The Crouzeix-Raviart Finite Element Method for a Nonconforming Formulation of the Rudin-Osher-Fatemi Model Problem

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Table of Contents

Table of Contents

Let U be an open subset of $\mathbb{R}^d$. A function $v \in L^1(U)$ is a function of bounded variation iff

$$|v|_{\mathrm{BV}(U)} := \sup_{\substack{\phi \in C_c^1(U;\mathbb{R}^d) \\ \|\phi\|_{L^\infty(U)} \leqslant 1}} \int_U v \operatorname{div}(\phi) \, \mathrm{d}x < \infty.$$

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We have $W^{1,1}(\Omega) \subset \mathrm{BV}(\Omega)$ with $\|v\|_{\mathrm{BV}(\Omega)} = \|v\|_{W^{1,1}(\Omega)}$ for all $v \in W^{1,1}(\Omega)$.

Hedy Attouch, Giuseppe Buttazzo, and Gérard Michaille. **Variational Analysis in Sobolev and BV Spaces. Applications to PDEs and Optimization**. Second Edition. Vol. 17. MOS-SIAM Series on Optimization. Philadelphia: Society for Industrial and Applied Mathematics, Mathematical Optimization Society, 2014. ISBN: 978-1-611973-47-1

Lawrence C. Evans and Ronald F. Gariepy. **Measure Theory and Fine Properties of Functions**. CRC Press, 1992. ISBN: 0-8493-7157-0

Table of Contents

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Rudin-Osher-Fatemi (ROF) model problem

For a parameter $\alpha \in \mathbb{R}_+$ and an input signal $g \in L^2(\Omega)$ minimize the functional

$$I(v) := |v|_{\mathrm{BV}(\Omega)} + \frac{\alpha}{2}\|v - g\|^2_{L^2(\Omega)}$$

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Original picture[0]

[0] https://homepages.cae.wisc.edu/~ece533/images/cameraman.tif

Original picture[0]

Input signal

The input signal was created by adding AWGN with a SNR of 20 to the original picture.

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Input signal

$\alpha = 10^5$

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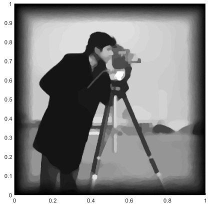

$\alpha = 10^3$

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$$I(v) := |v|_{BV(\Omega)} + \frac{\alpha}{2}\|v - g\|^2_{L^2(\Omega)}$$

Original picture

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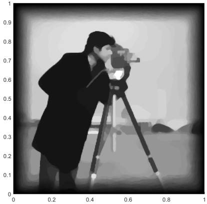

$\alpha = 10^3$

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Pascal Getreuer. “Rudin-Osher-Fatemi Total Variation Denoising using Split Bregman”. In: **Image Processing On Line** 2 (2012), pp. 74–95. URL: https://doi.org/10.5201/ipol.2012.g-tvd

Table of Contents

Continuous problem

For a parameter $\alpha \in \mathbb{R}_+$ and an input signal $f \in L^2(\Omega)$ minimize the functional

$$E(v) := \frac{\alpha}{2}\|v\|^2 + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \int_\Omega fv\,\mathrm{d}x$$

amongst all $v \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$.

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For $f = \alpha g$ the functional E has the same minimizers as

$$I(v) = |v|_{\mathrm{BV}(\Omega)} + \frac{\alpha}{2}\|v - g\|^2_{L^2(\Omega)}$$

in $\left\{v \in \mathrm{BV}(\Omega) \cap L^2(\Omega) \mid \|v\|_{L^1(\partial\Omega)} = 0\right\}$.

Theorem (Existence and uniqueness of a minimizer)

There exists a unique minimizer $u \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$ *for* $E(v) = \frac{\alpha}{2}\|v\|^2 + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \int_\Omega fv \,\mathrm{d}x$ *amongst all* $v \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$.

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Lemma

Let $v \in \mathrm{BV}(\Omega)$. *For all* $x \in \mathbb{R}^d$, *define*

$$\tilde{v}(x) := \begin{cases} v(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^d \setminus \overline{\Omega}. \end{cases}$$

Then $\tilde{v} \in \mathrm{BV}(\mathbb{R}^d)$ *and* $|\tilde{v}|_{\mathrm{BV}(\mathbb{R}^d)} = |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)}$.

Let U be an open subset of $\mathbb{R}^d$.

Definition (Weak convergence in $\mathrm{BV}(U)$)

Let $(v_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(U)$ and $v \in \mathrm{BV}(U)$ with $v_n \to v$ in $L^1(U)$ as $n \to \infty$. Then $(v_n)_{n\in\mathbb{N}}$ converges weakly to v in $\mathrm{BV}(U)$ iff, for all $\phi \in C_0(U;\mathbb{R}^d)$, it holds

$$\int_U v_n \operatorname{div}(\phi)\,\mathrm{d}x \to \int_U v \operatorname{div}(\phi)\,\mathrm{d}x \quad \text{as } n \to \infty.$$

We write $v_n \rightharpoonup v$ as $n \to \infty$.

Theorem

Let $v \in L^1(U)$ *and* $(v_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(U)$ *with* $\sup_{n\in\mathbb{N}} |v_n|_{\mathrm{BV}(U)} < \infty$ *and* $v_n \to v$ *in* $L^1(U)$ *as* $n \to \infty$. *Then* $v \in \mathrm{BV}(U)$ *and* $|v|_{\mathrm{BV}(U)} \leqslant \liminf_{n\to\infty} |v_n|_{\mathrm{BV}(U)}$. *Furthermore,* $v_n \rightharpoonup v$ *in* $\mathrm{BV}(U)$.

Theorem

Let $v \in L^1(U)$ and $(v_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(U)$ with $\sup_{n\in\mathbb{N}} |v_n|_{\mathrm{BV}(U)} < \infty$ and $v_n \to v$ in $L^1(U)$ as $n \to \infty$. Then $v \in \mathrm{BV}(U)$ and $|v|_{\mathrm{BV}(U)} \leqslant \liminf_{n\to\infty} |v_n|_{\mathrm{BV}(U)}$. Furthermore, $v_n \rightharpoonup v$ in $\mathrm{BV}(U)$.

Let U be a bounded Lipschitz domain.

Theorem

Let $(v_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(U)$ be bounded. Then there exists some subsequence $(v_{n_k})_{k\in\mathbb{N}}$ of $(v_n)_{n\in\mathbb{N}}$ and $v \in \mathrm{BV}(U)$ such that $v_{n_k} \to v$ in $L^1(U)$ as $k \to \infty$.

Theorem (Stability)

Let $f_1, f_2 \in L^2(\Omega)$. For $\ell \in \{1,2\}$, let $u_\ell \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$ minimize

$$E_\ell(v) := \frac{\alpha}{2}\|v\|^2 + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \int_\Omega f_\ell v \,\mathrm{d}x$$

amongst all $v \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$. Then

$$\|u_1 - u_2\| \leqslant \frac{1}{\alpha}\|f_1 - f_2\|.$$

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Sören Bartels. **Numerical Methods for Nonlinear Partial Differential Equations**. Vol. 47. Springer Series in Computational Mathematics. Springer International Publishing, 2015. ISBN: 978-3-319-13796-4. DOI: 10.1007/978-3-319-13797-1, Chapter 10, p. 297-319.

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Table of Contents

Let $\mathcal{T}$ be a regular triangulation of Ω.

For all $v_{\mathrm{CR}} \in \mathrm{CR}^1(\mathcal{T})$,

$$|v_{\mathrm{CR}}|_{\mathrm{BV}(\Omega)} = \|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^1(\Omega)} + \sum_{F \in \mathcal{E}(\Omega)} \|[v_{\mathrm{CR}}]_F\|_{L^1(F)}.$$

In particular, $\mathrm{CR}^1(\mathcal{T}) \subset \mathrm{BV}(\Omega)$.

$$E(v_{\mathrm{CR}}) = \frac{\alpha}{2}\|v_{\mathrm{CR}}\|^2 + |v_{\mathrm{CR}}|_{\mathrm{BV}(\Omega)} + \|v_{\mathrm{CR}}\|_{L^1(\partial\Omega)} - \int_\Omega f v_{\mathrm{CR}}\,\mathrm{d}x$$

$$|v_{\mathrm{CR}}|_{\mathrm{BV}(\Omega)} + \|v_{\mathrm{CR}}\|_{L^1(\partial\Omega)} = \|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^1(\Omega)} + \sum_{F\in\mathcal{E}} \|[v_{\mathrm{CR}}]_F\|_{L^1(F)}$$

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Discrete problem

For a parameter $\alpha \in \mathbb{R}_+$ and an input signal $f \in L^2(\Omega)$ minimize the functional

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amongst all $v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$.

Theorem (Existence and uniqueness of a minimizer)

There exists a unique minimizer $u_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ *for* $E_{\mathrm{NC}}(v_{\mathrm{CR}}) := \frac{\alpha}{2}\|v_{\mathrm{CR}}\|^2 + \|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^1(\Omega)} - \int_\Omega f v_{\mathrm{CR}}\,\mathrm{d}x$ *amongst all* $v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$.

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Let $K := \left\{\Lambda \in L^\infty(\Omega;\mathbb{R}^2) \,\middle|\, |\Lambda(\bullet)| \leqslant 1 \text{ a.e. in } \Omega\right\}$ and, for all $(v_{\mathrm{CR}}, \Lambda_0) \in \mathrm{CR}^1_0(\mathcal{T}) \times P_0(\mathcal{T};\mathbb{R}^2)$,

$$L(v_{\mathrm{CR}}, \Lambda_0) := \int_\Omega \Lambda_0 \cdot \nabla_{\mathrm{NC}} v_{\mathrm{CR}}\,\mathrm{d}x + \frac{\alpha}{2}\|v_{\mathrm{CR}}\|^2 - \int_\Omega f v_{\mathrm{CR}}\,\mathrm{d}x - I_K(\Lambda_0).$$

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Minimax problem

Find $(\tilde{u}_{\mathrm{CR}}, \bar{\Lambda}_0) \in \mathrm{CR}_0^1(\mathcal{T}) \times P_0(\mathcal{T};\mathbb{R}^2)$ such that

$$L(\tilde{u}_{\mathrm{CR}}, \bar{\Lambda}_0) = \inf_{v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})} \sup_{\Lambda_0 \in P_0(\mathcal{T};\mathbb{R}^2)} L(v_{\mathrm{CR}}, \Lambda_0).$$

$$L(v_{\mathrm{CR}},\Lambda_0) := \int_\Omega \Lambda_0 \cdot \nabla_{\mathrm{NC}} v_{\mathrm{CR}} \,\mathrm{d}x + \frac{\alpha}{2}\|v_{\mathrm{CR}}\|^2 - \int_\Omega f v_{\mathrm{CR}} \,\mathrm{d}x - I_K(\Lambda_0)$$

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Minimax problem

Find $\left(\tilde{u}_{\mathrm{CR}}, \bar{\Lambda}_0\right) \in \mathrm{CR}_0^1(\mathcal{T}) \times P_0\left(\mathcal{T}; \mathbb{R}^2\right)$ such that

$$L(\tilde{u}_{\mathrm{CR}}, \bar{\Lambda}_0) = \inf_{v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})} \sup_{\Lambda_0 \in P_0(\mathcal{T}; \mathbb{R}^2)} L(v_{\mathrm{CR}}, \Lambda_0).$$

This problem has a solution $\left(\tilde{u}_{\mathrm{CR}}, \bar{\Lambda}_0\right) \in \mathrm{CR}_0^1(\mathcal{T}) \times \left(P_0\left(\mathcal{T}; \mathbb{R}^2\right) \cap K\right)$.

$$L(v_{\mathrm{CR}}, \Lambda_0) := \int_\Omega \Lambda_0 \cdot \nabla_{\mathrm{NC}} v_{\mathrm{CR}} \,\mathrm{d}x + \frac{\alpha}{2}\|v_{\mathrm{CR}}\|^2 - \int_\Omega f v_{\mathrm{CR}} \,\mathrm{d}x - I_K(\Lambda_0)$$

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$$L(\tilde{u}_{\mathrm{CR}}, \bar{\Lambda}_0) = \inf_{v_{\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T})} \sup_{\Lambda_0 \in P_0(\mathcal{T};\mathbb{R}^2)} L(v_{\mathrm{CR}}, \Lambda_0).$$

This problem has a solution
$\left(\tilde{u}_{\mathrm{CR}}, \bar{\Lambda}_0\right) \in \mathrm{CR}^1_0(\mathcal{T}) \times \left(P_0\left(\mathcal{T}; \mathbb{R}^2\right) \cap K\right)$.

R. Tyrrell Rockafellar. **Convex Analysis**. New Jersey: Princeton University Press, 1970. ISBN: 0-691-08069-0

Theorem (Equivalent characterizations)

For a function $\tilde{u}_{\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T})$ *the following statements are equivalent.*

(i) $\tilde{u}_{\mathrm{CR}}$ *solves the discrete problem.*

(ii) There exists $\bar{\Lambda}_0 \in P_0\big(\mathcal{T};\mathbb{R}^2\big)$ *with* $\big|\bar{\Lambda}_0(\bullet)\big| \leqslant 1$ *a.e. in* Ω *s.t.*

$$\bar{\Lambda}_0(\bullet)\cdot\nabla_{\mathrm{NC}}\tilde{u}_{\mathrm{CR}}(\bullet) = |\nabla_{\mathrm{NC}}\tilde{u}_{\mathrm{CR}}(\bullet)| \quad \textit{a.e. in } \Omega$$

and

$$\big(\bar{\Lambda}_0, \nabla_{\mathrm{NC}} v_{\mathrm{CR}}\big) = (f - \alpha\tilde{u}_{\mathrm{CR}}, v_{\mathrm{CR}}) \quad \textit{for all } v_{\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T}).$$

(iii) For all $v_{\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T})$,

$$(f - \alpha\tilde{u}_{\mathrm{CR}}, v_{\mathrm{CR}} - \tilde{u}_{\mathrm{CR}}) \leqslant \|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^1(\Omega)} - \|\nabla_{\mathrm{NC}}\tilde{u}_{\mathrm{CR}}\|_{L^1(\Omega)}.$$

Table of Contents

Algorithm (Primal-dual iteration)

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Input: $(u_0, \Lambda_0) \in \mathrm{CR}^1_0(\mathcal{T}) \times P_0\big(\mathcal{T}; \overline{B_{\mathbb{R}^2}}\big)$, $\tau > 0$, $\varepsilon_{\mathrm{stop}} > 0$

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Theorem (Convergence of the primal-dual iteration)

Let $u_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ *solve the discrete problem,* $\bar{\Lambda}_0 \in P_0\big(\mathcal{T};\mathbb{R}^2\big)$ *satisfy* $\big|\bar{\Lambda}_0(\bullet)\big| \leqslant 1$ *a.e. in* Ω *as well as*

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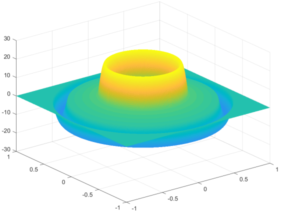
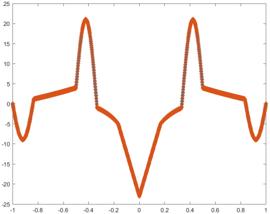
Table of Contents

Let $\Omega = (-1,1)^2$. Define $f \in H_0^1(\Omega)$ by $f(x) = \tilde{f}(|x|)$ for all $x \in \Omega$ with

$$\tilde{f}(r) := \begin{cases} \alpha - 12(2-9r) & \text{if } 0 \leqslant r \leqslant \frac{1}{6}, \\ 6r\alpha - \frac{1}{r} & \text{if } \frac{1}{6} \leqslant r \leqslant \frac{1}{3} \\ 2\alpha + 6\pi \sin(\pi(6r-2)) - \frac{1}{r}\cos(\pi(6r-2)) & \text{if } \frac{1}{3} \leqslant r \leqslant \frac{1}{2}, \\ \alpha(5-6r) + \frac{1}{r} & \text{if } \frac{1}{2} \leqslant r \leqslant \frac{5}{6}, \\ -3\pi \sin(\pi(6r-5)) + \frac{1+\cos(\pi(6r-5))}{2r} & \text{if } \frac{5}{6} \leqslant r \leqslant 1. \end{cases}$$



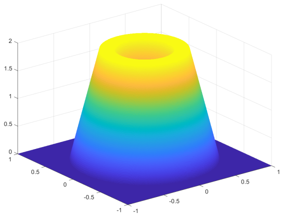
f for $\alpha = 1$



f for $\alpha = 1$ along the axes

Then the solution to the continuous problem with input signal f is given by $u \in H^1_0(\Omega)$ defined by $u(x) = \tilde{u}(|x|)$ for all $x \in \Omega$ with

$$\tilde{u}(r) := \begin{cases} 1 & \text{if } 0 \leqslant r \leqslant \frac{1}{6}, \\ 6r & \text{if } \frac{1}{6} \leqslant r \leqslant \frac{1}{3}, \\ 2 & \text{if } \frac{1}{3} \leqslant r \leqslant \frac{1}{2}, \\ 5 - 6r & \text{if } \frac{1}{2} \leqslant r \leqslant \frac{5}{6}, \\ 0 & \text{if } \frac{5}{6} \leqslant r. \end{cases}$$



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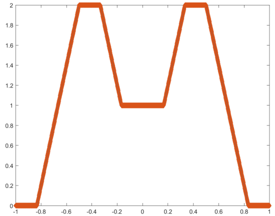

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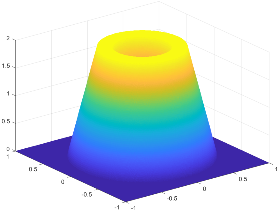

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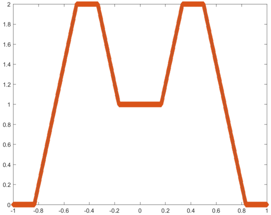


u along the axes

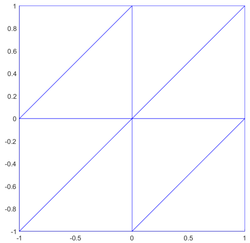
It holds $E(u) \approx -2.058034062391$.

For $\alpha = 10000$ let the input signal represent the grayscale of an image in $[0,1]^{256\times 256}$ multiplied with α scaled to the domain $\Omega = (0,1)^2$.

Image cameraman

Initial Triangulations for the Input Signals

Input signal f



$\Omega = (-1, 1)^2$

Input signal cameraman

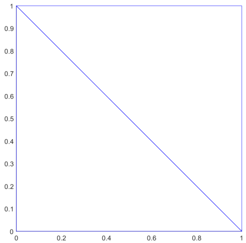


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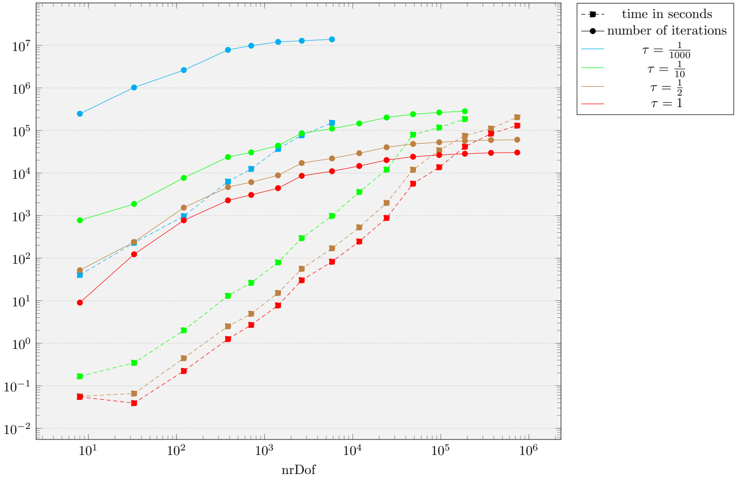
Table of Contents

Choice of τ

For the rest of the presentation (unless otherwise specified) let the bulk parameter be $\theta = 0.5$, and $\varepsilon_{\text{stop}} = 10^{-4}$.



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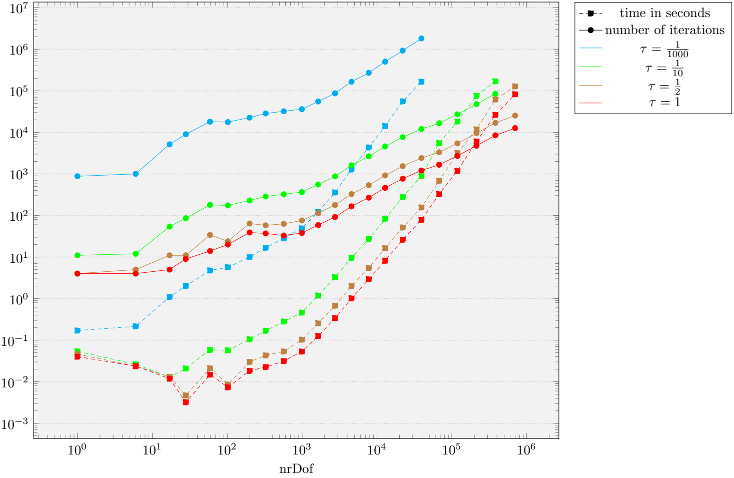

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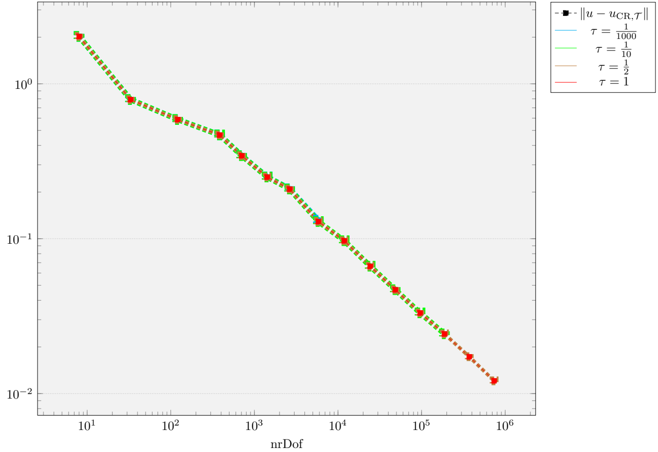

Input signal f

Conclusion and Hypothesis

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For the rest of the presentation $\tau = 1$.

Convergence of the iterates of the primal-dual iteration to the discrete solution u_{CR} followed from

$$\sum_{j=1}^{\infty} \|u_{\mathrm{CR}} - u_j\|^2 \leqslant \frac{1}{2\alpha\tau} \left(|||u_{\mathrm{CR}} - u_0|||_{\mathrm{NC}}^2 + \|\bar{\Lambda}_0 - \Lambda_0\|^2 \right).$$

Settings with $\tau = 1.2$ and no convergence were observed.

For $v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$, define $J_1 : \mathrm{CR}_0^1(\mathcal{T}) \to P_1(\mathcal{T}) \cap C_0(\Omega)$ by

$$J_1 v_{\mathrm{CR}}(z) := |\mathcal{T}(z)|^{-1} \sum_{T \in \mathcal{T}(z)} v_{\mathrm{CR}}|_T(z) \quad \text{for all } z \in \mathcal{N}(\Omega).$$

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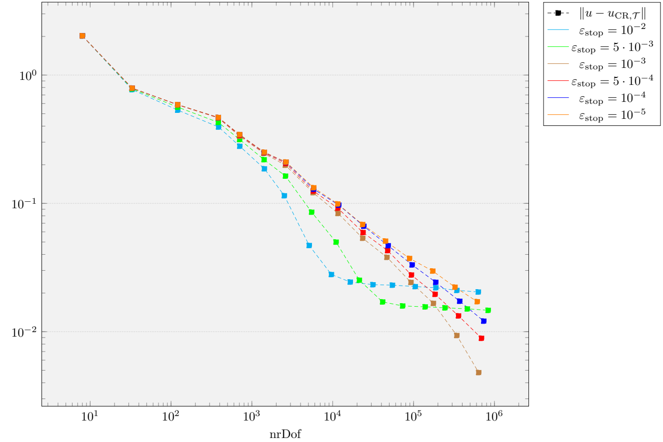
Use $\hat{u}_0 := J_1 u_{\mathrm{CR},\mathcal{T}} \in P_1(\mathcal{T}) \cap C_0(\Omega) \subseteq P_1(\hat{\mathcal{T}}) \cap C_0(\Omega) \subseteq \mathrm{CR}_0^1(\hat{\mathcal{T}})$ as input for the iteration on the refined triangulation $\hat{\mathcal{T}}$.

Choice of $\varepsilon_{\text{stop}}$

With $\tau = 1$ the stopping criterion reads $|||u_j - u_{j-1}|||_{\text{NC}} < \varepsilon_{\text{stop}}$.



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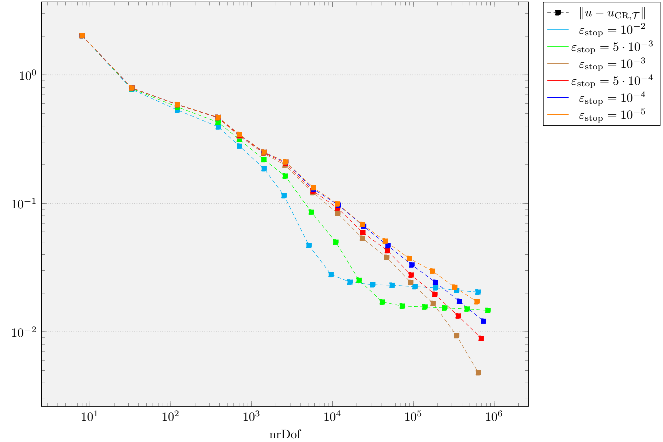

For the rest of the presentation $\varepsilon_{\text{stop}} = 10^{-4}$.

Table of Contents

Theorem (Guaranteed lower energy bound)

Let Ω be convex. Let $f \in H^1_0(\Omega)$ be the input signal of the continuous (discrete) problem with solution $u \in H^1_0(\Omega)$ ($u_{\mathrm{CR}} \in \mathrm{CR}^1_0(\Omega)$). Then

$$E_{\mathrm{NC}}(u_{\mathrm{CR}}) + \frac{\alpha}{2}\|u - u_{\mathrm{CR}}\|^2 - \frac{\kappa_{\mathrm{CR}}}{\alpha}\|h_{\mathcal{T}}(f - \alpha u_{\mathrm{CR}})\|\|\nabla f\| \leqslant E(u)$$

with $\kappa_{\mathrm{CR}} := \sqrt{1/48 + 1/j_{1,1}^2} \approx 0.298217419$.

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with $\kappa_{\mathrm{CR}} := \sqrt{1/48 + 1/j_{1,1}^2} \approx 0.298217419$*. In particular*

$$E_{\mathrm{GLEB}} := E_{\mathrm{NC}}(u_{\mathrm{CR}}) - \frac{\kappa_{\mathrm{CR}}}{\alpha}\|h_{\mathcal{T}}(f - \alpha u_{\mathrm{CR}})\|\|\nabla f\|$$

satisfies $E_{\mathrm{NC}}(u_{\mathrm{CR}}) \geqslant E_{\mathrm{GLEB}}$ *and* $E(u) \geqslant E_{\mathrm{GLEB}}$.

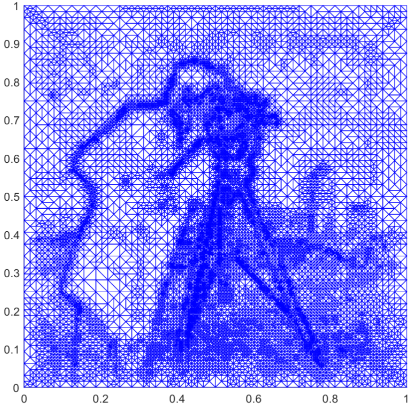
Definition (Refinement indicator)

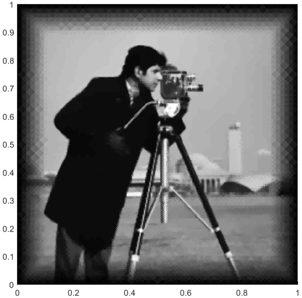
Let $0 < \gamma \leqslant 1$ (in this presentation $\gamma = 1$). For all $T \in \mathcal{T}$ and $u_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$, define

$$\begin{aligned}
\eta_{\mathrm{V}}(T) &:= |T| \|f - \alpha u_{\mathrm{CR}}\|_{L^2(T)}^2, \\
\eta_{\mathrm{J}}(T) &:= |T|^{\gamma/2} \sum_{F \in \mathcal{E}(T)} \|[u_{\mathrm{CR}}]_F\|_{L^1(F)}, \quad \text{and} \\
\eta(T) &:= \eta_{\mathrm{V}}(T) + \eta_{\mathrm{J}}(T).
\end{aligned}$$

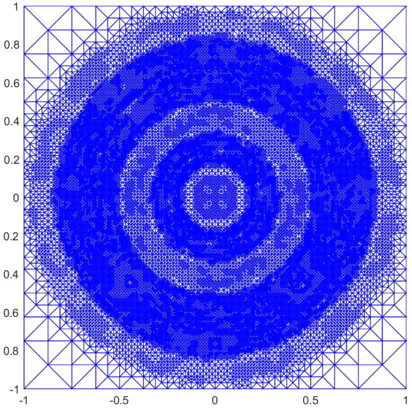
With that define the refinement indicator $\eta := \sum_{T \in \mathcal{T}} \eta(T)$.

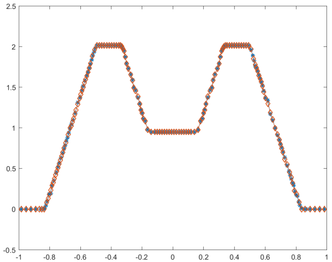
The mesh with 39232 degrees of freedom for level 16 of the adaptive algorithm with $\theta = 0.5$ and input signal cameraman and the solution of the iteration on this mesh.



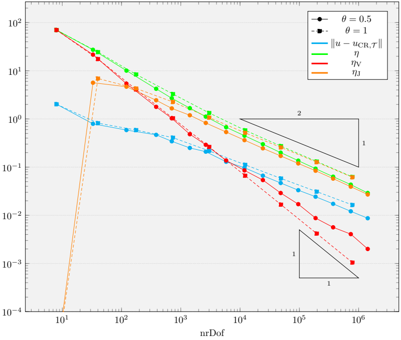


The mesh with 95865 degrees of freedom for level 11 of the adaptive algorithm with $\theta = 0.5$ and input signal f and the solution of the iteration on this mesh along the axes.

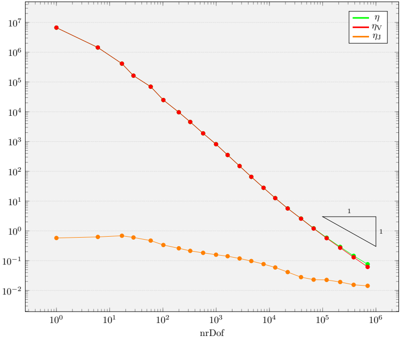




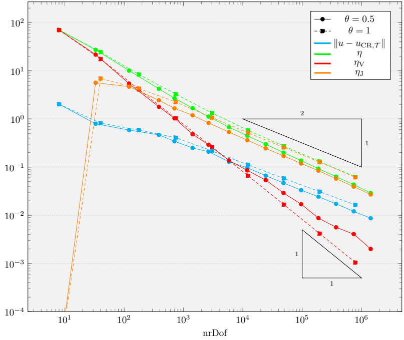
Input signal f



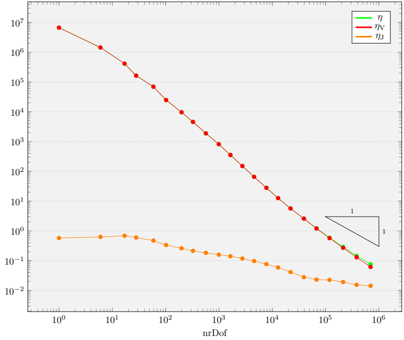
Input signal cameraman



Input signal f



Input signal cameraman

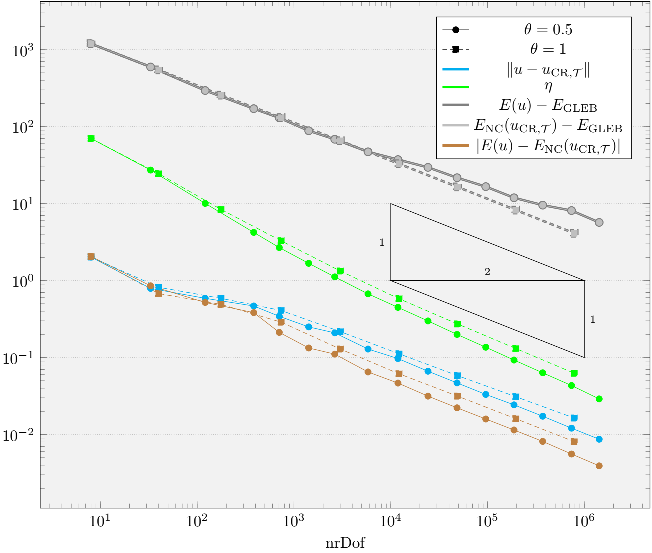


[Bar15, p. 309, Thm. 10.7]

For the conforming discretization of the ROF model problem with the Courant FEM it holds

$$\frac{\alpha}{2}\|u - u_{\mathrm{C}}\|^2 \lesssim h^{1/2}.$$

Input signal f



Let $u \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$ ($u_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$) solve the continuous (discrete) problem with input signal f. Then

$$\begin{aligned}\frac{\alpha}{2}\|u - u_{\mathrm{CR}}\|^2 &\leqslant E(u_{\mathrm{CR}}) - E(u)\\ &= E_{\mathrm{NC}}(u_{\mathrm{CR}}) + \sum_{F\in\mathcal{E}} \|[u_{\mathrm{CR}}]_F\|_{L^1(F)} - E(u).\end{aligned}$$

Let $u \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$ ($u_{\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T})$) solve the continuous (discrete) problem with input signal f. Then

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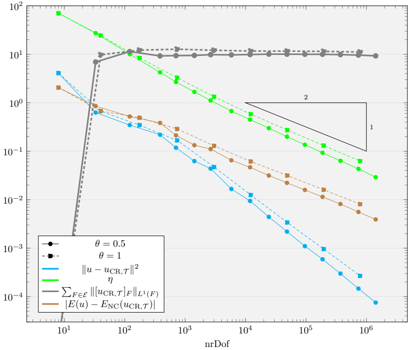
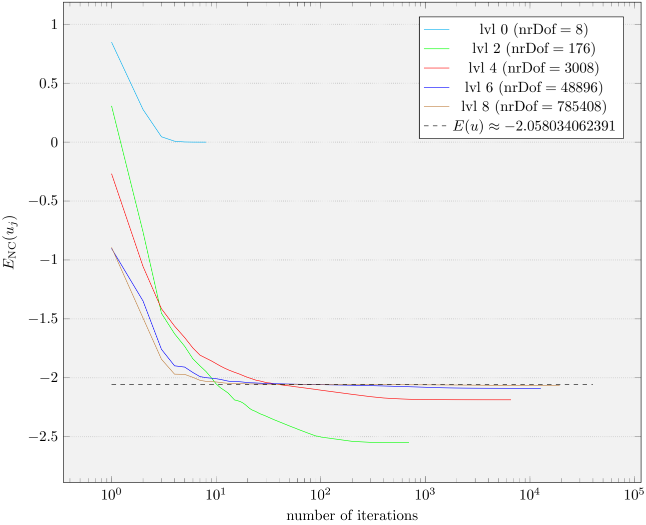
$$= E_{\mathrm{NC}}(u_{\mathrm{CR}}) + \sum_{F \in \mathcal{E}} \|[u_{\mathrm{CR}}]_F\|_{L^1(F)} - E(u).$$



Input signal f

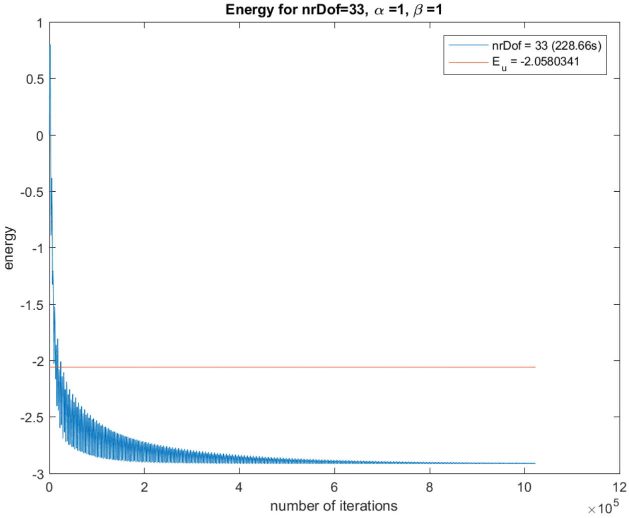
Table of Contents

Input Signal f, $\theta = 0.5$



$\tau = 1$

Input Signal f, $\theta = 0.5$

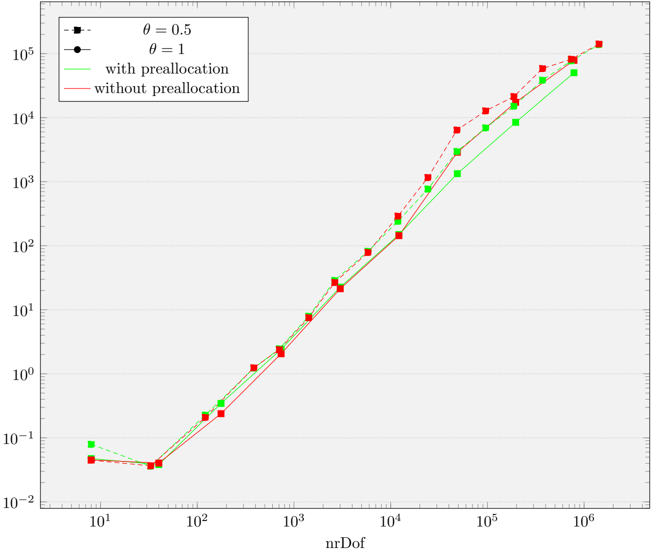


$\tau = 10^{-3}$

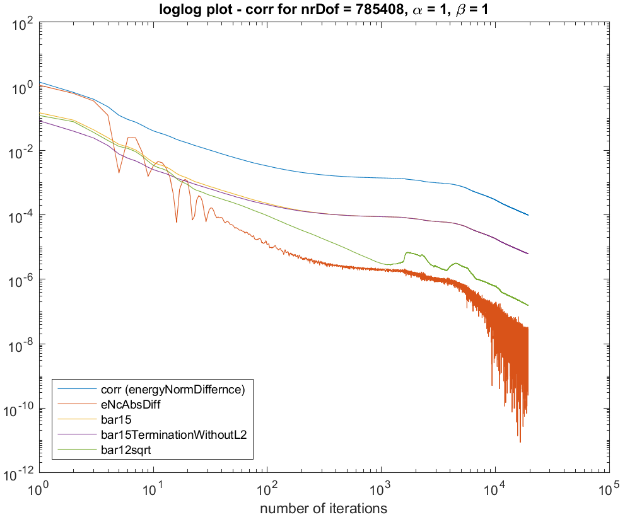
Table of Contents

Thank you for your attention.

Input Signal f



Input Signal f, $\theta = 1$



Let $u_P : [0,\infty) \to \mathbb{R}$ with $u_P(r) = 0$ for $r \geqslant 1$, and, for all $x \in \Omega$, $u(x) = u_P\big(|x|\big)$. Furthermore, assume the existence of $\partial_r u_P$ a.e. in $[0,\infty)$, the existence of the derivative of

$$\operatorname{sgn}\left(\partial_r u_P(r)\right) := \begin{cases} -1 & \text{für } \partial_r u_P(r) < 0, \\ x \in [0,1] & \text{für } \partial_r u_P(r) = 0, \\ 1 & \text{für } \partial_r u_P(r) > 0. \end{cases}$$

a.e. in $[0,\infty)$, and that $\operatorname{sgn}\left(\partial_r u_P(r)\right)/r \to 0$ as $r \to 0$. For all $r \in [0,\infty)$, define

$$f_P(r) := \alpha u_P(r) - \partial_r\left(\operatorname{sgn}\left(\partial_r u_P(r)\right)\right) - \frac{\operatorname{sgn}\left(\partial_r u_P(r)\right)}{r}$$

Then u solves the continuous problem on $\Omega \supseteq \left\{ w \in \mathbb{R}^2 \,\middle|\, |w| \leqslant 1 \right\}$ if the input signal is $f(x) := f_P\big(|x|\big)$.