^2 |\nabla u_\varepsilon|^2 \, dx \geq \int_{B_\rho(x_0)\cap K} s_\varepsilon^2 |\nabla u_\varepsilon|^2 \, dx$$
$$\geq \inf_K (s_\varepsilon^2) \int_{B_\rho(x_0)\cap K} |\nabla u_\varepsilon|^2 \, dx.$$

Therefore

$$C \geq \liminf_{\varepsilon\to 0} \int_{B_\rho(x_0)} s_\varepsilon^2 |\nabla u_\varepsilon|^2 \, dx \geq (1-\delta^{\frac{1}{2}})^2 \liminf_{\varepsilon\to 0} \int_{B_\rho(x_0)\cap K} |\nabla u_\varepsilon|^2 \, dx,$$

and the weak convergence of u_ε to u in $W^{1,2}(B_\rho(x_0)\cap K)$ yields, by lower semicontinuity,

$$\liminf_{\varepsilon\to 0} \int_{B_\rho(x_0)} s_\varepsilon^2 |\nabla u_\varepsilon|^2 \, dx \geq (1-\delta^{\frac{1}{2}})^2 \int_{B_\rho(x_0)\cap K} |\nabla u|^2 \, dx.$$

The conclusion (ii) follows after letting K to I and $\delta \to 0$.

We are now going to establish (12.64). We claim that $\sup_\varepsilon \int_I |\nabla v_\varepsilon| \, dx < +\infty$. Indeed, by convexity

$$+\infty \geq 2M_\varepsilon(s_\varepsilon, I) \geq \int_I 2(1-s_\varepsilon)|\nabla s_\varepsilon| \, dx$$
$$= \int_I |\nabla v_\varepsilon| \, dx.$$

Let $\sigma > 0$ satisfying $\delta > \sigma$ and consider for all t in $\mathbf{R}$, the sets $A_\varepsilon^t := [v_\varepsilon \leq t]$. According to the classical coarea formula, more precisely to Corollary 4.2.2, we have

$$C \geq \int_I |\nabla v_\varepsilon| \, dx = \int_{-\infty}^{+\infty} \mathcal{H}^0([v_\varepsilon = t]) \, dt$$
$$\geq \int_\sigma^\delta \mathcal{H}^0([v_\varepsilon = t]) \, dt.$$

Therefore, there exists $t_\varepsilon \in]\sigma, \delta[$ such that $\mathcal{H}^0([v_\varepsilon = t_\varepsilon]) \leq \frac{C}{\delta-\sigma}$. The set $A_\varepsilon^{t_\varepsilon}$ has then at most $k = [\frac{C}{\delta-\sigma}]$ connected components with k independent of ε: more precisely, there exists a family $(I_\varepsilon^i)_{i=1,\dots,k}$ of intervals (possibly empty) such that $A_\varepsilon^{t_\varepsilon} = \bigcup_{i=1}^k I_\varepsilon^i$. For every $i = 1, \dots, k$, consider the interval $I_\infty^i = \bigcup_N \bigcap_{n\geq N} I_{\varepsilon_n}^i$. The complementary of the union of k intervals $I_\infty := \bigcup_{i=1}^k \overset{\circ}{I}{}_\infty^i$ is the required finite part I_δ of I. Indeed, since v_ε converges a.e. to zero,

$$\operatorname{meas}(I_\infty) = \operatorname{meas}\left(\bigcup_{i=1}^k \bigcup_N \bigcap_{n\geq N} I_{\varepsilon_n}^i\right) = \operatorname{meas}\left(\bigcup_N \bigcap_{n\geq N} A_{\varepsilon_n}^{t_n}\right)$$
$$\geq \operatorname{meas}\left(\bigcup_N \bigcap_{n\geq N} [v_{\varepsilon_n} \leq \sigma]\right)$$
$$= \operatorname{meas}(I)$$

so that $I_\delta = I \setminus I_\infty$ possesses k elements. Finally, if K is a compact set included in I_δ, arguing on each interval $\overset{\circ}{I^i_\infty}$, we have $K \cap \overset{\circ}{I^i_\infty} \subset\subset I^i_\infty$ and, for N large enough

$$K \cap \overset{\circ}{I^i_\infty} \subset \bigcap_{n \geq N} I^i_{\varepsilon_n} \subset \bigcap_{n \geq N} [v_{\varepsilon_n} \leq \delta].$$

Second step. We establish Proposition 12.5.1 in the N-dimensional case, $N > 1$. We will use the same notation for the functionals considered in the one-dimensional and the N-dimensional case.

We begin by assuming $u \in L^\infty(\Omega)$. Let $(u_\varepsilon, s_\varepsilon)$ be a sequence in X converging to $(u, 1)$ such that $\liminf_{\varepsilon \to 0} F_\varepsilon((u_\varepsilon, s_\varepsilon), \Omega) < +\infty$ and A any open subset of Ω. With the notation and definitions of Theorem 10.5.2 for all $\nu \in S^{N-1}$ and for a subsequence not relabeled, $(u_{\varepsilon,x}, s_{\varepsilon,x})$ strongly converges in $L^1(A_x) \times L^1(A_x, [0,1])$ for $\mathcal{H}^{N-1}$ a.e. x in A_ν. (It's an easy consequence of Fubini's theorem.) On the other hand,

$$\begin{aligned}\int_{A_\nu} \liminf_{\varepsilon \to 0} F_\varepsilon((u_{\varepsilon,x}, s_{\varepsilon,x}), A_x)\, d\mathcal{H}^{N-1} &\leq \liminf_{\varepsilon \to 0} \int_{A_\nu} F_\varepsilon((u_{\varepsilon,x}, s_{\varepsilon,x}), A_x)\, d\mathcal{H}^{N-1} \\ &= \liminf_{\varepsilon \to 0} F_\varepsilon((u_\varepsilon, s_\varepsilon), A) < +\infty.\end{aligned}$$

Thus, for $\mathcal{H}^{N-1}$ a.e. x in A_ν, $\liminf_{\varepsilon \to 0} F_\varepsilon((u_{\varepsilon,x}, s_{\varepsilon,x}), A_x) < +\infty$. One may apply the result of the first step: for $\mathcal{H}^{N-1}$ a.e. x in A_ν, u_x belongs to $SBV(A_x) \cap L^\infty(A_x)$ and

$$\liminf_{\varepsilon \to 0} F_\varepsilon((u_{\varepsilon,x}, s_{\varepsilon,x}), A_x) \geq \int_{A_x} |\nabla u_x|^2 + \mathcal{H}^0(S_{u_x} \cap A_x).$$

Integrating this inequality over A_ν, according to Theorem 10.5.2, we deduce that for all open subset A of Ω, $u \in SBV(A)$ and

$$\begin{aligned}\liminf_{\varepsilon \to 0} F_\varepsilon((u_\varepsilon, s_\varepsilon), A) &\geq \int_{A_\nu} \liminf_{\varepsilon \to 0} F_\varepsilon((u_{\varepsilon,x}, s_{\varepsilon,x}), A_x)\, d\mathcal{H}^{N-1} \\ &\geq \int_{A_\nu} \int_{A_x} |\nabla u_x|^2\, dt\, d\mathcal{H}^{N-1}(x) + \int_{A_\nu} \mathcal{H}^0(S_{u_x} \cap A_x)\, d\mathcal{H}^{N-1}(x) \\ &= \int_A |\nabla u . \nu|^2\, dx + \int_A |\nu_u . \nu|\, d\mathcal{H}^{N-1} \lfloor_{S_u}.\end{aligned}$$

We conclude thanks to Lemma 4.2.2 and Example 4.2.2.

If now u is not assumed to belong to $L^\infty(\Omega)$, by a truncation argument we have

$$\begin{aligned}\Gamma - \liminf_{\varepsilon \to 0} F_\varepsilon((u, s), \Omega) &\geq \Gamma - \liminf_{\varepsilon \to 0} F_\varepsilon((N \wedge u \vee (-N), s), \Omega) \\ &\geq F((N \wedge u \vee (-N), s), \Omega).\end{aligned}$$

Letting $N \to +\infty$ gives the thesis. ☐

Proposition 12.5.2. *For all $(u, s) \in X$ there exists a subsequence $((u_\varepsilon, s_\varepsilon))_\varepsilon$ τ-converging to (u, s) such that $F(u, s) \geq \limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, s_\varepsilon)$ or, equivalently, $F \geq \Gamma - \limsup_{\varepsilon \to 0} F_\varepsilon$.*

PROOF. One may assume u in $SBV(\Omega) \cap L^\infty(\Omega)$. Indeed, if $u \in GSBV(\Omega)$, an easy truncation argument gives the thesis. For a given $u \in SBV(\Omega) \cap L^\infty(\Omega)$, it suffices to construct $(u_\varepsilon, s_\varepsilon)$ in $H^1(\Omega) \times H^1(\Omega, [0,1]) \cap X$, τ-converging to $(u, 1)$ in X and satisfying

$$F(u, s) \geq \limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, s_\varepsilon).$$

The expression of F_ε has indeed a sense in $H^1(\Omega)\times H^1(\Omega,[0,1])\cap X$. Moreover, $\mathbf{C}^1(\Omega)\times \mathbf{C}^1(\Omega,[0,1])$ is dense in $H^1(\Omega)\times H^1(\Omega,[0,1])$ equipped with its strong topology and F_ε is continuous for this topology which is stronger than τ. The conclusion will follow by a diagonalization argument.

First step. Let $a_\varepsilon, b_\varepsilon, c_\varepsilon$ be three sequences in $\mathbf{R}^+$ going to zero which will be adjusted later in a suitable way. The idea consists in modifying $(u,1)$ in a neighborhood of S_u to obtain, from the expression of $F_\varepsilon(u_\varepsilon, s_\varepsilon)$, an equivalent of $\mathscr{H}^{N-1}(S_u)$. We begin by assuming the following regularity condition on S_u:

$$\lim_{\rho\to 0}\frac{\text{mes}(\Omega\cap(S_u)_\rho)}{2\rho}=H_{N-1}(S_u), \tag{12.65}$$

where $(S_u)_\rho$ is the tubular neighborhood $\{x\in\mathbf{R}^N:\ \text{d}(x,S_u)<\rho\}$ of order ρ of S_u. In what follows, for any $t\in\mathbf{R}^+$, $(S_u)_t$ denotes a tubular neighborhood of order $t>0$ of S_u. We construct u_ε in $H^1(\Omega)$ satisfying $u_\varepsilon=u$ in $\Omega\setminus(S_u)_{a_\varepsilon}$ and such that its gradient satisfies

$$|\nabla u_\varepsilon(x)|\le\frac{C}{a_\varepsilon} \tag{12.66}$$

a.e. in $(S_u)_{a_\varepsilon}$. Consider now s_ε in $H^1(\Omega,[0,1])$ such that

$$s_\varepsilon=\begin{cases}0 & \text{on } (S_u)_{a_\varepsilon},\\ 1-c_\varepsilon & \text{on } \Omega\setminus(S_u)_{a_\varepsilon+b_\varepsilon}.\end{cases}$$

The positive constant c_ε is introduced for technical reasons and, as said before, will be adjusted later. We have $(u_\varepsilon,s_\varepsilon)\in H^1(\Omega)\times H^1(\Omega,[0,1])\cap X$ and

$$\begin{aligned}F_\varepsilon((u_\varepsilon,s_\varepsilon),\Omega)=&\int_{\Omega\setminus(S_u)_{a_\varepsilon}}(s_\varepsilon^2+\varepsilon^2)|\nabla u|^2\,dx+\int_{(S_u)_{a_\varepsilon}}\varepsilon^2|\nabla u_\varepsilon|^2\,dx\\&+\frac{c_\varepsilon^2}{4\varepsilon}\text{meas}(\Omega\setminus(S_u)_{a_\varepsilon+b_\varepsilon})+\frac{1}{4\varepsilon}\text{meas}((S_u)_{a_\varepsilon})\\&+M_\varepsilon(s_\varepsilon,(S_u)_{a_\varepsilon+b_\varepsilon}\setminus(S_u)_{a_\varepsilon}).\end{aligned}$$

The first term trivially goes to $\int_\Omega|\nabla u|^2\,dx$ when ε goes to zero. We adjust a_ε so that the second and fourth terms go to zero. For this, it suffices, thanks to (12.66), to select an intermediate power of ε between ε^2 and ε, for instance, $a_\varepsilon=\varepsilon^{3/2}$. To make the third term vanish, it suffices to choose $c_\varepsilon\le\sqrt{\varepsilon}$, for instance, $c_\varepsilon=\varepsilon^{3/4}$. We are reduced to finding s_ε satisfying

$$\limsup_{\varepsilon\to 0}M_\varepsilon(s_\varepsilon,(S_u)_{a_\varepsilon+b_\varepsilon}\setminus(S_u)_{a_\varepsilon})\le H_{N-1}(S_u).$$

Let us denote the map $d(.)=\text{dist}(.,S_u)$ by d. We try to find s_ε of the form $\sigma_\varepsilon\circ d$. Applying the coarea formula Theorem 4.2.5 to the function $g=\varepsilon|\sigma_\varepsilon'\circ d|^2+\frac{(1-\sigma_\varepsilon\circ d)^2}{4\varepsilon}$ and to the truncated function $f=a_\varepsilon\vee d\wedge(a_\varepsilon+b_\varepsilon)$ of d, we obtain

$$M_\varepsilon(s_\varepsilon,(S_u)_{a_\varepsilon+b_\varepsilon}\setminus(S_u)_{a_\varepsilon})=\int_{a_\varepsilon}^{a_\varepsilon+b_\varepsilon}\left(\varepsilon|\sigma_\varepsilon'(t)|^2+\frac{(1-\sigma_\varepsilon(t))^2}{4\varepsilon}\right)H_{N-1}([d=t])\,dt.$$

Consider $h(t)=\text{meas}([d<t])$. Then, according to Corollary 4.2.3, $h'(t)=H_{N-1}([d=t])$, and

$$M_\varepsilon(s_\varepsilon,(S_u)_{a_\varepsilon+b_\varepsilon}\setminus(S_u)_{a_\varepsilon})=\int_{a_\varepsilon}^{a_\varepsilon+b_\varepsilon}\left(\varepsilon|\sigma_\varepsilon'(t)|^2+\frac{(1-\sigma_\varepsilon(t))^2}{4\varepsilon}\right)h'(t)\,dt.$$

The function σ_ε is chosen as the solution of the ordinary boundary value problem

$$\begin{cases} \sigma_\varepsilon' = \frac{1-\sigma_\varepsilon}{2\varepsilon}, \\ \sigma_\varepsilon(a_\varepsilon) = 0, \ \sigma_\varepsilon(a_\varepsilon + b_\varepsilon) = 1 - c_\varepsilon, \end{cases}$$

that is, $\sigma_\varepsilon(t) = 1 - \exp(\frac{a_\varepsilon - t}{2\varepsilon})$ when we choose $b_\varepsilon = -\varepsilon \ln(\varepsilon^{3/2})$. On the other hand, the regularity assumption on S_u yields for all $\eta > 0$ the existence of ε_0 such that for all $\varepsilon < \varepsilon_0$ and all $t < a_\varepsilon + b_\varepsilon$, one has $h(t) \leq 2t(H_{N-1}(S_u) + \eta)$. Thanks to this estimate, the conclusion then follows by integrating by parts the expression $M_\varepsilon(s_\varepsilon, (S_u)_{a_\varepsilon + b_\varepsilon} \setminus (S_u)_{a_\varepsilon})$. (For details see [30, Proposition 5.1].)

Second step. We do not assume hypothesis (12.65). To apply the first step, we construct a sequence u_η converging to u in $L^2(\Omega)$ such that S_{u_η} satisfies (12.65) and such that $F(u) = \lim_{\eta\to 0} F(u_\eta)$. Afterwards, it will suffice to apply the procedure of the first step to the function u_η and to conclude by a diagonalization argument.

For constructing u_η, the idea consists in finding u_η as a solution of the Mumford–Shah problem

$$\inf\left\{ \int_{\Omega'} |\nabla v|^2 + \mathcal{H}^{N-1}(S_v) + \frac{1}{\eta}\int_{\Omega'} |v - \overline{u}|^2\, dx : v \in SBV(\Omega') \right\}, \qquad (\mathcal{P}_\eta)$$

where $\Omega' = \Omega \cup U$ and $\overline{u}$ is the extension of u on Ω' defined by

$$\overline{u}(x) = \begin{cases} u(\varphi^{-1}(x)) \text{ if } x \in U \setminus \overline{\Omega}, \\ \gamma_0(u) \text{ if } x \in \partial\Omega, \\ u(x) \text{ if } x \in \Omega. \end{cases}$$

U and φ are given by the regularity condition $(\mathcal{R})$ fulfilled by Ω and γ_0 denotes the trace operator.

We next use the following regularity property related to Mumford–Shah solutions (see Ambrosio and Tortorelli [30] and De Giorgi, Carriero, and Leaci [196]): $\mathcal{H}^{N-1}(\overline{S}_{u_\eta} \cap \Omega' \setminus S_{u_\eta}) = 0$ and, for all compact set K included in $\overline{S}_{u_\eta} \cap \Omega'$,

$$\lim_{\rho\to 0} \frac{\text{meas}((K)_\rho)}{2\rho} = \mathcal{H}^{N-1}(K).$$

Taking $K = \overline{\Omega} \cap \overline{S_{u_\eta}}$, we obtain the required regularity on u_η in Ω.

Obviously, u_η converges to u in $L^2(\Omega)$ thanks to the penalization parameter $1/\eta$ in $(\mathcal{P}_\eta)$. It remains to establish the convergence of $F(u_\eta)$ to $F(u)$. Consider the two Borel measures μ_η and μ in $\mathbf{M}^+(\Omega')$ defined for all Borel set B in Ω', by

$$\begin{cases} \mu_\eta(B) := \displaystyle\int_B |\nabla u_\eta|^2\, dx + \mathcal{H}^{N-1}(B \cap S_{u_\eta}), \\ \mu(B) := \displaystyle\int_B |\nabla \overline{u}|^2\, dx + \mathcal{H}^{N-1}(B \cap S_{\overline{u}}). \end{cases}$$

It is worth noticing that $\overline{u}$ has no jump through $\partial\Omega$ so that $\mu(\partial\Omega) = 0$. Taking $v = \overline{u}$ as a test function in $(\mathcal{P}_\eta)$, we obtain

$$\limsup_{\eta\to 0} \mu_\eta(\Omega') \leq \mu(\Omega').$$

On the other hand, according to Theorem 13.4.3, which we will establish in the next chapter, we have for all open subset A of Ω'

$$\mu(A) \leq \liminf_{\eta \to 0} \mu_\eta(A).$$

According to Proposition 4.2.5, we deduce that μ_η narrow converges to μ. Since $\mu(\partial\Omega) = 0$, we have $\mu_\eta(\Omega) \to \mu(\Omega)$. $\square$