Chapter 14

Application in mechanics and computer vision

14.1 ▪ Problems in pseudoplasticity

14.1.1 ▪ Introduction

This section is devoted to the study of the equilibrium of a three-dimensional elastoplastic material occupying a bounded domain $\Omega \subset \mathbf{R}^3$ as reference configuration and subjected to body and surface forces. The unknown displacement vector field u solves a minimization problem of the form

$$\inf\left\{ \int_\Omega W(\varepsilon(v))\,dx - L(v) : v \in \mathscr{A} \right\},$$

where $\varepsilon(v)$ denotes the linearized strain tensor $\varepsilon_{i,j}(v) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$. The linear mapping $v \mapsto L(v)$ accounts for the exterior loading and is of the form

$$L(v) = \int_\Omega f \cdot v\; dx + \int_{\Gamma_1} g \cdot v\; d\mathscr{H}^2,$$

where f denotes the body forces and g the surface forces on a part Γ_1 of the boundary. The body is assumed to be clamped on $\Gamma_0 = \Gamma \setminus \overline{\Gamma}_1$ with $\mathscr{H}^2(\Gamma_0) > 0$. We will denote the space of 3×3 symmetric matrices by $\mathbf{M}_S$ and its subspace of matrices with null trace by $\mathbf{M}_S^D$, so that $\mathbf{M}_S = \mathbf{M}_S^D \oplus \mathbf{R}I$. The constitutive equation of the material is such that the restriction W^D of the stored energy density W to $\mathbf{M}_S^D$ satisfies a linear growth at infinity. The admissible set $\mathscr{A}$ of displacement fields is a subset of the space $\{v \in LD(\Omega) \;:\; v = 0 \text{ on } \Gamma_0\}$, where

$$LD(\Omega) = \{v \in L^1(\Omega, \mathbf{R}^3) : \varepsilon(v) \in L^1(\Omega, \mathbf{M}_S)\}.$$

From a mathematical point of view, due to the linear growth of W^D at infinity, two difficulties may appear:

• The value of the infimum may be infinite. This problem leads to the theory of *yield design* (or *limit load*), which consists in analyzing the set of $\lambda \in \mathbf{R}^+$ such that

$$\inf\left\{ \int_\Omega W(\varepsilon(v))\,dx - \lambda L(v) \;:\; v \in \mathscr{A} \right\} > -\infty.$$

From the mechanical point of view, this analysis predicts the load capacity of the structure. For more details about limit analysis, see Section 15.4.

• Even if the infimum is finite, the problem has generally no solution. This is not surprising from a mechanical point of view, since the observed displacements are sometimes discontinuous on surfaces. A well-adapted space must contain admissible displacements with discontinuity on two-dimensional surfaces. This space, derived from $BV(\Omega,\mathbf{R}^3)$ where the measure $\varepsilon(u)$ plays the role of the measure Du, is precisely

$$BD(\Omega) := \left\{v \in L^1(\Omega,\mathbf{R}^3) \ : \ \varepsilon(u) \in \mathbf{M}(\Omega,\mathbf{M}_S)\right\}.$$

It will be described in detail in Subsection 14.1.3. The integral functional $\int_\Omega W(\varepsilon(u))$ of the measure $\varepsilon(u)$ will be defined in Subsection 14.1.2. Unfortunately, the new problem

$$\inf\left\{ \int_\Omega W(\varepsilon(v)) - L(v) \ : \ v \in \tilde{\mathscr{A}} \right\},$$

where now the set $\tilde{\mathscr{A}}$ of admissible displacements is a subset of the space

$$\{v \in BD(\Omega) \ : \ v = 0 \text{ on } \Gamma_0\},$$

has generally no solution. Indeed, plastification phenomena may appear on the boundary, and discontinuities are sometimes observed on the part Γ_0 of the boundary. The boundary condition $v = 0$ in the trace sense must be replaced by a surface energy of the form

$$\int_{\Gamma_0} W^\infty(\gamma_0(v)_\tau \otimes_s \nu)\, d\mathscr{H}^2.$$

We have denoted by γ_0 the trace operator and by ν the exterior unit normal to Γ_0. The symmetric matrix field $\gamma_0(v)_\tau \otimes_s \nu$ will be defined further. Roughly, the function W^∞ describes the behavior of W at infinity on straight lines generated by $\gamma_0(v)_\tau \otimes_s \nu$. The set of admissible displacement fields, still denoted by $\tilde{\mathscr{A}}$, is now a subset of $BD(\Omega)$ whose elements satisfy a weaker boundary condition and will be described further.

The mathematical theory of relaxation introduced in Chapter 11 allows us to sum up this discussion as follows:

$$\inf\left\{ \int_\Omega W(\varepsilon(v)) + \int_{\Gamma_0} W^\infty(\gamma_0(v)_\tau \otimes_s \nu)\, d\mathscr{H}^2 - L(v) \ : \ v \in \tilde{\mathscr{A}} \right\} \tag{$\overline{\mathscr{P}}$}$$

is the relaxed problem of

$$\inf\left\{ \int_\Omega W(\varepsilon(v))\, dx - L(v) \ : \ v \in \mathscr{A} \right\}, \tag{$\mathscr{P}$}$$

that is,

$$v \mapsto \int_\Omega W(\varepsilon(v)) + \int_{\Gamma_0} W^\infty(\gamma_0(v)_\tau \otimes_s \nu)\, d\mathscr{H}^2 + \mathrm{Ind}_{\tilde{\mathscr{A}}}$$

is the lower semicontinuous envelope of $v \mapsto \int_\Omega W(\varepsilon(v))\, dx + \mathrm{Ind}_{\mathscr{A}}$ when $BD(\Omega)$ is equipped with its weak convergence. Moreover $\inf(\mathscr{P}) = \min(\overline{\mathscr{P}})$. The next sections are devoted to a precise description and to a proof of this relaxation scheme.

14.1.2 ▪ The Hencky model

To illustrate the previous general considerations, we deal with the description of the Hencky model. The reference configuration Ω is assumed to have a boundary of class $\mathbf{C}^1$. Let λ and μ be two given positive constants, namely, the Lamé coefficients of the material, and set $k = \lambda + 2\mu/3$, the compression stiffness. The constitutive equation of the material is such that there exists a potential W of the form

$$W(E) = W^D(E^D) + \frac{k}{2}\left(\mathrm{tr}(E)\right)^2$$

for all $E = E^D + (1/3)\,\mathrm{tr}(E)I$ in $\mathbf{M}_S = \mathbf{M}_S^D \otimes \mathbf{R}I$. The density W^D is more precisely defined by $W^D(E^D) = \phi(|E^D|)$, where $\Phi : \mathbf{R}^+ \longrightarrow \mathbf{R}$ has a quadratic growth up to $\frac{k}{\mu\sqrt{2}}$ and a linear growth beyond this threshold. More precisely,

$$\phi(s) = \begin{cases} \mu s^2 \text{ if } s \le \dfrac{k}{\mu\sqrt{2}}, \\[2ex] s\, k\sqrt{2} - \dfrac{k^2}{2\mu} \text{ if } s \ge \dfrac{k}{\mu\sqrt{2}}. \end{cases}$$

The proof of Lemma 14.1.1 may be easily established and is left to the reader.

Lemma 14.1.1. *The function W^D is convex and fulfills the three conditions* (13.23), (13.24), *and* (13.25).

The set of admissible displacement fields is the set of finite energy, that is,

$$\mathscr{A} = \{v \in LD(\Omega) : \mathrm{div}(v) \in L^2(\Omega),\ v = 0 \text{ on } \Gamma_0\},$$

and problem $(\mathscr{P})$ is precisely

$$\inf\left\{ \int_\Omega W^D(\varepsilon^D(v))\,dx + \frac{k}{2}\int_\Omega (\mathrm{div}(v))^2\,dx - L(v) \;:\; v \in \mathscr{A} \right\}.$$

Let us recall that for a function $v : \Omega \mapsto \mathbf{R}^N$, its divergence in the distributional sense is given by div $v := \sum_{i=1}^N \frac{\partial v_i}{\partial x_i}$. The boundary condition $v = 0$ on Γ_0 must be taken in the trace sense. The trace operator is indeed well defined from $LD(\Omega)$ into $L^1(\Gamma_0, \mathbf{R}^3)$. For a proof, it suffices to adapt the trace Theorem 10.2.1 (see Remark 10.2.2 or see, for instance, Temam [348]).

We define now the relaxed problem. For all a in $\mathbf{R}^3$ and every unit vector ν in $\mathbf{R}^3$, we denote the tangential and normal components of a relative to ν, by a_τ and a_ν, respectively. In other words $a_\tau = a - (a\cdot\nu)\,\nu$, where $a\cdot\nu$ denotes the scalar product of a and ν in $\mathbf{R}^3$. For all a and b in $\mathbf{R}^3$, we define their symmetric tensor product by $a \otimes_s b := 1/2(a_i b_j + a_j b_i)_{i,j}$. The relaxed problem is precisely

$$(\overline{\mathscr{P}}) \qquad \inf\left\{ \int_\Omega W^D(\varepsilon^D(v)) + \frac{k}{2}\int_\Omega (\mathrm{div}(v))^2\,dx \right. \\ \left. + \int_{\Gamma_0} W^\infty(\gamma_0(v)_\tau \otimes_s \nu)\,d\mathscr{H}^2 - L(v) \;:\; v \in \tilde{\mathscr{A}} \right\}, \tag{14.1}$$

where γ_0 is the trace operator from $BD(\Omega)$ into $L^1(\Gamma)$ and the set of admissible displacements is

$$\tilde{\mathscr{A}} = \{v \in BD(\Omega) \,:\, \text{div } v \in L^2(\Omega),\ v_\nu = 0 \text{ on } \Gamma_0\}.$$

Note that the boundary condition $v = 0$ (taken in the trace sense) on Γ_0 in problem $(\mathscr{P})$ has been relaxed, in problem $(\overline{\mathscr{P}})$, by the surface energy

$$\int_{\Gamma_0} W^\infty(\gamma_0(v)_\tau \otimes_s \nu)\, d\mathscr{H}^2$$

and the weaker boundary condition $v_\nu = 0$ on Γ_0. This last condition must be taken in the trace sense $\gamma_\nu(v) = 0$, where γ_ν is a linear continuous operator from the space

$$\{v \in L^1(\Omega, \mathbf{R}^3) : \text{div } v \in L^2(\Omega)\}$$

into the dual $\mathbf{C}^1(\Gamma)'$ of $\mathbf{C}^1(\Gamma)$. The existence of this trace operator γ_ν will be established in Theorem 14.1.3.

The integral $\int_\Omega W^D(\varepsilon^D(v))$ must be taken in the sense of measures defined in Sections 13.3 and 13.4.1. Let us recall that the measure $W^D(\varepsilon^D(v))$ denotes the Borel measure $W^D(e^D)\,\mathscr{L}^3\lfloor\Omega + (W^D)^\infty(\varepsilon^D(v)^S)$, where $e^D\,\mathscr{L}^3\lfloor\Omega + \varepsilon^D(v)^S$ is the Lebesgue–Nikodým decomposition of the measure $\varepsilon^D(v)$, and that the recession function $(W^D)^\infty$ of W^D is defined for all E in $\mathbf{M}_S^D$ by

$$(W^D)^\infty(E) = \lim_{t \to +\infty} \frac{W^D(tE)}{t}.$$

Consequently, by definition one has

$$\int_\Omega W^D(\varepsilon^D(v)) := \int_\Omega W^D(e^D)\, dx + \int_\Omega (W^D)^\infty(\varepsilon^D(v)^S).$$

When the singular part $\varepsilon^D(v)^S$ of $\varepsilon^D(v)$ vanishes, we also denote the measure $e^D(v)\mathscr{L}^3\lfloor\Omega$ by $\varepsilon^D(v)\mathscr{L}^3\lfloor\Omega$. In the same spirit, if $e\ \mathscr{L}^3\lfloor\Omega + \varepsilon(v)^S$ is the Lebesgue–Nikodým decomposition of the measure $\varepsilon(v)$, we denote the measure $e(v)\mathscr{L}^3\lfloor\Omega$ by $\varepsilon(v)\mathscr{L}^3\lfloor\Omega$ when $\varepsilon(v)^S = 0$.

14.1.3 ▪ The spaces $BD(\Omega)$, $M(\text{div})$, and $U(\Omega)$

Unless differently specified, the set Ω is, for the moment, a bounded open subset of $\mathbf{R}^3$. As said before, a well-adapted space for relaxing the above model is the space defined below.

Definition 14.1.1. *The subspace*

$$BD(\Omega) := \{v \in L^1(\Omega, \mathbf{R}^3) : \varepsilon(v) \in \mathbf{M}(\Omega, \mathbf{M}_S)\}$$

of $L^1(\Omega, \mathbf{R}^3)$ is called the space of bounded deformations. The measure $\varepsilon(v) \in \mathbf{C}_0'(\Omega, \mathbf{M}_S)$ is defined by its action on all φ in $\mathbf{C}_0(\Omega, \mathbf{M}_S)$:

$$\langle \varepsilon(u), \varphi \rangle = \sum_{i,j} \langle \varepsilon(v)_{i,j}, \varphi_{i,j} \rangle,$$

where the brackets on the right-hand side denote the action of the signed measure $\varepsilon(v)_{i,j}$ on the scalar function $\varphi_{i,j}$ for the duality $(\mathbf{C}_0'(\Omega), \mathbf{C}_0(\Omega))$.

Remark 14.1.1. The action of $\varepsilon(v)$ on φ will also be written as $\int_\Omega \varphi\ \varepsilon(v)$, which is also well defined on bounded Du-integrable functions φ. Note that when v is regular (for instance, belongs to $W^{1,1}(\Omega,\mathbf{R}^3)$),

$$\begin{aligned}\int_\Omega \varphi\ \varepsilon(v) &= \int_\Omega \varphi(x)\ \varepsilon(v)(x)\,dx\\ &:= \int_\Omega \varphi(x)\ :\ \varepsilon(v)(x)\,dx,\end{aligned}$$

where for two 3×3 matrices A and B, $A:B$ denotes their Hilbert–Schmidt scalar product defined by $A:B := \operatorname{trace}(A^T B)$. This is why we also write $\int_\Omega \varphi : \varepsilon(v)$ for the integral $\int_\Omega \varphi\,\varepsilon(v)$ with respect to the measure $\varepsilon(v)$.

Under these considerations, we leave the reader to adapt the definitions of the weak and intermediate convergences and the proof of the approximating Theorem 10.1.2 or its generalization, Theorem 13.4.1. It suffices to argue with the components of u and to replace everywhere Du by $\varepsilon(u)$ (see also Remark 10.2.2). Because of its importance in Subsection 14.1.4, we state the approximating theorem.

Theorem 14.1.1. *Let $f:\mathbf{M}_S\to\mathbf{R}^+$ be a convex function satisfying* (13.23) *and* (13.24). *The space $\mathbf{C}^\infty(\Omega,\mathbf{M}_S)\cap BD(\Omega)$ is dense in $BD(\Omega)$ equipped with the intermediate convergence associated with f. More precisely, for all u in $BD(\Omega)$, there exists u_n in $\mathbf{C}^\infty(\Omega,\mathbf{M}_S)\cap BD(\Omega)$ such that*

$$\begin{cases} u_n\to u \quad \text{strongly in } L^1(\Omega,\mathbf{R}^3);\\ \displaystyle\int_\Omega |\varepsilon(u_n)|\,dx \to \int_\Omega |\varepsilon(u)|;\\ \displaystyle\int_\Omega f(\varepsilon(u_n))\,dx \to \int_\Omega f(\varepsilon(u));\\ \displaystyle\int_\Omega f(\varepsilon^D(u_n))\,dx \to \int_\Omega f(\varepsilon^D(u)).\end{cases}$$

In the same spirit, we state without proof the trace theorem.

Theorem 14.1.2. *Let Ω be a Lipschitz open bounded subset of $\mathbf{R}^3$. There exists a linear continuous map γ_0 from $BD(\Omega)$ onto $L^1_{\mathcal{H}^2}(\Gamma,\mathbf{R}^3)$ satisfying*

(i) *for all $u\in\mathbf{C}(\overline{\Omega},\mathbf{R}^3)\cap BD(\Omega)$, $\gamma_0(u)=u\lfloor_\Gamma$;*

(ii) *for all $\varphi\in\mathbf{C}(\overline{\Omega},\mathbf{M}_S)$*

$$\int_\Omega \varphi:\varepsilon(u) = -\int_\Omega u\cdot\operatorname{div}\ \varphi\,dx + \int_\Gamma \gamma_0(u)\otimes_S \nu:\varphi\,d\mathcal{H}^2,$$

where ν is the outer unit normal at $\mathcal{H}^2$-almost all x on Γ and div φ *denotes the vector valued function defined by* $(\operatorname{div}\ \varphi)_i := \sum_{j=1}^3 \frac{\partial\varphi_{i,j}}{\partial x_j}$, $i=1,\dots,3$.

As a consequence of the Green's formula (ii) above, one can adapt the two first examples of Section 10.2.

Example 14.1.1. Consider two disjoint Lipschitz open bounded subsets Ω_1 and Ω_2 of an open bounded subset Ω of $\mathbf{R}^3$ such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ and set $\Gamma_{1,2} := \partial\Omega_1 \cap \partial\Omega_2$, which is assumed to satisfy $\mathcal{H}^2(\Gamma_{1,2}) > 0$. We denote the trace operators from $BD(\Omega_1)$ onto $L^1_{\mathcal{H}^2}(\partial\Omega_1, \mathbf{R}^3)$ and $BD(\Omega_2)$ onto $L^1_{\mathcal{H}^2}(\partial\Omega_2, \mathbf{R}^3)$ by γ_1 and γ_2, respectively. Let u_1 and u_2 be, respectively, two elements of $BD(\Omega_1)$ and $BD(\Omega_2)$ and define

$$u = \begin{cases} u_1 & \text{in } \Omega_1, \\ u_2 & \text{in } \Omega_2. \end{cases}$$

Then u belongs to $BD(\Omega)$ and

$$\varepsilon(u) = \varepsilon(u_1)\lfloor_{\Omega_1} + \varepsilon(u_2)\lfloor_{\Omega_2} + [u] \otimes_s \nu \mathcal{H}^{N-1}\lfloor_{\Gamma_{1,2}},$$

where $[u] = \gamma_1(u_1) - \gamma_2(u_2)$ and $\nu(x)$ is the unit inner normal at x to $\Gamma_{1,2}$, considered as a part of the boundary of Ω_1.

Example 14.1.2. By slightly modifying the previous example, if we set

$$v = \begin{cases} u & \text{in } \Omega, \\ 0 & \text{in } \mathbf{R}^3 \setminus \overline{\Omega}, \end{cases}$$

where Ω is a Lipschitz bounded open subset of $\mathbf{R}^3$ and $u \in BD(\Omega)$, we see that v belongs to $BD(\mathbf{R}^3)$ and that

$$\varepsilon(v) = \varepsilon(u)\lfloor_{\Omega} + u^+ \otimes_s \nu\, \mathcal{H}^{N-1}\lfloor_{\Gamma},$$

where Γ is the boundary of Ω, ν denotes the inner unit vector normal to Γ, and u^+ denotes the trace of u on Γ.

Remark 14.1.2. Remark 10.2.1 also holds in this situation. More precisely, let Ω be a Lipschitz open bounded subset of $\mathbf{R}^3$. The approximating Theorem 14.1.1 may easily be improved as follows: the regular approximating functions of $u \in BD(\Omega)$ have all their traces equal to that of u on the boundary of Ω. For a proof, see, for instance, [348].

We define now the space $\mathbf{M}(\mathrm{div})$ which is involved in the definition of the admissible set of displacement fields in the Hencky model.

Definition 14.1.2. *We denote the space of all the functions in* $L^1(\Omega, \mathbf{R}^3)$ *such that their divergence belongs to* $L^2(\Omega)$, *by* $\mathbf{M}(\mathrm{div})$*:*

$$\mathbf{M}(\mathrm{div}) := \{v \in L^1(\Omega, \mathbf{R}^3) \,:\, \mathrm{div}\ v \in L^2(\Omega)\}.$$

On $\mathbf{M}(\mathrm{div})$, we can define a trace notion.

Theorem 14.1.3. *Let* Ω *be a* $\mathbf{C}^2$ *open bounded subset of* $\mathbf{R}^3$. *There exists a linear continuous map* γ_ν *from* $\mathbf{M}(\mathrm{div})$ *into* $\mathbf{C}^1(\Gamma)'$ *satisfying the following:*

(i) *for all* $u \in \mathbf{C}(\overline{\Omega}, \mathbf{R}^3) \cap BD(\Omega)$, $\gamma_\nu(u) = u \cdot \nu\lfloor_\Gamma$*;*

(ii) *for all* u *in* $\mathbf{M}(\mathrm{div})$, *all* $\varphi \in \mathbf{C}^1(\Gamma)$, *and all* $\Phi \in \mathbf{C}^1(\overline{\Omega})$ *such that* $\Phi\lfloor_\Gamma = \varphi$,

$$\langle \gamma_\nu(u), \varphi \rangle = \int_\Omega u \cdot D\Phi\, dx + \int_\Omega \Phi \ \mathrm{div}\ u\, dx.$$

For a proof, consult Temam [348, Proposition 7.2]. Note that this theorem also makes sense when div u is a Borel measure. In this case, the space $\mathbf{M}(\mathrm{div})$ is defined by

$$\mathbf{M}(\mathrm{div}) := \{v \in L^1(\Omega, \mathbf{R}^3) \ : \ \mathrm{div}\ v \in \mathbf{M}(\Omega)\}.$$

For our application we consider only the case when div $v \in L^2(\Omega)$.

We consider now the space $U(\Omega) := BD(\Omega) \cap \mathbf{M}(\mathrm{div})$ equipped with two convergences. Precisely, for all sequence $(u_n)_{n\in\mathbf{N}}$ in $U(\Omega)$, one defines

• the weak convergence, defined by the weak convergence of u_n to u in $BD(\Omega)$ and the strong convergence of div u_n to div u in $L^2(\Omega)$;

• the intermediate convergence associated with a convex function f, defined by the intermediate convergence of u_n to u in $BD(\Omega)$ associated with f together with the strong convergence of div u_n to div u in $L^2(\Omega)$.

The following result completes the approximating Theorem 14.1.1 in the spirit of Remark 14.1.2. For a proof, consult Theorems 3.4 and 5.3 in [348].

Theorem 14.1.4. *Let $f : \mathbf{M}_S \to \mathbf{R}$ be a convex function satisfying*

$$\alpha(|A| - 1) \leq f(A) \leq \beta(1 + |A|)$$

for all A in $\mathbf{M}_S$. Then for all u in $U(\Omega)$, there exists u_n in $\mathbf{C}^\infty(\Omega, \mathbf{M}_S) \cap BD(\Omega)$ such that

$$\begin{cases} u_n \to u \quad \textit{strongly in } L^1(\Omega, \mathbf{R}^3); \\ \displaystyle\int_\Omega |\varepsilon(u_n)|\, dx \to \int_\Omega |\varepsilon(u)|; \\ \displaystyle\int_\Omega f(\varepsilon(u_n))\, dx \to \int_\Omega f(\varepsilon(u)); \\ \mathrm{div}\ u_n \to \mathrm{div}\ u \quad \textit{strongly in } L^2(\Omega); \\ \displaystyle\int_\Omega f(\varepsilon^D(u_n))\, dx \to \int_\Omega f(\varepsilon^D(u)); \\ \gamma_0(u_n) = \gamma_0(u). \end{cases}$$

14.1.4 ▪ Relaxation of the Hencky model

In this subsection we show that the lower semicontinuous envelope of the Hencky functional energy for the weak topology of $U(\Omega)$ is the functional of $(\overline{\mathscr{P}})$ in (14.1). We moreover establish the convergence of the corresponding energy to the energy of the relaxed functional. Here, Ω is a bounded open subset of $\mathbf{R}^3$ of class $\mathbf{C}^2$.

Theorem 14.1.5. *The lower semicontinuous envelope for the weak convergence of $U(\Omega)$, of the integral functional defined on $U(\Omega)$ by*

$$F(v) = \begin{cases} \displaystyle\int_\Omega W^D(\varepsilon^D(v))\, dx + \frac{k}{2}\int_\Omega (\mathrm{div}(v))^2\, dx \ \textit{if } v \in LD(\Omega) \cap \mathbf{M}(\mathrm{div}), \quad v = 0 \textit{ on } \Gamma_0, \\ +\infty \quad \textit{otherwise}, \end{cases}$$

is the functional defined on $U(\Omega)$ by

$$\overline{F}(v)=\begin{cases}\displaystyle\int_\Omega W^D(\varepsilon^D(v))+\frac{k}{2}\int_\Omega(\operatorname{div}(v))^2\,dx+\int_{\Gamma_0}(W^D)^\infty(\gamma_0(v)_\tau\otimes_s\nu)\,d\mathcal{H}^2 & \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{if } v\in U,\ \gamma_\nu(v)=0 \textit{ on } \Gamma_0, & \\ +\infty \quad \textit{otherwise.} & \end{cases}$$

SKETCH OF THE PROOF. Arguing exactly as in the proof of Theorem 13.4.2, where Theorem 13.4.1 is replaced by Theorem 14.1.4, we obtain

$$\overline{F}(v)=\begin{cases}\displaystyle\int_\Omega W^D(\varepsilon^D(v))+\frac{k}{2}\int_\Omega(\operatorname{div}(v))^2\,dx+\int_{\Gamma_0}(W^D)^\infty(\gamma_0(v)\otimes_s\nu)\,d\mathcal{H}^2 & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } v\in U,\ v_\nu=0 \text{ on } \Gamma_0, & \\ +\infty \quad \text{otherwise,} & \end{cases}$$

where condition $v_\nu = 0$ on Γ_0 is intended in the trace sense: for H^2-a.e. x on Γ_0, $\gamma_\nu(v)(x)=0$. Indeed, the map $v\mapsto\int_\Omega(\operatorname{div}(v))^2\,dx$ is continuous for the weak convergence of $U(\Omega)$. Note also that according to Theorem 14.1.3, the trace operator defined from $\mathbf{M}(\operatorname{div})$ into $\mathbf{C}^1(\Gamma)'$ is continuous when the two spaces are equipped with their weak convergences. Finally, for $\mathcal{H}^2$-a.e. x on Γ_0, since $v_\nu=0$ in the trace sense on Γ_0, we have $\gamma_0(v)_\tau\otimes_s\nu(x)=\gamma_0(v)\otimes_s\nu(x)$. Indeed, it is easily seen that $\gamma_0(v)\cdot\nu=\gamma_\nu(v)\ \mathcal{H}^2_{\lfloor\Gamma_0}$-a.e. □

Corollary 14.1.1. *The energy of the Hencky model*

$$\inf\left\{\int_\Omega W^D(\varepsilon^D(v))\,dx+\frac{k}{2}\int_\Omega(\operatorname{div}(v))^2\,dx-L(v)\ :\ v\in\mathcal{A}\right\},$$

where $\mathcal{A}=\{v\in LD(\Omega):\operatorname{div}(v)\in L^2(\Omega),\ v=0 \textit{ on } \Gamma_0\}$, *relaxes to*

$$\min\left\{\int_\Omega W^D(\varepsilon^D(v))+\frac{k}{2}\int_\Omega(\operatorname{div}(v))^2\,dx+\int_{\Gamma_0}W^\infty(\gamma_0(v)_\tau\otimes_s\nu)\,dH^2-L(v)\ :\ v\in\tilde{\mathcal{A}}\right\},$$

where $\tilde{\mathcal{A}}=\{v\in BD(\Omega)\ :\ \operatorname{div}(v)\in L^2(\Omega),\ v_\nu=0 \textit{ on } \Gamma_0\}$.

SKETCH OF THE PROOF. According to the general theory of relaxation (see Theorem 11.1.2), it suffices to prove that any minimizing sequence related to the Hencky energy possesses a subsequence weakly converging in the space U. This assertion may be easily established thanks to coercivity condition (13.25). □

14.2 ▪ Some variational models in fracture mechanics

14.2.1 ▪ A few considerations in fracture mechanics

Let us consider an elastic brittle medium whose reference configuration is a bounded domain Ω of $\mathbf{R}^3$. Griffith's theory of fracture mechanics asserts that the energy necessary to produce a crack K included in Ω is proportional to the crack area $\mathcal{H}^2(K)$. Consequently, the elastic deformation energy outside the crack must be completed by an additional energy whose simplest form is $\lambda\mathcal{H}^2(K)$. The constant λ is the Griffith coefficient, introduced for fracture initiation (see [232], [324]). The elastic energy of the deformable body under consideration then takes the form

$$E(u,K)=\int_{\Omega\setminus K} f(\nabla u)\,dx+\lambda\mathcal{H}^2(K),$$

where u denotes the deformation vector field and ∇u the deformation gradient. Under suitable conditions on f, the functional E makes sense in a classical way if, for instance, K is a closed set, and u belongs to $\mathbf{C}^1(\Omega\setminus K,\mathbf{R}^3)$. From the inequality $\mathcal{H}^2(K)\le\lambda^{-1}E(u,K)$, we see that when $E(u,K)<+\infty$, the crack surface K is a two-Hausdorff-dimensional closed set of Ω and its Lebesgue measure is zero. Thus, the crack surface K can be seen as the set of discontinuity points for the measurable function u, more precisely, the measurable representative of u defined on Ω, satisfying the convention of Remark 10.3.2. It is worth noticing the analogy between this model and the strong model introduced in image segmentation by Mumford and Shah and discussed in Section 12.5 (see also Section 14.3 for complements). Following the idea developed in Section 12.5, one may define a weak formulation in the setting of SBV functions introduced in Section 10.5 (completed by Remark 10.5.1) by considering the functional

$$E(u)=\int_\Omega f(\nabla u)\,dx+\lambda\mathcal{H}^2(S_u),\ u\in SBV(\Omega,\mathbf{R}^3),$$

where ∇u is the density of the regular part of the measure Du and S_u the jump set of u. Actually, one can deal with functionals of the more general form

$$E(u)=\int_\Omega f(x,\nabla u)\,dx+\int_\Omega g(x,u^+(x),u^-(x),\nu_u(x))\,d\mathcal{H}^2\lfloor S_u.$$

The bulk energy density f accounts for the elastic deformation outside the crack and g for the density energy necessary to produce a crack of surface S_u. The meaning of the presence of the terms $u^+(x)$, $u^-(x)$, and $\nu_u(x)$ is the following: the fracture energy may depend on the crack opening and, for nonisotropic materials, on the crack surface orientation.

According to Theorem 13.4.5 and Remark 13.4.3, suitable conditions on f and g ensure semicontinuity of E so that direct methods of the calculus of variations provide the existence of solutions for optimization problems related to the energy functionals of the type E. More precisely, we consider a lower semicontinuous symmetric and subadditive function $g:\mathbf{R}^3\times\mathbf{R}^3\to\mathbf{R}^+$, i.e.,

$$g(a,b)=g(b,a)\le g(b,c)+g(c,a)\quad\forall\ a,\ b,\ c\in\mathbf{R}^3.$$

We assume moreover that $g(a,b)\ge\psi(|a-b|)$ for all $a,b\in\mathbf{R}^3$, where $\psi:[0,+\infty)\to[0,+\infty]$, satisfies $\psi(t)/t\to+\infty$ as $t\to 0$. Subadditivity assumption on the function g forces the crack material to possess a minimal number of connected components. We finally consider a convex, even function $h:\mathbf{R}^3\to[0,+\infty)$, positively homogeneous of degree 1 and satisfying for all $\nu\in\mathbf{R}^N$, $h(\nu)\ge c|\nu|$, where c is a given positive constant.

Theorem 14.2.1. *Let $f:\mathbf{R}\times M^{3\times3}\to\mathbf{R}$ be a quasi-convex function satisfying the following growth conditions of order $p>1$: there exist two positive constants α and β such that for all $A\in M^{3\times3}$,*

$$\alpha|A|^p\le f(x,A)\le\beta(1+|A|^p).$$

Let g and h be two functions satisfying the conditions introduced above, and let $L\in L^\infty(\Omega,\mathbf{R}^3)$ (the exterior loading). Then, given a nonempty compact subset K of $\mathbf{R}^3$, there exists a solution *of the minimum problem*

$$\min\left\{\int_\Omega f(\nabla u)\,dx+\int_\Omega g(u^+,u^-)h(\nu_u)\,d\mathcal{H}^2\lfloor S_u+\int_\Omega L\cdot u\,dx\;:\;u\in SBV(\Omega,\mathbf{R}^3),\right.$$
$$\left.u(x)\in K\ \textit{for a.e. } x\right\}.$$

PROOF. Following the classical direct methods of the calculus of variation, the conclusion is a straightforward consequence of the lower semicontinuity of the functional energy, which has been established in Theorem 13.4.5 and Remark 13.4.3. The coercivity easily follows from the confinement condition $u(x)\in K$ for a.e. x. □

A similar type of result is described in detail in Section 14.3 for the Mumford–Shah model in image segmentation. In the second assertion of Theorem 14.2.1, the confinement condition $u\in K$ does not seem to be natural for all problems in fracture mechanics. To remove such a condition, we have in general to state the problems in the space $GSBV$ of functions whose truncations are in SBV (see [21], [122], and references therein). A similar statement can be given for boundary value problems. In this case we know (see the previous section or Theorem 11.3.1) that the Dirichlet boundary conditions $u=u_0$ on a subset Γ_0 of the boundary Γ of Ω, assumed to be Lipschitz, are relaxed into a surface energy at the boundary. Taking, for instance, $g=1$, we have to consider minimization problems of the form

$$\min\left\{\int_\Omega f(\nabla u)\,dx+\int_\Omega h(\nu_u)\,d\mathcal{H}^2\lfloor S_u+\int h(\nu)\,d\mathcal{H}^2\lfloor\Gamma_0+\int_\Omega L\cdot u\,dx\;:\;u\in SBV(\Omega,\mathbf{R}^3),\right.$$
$$\left.u(x)\in K \text{ for a.e. } x\right\},$$

where ν is the inner unit normal to Γ_0. Existence of a solution is obtained in a similar way. The argument above also works for more general minimum problems of the form (see, for instance, [184])

$$\min\left\{\int_\Omega f(\nabla u)\,dx+\int_\Omega g(u^+,u^-)h(\nu_u)\,d\mathcal{H}^2\lfloor S_u+\int_\Omega V(x,u)\,dx:u\in SBV(\Omega,\mathbf{R}^3)\right\},$$

where f,g,h are as above, and the potential V is integrable in x and satisfies

$$V(x,s)\geq\theta(|s|)-a(x)$$

with $a\in L^1(\Omega)$ and $\theta(t)/t\to+\infty$ as $t\to+\infty$. The case of Theorem 14.2.1 corresponds to $V(x,s)=L(x)\cdot s+\delta_K(s)$.

Functionals of the type E can be approximated by elliptic functionals via a variational procedure due to Ambrosio and Tortorelli and described in Section 12.5.3 for the Mumford–Shah energy. These approximating functionals are more adapted to a numerical treatment (see [89], [120]). In the one-dimensional case, the two models introduced in Subsections 14.2.2 and 14.2.3 provide an alternative and more direct way for defining a discrete variational approximation of E.

Other models, proposed by Barenblatt [83], can be similarly weakened in the space $SBV(\Omega,\mathbf{R}^3)$ by considering energies of the type

$$E(u)=\int_\Omega f(\nabla u)\,dx+\int_\Omega g(|u^+-u^-|)\,d\mathcal{H}^2\lfloor S_u,$$

where $g(t) \to 0$ as $t \to 0$. Optimization problems related to such energies functionals do not provide minimizing sequences satisfying Theorem 14.2.1. Indeed, we lose the control of the Hausdorff measure of the jump set. Therefore, minimizing sequences may converge to a function whose jump part is not concentrated on an $(N-1)$-Hausdorff dimensional set and, consequently, it does not belong to $SBV(\Omega, \mathbf{R}^3)$. To select minimizers, one can follow a singular perturbation approach, involving the notion of viscosity solutions (consult Attouch [39]). This procedure has already been described in detail for the phase transition model in Section 12.5.2. Precisely, the method consists in perturbing the functional E by the fracture initiation energy $\varepsilon \mathscr{H}^2(S_u)$ and introducing functionals of Griffith type

$$E_\varepsilon(u) = E(u) + \varepsilon \mathscr{H}^2(S_u)$$

($\varepsilon > 0$ intended to tend to zero) for which Theorem 14.2.1 applies. The cluster points of ε-minimizers related to E_ε then belong to $SBV(\Omega, \mathbf{R}^3)$ and minimize E among minimizers with jump set of minimal $\mathscr{H}^{N-1}$ measure (see [21] and [125]). For a specific study of problems involving fracture mechanics in the modern framework of SBV functions, we refer the reader to [21], [26], [219], [220], and references therein.

In the sections below, we would like to supply a justification of weak Griffith's models, in the one-dimensional case, by taking into account the microscopic scale and the statistical local energy distribution. More precisely, we deal with two discrete systems of material points, which, in the reference configuration, occupy the points of the lattice $\varepsilon \mathbf{Z}$, $\varepsilon = 1/n$ included in the interval $[0,1]$. For each material point placed at $x \in \varepsilon \mathbf{Z}$ in the reference configuration, $u(x) \in \mathbf{R}$ denotes its new position in the deformed state. Each point interacts only with its nearest neighbors. We aim at describing the continuous limit of the discrete energy in the sense of Γ-convergence. We show that the continuous variational limit of the second model takes the form of Griffith's model discussed above. In the first model, we show that convexity conditions satisfied by local density energies entail the presence of a Cantor-part energy: the Griffith's initiation energy is completed by an additional term $C\ u'_c(0,1)$, where u'_c is the Cantor part of u' and C a suitable constant. It is worth noticing that the discrete energies considered may also be viewed as discrete variational approximating functionals of Griffith's type in the one-dimensional case. This approach is very close to that of Chambolle consisting in approximating the Mumford–Shah functionals in the two-dimensional case (see Chambolle [168]) and to some recent works by Braides [124] and Braides and Gelli [129].

14.2.2 ▪ A first model in one dimension

The interaction between each pair $\{z\varepsilon, (z+1)\varepsilon\}$ of contiguous points is described by a random energy

$$\varepsilon W_{\omega_z}\left(\frac{u(\varepsilon(z+1)) - u(\varepsilon z)}{\varepsilon}\right),$$

where W_{ω_z} belongs to a finite set $\{W_j,\ j \in J\}$. Each density $W_j,\ j \in J$, mapping $\mathbf{R}$ into $[0, +\infty]$, is assumed to be of Hencky pseudoplastic type (see Section 14.1) and must take the value $+\infty$ if two neighboring points occupy the same location. The total energy of the interaction between points located in $[0,1] \cap \varepsilon \mathbf{Z}$ is

$$E_\varepsilon(\omega, u) = \sum_{z=0}^{n-1} \varepsilon\ W_{\omega_z}\left(\frac{u(\varepsilon(z+1)) - u(\varepsilon z)}{\varepsilon}\right),$$

where $\omega = (\omega_z)_{z\in\mathbf{Z}} \in J^{\mathbf{Z}}$. One may think that two neighboring material points are connected by randomly chosen nonlinear springs, and this model also describes a system of n springs which are randomly distributed. We still denote the continuous extension of u, which is affine on $(\varepsilon z, \varepsilon(z+1))$ for all $z \in \mathbf{Z}$, by u. Thus the discrete energy may be considered as defined on $L^1(0,1)$. This first model is described by the almost sure Γ-convergence of $u \mapsto E_\varepsilon(\omega, u)$ toward a deterministic energy functional living in $BV(0,1)$.

Let us give some specific notation about BV-functions in the one-dimensional case. For each function u in $BV(0,1)$, one writes $u' = u'_a\, dt + u'_s$ the Lebesgue–Nikodým decomposition of its distributional derivative u' with respect to the Lebesgue measure dt on $(0,1)$. The singular part with respect to dt has the following decomposition: $u'_s = \sum_{t\in S_u}(u^+ - u^-)\delta_t + u'_c$, where u^+ and u^- are, respectively, the approximate upper and lower limits of u, S_u is the jump set $\{t \in (0,1) : u^+(t) \neq u^-(t)\}$ of u, and u'_c is the singular diffuse part, also called the Cantor part of u'. Let $BV^+(0,1)$ be the subset of all the functions u in $BV(0,1)$ such that $u'_a > 0$ a.e. in $(0,1)$ and $u'_s \geq 0$. We will establish that the (deterministic) limit energy functional is defined on $BV^+(0,1)$ by

$$E(u) := \int_0^1 W^{hom}(u'_a)\, dt + C\, u'_s\big((0,1)\big),$$

where C is a positive constant. The density $e \mapsto W^{hom}(e)$ is obtained as the almost sure limit of a suitable subadditive ergodic process. This mathematical result expresses that the mechanical macroscopic behavior of a string can be interpreted as the variational limit of a discrete system at a microscopic scale. Moreover, we will express the duality principle $(W^{hom})^* = \sum_{j\in J} p_j W_j^*$, where p_j is the probability presence of each density W_j. For other models in a deterministic setting and some discussions about the possible existence of a fracture site related to these models, consult [127] and [21].

Let us give some more details on the random discrete model. As said above, we consider a finite set Λ of functions W_j, $j \in J$, of probability presence p_j, satisfying the three following conditions:

(i) $W_j : \mathbf{R} \to \mathbf{R}^+ \cup \{+\infty\}$ is convex, finite for $e > 0$, $W_j(1) = 0$, and there exists $\alpha > 0$ such that $\alpha(e-1) \leq W_j(e)$ for all $e \geq 0$;

(ii) there exists $\beta > 0$ such that $W_j(e) \leq \beta(1+e)$ for all $e > 1$;

(iii) $\lim_{e\to 0^+} W_j(e) = +\infty$ and $W_j(e) = +\infty$ when $e \leq 0$.

The assumption $W_j(1) = 0$ means that no energy is needed when no deformation occurs. Assumption (iii) means that an infinite amount of energy is needed to squeeze a pair of material points down to a single one and that there is no interpenetrability of the matter. These density functions are of Hencky pseudoplastic type owing to the convexity and the linear growth conditions.

The fundamental stochastic setting that we will need for describing the discrete energy and its asymptotic behavior is the discrete dynamical system $(\Omega, \mathscr{T}, \mathbf{P}, (T_z)_{z\in\mathbf{Z}})$: $\Omega = \Lambda^{\mathbf{Z}}$, $(\Omega, \mathscr{T}, \mathbf{P})$ is the product probability space of the Bernoulli probability space on Λ constructed from $p_j, j \in J$, the transformation T_z is the shift defined for all $z \in \mathbf{Z}$ by $T_z\big((\omega_s)_{s\in\mathbf{Z}}\big) = (\omega_{s+z})_{s\in\mathbf{Z}}$. The expectation operator will be denoted by $\mathbf{E}$. To write in a continuous form the random energy functional, we consider the random function defined for all $(\omega, t, e) \in \Omega \times \mathbf{R} \times \mathbf{R}$ by

$$W(\omega, t, e) = W_{\omega_z}(e) \quad \text{when } t \in [z, z+1).$$

Let $\mathscr{A}_\varepsilon(0,1)$ denote the space of continuous functions on $(0,1)$ which are affine on each interval $(\varepsilon z, \varepsilon(z+1))$ of $(0,1)$. More generally when $s=(b-a)/n$, $\mathscr{A}_s(a,b)$ will denote the space of continuous functions on (a,b) which are affine on each interval $(a+sz, a+s(z+1))$ of (a,b). The total energy due to the interactions between the points of $[0,1]\cap\varepsilon\mathbf{Z}$ is the functional defined in $L^1(0,1)$ by

$$E_\varepsilon(\omega,u)=\begin{cases}\displaystyle\sum_{z=0}^{n-1}\varepsilon\; W\Big(\omega,z,\frac{u(\varepsilon(z+1))-u(\varepsilon z)}{\varepsilon}\Big) & \text{if } u\in\mathscr{A}_\varepsilon(0,1),\\ +\infty & \text{otherwise,}\end{cases}$$

or, in a continuous form,

$$E_\varepsilon(\omega,u)=\begin{cases}\displaystyle\int_0^1 W\Big(\omega,\frac{t}{\varepsilon},u'(t)\Big)dt & \text{if } u\in\mathscr{A}_\varepsilon(0,1),\\ +\infty & \text{otherwise,}\end{cases}$$

where $t\mapsto u(t)$ also denotes the piecewise affine extension of u. Note that the domain of $E_\varepsilon(\omega,.)$ is the subset of all functions u in $\mathscr{A}_\varepsilon(0,1)$ whose distributional derivative u' is positive.

By using classical probabilistic arguments, it is easily seen that $(\Omega,\mathscr{T},\mathbf{P},(T_z)_{z\in\mathbf{Z}})$, previously defined, is an ergodic dynamical system. To solve the problem, we make use of the concept of the subadditive process developed in Section 12.4. Let us recall the following ergodic theorem taken from Theorem 12.4.4.

Theorem 14.2.2. *Let $(\Omega,\mathscr{T},\mathbf{P},(T_z)_{z\in\mathbf{Z}})$ be an ergodic dynamical system and $\mathscr{S}$ a discrete subadditive process. Suppose that*

$$\inf\left\{\int_\Omega\frac{\mathscr{S}_I(\omega)}{|I|}\mathbf{P}(d\omega):|I|\neq 0\right\}>-\infty$$

and let $(A_n)_{n\in\mathbf{N}}$ be a regular sequence of $\mathscr{J}$ satisfying $\lim_{n\to+\infty}\rho(A_n)=+\infty$. Then almost surely

$$\lim_{n\to+\infty}\frac{\mathscr{S}_{A_n}(\omega)}{|A_n|}=\inf_{m\in\mathbf{N}^*}\left\{E\frac{\mathscr{S}_{[0,m[^d}}{m^d}\right\}.$$

The space $L^1(0,1)$, equipped with its norm, plays the role of the metric space (X,d) in the Γ-convergence definition of Chapter 12.

Theorem 14.2.3. *The energy functional $E_\varepsilon(\omega,.)$ Γ-converges almost surely to the functional E defined in $L^1(0,1)$ by*

$$E(u)=\begin{cases}\displaystyle\int_0^1 W^{hom}(u'_a)\,dt+W^{hom,\infty}(1)\,u'_s\big((0,1)\big) & \textit{if } u\in BV^+(0,1),\\ +\infty & \textit{otherwise.}\end{cases}$$

The density W^{hom} is defined as follows: $W^{hom}(e)=+\infty$ if $e\leq 0$ and, if $e>0$, one has ω-a.s.,

$$\begin{aligned}W^{hom}(e)&=\lim_{n\to+\infty}\inf\left\{\frac{1}{n}\int_0^n W(\omega,t,e+v')\,dt\ :\ v\in W_0^{1,1}(0,n)\right\}\\ &=\inf_{n\in\mathbf{N}^*}E\left(\inf\left\{\frac{1}{n}\int_0^n W(.,t,e+v')\,dt\ :\ v\in W_0^{1,1}(0,n)\right\}\right).\end{aligned}$$

Moreover, W^{hom} verifies properties (i), (ii), *and* (iii) *of the functions W_j and its Legendre–Fenchel transform is given by* $(W^{hom})^* = \sum_{j\in J} p_j W_j^*$.

The proof is established by means of Propositions 14.2.1 and 14.2.2, each giving, respectively, the lower bound and the upper bound for a subsequence in the definition of Γ-convergence. Before stating Proposition 14.2.1, we introduce a parametrized subadditive process, i.e., a family of subadditive processes which will be used to define the limit problem. For this purpose, for $\delta \in (0,\delta_0]$, $j \in J$, we consider the truncated functions $T_\delta W_j = W_j \wedge L_\delta$, where L_δ is the affine function defined by $L_\delta(t) = W_j(\delta) + \tau_i(t-\delta)$, $\tau_j \in \partial W_j(\delta)$ (the subdifferential of W_j at δ) and set

$$\begin{cases} W_\delta(\omega,t,e) = T_\delta \omega_z(e) & \text{when } t \in [z,z+1), \\ W_0 = W. \end{cases}$$

The nonincreasing family $(W_\delta)_{\delta\in(0,\delta_0]}$ satisfies for $0<\delta\le\delta_0$, $\omega\in\Omega$, $t\in\mathbf{R}$, and $e\in\mathbf{R}$:

$$\begin{cases} W_\delta \le W,\ W_\delta(e) = W(e) \quad \text{for } e \ge \delta, \\ \lim_{\delta\to 0} W_\delta(\omega,t,e) = W(\omega,t,e), \\ e \mapsto W_\delta(\omega,t,e) \text{ is convex}, \\ \alpha(|e|-1) \le W_\delta(\omega,t,e) \le \beta_\delta(1+|e|), \end{cases}$$

where β_δ is a positive constant depending only on δ.

Let $\mathscr{F}(\Omega,\mathbf{R}^+\cup\{+\infty\})$ be the set of all the measurable functions from Ω into $\mathbf{R}^+\cup\{+\infty\}$ and consider the parametrized subadditive process $\mathscr{S}$ defined by

$$\mathscr{S}:\ \mathscr{J}\times[0,\delta_0]\times\mathbf{R} \longrightarrow \mathscr{F}(\Omega,\mathbf{R}^+\cup\{+\infty\}),\ (A,\delta,e)\mapsto \mathscr{S}_A(\delta,e,.),$$

where

$$\mathscr{S}_A(\delta,e,\omega) = \inf\left\{ \int_A W_\delta(\omega,t,e+v')\,dt\ :\ v \in W_0^{1,1}(\mathring{A}) \right\}.$$

It is worth noticing that the domain of $e\mapsto \mathscr{S}_A(\delta,e,\omega)$ is $\mathbf{R}$ when $\delta\in(0,\delta_0]$ while the one of $e\mapsto\mathscr{S}_A(0,e,\omega)$ is $]0,+\infty[$. Indeed, if $\mathscr{S}_A(0,e,\omega)<+\infty$, for $e\le 0$, there exists $v\in W_0^{1,1}(\mathring{A})$ such that $W(\omega,t,v'(t))<+\infty$ t a.e. in $\mathring{A}$. Therefore $v'(t)>-e$ for t a.e. in $\mathring{A}$ and $\int_A v'\,dt = 0 > -e$, a contradiction.

It is easily seen that for all fixed (δ,e) in $\big((0,\delta_0]\times\mathbf{R}\big)\cup\big(\{0\}\times(0,+\infty)\big)$, the map $A\mapsto\mathscr{S}_A(\delta,e,.)$ is a subadditive process satisfying $\mathscr{S}_A(\delta,e,\omega)\le|A|\max\{T_\delta W_j(e) : j\in J\}$ and that all conditions of Theorem 14.2.2 are fulfilled. Consequently, there exists a set Ω' of full probability such that for all (δ,e) in $\big((0,\delta_0]\cap\mathbf{Q}\big)\times\mathbf{R}$ and all $\omega\in\Omega'$,

$$W_\delta^{hom}(e) := \lim_{n\to+\infty} \frac{\mathscr{S}_{[0,n)}(\delta,e,\omega)}{n} = \inf_{m\in\mathbf{N}^*} \mathbf{E}\frac{\mathscr{S}_{[0,m)}}{m}(\delta,e,.). \tag{14.2}$$

On the other hand, for all $e>0$, there exists a set Ω_e of full probability such that

$$W^{hom}(e) := \lim_{n\to+\infty} \frac{\mathscr{S}_{[0,n)}(0,e,\omega)}{n} = \inf_{m\in\mathbf{N}^*} \left\{ \mathbf{E}\frac{\mathscr{S}_{[0,m)}}{m}(0,e,.) \right\}$$

and $W^{hom}(e)=+\infty$ if $e\le 0$. The independence of the first set Ω' with respect to e comes from the equi-Lipschitz property of $e\mapsto\mathscr{S}_{[0,n)}(\delta,e,\omega)$ (see [185] or [291]). The following lemma states the continuity at $\delta=0^+$ of the function $\delta\mapsto W_\delta^{hom}(e)$ when $e>0$.

Lemma 14.2.1. *For all fixed $e>0$, there exists $\overline{\delta}(e)>0$ such that for all $\delta\in(0,\overline{\delta}(e)]\cap\mathbf{Q}$, $W^{hom}_{\delta}(e)=W^{hom}(e)$.*

PROOF. *First step.* This step is also valid for all $e\in\mathbf{R}$ and all $\omega\in\Omega$. Let $\delta>0$, we show that there exists $u_{\delta,n}(\omega)\in\mathscr{A}_1(0,n)\cap W^{1,1}_0(0,n)$ such that

$$\frac{\mathscr{S}_{[0,n)}(\delta,e,\omega)}{n}=\frac{1}{n}\int_0^n W_\delta(\omega,t,e+u'_{\delta,n}(\omega))\,dt.$$

To shorten notation, we set $W_n(e)=\frac{\mathscr{S}_{[0,n)}(\delta,e,\omega)}{n}$. From a classical calculation, the Fenchel conjugate of $e\mapsto W_n(e)$ is given for all σ in $\mathbf{R}$ by

$$W^*_n(\sigma)=\frac{1}{n}\int_0^n W^*_\delta(\omega,t,\sigma)\,dt.$$

Set $A_j(\omega,n)=\{t\in[0,n]\ :\ W_\delta(\omega,t,.)=T_\delta W_j\}$ and $\lambda_j(\omega,n)=\text{meas}(A_j(\omega,n))$ for $j\in J$. Thus

$$W^*_n(\sigma)=\sum_{j\in J}\frac{\lambda_j(\omega,n)}{n}(T_\delta W_j)^*(\sigma) \tag{14.3}$$

and, by classical subdifferential rules (see Chapter 9),

$$\partial W^*_n(\sigma)=\sum_{j\in J}\frac{\lambda_j(\omega,n)}{n}\partial(T_\delta W_j)^*(\sigma). \tag{14.4}$$

Now let $\sigma\in\partial W_n(e)$. Thus $e\in\partial W^*_n(\sigma)$, and from (14.4) there exists $U_{j,\delta,n}(\omega)\in\partial(T_\delta W_j)^*(\sigma)$ such that

$$e=\sum_{j\in J}\frac{\lambda_j(\omega,n)}{n}U_{j,\delta,n}(\omega).$$

The function

$$u_{\delta,n}(\omega)(x)=\int_0^x\sum_{j\in J}\big(U_{j,\delta,n}(\omega)-e\big)1_{A_j(\omega,n)}(s)\,ds$$

answers the question. Indeed, $u_{\delta,n}$ belongs to $\mathscr{A}_1(0,n)\cap W^{1,1}_0(0,n)$. On the other hand, according to $U_{j,\delta,n}(\omega)\in\partial(T_\delta W_j)^*(\sigma)$, (14.3), and the fact that $e\in\partial W^*_n(\sigma)$,

$$\begin{aligned}\frac{1}{n}\int_0^n W_\delta(\omega,t,e+u'_{\delta,n}(\omega))\,dt&=\sum_{j\in J}\frac{\lambda_j(\omega,n)}{n}T_\delta W_j(U_{j,\delta,n}(\omega))\\&=\sum_{j\in J}\frac{\lambda_j(\omega,n)}{n}\big(\sigma U_{j,\delta,n}(\omega)-T_\delta W^*_j(\sigma)\big)\\&=\sigma e-W^*_n(\sigma)\\&=W_n(e).\end{aligned}$$

Second step. From classical probabilistic arguments, there exists a set Ω'' of full probability such that for all $\omega\in\Omega''$,

$$\lim_{n\to+\infty}\frac{\lambda_j(\omega,n)}{n}=p_j. \tag{14.5}$$

We pick up ω_0 in $\Omega_e \cap \Omega' \cap \Omega''$ and claim that $\inf_n U_{j,\delta,n}(\omega_0) \geq \delta$ for all $\delta \in \mathbf{Q}^{*+}$ small enough, say, $\delta \in (0, \overline{\delta}(e)] \cap \mathbf{Q}$. Otherwise there exist two sequences $(\delta_k)_k$ and $(n_k)_k$ converging to 0 and $+\infty$ such that $U_{j,\delta_k,n_k}(\omega_0) \leq \delta_k$. But, according to the first step

$$\begin{aligned}\frac{\mathscr{S}_{[0,n_k)}(0,e,\omega_0)}{n_k} &\geq \frac{\mathscr{S}_{[0,n_k)}(\delta_k,e,\omega_0)}{n_k}\\ &= \sum_{j\in J} \frac{\lambda_j(\omega_0,n_k)}{n_k} T_{\delta_k} W_j(U_{j,\delta_k,n_k}(\omega_0))\\ &\geq \sum_{j\in J} \frac{\lambda_j(\omega_0,n_k)}{n_k} W_j(\delta_k)\end{aligned}$$

and letting $k \to +\infty$, we obtain $W^{hom}(e) = +\infty$, a contradiction.

Last step. According to the two previous steps, for $\delta \in (0, \overline{\delta}(e)] \cap \mathbf{Q}$ one has

$$\begin{aligned}\frac{\mathscr{S}_{[0,n)}(0,e,\omega_0)}{n} &\geq \frac{\mathscr{S}_{[0,n)}(\delta,e,\omega_0)}{n}\\ &= \sum_{j\in J} \frac{\lambda_j(\omega_0,n)}{n} T_{\delta} W_j(U_{j,\delta,n}(\omega_0))\\ &= \sum_{j\in J} \frac{\lambda_j(\omega,n)}{n} W_j(U_{j,\delta,n}(\omega_0))\\ &= \frac{1}{n}\int_0^n W(\omega_0,t,u'_{\delta,n}(\omega_0))\,dt\\ &\geq \frac{\mathscr{S}_{[0,n[}(0,e,\omega_0)}{n}.\end{aligned}$$

Therefore $\frac{\mathscr{S}_{[0,n)}(0,e,\omega_0)}{n} = \frac{\mathscr{S}_{[0,n)}(\delta,e,\omega_0)}{n}$. Letting $n \to +\infty$, we obtain that $W^{hom}(e) = W_\delta^{hom}(e)$ as soon as $\delta \in (0, \overline{\delta}(e)] \cap \mathbf{Q}$. $\square$

Proposition 14.2.1. *Let Ω', Ω'' be the subsets of full probability defined in* (14.2) *and* (14.5) *and let u, u_ε in $L^1(0,1)$ be such that $u_\varepsilon \to u$ strongly in $L^1(0,1)$. Then for all ω in $\Omega' \cap \Omega''$,*

$$E(u) \leq \liminf_{\varepsilon\to 0} E_\varepsilon(\omega, u_\varepsilon).$$

Moreover the domain of $\Gamma - \liminf_{\varepsilon\to 0} E_\varepsilon(\omega,.)$ is included in $BV^+(0,1)$.

PROOF. We fix ω in $\Omega' \cap \Omega''$.

First step. If $\liminf_{\varepsilon\to 0} E_\varepsilon(\omega, u_\varepsilon) < +\infty$, by the coercivity condition on the functions W_j (property (i)), u belongs to $BV(0,1)$ and obviously $u' \geq 0$. Let us now consider, for $\delta \in]0, \delta_0] \cap \mathbf{Q}$, the truncated energy

$$E_{\varepsilon,\delta}(\omega,u) = \begin{cases} \displaystyle\int_0^1 W_\delta\left(\omega, \frac{t}{\varepsilon}, u'(t)\right) dt & \text{if } u \in \mathscr{A}_\varepsilon(0,1),\\ +\infty & \text{otherwise,}\end{cases}$$

and the corresponding energy with domain $W^{1,1}(0,1)$

$$\tilde{E}_{\varepsilon,\delta}(\omega,u)=\begin{cases}\displaystyle\int_0^1 W_\delta\left(\omega,\frac{t}{\varepsilon},u'(t)\right)dt & \text{if } u\in W^{1,1}(0,1),\\ +\infty & \text{otherwise,}\end{cases}$$

which satisfies all the properties of random integral functionals considered in [1]. From the inequalities $E_\varepsilon(\omega,.)\geq E_{\varepsilon,\delta}(\omega,.)\geq \tilde{E}_{\varepsilon,\delta}(\omega,.)$ and according to [1] we then deduce

$$\liminf_{\varepsilon\to 0} E_\varepsilon(\omega,u_\varepsilon)\geq \liminf_{\varepsilon\to 0} E_{\varepsilon,\delta}(\omega,u_\varepsilon)\geq E_\delta(u), \tag{14.6}$$

where

$$E_\delta(u)=\begin{cases}\displaystyle\int_0^1 W_\delta^{hom}(u'_a)\,dt+W_\delta^{hom,\infty}(1)u'_s\Big((0,1)\Big) & \text{if } u\in BV(0,1),\\ +\infty & \text{otherwise.}\end{cases}$$

Second step. It is not restrictive to assume $\liminf_{\varepsilon\to 0}E_\varepsilon(\omega,u_\varepsilon)<+\infty$. Obviously $u'_a\geq 0$ and $u'_s\geq 0$. We would like to let δ going to 0 in (14.6) and apply Lemma 14.2.1. It remains to prove that $u'_a>0$ a.e. in $(0,1)$. Otherwise, there exists a Borel set N of $(0,1)$ with $\text{meas}(N)\neq 0$, such that $u'_a=0$ on N. We have

$$+\infty>\liminf_{\varepsilon\to 0}E_\varepsilon(\omega,u_\varepsilon)\geq \text{meas}(N)W_\delta^{hom}(0), \tag{14.7}$$

where, from the first step in the proof of Lemma 14.2.1,

$$\begin{cases}\displaystyle W_\delta^{hom}(0)=\lim_{n\to+\infty}\sum_{j\in J}\frac{\lambda_j(\omega,n)}{n}T_\delta W_j(U_{j,\delta,n}(\omega)),\\ \displaystyle\sum_{j\in J}\frac{\lambda_j(\omega,n)}{n}U_{j,\delta,n}(\omega)=0.\end{cases}$$

By the coercivity assumption (i), an easy calculation leads to $\sup_n|U_{j,\delta,n}(\omega)|\leq C$. Thus there exists $U_{j,\delta}(\omega)$ in $\mathbf{R}$ satisfying, up to a subsequence, $\lim_{n\to+\infty}U_{j,\delta,n}(\omega)=U_{j,\delta}(\omega)$ and

$$\begin{cases}\displaystyle W_\delta^{hom}(0)=\sum_{j\in J}p_j T_\delta W_j(U_{j,\delta}(\omega)),\\ \displaystyle\sum_{j\in J}p_j U_{j,\delta}(\omega)=0.\end{cases}$$

The second equality yields the existence of an index j_δ such that $U_{j_\delta,\delta}\leq 0$ and the first equality gives

$$\begin{aligned}W_\delta^{hom}(0)&\geq p_{j_\delta}T_\delta W_{j_\delta}(U_{j_\delta,\delta}(\omega))\\ &\geq p_{j_\delta}T_\delta W j_\delta(0)\\ &\geq \min_j p_j \min_j T_\delta W_j(0)\end{aligned}$$

so that $\lim_{\delta\to 0}W_\delta^{hom}(0)=+\infty$. Letting $\delta\to 0$ in (14.7) leads to a contradiction.

Last step. Letting $\delta \to 0$ in (14.6), according to the monotone convergence theorem and Lemma 14.2.1, we finally obtain

$$\liminf_{\varepsilon \to 0} E_\varepsilon(\omega, u_\varepsilon) \geq \int_0^1 W^{hom}(u'_a)\,dt + W^{hom,\infty}(1)\, u'_s\big((0,1)\big).$$

The proof is then achieved. □

To establish the upper bound, we will apply the following lemma.

Lemma 14.2.2. (i) *Let $e > 0$ and let $i \in \mathbf{N}$ be a fixed integer. There exists $u_{i,n}(\omega)$ in $\mathscr{A}_1(in,(i+1)n) \cap W_0^{1,1}(in,(i+1)n)$ such that*

$$\mathscr{S}_{[in,(i+1)n)}(0,e,\omega) = \int_{in}^{(i+1)n} W(\omega, t, e + u'_{i,n}(\omega))\,dt.$$

(ii) *The map $W^{hom} : \mathbf{R} \to [0,+\infty]$ is convex, continuous, and $W^{hom}(1) = 0$.*

PROOF. For establishing (i), reproduce the first step of the proof of Lemma 14.2.1 with $(0,n)$ replaced by $(in,(i+1)n)$ and W_δ by W. Note also that there exist $U_{j,\delta} > 0$, $j = 1,2,3$ (depending on e), satisfying $\sup_n U_{j,n} < +\infty$, such that

$$\begin{cases} \dfrac{\mathscr{S}_{[0,n)}(0,e,\omega)}{n} = \displaystyle\sum_{j\in J} \frac{\lambda_j(\omega,n)}{n} W_j(U_{j,n}(\omega)), \\ \displaystyle\sum_{j\in J} \frac{\lambda_j(\omega,n)}{n} U_{j,n} = e,\ U_{j,n} > 0. \end{cases}$$

The convexity of the map $e \mapsto W^{hom}(e)$ is a consequence of Jensen's inequality fulfilled for $e > 0$ and established by a straightforward calculation. Consequently, this map is continuous on its domain $\mathbf{R}^{*+}$ and, to prove (ii), we must show that $\lim_{e\to 0^+} W^{hom}(e) = +\infty$. Let $e_k > 0$ tend to 0 and fix ω in $\Omega'' \cap_{k\in\mathbf{N}} \Omega_{e_k}$. Letting $n \to +\infty$ (up to a subsequence) in

$$\begin{cases} \dfrac{\mathscr{S}_{[0,n[}(0,e_k,\omega)}{n} = \displaystyle\sum_{j\in J} \frac{\lambda_j(\omega,n)}{n} W_j(U_{j,n,k}(\omega)), \\ \displaystyle\sum_{j\in J} \frac{\lambda_j(\omega,n)}{n} U_{j,n,k} = e_k, \end{cases}$$

we obtain the existence of numbers $U_{j,k} \geq 0$ such that

$$\begin{cases} W^{hom}(e_k) = \displaystyle\sum_{j\in J} p_j\, W_j(U_{j,k}(\omega)), \\ \displaystyle\sum_{j\in J} p_j\, U_{j,k} = e_k. \end{cases}$$

Note that this shows that $W^{hom}(1) = 0$. Obviously $U_{j,k} \to 0$ when $k \to +\infty$. Thus

$$\lim_{k\to+\infty} W^{hom}(e_k) = \sum_{j\in J} p_j \lim_{k\to+\infty} W_j(U_{j,k}) = +\infty$$

and the proof is complete. □

We establish now the upper bound.

Proposition 14.2.2. *There exists a subset Ω''' of full probability such that for all $\omega \in \Omega'''$, there exists a subsequence $(n_k)_{k\in\mathbf{N}}$ of positive integers satisfying the following: for all u in $L^1(0,1)$, there exists u_{n_k} in $L^1(0,1)$ possibly depending on ω, converging to u, and such that* $\limsup_{k\to+\infty} E_{1/n_k}(\omega, u_{n_k}) \le E(u)$.

PROOF. *First step.* We prove Proposition 14.2.2 for all u belonging to $SBV_\#(0,1)\cap BV^+(0,1)$ where $SBV_\#(0,1)$ denotes the subspace of all functions of $SBV(0,1)$ having a finite jump set.

It is not restrictive to assume that $S_u = \{t_0\}$. For $m \in \mathbf{N}^*$, consider the decomposition $(0,1) = \cup_{i=0}^{m-1}(i/m,(i+1)/m)$. Let us set i_0 to denote the integer such that $t_0 \in [i_0/m,(i_0+1)/m)$ and by u_m the interpolate function of u with respect to this decomposition. We set $e_{i,m} := u'_m\lfloor(i/m,(i+1)/m)$ if $i \neq i_0$ and $e_{i_0,m} = m\big(u((i_0+1)/m) - u(i_0/m)\big)$. For each $i \in \{1,\dots,m-1\}$ we consider a sequence $e_{i,m,\eta}$ in $\mathbf{Q}^{*+}$ such that $\lim_{\eta\to 0} e_{i,m,\eta} = e_{i,m}$. By continuity of W^{hom} and Jensen's inequality, we have

$$\begin{aligned}\lim_{\eta\to 0} \sum_{i=0,\ i\neq i_0}^{m-1} \frac{1}{m} W^{hom}(e_{i,m,\eta}) &\le \sum_{i=0,\ i\neq i_0}^{m-1} \int_{\frac{i}{m}}^{\frac{i+1}{m}} W^{hom}(u'_a)\,dt \\ &\le \int_0^1 W^{hom}(u'_a)\,dt\end{aligned}$$

and

$$\begin{aligned}\lim_{m\to+\infty}\lim_{\eta\to 0} \frac{1}{m} W^{hom}(e_{i_0,m,\eta}) &= \lim_{m\to+\infty} \frac{1}{m} W^{hom}\left(m\left(u\left(\frac{i_0+1}{m}\right) - u\left(\frac{i_0}{m}\right)\right)\right) \\ &= W^{hom,\infty}([u](t_0))\end{aligned}$$

so that

$$\lim_{m\to+\infty}\lim_{\eta\to 0}\sum_{i=0}^{m-1} \frac{1}{m} W^{hom}(e_{i,m,\eta}) \le E(u). \tag{14.8}$$

But Lemma 14.2.2 implies the existence of $u_{i,\eta,n}$ in $\mathscr{A}_1(in,(i+1)n) \cap W_0^{1,1}(in,(i+1)n)$ (we have dropped the dependence on ω) such that

$$\begin{aligned}\frac{\mathscr{S}_{[in,(i+1)n)}(0, e_{i,m,\eta}, \omega)}{n} &= \frac{1}{n}\int_{in}^{(i+1)n} W(\omega, t, e_{i,m,\eta} + u'_{i,\eta,n})\,dt \\ &= m\int_{i/m}^{(i+1)/m} W(\omega, mnt, e_{i,m} + u'_{i,\eta,n}(mnt))\,dt.\end{aligned}$$

Therefore, as the sequence of intervals $\big((in,(i+1)n)\big)_{n\in\mathbf{N}^*}$ is regular, according to Theorem 14.2.2, there exists $\Omega''' = \cap_{e\in\mathbf{Q}^{*+}}\Omega_e$ of full probability such that for all $\omega \in \Omega'''$

$$W^{hom}(e_{i,m,\eta}) = \lim_{n\to+\infty} m\int_{i/m}^{(i+1)/m} W(\omega, mnt, e_{i,m,\eta} + u'_{i,\eta,n}(mnt))\,dt. \tag{14.9}$$

Combining (14.8) and (14.9), we obtain

$$\lim_{m\to+\infty}\lim_{\eta\to 0}\lim_{n\to+\infty}\int_0^1 W(\omega, mnt, v'_{m,\eta,n})\,dt \le E(u),$$

where

$$v_{m,\eta,n}(t)=u_m(t)+\sum_{i=0}^{m-1}1_{(i/m,(i+1)/m)}(t)\frac{1}{mn}u_{i,\eta,n}(nmt).$$

An easy calculation shows that $\lim_{m\to+\infty}\lim_{\eta\to 0}\lim_{n\to+\infty}v_{m,\eta,n}=u$ strongly in $L^1(0,1)$ and that $v_{m,\eta,n}$ belongs to $\mathscr{A}_{1/nm}(0,1)$. Therefore, by using a diagonalization argument, there exists a map $n\mapsto(m(n),\eta(n))$ such that

$$\begin{cases}\lim_{n\to+\infty}E_{\frac{1}{m(n)n}}(\omega,v_{m(n),\eta(n),n})\le E(u),\\ \lim_{n\to+\infty}v_{m(n),\eta(n),n}=u.\end{cases}$$

We complete the proof by denoting $k\mapsto n_k$, the subsequence $n\mapsto m(n)n$, and setting $u_{nm(n)}:=v_{m(n),\eta(n),n}$.

Second step. We prove Proposition 14.2.2 for u belonging to $SBV^+(0,1)$. Let $S_u=\{t_0,\dots,t_i,\dots\}$ be the jump set of u and let u_l the function of $SBV_\#(0,1)$ with jump set $S_{u_l}=\{t_0,\dots,t_l\}$ defined by $u_l(0^+)=u(0^+)$ and $u_l'=u_a'+\sum_{i=0}^{l}[u](.)\delta_{t_i}$. According to the first step, there exists u_{l,n_k} strongly converging to u_l in $L^1(0,1)$ such that

$$\limsup_{k\to+\infty}E_{1/n_k}(\omega,u_{l,n_k})\le E(u_l).$$

Letting l tend to $+\infty$ and by using a diagonalization argument, there exists a map $k\mapsto l(k)$ such that the sequence $u_{l(k),n_k}$, still denoted by u_{n_k}, strongly converges to u in $L^1(0,1)$ and satisfies

$$\limsup_{k\to+\infty}E_{1/n_k}(\omega,u_{n_k})\le E(u).$$

Last step. According to the previous step, we have $\Gamma-\limsup_{k\to+\infty}E_{1/n_k}(\omega,.)\le\tilde{E}$, where

$$\tilde{E}(u)=\begin{cases}\displaystyle\int_0^1 W^{hom}(u_a')\,dt\ +W^{hom,\infty}(1)\sum_{t\in S_u}[u](t) & \text{if } u\in SBV^+(0,1),\\ +\infty\quad\text{otherwise.}\end{cases}$$

Its lower semicontinuous envelope for the strong topology of $L^1(0,1)$ is (see [112])

$$E(u)=\begin{cases}\displaystyle\int_0^1 W^{hom}(u_a')\,dt\ +W^{hom,\infty}(1)u_s'((0,1)) & \text{if } u\in BV^+(0,1),\\ +\infty\quad\text{otherwise,}\end{cases}$$

which ends the proof. $\square$

PROOF OF THEOREM 14.2.3. Propositions 14.2.1 and 14.2.2 and the last assertion of Theorem 12.1.1 imply that for all ω in the set $\Omega\cap\Omega''\cap\Omega'''$ of full probability, $E_\varepsilon(\omega,.)$ Γ-converges to E. It remains to show that W^{hom} satisfies the three properties of the functions W_j. Convexity, property (iii), and $W^{hom}(1)=0$ have been proved in Lemma 14.2.2(ii) and the growth conditions are trivially satisfied. Finally, we establish $(W^{hom})^*=\sum_{j\in J}p_j W_j^*$. We make precise the notation introduced in the proof of Lemma 14.2.1 by pointing out the dependence of W_n with respect to the parameter δ. We then set

$$W_n^\delta(e)=\frac{\mathscr{S}_{[0,n)}(\delta,e,\omega)}{n}.$$

Its Fenchel conjugate is given for all σ in $\mathbf{R}$ by

$$(W_n^\delta)^*(\sigma) = \frac{1}{n}\int_0^n W_\delta^*(\omega, t, \sigma)\,dt.$$

Equation (14.3) becomes now

$$(W_n^\delta)^*(\sigma) = \sum_{j\in J} \frac{\lambda_j(\omega, n)}{n}(T_\delta W_j)^*(\sigma). \tag{14.10}$$

According to the strong law of large numbers, the right-hand side of (14.10) tends almost surely to

$$\sum_{j\in J} p_j (T_\delta W_j)^*(\sigma)$$

when n goes to infinity. Note that this pointwise limit is also the Γ-limit of

$$\sigma \mapsto \sum_{j\in J} \frac{\lambda_j(\omega, n)}{n}(T_\delta W_j)^*(\sigma)$$

defined on $\mathbf{R}$.

On the other hand the pointwise limit of W_n^δ toward W_δ^{hom} obtained by the subadditive ergodic theorem is also the Γ-limit of $\sigma \mapsto W_n^\delta(\sigma)$ defined on $\mathbf{R}$. Indeed, the sequence of functions $(W_\delta^{hom})_{n\in\mathbf{N}^*}$ is equi-Lipschitz, so that pointwise and Γ-limit agree (see Corollary 2.59 in [37]). According to the continuity of the Fenchel conjugate with respect to the Mosco-convergence, hence here to the Γ-convergence, we deduce from (14.10)

$$(W_\delta^{hom})^*(\sigma) = \sum_{j\in J} p_j (T_\delta W_j)^*(\sigma) \quad \forall \sigma \in \mathbf{R}. \tag{14.11}$$

We would now like to go to the limit on δ in (14.11). Since the sequence $(W_\delta^{hom})_\delta$ of lower semicontinuous functions defined on $\mathbf{R}$ increases to the lower semicontinuous function W^{hom} when δ tends to 0, we have $W_\delta^{hom} \to W^{hom}$ in the sense of Γ-convergence for functionals defined on $\mathbf{R}$. The same argument implies that $T_\delta W_j$ Γ-converges to W_j. According to the continuity of the Fenchel conjugate with respect to the Γ-convergence (note that Mosco- and Γ-convergence agree), we finally deduce our result by going to the limit on δ in (14.11). $\square$

Remark 14.2.1. Concerning the upper bound, for all u in $SBV_\#(0,1) \cap BV^+(0,1)$, we have proved the existence of u_ε strongly converging to u in $L^1(0,1)$ and having the traces of u at 0^+ and 1^-.

It is possible to generalize the previous study when the density functions W_j satisfy a growth condition of order $p > 1$. In this case $W^{hom,\infty}(1) = +\infty$ and the limit functional given by Theorem 14.2.3 becomes

$$E(u) = \begin{cases} \displaystyle\int_0^1 W^{hom}(u')\,dt & \text{if } u' \in L^p(0,1),\ u' > 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Let us consider the functional

$$\tilde{E}_\varepsilon(\omega, u) = \begin{cases} \displaystyle\int_0^1 W\left(\omega, \frac{t}{\varepsilon}, u'\right) dt & \text{if } u \in W^{1,1}(0,1), \\ +\infty & \text{otherwise.} \end{cases}$$

Then, according to Proposition 14.2.2, we have $\Gamma - \limsup_{\varepsilon\to 0} \tilde{E}_\varepsilon \le E$ almost surely. On the other hand, the truncation argument of the first step in the proof of Proposition 14.2.2 implies that for $u_\varepsilon \to u$ in $L^1(0,1)$,

$$\liminf_{\varepsilon\to 0} \tilde{E}_\varepsilon(\omega, u_\varepsilon) \ge \liminf_{\varepsilon\to 0} \tilde{E}_{\varepsilon,\delta}(\omega, u_\varepsilon) \ge E_\delta(u).$$

Arguing as in the second step of this proof, we also obtain $\liminf_{\varepsilon\to 0} \tilde{E}_\varepsilon(\omega, u_\varepsilon) \ge E(u)$. Thus the functional $\tilde{E}(\omega, .)$ Γ-converges to the functional E.

14.2.3 ▪ A second model in one dimension

Keeping the same probabilistic setting, we study a new discrete model for which interaction between each pair of contiguous points is described by a random energy density which is no longer assumed to be convex but which satisfies the same conditions in a neighborhood of 0^+. This energy functional is precisely assumed to be subadditive beyond a random threshold e_ε satisfying $\lim_{\varepsilon\to 0} \varepsilon e_\varepsilon = 0$. In this case, the total energy is of the form

$$F_\varepsilon(\omega, u) = \sum_{z=0}^{n-1} \varepsilon\, W_\varepsilon\left(\omega, z, \frac{u_{z+1} - u_z}{\varepsilon}\right)$$

and almost surely Γ-converges to a deterministic energy functional defined on the subset $SBV^+(0,1) := SBV(0,1) \cap BV^+(0,1)$ by

$$F(u) = \int_0^1 f(u'_a)\, dt + \sum_{t\in S_u} g([u]).$$

More precisely, we consider a finite number $W_j, j \in J$ of density functions satisfying

(i) $W_j : \mathbf{R} \to \mathbf{R}^+ \cup \{+\infty\}$ is convex, finite for $e > 0$, $W_j(1) = 0$, and there exists $\alpha > 0$ such that $\alpha(e-1)^2 \le W_j(e)$ for all e in $[0, +\infty)$;

(ii) there exists $\beta > 0$ such that $W_j(e) \le \beta(1 + e^2)$ for all $e > 1$;

(iii) $\lim_{e\to 0^+} W_j(e) = +\infty$ and $W_j(e) = +\infty$ when $e \le 0$.

Note that (i), (ii), and (iii) are the conditions of the first model but with growth conditions of order 2. On the other hand, let g be a subadditive, continuous function with at most linear growth, mapping $[0, +\infty)$ into $(0, +\infty)$ and satisfying $\inf_{[0,+\infty)} g > 0$. We then consider the density functions $W_{j,\varepsilon}, j \in J$, from $\mathbf{R}$ into $[0, +\infty)$ defined by

$$W_{j,\varepsilon}(e) = \begin{cases} W_j(e) & \text{if } e \le 1, \\ W_j(e) \wedge \dfrac{1}{\varepsilon} g(\varepsilon(e-1)) & \text{if } e > 1. \end{cases}$$

According to growth conditions, it is easily seen that there exists $e_{j,\varepsilon} > 1$ satisfying $\lim_{\varepsilon\to 0} e_{j,\varepsilon} = +\infty$, $\lim_{\varepsilon\to 0} \varepsilon e_{j,\varepsilon} = 0$, and such that

$$W_{j,\varepsilon}(e) = \begin{cases} W_j(e) & \text{if } e \le e_{j,\varepsilon}, \\ \dfrac{1}{\varepsilon} g(\varepsilon(e-1)) & \text{if } e > e_{j,\varepsilon}. \end{cases}$$

An example of such functions is given by $W_j = (e-1)^2$ when $e \geq 1$, $W_j(e) = -\ln e$ when $0 \leq e \leq 1$, and $g(e) = 1+e$ or $g(e) = 1+\sqrt{e}$ or $g(e) = \sqrt{1+e}$.

As said above, the subadditivity assumption on g forces the crack to possess a minimal number of connected components. Denoting by $e_{W_j,\varepsilon}$ every threshold $e_{j,\varepsilon}$, we now can define the random threshold $\omega \mapsto e_\varepsilon(\omega)$ by $e_\varepsilon(\omega) = (e_{\omega_z,\varepsilon})_{z\in\mathbf{Z}}$ and the random function W_ε for all $t \in [z, z+1)$ by

$$W_\varepsilon(\omega, t, e) = \begin{cases} \omega_z(e) & \text{if } e \leq e_{\omega_z,\varepsilon}, \\ \dfrac{1}{\varepsilon} g(\varepsilon(e-1)) & \text{if } e > e_{\omega_z,\varepsilon}. \end{cases}$$

The total energy modeling interactions between contiguous points of $[0,1] \cap \varepsilon\mathbf{Z}$ is the functional defined on $L^1(0,1)$ by

$$F_\varepsilon(\omega, u) = \begin{cases} \displaystyle\sum_{z=0}^{n-1} \varepsilon\, W_\varepsilon\left(\omega, z, \frac{u(\varepsilon(z+1)) - u(\varepsilon z)}{\varepsilon}\right) & \text{if } u \in \mathscr{A}_\varepsilon(0,1), \\ +\infty & \text{otherwise,} \end{cases}$$

or, in a continuous form, by

$$F_\varepsilon(\omega, u) = \begin{cases} \displaystyle\int_0^1 W_\varepsilon\left(\omega, \frac{t}{\varepsilon}, u'\right) dt & \text{if } u \in \mathscr{A}_\varepsilon(0,1), \\ +\infty & \text{otherwise.} \end{cases}$$

We equip $L^1(0,1)$ with its strong convergence. The main result is given below.

Theorem 14.2.4. *The functional F_ε Γ-converges almost surely to the functional F defined in $L^1(0,1)$ by*

$$F(u) = \begin{cases} \displaystyle\int_0^1 W^{hom}(u'_a)\, dt + \sum_{t\in S_u} g([u](t)) & \textit{if } u \in SBV^+(0,1), \\ +\infty & \textit{otherwise,} \end{cases}$$

where W^{hom} is the limit density defined in the first model.

The proof follows the lines of the first model proof: first we establish the lower bound, then the upper one for a subsequence. For more general models, but in a deterministic setting, see [128].

We would like to write the functional F_ε so that the contribution of W_j and g are separated. To this end, we consider the space $SBV_\varepsilon(0,1)$ of all the functions of $SBV(0,1)$ whose restriction to each interval $(\varepsilon z, \varepsilon(z+1))$ included in $(0,1)$ is affine, and we associate to each function u of $\mathscr{A}_\varepsilon(0,1)$ the function $\tilde{u}$ in $SBV_\varepsilon(0,1)$ defined for all $t \in \varepsilon[z, z+1)$ by

$$\tilde{u}(t) = \begin{cases} u(t) & \text{if } \dfrac{u(\varepsilon(z+1)) - u(\varepsilon z)}{\varepsilon} \leq e_{\omega_z,\varepsilon}, \\ t - \varepsilon z + u(\varepsilon z) & \text{otherwise.} \end{cases}$$

Note that actually $\tilde{u}$ is a random function, but we have dropped the dependence on ω to shorten notations. Then, for all $u \in \mathscr{A}_\varepsilon(0,1)$, one has

$$
\begin{aligned}
F_\varepsilon(\omega,u) &= \sum_{\{z\,:\,u(\varepsilon(z+1))-u(\varepsilon z)\leq \varepsilon e_{\omega_z,\varepsilon}\}} \varepsilon W\left(\omega, z, \frac{u(\varepsilon(z+1))-u(\varepsilon z)}{\varepsilon}\right) \\
&\quad + \sum_{\{z\,:\,u(\varepsilon(z+1))-u(\varepsilon z)> \varepsilon e_{\omega_z,\varepsilon}\}} g\left(\varepsilon\left(\frac{u(\varepsilon(z+1))-u(\varepsilon z)}{\varepsilon}-1\right)\right) \\
&\geq \int_0^1 W\left(\omega, \frac{t}{\varepsilon}, \tilde{u}'_a\right) dt + \sum_{t\in S_{\tilde{u}}} g([\tilde{u}](t)) := \tilde{F}_\varepsilon(\omega,\tilde{u}),
\end{aligned}
$$

where $\omega \mapsto W(\omega,t,e)$ is the random function defined in the first model. The proof of the lower bound in the definition of Γ-convergence is based on the following lemma.

Lemma 14.2.3. *Assume that* $\sup_\varepsilon F_\varepsilon(\omega,u_\varepsilon) < +\infty$ *and that* u_ε *strongly converges to some* u *in* $L^1(0,1)$. *Then* $\tilde{u}_\varepsilon$ *strongly converges to* u *in* $L^1_{loc}(0,1)$. *Moreover* u *belongs to* $SBV^+(0,1)$ *and*

$$\sum_{t\in S_u} g([u]) \leq \liminf_{\varepsilon\to 0} \sum_{t\in S_{\tilde{u}_\varepsilon}} g([\tilde{u}_\varepsilon]).$$

PROOF. In the proof of the first statement, the difficulty stems from the lack of coercivity of W_ε with respect to $(u'_\varepsilon - 1)^+$. Nevertheless, note that for all i, j in $\{1,\dots,n-1\}$,

$$\int_{i\varepsilon}^{j\varepsilon} |u_\varepsilon - \tilde{u}_\varepsilon|\, dt \leq \frac{\varepsilon}{2}\int_{i\varepsilon}^{j\varepsilon} (u'_\varepsilon - 1)^+\, dt$$

and that

$$\sup_\varepsilon \int_0^1 (u'_\varepsilon - 1)^-\, dt < +\infty.$$

Thus for $(a,b) \subset\subset (0,1)$, where a, b satisfy $\lim_{\varepsilon\to 0} u_\varepsilon(a) = u(a)$ and $\lim_{\varepsilon\to 0} u_\varepsilon(b) = u(b)$, one has

$$\sup_\varepsilon \int_a^b (u'_\varepsilon - 1)^+\, dt < +\infty.$$

Let now $(a,b) \subset\subset (0,1)$ as above. From $\sup_\varepsilon \tilde{F}_\varepsilon(\omega,\tilde{u}_\varepsilon) < +\infty$ and using the previous estimate, the coercivity assumption on W and the fact that $\inf_{[0,+\infty)} g > 0$, we have

$$
\begin{cases}
\sup_\varepsilon \|\tilde{u}_\varepsilon\|_{BV(a,b)} < +\infty; \\
(\tilde{u}'_{\varepsilon,a})_\varepsilon \text{ equi-integrable on } (a,b); \\
\sup_\varepsilon H^0(S_{\tilde{u}_\varepsilon\lfloor(a,b)}) < +\infty.
\end{cases}
$$

Thus, according to compactness Theorem 13.4.4, u belongs to $SBV(a,b)$ and there exists a subsequence (not relabeled) satisfying

$$
\begin{cases}
\tilde{u}_\varepsilon \rightharpoonup u \quad \text{weakly in } BV(a,b), \\
\tilde{u}'_{\varepsilon,a} \rightharpoonup u'_a \quad \text{weakly in } L^1(a,b), \\
\displaystyle\sum_{t\in S_{\tilde{u}_\varepsilon\lfloor(a,b)}} [\tilde{u}_\varepsilon\lfloor(a,b)]\delta_t \rightharpoonup \sum_{t\in S_{u\lfloor(a,b)}} [u\lfloor(a,b)]\delta_t \quad \text{weakly in } M(a,b), \\
H^0(S_{u\lfloor(a,b)}) \leq \liminf_{\varepsilon\to 0} H^0(S_{\tilde{u}_\varepsilon\lfloor(a,b)}).
\end{cases}
$$

Note that thanks to the coercivity assumption on W_j, $(\tilde{u}'_{\varepsilon,a})_\varepsilon$ is equi-integrable on $(0,1)$ and thus weakly converges to u'_a in $L^1(0,1)$. Moreover, u' is a Borel measure on $(0,1)$ as a nonnegative distribution, so that u belongs to $BV(0,1)$. Therefore u belongs to $SBV(0,1)$ owing to the characterization of the space $SBV(0,1)$ (see Theorem 10.5.1). To prove the last assertion, it is enough to establish

$$\sum_{t\in S_{u\lfloor(a,b)}} g([u]) \leq \liminf_{\varepsilon\to 0} \sum_{t\in S_{\tilde{u}_\varepsilon}} g([\tilde{u}_\varepsilon])$$

and to let a go to 0 and b to 1. Set $\mu_\varepsilon = g([\tilde{u}_\varepsilon\lfloor(a,b)])\, H^0\lfloor S_{\tilde{u}_\varepsilon\lfloor(a,b)}$. From $\sup_\varepsilon \tilde{F}_\varepsilon(\omega,\tilde{u}_\varepsilon) < +\infty$, up to a subsequence, μ_ε weakly converges to a Borel measure $\mu \in M(a,b)$. Let $\mu = \theta\, H^0\lfloor S_{\tilde{u}\lfloor(a,b)} + \mu_s$ its Lebesgue–Nikodým decomposition with respect to the Borel measure $H^0\lfloor S_{\tilde{u}\lfloor(a,b)}$. It suffices now to establish $\theta(t_0) \geq g([u](t_0))$ for $H^0\lfloor S_{\tilde{u}\lfloor(a,b)}$ a.e. t_0 in $(0,1)$. For $H^0\lfloor S_{\tilde{u}\lfloor(a,b)}$ a.e. t_0 in $(0,1)$ and for a.e. $\rho > 0$, we have

$$\begin{aligned}
\theta(t_0) &= \lim_{\rho\to 0}\lim_{\varepsilon\to 0} \frac{\mu_\varepsilon(B_\rho(x_0))}{H^0\lfloor S_{\tilde{u}\lfloor(a,b)}(B_\rho(x_0))}\\
&= \lim_{\rho\to 0}\lim_{\varepsilon\to 0} \mu_\varepsilon(B_\rho(x_0))\\
&= \lim_{\rho\to 0}\lim_{\varepsilon\to 0} \sum_{t\in B_\rho(x_0)\cap S_{\tilde{u}_\varepsilon\lfloor(a,b)}} g([\tilde{u}_\varepsilon](t))\\
&\geq \liminf_{\rho\to 0}\liminf_{\varepsilon\to 0} g\Big(\sum_{t\in B_\rho(x_0)\cap S_{\tilde{u}_\varepsilon\lfloor(a,b)}} [\tilde{u}_\varepsilon](t)\Big)\\
&\geq \liminf_{\rho\to 0} g\Big(\sum_{t\in B_\rho(x_0)\cap S_{u\lfloor(a,b)}} [\tilde{u}](t)\Big)\\
&= g([u](t_0)),
\end{aligned}$$

where we have used the subadditivity and continuity assumptions on g. □

We now establish the lower bound.

Proposition 14.2.3. *There exists a set Ω' of full probability such that for all ω in Ω' and u, u_ε in $L^1(0,1)$ satisfying $u_\varepsilon \to u$ strongly in $L^1(0,1)$,*

$$F(u) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(\omega,u_\varepsilon).$$

PROOF. We assume $\liminf_{\varepsilon\to 0} F_\varepsilon(\omega,u_\varepsilon) < +\infty$. Define v_ε in $W^{1,1}(0,1)$ by $v_\varepsilon(t) = \int_0^t \tilde{u}'_{a,\varepsilon}\, ds$. Since $\tilde{u}'_{a,\varepsilon}$ weakly converges to u'_a in $L^2(0,1)$, v_ε strongly converges in $L^2(0,1)$ to the function v of $L^2(0,1)$ defined by $v(t) = \int_0^t u'_a\, ds$. Therefore, according to Remark 14.2.1 seen for the first model, there exists a set Ω' of full probability such that, for all $\omega \in \Omega'$,

$$\liminf_{\varepsilon\to 0} \int_0^1 W\left(\omega, \frac{t}{\varepsilon}, v'_\varepsilon\right) dt \geq \int_0^1 W^{hom}(v')\, dt.$$

Therefore

$$\liminf_{\varepsilon\to 0} \int_0^1 W\left(\omega, \frac{t}{\varepsilon}, \tilde{u}'_{a,\varepsilon}\right) dt \geq \int_0^1 W^{hom}(u'_a)\, dt,$$

and $u'_a > 0$. We end the proof by applying Lemma 14.2.3. □

To conclude the proof of Theorem 14.2.4, we now establish the upper bound.

Proposition 14.2.4. *There exists a subset Ω'' of full probability such that for all $\omega \in \Omega''$, there exists a subsequence $(n_k)_{k\in\mathbf{N}}$ of positive integers satisfying: for all u in $L^1(0,1)$, there exists u_{n_k} in $L^1(0,1)$ possibly depending on ω, converging to u and such that* $\limsup_{k\to+\infty} F_{1/n_k}(\omega, u_{n_k}) \le F(u)$.

PROOF. Without loss of generality, one may assume $F(u) < +\infty$ and $S_u = \{t_0\}$. Let $m \in \mathbf{N}^*$ and $i_0 \in \mathbf{N}$ be such that $t_0 \in [i_0/m, (i_0+1)/m)$ and set $I_m = (0, i_0/m) \cup ((i_0+1)/m, 1)$. According to Remark 14.2.1, where $(0,1)$ is replaced by $(0, i_0/m)$ or $((i_0+1)/m, 1)$, there exists a subset Ω'' of full probability which can be chosen independent of m, such that for all $\omega \in \Omega''$, there exists $w_n \in \mathscr{A}_{1/mn}(0, i_0/m) \cap \mathscr{A}_{1/mn}((i_0+1)/m, 1)$ satisfying

$$\begin{cases} \lim\limits_{n\to+\infty} w_n = u \quad \text{strongly in } L^1(I_m), \\ w_n\left(\dfrac{i_0}{m}\right) = u\left(\dfrac{i_0}{m}\right) \text{ and } w_n\left(\dfrac{i_0+1}{m}\right) = u\left(\dfrac{i_0+1}{m}\right), \\ \limsup\limits_{n\to+\infty} \displaystyle\int_{I_m} W(\omega, nmt, w_n')\, dt \le \int_{I_m} W^{hom}(u_a')\, dt \le \int_0^1 W^{hom}(u_a')\, dt. \end{cases}$$

Noticing that $W_{1/mn} \le W$, we obtain

$$\limsup_{n\to+\infty} \int_{I_m} W_{1/mn}(\omega, nmt, w_n')\, dt \le \int_0^1 W^{hom}(u_a')\, dt. \tag{14.12}$$

Consider now the function $w_{n,m}$ defined as follows:

$$\begin{cases} w_{n,m} = w_n \quad \text{on } I_m; \\ w_{n,m}\left(\dfrac{i_0}{m}\right) = u\left(\dfrac{i_0}{m}\right), \quad w_{n,m}\left(\dfrac{i_0+1}{m}\right) = u\left(\dfrac{i_0+1}{m}\right); \\ w_{n,m} \text{ is affine on } \left(\dfrac{i_0}{m}, \dfrac{i_0+1}{m} - \dfrac{1}{nm}\right) \text{ with } w_{n,m}' = 1; \\ w_{n,m} \text{ is affine on } \left(\dfrac{i_0+1}{m} - \dfrac{1}{nm}, \dfrac{i_0+1}{m}\right). \end{cases}$$

Clearly $w_{n,m}$ belongs to $\mathscr{A}_{1/mn}(0,1)$ and the slope e of its restriction to $(\frac{i_0+1}{m} - \frac{1}{nm}, \frac{i_0+1}{m})$ satisfies

$$\frac{1}{mn} e = u\left(\frac{i_0+1}{m}\right) - u\left(\frac{i_0}{m}\right) - \frac{n-1}{mn} > u\left(\frac{i_0+1}{m}\right) - u\left(\frac{i_0}{m}\right) - \frac{1}{m},$$

where the last term tends to $[u](t_0) > 0$. Since $\frac{1}{mn} e_{\omega_{i_0+1}, 1/mn}$ tends to 0, for n large enough we have $e > e_{\omega_{i_0+1}, 1/mn}$. A straightforward calculation then yields

$$\int_{(0,1)\setminus \overline{I}_m} W_{1/mn}(\omega, nmt, w_{n,m}')\, dt = g\left(u\left(\frac{i_0+1}{m}\right) - u\left(\frac{i_0}{m}\right) - \frac{1}{m}\right)$$

so that

$$\limsup_{m\to+\infty} \limsup_{n\to+\infty} \int_{(0,1)\setminus \overline{I}_m} W_{1/mn}(\omega, nmt, w_{n,m}')\, dt = g([u](t_0)). \tag{14.13}$$

Finally, combining (14.12) and (14.13), we obtain

$$\limsup_{m\to+\infty}\limsup_{n\to+\infty}\int_0^1 W_{1/mn}(\omega, nmt, w'_{n,m})\,dt \le F(u).$$

On the other hand, clearly

$$\limsup_{m\to+\infty}\limsup_{n\to+\infty} w_{n,m} = u \text{ strongly in } L^1(0,1)$$

and we complete the proof by a diagonalization argument as in Proposition 14.2.2. □

Remark 14.2.2. One can slightly generalize this model by assuming the function g of random type (see [249], [250]). More precisely, let $\{g_j,\ j\in J\}$ be a finite set of Lipschitz functions mapping $[0,+\infty)$ into $(0,+\infty)$, satisfying $\inf_{[0,+\infty)} g > 0$, and not necessarily subadditive. We set $\Omega = \{(w_j, g_j),\ j\in J\}^{\mathbf{Z}}$ and

$$W_{j,\varepsilon}(e) = \begin{cases} W_j(e) & \text{if } e\le 1,\\ W_j(e)\wedge \dfrac{1}{\varepsilon} g_j(\varepsilon(e-1)) & \text{if } e>1, \end{cases}$$

which, as previously, leads to the random function defined by

$$W_\varepsilon(\omega,t,e) = \begin{cases} \omega_z^1(e) & \text{if } e\le e_{\omega_z,\varepsilon},\\ \dfrac{1}{\varepsilon}\omega_z^2(\varepsilon(e-1)) & \text{if } e> e_{\omega_z,\varepsilon} \end{cases}$$

for all $\omega = ((\omega_z^1,\omega_z^2))_{z\in\mathbf{Z}}$ in Ω, t in $[z,z+1)$, and e in $\mathbf{R}$. Then, the corresponding total energy almost surely Γ-converges to the functional

$$F(u) = \begin{cases} \displaystyle\int_0^1 W^{hom}(u'_a)\,dt + \sum_{t\in S_u} g^{hom}([u](t)) & \text{if } u\in SBV^+(0,1),\\ +\infty & \text{otherwise.} \end{cases}$$

Setting $g(\omega,t,.) = \omega_z^2$ for all t in $[z,z+1)$, the density $g^{hom}(a)$ at $a\in\mathbf{R}$ is the almost sure deterministic limit of the process

$$\begin{cases} \mathscr{G}_{(-T,T)}(a) = \inf\left\{ \displaystyle\sum_{t\in(-T,T)\cap S_v} g(\omega,t,[v](t)) \,:\, v\in SBV^+_{0,a}(-T,T)\right\},\\ SBV^+_{0,a}(-T,T) = \{v\in SBV^+(-T,T) \,:\, v'_a = 0,\ v(-T)=0,\ v(T)=a\} \end{cases}$$

when T goes to $+\infty$. It is easily seen that $a\mapsto g^{hom}(a)$ is subadditive.

14.3 ▪ The Mumford–Shah model

Let us recall the Mumford–Shah model discussed in Section 12.5. Let Ω be a bounded open subset of $\mathbf{R}^N$ and g a given function in $L^\infty(\Omega)$. Denoting by $\mathscr{F}$ the class of the closed sets of Ω, for all K in $\mathscr{F}$ and all u in $\mathbf{C}^1(\Omega\setminus K)$ we define the functional

$$E(u,K) := \int_\Omega |u-g|^2\,dx + \int_{\Omega\setminus K} |\nabla u|^2\,dx + \mathscr{H}^{N-1}(K)$$

and the associated optimization problem which is the strong formulation of the Mumford–Shah model in image segmentation:

$$\inf\{E(u,K) \,:\, (u,K) \in \mathbf{C}^1(\Omega \setminus K) \times \mathscr{F}\}. \tag{14.14}$$

When Ω is a rectangle in $\mathbf{R}^2$ and $g(x)$ is the light signal striking Ω at a point x, (14.14) is the Mumford–Shah model of image segmentation: K may be considered as the outline of the given light image in computer vision. In Section 12.5 we introduced the corresponding weak formulation,

$$m_w := \inf\left\{ \int_\Omega |u-g|^2\,dx + \int_\Omega |\nabla u|^2\,dx + \mathscr{H}^{N-1}(S_u) \,:\, u \in SBV(\Omega) \right\}, \tag{14.15}$$

where ∇u denotes the density of the Lebesgue part of Du. One may now establish the weak existence result.

Theorem 14.3.1. *There exists at least a solution of the weak problem* (14.15).

PROOF. We adopt the strategy of the so-called direct methods in the calculus of variations. From the hypothesis $g \in L^\infty(\Omega)$, we may assume that all the admissible functions u in (14.15) are uniformly bounded in $L^\infty(\Omega)$ by $|g|_{L^\infty(\Omega)}$. Indeed, let $c = |g|_{L^\infty(\Omega)}$ and consider the truncated function $u_c = c \wedge u \vee (-c)$. It is easily seen that u_c belongs to $SBV(\Omega)$ and satisfies

$$\begin{cases} S_{u_c} \subset S_u, \\ \displaystyle\int_\Omega |\nabla u_c|^2\,dx \le \int_\Omega |\nabla u|^2\,dx, \\ \displaystyle\int_\Omega |u_c - g|^2\,dx \le \int_\Omega |u-g|^2\,dx. \end{cases}$$

(Note that only the last inequality requires the explicit value of c.) Let $(u_n)_n$ be a minimizing sequence of (14.15). It obviously satisfies

$$|u_n|_\infty + \int_\Omega |\nabla u_n|^2\,dx + H^{N-1}(S_{u_n}) \le C,$$

where C is a constant which does not depend on n. According to Theorem 13.4.3, there exists a subsequence $(u_{n_k})_k$ and a function u^* in $SBV(\Omega)$ such that

$$\begin{cases} u_{n_k} \to u^* \quad \text{in } L^1_{loc}(\Omega), \\ \nabla u_{n_k} \rightharpoonup \nabla u^* \quad \text{in } L^2(\Omega, \mathbf{R}^N), \\ \mathscr{H}^{N-1}(S_{u^*}) \le \liminf\limits_{k \longrightarrow +\infty} H^{N-1}(S_{u_{n_k}}). \end{cases}$$

According to Fatou's lemma and to the weak lower semicontinuity of the norm of the space $L^2(\Omega, \mathbf{R}^N)$, we deduce

$$\begin{aligned} &\int_\Omega |u^* - g|^2\,dx + \int_\Omega |\nabla u^*|^2\,dx + \mathscr{H}^{N-1}(S_{u^*}) \\ &\le \liminf_{k\to+\infty} \int_\Omega |u_{n_k} - g|^2\,dx + \liminf_{k\to+\infty} \int_\Omega |\nabla u_{n_k}|^2\,dx + \liminf_{k\to+\infty} \mathscr{H}^{N-1}(S_{u_{n_k}}) \\ &\le \liminf_{k\to+\infty} \left(\int_\Omega |u_{n_k} - g|^2\,dx + \int_\Omega |\nabla u_{n_k}|^2\,dx + \mathscr{H}^{N-1}(S_{u_{n_k}}) \right) = m_w, \end{aligned}$$

which proves that u^* is a solution of (14.15). □

We establish now the existence of a solution for the strong problem (14.14).

Theorem 14.3.2. *There exists at least a solution of the strong problem* (14.14).

PROOF. The proof proceeds in three steps.

First step. Let m_s be the value of the infimum of problem (14.14). We begin by showing that $m_w \leq m_s$. Indeed, arguing as in the previous proof, one may assume that all the admissible functions of (14.14) are uniformly bounded in $L^\infty(\Omega)$. Moreover, according to Example 10.5.1, for all K in $\mathscr{F}$ such that $\mathscr{H}^{N-1}(K) < +\infty$, the space $W^{1,1}(\Omega \setminus K) \cap L^\infty(\Omega)$ is included in $SBV(\Omega)$ and all its elements satisfy $\mathscr{H}^{N-1}(S_u \setminus K) = 0$.

Second step. We establish that any solution u^* of (14.15) satisfies

$$\begin{cases} u^* \in \mathbf{C}^1(\Omega \setminus \overline{S}_{u^*}), \\ \mathscr{H}^{N-1}(\overline{S}_{u^*} \cap \Omega \setminus S_{u^*}) = 0. \end{cases}$$

We only prove the first assertion. The second is more involved and we refer the reader to the paper of De Giorgi, Carriero, and Leaci [196]. Let $B_\rho(x)$ be the open ball centered at x, with radius ρ small enough so that $B_\rho(x) \subset \Omega \setminus \overline{S}_{u^*}$. Then u^* belongs to $W^{1,2}(B_\rho(x))$ and minimizes the problem

$$\inf \left\{ \int_{B_\rho(x)} |\nabla v|^2 \, dx + \int_{B_\rho(x)} |v - g|^2 \, dx \; : \; v \in u^* + W_0^{1,2}(B_\rho(x)) \right\}.$$

Thus u^* is a solution of the Dirichlet problem

$$\begin{cases} -\Delta v + v = g & \text{in } B_\rho(x), \\ v = u^* & \text{on } \partial B_\rho(x). \end{cases}$$

According to classical results on regularity properties of the solutions of Dirichlet problems (see, for instance, [137], [310]) we have $u^* \in \mathbf{C}^1(B_\rho(x))$.

Last step. Collecting the two previous steps we straightforwardly deduce that $m_w = m_s$ and that $(u^*, \overline{S}_{u^*} \cap \Omega)$ is a solution of (14.14). ☐

For other variational models in computer vision and image processing, see [300], [60], [61], [62], and references therein.