Chapter 16

An introduction to shape optimization problems

In this section we give a quick introduction to shape optimization problems in a rather general framework, and we discuss some of their features, especially in relation to the existence of an optimal solution. Our goal is not to give here a detailed presentation of the many problems and results in this very wide field, but only to show that several optimization problems, often very important for applications in mechanics and engineering, cannot be formulated by taking a Banach function space of the competing admissible choices: a more appropriate framework consists in taking as admissible controls the elements of a class of domains. We refer the reader interested in a deeper knowledge and analysis of this fascinating field to one of the several books on the subject [239], [11], [322], [337], to the notes by Tartar [347], or to the collection of lecture notes by Bucur and Buttazzo [144], [145].

A shape optimization problem is a minimization problem where the unknown variable runs over a class of domains; then every shape optimization problem can be written in the form

$$\min\left\{F(A) \,:\, A\in\mathscr{A}\right\}, \tag{16.1}$$

where $\mathscr{A}$ is the class of admissible domains and F is the cost function that one has to minimize over $\mathscr{A}$.

It must be noted that the class $\mathscr{A}$ of admissible domains does not have any linear or convex structure, so in shape optimization problems it is meaningless to speak of convex functionals and similar notions. Moreover, even if several topologies on families of domains are available, in general there is not an a priori choice of a topology which allows us to apply the direct methods of the calculus of variations for obtaining the existence of at least an optimal domain.

We want to stress that, as also happens in other kinds of optimal control problems, in several situations an optimal domain does not exist; this is mainly because in these cases the minimizing sequences are highly oscillating and converge to a limit object only in a suitable "relaxed" sense. Then we may have, in these cases, only the existence of a relaxed solution that in general is not a domain, and whose characterization may change from problem to problem.

A general procedure to relax optimal control problems can be successfully developed by using the Γ-convergence scheme which provides the right topology that has to be used for sequences of admissible controls. In particular, for shape optimization problems, this provides the right notion of convergence for sequences of domains. Presenting in a detailed way the abstract framework of relaxed optimal control problems through the

Γ-convergence would require us to develop several preliminary tools as background. This goes beyond our purposes, so we simply refer the interested reader to [92], where this framework was first introduced, or to [144], [151].

Coming back to the minimization problem (16.1), in general, unless some geometrical constraints on the admissible sets are assumed, or some very special cases of cost functionals are considered, the existence of an optimal domain may fail. In these situations the discussion will then be focused on the relaxed solutions that always exist.

As usually happens in all optimization problems, to give a qualitative description of the optimal solutions of a shape optimization problem, it is important to derive the so-called necessary conditions of optimality. These conditions have to be derived from the comparison of the cost of an optimal solution A_{opt} to the cost of other suitable admissible choices, close enough to A_{opt}. This procedure is what is usually called a *variation* near the solution. The difficulty in obtaining necessary conditions of optimality for shape optimization problems consists in the fact that, since the unknown variables are domains, the notion of neighborhood is not a priori clear; the possibility of choosing a domain variation could then be rather wide. The same method can be applied, when no classical solution exists, to relaxed solutions, and this will provide some qualitative information about the behavior of the minimizing sequences of the original problem.

Finally, for some particular problems presenting special behaviors or symmetries, one would like to exhibit explicit solutions (balls, ellipsoids, ...). This could be very difficult, even for simple problems, and often, instead of having established results, one can give only conjectures.

In general, since the explicit computations are difficult, one should develop efficient numerical schemes to produce approximated solutions. This is a challenging field we will not enter; we refer the interested reader to the books and papers available on the subject [11], [322], [337].

In the following examples we show that several classical optimization problems can be written in the form (16.1).

16.1 ▪ The isoperimetric problem

The isoperimetric problem is certainly the oldest shape optimization problem; it seems to go back to the Greek golden age of mathematics (Archimedes, Zenodorus, etc.), and a legend about Queen Dido shows that the question was clearly formulated long ago. The problem with constraint Q (see, for instance, [84]) consists in finding among all Borel subsets A of a given closed set $Q \subset \mathbf{R}^N$ the one which minimizes the perimeter, once its Lebesgue measure, or more generally the quantity $\int_A f(x)\,dx$ for a given function $f \in L^1_{loc}(\mathbf{R}^N)$, is prescribed. With this notation the isoperimetric problem can be then formulated in the form (16.1) if we take

$$F(A) = \mathrm{Per}(A),$$
$$\mathscr{A} = \left\{ A \subset Q \;:\; \int_A f(x)\,dx = c \right\}.$$

Here the perimeter of a Borel set is the one defined in Chapter 10 as

$$\mathrm{Per}(A) = \int |D1_A| = \mathscr{H}^{N-1}(\partial^* A),$$

where $D1_A$ is the distributional derivative of the characteristic function of A, $\mathscr{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure introduced in Section 4.1, and $\partial^* A$ is the reduced boundary defined in Section 10.3. By using the properties of the BV spaces seen in Chapter 10, when Q is bounded we obtain the lower semicontinuity and the coercivity of the perimeter for the L^1 convergence, which enables us to apply the direct methods of the calculus of variations of Section 3.2 and to obtain straightforwardly the existence of an optimal solution for the problem

$$\min\left\{ \operatorname{Per}(A) \,:\, A \subset Q, \int_A f\,dx = c \right\}. \tag{16.2}$$

It is also very simple to show that in general the problem above may have no solution if we drop the assumption that Q is bounded (see, for instance, [144]). Take indeed $f \equiv 1$, $c = \pi$, and Q the countable union of all closed disks in $\mathbf{R}^2$ of the form $\overline{B(x_n, r_n)}$, where $x_n = (2n, 0)$ and $r_n = 1 - 1/n$ (see Figure 16.1). It is then easy to see that the infimum of problem (16.2) is 2π, whereas no admissible domain in $\mathscr{A}$ provides the value 2π to the cost functional.

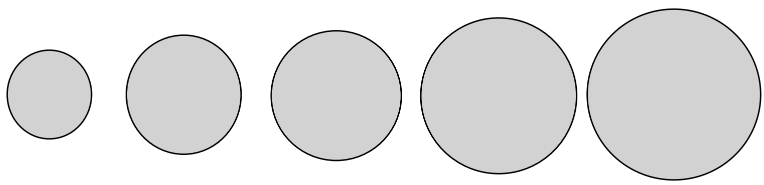

Figure 16.1. *An unbounded set Q.*

On the other hand, it is very well known that the classical isoperimetric problem, with $Q = \mathbf{R}^N$ and $f \equiv 1$, admits a solution which is any ball of measure c, even if the complete proof of this fact requires very delicate tools, especially when the dimension N is larger than 2. A complete characterization of pairs (Q, f) which provide the existence of a solution for the problem (16.2) seems to be difficult.

A variant of the isoperimetric problem consists in not counting some parts of the boundary ∂A in the cost functional. More precisely, if Q is the closure of an open set Ω with a Lipschitz boundary, we may consider problem (16.2) with $\operatorname{Per}(A)$ replaced by the cost functional

$$\operatorname{Per}_\Omega(A) = \int_\Omega |D1_A| = \mathscr{H}^{N-1}(\Omega \cap \partial^* A)$$

which does not count the part of ∂A which is included in $\partial\Omega$. The existence of a solution when Ω is bounded still holds, as above, together with nonexistence examples when this boundedness condition is dropped. Indeed, it is enough to take f, c, Q as above and to observe that the infimum of problem (16.2) is in this case zero, whereas no admissible domain provides the value zero to the cost functional.

16.2 ▪ The Newton problem

Another classical question which can be considered as a shape optimization problem is the determination of the best aerodynamic profile for a body in a fluid stream under some constraints on its size. This problem, at least within the class of radially symmetric bodies, which makes the problem one-dimensional, was first considered by Newton, who gave a rather simple variational expression for the aerodynamic resistance of a convex body in a fluid stream. Here are his words (from *Principia Mathematica*):

If in a rare medium, consisting of equal particles freely disposed at equal distances from each other, a globe and a cylinder described on equal diameter move with equal velocities in the direction of the axis of the cylinder, (then) the resistance of the globe will be half as great as that of the cylinder.... I reckon that this proposition will be not without application in the building of ships.

Indeed, if we make the assumption that the resistance is due to the impact of fluid particles against the body surface, if all the particles are supposed independent (which is quite reasonable if the fluid is rarefied), and if the tangential friction is neglected, simple geometric considerations lead us to obtain for the resistance along the direction of the fluid stream the expression

$$R(u) = \int_\Omega \frac{1}{1+|Du|^2}\, dx, \tag{16.3}$$

where we normalize to one all the physical multiplicative constants involving the density and the velocity of the fluid. Here Ω represents the cross section of the body at the basis level, and $u(x)$ is a function whose graph is the body upper boundary. Since the validity of the model requires that all particles hit the body at most once, we consider only convex bodies, which turns out to require Ω convex and $u : \Omega \to [0, +\infty[$ concave.

Note that the integral functional F above is neither convex nor coercive. Therefore, obtaining an existence theorem for minimizers via the usual direct method in the calculus of variations may fail. Indeed, if we do not impose any further constraint on the competing functions u, the infimum of the functional in (16.3) turns out to be zero, as immediately seen by taking, for instance,

$$u_n(x) = n\, \mathrm{dist}(x, \partial\Omega)$$

for every $n \in \mathbf{N}$ and by letting $n \to +\infty$. Therefore, no function u can minimize the functional F, because $F(u) > 0$ for every function u.

A complete discussion of the problem can be found in [144], where all the relevant references are quoted. Here we simply recall that the problem

$$\min \left\{ \int_\Omega \frac{1}{1+|Du|^2}\, dx \ : \ u \text{ concave},\ 0 \le u \le M \right\}$$

admits a solution u_{opt}. Some interesting necessary conditions of optimality can be deduced: for instance (see [269]), it can be proved that on every open set ω where u_{opt} is of class C^2 we obtain

$$\det D^2 u(x) = 0 \qquad \forall x \in \omega.$$

In particular, this excludes that in the case $\Omega = B(0, R)$ the solution u_{opt} is radially symmetric. The optimal radially symmetric profile and a nonsymmetric profile which is better that all the radial ones are shown, respectively, in Figures 16.2 and 16.3.

It is interesting to notice that with simple calculations, one can write the optimization problem above in the form (16.1) by taking the cost functional as a boundary integral,

$$F(A) = \int_{\partial A} j\big(x, \nu(x)\big)\, d\mathscr{H}^{N-1},$$

for a suitable integrand $j(x, s)$, being $\nu(x)$ the exterior normal unit vector to ∂A at x and $\mathscr{H}^{N-1}$ the Hausdorff $(N-1)$-dimensional measure (see [155]).

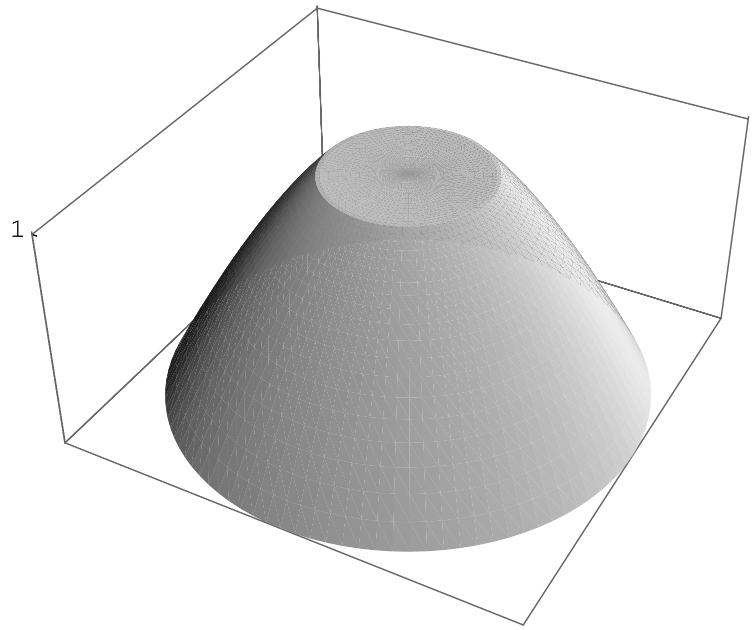

Figure 16.2. *The optimal radial shape for $M = R$.*

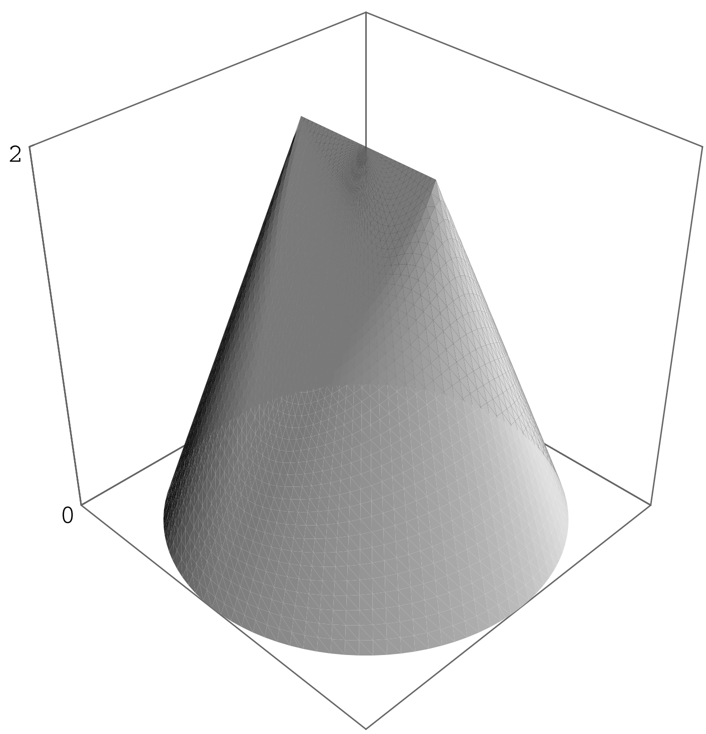

Figure 16.3. *A nonradial profile better than all radial ones for $M = 2R$.*

16.3 ▪ Optimal Dirichlet free boundary problems

We consider now the model example of a Dirichlet problem over an unknown domain, which has to be optimized according to a given cost functional. More precisely, we consider a given bounded open subset Ω of $\mathbf{R}^N$, an admissible class $\mathscr{A}$ of subsets of Ω, a given function $f \in L^2(\Omega)$, and a cost functional of the form

$$F(A) = \int_\Omega j(x, u_A)\, dx. \tag{16.4}$$

Here the integrand $j : \Omega \times \mathbf{R} \to \mathbf{R}$ is given, and we denote by u_A the unique solution of the elliptic problem

$$\begin{cases} -\Delta u = f & \text{in } A, \\ u \in H_0^1(A), \end{cases} \tag{16.5}$$

extended by zero to $\Omega \setminus A$.

It is well known that in general one should not expect the existence of an optimal solution; below we show an example where the existence of an optimal domain does not occur. The problem we consider is of the form (16.1), where the admissible class $\mathscr{A}$ consists of all subdomains of a given bounded open subset Ω of $\mathbf{R}^N$ and the cost functional F is of the form (16.4) with

$$j(x, s) = |s - \overline{u}(x)|^2$$

for a prescribed desired state $\overline{u}$. In the thermostatic model the shape optimization problem (16.1) with the choices above consists in finding an optimal distribution, inside Ω, of the Dirichlet region $\Omega \setminus A$ to achieve a temperature which is as close as possible to the desired temperature $\overline{u}$, once the heat sources f are prescribed.

For simplicity, we consider a uniformly distributed heat source, that is, we take $f \equiv 1$, and we take the desired temperature $\overline{u}$ constantly equal to $c > 0$. Therefore, problem (16.1) becomes

$$\min \left\{ \int_\Omega |u_A - c|^2\, dx \;:\; -\Delta u_A = 1 \text{ in } A,\ u_A \in H_0^1(A) \right\}. \tag{16.6}$$

We will actually show that for small values of the constant c no regular domain A can solve problem (16.6) above; the proof of nonexistence of any domain is slightly more delicate and requires additional tools like the capacitary form of necessary conditions of optimality (see, for instance, [144], [148], [149], [171]).

Proposition 16.3.1. *If $c > 0$ is small enough, then problem* (16.6) *has no smooth solutions.*

PROOF. The nonexistence proof can be obtained by contradiction. Assume indeed that a regular domain A solves the optimization problem (16.6) and that A does not coincide with the whole set Ω. Take a point $x_0 \in \Omega \setminus \overline{A}$ and a small ball B_ε of radius ε sufficiently small, centered at x_0. If u_A denotes the solution of (16.5) corresponding to A, and if ε is small enough so that B_ε does not intersect A, then the solution $u_{A\cup B_\varepsilon}$, corresponding to the admissible choice $A \cup B_\varepsilon$, is given by

$$u_{A\cup B_\varepsilon}(x) = \begin{cases} u_A(x) & \text{if } x \in A, \\ (\varepsilon^2 - |x - x_0|^2)/4 & \text{if } x \in B_\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we obtain

$$F(A)=\int_A |u_A-c|^2\,dx+\int_{B_\varepsilon} c^2\,dx+\int_{\Omega\setminus(A\cup B_\varepsilon)} c^2\,dx,$$

$$F(A\cup B_\varepsilon)=\int_A |u_A-c|^2\,dx+\int_{B_\varepsilon}\left|\frac{\varepsilon^2-|x-x_0|^2}{4}-c\right|^2 dx+\int_{\Omega\setminus(A\cup B_\varepsilon)} c^2\,dx.$$

By using the minimality of A this then yields

$$\begin{aligned} c^2\,\mathrm{meas}(B_\varepsilon) &\le \int_{B_\varepsilon}\left|\frac{\varepsilon^2-|x-x_0|^2}{4}-c\right|^2 dx \\ &= N\varepsilon^{-N}\,\mathrm{meas}(B_\varepsilon)\int_0^\varepsilon \left|\frac{\varepsilon^2-r^2}{4}-c\right|^2 r^{N-1}\,dr \\ &= c^2\,\mathrm{meas}(B_\varepsilon)+\frac{1}{16}\int_0^\varepsilon (\varepsilon^2-r^2)(\varepsilon^2-r^2-8c)r^{N-1}\,dr, \end{aligned}$$

which, for a fixed $c>0$, turns out to be false if ε is small enough.

Thus all regular domains $A\neq\Omega$ are ruled out by the argument above. We can now exclude also the case $A=\Omega$ if c is small enough, by comparing the full domain Ω to the empty set $\emptyset$. This gives, taking into account that $u_\emptyset\equiv 0$,

$$F(\Omega)=\int_\Omega |u_\Omega-c|^2\,dx,$$

$$F(\emptyset)=\int_\Omega c^2\,dx,$$

so that we have $F(\emptyset)<F(\Omega)$ if c is small enough. Hence all regular subdomains of Ω are excluded, and the nonexistence proof is achieved. □

By a more refined proof we could exclude all kinds of domains, so we may conclude that the shape optimization problem (16.6) has no solutions at all. In fact, we could consider the relaxed problem

$$\min\left\{\int_\Omega |u_\mu-c|^2\,dx\ :\ -\Delta u_\mu+\mu u=1 \text{ in } \Omega,\ u_\mu\in H_0^1(\Omega)\right\}, \tag{16.7}$$

where now μ varies in the class $\mathbf{M}_0(\Omega)$ of all capacitary measures introduced in Section 5.8.4. By the compactness properties of $\mathbf{M}_0(\Omega)$ with respect to the γ-convergence the existence of a relaxed solution μ is straightforward; by an argument developed in [171] it is possible to prove that this solution is also unique.

The example above proves that even very simple shape optimization problems do not admit a domain as an optimal solution. Nevertheless, the existence of an optimal domain occurs for problem (16.1) in some particular cases:

(i) when severe geometrical constraints on the class of admissible domains are imposed;

(ii) when the cost functional fulfills some particular qualitative assumptions;

(iii) when the problem is of a very special type, involving, for instance, only the eigenvalues of the Laplace operator, and where neither geometrical constraints nor monotonicity of the cost is required.

See the lecture notes of Bucur and Buttazzo [144], [145] for a more complete discussion on this topic; here, for the sake of simplicity, we list only some situations when conditions (i), (ii), and (iii) occur.

Concerning case (i), a rather simple situation when the existence of an optimal solution occurs is when the class of admissible domains fulfills the geometrical constraint which is called the *exterior cone condition*. It consists in requiring that there exists a fixed height h and opening ω such that for every domain A of the admissible class $\mathscr{A}$ and every point $x_0 \in \partial A$ a cone with height h, opening ω, and vertex at x_0 is contained in $\Omega \backslash A$. This condition is weaker than an equi-Lipschitz conditions on the class of admissible domains; for instance, a domain with an exterior cusp is not Lipschitz but verifies the exterior cone condition. Conditions weaker than the exterior cone but which still imply the existence of an optimal domain can be given in terms of capacity (see [144], [145]).

Concerning case (ii), we consider the problem

$$\min\left\{F(A) \,:\, A \in \mathscr{A},\ \mathrm{meas}(A) \le m\right\}, \tag{16.8}$$

where the volume constraint $\mathrm{meas}(A) \le m$ has been added. The cost functional F is assumed to be monotone nonincreasing with respect to the set inclusion, that is,

$$A_1 \subset A_2 \quad \Rightarrow \quad F(A_2) \le F(A_1).$$

Moreover, F is assumed to be lower semicontinuous with respect to the γ-convergence on the class of domains, defined by

$$A_n \to A \text{ in the } \gamma\text{-convergence} \quad \Longleftrightarrow \quad u_{A_n} \to u_A \text{ weakly in } H_0^1(\Omega),$$

where u_{A_n} and u_A are the solutions of (16.5) in A_n and A, respectively, with the right-hand side $f \equiv 1$. Under the assumptions above, Buttazzo and Dal Maso in [150] showed that problem (16.5) admits an optimal solution. It is important to stress that several optimal shape problems can be written in the form (16.8) with a cost functional F which is nonincreasing for the set inclusion and γ-lower semicontinuous. For instance, if L denotes a second-order elliptic operator of the form

$$Lu = -\mathrm{div}\big(a(x)Du\big)$$

with the $N \times N$ matrix $a(x)$ symmetric, uniformly elliptic, and with bounded and measurable coefficients, we may consider the spectrum $\Lambda(A)$ of L associated to the Dirichlet boundary conditions on A:

$$Lu = \lambda u, \qquad u \in H_0^1(A).$$

It is known that $\Lambda(A)$ is given by a nonnegative sequence $\lambda_k(A)$ which tends to $+\infty$, that every $\lambda_k(A)$ is a nonincreasing set function with respect to A, and that the mappings $\lambda_k(A)$ are γ-continuous. Therefore the existence result above applies and we obtain that the problem

$$\min\left\{\Phi\big(\Lambda(A)\big) \,:\, A \in \mathscr{A},\ \mathrm{meas}(A) \le m\right\}$$

admits an optimal solution whenever the mapping $\Phi : \mathbf{R^N} \to [0, +\infty]$ is

- nondecreasing in the sense that

$$\Lambda_k^1 \le \Lambda_k^2 \quad \forall k \quad \Rightarrow \quad \Phi(\Lambda^1) \le \Phi(\Lambda^2);$$

- lower semicontinuous in the sense that

$$\Lambda_k^n \to \Lambda_k \quad \forall k \quad \Rightarrow \quad \Phi(\Lambda) \le \liminf_{n\to+\infty} \Phi(\Lambda^n).$$

In particular, for a fixed integer k, all cost functionals of the form

$$F(A) = \phi\big(\lambda_1(A), \dots, \lambda_k(A)\big)$$

with a mapping $\phi : \mathbf{R}^k \to \mathbf{R}$ continuous and nondecreasing in each variable verify the assumptions above.

Concerning case (iii), we may still prove the existence of an optimal domain for problem (16.8) for some special form of the cost functional. The case which has been considered in [146] (see also [144]) is when

$$F(A) = \phi\big(\lambda_1(A), \lambda_2(A)\big),$$

where $\lambda_1(A)$ and $\lambda_2(A)$ are the first two eigenvalues of the Laplace operator $-\Delta$ on $H_0^1(A)$. Due to the special form of the cost functional and to the fact that the operator is $-\Delta$, it is possible to show that an optimal solution exists without any monotonicity assumption on ϕ by requiring only that ϕ is a lower semicontinuous function on $\mathbf{R}^2$.

16.4 ▪ Optimal distribution of two conductors

Another interesting case of a shape optimization problem is the optimal distribution of two given conductors into a given set. If Ω denotes a bounded open subset of $\mathbf{R}^N$ (the prescribed container), denoting by α and β the conductivities of the two materials, the problem consists in filling Ω with the two materials in the best performing way according to some given cost functional. The volume of each material can also be prescribed. It is convenient to denote by A the domain where the conductivity is α and by $a_A(x)$ the conductivity coefficient

$$a_A(x) = \alpha 1_A(x) + \beta 1_{\Omega\setminus A}(x).$$

In this way the state equation becomes

$$\begin{cases} -\operatorname{div}\big(a_A(x)Du\big) = f & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{16.9}$$

where f is the (given) source density, and we denote by u_A its unique solution.

It is well known (see, for instance, [261], [309]) that if we take as a cost functional an integral of the form

$$\int_\Omega j(x, 1_A, u_A, Du_A)\, dx,$$

in general an optimal configuration does not exist. However, the addition of a perimeter penalization is enough to imply the existence of classical optimizers. In other words, if we take as a cost the functional

$$J(u,A) = \int_\Omega j(x, 1_A, u, Du)\, dx + \sigma \operatorname{Per}_\Omega(A),$$

where $\sigma > 0$, the problem can be written as an optimal control problem in the form

$$\min\big\{J(u,A) \,:\, A \subset \Omega,\ u \text{ solves (16.9)}\big\}. \tag{16.10}$$

A volume constraint of the form $\mathrm{meas}(A)=m$ could also be present. The main ingredient for the proof of the existence of an optimal classical solution is the following result.

Theorem 16.4.1. *Let $a_n(x)$ be a sequence of $N\times N$ symmetric matrices with measurable coefficients, such that the uniform ellipticity condition*

$$c_0|z|^2 \le a_n(x)z\cdot z \le c_1|z|^2 \qquad \forall x\in\Omega,\ \forall z\in\mathbf{R}^N \tag{16.11}$$

holds with $0<c_0\le c_1$. Given $f\in H^{-1}(\Omega)$ we denote by u_n the unique solution of the problem

$$-\operatorname{div}\big(a_n(x)Du\big)=f, \qquad u_n\in H_0^1(\Omega). \tag{16.12}$$

If $a_n(x)\to a(x)$ a.e. in Ω, then $u_n\to u$ weakly in $H_0^1(\Omega)$, where u is the solution of (16.12) *with a_n replaced by a.*

PROOF. By the uniform ellipticity condition (16.11) we have

$$c_0\int_\Omega |Du_n|^2\,dx \le \int_\Omega f\,u_n\,dx,$$

and by the Poincaré inequality we have that u_n are bounded in $H_0^1(\Omega)$ so that a subsequence (still denoted by the same indices) converges weakly in $H_0^1(\Omega)$ to some v. All we have to show is that $v=u$, or equivalently that

$$-\operatorname{div}\big(a(x)Dv\big)=f. \tag{16.13}$$

This means that for every smooth test function ϕ we have

$$\int_\Omega a(x)DvD\phi\,dx=\langle f,\phi\rangle.$$

Then it is enough to show that for every smooth test function ϕ we have

$$\lim_{n\to+\infty}\int_\Omega a_n(x)Du_nD\phi\,dx=\int_\Omega a(x)DvD\phi\,dx.$$

This is an immediate consequence of the fact that ϕ is smooth, $Du_n\to Dv$ weakly in $L^2(\Omega)$, and $a_n\to a$ a.e. in Ω remaining bounded.

Another way to show that (16.13) holds is to verify that v minimizes the functional

$$F(w)=\int_\Omega a(x)DwDw\,dx-2\langle f,w\rangle, \qquad w\in H_0^1(\Omega). \tag{16.14}$$

Since the function $\alpha(s,z)=sz\cdot z$, which is defined for all $z\in\mathbf{R}^N$ and for symmetric positive definite $N\times N$ matrices s, is convex in z and lower semicontinuous in s, the functional

$$\Phi(a,\xi)=\int_\Omega a(x)\xi\cdot\xi\,dx$$

is sequentially lower semicontinuous with respect to the strong L^1-convergence on a and the weak L^1-convergence on ξ (see, for instance, Theorem 13.1.1 and [147]). Therefore we have

$$F(v)=\Phi(a,Dv)-2\langle f,v\rangle \le \liminf_{n\to+\infty}\Phi(a_n,Du_n)-2\langle f,u_n\rangle=\liminf_{n\to+\infty}F(u_n).$$

Since u_n minimizes the functional F_n defined as in (16.14) with a replaced by a_n, we also have for every $w \in H_0^1(\Omega)$

$$F_n(u_n) \le F_n(w) = \int_\Omega a_n(x) Dw Dw \, dx - 2\langle f, w\rangle,$$

so that taking the limit as $n \to +\infty$ and using the convergence $a_n \to a$ we obtain

$$\liminf_{n\to+\infty} F_n(u_n) \le \int_\Omega a(x) Dw Dw \, dx - 2\langle f, w\rangle = F(w).$$

Thus $F(v) \le F(w)$, which shows what is required. □

Remark 16.4.1. The result above can be rephrased in terms of G-convergence by saying that for uniformly elliptic operators of the form $-\operatorname{div}\big(a(x)Du\big)$ the G-convergence is weaker than the L^1-convergence of coefficients. Analogously, we can say that the functionals

$$G_n(w) = \int_\Omega a_n(x) Dw Dw \, dx$$

Γ-converge, with respect to the $L^2(\Omega)$-convergence, to the functional G defined in the same way with a in the place of a_n.

Corollary 16.4.1. *If $A_n \to A$ in $L^1(\Omega)$, then $u_{A_n} \to u_A$ weakly in $H_0^1(\Omega)$.*

A more careful inspection of the proof of Theorem 16.4.1 shows that the following stronger result holds.

Theorem 16.4.2. *Under the same assumptions of Theorem* 16.4.1, *the convergence of u_n is actually strong in $H_0^1(\Omega)$.*

PROOF. We have already seen that $u_n \to u$ weakly in $H_0^1(\Omega)$, which gives $Du_n \to Du$ weakly in $L^2(\Omega)$. Denoting by $c_n(x)$ and $c(x)$ the square root matrices of $a_n(x)$ and $a(x)$, respectively, we have that $c_n \to c$ a.e. in Ω remaining equibounded. Then $c_n(x)Du_n$ converges to $c(x)Du$ weakly in $L^2(\Omega)$. Multiplying (16.4) by u_n and integrating by parts we obtain

$$\begin{aligned}\int_\Omega a(x) Du Du \, dx &= \langle f, u\rangle = \lim_{n\to+\infty} \langle f, u_n\rangle \\ &= \lim_{n\to+\infty} \int_\Omega a_n(x) Du_n Du_n \, dx.\end{aligned}$$

This implies that

$$c_n(x) Du_n \to c(x) Du \qquad \text{strongly in } L^2(\Omega).$$

Multiplying now by $\big(c_n(x)\big)^{-1}$ we finally obtain the strong convergence of Du_n to Du in $L^2(\Omega)$. □

We are now in a position to obtain an existence result for the optimization problem (16.2). On the function j we only assume that it is nonnegative, Borel measurable, and such that $j(x, s, z, w)$ is lower semicontinuous in (s, z, w) for a.e. $x \in \Omega$.

Theorem 16.4.3. *Under the assumptions above, the minimum problem* (16.2) *admits at least a solution.*

PROOF. Let (A_n) be a minimizing sequence; then $\mathrm{Per}_\Omega(A_n)$ are bounded so that, up to extracting subsequences, we may assume (A_n) is strongly convergent in the L^1_{loc} sense to some set $A \subset \Omega$. We claim that A is a solution of problem (16.2). Let us denote by u_n a solution of problem (16.1) associated to A_n; by Theorem 16.4.2, (u_n) converges strongly in $H^1_0(\Omega)$ to some $u \in H^1_0(\Omega)$. Then by the lower semicontinuity of the perimeter (see Proposition 10.1.1) and by Fatou's lemma we have

$$J(u,A) \leq \liminf_{n\to+\infty} J(u_n,A_n),$$

which proves the optimality of A. □

Remark 16.4.2. The same proof works when volume constraints of the form $\mathrm{meas}(A) = m$ are present. Indeed this constraint passes to the limit when $A_n \to A$ strongly in $L^1(\Omega)$.

The existence result above shows the existence of a classical solution for the optimization problem (16.2). This solution is simply a set with finite perimeter and additional assumptions have to be made to prove further regularity. For instance, in [22] Ambrosio and Buttazzo considered the similar problem

$$\min\Big\{E(u,A) + c\,\mathrm{Per}_\Omega(A)\ :\ u \in H^1_0(\Omega),\ A \subset \Omega\Big\},$$

where $c > 0$ and

$$E(u,A) = \int_\Omega \Big[a_A(x)|Du|^2 + 1_A(x)g_1(x,u) + 1_{\Omega\setminus A}g_2(x,u)\Big]\,dx.$$

They showed that every solution A is actually an open set provided g_1 and g_2 are Borel measurable and satisfy inequalities

$$g_i(x,s) \geq \gamma(x) - k|s|^2, \qquad i = 1,2,$$

where $\gamma \in L^1(\Omega)$ and $k < \alpha\lambda_1$, λ_1 being the first eigenvalue of $-\Delta$ on Ω.

16.5 ▪ Optimal potentials for elliptic operators

In this section we consider the Schrödinger operator $-\Delta + V(x)$ in a bounded domain Ω of $\mathbf{R}^N$ with homogeneous Dirichlet conditions on $\partial\Omega$. We consider nonnegative potentials $V(x)$ and our goal is to show the existence of optimal potentials for some suitable cost functionals F and admissible classes $\mathcal{V}$. Then our problem will be of the form

$$\min\big\{F(V)\ :\ V \in \mathcal{V}\big\}.$$

Problems of this kind have been treated in [152] and in [238], to which we refer the reader for a complete list of references in the field.

The following proposition links the weak $L^1(\Omega)$-convergence to the γ-convergence introduced in Section 5.8.4.

Proposition 16.5.1. *Let $V_n \in L^1(\Omega)$ be a sequence weakly converging in $L^1(\Omega)$ to a function V. Then the capacitary measures $V_n\,dx$ γ-converge to $V\,dx$.*

PROOF. We have to prove that the solutions $u_n = R_{V_n}(1)$ of

$$\begin{cases} -\Delta u + V_n(x)u = 1, \\ u \in H_0^1(\Omega) \end{cases}$$

weakly converge in $H_0^1(\Omega)$ to the solution $u = R_V(1)$ of

$$\begin{cases} -\Delta u + V(x)u = 1, \\ u \in H_0^1(\Omega), \end{cases}$$

or equivalently that the functionals

$$J_n(u) = \int_\Omega |\nabla u|^2\,dx + \int_\Omega V_n(x)u^2\,dx$$

$\Gamma\big(L^2(\Omega)\big)$-converge to the functional

$$J(u) = \int_\Omega |\nabla u|^2\,dx + \int_\Omega V(x)u^2\,dx.$$

The Γ-liminf inequality is immediate since, if $u_n \to u$ in $L^2(\Omega)$, we have

$$\int_\Omega |\nabla u|^2\,dx \le \liminf_n \int_\Omega |\nabla u_n|^2\,dx$$

by the lower semicontinuity of the $H^1(\Omega)$ norm with respect to the $L^2(\Omega)$-convergence, and

$$\int_\Omega V(x)u^2\,dx \le \liminf_n \int_\Omega V_n(x)u_n^2\,dx$$

by the strong-weak lower semicontinuity theorem for integral functionals (see Theorem 13.1.1).

Let us now prove the Γ-limsup inequality which consists, given $u \in H_0^1(\Omega)$, in constructing a sequence $u_n \to u$ in $L^2(\Omega)$ such that

$$\limsup_n \int_\Omega |\nabla u_n|^2\,dx + \int_\Omega V_n(x)u_n^2\,dx \le \int_\Omega |\nabla u|^2\,dx + \int_\Omega V(x)u^2\,dx. \tag{16.15}$$

For every $t > 0$ let $u^t = (u \wedge t) \vee (-t)$; then for t fixed we have

$$\lim_n \int_\Omega V_n(x)|u^t|^2\,dx = \int_\Omega V(x)|u^t|^2\,dx$$

and

$$\lim_{t\to+\infty} \int_\Omega V(x)|u^t|^2\,dx = \int_\Omega V(x)|u|^2\,dx.$$

Then, by a diagonal argument, we can find a sequence $t_n \to +\infty$ such that

$$\lim_n \int_\Omega V_n(x)|u^{t_n}|^2\,dx = \int_\Omega V(x)|u|^2\,dx.$$

Taking now $u_n = u^{t_n}$, and noticing that for every $t > 0$

$$\int_\Omega |\nabla u^t|^2\,dx \le \int_\Omega |\nabla u|^2\,dx,$$

we obtain (16.15) and so the proof is complete. □

We are now in a position to prove a rather general existence result for an optimal potential.

Theorem 16.5.1. *Let $\mathcal{V}$ be a subset of $L^1_+(\Omega)$, weakly compact for the $L^1(\Omega)$-convergence, and let $F : \mathcal{V} \to \overline{\mathbf{R}}$ be a functional which is γ-lsc. Then the optimization problem*

$$\min\left\{F(V)\,:\,V \in \mathcal{V}\right\} \tag{16.16}$$

admits a solution.

PROOF. Let (V_n) be a minimizing sequence for the problem (16.16); by the weak $L^1(\Omega)$ compactness assumption on $\mathcal{V}$ we may extract a subsequence (still denoted by (V_n)) which converges weakly in $L^1(\Omega)$ to some function $V \in \mathcal{V}$. By Proposition 16.5.1 we obtain that $V_n\,dx \to_\gamma V\,dx$ as capacitary measures, and so the γ-lower semicontinuity of F allows us to conclude that V is an optimal potential for problem (16.16). □

Remark 16.5.1. We recall that, by the De La Vallée–Poussin theorem, Theorem 2.4.4, the set $\mathcal{V}$ is weakly compact in $L^1(\Omega)$ if

$$\int_\Omega \theta(V)\,dx \le 1 \qquad \forall V \in \mathcal{V},$$

where $\theta : \mathbf{R} \to \mathbf{R}$ is a superlinear function, that is,

$$\lim_{t\to+\infty} \frac{\theta(t)}{t} = +\infty.$$

Remark 16.5.2. Since the γ-convergence is rather strong, many cost functionals are γ-lsc, for instance, the integral functionals and the spectral functionals shown in Section 5.8.4 of the following form:

- the integral functionals

$$F(V) = \int_\Omega j(x, u_V, \nabla u_V)\,dx,$$

where $u_V = R_V(f)$ is the solution of

$$\begin{cases} -\Delta u + V(x)u = f & \text{in } \Omega, \\ u \in H^1_0(\Omega) \end{cases} \tag{16.17}$$

and the integrand $j(x,s,z)$ is measurable in x, lower semicontinuous in (s,z), convex in z, and such that

$$j(x,s,z) \ge -a(x) - c|s|^p,$$

where $a \in L^1(\Omega)$, $p < 2N/(N-2)$ ($p < +\infty$ if $N = 2$), $c \in \mathbf{R}$;

- the spectral functionals
$$F(V) = \Phi\big(\Lambda(V)\big),$$
where $\Lambda(V)$ is the spectrum of the Schrödinger operator $-\Delta + V(x)$ on $H^1_0(\Omega)$ and $\Phi : \mathbf{R}^\mathbf{N} \to \overline{\mathbf{R}}$ is lower semicontinuous in the sense that
$$\lambda^n_k \to \lambda_k \ \forall k \quad \Rightarrow \quad \Phi(\Lambda) \leq \liminf_{n\to+\infty} \Phi(\Lambda^n).$$

Example 16.5.1. Consider the cost functional given by the energy
$$\mathcal{E}_f(V) = \min\left\{\int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}V(x)u^2 - f(x)u\right)dx \ : \ u \in H^1_0(\Omega)\right\}$$
and the admissible class
$$\mathcal{V} = \left\{V \geq 0 \ : \ \int_\Omega V^p\,dx \leq 1\right\}$$
with $p > 1$. The functional $\mathcal{E}_f$ turns out to be γ-continuous; therefore the existence Theorem 16.5.1 applies to $F(V) = -\mathcal{E}_f(V)$ and we obtain that the problem
$$\max\left\{\mathcal{E}_f(V) \ : \ V \geq 0, \ \int_\Omega V^p\,dx \leq 1\right\} \tag{16.18}$$
admits a solution. Notice that, $\mathcal{E}_f$ being the infimum of linear functions with respect to V, the cost $F(V)$ is convex on $\mathcal{V}$. The functional $\mathcal{E}_f$ is an integral functional: in fact, from the PDE (16.17) we find, multiplying by u_V and integrating by parts,
$$\int_\Omega \left[|\nabla u_V|^2 + V u_V^2\right]dx = \int_\Omega f\,u_V\,dx.$$
Therefore we obtain
$$\mathcal{E}_f(V) = -\frac{1}{2}\int_\Omega f\,u_V\,dx,$$
which corresponds to the integral functional above with
$$j(x,s,z) = -\frac{1}{2}f(x)s.$$

In this particular case some more explicit computations can be made: indeed, interchanging the max and the min in (16.18) we obtain
$$\begin{aligned}\max_{V\in\mathcal{V}} \min_{u\in H^1_0(\Omega)} &\int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}V(x)u^2 - f(x)u\right)dx \\ &\leq \min_{u\in H^1_0(\Omega)} \max_{V\in\mathcal{V}} \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}V(x)u^2 - f(x)u\right)dx.\end{aligned}$$

The max in the last term can be explicitly computed and we obtain easily
$$\max_{V\in\mathcal{V}} \int_\Omega \frac{1}{2}V(x)u^2\,dx = \frac{1}{2}\left(\int_\Omega |u|^{2p/(p-1)}\,dx\right)^{1-1/p},$$

reached at

$$V = |u|^{2/(p-1)} \left(\int_\Omega |u|^{2p/(p-1)} \, dx \right)^{-1/p}.$$

This gives

$$\max_{V \in \mathcal{V}} \mathcal{E}_f(V) \leq \min_{u \in H_0^1(\Omega)} \int_\Omega \frac{1}{2} |\nabla u|^2 \, dx + \frac{1}{2} \left(\int_\Omega |u|^{2p/(p-1)} \, dx \right)^{1-1/p} - \int_\Omega f(x) u \, dx.$$

Due to the assumption $p > 1$, the functional appearing in the right-hand side is coercive and strictly convex, so that the minimum in the right-hand side is attained at a unique function $\overline{u} \in H_0^1(\Omega)$ which verifies the PDE

$$-\Delta u + C_u |u|^{2/(p-1)} u = f \qquad \text{with } C_u = \left(\int_\Omega |u|^{2p/(p-1)} \, dx \right)^{-1/p}. \tag{16.19}$$

Now, taking

$$\overline{V} = |\overline{u}|^{2/(p-1)} \left(\int_\Omega |\overline{u}|^{2p/(p-1)} \, dx \right)^{-1/p} \tag{16.20}$$

we have $\overline{V} \in \mathcal{V}$, hence

$$\max_{V \in \mathcal{V}} \mathcal{E}_f(V) \geq \mathcal{E}_f(\overline{V}).$$

The term $\mathcal{E}_f(\overline{V})$ can be computed through its Euler–Lagrange equation

$$\begin{cases} -\Delta u + \overline{V}(x) u = f & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases}$$

which is uniquely solved by $\overline{u}$. Therefore, all the inequalities above become equalities and we may conclude that problem (16.18) admits a unique solution $\overline{V}$ given by (16.20), where $\overline{u}$ is the solution of the minimum problem

$$\min_{u \in H_0^1(\Omega)} \int_\Omega \frac{1}{2} |\nabla u|^2 \, dx + \frac{1}{2} \left(\int_\Omega |u|^{2p/(p-1)} \, dx \right)^{1-1/p} - \int_\Omega f(x) u \, dx,$$

which corresponds to the PDE (16.19). Similar computations can be made (see [238]) in the case of the cost functional

$$\lambda_1(V) = \min \left\{ \int_\Omega \left(|\nabla u|^2 + V(x) u^2 \right) dx \ : \ u \in H_0^1(\Omega), \ ||u||_{L^2(\Omega)} = 1 \right\}.$$

Changing the integral constraint which defines the admissible class and considering the new class

$$\mathcal{V} = \left\{ V \geq 0 \ : \ \int_\Omega V^{-p} \, dx \leq 1 \right\}$$

with $p > 0$, the meaningful problem for the energy cost $\mathcal{E}_f(V)$ (similarly for the first eigenvalue $\lambda_1(V)$) becomes the minimization problem

$$\min \left\{ \mathcal{E}_f(V) \ : \ V \geq 0, \ \int_\Omega V^{-p} \, dx \leq 1 \right\}. \tag{16.21}$$

Note that in the present situation the set $\mathcal{V}$ is unbounded in every $L^p(\Omega)$; actually, in this case the value $+\infty$ is admissible for a potential V, intending that on the set $V=+\infty$ the Sobolev functions entering in the definition of $\mathcal{E}_f(V)$ must vanish q.e. on the set $u_V=0$.

By repeating the calculations made in Example 16.5.1 we end up with the conclusion that for every $p>0$ the minimization problem (16.21) admits a solution $\overline{V}$, given by

$$\overline{V}=|\overline{u}|^{-2/(p+1)}\left(\int_\Omega|\overline{u}|^{2p/(p+1)}\,dx\right)^{1/p}.$$

Here $\overline{u}$ solves the minimum problem

$$\min_{u\in H_0^1(\Omega)}\int_\Omega\frac{1}{2}|\nabla u|^2\,dx+\frac{1}{2}\left(\int_\Omega|u|^{2p/(p+1)}\,dx\right)^{1+1/p}-\int_\Omega f(x)u\,dx,$$

which corresponds to the PDE

$$-\Delta u+C_u|u|^{-2/(p+1)}u=f\qquad\text{with } C_u=\left(\int_\Omega|u|^{2p/(p+1)}\,dx\right)^{1/p}.$$

In order to handle more general optimization problems of the form

$$\min\{F(V)\,:\,V\in\mathcal{V}\},\tag{16.22}$$

we consider a function $\Psi:[0,+\infty]\to[0,+\infty]$ and the admissible class

$$\mathcal{V}=\left\{V:\Omega\to[0,+\infty]\,:\,V\text{ Lebesgue measurable, }\int_\Omega\Psi(V)\,dx\le1\right\}.$$

On the function Ψ we make the following assumptions:

(i) Ψ is strictly decreasing;

(ii) there exists $p>1$ such that the function $s\mapsto\Psi^{-1}(s^p)$ is convex.

Note that the conditions above are, for instance, satisfied by the following functions:

$$\begin{aligned}&\Psi(s)=s^{-p}\qquad\text{for any } p>0;\\&\Psi(s)=e^{-\alpha s}\qquad\text{for any } \alpha>0.\end{aligned}$$

The general existence result we may obtain in this framework is the following.

Theorem 16.5.2. *Let $\Omega\subset\mathbf{R}^N$ be a bounded open set and let $\Psi:[0,+\infty]\to[0,+\infty]$ be a function satisfying the conditions* (i) *and* (ii) *above. Then, for any functional $F(V)$ which is monotone increasing and lower semicontinuous with respect to the γ-convergence, the problem* (16.22) *admits a solution.*

PROOF. Let $V_n\in\mathcal{V}$ be a minimizing sequence for problem (16.22). Then, $v_n:=\big(\Psi(V_n)\big)^{1/p}$ is a bounded sequence in $L^p(\Omega)$ and so, up to a subsequence, v_n converges weakly in $L^p(\Omega)$ to some function v. We will prove that $V:=\Psi^{-1}(v^p)$ is a solution of (16.22). Clearly $V\in\mathcal{V}$ and so it remains to prove that $F(V)\le\liminf_n F(V_n)$. In view of the compactness of the γ-convergence on the class $\mathbf{M}_0(\Omega)$ of capacitary measures (see Section 5.8.4), we can suppose that, up to a subsequence, V_n γ-converges to a capacitary measure $\mu \in \mathbf{M}_0(\Omega)$. We claim that the following inequalities hold true:

$$F(V) \leq F(\mu) \leq \liminf_{n\to\infty} F(V_n). \tag{16.23}$$

In fact, the second inequality in (16.23) is the lower semicontinuity of F with respect to the γ-convergence, while the first needs a more careful examination. By the definition of γ-convergence, we have that for any $u \in H_0^1(\Omega)$, there is a sequence $u_n \in H_0^1(\Omega)$ which converges to u in $L^2(\Omega)$ and is such that

$$\begin{aligned}
\int_\Omega |\nabla u|^2\,dx + \int_\Omega u^2\,d\mu &= \lim_{n\to\infty} \int_\Omega |\nabla u_n|^2\,dx + \int_\Omega u_n^2 V_n\,dx \\
&= \lim_{n\to\infty} \int_\Omega |\nabla u_n|^2\,dx + \int_\Omega u_n^2 \Psi^{-1}(v_n^p)\,dx \\
&\geq \int_\Omega |\nabla u|^2\,dx + \int_\Omega u^2 \Psi^{-1}(v^p)\,dx \\
&= \int_\Omega |\nabla u|^2\,dx + \int_\Omega u^2 V\,dx,
\end{aligned} \tag{16.24}$$

where the inequality in (16.24) is due to strong-weak lower semicontinuity of integral functionals (see Theorem 13.1.1). Thus, for any $u \in H_0^1(\Omega)$, we have

$$\int_\Omega u^2\,d\mu \geq \int_\Omega u^2 V\,dx,$$

which gives $V \leq \mu$. Since F is assumed to be monotone increasing, we obtain the first inequality in (16.23) and so the conclusion. $\square$

Taking for instance $\Psi(s) = e^{-\alpha s}$, we can repeat the explicit computation made for problem (16.21), and we find that the optimization problem

$$\min\left\{\mathscr{E}_f(V) \,:\, V \geq 0,\ \int_\Omega e^{-\alpha V}\,dx \leq 1\right\}$$

admits a solution $\overline{V}$ given by

$$\overline{V} = \frac{1}{\alpha}\left(\log\left(\int_\Omega |\overline{u}|^2\,dx\right) - \log\left(|\overline{u}|^2\right)\right).$$

Here $\overline{u}$ solves the minimum problem

$$\min_{u\in H_0^1(\Omega)} \int_\Omega \frac{1}{2}|\nabla u|^2\,dx + \frac{1}{2\alpha}\left(\int_\Omega u^2\,dx \int_\Omega \log\left(u^2\right)dx - \int_\Omega u^2 \log\left(u^2\right)dx\right) - \int_\Omega f u\,dx,$$

which corresponds to the PDE

$$-\Delta u + \frac{1}{\alpha}\left(A_u u + \frac{B_u}{u} - u\left(1 + \log(u^2)\right)\right) = f \qquad \text{with } A_u = \int_\Omega \log(u^2)\,dx,\ B_u = \int_\Omega u^2\,dx.$$

It is interesting to notice that the case of constraints of the form $\int_\Omega e^{-\alpha V}\,dx \le 1$ can be used to efficiently approximate shape optimization problems, in which the main unknown is a domain $A \subset \Omega$, that is, a capacitary measure of the form $\infty_{\Omega\setminus A}$ (see Section 5.8.4). Indeed, if $V = \infty_{\Omega\setminus A}$ we have that $e^{-\alpha V} = 1_A$ and the constraint $\int_\Omega e^{-\alpha V}\,dx \le 1$ becomes the measure constraint $|A| \le 1$, while the Schrödinger operator $-\Delta + V$ corresponds to the Dirichlet–Laplacian on the set A. Buttazzo et al. [152] prove the Γ-convergence, as $\alpha \to 0$, of the Schrödinger problems with constraint $\int_\Omega e^{-\alpha V}\,dx \le 1$ to the shape optimization problem with measure constraint $|A| \le 1$.