The Crouzeix-Raviart Finite Element Method for a Nonconforming Formulation of the Rudin-Osher-Fatemi Model Problem

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Let U be an open subset of $\mathbb{R}^d$. A function $v \in L^1(U)$ is a function of bounded variation iff

$$|v|_{\mathrm{BV}(U)} := \sup_{\substack{\phi \in C_c^1(U;\mathbb{R}^d) \\ \|\phi\|_{L^\infty(U)} \leqslant 1}} \int_U v \operatorname{div}(\phi)\,\mathrm{d}x < \infty.$$

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We have $W^{1,1}(\Omega) \subset \mathrm{BV}(\Omega)$ with $\|v\|_{\mathrm{BV}(\Omega)} = \|v\|_{W^{1,1}(\Omega)}$ for all $v \in W^{1,1}(\Omega)$.

Hedy Attouch, Giuseppe Buttazzo, and Gérard Michaille. **Variational Analysis in Sobolev and BV Spaces. Applications to PDEs and Optimization**. Second Edition. Vol. 17. MOS-SIAM Series on Optimization. Philadelphia: Society for Industrial and Applied Mathematics, Mathematical Optimization Society, 2014. ISBN: 978-1-611973-47-1

Lawrence C. Evans and Ronald F. Gariepy. **Measure Theory and Fine Properties of Functions**. CRC Press, 1992. ISBN: 0-8493-7157-0

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Rudin-Osher-Fatemi (ROF) model problem

For a parameter $\alpha \in \mathbb{R}_+$ and an input signal $g \in L^2(\Omega)$ minimize the functional

$$I(v) := |v|_{\mathrm{BV}(\Omega)} + \frac{\alpha}{2}\|v - g\|^2_{L^2(\Omega)}$$

amongst all $v \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$.

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Original picture[0]

[0] https://homepages.cae.wisc.edu/~ece533/images/cameraman.tif

Original picture[0]

Input signal

The input signal was created by adding AWGN with a SNR of 20 to the original picture.

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$\alpha = 10^5$

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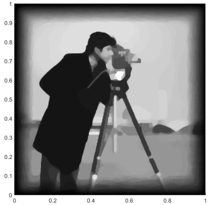

$\alpha = 10^3$

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Original picture

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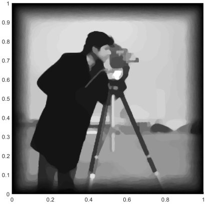

$\alpha = 10^3$

$\alpha = 10^4$

$\alpha = 10^5$

Pascal Getreuer. “Rudin-Osher-Fatemi Total Variation Denoising using Split Bregman”. In: **Image Processing On Line** 2 (2012), pp. 74–95. URL: https://doi.org/10.5201/ipol.2012.g-tvd

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Continuous problem

For a parameter $\alpha \in \mathbb{R}_+$ and an input signal $f \in L^2(\Omega)$ minimize the functional

$$E(v) := \frac{\alpha}{2}\|v\|^2 + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \int_\Omega fv \,\mathrm{d}x$$

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For $f = \alpha g$ the functional E has the same minimizers as

$$I(v) = |v|_{\mathrm{BV}(\Omega)} + \frac{\alpha}{2}\|v - g\|^2_{L^2(\Omega)}$$

in $\left\{v \in \mathrm{BV}(\Omega) \cap L^2(\Omega) \mid \|v\|_{L^1(\partial\Omega)} = 0\right\}$.

Theorem (Existence and uniqueness of a minimizer)

There exists a unique minimizer $u \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$ *for* $E(v) = \frac{\alpha}{2}\|v\|^2 + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \int_\Omega fv \,\mathrm{d}x$ *amongst all* $v \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$.

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Lemma

Let $v \in \mathrm{BV}(\Omega)$. *For all* $x \in \mathbb{R}^d$, *define*

$$\tilde{v}(x) := \begin{cases} v(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^d \setminus \overline{\Omega}. \end{cases}$$

Then $\tilde{v} \in \mathrm{BV}(\mathbb{R}^d)$ *and* $|\tilde{v}|_{\mathrm{BV}(\mathbb{R}^d)} = |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)}$.

Let U be an open subset of $\mathbb{R}^d$.

Definition (Weak convergence in $\mathrm{BV}(U)$)

Let $(v_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(U)$ and $v \in \mathrm{BV}(U)$ with $v_n \to v$ in $L^1(U)$ as $n \to \infty$. Then $(v_n)_{n\in\mathbb{N}}$ converges weakly to v in $\mathrm{BV}(U)$ iff, for all $\phi \in C_0(U;\mathbb{R}^d)$, it holds

$$\int_U v_n \operatorname{div}(\phi)\,\mathrm{d}x \to \int_U v \operatorname{div}(\phi)\,\mathrm{d}x \quad \text{as } n \to \infty.$$

We write $v_n \rightharpoonup v$ as $n \to \infty$.

Theorem

Let $v \in L^1(U)$ *and* $(v_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(U)$ *with* $\sup_{n\in\mathbb{N}} |v_n|_{\mathrm{BV}(U)} < \infty$ *and* $v_n \to v$ *in* $L^1(U)$ *as* $n \to \infty$*. Then* $v \in \mathrm{BV}(U)$ *and* $|v|_{\mathrm{BV}(U)} \leqslant \liminf_{n\to\infty} |v_n|_{\mathrm{BV}(U)}$*. Furthermore,* $v_n \rightharpoonup v$ *in* $\mathrm{BV}(U)$*.*

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Let U be a bounded Lipschitz domain.

Theorem

Let $(v_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(U)$ *be bounded. Then there exists some subsequence* $(v_{n_k})_{k\in\mathbb{N}}$ *of* $(v_n)_{n\in\mathbb{N}}$ *and* $v \in \mathrm{BV}(U)$ *such that* $v_{n_k} \to v$ *in* $L^1(U)$ *as* $k \to \infty$*.*

Theorem (Stability)

Let $f_1, f_2 \in L^2(\Omega)$*. For* $\ell \in \{1,2\}$*, let* $u_\ell \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$ *minimize*

$$E_\ell(v) := \frac{\alpha}{2}\|v\|^2 + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \int_\Omega f_\ell v \,\mathrm{d}x$$

amongst all $v \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$*. Then*

$$\|u_1 - u_2\| \leqslant \frac{1}{\alpha}\|f_1 - f_2\|.$$

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Sören Bartels. **Numerical Methods for Nonlinear Partial Differential Equations**. Vol. 47. Springer Series in Computational Mathematics. Springer International Publishing, 2015. ISBN: 978-3-319-13796-4. DOI: 10.1007/978-3-319-13797-1, Chapter 10, p. 297-319.

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Let $\mathcal{T}$ be a regular triangulation of Ω.

For all $v_{\mathrm{CR}} \in \mathrm{CR}^1(\mathcal{T})$,

$$|v_{\mathrm{CR}}|_{\mathrm{BV}(\Omega)} = \|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^1(\Omega)} + \sum_{F \in \mathcal{E}(\Omega)} \|[v_{\mathrm{CR}}]_F\|_{L^1(F)}.$$

In particular, $\mathrm{CR}^1(\mathcal{T}) \subset \mathrm{BV}(\Omega)$.

$$E(v_{\mathrm{CR}}) = \frac{\alpha}{2}\|v_{\mathrm{CR}}\|^2 + |v_{\mathrm{CR}}|_{\mathrm{BV}(\Omega)} + \|v_{\mathrm{CR}}\|_{L^1(\partial\Omega)} - \int_\Omega f v_{\mathrm{CR}}\,\mathrm{d}x$$

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Discrete problem

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Theorem (Existence and uniqueness of a minimizer)

There exists a unique minimizer $u_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ *for* $E_{\mathrm{NC}}(v_{\mathrm{CR}}) := \frac{\alpha}{2}\|v_{\mathrm{CR}}\|^2 + \|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^1(\Omega)} - \int_\Omega f v_{\mathrm{CR}}\,\mathrm{d}x$ *amongst all* $v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$.

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Let $K := \left\{\Lambda \in L^\infty(\Omega;\mathbb{R}^2) \;\middle|\; |\Lambda(\bullet)| \leqslant 1 \text{ a.e. in } \Omega\right\}$ and, for all $(v_{\mathrm{CR}}, \Lambda_0) \in \mathrm{CR}^1_0(\mathcal{T}) \times P_0(\mathcal{T};\mathbb{R}^2)$,

$$L(v_{\mathrm{CR}}, \Lambda_0) := \int_\Omega \Lambda_0 \cdot \nabla_{\mathrm{NC}} v_{\mathrm{CR}}\,\mathrm{d}x + \frac{\alpha}{2}\|v_{\mathrm{CR}}\|^2 - \int_\Omega f v_{\mathrm{CR}}\,\mathrm{d}x - I_K(\Lambda_0).$$

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Minimax problem

Find $(\tilde{u}_{\mathrm{CR}}, \bar{\Lambda}_0) \in \mathrm{CR}_0^1(\mathcal{T}) \times P_0(\mathcal{T};\mathbb{R}^2)$ such that

$$L(\tilde{u}_{\mathrm{CR}}, \bar{\Lambda}_0) = \inf_{v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})} \sup_{\Lambda_0 \in P_0(\mathcal{T};\mathbb{R}^2)} L(v_{\mathrm{CR}}, \Lambda_0).$$

$$L(v_{\mathrm{CR}},\Lambda_0) := \int_\Omega \Lambda_0 \cdot \nabla_{\mathrm{NC}} v_{\mathrm{CR}}\,\mathrm{d}x + \frac{\alpha}{2}\|v_{\mathrm{CR}}\|^2 - \int_\Omega f v_{\mathrm{CR}}\,\mathrm{d}x - I_K(\Lambda_0)$$

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Minimax problem

Find $(\tilde{u}_{\mathrm{CR}}, \bar{\Lambda}_0) \in \mathrm{CR}_0^1(\mathcal{T}) \times P_0(\mathcal{T}; \mathbb{R}^2)$ such that

$$L(\tilde{u}_{\mathrm{CR}}, \bar{\Lambda}_0) = \inf_{v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})} \sup_{\Lambda_0 \in P_0(\mathcal{T}; \mathbb{R}^2)} L(v_{\mathrm{CR}}, \Lambda_0).$$

This problem has a solution $(\tilde{u}_{\mathrm{CR}}, \bar{\Lambda}_0) \in \mathrm{CR}_0^1(\mathcal{T}) \times \left(P_0(\mathcal{T}; \mathbb{R}^2) \cap K\right)$.

$$L(v_{\mathrm{CR}},\Lambda_0) := \int_\Omega \Lambda_0\cdot\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\,\mathrm{d}x + \frac{\alpha}{2}\|v_{\mathrm{CR}}\|^2 - \int_\Omega f v_{\mathrm{CR}}\,\mathrm{d}x - I_K(\Lambda_0)$$

Minimax problem

Find $\left(\tilde{u}_{\mathrm{CR}},\bar{\Lambda}_0\right) \in \mathrm{CR}^1_0(\mathcal{T}) \times P_0\left(\mathcal{T};\mathbb{R}^2\right)$ such that

$$L(\tilde{u}_{\mathrm{CR}},\bar{\Lambda}_0) = \inf_{v_{\mathrm{CR}}\in\mathrm{CR}^1_0(\mathcal{T})}\ \sup_{\Lambda_0\in P_0(\mathcal{T};\mathbb{R}^2)} L(v_{\mathrm{CR}},\Lambda_0).$$

This problem has a solution $\left(\tilde{u}_{\mathrm{CR}},\bar{\Lambda}_0\right) \in \mathrm{CR}^1_0(\mathcal{T}) \times \left(P_0\left(\mathcal{T};\mathbb{R}^2\right)\cap K\right)$.

R. Tyrrell Rockafellar. **Convex Analysis**. New Jersey: Princeton University Press, 1970. ISBN: 0-691-08069-0

Theorem (Equivalent characterizations)

For a function $\tilde{u}_{\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T})$ *the following statements are equivalent.*

(i) $\tilde{u}_{\mathrm{CR}}$ *solves the discrete problem.*

(ii) There exists $\bar{\Lambda}_0 \in P_0(\mathcal{T};\mathbb{R}^2)$ *with* $\left|\bar{\Lambda}_0(\bullet)\right| \leqslant 1$ *a.e. in* Ω *s.t.*

$$\bar{\Lambda}_0(\bullet) \cdot \nabla_{\mathrm{NC}} \tilde{u}_{\mathrm{CR}}(\bullet) = |\nabla_{\mathrm{NC}} \tilde{u}_{\mathrm{CR}}(\bullet)| \quad \textit{a.e. in } \Omega$$

and

$$\left(\bar{\Lambda}_0, \nabla_{\mathrm{NC}} v_{\mathrm{CR}}\right) = (f - \alpha \tilde{u}_{\mathrm{CR}}, v_{\mathrm{CR}}) \quad \textit{for all } v_{\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T}).$$

(iii) For all $v_{\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T})$,

$$(f - \alpha \tilde{u}_{\mathrm{CR}}, v_{\mathrm{CR}} - \tilde{u}_{\mathrm{CR}}) \leqslant \|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^1(\Omega)} - \|\nabla_{\mathrm{NC}} \tilde{u}_{\mathrm{CR}}\|_{L^1(\Omega)}.$$

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Algorithm (Primal-dual iteration)

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Input: $(u_0, \Lambda_0) \in \mathrm{CR}^1_0(\mathcal{T}) \times P_0\big(\mathcal{T}; \overline{B_{\mathbb{R}^2}}\big)$, $\tau > 0$, $\varepsilon_{\mathrm{stop}} > 0$

Initialize $v_0 := 0$ *in* $\mathrm{CR}^1_0(\mathcal{T})$.

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$$v_j := \frac{u_j - u_{j-1}}{\tau}.$$ *Terminate iteration if* $|||v_j||| < \varepsilon_{\mathrm{stop}}$.

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Theorem (Convergence of the primal-dual iteration)

Let $u_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ *solve the discrete problem,* $\bar{\Lambda}_0 \in P_0(\mathcal{T};\mathbb{R}^2)$ *satisfy* $|\bar{\Lambda}_0(\bullet)| \leqslant 1$ *a.e. in* Ω *as well as*

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Let $u_P : [0, \infty) \to \mathbb{R}$ with $u_P(r) = 0$ for $r \geqslant 1$, and, for all $x \in \Omega$, $u(x) = u_P\big(|x|\big)$.

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$$\operatorname{sgn}\big(\partial_r u_P(r)\big) := \begin{cases} -1 & \text{für } \partial_r u_P(r) < 0, \\ x \in [0,1] & \text{für } \partial_r u_P(r) = 0, \\ 1 & \text{für } \partial_r u_P(r) > 0. \end{cases}$$

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Then u solves the continuous problem on $\Omega \supseteq \big\{w \in \mathbb{R}^2 \,\big|\, |w| \leqslant 1\big\}$ if the input signal is $f(x) := f_P\big(|x|\big)$.

f01 mit exakter Lösung beschreiben und vlt auch Plots zeigen und außerdem die potentiellen Bilderinputs whiteSquare und cameraman kurz zeigen als Exps ohne exakte Lösung

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choice of tau plots for f01 (mentioning that the same behaviour was seen for the other experiments). Also show inequaltiy from convergence proof again talking about the upper bound choice of epsStop plots for f01 (only quickly show the stop of reduction of the l2 error at certain points

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drüber nachdenken, was hier gezeigt werden soll. Idealerweise viele subsections mit Themenbereichen (f01, cam, termCrit, tau. . .)
termination criteria experiments only in the end if questions arise, only mention the possible termination criteria and that they seem equally valid (except for energy difference)
show tau experiments
energy during a iteration (convergence of subsequences from above, i.e. also choose one exampe with osscilating convergence)
find good alpha for denoising
show adaptive mesh for camerman and maybe for square to show the working of the refinement indicator
vom Kapitel continuous problem auch die Konstruktion einer exakten Lösung anreißen
L2 Sprünge vielleicht auswerten (bleiben sie konstant. . . , if we consider them, it becomes conforming
Verfeinerungsindikator, strikte Konvexität, EGLEB alles hier genau dann, wenn danach ein Plot dazu kommen soll.
Probably etaJumps and etaVol Vergleich und eta und Fehler in einem getrennten Plot, in einem gesamt Plot dann irgendwann, wo

tien schicken spätestens am Wochenende vor der Präsi, CC vor der Präsi die fertige Präsi + akuteller Stand der Arbeit schicken