HUMBOLDT-UNIVERSITÄT ZU BERLIN

Minimization of a Functional on the Space of BV Functions and Nonconforming Discretization of the Problem

I. Theoretical Basics and Characterization of Minimizers

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Sören Bartels. **Numerical Methods for Nonlinear Partial Differential Equations**. Vol. 47. Springer Series in Computational Mathematics. Springer International Publishing, 2015. ISBN: 978-3-319-13796-4. DOI: 10.1007/978-3-319-13797-1, Chapter 10, p. 297-319

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Let $\Omega \subset \mathbb{R}^n$ be a bounded polyhedral Lipschitz domain.

For given $g \in L^2(\Omega)$ and $\alpha > 0$ minimize the functional

$$I(v) = |v|_{\mathrm{BV}(\Omega)} + \frac{\alpha}{2}\|v - g\|^2$$

amongst all $v \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$.

Functions of Bounded Variation

A function $v \in L^1(\Omega)$ with distributional derivative $Dv : C_C^\infty(\Omega;\mathbb{R}^n) \to \mathbb{R}$ is said to be of bounded variation if there exists $c > 0$ such that

$$\langle Dv, \phi \rangle := -\int_\Omega v \operatorname{div}(\phi)\,\mathrm{d}x \leqslant c\|\phi\|_{L^\infty(\Omega)}$$

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The minimal constant $c \geqslant 0$ satisfying this property is called total variation of Dv and is given by

$$|v|_{\mathrm{BV}(\Omega)} = \sup_{\substack{\phi \in C_C^1(\Omega;\mathbb{R}^n) \\ \|\phi\|_{L^\infty(\Omega)} \leqslant 1}} -\int_\Omega v \operatorname{div}(\phi)\,\mathrm{d}x.$$

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The space of all such functions is denoted by $\mathrm{BV}(\Omega)$.

Properties of $\mathrm{BV}(\Omega)$

$\mathrm{BV}(\Omega)$ is a Banach space equipped with the norm

$$\|v\|_{\mathrm{BV}(\Omega)} := \|v\|_{L^1(\Omega)} + |v|_{\mathrm{BV}(\Omega)} \quad \text{for all } v \in \mathrm{BV}(\Omega).$$

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$W^{1,1}(\Omega) \subset \mathrm{BV}(\Omega)$ with $\|v\|_{\mathrm{BV}(\Omega)} = \|v\|_{W^{1,1}(\Omega)}$ for all $v \in W^{1,1}(\Omega)$.

Notions of convergence on $\mathrm{BV}(\Omega)$

Let $(v_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(\Omega)$ and $v \in \mathrm{BV}(\Omega)$ such that $v_n \to v$ in $L^1(\Omega)$ as $n \longrightarrow \infty$.

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(i) $(v_n)_{n\in\mathbb{N}}$ converges intermediately or strictly to v if $|v_n|_{\mathrm{BV}(\Omega)} \longrightarrow |v|_{\mathrm{BV}(\Omega)}$ as $n \longrightarrow \infty$.

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(i) $(v_n)_{n\in\mathbb{N}}$ converges intermediately or strictly to v if $|v_n|_{\mathrm{BV}(\Omega)} \longrightarrow |v|_{\mathrm{BV}(\Omega)}$ as $n \to \infty$.

(ii) $(v_n)_{n\in\mathbb{N}}$ converges weakly to v if $\langle Dv_n, \phi\rangle \to \langle Dv, \phi\rangle$ for all $\phi \in C_0(\Omega; \mathbb{R}^n)$ as $n \to \infty$.

Further Properties of $\mathrm{BV}(\Omega)$

$C^\infty(\overline{\Omega})$ and $C^\infty(\Omega) \cap \mathrm{BV}(\Omega)$ are dense in $\mathrm{BV}(\Omega)$ with respect to intermediate convergence.

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There exists a linear operator $T : \mathrm{BV}(\Omega) \to L^1(\partial\Omega)$ such that $T(v) = v|_{\partial\Omega}$ for all $v \in \mathrm{BV}(\Omega) \cap C(\overline{\Omega})$.
T is continuous with respect to intermediate convergence in $\mathrm{BV}(\Omega)$ but not with respect to weak convergence in $\mathrm{BV}(\Omega)$.

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For given $f \in L^2(\Omega)$ and $\alpha > 0$ minimize the functional

$$E(v) := \frac{\alpha}{2}\|v\|_{L^2(\Omega)}^2 + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \int_\Omega f\, v \,\mathrm{d}x$$

amongst all $v \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$.

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For $f = \alpha g$ we have

$$I(v) = |v|_{\mathrm{BV}(\Omega)} + \frac{\alpha}{2}\|v - g\|^2 = E(v) - \|v\|_{L^1(\partial\Omega)} + \frac{\alpha}{2}\|g\|_{L^2(\Omega)}^2$$

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for all $v \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$.

I and E have the same minimizers in
$\big\{v \in \mathrm{BV}(\Omega) \cap L^2(\Omega) \mid \|v\|_{L^1(\partial\Omega)} = 0\big\}$.

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E(v) &= \frac{\alpha}{2}\|v\|^2_{L^2(\Omega)} + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \int_\Omega fv \,\mathrm{d}x \\
&\geqslant \frac{\alpha}{2}\|v\|^2_{L^2(\Omega)} + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}
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$$\int_\Omega u_n w \,\mathrm{d}x \to \int_\Omega \bar{u} w \,\mathrm{d}x \text{ as } n \to \infty$$

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$(u_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(\Omega) \cap L^2(\Omega)$ is a bounded infimizing sequence of E.

Let $(u_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(\Omega)$ be bounded. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ and $u \in \mathrm{BV}(\Omega)$ such that u_{n_k} converges weakly to u in $\mathrm{BV}(\Omega)$ as $k \to \infty$.

- $(u_n)_{n\in\mathbb{N}}$ (w.l.o.g.) converges weakly to $u \in \mathrm{BV}(\Omega)$in $L^1(\Omega)$.
- $(u_n)_{n\in\mathbb{N}}$ (w.l.o.g.) converges weakly to $\bar{u} \in L^2(\Omega)$ in $L^2(\Omega)$.

$$\forall w \in (L^2(\Omega))^* \cong L^2(\Omega) \supset L^\infty(\Omega) \cong (L^1(\Omega))^* :$$
$$\int_\Omega u_n w \,\mathrm{d}x \to \int_\Omega \bar{u} w \,\mathrm{d}x \text{ as } n \to \infty$$

- $(u_n)_{n\in\mathbb{N}}$ converges weakly to $\bar{u}$ in $L^1(\Omega)$.
- $u = \bar{u} \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$.

Lawrence C. Evans and Ronald F. Gariepy. **Measure Theory and Fine Properties of Functions**. CRC Press, 1992. ISBN: 0-8493-7157-0, p. 183, Theorem 1

Let $v \in \mathrm{BV}(\Omega)$. For all $x \in \mathbb{R}^n$ define

$$\tilde{v}(x) := \begin{cases} v(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \backslash \Omega. \end{cases}$$

Then $\tilde{v} \in \mathrm{BV}(\mathbb{R}^n)$ and $|\tilde{v}|_{\mathrm{BV}(\mathbb{R}^n)} = |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)}$.

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- $(|\tilde{u}_n|_{\mathrm{BV}(\mathbb{R}^n)})_{n\in\mathbb{N}} = (|u_n|_{\mathrm{BV}(\Omega)} + \|u_n\|_{L^1(\partial\Omega)})_{n\in\mathbb{N}}$ is bounded since $(u_n)_{n\in\mathbb{N}}$ is infimizing sequence of E and $E(v) \geqslant \frac{\alpha}{4}\|v\|^2_{L^2(\Omega)} + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \frac{1}{\alpha}\|f\|^2_{L^2(\Omega)}$.

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- $\tilde{u}_n \to \tilde{u}$ in $L^1(\mathbb{R}^n)$ as $n \to \infty$ since $u_n \to u$ in $L^1(\Omega)$.

- $\left(|\tilde{u}_n|_{\mathrm{BV}(\mathbb{R}^n)}\right)_{n\in\mathbb{N}} = \left(|u_n|_{\mathrm{BV}(\Omega)} + \|u_n\|_{L^1(\partial\Omega)}\right)_{n\in\mathbb{N}}$ is bounded.

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Let $(v_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(\Omega)$ and $v \in L^1(\Omega)$ such that $|v_n|_{\mathrm{BV}(\Omega)} \leqslant c$ for some $c > 0$ and all $n \in \mathbb{N}$ and $v_n \to v$ in $L^1(\Omega)$ as $n \to \infty$. Then $v \in \mathrm{BV}(\Omega)$ and $|v|_{\mathrm{BV}(\Omega)} \leqslant \liminf_{n\to\infty} |v_n|_{\mathrm{BV}(\Omega)}$. Furthermore v_n converges weakly to v in $\mathrm{BV}(\Omega)$.

- $\left(|\tilde{u}_n|_{BV(\mathbb{R}^n)}\right)_{n\in\mathbb{N}} = \left(|u_n|_{BV(\Omega)} + \|u_n\|_{L^1(\partial\Omega)}\right)_{n\in\mathbb{N}}$ is bounded.
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$$
\begin{aligned}
|u|_{BV(\Omega)} + \|u\|_{L^1(\partial\Omega)} = |\tilde{u}|_{BV(\mathbb{R}^n)} &\leqslant \liminf_{n\to\infty} |\tilde{u}_n|_{BV(\mathbb{R}^n)} \\
&= \liminf_{n\to\infty} \left(|u_n|_{BV(\Omega)} + \|u_n\|_{L^1(\partial\Omega)}\right).
\end{aligned}
$$

- $|u|_{\mathrm{BV}(\Omega)} + \|u\|_{L^1(\partial\Omega)} \leqslant \liminf_{n\to\infty}\big(|u_n|_{\mathrm{BV}(\Omega)} + \|u_n\|_{L^1(\partial\Omega)}\big)$.

- $|u|_{\mathrm{BV}(\Omega)} + \|u\|_{L^1(\partial\Omega)} \leqslant \liminf_{n\to\infty}(|u_n|_{\mathrm{BV}(\Omega)} + \|u_n\|_{L^1(\partial\Omega)})$.
- $\frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} - \int_\Omega fu\,\mathrm{d}x \leqslant \liminf_{n\to\infty}\left(\frac{\alpha}{2}\|u_n\|^2_{L^2(\Omega)} - \int_\Omega fu_n\,\mathrm{d}x\right)$ since $\|\bullet\|^2_{L^2(\Omega)}$ and $-\int_\Omega f\bullet\,\mathrm{d}x$ are continuous and convex (and hence w.l.s.c.) on $L^2(\Omega)$ and $u_n \rightharpoonup u$ in $L^2(\Omega)$ as $n\to\infty$.

- $|u|_{\mathrm{BV}(\Omega)} + \|u\|_{L^1(\partial\Omega)} \leqslant \liminf_{n\to\infty}(|u_n|_{\mathrm{BV}(\Omega)} + \|u_n\|_{L^1(\partial\Omega)})$.
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$$\begin{aligned}
\inf_{v\in \mathrm{BV}(\Omega)\cap L^2(\Omega)} E(v) &\leqslant E(u) \\
&\leqslant \liminf_{n\to\infty} E(u_n) \\
&= \lim_{n\to\infty} E(u_n) \\
&= \inf_{v\in \mathrm{BV}(\Omega)\cap L^2(\Omega)} E(v),
\end{aligned}$$

i.e. $\min_{v\in \mathrm{BV}(\Omega)\cap L^2(\Omega)} E(v) = E(u)$. □

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Let $u_1, u_2 \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$ be minimizers of E with $f_1, f_2 \in L^2(\Omega)$ instead of f.
Then

$$\|u_1 - u_2\|_{L^2(\Omega)} \leqslant \frac{1}{\alpha}\|f_1 - f_2\|_{L^2(\Omega)}.$$

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Define convex functionals $F : \mathrm{BV}(\Omega) \cap L^2(\Omega) \to \mathbb{R}$ and $G_\ell : \mathrm{BV}(\Omega) \cap L^2(\Omega) \to \mathbb{R}$, $\ell = 1, 2$, via

$$F(u) := |u|_{\mathrm{BV}(\Omega)} + \|u\|_{L^1(\partial\Omega)}, \quad G_\ell(u) := \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} - \int_\Omega f_\ell u \,\mathrm{d}x.$$

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The Fréchet derivative $G_\ell'(u) : L^2(\Omega) \to \mathbb{R}$ of G_ℓ at $u \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$ is

$$G_\ell'(u) = \alpha(u, \bullet)_{L^2(\Omega)} - \int_\Omega f_\ell \bullet \,\mathrm{d}x = (\alpha u - f_\ell, \bullet)_{L^2(\Omega)}.$$

Eberhard Zeidler. **Nonlinear Functional Analysis and its Applications. III: Variational Methods and Optimization**. New York: Springer Science+Business Media, LLC, 1985. ISBN: 978-1-4612-9529-7

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Let $(X, \| \bullet \|_X)$ be a real Banach space and $H : X \to [-\infty, \infty]$.

The subdifferential of H at some $u \in X$ with $H(u) \neq \pm\infty$ is

$$\partial H(u) \coloneqq \{u^* \in X^* \mid \forall v \in X \quad H(v) \geqslant H(u) + \langle u^*, v - u \rangle\}$$

Define $\partial H(u) \coloneqq \emptyset$ if $H(u) = \pm\infty$.

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$u^* \in \partial H(u)$ is called subgradient of H at u.

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$u^* \in \partial H(u)$ is called subgradient of H at u.

If $H : X \to (-\infty, \infty]$ such that $H \not\equiv \infty$, then $H(u) = \inf_{v \in X} H(v)$ if and only if $0 \in \partial H(u)$.

If H convex and Gâteaux differentiable at $u \in X$ with Gâteaux derivative $H'(u)$, then $\partial H(u) = \{H'(u)\}$.

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$H_1, H_2, \ldots, H_n : X \to (-\infty, \infty]$ and the summation of sets in X^* commute, [Zei85, S. 389, Theorem 47.B] implies the following statement.

If the functionals $H_1, H_2, \ldots, H_n : X \to (-\infty, \infty]$, $n \geqslant 2$, are convex and there exists $u_0 \in X$ and $j \in \{1, 2, \ldots, n\}$ such that $H_k(u_0) < \infty$ for all $k \in \{1, 2 \ldots, n\}$ and H_k continuous at u_0 for all $k \in \{1, 2, \ldots, n\} \backslash \{j\}$, then

$$\partial(H_1 + H_2 + \ldots + H_n)(u) = \partial H_1(u) + \partial H_2(u) + \ldots + \partial H_n(u)$$

for all $u \in X$.

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$$0 \in \partial E_\ell(u_\ell) = \partial F(u_\ell) + \partial G_\ell(u_\ell) = \partial F(u_\ell) + \{G'_\ell(u_\ell)\} \text{ for } \ell = 1, 2.$$

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Let $H : X \to (-\infty, \infty]$ convex and lower semi-continuous with $H \not\equiv \infty$.
Then $\partial H(\bullet)$ is monoton, i.e.

$$\langle u^* - v^*, u - v \rangle \geqslant 0 \quad \text{for all } u, v \in X, u^* \in \partial H(u), v^* \in \partial H(v).$$

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Hence $\big(-(\alpha u_1 - f_1) + (\alpha u_2 - f_2), u_1 - u_2 \big)_{L^2(\Omega)} \geqslant 0.$

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Hence $\big(-(\alpha u_1 - f_1) + (\alpha u_2 - f_2), u_1 - u_2\big)_{L^2(\Omega)} \geqslant 0.$

With the Cauchy-Schwarz inequality this implies

$$\begin{aligned}\alpha \|u_1 - u_2\|^2_{L^2(\Omega)} &\leqslant \big(f_1 - f_2, u_1 - u_2\big)_{L^2(\Omega)} \\ &\leqslant \|f_1 - f_2\|_{L^2(\Omega)} \|u_1 - u_2\|_{L^2(\Omega)}.\end{aligned}$$

□

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Let $\mathcal{T}$ be a regular triangulation of Ω.
For given $f \in L^2(\Omega)$ and $\alpha > 0$ minimize the functional

$$E_{\mathrm{NC}}(v_{\mathrm{CR}}) := \frac{\alpha}{2}\|v_{\mathrm{CR}}\|_{L^2(\Omega)}^2 + \|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^1(\Omega)} - \int_\Omega f v_{\mathrm{CR}} \,\mathrm{d}x$$

amongst all $v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\Omega)$.

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It holds

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Refinement indicator, for some $0 < \beta \leqslant 1$, $\eta := \sum_{T\in\mathcal{T}} \eta(T)$ with

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If $u \in L^1(\Omega)$, $\Omega_1 \cap \Omega_2 = \emptyset$, $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, $\partial\Omega_1 \cap \partial\Omega_2 = \Sigma$, and $u|_{\Omega_j} \in W^{1,1}(\Omega_j)$, then $u \in \mathrm{BV}(\Omega)$ and
$Du = \nabla_{\mathrm{NC}} u \otimes \mathrm{d}x - [un] \otimes \mathrm{d}s|_\Sigma$.

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For $v_{\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T})$ and $\Lambda \in \mathbb{P}_0(\mathcal{T};\mathbb{R}^n) \subset L^\infty(\Omega;\mathbb{R}^n)$ define

$$K_1(0) := \{\Lambda \in L^\infty(\Omega;\mathbb{R}^n) \mid |\Lambda(\bullet)| \leqslant 1 \text{ a.e. in } \Omega\},$$

$$I_{K_1(0)}(\Lambda) := \begin{cases} \infty & \text{if } \Lambda \notin K_1(0), \\ 0 & \text{if } \Lambda \in K_1(0), \end{cases}$$

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and $\mathcal{L}_h : \mathrm{CR}^1_0(\mathcal{T}) \times \mathbb{P}_0(\mathcal{T};\mathbb{R}^n) \to [-\infty,\infty)$ by

$$\mathcal{L}_h(v_{\mathrm{CR}},\Lambda) := \int_\Omega \Lambda \cdot \nabla_{\mathrm{NC}} v_{\mathrm{CR}} \,\mathrm{d}x + \frac{\alpha}{2}\|v_{\mathrm{CR}}\|^2_{L^2(\Omega)} - \int_\Omega f v_{\mathrm{CR}} \,\mathrm{d}x - I_{K_1(0)}(\Lambda).$$

For $v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ and $\Lambda \in \mathbb{P}_0(\mathcal{T};\mathbb{R}^n) \subset L^\infty(\Omega;\mathbb{R}^n)$ define

$$K_1(0) := \{\Lambda \in L^\infty(\Omega;\mathbb{R}^n) \mid |\Lambda(\bullet)| \leqslant 1 \text{ a.e. in } \Omega\},$$

$$I_{K_1(0)}(\Lambda) := \begin{cases} \infty & \text{if } \Lambda \notin K_1(0), \\ 0 & \text{if } \Lambda \in K_1(0), \end{cases}$$

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$\mathcal{L}_h(v_{\mathrm{CR}},\Lambda) > -\infty \Leftrightarrow \Lambda \in K_1(0)$

$$E_{\mathrm{NC}}(v_{\mathrm{CR}}) := \frac{\alpha}{2}\|v_{\mathrm{CR}}\|_{L^2(\Omega)}^2 + \|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^1(\Omega)} - \int_\Omega f v_{\mathrm{CR}} \,\mathrm{d}x$$

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For any $\Lambda \in \mathbb{P}_0(\mathcal{T};\mathbb{R}^n) \cap K_1(0)$ the Cauchy-Schwarz inequality implies

$$\begin{aligned}\int_\Omega \Lambda \cdot \nabla_{\mathrm{NC}} v_{\mathrm{CR}}\,\mathrm{d}x &\leqslant \int_\Omega |\Lambda \cdot \nabla_{\mathrm{NC}} v_{\mathrm{CR}}|\,\mathrm{d}x \leqslant \int_\Omega |\Lambda||\nabla_{\mathrm{NC}} v_{\mathrm{CR}}|\,\mathrm{d}x \\ &\leqslant \int_\Omega 1|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}|\,\mathrm{d}x = \|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^1(\Omega)}.\end{aligned}$$

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Hence

$$\sup_{\Lambda\in\mathbb{P}_0(\mathcal{T};\mathbb{R}^n)} \mathcal{L}_h(v_{\mathrm{CR}},\Lambda) = \sup_{\Lambda\in\mathbb{P}_0(\mathcal{T};\mathbb{R}^n)\cap K_1(0)} \mathcal{L}_h(v_{\mathrm{CR}},\Lambda) \leqslant E_{\mathrm{NC}}(v_{\mathrm{CR}}).$$

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Let

$$\operatorname{sign}(x) = \begin{cases} \{x/|x|\} & \text{if } x \in \mathbb{R}^n \setminus \{0\}, \\ \overline{B(0;1)} & \text{else.} \end{cases}$$

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Let $\Lambda\in\mathrm{sign}(\nabla_{\mathrm{NC}} v_{\mathrm{CR}}) \subset \mathbb{P}_0(\mathcal{T};\mathbb{R}^n)\cap K_1(0)$, then $\mathcal{L}_h(v_{\mathrm{CR}},\Lambda) = E_{\mathrm{NC}}(v_{\mathrm{CR}})$.

$$\sup_{\Lambda\in\mathbb{P}_0(\mathcal{T};\mathbb{R}^n)} \mathcal{L}_h(v_{CR},\Lambda) \leqslant E_{NC}(v_{CR})$$

$$E_{NC}(v_{CR}) := \frac{\alpha}{2}\|v_{CR}\|^2_{L^2(\Omega)} + \|\nabla_{NC} v_{CR}\|_{L^1(\Omega)} - \int_\Omega f v_{CR}\,\mathrm{d}x$$

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Altogether we obtain

$$\inf_{v_{\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T})} E_{\mathrm{NC}}(v_{\mathrm{CR}}) = \inf_{v_{\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T})} \sup_{\Lambda \in \mathbb{P}_0(\mathcal{T};\mathbb{R}^n)} \mathcal{L}_h(v_{\mathrm{CR}}, \Lambda).$$

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(i) u_{CR} minimizes $E_{\mathrm{NC}}(v_{\mathrm{CR}})$ amongst all $v_{\mathrm{CR}} \in \mathrm{CR}^1_0(\Omega)$.

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(iii) u_{CR} satisfies

$$(f - \alpha u_{\mathrm{CR}}, v_{\mathrm{CR}} - u_{\mathrm{CR}})_{L^2(\Omega)} \leqslant \|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^1(\Omega)} - \|\nabla_{\mathrm{NC}} u_{\mathrm{CR}}\|_{L^1(\Omega)}$$

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- strict convexity of E_{NC} implies uniqueness

Since

$$\inf_{v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})} E_{\mathrm{NC}}(v_{\mathrm{CR}}) = E_{\mathrm{NC}}(u_{\mathrm{CR}}) = \sup_{\Lambda \in \mathbb{P}_0(\mathcal{T};\mathbb{R}^n)} \mathcal{L}_h(u_{\mathrm{CR}}, \Lambda)$$
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R. Tyrrell Rockafellar. **Convex Analysis**. New Jersey: Princeton University Press, 1970. ISBN: 0-691-08069-0

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It holds

$$\inf_{v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})} \sup_{\Lambda \in \mathbb{P}_0(\mathcal{T};\mathbb{R}^n)} \mathcal{L}_h(v_{\mathrm{CR}}, \Lambda) \geqslant \sup_{\Lambda \in \mathbb{P}_0(\mathcal{T};\mathbb{R}^n)} \inf_{v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})} \mathcal{L}_h(v_{\mathrm{CR}}, \Lambda)$$

and hence

$$\begin{aligned} \inf_{v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})} \sup_{\Lambda \in \mathbb{P}_0(\mathcal{T};\mathbb{R}^n)} \mathcal{L}_h(v_{\mathrm{CR}}, \Lambda) &= \mathcal{L}_h\left(u_{\mathrm{CR}}, \bar{\Lambda}\right) \\ &= \sup_{\Lambda \in \mathbb{P}_0(\mathcal{T};\mathbb{R}^n)} \inf_{v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})} \mathcal{L}_h(v_{\mathrm{CR}}, \Lambda). \end{aligned}$$

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$$\mathcal{L}_h(v_{\mathrm{CR}}, \Lambda)$$

$$:= \int_\Omega \Lambda \cdot \nabla_{\mathrm{NC}} v_{\mathrm{CR}} \,\mathrm{d}x + \frac{\alpha}{2} \|v_{\mathrm{CR}}\|_{L^2(\Omega)}^2 - \int_\Omega f v_{\mathrm{CR}} \,\mathrm{d}x - I_{K_1(0)}(\Lambda)$$

$\left(u_{\mathrm{CR}}, \bar{\Lambda}\right) \in \mathrm{CR}_0^1(\mathcal{T}) \times (\mathbb{P}_0(\mathcal{T};\mathbb{R}^n) \cap K_1(0))$ is saddle-point of $\mathcal{L}_h$ w.r.t. maximizing over $\mathbb{P}_0(\mathcal{T};\mathbb{R}^n)$ and minimizing over $\mathrm{CR}_0^1(\mathcal{T})$.

$$\inf_{v_{\mathrm{CR}}\in \mathrm{CR}_0^1(\mathcal{T})} \sup_{\Lambda\in\mathbb{P}_0(\mathcal{T};\mathbb{R}^n)} \mathcal{L}_h(v_{\mathrm{CR}},\Lambda) = \mathcal{L}_h\left(u_{\mathrm{CR}},\bar\Lambda\right)$$

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$$\mathcal{L}_h(v_{\mathrm{CR}},\Lambda) := \int_\Omega \Lambda\cdot\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\,\mathrm{d}x + \frac{\alpha}{2}\|v_{\mathrm{CR}}\|^2_{L^2(\Omega)} - \int_\Omega f v_{\mathrm{CR}}\,\mathrm{d}x - I_{K_1(0)}(\Lambda)$$

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In particular, u_{CR} minimizes $\mathcal{L}_h(\bullet,\bar\Lambda)$ in $\mathrm{CR}_0^1(\mathcal{T})$ and $\bar\Lambda$ maximizes $\mathcal{L}_h(u_{\mathrm{CR}},\bullet)$ in $\mathbb{P}_0(\mathcal{T};\mathbb{R}^n)$.

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$-\bar\Lambda$ minimizes the convex functional $-\mathcal{L}_h(u_{\mathrm{CR}},\bullet)$ in $\mathbb{P}_0(\mathcal{T};\mathbb{R}^n)$.

$$\mathcal{L}_h(v_{\mathrm{CR}},\Lambda)$$
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(i) u_{CR} minimizes $E_{\mathrm{NC}}(v_{\mathrm{CR}})$ amongst all $v_{\mathrm{CR}} \in \mathrm{CR}^1_0(\Omega)$.

(ii) There exists $\bar{\Lambda} \in \mathbb{P}_0(\mathcal{T}; \mathbb{R}^n)$ with $|\bar{\Lambda}(\bullet)| \leqslant 1$ almost everywhere in Ω such that $\bar{\Lambda}(\bullet) \cdot \nabla_{\mathrm{NC}} u_{\mathrm{CR}}(\bullet) = |\nabla_{\mathrm{NC}} u_{\mathrm{CR}}(\bullet)|$ almost everywhere in Ω and $\left(\bar{\Lambda}, \nabla_{\mathrm{NC}} v_{\mathrm{CR}}\right)_{L^2(\Omega)} = (f - \alpha u_{\mathrm{CR}}, v_{\mathrm{CR}})_{L^2(\Omega)}$ for all $v_{\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T})$.

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$$\begin{aligned}&\mathcal{L}_h(v_{\mathrm{CR}},\Lambda)\\&:=\int_\Omega \Lambda\cdot\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\,\mathrm{d}x+\frac{\alpha}{2}\|v_{\mathrm{CR}}\|^2_{L^2(\Omega)}-\int_\Omega f v_{\mathrm{CR}}\,\mathrm{d}x-I_{K_1(0)}(\Lambda)\end{aligned}$$

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(iii) u_{CR} satisfies
$(f-\alpha u_{\mathrm{CR}},v_{\mathrm{CR}}-u_{\mathrm{CR}})_{L^2(\Omega)}\leqslant\|\nabla_{\mathrm{NC}}v_{\mathrm{CR}}\|_{L^1(\Omega)}-\|\nabla_{\mathrm{NC}}u_{\mathrm{CR}}\|_{L^1(\Omega)}$
for all $v_{\mathrm{CR}}\in\mathrm{CR}^1_0(\mathcal{T})$.