Chapter 10

Spaces BV and SBV

The modelization of a large number of problems in physics, mechanics, or image processing requires the introduction of new functional spaces permitting discontinuities of the solution. In phase transitions, image segmentation, plasticity theory, the study of cracks and fissures, the study of the wake in fluid dynamics, and so forth, the solution of the problem presents discontinuities along one-codimensional manifolds. Its first distributional derivatives are now measures which may charge zero Lebesgue measure sets, and the solution of these problems cannot be found in classical Sobolev spaces. Thus, the classical theory of Sobolev spaces must be completed by the new spaces $BV(\Omega)$ and $SBV(\Omega)$.

10.1 ▪ The space $BV(\Omega)$: Definition, convergences, and approximation

In this section Ω is an open subset of $\mathbf{R}^N$. Let us recall (see Chapter 4) that $\mathbf{M}(\Omega,\mathbf{R}^N)$ denotes the space of all $\mathbf{R}^N$-valued Borel measures, which is also, according to the Riesz theory, the dual of the space $\mathbf{C}_0(\Omega,\mathbf{R}^N)$ of all continuous functions φ vanishing at infinity, equipped with the uniform norm $||\varphi||_\infty = \left(\sum_{i=1}^N \sup_{x\in\Omega} |\varphi_i(x)|^2\right)^{1/2}$. Note that $\mathbf{M}(\Omega,\mathbf{R}^N)$ is isomorphic to the product space $\mathbf{M}^N(\Omega)$ and that

$$\mu = (\mu_1,\dots,\mu_N) \in \mathbf{M}(\Omega,\mathbf{R}^N) \iff \mu_i \in \mathbf{C}_0'(\Omega), \quad i=1,\dots,N.$$

Definition 10.1.1. *We say that a function $u:\Omega\to\mathbf{R}$ is a function of bounded variation iff it belongs to $L^1(\Omega)$ and its gradient Du in the distributional sense belongs to $\mathbf{M}(\Omega,\mathbf{R}^N)$. We denote the set of all functions of bounded variation by $BV(\Omega)$. The four following assertions are then equivalent:*

(i) $u \in BV(\Omega)$;

(ii) $u\in L^1(\Omega)$ *and* $\forall i=1,\dots,N$, $\frac{\partial u}{\partial x_i} \in \mathbf{M}(\Omega)$;

(iii) $u \in L^1(\Omega)$ *and* $||Du|| := \sup\{\langle Du,\varphi\rangle \,:\, \varphi\in \mathbf{C}_c(\Omega,\mathbf{R}^N),\ ||\varphi||_\infty \le 1\} < +\infty$;

(iv) $u\in L^1(\Omega)$ *and* $||Du|| = \sup\{\int_\Omega u \ \mathrm{div}\ \varphi\, dx : \varphi \in \mathbf{C}_c^1(\Omega,\mathbf{R}^N),\ ||\varphi||_\infty \le 1\} < +\infty$, *where the bracket* $\langle , \rangle$ *in* (iii) *is defined by*

$$\langle Du, \varphi \rangle := \sum_{i=1}^{N} \int_\Omega \varphi_i \, \frac{\partial u}{\partial x_i}.$$

Equivalence between (ii) and (iii) is a direct consequence of the density of the space $\mathbf{C}_c(\Omega, \mathbf{R}^N)$ in $\mathbf{C}_0(\Omega, \mathbf{R}^N)$ equipped with the uniform norm. Equivalence between (iii) and (iv) can easily be established by the density of $\mathbf{C}_c^\infty(\Omega, \mathbf{R}^N)$ in $\mathbf{C}_c(\Omega, \mathbf{R}^N)$ and $\mathbf{C}_c^1(\Omega, \mathbf{R}^N)$.

Remark 10.1.1. According to the vectorial version of the Riesz–Alexandroff representation theorem, Theorem 2.4.7, the dual norm $||Du||$ is also the total mass $|Du|(\Omega) = \int_\Omega |Du|$ of the total variation $|Du|$ of the measure Du. Moreover, from classical integration theory, the integral $\int_\Omega f \, Du$ can be defined for all Du-integrable functions f from Ω into $\mathbf{R}^N$ as, for example, for functions in $\mathbf{C}_b(\Omega, \mathbf{R}^N)$. For the same reasons, $\int_\Omega f \, |Du|$ is well defined for all $|Du|$-integrable real-valued functions f as, for example, for functions in $\mathbf{C}_b(\Omega)$.

According to the Radon–Nikodým theorem, Theorem 4.2.1, there exist $\nabla u \in L^1(\Omega, \mathbf{R}^N)$ and a measure $D_s u$, singular with respect to the N-dimensional Lebesgue measure $\mathscr{L}^N \lfloor \Omega$ restricted to Ω, such that $Du = \nabla u \, \mathscr{L}^N \lfloor \Omega + D_s u$. Consequently, $W^{1,1}(\Omega)$ is a subspace of the vectorial space $BV(\Omega)$ and $u \in W^{1,1}(\Omega)$ iff $Du = \nabla u \, \mathscr{L}^N \lfloor \Omega$. For functions in $W^{1,1}(\Omega)$, we will sometimes write ∇u for Du. The space $BV(\Omega)$ is equipped with the following norm, which extends the classical norm in $W^{1,1}(\Omega)$:

$$||u||_{BV(\Omega)} := |u|_{L^1(\Omega)} + ||Du||.$$

We will define two weak convergence processes in $BV(\Omega)$. The first is too weak to ensure continuity of the trace operator defined in Section 10.2 but is sufficient to provide compactness of bounded sequences. The second is an intermediate convergence between the weak and the strong convergence associated with the norm.

Definition 10.1.2. *A sequence* $(u_n)_{n \in \mathbf{N}}$ *in* $BV(\Omega)$ *weakly converges to some* u *in* $BV(\Omega)$, *and we write* $u_n \rightharpoonup u$ *iff the two following convergences hold:*

$$\begin{aligned} &u_n \to u \text{ strongly in } L^1(\Omega); \\ &Du_n \rightharpoonup Du \text{ weakly in } \mathbf{M}(\Omega, \mathbf{R}^N). \end{aligned}$$

We will see later that when Ω is regular, the boundedness of a sequence in $BV(\Omega)$ is sufficient to ensure the existence of a weak cluster point (Theorem 10.1.4). In the proposition below we establish a compactness result related to this convergence, together with the lower semicontinuity of the total mass.

Proposition 10.1.1. *Let* $(u_n)_{n \in \mathbf{N}}$ *be a sequence in* $BV(\Omega)$ *strongly converging to some* u *in* $L^1(\Omega)$ *and satisfying* $\sup_{n \in \mathbf{N}} \int_\Omega |Du_n| < +\infty$. *Then*

(i) $u \in BV(\Omega)$ *and* $\int_\Omega |Du| \le \liminf_{n \to +\infty} \int_\Omega |Du_n|$;

(ii) u_n *weakly converges to* u *in* $BV(\Omega)$.

PROOF. For all φ in $\mathbf{C}_c^1(\Omega, \mathbf{R}^N)$ such that $||\varphi||_\infty \le 1$, we have

$$\int_\Omega u \, \operatorname{div} \varphi \, dx = \lim_{n \to +\infty} \int_\Omega u_n \, \operatorname{div} \varphi \, dx \le \liminf_{n \to +\infty} \int_\Omega |Du_n|$$

and assertion (i) is proved by taking the supremum in the first member, over all the elements φ in $\mathbf{C}_c^1(\Omega,\mathbf{R}^N)$ satisfying $||\varphi||_\infty \le 1$.

We now establish (ii). Since u_n strongly converges to u in $L^1(\Omega)$, for all $\varphi \in \mathbf{C}_c^\infty(\Omega,\mathbf{R}^N)$, one has

$$\langle Du_n,\varphi\rangle = -\int_\Omega u_n \text{ div } \varphi\, dx \to -\int_\Omega u \text{ div } \varphi\, dx = \langle Du,\varphi\rangle.$$

By using the density of $\mathbf{C}_c^\infty(\Omega,\mathbf{R}^N)$ in $\mathbf{C}_0(\Omega,\mathbf{R}^N)$ for the uniform norm and the boundedness of $(Du_n)_{n\in\mathbf{N}}$, we easily conclude that the sequence $(Du_n)_{n\in\mathbf{N}}$ weakly converges to Du. □

As a consequence of the semicontinuity property (i), $BV(\Omega)$ is a complete normed space.

Theorem 10.1.1. *Equipped with its norm, $BV(\Omega)$ is a Banach space.*

PROOF. Let $(u_n)_{n\in\mathbf{N}}$ be a Cauchy sequence in $BV(\Omega)$. Then $(u_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in $L^1(\Omega)$ and for all $\varepsilon > 0$ there exists N_ε in $\mathbf{N}$ such that

$$\forall p,q > N_\varepsilon \quad \int_\Omega |Du_p - Du_q| < \varepsilon. \tag{10.1}$$

Thus, there exists $u \in L^1(\Omega)$ such that $u_n \to u$ strongly in $L^1(\Omega)$. In particular $u_p - u_q \to u - u_q$ strongly in $L^1(\Omega)$ when p goes to $+\infty$. According to the lower semicontinuity property (i) of Proposition 10.1.1, (10.1) yields, for $q > N_\varepsilon$,

$$\int_\Omega |D(u-u_q)| \le \liminf_{p\to+\infty} \int_\Omega |D(u_p - u_q)| \le \varepsilon.$$

This estimate yields first $u \in BV(\Omega)$ then $\lim_{q\to+\infty} \int_\Omega |D(u-u_q)| = 0$, so that $u_n \to u$ in $BV(\Omega)$. □

To define the second weak convergence process, let us recall the notion of narrow convergence defined in Section 4.2.2. As said in Remark 10.1.1, the integral $\int_\Omega f\ |Du|$ is well defined for all f in the space $\mathbf{C}_b(\Omega)$ of bounded continuous functions on Ω. Thus $|Du|$ may be considered as an element of $\mathbf{C}'_b(\Omega)$. We now say that a sequence $(|Du_n|)_{n\in\mathbf{N}}$ narrowly converges to μ in $\mathbf{M}(\Omega)$ iff $|Du_n| \rightharpoonup \mu$ for the $\sigma(\mathbf{C}'_b(\Omega),\mathbf{C}_b(\Omega))$ convergence (see Section 4.2.2).

Definition 10.1.3. *Let $(u_n)_{n\in\mathbf{N}}$ be a sequence in $BV(\Omega)$ and $u \in BV(\Omega)$. We say that u_n converges to u in the sense of the intermediate convergence iff*

$$u_n \to u \quad \text{strongly in } L^1(\Omega),$$
$$\int_\Omega |Du_n| \to \int_\Omega |Du|.$$

The term *intermediate convergence* is due to Temam [348] and is also called strict convergence. Let us notice that according to Proposition 10.1.1(i), when u_n strongly converges to u in $L^1(\Omega)$, there is in general loss of the total mass at the limit. The proposition below states that this convergence is stronger than the weak convergence, therefore justifying the terminology.

Proposition 10.1.2. *The three following assertions are equivalent:*

(i) $u_n \rightharpoonup u$ *in the sense of the intermediate convergence;*

(ii) $\begin{cases} u_n \rightharpoonup u \text{ weakly in } BV(\Omega), \\ \int_\Omega |Du_n| \to \int_\Omega |Du|; \end{cases}$

(iii) $\begin{cases} u_n \to u \text{ strongly in } L^1(\Omega), \\ |Du_n| \rightharpoonup |Du| \text{ narrowly in } \mathbf{M}(\Omega). \end{cases}$

PROOF. (i) $\Longrightarrow$ (ii) This implication is a straightforward consequence of Proposition 10.1.1. We are going to prove (ii) $\Longrightarrow$ (iii). Let us recall (cf. Proposition 4.2.5) that for nonnegative Borel measures μ_n and μ in $\mathbf{M}(\Omega)$, there is equivalence between $\mu_n \rightharpoonup \mu$ narrowly and

$$\mu_n(\Omega) \to \mu(\Omega),$$

$$\mu(U) \leq \liminf_{n\to+\infty} \mu_n(U) \quad \forall \text{ open subset } U \text{ of } \Omega.$$

Set $\mu_n = |Du_n|$ and $\mu = |Du|$. At first we have $\mu_n(\Omega) = \int_\Omega |Du_n| \to \mu(\Omega) = \int_\Omega |Du|$. Now let U be any open subset of Ω. Obviously, u_n and u belong to $BV(U)$ and $u_n \to u$ strongly in $L^1(U)$. Applying Proposition 10.1.1 with $\Omega = U$ (note that $\sup_{n\in\mathbf{N}} \int_U |Du_n| \leq \sup_{n\in\mathbf{N}} \int_\Omega |Du_n| < +\infty$), we obtain

$$\int_U |Du| \leq \liminf_{n\to+\infty} \int_U |Du_n|.$$

Implication (iii) $\Longrightarrow$ (i) is straightforward. □

Remark 10.1.2. It results from Propositions 4.2.5 and 10.1.2 that if $u_n \rightharpoonup u$ in the sense of the intermediate convergence, for all Borel subset B of Ω such that $\int_{\partial B} |Du| = 0$, one has

$$\int_B |Du_n| \to \int_B |Du|.$$

More generally, according to Proposition 4.2.6, for all bounded Borel function $f : \Omega \to \mathbf{R}$ such that the set of its discontinuity points has a null $|Du|$-measure, one has

$$\int_\Omega f|Du_n| \to \int_\Omega f|Du|.$$

Remark 10.1.3. The intermediate convergence is strictly finer than the weak convergence in $BV(\Omega)$. Indeed, the sequence of functions in $BV(0,1)$ defined by

$$u_n(x) = \begin{cases} nx & \text{if } 0 < x \leq \frac{1}{n}, \\ 1 & \text{if } x > \frac{1}{n} \end{cases}$$

weakly converges to the function 1 in $BV(0,1)$ and does not converge in the sense of the intermediate convergence. Indeed, the total mass $|Du_n|(0,1)$ is the constant 1.

The space $\mathbf{C}^\infty(\overline{\Omega})$ is not dense in $BV(\Omega)$ when $BV(\Omega)$ is equipped with its strong convergence. Indeed, its closure is the space $W^{1,1}(\Omega)$ (see Proposition 5.4.1). Nevertheless, one can approximate every element of $BV(\Omega)$ by a function of $\mathbf{C}^\infty(\overline{\Omega})$ in the sense of the intermediate convergence.

Theorem 10.1.2. *The space* $\mathbf{C}^\infty(\Omega) \cap BV(\Omega)$ *is dense in* $BV(\Omega)$ *equipped with the intermediate convergence. Consequently* $\mathbf{C}^\infty(\overline{\Omega})$ *is also dense in* $BV(\Omega)$ *for the intermediate convergence.*

PROOF. First notice that $\mathbf{C}^\infty(\Omega) \cap BV(\Omega) = \mathbf{C}^\infty(\Omega) \cap W^{1,1}(\Omega)$. The second assertion is then a straightforward consequence of the density of the space $\mathbf{C}^\infty(\overline{\Omega})$ in $W^{1,1}(\Omega)$ equipped with its strong convergence (Proposition 5.4.1) which is finer than the intermediate convergence.

Let $\varepsilon > 0$, intended to go to zero, and $u \in BV(\Omega)$. We are going to construct u_ε in $\mathbf{C}^\infty(\Omega) \cap W^{1,1}(\Omega)$ such that

$$\int_\Omega |u - u_\varepsilon|\, dx < \varepsilon \quad \text{and} \quad \left| \int_\Omega |Du_\varepsilon| - \int_\Omega |Du| \right| < 4\varepsilon. \tag{10.2}$$

The following construction is similar to the proof of the Meyers–Serrin theorem, Theorem 5.1.4. Let us consider a family $(\Omega_i)_{i \in \mathbf{N}}$ of open subsets of Ω such that

$$\begin{aligned} &\int_{\Omega \setminus \Omega_0} |Du| < \varepsilon; \\ &\Omega_i \subset\subset \Omega_{i+1}; \\ &\Omega = \bigcup_{i=0}^{\infty} \Omega_i. \end{aligned} \tag{10.3}$$

We construct the open covering $(C_i)_{i \in \mathbf{N}^*}$ of Ω as follows: set $C_1 = \Omega_2$ and, for $i \geq 2$, $C_i = \Omega_{i+1} \setminus \overline{\Omega}_{i-1}$. Let now $(\varphi_i)_{i \in \mathbf{N}^*}$ be a partition of unity subordinate to the covering $(C_i)_{i \in \mathbf{N}^*}$. The functions φ_i satisfy $\varphi_i \in \mathbf{C}_c^\infty(C_i)$, $0 \leq \varphi_i \leq 1$, $\sum_{i=1}^{\infty} \varphi_i = 1$. Note that $\varphi_1 = 1$ on Ω_1. For each i, choose $\varepsilon_i > 0$ such that

$$\operatorname{spt}(\rho_{\varepsilon_i} * \varphi_i u) \subset C_i, \tag{10.4}$$

$$\left| \int_\Omega |\rho_{\varepsilon_1} * (\varphi_1 Du)|\, dx - \int_\Omega |\varphi_1 Du| \right| < \varepsilon, \tag{10.5}$$

$$\int_\Omega |\rho_{\varepsilon_i} * (u\varphi_i) - u\varphi_i|\, dx < \varepsilon 2^{-i}, \tag{10.6}$$

$$\int_\Omega |\rho_{\varepsilon_i} * (uD\varphi_i) - uD\varphi_i|\, dx < \varepsilon 2^{-i}, \tag{10.7}$$

where ρ_{ε_i} are the regularizers defined in Theorem 4.2.2. Estimate (10.5) is obtained by applying Theorem 4.2.2(iii) to the measure $\varphi_1 Du$. Estimates (10.6) and (10.7) are straightforward consequences of the convergence of $\rho_\varepsilon * (u\varphi_i)$ and $\rho_\varepsilon * (uD\varphi_i)$, respectively, to $u\varphi_i$ and $uD\varphi_i$ in $L^1(\Omega)$ (see Proposition 2.2.4). We define u_ε by

$$u_\varepsilon = \sum_{i=1}^{\infty} \rho_{\varepsilon_i} * (u\varphi_i).$$

Note that each x in Ω belongs to at most two of the sets C_i and that, by (10.4), u_ε is well defined and clearly belongs to $\mathbf{C}^\infty(\Omega)$. From (10.6) we obtain

$$\int_\Omega |u-u_\varepsilon|\,dx \leq \sum_{i=1}^\infty \int_\Omega |\rho_{\varepsilon_i} * (u\varphi_i) - u\varphi_i|\,dx < \varepsilon.$$

We are going to establish the last estimate of (10.2). In the distributional sense, we have $D(u\varphi_i) = \varphi_i Du + uD\varphi_i \mathscr{L}\lfloor\Omega$ so that

$$\begin{aligned} Du_\varepsilon &= \sum_{i=1}^\infty D(\rho_{\varepsilon_i} * (u\varphi_i)) = \sum_{i=1}^\infty \rho_{\varepsilon_i} * (D(u\varphi_i)) \\ &= \sum_{i=1}^\infty \rho_{\varepsilon_i} * (\varphi_i Du) + \sum_{i=1}^\infty \rho_{\varepsilon_i} * (uD\varphi_i) \\ &= \sum_{i=1}^\infty \rho_{\varepsilon_i} * (\varphi_i Du) + \sum_{i=1}^\infty \Big(\rho_{\varepsilon_i} * (uD\varphi_i) - uD\varphi_i\Big). \end{aligned}$$

Therefore, according to (10.7), Theorem 4.2.2(ii), and (10.3),

$$\begin{aligned} \left| \int_\Omega |\rho_{\varepsilon_1} * (\varphi_1 Du)|\,dx - \int_\Omega |Du_\varepsilon| \right| &\leq \sum_{i=2}^\infty \int_\Omega |\rho_{\varepsilon_i} * (\varphi_i Du)|\,dx \\ &\quad + \sum_{i=1}^\infty \int_\Omega |\rho_{\varepsilon_i} * (uD\varphi_i) - uD\varphi_i|\,dx \\ &\leq \sum_{i=2}^\infty \int_\Omega |\rho_{\varepsilon_i} * (\varphi_i Du)|\,dx + \varepsilon \\ &\leq \sum_{i=2}^\infty \int_\Omega |\varphi_i Du| + \varepsilon \\ &\leq \int_{\Omega\setminus\Omega_0} |Du| + \varepsilon < 2\varepsilon. \end{aligned} \tag{10.8}$$

On the other hand, from (10.5), (10.3) and because $\varphi_1 = 1$ on Ω_1,

$$\begin{aligned} \left| \int_\Omega |\rho_{\varepsilon_1} * (\varphi_1 Du)|\,dx - \int_\Omega |Du| \right| &\leq \varepsilon + \int_\Omega (1-\varphi_1)|Du| \\ &\leq \varepsilon + \int_{\Omega\setminus\Omega_0} |Du| \leq 2\varepsilon. \end{aligned} \tag{10.9}$$

Collecting (10.8), (10.9) we obtain

$$\left| \int_\Omega |Du_\varepsilon| - \int_\Omega |Du| \right| < 4\varepsilon,$$

which completes the proof of (10.2). □

Theorem 10.1.2 allows us to extend Sobolev's inequalities and compactness embedding results on $W^{1,1}(\Omega)$ (see Section 5.7) to the space $BV(\Omega)$.

Theorem 10.1.3. *Let Ω be a* 1*-regular open bounded subset of* $\mathbf{R}^N$*. For all p, $1 \le p \le \frac{N}{N-1}$, the embedding*

$$BV(\Omega) \hookrightarrow L^p(\Omega)$$

is continuous. More precisely, there exists a constant C which depends only on Ω, p, and N such that for all u in $BV(\Omega)$,

$$\left(\int_\Omega |u|^p \, dx\right)^{\frac{1}{p}} \le C|u|_{BV(\Omega)}.$$

PROOF. Let $(u_n)_{n\in\mathbf{N}}$ be a sequence of functions in $\mathbf{C}^\infty(\Omega)\cap BV(\Omega)$ which converges to some u in $BV(\Omega)$ for the intermediate convergence. Since the embedding $W^{1,1}(\Omega) \hookrightarrow L^p(\Omega)$ is continuous for $1 \le p \le \frac{N}{N-1}$, there exists a constant C, which depends only on Ω, p, and N such that

$$\left(\int_\Omega |u_n|^p \, dx\right)^{\frac{1}{p}} \le C\left(|u_n|_{L^1(\Omega)} + \int_\Omega |Du_n| \, dx\right) < +\infty.$$

We deduce that $u_n \rightharpoonup u$ in $L^p(\Omega)$ and, according to the weak lower semicontinuity of the norm of $L^p(\Omega)$,

$$\begin{aligned}\left(\int_\Omega |u|^p \, dx\right)^{\frac{1}{p}} &\le \liminf_{n\to+\infty}\left(\int_\Omega |u_n|^p \, dx\right)^{\frac{1}{p}} \\ &\le \liminf_{n\to+\infty} C\left(|u_n|_{L^1(\Omega)} + \int_\Omega |Du_n| \, dx\right) \\ &= C\|u\|_{BV(\Omega)},\end{aligned}$$

where we have used the intermediate convergence in the last equality. □

Theorem 10.1.4. *Let Ω be a* 1*-regular open bounded subset of* $\mathbf{R}^N$*. Then for all p, $1 \le p < \frac{N}{N-1}$ the embedding*

$$BV(\Omega) \hookrightarrow L^p(\Omega)$$

is compact.

PROOF. According to Theorem 10.1.3, every element of $BV(\Omega)$ belongs to $L^p(\Omega)$ for $1 \le p \le \frac{N}{N-1}$, so that we can slightly improve the density Theorem 10.1.2 as follows: for all $u \in BV(\Omega)$, there exists $u_n \in \mathbf{C}^\infty(\Omega)\cap BV(\Omega)$ satisfying

$$\begin{cases} u_n \to u & \text{in } L^p(\Omega), \\ \displaystyle\int_\Omega |Du_n| \to \int_\Omega |Du|. \end{cases}$$

We conclude thanks to the compactness of the embedding of $W^{1,1}(\Omega) \hookrightarrow L^p(\Omega)$. Indeed, let u_n be such that $\|u_n\|_{BV(\Omega)} \le 1$ and $v_n \in \mathbf{C}^\infty(\Omega)\cap BV(\Omega)$ be such that

$$\begin{cases} |v_n - u_n|_{L^p(\Omega)} \le \frac{1}{n}, \\ \displaystyle\int_\Omega |Dv_n| \, dx \le 2. \end{cases}$$

Since $||v_n||_{W^{1,1}(\Omega)} \leq 4$ for n large enough there exists a subsequence $(v_{n_k})_{k\in\mathbf{N}}$ and u in $L^p(\Omega)$ such that

$$v_{n_k} \to u \quad \text{strongly in } L^p(\Omega),$$

and thus

$$u_{n_k} \to u \quad \text{strongly in } L^p(\Omega).$$

By the lower semicontinuity of the total mass, and since u_{n_k} strongly converges to u in $L^1(\Omega)$, we obtain

$$\begin{aligned} |u|_{L^1(\Omega)} + \int_\Omega |Du| &\leq \lim_{k\to+\infty} |u_{n_k}|_{L^1(\Omega)} + \liminf_{k\to+\infty} \int_\Omega |Du_{n_k}| \\ &\leq \liminf_{k\to+\infty} ||u_{n_k}||_{BV(\Omega)} \leq 1 \end{aligned}$$

and the proof is complete. □

10.2 ▪ The trace operator, the Green's formula, and its consequences

Throughout this section, Ω is a domain of $\mathbf{R}^N$ with a Lipschitz boundary Γ (i.e., a Lipschitz domain). Under a weaker hypothesis on the regularity of the set Ω, we extend in the space $BV(\Omega)$ the notion of trace developed for Sobolev functions in Section 5.6. It is worth noticing that the method used for establishing the trace theorem below also applies to Sobolev functions.

Theorem 10.2.1. *There exists a linear continuous map γ_0 from $BV(\Omega)$ onto $L^1_{\mathcal{H}^{N-1}}(\Gamma)$ satisfying*

(i) *for all u in $\mathbf{C}(\overline{\Omega}) \cap BV(\Omega)$, $\gamma_0(u) = u\lfloor\Gamma$;*

(ii) *the generalized Green's formula holds: for all $\varphi \in \mathbf{C}^1(\overline{\Omega}, \mathbf{R}^N)$,*

$$\int_\Omega \varphi Du = -\int_\Omega u \ \operatorname{div} \varphi \, dx + \int_\Gamma \gamma_0(u) \, \varphi.\nu \, d\mathcal{H}^{N-1},$$

where $\nu(x)$ is the outer unit normal at $\mathcal{H}^{N-1}$ almost all x in Γ.

PROOF. Each generic element x in $\mathbf{R}^N$ will be denoted by $x = (\tilde{x}, x_N)$, where

$$\tilde{x} = (x_1, \dots, x_{N-1}) \in \mathbf{R}^{N-1}$$

and $x_N \in \mathbf{R}$. Let us consider a finite cover of Γ by the open cylinders

$$C_R(y) = S_R(\tilde{y}) \times (y_N - R, y_N + R), \ y = (\tilde{y}, y_N) \in \Gamma,$$

where $S_R(\tilde{y})$ is the open ball of $\mathbf{R}^{N-1}$ with radius R centered at $\tilde{y} = (y_1, \dots, y_{N-1})$. Since Γ is Lipschitz regular, relabeling the coordinate axes if necessary, there exists $\varepsilon_0 > 0$ and a Lipschitz function f such that $\Omega \cap C_R(y)$ contains the open set

$$C_{R,\varepsilon_0}(y) := \{x \in \mathbf{R}^N : \tilde{x} \in S_R(\tilde{y}), \ f(\tilde{x}) - \varepsilon_0 < x_N < f(\tilde{x})\},$$

and

$$\Sigma(y) := \{(\tilde{x}, x_N) : \tilde{x} \in S_R(\tilde{y}), \ x_N = f(\tilde{x})\}$$

is a neighborhood of y in Γ. Let u be a fixed function in $BV(\Omega)$. According to Lemma 4.2.2, since $\int_{C_{R,\varepsilon_0}(y)} |Du| < +\infty$, R and ε_0 can be chosen so that the measure $|Du|$ does not charge $\partial C_{R,\varepsilon_0}(y) \setminus \Sigma(y)$, i.e., $\int_{\partial C_{R,\varepsilon_0}(y)\setminus\Sigma(y)} |Du| = 0$.

First step. We fix y in Γ and, to shorten notation, we denote the sets $\Sigma(y)$, $S_R(\tilde{y})$, and $C_{R,\varepsilon_0}(y)$ by Σ, S_R, and C_{R,ε_0}, respectively. Since Γ is Lipschitz, according to Rademacher's theorem (see [211]), the outer unit normal $\nu(x)$ exists at $\mathcal{H}^{N-1}$ a.e. x on Γ. This step is devoted to the existence of u^+ in $L^1_{\mathcal{H}^{N-1}}(\Sigma)$ satisfying

$$\begin{cases} \displaystyle\int_\Sigma |u^+| \, d\mathcal{H}^{N-1} \le C\left(\int_{C_{R,\varepsilon_0}} |u| \, dx + \int_{C_{R,\varepsilon_0}} |Du| \right) \\ \displaystyle\forall \varphi \in \mathbf{C}^1_c(C_{R,\varepsilon_0} \cup \Sigma, \mathbf{R}^N) \int_{C_{R,\varepsilon_0}} \varphi Du = -\int_{C_{R,\varepsilon_0}} u \ \mathrm{div} \ \varphi \, dx + \int_\Sigma u^+ \varphi \nu \, d\mathcal{H}^{N-1}, \end{cases} \tag{10.10}$$

where C is a positive constant depending only on f.

For all regular function v defined on C_{R,ε_0} and all ε in $]0, \varepsilon_0[$, we adopt the notation $v^\varepsilon(\tilde{x}) := v(\tilde{x}, f(\tilde{x}) - \varepsilon)$. Consider now a sequence $(u_n)_{n\in\mathbf{N}}$ in $\mathbf{C}^\infty(C_{R,\varepsilon_0}) \cap BV(C_{R,\varepsilon_0})$ which converges to u in $BV(C_{R,\varepsilon_0})$ in the sense of intermediate convergence (see Theorem 10.1.2). We have

$$\begin{cases} \displaystyle\int_{C_{R,\varepsilon_0}} |u_n - u| \, dx \to 0, \\ \displaystyle\int_{C_{R,\varepsilon_0}} |Du_n| \, dx \to \int_{C_{R,\varepsilon_0}} |Du|. \end{cases} \tag{10.11}$$

Since the function u_n is smooth, for $\varepsilon > \varepsilon'$ in $]0, \varepsilon_0[$ one has

$$u_n^\varepsilon(\tilde{x}) - u_n^{\varepsilon'}(\tilde{x}) = \int_{-\varepsilon'}^{-\varepsilon} \frac{\partial u_n}{\partial x_N}(\tilde{x}, f(\tilde{x}) + s) \, ds,$$

so that

$$|u_n^\varepsilon(\tilde{x}) - u_n^{\varepsilon'}(\tilde{x})| \le \int_{\varepsilon'}^{\varepsilon} |Du_n(\tilde{x}, f(\tilde{x}) - s)| \, ds.$$

Thus, according to Proposition 4.1.6 and Remark 4.1.5 applied to the map $S_R \subset \mathbf{R}^{N-1} \to \mathbf{R}^N$, $\tilde{x} \mapsto (\tilde{x}, f(\tilde{x}))$, we deduce

$$\begin{aligned} \int_\Sigma |u_n^\varepsilon(\tilde{x}) - u_n^{\varepsilon'}(\tilde{x})| \, d\mathcal{H}^{N-1}(x) &\le \int_{S_R} |u_n^\varepsilon(\tilde{x}) - u_n^{\varepsilon'}(\tilde{x})| \sqrt{1 + |Df(\tilde{x})|^2} \, d\tilde{x} \\ &\le C \int_{S_R} \int_{\varepsilon'}^{\varepsilon} |Du_n(\tilde{x}, f(\tilde{x}) - s)| \, ds \, d\tilde{x} \\ &= C \int_{C_{R,\varepsilon,\varepsilon'}} |Du_n(x)| \ dx, \end{aligned}$$

where C is a positive constant depending only on f and

$$C_{R,\varepsilon,\varepsilon'} = \{x \in \mathbf{R}^N : \tilde{x} \in S_R, \ f(\tilde{x}) - \varepsilon < x_N < f(\tilde{x}) - \varepsilon'\}.$$

Therefore, with the notation made precise above,

$$\int_\Sigma |u_n^\varepsilon - u_n^{\varepsilon'}|\, d\mathscr{H}^{N-1} \le C \int_{C_{R,\varepsilon,\varepsilon'}} |Du_n|\, dx. \tag{10.12}$$

We intend to go to the limit on n in (10.12). From the coarea formula (Theorem 4.2.5), one has

$$\int_{-\varepsilon_0}^0 \left(\int_\Sigma |u_n^\varepsilon(\tilde{x}) - u^\varepsilon(\tilde{x})|\, d\mathscr{H}^{N-1}(x) \right) d\varepsilon \le C \int_{C_{R,\varepsilon_0}} |u_n(x) - u(x)|\, dx,$$

and the first limit in (10.11) yields the existence of a subsequence on n (not relabeled) such that for a.e. ε in $]0,\varepsilon_0[$, $u_n^\varepsilon \to u^\varepsilon$ in $L^1_{\mathscr{H}^{N-1}}(\Sigma)$. On the other hand, according to Proposition 10.1.2, (10.11) ensures the narrow convergence of the measure $|Du_n|$ to the measure $|Du|$ in $\mathbf{M}^+(C_{R,\varepsilon_0})$. Let us consider ε, ε' such that $\int_{\partial C_{R,\varepsilon,\varepsilon'}} |Du| = 0$. This choice is indeed valid in the complementary I of a countable subset of $]0,\varepsilon_0[$, by using Lemma 4.2.2 and because $|Du|$ does not charge $\partial C_{R,\varepsilon_0} \setminus \Sigma$. According to the properties of the narrow convergence (cf. Proposition 4.2.5), we have, for ε and ε' in I,

$$\int_{C_{R,\varepsilon,\varepsilon'}} |Du_n| \to \int_{C_{R,\varepsilon,\varepsilon'}} |Du|.$$

Going to the limit on n in (10.12), we finally obtain, for ε in $]0,\varepsilon_0[\setminus\mathscr{N}$, where $\mathscr{N}$ is an $\mathscr{L}^1$-negligible set,

$$\int_\Sigma |u^\varepsilon - u^{\varepsilon'}|\, d\mathscr{H}^{N-1} \le C \int_{C_{R,\varepsilon,\varepsilon'}} |Du|. \tag{10.13}$$

From now on, ε denotes a sequence of numbers in $]0,\varepsilon_0[\setminus\mathscr{N}$, going to zero, and (10.13) must be taken in the sense

$$\int_\Sigma |u^{\varepsilon_p} - u^{\varepsilon_q}|\, d\mathscr{H}^{N-1} \le C \int_{C_{R,\varepsilon_p,\varepsilon_q}} |Du|.$$

From (10.13), $(u^\varepsilon)_\varepsilon$ is a Cauchy sequence in $L^1_{\mathscr{H}^{N-1}}(\Sigma)$, then strongly converges to some function u^+ in $L^1(\Sigma)$. It remains to prove that u^+ satisfies (10.10). Letting $\varepsilon' \to 0$ in (10.13) and integrating over $]-\varepsilon_0, 0[$, we obtain

$$\int_\Sigma |u^+|\, d\mathscr{H}^{N-1} \le C \left(\int_{C_{R,\varepsilon_0}} |u|\, dx + \int_{C_{R,\varepsilon_0}} |Du| \right). \tag{10.14}$$

On the other hand, since $u_n \in \mathbf{C}^\infty(C_{R,\varepsilon_0})$ and $\varphi \in \mathbf{C}^1_c(C_{R,\varepsilon_0} \cup \Sigma, \mathbf{R}^N)$, going to the limit on n in the classical Green's formula

$$\int_{C_{R,\varepsilon_0,\varepsilon}} u_n \operatorname{div} \varphi\, dx = -\int_{C_{R,\varepsilon_0,\varepsilon}} Du_n.\varphi\, dx + \int_\Sigma u_n^\varepsilon \varphi^\varepsilon.\nu\, d\mathscr{H}^{N-1},$$

where

$$C_{R,\varepsilon_0,\varepsilon} := \{x \in \mathbf{R}^N : \tilde{x} \in S_R(\tilde{y}),\ f(\tilde{x}) - \varepsilon_0 < x_N < f(\tilde{x}) - \varepsilon\},$$

we claim that

$$\int_{C_{R,\varepsilon_0,\varepsilon}} u \operatorname{div} \varphi\, dx = -\int_{C_{R,\varepsilon_0,\varepsilon}} \varphi Du + \int_\Sigma u^\varepsilon \varphi^\varepsilon.\nu\, d\mathscr{H}^{N-1}. \tag{10.15}$$

We must justify the convergence

$$\int_{C_{R,\varepsilon_0,\varepsilon}} Du_n.\varphi\, dx \to \int_{C_{R,\varepsilon_0,\varepsilon}} \varphi Du. \tag{10.16}$$

The two others are straightforward. We reason by truncation: let $\varepsilon' < \varepsilon$ and consider $\tilde{\varphi} = \varphi\theta$ in $\mathbf{C}_c^1(C_{R,\varepsilon_0,\varepsilon'}, \mathbf{R}^N)$, where the scalar function θ belongs to $\mathbf{C}_c(C_{R,\varepsilon_0,\varepsilon'})$ and satisfies $\theta = 1$ in $C_{R,\varepsilon_0,\varepsilon}$, with $\|\theta\|_\infty \le 1$. We have

$$\int_{C_{R,\varepsilon_0,\varepsilon}} Du_n\varphi\, dx = \int_{C_{R,\varepsilon_0,\varepsilon'}} Du_n\tilde{\varphi}\, dx - \int_{C_{R,\varepsilon,\varepsilon'}} Du_n\tilde{\varphi}\, dx. \tag{10.17}$$

From

$$\left|\int_{C_{R,\varepsilon,\varepsilon'}} Du_n\tilde{\varphi}\, dx\right| \le \|\tilde{\varphi}\|_\infty \int_{C_{R,\varepsilon,\varepsilon'}} |Du_n|$$

and according to the narrow convergence of $|Du_n|$ to $|Du|$, and because $|Du|$ does not charge $\partial C_{R,\varepsilon,\varepsilon'}$, for ε and ε' in $]0,\varepsilon_0[\setminus\mathcal{N}$, we obtain

$$\lim_{\varepsilon'\to\varepsilon} \limsup_{n\to+\infty} \left|\int_{C_{R,\varepsilon,\varepsilon'}} Du_n\tilde{\varphi}\, dx\right| = 0.$$

The convergence in (10.16) is obtained by letting $n \to +\infty$ and $\varepsilon' \to \varepsilon$ in (10.17) and the claim is proved.

Finally going to the limit on ε in (10.15) we obtain

$$\int_{C_{R,\varepsilon_0}} u \ \operatorname{div}\ \varphi\, dx = -\int_{C_{R,\varepsilon_0}} \varphi Du - \int_{S_R} u^+\varphi.e_N\ \mathcal{H}^{N-1}.$$

Second step. According to the first step and from a straightforward argument using a partition of unity subordinate to a finite cover $(C_R(y_i))_{i=1,\dots,r}$ of Γ, there exists $\gamma_0(u)$ in $L^1_{\mathcal{H}^{N-1}}(\Gamma)$ satisfying

$$\int_\Gamma |\gamma_0(u)|\, d\mathcal{H}^{N-1} \le C\Big(\int_\Omega |u|\, dx + \int_\Omega |Du|\Big) \tag{10.18}$$

and such that for all $\varphi \in \mathbf{C}(\overline{\Omega}, \mathbf{R}^N)$,

$$\int_\Omega \varphi Du = -\int_\Omega u\ \operatorname{div}\ \varphi\, dx + \int_\Gamma \gamma_0(u)\, \varphi.\nu\, d\mathcal{H}^{N-1}, \tag{10.19}$$

where $\nu(x)$ is the outer normal unit at $\mathcal{H}^{N-1}$ almost all x in Γ and C a positive constant depending only on Ω. The operator γ_0 is well defined by $\gamma_0(u) = u_i^+$, in Σ_i, $i = 1,\dots,r$. Indeed, Green's formula (10.10) established in the first step yields

$$\int_{\Sigma_i\cap\Sigma_j} (u_i^+ - u_j^+)\varphi.\nu\, d\mathcal{H}^{N-1} = 0$$

for all functions φ in $\mathbf{C}_c^1((C_{R,\varepsilon_0}(y_i)\cap C_{R,\varepsilon_0}(y_j))\cup(\Sigma_i\cap\Sigma_j), \mathbf{R}^N)$ so that $u_i^+ = u_j^+$ in $\Sigma_i\cap\Sigma_j$ up to sets of $\mathcal{H}^{N-1}$ measure zero.

The generalized Green's formula (10.19) yields the linearity of γ_0. The continuity is a consequence of (10.18). The identity $\gamma_0(u) = u\lfloor S_R$ for all u in $\mathbf{C}(\overline{\Omega}) \cap BV(\Omega)$ is a straightforward consequence of the definition of $\gamma_0(u)$. The operator γ_0 then agrees with the trace operator defined in $W^{1,1}(\Omega)$. Since $\gamma_0(W^{1,1}(\Omega)) = L^1(\Omega)$, we also have $\gamma_0(BV(\Omega)) = L^1(\Omega)$. $\square$

Remark 10.2.1. Let Ω be a Lipschitz open bounded subset of $\mathbf{R}^N$. The density theorem, Theorem 10.1.2, may be slightly improved in the sense that one may further assert that the trace of each regular approximating function of $u \in BV(\Omega)$, belonging to $\mathbf{C}^\infty(\Omega) \cap BV(\Omega)$, coincides with the trace of u on the boundary of Ω. Indeed, it is easily seen, with the notation of Theorem 10.1.2, that $u_\varepsilon - u$ is the strong limit in $BV(\Omega)$ of the functions

$$u_{\varepsilon,n} - u_n := \sum_{i=0}^{n} \big(\rho_{\varepsilon_i} * (u\varphi_i) - u\varphi_i\big)$$

whose traces on $\Gamma := \partial\Omega$ are the function null. The result then follows from the strong continuity of the trace operator.

Remark 10.2.2. One may define the space $BV(\Omega, \mathbf{R}^m)$ as the space of all functions $u : \Omega \to \mathbf{R}^m$ in $L^1(\Omega, \mathbf{R}^m)$ whose distributional derivative Du belongs to the space $\mathbf{M}(\Omega, M^{m\times N})$ of $m \times N$ matrix-valued measures. Arguing as in the proof of Theorem 10.2.1 with each component of u, one may prove the existence of the trace operator γ_0 from $BV(\Omega, \mathbf{R}^m)$ onto $L^1_{\mathscr{H}^{N-1}}(\Gamma, \mathbf{R}^m)$ satisfying

(i) $\forall u \in \mathbf{C}(\overline{\Omega}, \mathbf{R}^m) \cap BV(\Omega, \mathbf{R}^m)$, $\gamma_0(u) = u\lfloor\Gamma$;

(ii) the Green's formula holds: for all $\varphi \in \mathbf{C}^1(\overline{\Omega}, M^{m\times N})$

$$\int_\Omega \varphi : Du = -\int_\Omega u.\mathrm{div}\ \varphi\, dx + \int_\Gamma \gamma_0(u) \otimes \nu : \varphi\, d\mathscr{H}^{N-1},$$

where $\nu(x)$ is the outer unit normal at $\mathscr{H}^{N-1}$ almost all x in Γ and $\gamma_0(u) \otimes \nu$ is the $M^{m\times N}$ valued function $(\gamma_0(u)_i \nu_j)_{i=1\ldots m, j=1\ldots N}$. The integral with respect to the measure Du in the first member is defined by

$$\int_\Omega \varphi : Du := \sum_{i=1}^{N}\sum_{j=1}^{m} \int_\Omega \varphi_{i,j} \frac{\partial u_j}{\partial x_i}$$

and

$$\gamma_0(u) \otimes \nu : \varphi := \sum_{i=1}^{m}\sum_{j=1}^{N} \gamma_0(u)_i \nu_j\ \varphi_{i,j}.$$

The divergence of φ is the vector valued distribution $(\mathrm{div}\ \varphi)_j := \sum_{i=1}^N \frac{\partial u_j}{\partial x_i}$, $j = 1, \ldots, m$.

The density theorem, Theorem 10.1.2, and Remark 10.2.1 also hold in $BV(\Omega, \mathbf{R}^m)$.

We now give some consequences of Theorem 10.2.1.

Example 10.2.1. Consider two disjoint Lipschitz domains Ω_1 and Ω_2, included in an open bounded subset Ω of $\mathbf{R}^N$, such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, and set $\Gamma_{1,2} := \partial\Omega_1 \cap \partial\Omega_2$ which

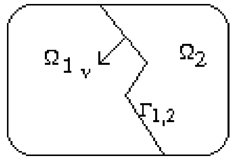

Figure 10.1. *The set Ω.*

is assumed to satisfy $\mathcal{H}^{N-1}(\Gamma_{1,2}) > 0$ (see Figure 10.1). We respectively denote the trace operators from $BV(\Omega_1)$ onto $L^1(\partial\Omega_1)$ and $BV(\Omega_2)$ onto $L^1(\partial\Omega_2)$ by γ_1 and γ_2. Let u_1 and u_2 be, respectively, two functions in $BV(\Omega_1)$ and $BV(\Omega_2)$ and define

$$u = \begin{cases} u_1 & \text{in } \Omega_1, \\ u_2 & \text{in } \Omega_2. \end{cases}$$

Then u belongs to $BV(\Omega)$ and

$$Du = Du_1 \lfloor \Omega_1 + Du_2 \lfloor \Omega_2 + [u]\nu \mathcal{H}^{N-1} \lfloor \Gamma_{1,2},$$

where $[u] = \gamma_1(u_1) - \gamma_2(u_2)$ and $\nu(x)$ is the unit inner normal at x to $\Gamma_{1,2}$ considered as a part of the boundary of Ω_1 (see Figure 10.1). In particular, if $u \in W^{1,1}(\Omega \setminus \Gamma_{1,2})$

$$Du = \nabla u \, \mathcal{L}_N + [u]\nu \mathcal{H}^{N-1} \lfloor \Gamma_{1,2},$$

where ∇u is the gradient of u in $L^1(\Omega)$.

PROOF. For all $\varphi \in \mathbf{C}^1_c(\Omega, \mathbf{R}^N)$,

$$\langle Du, \varphi \rangle = -\int_\Omega u \ \operatorname{div} \varphi \, dx = -\int_{\Omega_1} u_1 \ \operatorname{div} \varphi \, dx - \int_{\Omega_2} u_2 \ \operatorname{div} \varphi \, dx.$$

Since φ belongs to $\mathbf{C}^1(\overline{\Omega}_1, \mathbf{R}^N) \cap \mathbf{C}^1(\overline{\Omega}_2, \mathbf{R}^N)$, applying the generalized Green's formula in $BV(\Omega_1)$ and $BV(\Omega_2)$, we have

$$\begin{cases} \displaystyle\int_{\Omega_1} u_1 \operatorname{div} \varphi \, dx = -\int_{\Omega_1} \varphi \, Du_1 + \int_{\Gamma_{1,2}} \gamma_1(u_1) \varphi.(-\nu) \, \mathcal{H}^{N-1}, \\ \displaystyle\int_{\Omega_2} u_2 \ \operatorname{div} \varphi \, dx = -\int_{\Omega_2} \varphi \, Du_2 + \int_{\Gamma_{1,2}} \gamma_2(u_2) \varphi.\nu \, \mathcal{H}^{N-1}. \end{cases}$$

By summing these two equalities, we obtain

$$\langle Du, \varphi \rangle = \int_{\Omega_1} \varphi \, Du_1 + \int_{\Omega_2} \varphi \, Du_2 + \int_{\Gamma_{1,2}} \big(\gamma_1(u_1) - \gamma_2(u_2)\big) \varphi.\nu \mathcal{H}^{N-1}. \tag{10.20}$$

Assume now $\|\varphi\|_\infty \leq 1$. According to the continuity of γ_1 and γ_2, there exists a positive constant C depending only on Ω_1 and Ω_2 such that

$$\left| \int_{\Gamma_{1,2}} \big(\gamma_1(u_1) - \gamma_2(u_2)\big) \varphi.\nu \mathcal{H}^{N-1} \right| \leq C\big(\|u_1\|_{BV(\Omega_1)} + \|u_2\|_{BV(\Omega_2)}\big),$$

so that u belongs to $BV(\Omega)$ and

$$\|Du\| := \sup\{\langle Du,\varphi\rangle \ : \varphi\in \mathbf{C}_c(\Omega,\mathbf{R}^N),\ \|\varphi\|_\infty \le 1\} \le C\big(\|u_1\|_{BV(\Omega_1)}+\|u_2\|_{BV(\Omega_2)}\big).$$

Finally, (10.20) can be written

$$\langle Du,\varphi\rangle = \int_\Omega \varphi\,\left(Du_1\lfloor\Omega_1 + Du_2\lfloor\Omega_2\right) + \int_\Omega \big(\gamma_1(u_1)-\gamma_2(u_2)\big)\varphi.\nu\mathscr{H}^{N-1}\lfloor\Gamma_{1,2}.$$

This shows that $Du = Du_1\lfloor\Omega_1 + Du_2\lfloor\Omega_2 + \big(\gamma_1(u_1)-\gamma_2(u_2)\big)\nu\mathscr{H}^{N-1}\lfloor\Gamma_{1,2}$ on $\mathbf{C}^1_c(\Omega,\mathbf{R}^N)$ and thus are equal by density. □

Example 10.2.2. Let us slightly modify the previous example by considering the function

$$v = \begin{cases} u & \text{in } \Omega, \\ 0 & \text{in } \mathbf{R}^N\setminus\overline{\Omega}, \end{cases}$$

where Ω is a Lipschitz domain of $\mathbf{R}^N$ and $u\in BV(\Omega)$. We see that v belongs to $BV(\mathbf{R}^N)$ and

$$Dv = Du\lfloor\Omega + u^+\ \nu\ \mathscr{H}^{N-1}\lfloor\Gamma,$$

where Γ is the boundary of Ω, ν denotes the inner unit vector normal to Γ, and u^+ the trace of u on Γ. Thus, taking for instance u equal to the constant 1 in Ω, the characteristic function 1_Ω of Ω belongs to $BV(\mathbf{R}^N)$ (more precisely in a subspace $SBV(\mathbf{R}^N)$ introduced in Section 10.5) and

$$D1_\Omega = \nu\ \mathscr{H}^{N-1}\lfloor\Gamma.$$

This formula will be generalized in Section 10.3 when Ω possesses a *reduce boundary* $\partial_r\Omega$ (which in general does not coincides with the topological boundary) and a generalized unit normal ν to $\partial_r\Omega$. More precisely, for 1_Ω belonging to $BV(\mathbf{R}^N)$, we will obtain

$$D1_\Omega = \nu\ \mathscr{H}^{N-1}\lfloor\partial_r\Omega.$$

Example 10.2.3. *Jump through a family of hypersurfaces.* Let Ω be a Lipschitz domain of $\mathbf{R}^N$, u a function in $BV(\Omega)$, and $(\Sigma_t)_{t\in I}$ a family of Lipschitz hypersurfaces such that $\Sigma_t\subset\Omega$ is the boundary of a Lipschitz open bounded subset Ω_t of Ω with $t<t'\implies \Omega_t\subset\subset\Omega_{t'}$. As a consequence of Example 10.2.1, one has for all but countably many t in I that the jumps $[u]_t$ of u through Σ_t are null.

PROOF. Indeed, from Example 10.2.1,

$$\int_{\Sigma_t} |Du| = \int_{\Sigma_t} |[u]_t|\, d\mathscr{H}^{N-1},$$

and we conclude by Lemma 4.2.1. □

Example 10.2.4. Let Ω be a Lipschitz domain of $\mathbf{R}^N$ and $u\in BV(\Omega)$. For all $t>0$, consider the Lipschitz domains $\Omega_t=\{x\in\Omega : d(x,\mathbf{R}^N\setminus\Omega)>t\}$ with boundary Γ_t and denote, respectively, by u_t^+ and u_t^- the traces of u when u is considered as an element of $BV(\Omega_t)$ or $BV(\Omega\setminus\overline{\Omega}_t)$. We have for all but countably many t in $\mathbf{R}^+$: the traces $u_t^+(x)$ and $u_t^-(x)$ agree with $u(x)$ for $\mathscr{H}^{N-1}$ almost x in Γ_t.

PROOF. By using arguments of the proof of Theorem 10.2, one may assume that Ω is a cylinder. With the notation of this theorem, for all t in a complementary of a countable subset of $\mathbf{R}^+$, estimate (10.13) becomes

$$\int_\Sigma |u^{t+\varepsilon} - u^t|\, d\mathcal{H}^{N-1} \le C \int_{C_{R,t+\varepsilon,t}} |Du|.$$

We deduce, for these t, that $u^{t+\varepsilon}$ tends to u^t in $L^1(\Sigma)$ when ε goes to 0^+. On the other hand, according to the proof of the trace theorem, $u^{t+\varepsilon}$ tends to u_t^+ in $L^1(\Sigma_t)$ when ε goes to 0^+. Therefore, for $\mathcal{H}^{N-1}$ almost all x in Σ_t, $u_t^+ = u$. With the same arguments, but now reasoning on $C_{R,t,t+\varepsilon}$, we obtain that for $\mathcal{H}^{N-1}$ almost all x in Γ_t, $u_t^- = u$.

We are going to establish the continuity of the trace operator with respect to the intermediate convergence. Let us recall that the trace operator is continuous from $W^{1,p}(\Omega)$ into $L^p(\Gamma)$, $p \ge 1$ when these two spaces are equipped with their weak topology. Indeed, this is a consequence of the following property (cf. [137, Proposition III 9]): if T is a continuous linear operator from a Banach space E into a Banach space F, then T is continuous from E equipped with the $\sigma(E,E')$ topology into F equipped with the $\sigma(F,F')$ topology. We cannot apply this general principle to the trace operator defined in $BV(\Omega)$. Indeed, the weak topology in $BV(\Omega)$ is not the $\sigma(BV(\Omega), BV(\Omega)')$ topology and we must be careful with the terminology "weak" convergence in $BV(\Omega)$.

The example of Remark 10.1.3 shows that γ_0 is not continuous from $BV(\Omega)$ into $L^1(\Gamma)$ when $BV(\Omega)$ and $L^1(\Gamma)$ are equipped with their weak convergence: define $u_n \in BV(\Omega)$, $\Omega = (0,1)$ by $u_n(x) = nx$ if $x \in (0, \frac{1}{n}]$ and $u(x) = 1$ if $x \in [\frac{1}{n}, 1)$. It is easily seen that u_n weakly converges to the constant 1 in $BV(\Omega)$, whereas $u_n(x) = 0$ for $x \in \{0\}$. Note that in this example, u_n does not converge to $u = 1$ in the sense of intermediate convergence because there is "loss of total mass." Indeed $\int_\Omega |Du_n| = 1$ but $\int_\Omega |Du| = 0$. When the total mass is preserved at the limit, the trace operator is continuous.

Theorem 10.2.2. *Let Ω be a Lipschitz domain of $\mathbf{R}^N$. The trace operator γ_0 is continuous from $BV(\Omega)$ equipped with the intermediate convergence onto $L^1(\Gamma)$ equipped with the strong convergence.*

PROOF. For each $t > 0$, consider the Lipschitz domain $\Omega_t = \{x \in \Omega : d(x, \mathbf{R}^N \setminus \Omega) > t\}$ with Lipschitz boundary Γ_t, and $(u_n)_{n\in\mathbf{N}}$ and u in $BV(\Omega)$ such that

$$\begin{cases} \displaystyle\int_\Omega |u_n - u \mid dx \to 0, \\ \displaystyle\int_\Omega |Du_n| \to \int_\Omega |Du|. \end{cases} \tag{10.21}$$

Possibly passing to a subsequence on n, and for almost all $t \in \mathbf{R}^+$, we have

$$\begin{cases} \displaystyle\int_{\Gamma_t} |Du| = 0, \\ u_n - u = (u_n - u)_t \quad \text{for } \mathcal{H}^{N-1} \text{ a.e. } x \text{ on } \Gamma_t, \\ \displaystyle\lim_{n\to+\infty} \int_{\Gamma_t} |u_n - u|\, d\mathcal{H}^{N-1} \to 0, \end{cases} \tag{10.22}$$

where $(u_n - u)_t$ denotes the trace on Γ_t of the function $u_n - u$ in $BV(\Omega_t)$. Indeed, the two first assertions are satisfied for all but countably many t in $\mathbf{R}^+$, thanks to Lemma 4.2.1

and to Example 10.2.4. The last assertion is a consequence of the curvilinear Fubini theorem (cf. Corollary 4.2.2):

$$\int_0^{+\infty}\int_{\Gamma_t}|u_n-u|\,d\mathcal{H}^{N-1}\,dt=\int_0^{+\infty}\int_{[d(x,\mathbf{R}^N\setminus\overline{\Omega})=t]}|u_n-u|\,d\mathcal{H}^{N-1}\,dt$$
$$=\int_\Omega|u_n-u|\,dx.$$

For a fixed t, for which these assertions hold, let us define the function $u_{n,t}$ in $BV(\Omega)$ by

$$u_{n,t}=\begin{cases}u_n-u & \text{in } \Omega\setminus\overline{\Omega}_t,\\ 0 \quad \text{in } \Omega_t. & \end{cases}$$

We have

$$Du_{n,t}=D(u_n-u)\lfloor(\Omega\setminus\overline{\Omega}_t)+(u_n-u)_t\ \nu_t\ \mathcal{H}^{N-1}\lfloor\Gamma_t,$$

where ν_t is the outer unit normal vector to Γ_t. According to the strong continuity of the trace operator γ_0 from $BV(\Omega)$ onto $L^1(\Gamma)$, we finally deduce

$$\int_\Gamma|\gamma_0(u_n-u)|\,d\mathcal{H}^{N-1}=\int_\Gamma|\gamma_0(u_{n,t})|\,d\mathcal{H}^{N-1}$$
$$\leq C\Big(\int_{\Omega\setminus\Omega_t}|u_n-u|\,dx+\int_{\Omega\setminus\overline{\Omega}_t}|Du_n-Du|\,dx$$
$$+\int_{\Gamma_t}|u_n-u|\,d\mathcal{H}^{N-1}\Big).$$

Letting $n\to\infty$, (10.21) and (10.22) yield

$$\limsup_{n\to+\infty}\int_\Gamma|\gamma_0(u_n-u)|\,d\mathcal{H}^{N-1}\leq 2C\int_{\Omega\setminus\overline{\Omega}_t}|Du|\,dx.$$

We have used the narrow convergence of $|Du_n|$ to $|Du|$ and $\int_{\Gamma_t}|Du|=0$ to assert that $\int_{\Omega\setminus\overline{\Omega}_t}|Du_n|$ tends to $\int_{\Omega\setminus\overline{\Omega}_t}|Du|$ (see Proposition 4.2.5). We complete the proof by letting t go to zero. □

10.3 ▪ The coarea formula and the structure of BV functions

It is well known that each real-valued function u of bounded variation on an interval I of $\mathbf{R}$ is the difference between two monotonous functions and, consequently, possesses at every point $x_0\in I$ the two limits $u(x_0-0)$ and $u(x_0+0)$. One can then define its jump set $S_u:=\{x\in\Omega : u(x-0)\neq u(x+0)\}$. The main objective of this section is to establish that one can associate a set S_u to each u in $BV(\Omega)$, which generalize in any dimension the jump set of u in one dimension. In the particular case of a simple function $u=\chi_\Omega\in BV(\Omega)$, we will see that S_u is a part $\partial_M\Omega$ of the topological boundary $\partial\Omega$, which may differ from $\partial\Omega$ of a set of null $\mathcal{H}^{N-1}$ measure. The structure theorem, Theorem 10.3.4, will be a straightforward consequence of this property thanks to the generalized coarea formula in Section 10.3.3. For a first reading, the reader is advised to go directly to the definitions of the approximate limit sup and approximate limit inf (Definition 10.3.4) and to Theorem 10.3.4.

10.3.1 ▪ Notion of density and regular points

In this subsection, we generalize the notions of interior, exterior of subsets of $\mathbf{R}^N$ as well as the notions of limit, continuity, and jump for measurable functions. Indeed, we intend to extend the expression $D\chi_\Omega = \nu\, \mathcal{H}^{N-1}\lfloor\Gamma$ previously obtained for Lipschitz domains Ω thanks to the theory of traces (see Example 10.2.3) to sets Ω whose characteristic function χ_Ω belongs to $BV(\mathbf{R}^N)$. Actually, for these sets, we will obtain $D\chi_\Omega = \nu\, \mathcal{H}^{N-1}\lfloor\partial_M\Omega$, where the generalized boundary $\partial_M\Omega$ (the measure theoretical boundary) is, up to a set of $\mathcal{H}^{N-1}$ measure zero, the set of all points that are neither in the generalized interior nor in the generalized exterior of Ω.

In what follows, $B_\rho(x_0)$ denotes the open ball of $\mathbf{R}^N$ with radius $\rho > 0$ and centered at x_0.

Definition 10.3.1. *Let E be a Borel subset of $\mathbf{R}^N$. A point x_0 in $\mathbf{R}^N$ is a density point of E iff*

$$\lim_{\rho\to 0} \frac{\mathcal{L}^N(B_\rho(x_0)\cap E)}{\mathcal{L}^N(B_\rho(x_0))} = 1.$$

A point x_0 is a rarefaction point of E iff

$$\lim_{\rho\to 0} \frac{\mathcal{L}^N(B_\rho(x_0)\cap E)}{\mathcal{L}^N(B_\rho(x_0))} = 0.$$

The set of all density points and all rarefaction points of E are respectively called measure theoretical interior and measure theoretical exterior of E and denoted by E_* and E^*.

Example 10.3.1. When O is an open subset of $\mathbf{R}^N$, it is easily seen that all the points of O are density points and all the points of $\mathbf{R}^N \setminus \overline{O}$ are rarefaction points. Nevertheless, if x_0 belongs to the boundary Γ of O, various situations may occur, as shown in Figure 10.2.

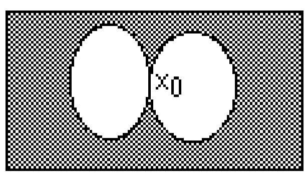

Figure 10.2. *The point x_0 is a density point of the union of the two discs but is a rarefaction point of the complementary of this union.*

We generalize these definitions relatively to a fixed Borel subset F of $\mathbf{R}^N$ as follows.

Definition 10.3.2. *Let F and E be two Borel subsets of $\mathbf{R}^N$ and assume that x_0 in $\mathbf{R}^N$ is such that $\mathcal{L}^N(B_\rho(x_0)\cap F) > 0$ for all $\rho > 0$ small enough. The point x_0 is an F-density point of E iff*

$$\lim_{\rho\to 0} \frac{\mathcal{L}^N(B_\rho(x_0)\cap F\cap E)}{\mathcal{L}^N(B_\rho(x_0)\cap F)} = 1.$$

The point x_0 is an F-rarefaction point of E iff

$$\lim_{\rho\to 0} \frac{\mathscr{L}^N(B_\rho(x_0)\cap F\cap E)}{\mathscr{L}^N(B_\rho(x_0)\cap F)} = 0.$$

These definitions allow us to adopt the following notion of boundary.

Definition 10.3.3. *Let E be a Borel subset of $\mathbf{R}^N$. The measure theoretical boundary of E is the subset of $\mathbf{R}^N$ denoted by $\partial_M E$, made up of all the elements of $\mathbf{R}^N$ which are neither density points nor rarefaction points of E.*

As a straightforward consequence of this definition, one can easily establish that the measure theoretical boundary is a part of the classical topological boundary as stated in the following proposition. The proof is left to the reader.

Proposition 10.3.1. *The measure theoretical boundary of E is the subset of the topological boundary ∂E defined by*

$$\partial_M E = \left\{ x\in \mathbf{R}^N : \limsup_{\rho\to 0} \frac{\mathscr{L}^N(B_\rho(x)\cap E)}{\mathscr{L}^N(B_\rho(x))} > 0 \textit{ and } \limsup_{\rho\to 0} \frac{\mathscr{L}^N(B_\rho(x)\setminus E)}{\mathscr{L}^N(B_\rho(x))} > 0 \right\}.$$

Remark 10.3.1. The measure theoretical boundary may differ from the topological boundary of a set of nonnull $\mathscr{H}^{N-1}$ measure. Indeed, consider $E = \mathbf{B}\setminus [(0,0),(0,1)[$, where $\mathbf{B}$ is the unit open ball of $\mathbf{R}^2$. The measure theoretical boundary is the sphere but the (topological) boundary is the union of the sphere and the interval $[(0,0),(0,1)]$.

Let E be an open subset of $\mathbf{R}^N$ satisfying the following property: for all point x_0 of ∂E, there exists a normal vector $\nu(x_0)$ to ∂E such that E is included in the half-space $\pi_\nu(x_0) := \{x\in \mathbf{R}^N : \langle x-x_0, \nu(x_0)\rangle > 0\}$, where $\langle\ ,\ \rangle$ denotes the scalar product in $\mathbf{R}^N$. Then one has $\partial E = \partial_M E$. Indeed, it is easily seen that $\chi_{\frac{1}{\rho}(E-x_0)}$ strongly converges to the characteristic function $\chi_{\pi_\nu(x_0)}$ of $\pi_\nu(x_0)$ in $L^1_{loc}(\mathbf{R}^N)$ so that for x_0 in ∂E,

$$\begin{aligned}\lim_{\rho\to 0}\frac{\mathscr{L}^N(B_\rho(x_0)\cap E)}{\mathscr{L}^N(B_\rho(x_0))} &= \lim_{\rho\to 0}\frac{\mathscr{L}^N(B_1(x_0)\cap \frac{1}{\rho}(E-x_0))}{\mathscr{L}^N(B_1(x_0))}\\ &= 1/2 = \lim_{\rho\to 0}\frac{\mathscr{L}^N(B_\rho(x_0)\setminus (E-x_0))}{\mathscr{L}^N(B_\rho(x_0))}.\end{aligned}$$

We now look into the notion of approximate limit.

Definition 10.3.4. *Let $f:\mathbf{R}^N \longrightarrow \mathbf{R}$ be a measurable function and $x_0\in\mathbf{R}^N$. A real number α is the approximate limit of f at x_0 iff*

$$\forall \varepsilon > 0 \quad x_0 \textit{ is a density point of the set } [\ |f-\alpha| < \varepsilon\ ],$$

or equivalently

$$\forall \varepsilon > 0 \quad x_0 \textit{ is a rarefaction point of the set } [\ |f-\alpha| > \varepsilon\ ].$$

We then write $\alpha = ap\lim_{x\to x_0} f(x)$.

More generally, we define in $\overline{\mathbf{R}}$ the approximate limit sup and approximate limit inf of f at x_0 by

$$ap\limsup_{x\to x_0} f(x) = \inf\left\{ t \in \mathbf{R} : \lim_{\rho\to 0} \frac{\mathscr{L}^N(B_\rho(x_0)\cap [f > t])}{\mathscr{L}^N(B_\rho(x_0))} = 0 \right\},$$

$$ap\liminf_{x\to x_0} f(x) = \sup\left\{ t \in \mathbf{R} : \lim_{\rho\to 0} \frac{\mathscr{L}^N(B_\rho(x_0)\cap [f < t])}{\mathscr{L}^N(B_\rho(x_0))} = 0 \right\}.$$

Let F be a fixed Borel subset of $\mathbf{R}^N$. A real number α is called the F-approximate limit of f at x_0 iff

$$\forall \varepsilon > 0 \quad x_0 \text{ is an } F\text{-density point of } [\ |f-\alpha| < \varepsilon\ ]$$

and we write $\alpha = ap\lim_{x\to x_0, x\in F} f(x)$.

Example 10.3.2. If $\alpha = \lim_{x\to x_0} f(x)$, it is straightforward to show that $\alpha = \text{ap}\lim_{x\to x_0} f(x)$. When f is the characteristic function $\chi_{D_1\cup D_2}$ of the union of the two discs in Example 10.3.1, $[\ |f-1| < \varepsilon\ ] = D_1 \cup D_2$, so that $\text{ap}\lim_{x\to x_0} f(x) = 1$ although the classical limit does not exist. When f is the characteristic function of the complementary of the union of these two discs, the approximate limit of f at x_0 is zero.

It is easy to establish uniqueness of the approximate limit when it exists. For further details related to these notions, see [245]. We only give, without proof, four elementary useful properties.

Proposition 10.3.2. *For all Borel subsets A and B of $\mathbf{R}^N$, the four following assertions hold:*

(i) *Let $C = A \cup B$. If the approximate limits*

$$ap \lim_{x\to x_0,\ x\in A} f(x) \quad \text{and} \quad ap \lim_{x\to x_0,\ x\in B} f(x)$$

exist and coincide, then $ap\lim_{x\to x_0,\ x\in C} f(x)$ exists.

(ii) *If x_0 is not a rarefaction point of A, $A \subset B$, and if x_0 is a B-rarefaction point of $B \setminus A$, then the existence of*

$$ap \lim_{x\to x_0,\ x\in A} f(x)$$

implies the existence of

$$ap \lim_{x\to x_0,\ x\in B} f(x).$$

(iii) *If $A \subset B$ and if x_0 is not a rarefaction point of A, then the existence of*

$$ap \lim_{x\to x_0,\ x\in B} f(x)$$

implies the existence of

$$ap \lim_{x\to x_0,\ x\in A} f(x).$$

(iv) *If x_0 is not a rarefaction point of $A \cap B$, then the existence of*

$$ap \lim_{x \to x_0,\ x \in A} f(x) \quad and \quad ap \lim_{x \to x_0,\ x \in B} f(x)$$

implies equality of these two approximate limits.

In the following proposition, we show that the approximate limit at x_0 is a classical limit for the restriction of f to a suitable Borel set. Moreover, when the approximate lim inf and approximate lim sup coincide, their common value is the approximate limit.

Proposition 10.3.3. *A measurable function $f : \mathbf{R}^N \longrightarrow \mathbf{R}$ possesses an approximate limit α at x_0 iff there exists a Borel subset B of $\mathbf{R}^N$ such that x_0 is a rarefaction point of $\mathbf{R}^N \setminus B$ and such that the restriction $f\lfloor B$ of f to B possesses the classical limit α at x_0. On the other hand, one always has*

$$ap \liminf_{x \to x_0} f \leq ap \limsup_{x \to x_0} f$$

in $\overline{\mathbf{R}}$ and $ap \liminf_{x \to x_0} f = ap \limsup_{x \to x_0} f := \alpha \in \mathbf{R}$ iff $ap \lim_{x \to x_0} f = \alpha$.

PROOF. Let us prove the first assertion. It is easily seen that the given condition is sufficient. Conversely, assume that f possesses an approximate limit α at x_0. Without loss of generality one may assume $\alpha = 0$. For all integer i, x_0 is then a rarefaction point of

$$A_i := \mathbf{R}^N \setminus \{x \in \mathbf{R}^N : |f(x)| < 1/i\}.$$

Consider a nonincreasing sequence $\rho_1 > \rho_2 > \cdots > \rho_i > \cdots$ in $\mathbf{R}^+$ such that

$$\frac{\mathscr{L}^N(B_\rho(x_0) \cap A_i)}{\mathscr{L}^N(B_\rho(x_0))} \leq 2^{-i}$$

when $0 \leq \rho \leq \rho_i$, and denote the complementary set of $\cup_{i \in \mathbf{N}^*}(A_i \cap B_{\rho_i}(x_0))$ by B. It is straightforward to show that $f\lfloor B$ has the limit zero at x_0. We now show that x_0 is a rarefaction point of $\cup_{i \in \mathbf{N}^*}(A_i \cap B_{\rho_i}(x_0))$. Let $\rho_i > \rho > \rho_{i+1}$. Then

$$\begin{aligned}\mathscr{L}^N(B_\rho(x_0) \cap (\mathbf{R}^N \setminus B)) &\leq \mathscr{L}^N(B_\rho(x_0) \cap A_i) + \sum_{k=1}^{\infty} \mathscr{L}^N(B_{\rho_{i+k}}(x_0) \cap A_{i+k}) \\ &\leq C\rho^N\, 2^{-(i-1)}.\end{aligned}$$

We conclude the proof of the assertion by letting ρ go to zero and then i go to $+\infty$ in inequality

$$\frac{\mathscr{L}^N(B_\rho(x_0) \cap (\mathbf{R}^N \setminus B))}{\mathscr{L}^N(B_\rho(x_0))} \leq C 2^{-(i-1)}.$$

We are going to establish the second assertion. Assume that $\mathrm{ap} \liminf_{x \to x_0} f(x) = \mathrm{ap} \limsup_{x \to x_0} f(x) = \alpha$. Let $\varepsilon > 0$, t_ε, t'_ε be such that $\alpha \leq t_\varepsilon < \alpha + \varepsilon$, $\alpha - \varepsilon < t'_\varepsilon \leq \alpha$ and

$$\lim_{\rho \to 0} \frac{\mathscr{L}^N(B_\rho(x_0) \cap [f > t_\varepsilon])}{\mathscr{L}^N(B_\rho(x_0))} = 0, \ \lim_{\rho \to 0} \frac{\mathscr{L}^N(B_\rho(x_0) \cap [f < t'_\varepsilon])}{\mathscr{L}^N(B_\rho(x_0))} = 0.$$

Since

$$[|f-\alpha|>\varepsilon]=[f>\alpha+\varepsilon]\cup[f<\alpha-\varepsilon]$$

and

$$[f>\alpha+\varepsilon]\subset[f>t_\varepsilon],\quad [f<\alpha-\varepsilon]\subset[f<t'_\varepsilon],$$

we infer

$$\lim_{\rho\to 0}\frac{\mathscr{L}^N(B_\rho(x_0)\cap[|f-\alpha|>\varepsilon])}{\mathscr{L}^N(B_\rho(x_0))}=0,$$

and thus $\alpha=\operatorname{ap}\lim_{x\to x_0}f(x)$. Conversely, assume that $\alpha=\operatorname{ap}\lim_{x\to x_0}f(x)$, i.e., for all $\varepsilon>0$

$$\lim_{\rho\to 0}\frac{\mathscr{L}^N(B_\rho(x_0)\cap[|f-\alpha|>\varepsilon])}{\mathscr{L}^N(B_\rho(x_0))}=0.$$

We deduce that for all $\varepsilon>0$

$$\lim_{\rho\to 0}\frac{\mathscr{L}^N(B_\rho(x_0)\cap[f>\alpha+\varepsilon])}{\mathscr{L}^N(B_\rho(x_0))}=0\quad\text{and}\quad\lim_{\rho\to 0}\frac{\mathscr{L}^N(B_\rho(x_0)\cap[f<\alpha-\varepsilon])}{\mathscr{L}^N(B_\rho(x_0))}=0.$$

Therefore

$$\alpha+\varepsilon\le\operatorname{ap}\liminf_{x\to x_0}f(x)\text{ and }\alpha-\varepsilon\ge\operatorname{ap}\limsup_{x\to x_0}f(x),$$

and thus

$$\operatorname{ap}\liminf_{x\to x_0}f(x)=\operatorname{ap}\limsup_{x\to x_0}f(x),$$

provided that we have established $\operatorname{ap}\liminf_{x\to x_0}f(x)\le\operatorname{ap}\liminf_{x\to x_0}f(x)$.

Let us show $\operatorname{ap}\liminf_{x\to x_0}f(x)\le\operatorname{ap}\limsup_{x\to x_0}f(x)$. One may assume $\operatorname{ap}\limsup_{x\to x_0}f(x)<+\infty$. Let any $\tau\in\mathbf{R}$ be such that

$$\lim_{\rho\to 0}\frac{\mathscr{L}^N(B_\rho(x_0)\cap[f<\tau])}{\mathscr{L}^N(B_\rho(x_0))}=0.$$

We claim that $\tau\le\operatorname{ap}\limsup_{x\to x_0}f(x)$. Assume, on the contrary, that $\tau>\operatorname{ap}\limsup_{x\to x_0}f(x)$ and set $\varepsilon=\tau-\operatorname{ap}\limsup_{x\to x_0}f(x)$. We deduce that there exists t_ε such that $\tau>t_\varepsilon$ and

$$\lim_{\rho\to 0}\frac{\mathscr{L}^N(B_\rho(x_0)\cap[f>t_\varepsilon])}{\mathscr{L}^N(B_\rho(x_0))}=0.$$

The inclusion $[f\ge\tau]\subset[f>t_\varepsilon]$ then yields

$$\lim_{\rho\to 0}\frac{\mathscr{L}^N(B_\rho(x_0)\cap[f\ge\tau])}{\mathscr{L}^N(B_\rho(x_0))}=0,$$

which is in contradiction with

$$\lim_{\rho\to 0}\frac{\mathscr{L}^N(B_\rho(x_0)\cap[f\ge\tau])}{\mathscr{L}^N(B_\rho(x_0))}=1.\qquad\square$$

Remark 10.3.2. Every measurable function $f : \mathbf{R}^N \longrightarrow \mathbf{R}$ possesses an approximate limit almost everywhere. For a proof, consult Morgan [301].

Let f be a function in $L^1_{loc}(\mathbf{R}^N)$. Then each Lebesgue point of f is a point of approximate limit. This property is indeed the straightforward consequence of

$$\frac{\mathscr{L}^N(B_\rho(x_0)\cap[|f-f(x_0)|>\varepsilon])}{\mathscr{L}^N(B_\rho(x_0))} \leq \frac{1}{\varepsilon\, \mathscr{L}^N(B_\rho(x_0))}\int_{B_\rho(x_0)} |f(x)-f(x_0)|\, dx.$$

Nevertheless, there exist points of approximate limit which are not Lebesgue points. Consult, for instance, Morgan [301, Exercise 2.7].

In the next sections, for all functions f in $L^1(\Omega)$, Ω open subset of $\mathbf{R}^N$, we will adopt the following convention: we choose a representative of f, still denoted f, such that at every point x_0 of approximate limit, $f(x_0) = ap\lim_{x\to x_0} f(x)$. Such a representative is said to be approximately continuous at its points of approximate limit.

We now generalize the concept of left and right limits $u(x_0-0)$ and $u(x_0+0)$ for functions defined on $\mathbf{R}^N$. We denote the unit sphere of $\mathbf{R}^N$ by S^{N-1} and for all a in S^{N-1} and all x_0 in $\mathbf{R}^N$, $\pi_a(x_0)$ denotes the open half-space

$$\pi_a(x_0) := \{x \in \mathbf{R}^N : \langle x - x_0, a\rangle > 0\}.$$

We also denote the hemiball $\pi_a(x_0)\cap B_\rho(x_0)$ by $H_{\rho,a}(x_0)$.

Definition 10.3.5. *A point x_0 in $\mathbf{R}^N$ is called a regular point for the measurable function $f : \mathbf{R}^N \to \mathbf{R}$ iff there exists $a \in S^{N-1}$ such that the two following approximate limits exist:*

$$f_a(x) := ap \lim_{x\to x_0,\ x\in\pi_a(x_0)} f(x) \quad \textit{and} \quad f_{-a}(x) := ap \lim_{x\to x_0,\ x\in\pi_{-a}(x_0)} f(x).$$

Example 10.3.3. Let us consider Example 10.3.1 and take for f the characteristic function of the union of the two discs. The point x_0 is a regular point of f and satisfies $1 = f_a(x_0) = f_{-a}(x_0)$, where a is one of the two unit vectors orthogonal to the common tangent hyperplane at the two discs at x_0. One could say that x_0 is a point of approximate continuity for f. Let us point out that we also have $1 = ap\lim_{x\to x_0} f(x)$.

Consider now the characteristic function f of one of the two discs. The point x_0 is also a regular point of f and satisfies $f_a(x_0) \neq f_{-a}(x_0)$.

The following theorem asserts that regular points always satisfy the alternative of Example 10.3.3.

Theorem 10.3.1 (structure of the set of regular points, jump points, and jump sets). *Let x_0 be a regular point of a measurable function $f : \mathbf{R}^N \to \mathbf{R}$. We have the following alternative:*

(i) *If $f_a(x_0) = f_{-a}(x_0)$, then*

$$ap\lim_{x\to x_0} f(x)$$

exists and for all b in S^{N-1}, $f_b(x_0) = ap\lim_{x\to x_0} f(x)$. The point x_0 is called a point of approximate continuity of f.

(ii) *If $f_a(x_0) \neq f_{-a}(x_0)$, then $\pm a$ is the unique element of S^{N-1} such that f_a and f_{-a} exist. The point x_0 is called a jump point of f. The real number $|f_a(x_0) - f_{-a}(x_0)|$ is the jump of f at x_0 and $\big(f_a(x_0) - f_{-a}(x_0)\big)a$ is the oriented jump. The set of all jump points of f is called the jump set of f and denoted by S_f.*

PROOF. Assume that $f_a(x_0) = f_{-a}(x_0)$. The existence of $ap\ \lim_{x\to x_0} f(x)$ is a consequence of Proposition 10.3.2(i) with $A = \pi_a(x_0)$ and $B = \pi_{-a}(x_0)$. Thus, according to Proposition 10.3.2(iii), with $B = \mathbf{R}^N$ and $A = \pi_b(x_0)$, $f_b(x_0)$ exists for all $b \in S^{N-1}$. According to Proposition 10.3.2(iv), we obtain $f_b(x_0) = ap\ \lim_{x\to x_0} f(x)$.

We finally establish (ii). Let b be any element of S^{N-1}. We show that $f_b(x_0)$ does not exist if $b \neq \pm a$. Otherwise, as x_0 is not a rarefaction point of the sets $\pi_a(x_0) \cap \pi_b(x_0)$ and $\pi_{-a}(x_0) \cap \pi_b(x_0)$, from Proposition 10.3.2(iv), $f_a(x_0) = f_b(x_0)$ and $f_{-a}(x_0) = f_b(x_0)$, a contradiction. $\square$

Example 10.3.4. Let us consider the function f defined by

$$f = \begin{cases} \alpha & \text{if } |x| \le 1, \\ \beta & \text{if } |x| > 1, \end{cases}$$

where $\alpha \neq \beta$. The jump set is the sphere S^{N-1}. In Example 10.3.1, if $f = \chi_{D_1 \cup D_2}$, the jump set is the boundary (topological) except the point x_0.

The next proposition characterizes the jump set of simple functions.

Proposition 10.3.4 (inner measure theoretic normal). *Let E be a Borel subset of $\mathbf{R}^N$ and χ its characteristic function. The jump set S_χ of χ is the set of all points x_0 of $\mathbf{R}^N$ for which there exists a in S^{N-1} satisfying*

$$\begin{cases} \lim_{\rho\to 0} \dfrac{\mathscr{L}^N\big(H_{\rho,a}(x_0) \cap E\big)}{\mathscr{L}^N\big(H_{\rho,a}(x_0)\big)} = 1, \\ \lim_{\rho\to 0} \dfrac{\mathscr{L}^N\big(H_{\rho,-a}(x_0) \cap E\big)}{\mathscr{L}^N\big(H_{\rho,a}(x_0)\big)} = 0. \end{cases}$$

For such points, the unit vector a is unique and called the inner measure theoretic normal to E at x_0. Moreover, $S_\chi \subset \partial_M E$.

PROOF. From the definition, every point satisfying the two above properties is a jump point of χ, and the uniqueness of a is a consequence of Theorem 10.3.1(ii).

Conversely, if x_0 is a jump point of χ, there exists a unique $\pm a$ in S^{N-1} such that $\chi_a(x_0) \neq \chi_{-a}(x_0)$. It is easily seen that $\chi_a(x_0)$ and $\chi_{-a}(x_0)$ belong to $\{0,1\}$. Then, exchanging, if necessary, a and $-a$, we have $\chi_a(x_0) = 1$ and $\chi_{-a}(x_0) = 0$, which gives the two required limits.

We must now prove $S_{\chi_E} \subset \partial_M E$. Let $x_0 \in S_{\chi_E}$. One has

$$\begin{aligned} \frac{\mathscr{L}^N(B_\rho(x_0) \cap E)}{\mathscr{L}^N(B_\rho(x_0))} &= \frac{\mathscr{L}^N(B_\rho(x_0) \cap E)}{\mathscr{L}^N(B_\rho(x_0) \cap E \cap \pi_a(x_0))} \\ &\quad \times \frac{\mathscr{L}^N(B_\rho(x_0) \cap E \cap \pi_a(x_0))}{\mathscr{L}^N(B_\rho(x_0) \cap \pi_a(x_0))} \\ &\quad \times \frac{\mathscr{L}^N(B_\rho(x_0) \cap \pi_a(x_0))}{\mathscr{L}^N(B_\rho(x_0))} \\ &\ge \frac{1}{2} \frac{\mathscr{L}^N(B_\rho(x_0) \cap E \cap \pi_a(x_0))}{\mathscr{L}^N(B_\rho(x_0) \cap \pi_a(x_0))}. \end{aligned}$$

Since the second factor tends to 1 when $\rho \to 0$, the inequality above yields

$$\limsup_{\rho\to 0} \frac{\mathscr{L}^N(B_\rho(x_0)\cap E)}{\mathscr{L}^N(B_\rho(x_0))} \geq \frac{1}{2} > 0.$$

Exchanging the roles of a and $-a$ and E and $\mathbf{R}^N \setminus E$, we also obtain

$$\limsup_{\rho\to 0} \frac{\mathscr{L}^N(B_\rho(x_0)\setminus E)}{\mathscr{L}^N(B_\rho(x_0))} \geq \frac{1}{2} > 0;$$

thus, according to Proposition 10.3.1, $x_0 \in \partial_M E$. □

10.3.2 ▪ Sets of finite perimeter, structure of simple BV functions

To clarify the structure of BV functions, we now establish that up to a set of $\mathscr{H}^{N-1}$ measure zero, the jump set of all simple function χ_E which belongs to $BV(\mathbf{R}^N)$ is essentially the measure theoretical boundary of E.

Definition 10.3.6. *A Borel subset E of $\mathbf{R}^N$ is called a set of finite perimeter in Ω iff its characteristic function χ_E belongs to $BV(\Omega)$. The total mass $|D_{\chi_E}|(\Omega)$ is called the perimeter of E in Ω and is denoted by $P(E,\Omega)$ or $P(E)$ when $\Omega = \mathbf{R}^N$. A Borel subset E of $\mathbf{R}^N$ is called a set of locally finite perimeter if it is a set of finite perimeter in U for all bounded open subset U of $\mathbf{R}^N$.*

Remark 10.3.3. When E is a Lipschitz open bounded subset of Ω, according to the trace Theorem 10.2.1 and to Example 10.2.2, we have $P(E,\Omega) = \mathscr{H}^{N-1}(\Omega \cap \partial E)$.

Theorem 10.3.2 (structure of simple functions of $BV(\Omega)$). *Let E be a set of finite perimeter in Ω. Then the following hold:*

(i) *Up to a Borel subset of $\mathscr{H}^{N-1}\lfloor\Omega$-zero measure, $\partial_M E \cap \Omega$ is the jump set of χ_E.*

(ii) *The set $\partial_M E \cap \Omega$ is countably $N-1$-rectifiable, i.e., $\partial_M E \cap \Omega \subset \bigcup_{i\in\mathbf{N}} A_n$, where*

$$\mathscr{H}^{N-1}\lfloor\Omega(A_0) = 0,$$

and for each $i = 1,\dots,+\infty$ there exists a Lipschitz function $f_i : \mathbf{R}^{N-1} \to \mathbf{R}^N$ such that $A_i = f_i(\mathbf{R}^{N-1})$.

(iii) *The following generalized Gauss–Green formula holds: for $\mathscr{H}^{N-1}$ almost all x in Ω, there exists $\nu(x) \in S^{N-1}$, called the generalized inner normal vector to E at x, such that for all φ in $\mathbf{C}^1_c(\Omega,\mathbf{R}^N)$,*

$$\int_\Omega \chi_E \ \mathrm{div}\ \varphi \, dx = \int_{\partial_M E\cap\Omega} \varphi.(-\nu) d\mathscr{H}^{N-1},$$

that is, $D\chi_E = \nu\ \mathscr{H}^{N-1}\lfloor\partial_M E \cap \Omega$.

PROOF. The proof is divided into the five following steps:

Step 1. We define the generalized inner normal vector $\nu(x)$ to E at x and the reduced boundary $\partial_r E$ of E.

Step 2. This step consists of establishing $\partial_r E \subset S_{\chi_E} \cap \Omega \subset \partial_M E \cap \Omega$.

Step 3. We prove that $\partial_M E \cap \Omega$, $S_{\chi_E} \cap \Omega$, and $\partial_r E$ are essentially the same sets; more precisely, $\mathscr{H}^{N-1}(\partial_M E \cap \Omega \setminus \partial_r E) = 0$.

Step 4. We establish that $S_{\chi_E} \cap \Omega$ is countably $(N-1)$-rectifiable.

Step 5. We prove the generalized Gauss–Green formula.

Step 1. Contrary to the previous notions of boundary $\partial_M E$ and S_{χ_E}, the definition below is specifically defined for subsets E of Ω such that $D\chi_E$ belongs to $\mathbf{M}(\Omega, \mathbf{R}^N)$.

Definition 10.3.7. *Let E be a subset of finite perimeter in Ω. The generalized unit inner normal to E is the Radon–Nikodým derivative of $D\chi_E$ with respect to the measure $|D\chi_E|$. In other words,*

$$\text{for } |D\chi_E| \text{ a.e. } x \text{ in } \Omega, \quad \nu(x) = \lim_{\rho \to 0} \frac{\int_{B_\rho(x)} D\chi_E}{\int_{B_\rho(x)} |D\chi_E|}.$$

The reduced boundary $\partial_r E$ consists of all points x in Ω such that the limit above exists.

Let us remark that according to the trace theory, when E is a Lipschitz domain of $\mathbf{R}^N$ with boundary Γ, we have $D\chi_E = \nu \mathscr{H}^{N-1} \lfloor \Gamma$, where $\nu(x)$ is the inner normal to E at $\mathscr{H}^{N-1}$ a.e. x in Γ. Consequently $\frac{D\chi_E}{|D\chi_E|}(x) = \nu(x)$ for $\mathscr{H}^{N-1}$ a.e. x in Γ and $\mathscr{H}^{N-1}(\Gamma \setminus \partial_r E) = 0$.

Step 2. We establish that $\partial_r E \subset S_{\chi_E} \cap \Omega$. The inclusion $S_{\chi_E} \cap \Omega \subset \partial_M E \cap \Omega$ has been proved in Proposition 10.3.4. The key of the proof is the following blow-up lemma.

Lemma 10.3.1. *Let $x_0 \in \partial_r E$, $\nu(x_0)$ be the generalized unit inner normal to E at x_0, E_ρ the homothetic subset $\{x \in \mathbf{R}^N : \rho(x - x_0) \in E\}$ of E, and $\pi_\nu(x_0)$ the half-space $\{x \in \mathbf{R}^N : \langle x - x_0, \nu(x_0) \rangle > 0\}$. Then χ_{E_ρ} weakly converges to $\chi_{\pi_\nu(x_0)}$ in $BV(B_1(x_0))$.*

SKETCH OF THE PROOF OF LEMMA 10.3.1. We admit the three following estimates: for each $x \in \partial_r E$ there exits a positive constant C such that for all sufficiently small $r > 0$,

$$C \le r^{N-1} |D\chi_E|(B_r(x)) \le C^{-1}, \tag{10.23}$$

$$C \le r^{-N} \mathscr{L}^N(B_r(x) \cap E), \tag{10.24}$$

$$C \le r^{-N} \mathscr{L}^N(B_r(x) \setminus E). \tag{10.25}$$

For a proof, we refer the reader to [366, Lemma 5.5.4].

Without loss of generality, one may assume $x_0 = 0$ and $\nu(0) = (0, \dots, 0, 1)$. Moreover, it is enough to establish that for each sequence $(\rho_h)_{h \in \mathbf{N}}$, there exists a subsequence (not relabeled) satisfying

$$\chi_{E_{\rho_h}} \rightharpoonup \chi_{\pi_\nu(0)} \text{ in } BV(B_1(x_0)).$$

Let us fix $r > 0$. Reasoning with a smooth approximating sequence in the sense of the intermediate convergence (Theorem 10.1.2), and changing scale, a straightforward calculation gives

$$D\chi_{E_{\rho_h}}(B_r(0)) = \rho_h^{1-N} D\chi_E(B_{\rho_h r}(0)), \tag{10.26}$$

$$|D\chi_{E_{\rho_h}}|(B_r(0)) = \rho_h^{1-N} |D\chi_E|(B_{\rho_h r}(0)). \tag{10.27}$$

Collecting (10.23) and (10.27), we obtain, for ρ_h small enough,

$$C\,r^{N-1} \le |D\chi_{E_{\rho_h}}|(B_r(0)) \le C^{-1} r^{N-1}. \tag{10.28}$$

Thanks to (10.28) with $r=1$, applying Theorem 10.1.4 in $BV(B_1(0))$, there exist a subsequence (not relabeled) and a subset $F \subset \mathbf{R}^N$ of finite perimeter in $B_1(0)$ such that

$$\chi_{E_{\rho_h}} \to \chi_F \quad \text{strongly in } L^1(B_1(0)), \tag{10.29}$$

$$D\chi_{E_{\rho_h}} \rightharpoonup D\chi_F \quad \text{weakly in } \mathbf{M}(B_1(0),\mathbf{R}^N). \tag{10.30}$$

From now on we reason in $BV(B_1(0))$. It remains to establish that $F = \pi_\nu(0)$. According to Theorem 4.2.1 and Lemma 4.2.1, for all but countably many $0<\rho<1$

$$D\chi_{E_{\rho_h}}(B_\rho(0)) \to D\chi_F(B_\rho(0)). \tag{10.31}$$

Thus, from (10.27), (10.26), according to the definition of $\nu(0) = (0,\dots,0,1)$, and from (10.31) we deduce

$$\begin{aligned}
\lim_{h\to+\infty} |D\chi_{E_{\rho_h}}|(B_\rho(0)) &= \lim_{h\to+\infty} \frac{|D\chi_{E_{\rho_h}}|(B_\rho(0))}{\frac{\partial}{\partial x_N}\chi_{E_{\rho_h}}(B_\rho(0))} \lim_{h\to+\infty} \frac{\partial}{\partial x_N}\chi_{E_{\rho_h}}(B_\rho(0))\\
&= \lim_{h\to+\infty} \frac{|D\chi_E|(B_{\rho_h\rho}(0))}{\frac{\partial}{\partial x_N}\chi_E(B_{\rho_h\rho}(0))} \lim_{h\to+\infty} \frac{\partial}{\partial x_N}\chi_{E_{\rho_h}}(B_\rho(0))\\
&= \lim_{h\to+\infty} \frac{\partial}{\partial x_N}\chi_{E_{\rho_h}}(B_\rho(0)) = \frac{\partial}{\partial x_N}\chi_F(B_\rho(0)).
\end{aligned} \tag{10.32}$$

The lower semicontinuity of the total variation (Proposition 10.1.1) and (10.30), (10.32) finally yield

$$|D\chi_F|(B_\rho(0)) \le \frac{\partial}{\partial x_N}\chi_F(B_\rho(0)),$$

hence equality. Let us consider the density ν_N of $\frac{\partial}{\partial x_N}\chi_F$ with respect to $|D\chi_F|$ (cf. the Radon–Nikodým theorem, Theorem 4.2.1). From above one has

$$|D\chi_F|(B_\rho(0)) = \frac{\partial}{\partial x_N}\chi_F(B_\rho(0)) = \int_{B_\rho(0)} \nu_N(x)\, d|D\chi_F|(x).$$

This shows that $\nu_N(x)=1$ for $|D\chi_F|$ a.e. $x\in B_1(0)$; hence $\frac{\partial}{\partial x_N}\chi_F = |D\chi_F|$ is a nonnegative Radon measure. Note also that from (10.32) and (10.28), $\frac{\partial}{\partial x_N}\chi_F(B_\rho(0)) \ge C\rho^{N-1}$ so that $\frac{\partial}{\partial x_N}\chi_F \not\equiv 0$.

On the other hand, from (10.31), (10.26), (10.27), and the definition of $\nu(0)$, for $i = 1,\dots,N-1$, and for all but countably many $0<\rho<1$, we have

$$\frac{\frac{\partial}{\partial x_i}\chi_F(B_\rho(0))}{|D\chi_F|(B_\rho(0))} = \lim_{h\to+\infty} \frac{\frac{\partial}{\partial x_i}\chi_{E_{\rho_h}}(B_\rho(0))}{|D\chi_{E_{\rho_h}}|(B_\rho(0))} = \lim_{h\to+\infty} \frac{\frac{\partial}{\partial x_i}\chi_E(B_{\rho_h\rho}(0))}{|D\chi_E|(B_{\rho_h\rho}(0))} = 0. \tag{10.33}$$

We deduce from (10.33) that $\frac{\partial}{\partial x_i}\chi_F(B_\rho(0)) = 0$ for $i = 1,\dots,N-1$. Let us consider the density ν_i of $\frac{\partial}{\partial x_i}\chi_F$ with respect to $|D\chi_F|$. Since

$$0 = \frac{\partial}{\partial x_i}\chi_F(B_\rho(0)) = \int_{B_\rho(0)} \nu_i(x)\, d|D\chi_F|(x),$$

$\nu_i(x) = 0$ for $|D\chi_F|$ a.e. $x \in B_1(0)$, and hence $\frac{\partial}{\partial x_i}\chi_F = 0$.

The function χ_F depends only on the variable x_N, is nondecreasing, and, from (10.29), takes only the two values 0 and 1. Now set $\alpha = \sup\{x_N : \chi_F(x_N) = 0\}$. Note that $\alpha \neq +\infty$; otherwise $D\chi_F \equiv 0$. For proving $F = \pi_\nu(0)$, it suffices to establish $\alpha = 0$. Assuming $\alpha > 0$ gives $B_\rho(0) \subset \mathbf{R}^N \setminus F$ for $\rho < \alpha$ so that, by (10.29),

$$\begin{aligned} 0 = \mathscr{L}^N(B_\rho(0)\cap F) &= \lim_{h\to+\infty} \mathscr{L}^N(B_\rho(0)\cap E_{\rho_h}) \\ &= \lim_{h\to+\infty} \rho_h^{-N}\mathscr{L}^N(B_{\rho\rho_h}(0)\cap E), \end{aligned}$$

which contradicts (10.24). If $\alpha < 0$, then $B_\rho(0) \subset F$ for $\rho < -\alpha$ and a similar argument gives

$$0 = \lim_{h\to+\infty} \rho_h^{-N}\mathscr{L}^N(B_{\rho\rho_h}(0)\setminus E),$$

which contradicts (10.25). The proof of Lemma 10.3.1 is complete.

We are going to establish $\partial_r E \subset S_{\chi_E} \cap \Omega$. Let $x_0 \in \partial_r E$. We have

$$\frac{\mathscr{L}^N\,(E\cap H_{\rho,-\nu}(x_0))}{\mathscr{L}^N(H_{\rho,-\nu}(x_0))} = \frac{\mathscr{L}^N\,(E_\rho\cap B_1(x_0)\cap \pi_{-\nu}(x_0))}{\mathscr{L}^N(B_1(x_0)\cap \pi_{-\nu}(x_0))}.$$

The convergence of χ_{E_ρ} to $\chi_{\pi_\nu(x_0)}$ in $L^1(B_1(x_0))$ yields

$$\lim_{\rho\to 0}\frac{\mathscr{L}^N\,(E\cap H_{\rho,-\nu}(x_0))}{\mathscr{L}^N(H_{\rho,-\nu}(x_0))} = \frac{\mathscr{L}^N(\pi_\nu(x_0)\cap B_1(x_0)\cap \pi_{-\nu}(x_0))}{\mathscr{L}^N(B_1(x_0)\cap \pi_{-\nu}(x_0))} = 0.$$

Similarly,

$$\lim_{\rho\to 0}\frac{\mathscr{L}^N\,(E\cap H_{\rho,\nu}(x_0))}{\mathscr{L}^N(H_{\rho,\nu}(x_0))} = \frac{\mathscr{L}^N(\pi_\nu(x_0)\cap B_1(x_0)\cap \pi_{\nu}(x_0))}{\mathscr{L}^N(B_1(x_0)\cap \pi_{\nu}(x_0))} = 1$$

and, according to Proposition 10.3.4, $x_0 \in S_{\chi_E}$.

Step 3. The proof of $\mathscr{H}^{N-1}(\partial_M E\cap\Omega\setminus\partial_r E) = 0$ consists first in proving $\mathscr{H}^{N-1}(S_{\chi_E}\cap \Omega\setminus\partial_r E) = 0$; next, $\mathscr{H}^{N-1}(\partial_M E\cap\Omega\setminus S_{\chi_E}\cap\Omega) = 0$. Keys of the proof are the relative isoperimetric inequality and Lemma 4.2.3. This is summarized in the next lemma.

Lemma 10.3.2. *Let E be a set of finite perimeter in Ω. Then the following assertions hold:*

(i) *Relative isoperimetric inequality: there exists a positive constant C such that*

$$\min\{\mathscr{L}^N(B_r\cap E), \mathscr{L}^N(B_r\setminus E)\}^{\frac{N-1}{N}} \le C|D\chi_E|(B_r)$$

for all open ball B_r with radius R included in Ω.

(ii) *There exists a positive constant C such that for all $x \in S_{\chi_E} \cap \Omega$,*

$$\liminf_{\rho \to 0} \frac{|D\chi_E|(B_\rho(x))}{\rho^{N-1}} \geq C.$$

(iii) *For $\mathcal{H}^{N-1}$ almost all $x \in \Omega \setminus S_{\chi_E} \cap \Omega$,*

$$\limsup_{\rho \to 0} \frac{|D\chi_E|(B_\rho(x))}{\rho^{N-1}} = 0.$$

(iv) $\mathcal{H}^{N-1}(S_{\chi_E} \cap \Omega \setminus \partial_r E) = 0$.

(v) $\mathcal{H}^{N-1}(\partial_M E \cap \Omega \setminus S_{\chi_E} \cap \Omega) = 0$.

PROOF OF LEMMA 10.3.2. Assertion (i) is a straightforward consequence of the Poincaré–Wirtinger inequality

$$\left(\int_{B_r} |u - \overline{u}|^{\frac{N}{N-1}} \right)^{\frac{N-1}{N}} \leq C \int_{B_r} |Du|, \quad \text{where } \overline{u} = \frac{1}{\mathcal{L}^N(B_r)} \int_{B_r} u(x)\, dx,$$

applied to $u = \chi_E$. Indeed, every function u in $BV(\Omega)$ satisfies the Poincaré–Wirtinger inequality: consider a smooth approximating sequence in the sense of the intermediate convergence, and apply Corollary 5.4.1 and Theorem 10.1.2 as in the proof of Proposition 10.1.3.

Let us prove (ii). Let $x \in S_{\chi_E} \cap \Omega$; according to Theorem 10.3.4, an easy computation yields

$$\liminf_{\rho \to 0} \frac{\mathcal{L}^N(B_\rho(x) \cap E)}{\mathcal{L}^N(B_\rho(x))} \geq \frac{1}{2} \quad \text{and} \quad \liminf_{\rho \to 0} \frac{\mathcal{L}^N(B_\rho(x) \setminus E)}{\mathcal{L}^N(B_\rho(x))} \geq \frac{1}{2}$$

so that

$$\lim_{\rho \to 0} \frac{\mathcal{L}^N(B_\rho(x) \cap E)}{\mathcal{L}^N(B_\rho(x))} = \lim_{\rho \to 0} \frac{\mathcal{L}^N(B_\rho(x) \setminus E)}{\mathcal{L}^N(B_\rho(x))} = \frac{1}{2}.$$

The conclusion of assertion (ii) then follows from (i).

For proving (iii), it suffices to establish that for all $\delta > 0$, the set

$$A_\delta = (\Omega \setminus S_{\chi_E}) \cap \left\{ x \in \Omega : \limsup_{\rho \to 0} \frac{|D\chi_E|(B_\rho(x))}{\rho^{N-1}} > \delta \right\}$$

satisfies $\mathcal{H}^{N-1}(A_\delta) = 0$. From Lemma 4.2.3 applied to the Borel measure $\mu = |D\chi_E|$, one has $|D\chi_E|(A_\delta) \geq C\delta \mathcal{H}^{N-1}(A_\delta)$ and the conclusion follows from $|D\chi_E|(A_\delta) = 0$ which is a consequence of $\partial_r E \subset S_{\chi_E} \cap \Omega$ established in Step 2.

We establish assertion (iv). From assertion (ii) and Lemma 4.2.3 applied to the Borel measure $\mu = |D\chi_E|$, there exists a nonnegative constant C' depending only on N, such that, for all Borel set B included in S_{χ_E},

$$\mathcal{H}^{N-1}(B) \leq C' \, |D\chi_E|(B). \tag{10.34}$$

Obviously, by definition

$$|D\chi_E|(\Omega \setminus \partial_r E) = 0;$$

thus

$$|D\chi_E|(S_{\chi_E} \cap \Omega \setminus \partial_r E) = 0,$$

and the conclusion $\mathscr{H}^{N-1}(S_{\chi_E} \cap \Omega \setminus \partial_r E) = 0$ follows from (10.34).

We finally establish $\mathscr{H}^{N-1}(\partial_M E \cap \Omega \setminus S_{\chi_E} \cap \Omega) = 0$. According to (iii), we are reduced to proving that for each x in $\partial_M E \cap \Omega$,

$$\limsup_{\rho \to 0} \frac{|D\chi_E|(B_\rho(x))}{\rho^{N-1}} > 0.$$

From the definition of the measure theoretical boundary (Proposition 10.3.1), there exists $\delta > 0$ such that

$$\limsup_{\rho \to 0} \frac{\mathscr{L}^N(B_\rho(x) \cap E)}{\mathscr{L}^N(B_\rho(x))} > \delta \quad \text{and} \quad \liminf_{\rho \to 0} \frac{\mathscr{L}^N(B_\rho(x) \cap E)}{\mathscr{L}^N(B_\rho(x))} < 1 - \delta.$$

Consequently, choosing $\delta < 1/2$, for ρ small enough,

$$\delta < \frac{\mathscr{L}^N(B_\rho(x) \cap E)}{\mathscr{L}^N(B_\rho(x))} < 1 - \delta,$$

and, for such ρ, the relative isoperimetric inequality (i) yields

$$\left(\delta \mathscr{L}^N(B_\rho(x))\right)^{\frac{N-1}{N}} \leq C|D\chi_E|(B_\rho(x)),$$

which completes the proof of Lemma 10.3.2.

Step 4. We admit that $S_{\chi_E} \cap \Omega$ is countably $(N-1)$-rectifiable. For a proof, see Ziemer [366, paragraph 5.7, pp. 243–246]. Actually, thanks to Rademacher's theorem, one may assume more precisely that $S_{\chi_E} \cap \Omega \subset \cup_{i=0}^{+\infty} A_i$, where $\mathscr{H}^{N-1} \lfloor \Omega(A_0) = 0$ and A_i is a $N-1$ manifold of class C^1 for $i = 1, \ldots, +\infty$ (see Ziemer [366, Lemma 5.7.2]).

Step 5. We establish the generalized Gauss–Green formula. We begin by establishing

$$|D\chi_E|(A) = \mathscr{H}^{N-1}(A) \tag{10.35}$$

for all Borel set A included in $\partial_r E$. From (10.27), (10.32), and Lemma 10.3.1, for all $x \in \partial_r E$, and all but countably many $0 < r < 1$, we have

$$\begin{aligned}
\lim_{\rho \to 0} \frac{|D\chi_E|(B_\rho(x))}{\rho^{N-1}} &= r^{1-N} \lim_{\rho \to 0} |D\chi_{E_{\frac{\rho}{r}}}|(B_r(x)) \\
&= r^{1-N} |D\chi_{\pi_\nu(x)}|(B_r(x)) \\
&= |D\chi_{\pi_\nu(x)}|(B_1(x)) \\
&= \mathscr{H}^{N-1}(B_1(x) \cap \pi_\nu(x)) \\
&= \omega_{N-1},
\end{aligned}$$

where ω_{N-1} is the volume of the unit ball of $\mathbf{R}^{N-1}$. Thus

$$\forall x \in \partial_r E \quad \lim_{\rho \to 0} \frac{|D\chi_E|(B_\rho(x))}{\mathscr{H}^{N-1}(B_\rho(x))} = 1. \tag{10.36}$$

On the other hand, from Steps 3 and 4, one may assume that

$$A \subset \cup_{i=0}^{+\infty} A_i,$$

where $\mathscr{H}^{N-1}\lfloor\Omega(A_0) = 0$ and A_i is a $N-1$ manifold of class C^1. Since each set A_i is regular, one has

$$\forall x \in A \cap A_i \quad \lim_{\rho \to 0} \frac{\mathscr{H}^{N-1}\lfloor A_i(B_\rho(x))}{\mathscr{H}^{N-1}(B_\rho(x))} = 1$$

so that, from (10.36),

$$\forall x \in A \cap A_i \quad \lim_{\rho \to 0} \frac{\mathscr{H}^{N-1}\lfloor A_i(B_\rho(x))}{|D\chi_E|(B_\rho(x))} = 1,$$

which yields

$$\mathscr{H}^{N-1}(A \cap A_i) = |D\chi_E|(A \cap A_i)$$

and finally gives $|D\chi_E|(A) = \mathscr{H}^{N-1}(A)$.

We are now in a position to prove the generalized Gauss–Green formula. Let $\varphi \in \mathbf{C}_c^1(\Omega, \mathbf{R}^N)$; from (10.35) and the definition of $\partial_r E$ and ν, one has

$$\begin{aligned}\int_\Omega \chi_E \operatorname{div} \varphi\, dx &= -\int_\Omega \varphi\, D\chi_E \\ &= \int_{\partial_r E} \varphi.(-\nu)\, |D\chi_E| \\ &= \int_{\partial_r E} \varphi.(-\nu)\, d\mathscr{H}^{N-1}\end{aligned}$$

and the conclusion follows from Step 3. □

10.3.3 ▪ Structure of BV functions

The functions of $BV(\Omega)$ naturally inherit their properties from their level sets $[u > t] := \{x \in \Omega : u(x) > t\}$ when t varies in $\mathbf{R}$. The following property, established by Fleming and Richel in [217], generalizes the classical coarea formula (Theorem 4.2.5) to BV functions and states that for almost every t in $\mathbf{R}$, the level set $[u > t]$ of each BV function u has a finite perimeter in Ω. Consequently, we show that the jump set of $u \in BV(\Omega)$ inherits its structure from the one of the jump set of finite perimeter sets $[u > t]$, $t \in \mathbf{R}$, stated in Theorem 10.3.2.

Theorem 10.3.3 (coarea formula). *Let u be a given function in $BV(\Omega)$. Then, for a.e. t in $\mathbf{R}$, the level set $E_t = \{x \in \Omega : u(x) > t\}$ of u is a set of finite perimeter in Ω, and*

$$\begin{cases} Du = \displaystyle\int_{-\infty}^{+\infty} D\chi_{E_t}\, dt, \\ |Du|(\Omega) = \displaystyle\int_{-\infty}^{+\infty} \int_\Omega |D\chi_{E_t}|\, dt. \end{cases}$$

More generally for all Borel function $f : \Omega \to \mathbf{R}^+$,

$$\int_\Omega f|Du| = \int_{-\infty}^{+\infty} \int_\Omega f|D\chi_{E_t}|dt.$$

PROOF. Let us assume for the moment that for a.e. t in $\mathbf{R}$, $D\chi_{E_t}$ belongs to $\mathbf{M}(\Omega, \mathbf{R}^N)$. For all t in $\mathbf{R}$, set

$$f_t = \begin{cases} \chi_{E_t} & \text{if } t \geq 0, \\ -\chi_{\Omega \setminus E_t} & \text{if } t < 0. \end{cases}$$

It is easily seen that for all x in Ω, $u(x) = \int_{-\infty}^{+\infty} f_t(x)\, dt$. For all φ in $\mathbf{C}_c^1(\Omega, \mathbf{R}^N)$ we have

$$\begin{aligned} < Du, \varphi > &= -\int_\Omega u \text{ div } \varphi \, dx \\ &= -\int_\Omega dx \int_{-\infty}^{+\infty} f_t(x) \text{ div } \varphi \, dt \\ &= \int_{-\infty}^{0} dt \int_\Omega \chi_{\Omega \setminus E_t} \text{ div } \varphi \, dx - \int_0^{+\infty} dt \int_\Omega \chi_{E_t} \text{ div } \varphi \, dx \\ &= -\int_{-\infty}^{0} dt \int_\Omega \chi_{E_t} \text{ div } \varphi \, dx - \int_0^{+\infty} dt \int_\Omega \chi_{E_t} \text{ div } \varphi \, dx \\ &= -\int_{-\infty}^{+\infty} dt \int_\Omega \chi_{E_t} \text{ div } \varphi \, dx \\ &= \int_{-\infty}^{+\infty} < D\chi_{E_t}, \varphi > dt. \end{aligned}$$

Hence $Du = \int_{-\infty}^{+\infty} D\chi_{E_t} \, dt$ and $|Du|(\Omega) \leq \int_{-\infty}^{+\infty} \int_\Omega |D\chi_{E_t}| dt$.

We establish now the converse inequality $\int_{-\infty}^{+\infty} \int_\Omega |D\chi_{E_t}| \; dt \leq |Du|(\Omega)$, which also proves that $D\chi_{E_t}$ belongs to $\mathbf{M}(\Omega, \mathbf{R}^N)$ for a.e. t in $\mathbf{R}$.

Step 1. We assume that u belongs to the space $\mathscr{A}(\Omega)$ of piecewise linear and continuous functions in Ω. By linearity, one can assume that u is the linear function $u = a.x + b$ with $a \in \mathbf{R}^N$ and $b \in \mathbf{R}$ so that

$$\begin{aligned} \int_\Omega |D\chi_{E_t}| &= \mathscr{H}^{N-1}(\Omega \cap \partial E_t) \\ &= \mathscr{H}^{N-1}(\Omega \cap [a.x + b = t]). \end{aligned}$$

Consequently, according to the classical coarea formula (Theorem 4.2.5),

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_\Omega |D\chi_{E_t}| &= \int_{-\infty}^{+\infty} \int_{[a.x+b=t]} \chi_\Omega(x) \, d\mathscr{H}^{N-1}(x) \, dt \\ &= |a| \mathscr{L}^N(\Omega) = \int_\Omega |Du|. \end{aligned}$$

Step 2. We establish the inequality $\int_{-\infty}^{+\infty} \int_\Omega |D\chi_{E_t}| \; dt \leq |Du|(\Omega)$ for all $u \in BV(\Omega)$.

Let $(u_n)_{n \in \mathbf{N}}$ be a sequence in $\mathscr{A}(\Omega)$ such that $u_n \rightharpoonup u$ for the intermediate convergence. Such a sequence exists from Theorem 10.1.2 and the well-known density of the space $\mathscr{A}(\Omega)$ in $W^{1,1}(\Omega)$ equipped with its strong topology. For another and direct proof of this assertion, consult Ziemer [366, Exercise 5.2]. Let us set $E_{n,t} := \{x \in \Omega : u_n(x) > t\}$. According to the first step and to Fatou's lemma, we have

$$\begin{aligned}\int_\Omega |Du| &= \lim_{n\to+\infty} \int_\Omega |Du_n| \\ &= \lim_{n\to+\infty} \int_{-\infty}^{+\infty} \int_\Omega |D\chi_{E_{n,t}}|\, dt \\ &\geq \int_{-\infty}^{+\infty} \liminf_{n\to+\infty} \int_\Omega |D\chi_{E_{n,t}}|\, dt. \end{aligned} \tag{10.37}$$

On the other hand,

$$\int_\Omega |u_n - u|\, dx = \int_\Omega \int_{-\infty}^{+\infty} |\chi_{E_{n,t}} - \chi_{E_t}|\, dt\, dx = \int_{-\infty}^{+\infty} \left(\int_\Omega |\chi_{E_{n,t}} - \chi_{E_t}|\, dx \right) dt,$$

which converges to zero. Thus, for a subsequence (not relabeled), and for almost all t in $\mathbf{R}$,

$$\chi_{E_{n,t}} \to \chi_{E_t} \quad \text{strongly in } L^1(\Omega). \tag{10.38}$$

The lower semicontinuity of the total variation with respect to the strong convergence in $L^1(\Omega)$ (Proposition 10.1.1) and (10.37), (10.38) finally yield

$$\int_\Omega |Du| \geq \int_{-\infty}^{+\infty} \int_\Omega |D\chi_{E_t}|\, dt.$$

It is now easy to adapt the proof above for obtaining the coarea formula with $f = \chi_E$ for any Borel subset E of Ω. The general coarea formula with a Borel function $f : \Omega \to \mathbf{R}^+$ is then obtained by a classical density argument. □

We are now in a position to establish that $\mathscr{H}^{N-1}$ almost all points of Ω are regular for function in $BV(\Omega)$ and that their jump set (cf. Theorem 10.3.1) is countably $(N-1)$-rectifiable. For each function u in $BV(\Omega)$ whose representative satisfies the convention of Remark 10.3.2, we set

$$S_u = \{\, x \in \Omega : u^-(x) < u^+(x) \,\},$$

where $u^-(x) = \operatorname{ap}\liminf_{y\to x} u(y)$ and $u^+(x) = \operatorname{ap}\limsup_{y\to x} u(y)$.

Theorem 10.3.4. *Let u be a given function in $BV(\Omega)$. Then, for $\mathscr{H}^{N-1}$-almost all x in Ω, $u^-(x)$ and $u^+(x)$ are finite and S_u is countably $(N-1)$-rectifiable. Moreover, S_u is, up to a set of $\mathscr{H}^{N-1}\lfloor\Omega$ measure zero, the jump set of u, and $\mathscr{H}^{N-1}$ almost all x in Ω are regular for u.*

PROOF. We begin by proving that S_u is countably $(N-1)$-rectifiable. According to the coarea formula (Theorem 10.3.3), for almost all $t \in \mathbf{R}$, $E_t = \{x \in \Omega : u(x) > t\} := [u > t]$ is a set of finite perimeter in Ω. Now let D be a dense countable subset of $\{t \in \mathbf{R} : E_t \text{ is of finite perimeter}\}$ and set $S_{u,t} := \{\, x \in S_u : u^-(x) < t < u^+(x) \,\}$. We have $S_u = \bigcup_{t\in D} S_{u,t}$. On the other hand, from definitions of u^- and u^+, it is easy to establish that for all $x \in S_{u,t}$,

$$\begin{cases} \limsup_{\rho\to 0} \dfrac{\mathscr{L}^N(B_\rho(x)\cap[u>t])}{\mathscr{L}^N(B_\rho(x))} > 0, \\ \limsup_{\rho\to 0} \dfrac{\mathscr{L}^N(B_\rho(x)\cap[u<t])}{\mathscr{L}^N(B_\rho(x))} > 0, \end{cases}$$

so that $x \in \partial_M E_t$. Thus $S_u \subset \bigcup_{t\in D} \partial_M E_t$ and the conclusion follows from the structure theorem, Theorem 10.3.2.

We admit that $-\infty < u^-(x) \le u^+(x) < +\infty$ for $\mathscr{H}^{N-1}$ almost all x in Ω. For a proof, consult Evans and Gariepy [211, Theorem 2]. For establishing that up to an $\mathscr{H}^{N-1}\lfloor\Omega$-negligible set, S_u is the jump set of u and that $\mathscr{H}^{N-1}$ almost all x in Ω are regular for u, according to Proposition 10.3.3, Definition 10.3.5, and Theorem 10.3.1, it is enough to establish that for $\mathscr{H}^{N-1}$ almost all x in S_u, there exists $\nu(x)$ in S^{N-1} such that $u^-(x) = \text{ap}\ \lim_{y\to x,\ y\in\pi_{-\nu(x)}(x)} u(y)$ and $u^+(x) = \text{ap}\ \lim_{y\to x,\ y\in\pi_{\nu(x)}(x)} u(y)$. Let $x \in S_u$ such that $u^-(x)$ and $u^+(x)$ are finite and set $t = u^+(x) - \varepsilon$ with ε small enough so that $u^-(x) < t < u^+(x)$. Thus $x \in \partial_M E_t$ and, from Theorem 10.3.2, for $\mathscr{H}^{N-1}$-almost all such x in S_u, there exists $\nu(x)$ in S^{N-1} such that

$$\lim_{\rho\to 0} \frac{\mathscr{L}^N\big(H_{\rho,\nu(x)}(x)\cap[u>u^+(x)-\varepsilon]\big)}{\mathscr{L}^N\big(H_{\rho,\nu(x)}(x)\big)} = 1. \tag{10.39}$$

On the other hand, according to the definition of the approximate limsup,

$$\lim_{\rho\to 0} \frac{\mathscr{L}^N\big(B_\rho(x)\cap[u>u^+(x)+\varepsilon]\big)}{\mathscr{L}^N\big(B_\rho(x)\big)} = 0,$$

hence

$$\lim_{\rho\to 0} \frac{\mathscr{L}^N\big(H_{\rho,\nu(x)}(x)\cap[u>u^+(x)+\varepsilon]\big)}{\mathscr{L}^N\big(H_{\rho,\nu(x)}(x)\big)} = 0. \tag{10.40}$$

Combining (10.39) and (10.40) we obtain

$$\lim_{\rho\to 0} \frac{\mathscr{L}^N\big(H_{\rho,\nu(x)}(x)\cap[|u-u^+(x)|<\varepsilon]\big)}{\mathscr{L}^N\big(H_{\rho,\nu(x)}(x)\big)} = 1,$$

which proves that $u^+(x) = \text{ap}\ \lim_{y\to x,\ y\in\pi_{\nu(x)}(x)} u(y)$. The proof of

$$u^-(x) = \text{ap} \lim_{y\to x,\ y\in\pi_{-\nu(x)}(x)} u(y)$$

is similar. □

Remark 10.3.4. In the proof of Theorem 10.3.4 we have established that S_u possesses for $\mathscr{H}^{N-1}$-a.e. x in Ω, a normal unit vector $\nu_u(x)$ and that

$$\mathscr{H}^{N-1}\big(S_u \setminus \{x\in\mathbf{R}^N : u_{\nu_u(x)}(x) \ne u_{-\nu_u(x)}(x)\}\big) = 0.$$

Moreover, we have obtained that for almost every $t\in\mathbf{R}$ and for $\mathscr{H}^{N-1}$-a.e. x in $\partial_M E_t \cap S_u$, $\nu_u(x) = \nu_{E_t}$, where ν_{E_t} is the inner measure theoretic normal to E_t at x.

Remark 10.3.5. According to Theorem 10.3.4 and Proposition 10.3.3, there exist two Borel sets E^+ and E^- such that for H^{N-1}-almost all x in S_u

$$u^+(x) = \lim_{y\to x,\ y\in E^+\cap\pi_{\nu_u}(x)} u(y) \text{ and } u^-(x) = \lim_{y\to x,\ y\in E^-\cap\pi_{-\nu_u}(x)} u(y).$$

Remark 10.3.6. According to our convention (Remark 10.3.2) on the representative of L^1-functions, for $\mathscr{H}^{N-1}$ a.e. x in $\Omega\setminus S_u$, $u^+(x) = u^-(x) = u(x)$.

In the following proposition, we give some information on u when x belongs to S_u.

Proposition 10.3.5. *Let $u \in BV(\Omega)$. For $\mathscr{H}^{N-1}$-a.e. x in S_u letting $E_t := [u > t]$, one has*

$$(u^-(x), u^+(x)) \subset \{t \in \mathbf{R} : x \in \partial_M E_t \cap \Omega\} \subset [u^-(x), u^+(x)],$$

and for $\mathscr{H}^{N-1}$-a.e. x in $\partial_M E_t \cap \Omega \setminus S_u$, $u(x) = t$.

PROOF. For the first inclusion $(u^-(x), u^+(x)) \subset \{t \in \mathbf{R} : x \in \partial_M E_t \cap \Omega\}$, it suffices to note that $t \in (u^-(x), u^+(x))$ implies $x \in S_{u,t} \subset \partial_M E_t$. We establish now the second inclusion $\{t \in \mathbf{R} : x \in \partial_M E_t \cap \Omega\} \subset [u^-(x), u^+(x)]$. Let $t \in \mathbf{R}$ be such that $x \in \partial_M E_t \cap \Omega$, assume that $t > u^+(x)$, and take t_0 such that $t > t_0 > u^+(x)$ and

$$\lim_{\rho\to 0} \frac{\mathscr{L}^N(B_\rho(x) \cap [u > t_0])}{\mathscr{L}^N(B_\rho(x))} = 0.$$

Such a t_0 exists from the definition of the approximate limsup. We have

$$\lim_{\rho\to 0} \frac{\mathscr{L}^N(B_\rho(x) \cap [u > t])}{\mathscr{L}^N(B_\rho(x))} = 0,$$

which is in contradiction with $x \in \partial_M E_t$. Assuming $t < u^-(x)$ yields the same contradiction.

Since, up to a set of $\mathscr{H}^{N-1}\lfloor\Omega$ measure zero, S_u is the jump set of u, one has, for $\mathscr{H}^{N-1}$-a.e. x in $\partial_M E_t \cap \Omega \setminus S_u$, $u(x) = u^+(x) = u^-(x)$, so that $u(x) = \text{ap}\lim_{y\to x} u(y)$. For such x, we establish that $u(x) = t$. Otherwise, assume that $t > u(x)$ and set $\varepsilon = t - u(x)$. According to the definition of the approximate limit at x, we have

$$\lim_{\rho\to 0} \frac{\mathscr{L}^N(B_\rho(x) \cap [|u - u(x)| > \varepsilon])}{\mathscr{L}^N(B_\rho(x))} = 0,$$

which yields

$$\lim_{\rho\to 0} \frac{\mathscr{L}^N(B_\rho(x) \cap [u > \varepsilon + u(x)])}{\mathscr{L}^N(B_\rho(x))} = 0,$$

that is,

$$\lim_{\rho\to 0} \frac{\mathscr{L}^N(B_\rho(x) \cap [u > t])}{\mathscr{L}^N(B_\rho(x))} = 0,$$

which contradicts the hypothesis $x \in \partial_M E_t$. Using the same arguments, assumption $u < t$ would give the same contradiction. □

10.4 ▪ Structure of the gradient of BV functions

Let u be a given function in $BV(\Omega)$ and $Du = D^a u + D^s u$ the Lebesgue–Nikodým decomposition of the measure Du with respect to the N-dimensional Lebesgue measure $\mathscr{L}^N \lfloor \Omega$ restricted to Ω. Let us recall that the measure $D^a u$ denotes the absolutely continuous part of Du with respect to the measure $\mathscr{L}^N \lfloor \Omega$ and $D^s u$ its singular part. We will denote the density of $D^a u$ with respect to $\mathscr{L}^N \lfloor \Omega$ by ∇u, so that $D^a u = \nabla u \, \mathscr{L}^N \lfloor \Omega$. The theorem below makes precise the structure of the singular part $D^s u$.

Theorem 10.4.1. *Let us denote the two measures $D^s u \lfloor S_u$ and $D^s u \lfloor \Omega \setminus S_u$ by Ju and Cu, respectively, called the jump part and the Cantor part of Du. Then Ju is absolutely continuous with respect to the restriction of the $(N-1)$-dimensional Hausdorff measure to S_u. More precisely,*

$$Ju = (u^+ - u^-)\, \nu_u \, \mathscr{H}^{N-1} \lfloor S_u.$$

Moreover, Ju and Cu are mutually singular: for all Borel sets E of Ω

$$\mathscr{H}^{N-1}(E) < +\infty \Longrightarrow |Cu|(E) = 0.$$

Consequently, the Hausdorff dimension of the support $spt(Cu)$ of the measure Cu satisfies

$$N - 1 \leq dim_H(spt(Cu)) < N.$$

PROOF. According to the coarea formula, Theorem 10.3.3, and Theorem 10.3.2(iii), for all Borel set $E \subset S_u$

$$\begin{aligned} Ju(E) = Du(E) &= \int_{-\infty}^{+\infty} D\chi_{E_t}(E)\, dt \\ &= \int_{-\infty}^{+\infty} \left(\int_{E \cap \partial_M E_t} \nu_u(x)\, d\mathscr{H}^{N-1}(x) \right) dt \\ &= \int_E \left(\int_{-\infty}^{+\infty} \chi_{\{t \in \mathbf{R} \,:\, x \in \partial_M E_t\}}\, dt \right) \nu_u(x)\, d\mathscr{H}^{N-1}(x). \end{aligned}$$

Since $E \subset S_u$, according to Proposition 10.3.5, for $\mathscr{H}^{N-1}$ a.e. x in E, one has

$$\mathscr{L}^1(\{t \in \mathbf{R} : x \in \partial_M E_t\}) = u^+(x) - u^-(x),$$

thus $Ju = (u^+ - u^-)\, \nu_u \, \mathscr{H}^{N-1} \lfloor S_u$.

Now let E be any Borel set included in $\Omega \setminus S_u$, satisfying $\mathscr{H}^{N-1}(E) < +\infty$. According to Theorems 10.3.3 and 10.3.2(iii), one has

$$|Cu|(E) = |Du|(E) = \int_{-\infty}^{+\infty} \mathscr{H}^{N-1}(E \cap \partial_M E_t)\, dt. \tag{10.41}$$

With our convention (Remark 10.3.2), the points of $\Omega \setminus S_u$ are all points of approximate continuity for u; thus, according to Proposition 10.3.5, one has $E \cap \partial_M E_t \subset \{y \in E : u(y) = t\}$. Moreover, $\mathscr{H}^{N-1}(E) < +\infty$, so that, from Lemma 4.2.1, the set of all t such that $\mathscr{H}^{N-1}(\{y \in E : u(y) = t\}) > 0$ is at most countable, and (10.41) yields $|Cu|(E) = 0$. $\square$

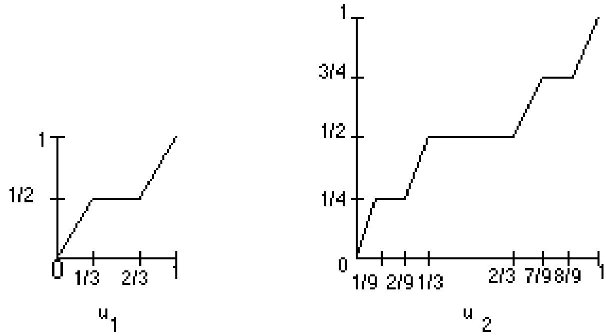

Figure 10.3. *Construction of the Cantor–Vitali function.*

The proposition below states that all functions u in $BV_{loc}(\mathbf{R}^N)$ possess an approximate derivative for almost all x in $\mathbf{R}^N$ in the following sense: there exists a linear function $L : \mathbf{R}^N \longrightarrow \mathbf{R}$ denoted by ap Du such that

$$\operatorname{ap}\lim_{y\to x} \frac{|u(y)-u(x)-L(y-x)|}{|y-x|} = 0.$$

For a proof, see [213, Theorem 4.5.9] or [211, Theorem 4].

Proposition 10.4.1. *Let u be a given function of $BV_{loc}(\mathbf{R}^N)$, i.e., $u \in BV(U)$ for all open bounded subset U of $\mathbf{R}^N$. Then for almost all x_0 in $\mathbf{R}^N$,*

$$\lim_{\rho\to 0} \frac{1}{\rho^d} \int_{B_\rho(x_0)} \frac{|\, u(x)-u(x_0)-\nabla u(x_0)\,.\,(x-x_0)\,|}{|\, x-x_0\,|} dx = 0.$$

Consequently for almost all x_0 in $\mathbf{R}^N$ ap$\lim_{x\to x_0} Du = \nabla u(x_0)$.

Example 10.4.1. We construct a BV-function whose gradient is reduced to its Cantor part: the Cantor–Vitali function. Let $\Omega = (0,1)$ and C be the classical triadic Cantor set $C = \bigcap_{n\in\mathbf{N}} C_n$, where C_n is the union of 2^n intervals of size 3^{-n}. We define

$$\begin{cases} f_n(x) := \left(\dfrac{2}{3}\right)^{-n} \chi_{C_n}, \\ u_n(x) := \displaystyle\int_0^x f_n(t)\,dt \quad \text{(see Figure 10.3).} \end{cases}$$

All the functions u_n belong to $\mathbf{C}([0,1])$ and if I is any of the 2^n intervals of C_n,

$$\begin{cases} \displaystyle\int_I f_n(t)\,dt = \int_I f_{n+1}(t)\,dt = 2^{-n}, \\ \forall x \in (0,1)\setminus C_n \quad u_n(x) = u_{n+1}(x). \end{cases}$$

Indeed $\int_I f_n(t)\,dt = (\frac{3}{2})^n \operatorname{mes}(I) = 2^{-n}$. We deduce that for all x in C_n,

$$|u_n(x) - u_{n+1}(x)| \le 2^{-(n-1)},$$

so that u_n uniformly converges to a continuous function u. According to the lower semicontinuity of the total variation, we then obtain

$$\int_{(0,1)} |Du| \leq \liminf_{n\to+\infty} \int_{(0,1)} |Du_n|\, dx = 1,$$

which proves that u belongs to $BV(0,1)$ and, since u is continuous, that $Ju = 0$. Finally, since u is locally constant on $(0,1)\setminus C$ and $\mathscr{L}^1(C) = 0$, one has $\nabla u = 0$ and $Du = Cu$. Moreover, the support of Cu is the Cantor set C whose Hausdorff dimension is $\ln(2)/\ln(3) \sim 0.632$ (see Example 4.1.1).

10.5 ▪ The space $SBV(\Omega)$

In some problems arising in image segmentation, or in mechanics in the study of cracks and fissures (see Chapters 12 and 14), the first distributional derivatives of the competing functions which operate in the models are often measures without singular Cantor part. The solutions of these problems may be found in a special space of functions of bounded variation.

10.5.1 ▪ Definition

Definition 10.5.1. *The special set of functions of bounded variation is the subset $SBV(\Omega)$ of $BV(\Omega)$ made up of all the functions of $BV(\Omega)$ whose gradient measures have no Cantor part in their Lebesgue decomposition, i.e.,*

$$u \in SBV(\Omega) \iff u \in L^1(\Omega) \text{ and } Du = \nabla u\, \mathscr{L}^N\lfloor\Omega + (u^+ - u^-)\, \nu_u\, \mathscr{H}^{N-1}\lfloor S_u.$$

Remark 10.5.1. Arguing as in Remark 10.2.2, one may define the space $SBV(\Omega,\mathbf{R}^m)$ as the space of all functions $u:\Omega\to\mathbf{R}^m$ which belong to $L^1(\Omega,\mathbf{R}^m)$ and whose distributional derivative Du is a $M^{m\times N}$-valued measure of the form

$$Du = \nabla u\, \mathscr{L}^N\lfloor\Omega + (u^+ - u^-)\otimes\nu_u\, \mathscr{H}^{N-1}\lfloor S_u.$$

Example 10.5.1. Let Ω be an open bounded subset of $\mathbf{R}^N$, K a closed subset of Ω such that $\mathscr{H}^{N-1}(K) < +\infty$, and $u \in W^{1,1}(\Omega\setminus K)\cap L^\infty(\Omega)$. We claim that u belongs to $SBV(\Omega)$ and that $S_u \subset K$.

We first assume that K is regular in the following sense: there exist a $\mathbf{C}^1$ hypersurface Σ such that $K\subset\Sigma$ and two disjoint subsets Ω_1 and Ω_2 of Ω such that $\partial\Omega_1\cap\partial\Omega_2 = \Sigma$ and $\mathscr{L}^N(\Omega\setminus\Omega_1\cup\Omega_2) = 0$. Then the result is a straightforward consequence of the trace theory (see Subsection 10.2 and Example 10.2.1). It is worth pointing out that in this case, the hypothesis $u\in L^\infty(\Omega)$ is unnecessary.

We now consider the general case. If $N = 1$, the result follows from the previous argument. We assume $N \geq 2$. Since $\mathscr{H}^{N-1}(K) < +\infty$ and K is a compact set, for all $n\in\mathbf{N}^*$ there exists a finite family of closed balls $(B^n_{\rho_i}(x_i))_{i\in I_n}$, covering K, with $\rho_i \leq \frac{1}{n}$ and such that

$$\sum_{i\in I_n} c_{N-1}(2\rho_i)^{N-1} \leq \mathscr{H}^{N-1}(K) + 1.$$

Therefore

$$\sum_{i\in I_n} \mathcal{H}^{N-1}(\partial B^n_{\rho_i}(x_i)) \le C(\mathcal{H}^{N-1}(K)+1),$$

where C is a positive constant depending only on the dimension N. We now consider the following functions u_n:

$$u_n(x) = \begin{cases} u(x) & \text{for } x \in \Omega \setminus \bigcup_{i\in I_n} B^n_{\rho_i}(x_i), \\ 0 & \text{elsewhere.} \end{cases}$$

Since $\mathcal{L}^N(\cup_{i\in I_n} B^n_{\rho_i}(x_i))$ tends to zero, $u_n \to u$ strongly in $L^1(\Omega)$ when $n \to +\infty$. Since moreover $\partial B^n_{\rho_i}(x_i))$ is a finite union of $\mathbf{C}^1$ hypersurfaces, reasoning on a neighborhood of each $\partial B^n_{\rho_i}(x_i))$, from the trace theory and the estimate $|u_n^+(x)| \le ||u||_{L^\infty(\Omega)}$ (note that according to Remark 10.3.4, u^+ is a classical limit in a Borel set E^+ of Ω), one has

$$\int_\Omega |Du_n| \le \int_{\Omega\setminus K} |\nabla u|\, dx + C||u||_{L^\infty(\Omega)}(\mathcal{H}^{N-1}(K)+1).$$

The semicontinuity of the total variation (Proposition 10.1.1) yields $u \in BV(\Omega)$. On the other hand, it is easily seen that $S_u \subset K$. Since $\mathcal{H}^{N-1}(K \setminus S_u) \le \mathcal{H}^{N-1}(K) < +\infty$, according to Theorem 10.4.1, $Cu(K \setminus S_u) = 0$. Finally $Cu = 0$ because $Cu(S_u) = 0$.

10.5.2 ▪ Properties

The following chain rule for the derivatives in $BV(\Omega)$ was established in Ambrosio [24].

Proposition 10.5.1. *Let u be a given function in $BV(\Omega)$ and φ in $\mathbf{C}^1_0(\mathbf{R})$. Then $v := \varphi \circ u$ belongs to $BV(\Omega)$ and, even if it means changing ν_u by $-\nu_u$,*

$$\begin{cases} Jv = \big(\varphi(u^+) - \varphi(u^-)\big)\nu_u\, \mathcal{H}^{N-1}\lfloor S_u, \\ \nabla v = \varphi'(u)\nabla u,\ Cv = \varphi'(u)Cu. \end{cases}$$

PROOF. Consider $u_n \in \mathbf{C}^\infty(\Omega) \cap BV(\Omega)$ converging to u for the intermediate convergence. Then $v_n := \varphi \circ u_n \to v := \varphi \circ u$ in $L^1(\Omega)$. On the other hand,

$$\begin{aligned} |Dv|(\Omega) &\le \liminf_{n\to+\infty} |Dv_n|(\Omega) \\ &\le ||\varphi'||_\infty \liminf_{n\to+\infty} |Du_n|(\Omega) \\ &= ||\varphi'||_\infty |Du|(\Omega) < +\infty. \end{aligned}$$

This proves that $v \in BV(\Omega)$.

We now show that $Jv = \big(\varphi(u^+) - \varphi(u^-)\big)\nu_u\, \mathcal{H}^{N-1}\lfloor S_u$. Since $\varphi \in \mathbf{C}^1_0(\mathbf{R})$, φ is the difference of two nondecreasing functions in $\mathbf{C}^1(\mathbf{R})$. One may then assume φ nondecreasing so that $v^+ = \varphi \circ u^+$, $v^- = \varphi \circ u^-$, $S_v = S_u$, and $\nu_v = \nu_u$.

It remains to establish $\nabla v = \varphi'(u)\nabla u$ and $Cv = \varphi'(u)Cu$ or, equivalently, $Dv\lfloor_{\Omega\setminus S_v} = \varphi'(u)Du\lfloor_{\Omega\setminus S_v}$. Consider a Borel set E of Ω included in $\Omega \setminus S_v$. From the coarea formula (Theorem 10.3.3) and the structure of simple functions of $BV(\Omega)$ (Theorem 10.3.2(iii)), one has

$$
\begin{aligned}
Dv(E) = \int_{-\infty}^{+\infty} D\chi_{[v>t]}(E)\,dt &= \int_{-\infty}^{+\infty} D\chi_{[u>\varphi^{-1}(t)]}(E)\,dt \\
&= \int_{-\infty}^{+\infty} D\chi_{[u>t]}(E)\,\varphi'(t)\,dt \\
&= \int_{-\infty}^{+\infty} \int_E D\chi_{[u>t]}\,\varphi'(t)\,dt \\
&= \int_{-\infty}^{+\infty} \mathscr{H}^{N-1}(\partial_M([u>t])\cap E)\,\varphi'(t)\,dt. \qquad (10.42)
\end{aligned}
$$

But, according to Proposition 10.3.5, for $\mathscr{H}^{N-1}$ a.e. x in Ω, $x \in \partial_M([u > t]) \cap E \Longrightarrow u(x) = t$ so that (10.42) yields

$$
\begin{aligned}
Dv(E) &= \int_{-\infty}^{+\infty} \left(\int_E \varphi'(u(x))\, d\mathscr{H}^{N-1}\lfloor \partial_M([u>t])(x) \right) dt \\
&= \int_{-\infty}^{+\infty} \left(\int_E \varphi'(u) D\chi_{[u>t]} \right) dt \\
&= \varphi'(u)\, Du(E),
\end{aligned}
$$

which completes the proof. □

The following criterion for a function u in $BV(\Omega)$ to belong to $SBV(\Omega)$ was established by Ambrosio in [17].

Theorem 10.5.1. *Let u be a given function in $BV(\Omega)$. Then u belongs to $SBV(\Omega)$ iff there exists a Borel measure μ in $\mathbf{M}(\Omega\times\mathbf{R}, \mathbf{R}^N)$ and a in $L^1(\Omega, \mathbf{R}^N)$ such that for all Φ in $\mathbf{C}_c^1(\Omega, \mathbf{R}^N)$ and all φ in $\mathbf{C}_0^1(\mathbf{R})$,*

$$
\int_{\Omega\times\mathbf{R}} \varphi(s)\Phi(x)\,\mu(dx,ds) = -\int_\Omega \Big(\varphi'(u)\, a\,.\,\Phi(x) + \varphi(u)\ \mathrm{div}\ \Phi(x) \Big)\, dx. \qquad (10.43)
$$

Moreover, $a = \nabla u$ a.e. and

$$
\begin{cases}
\mu = \Lambda^+_\#(\nu_u\ \mathscr{H}^{N-1}\lfloor S_u) - \Lambda^-_\#(\nu_u\ \mathscr{H}^{N-1}\lfloor S_u), \\
|\mu|(\Omega\times\mathbf{R}) = 2\,\mathscr{H}^{N-1}(S_u),
\end{cases}
$$

where $\Lambda^+ : \Omega \longrightarrow \Omega\times\mathbf{R},\ x \mapsto (x, u^+(x))$ and $\Lambda^- : \Omega \longrightarrow \Omega\times\mathbf{R},\ x \mapsto (x, u^-(x))$.

PROOF. Let us assume that u belongs to $SBV(\Omega)$. According to Proposition 17.2.5, for all $\varphi \in \mathbf{C}_0^1(\mathbf{R})$, $\varphi(u)$ belongs to $SBV(\Omega)$ and $D(\varphi(u)) = \varphi'(u)\nabla u \mathscr{L}\lfloor\Omega + \big(\varphi(u^+) - \varphi(u^-)\big)\nu_u\ \mathscr{H}^{N-1}\lfloor S_u$. Consequently, for all $\Phi \in \mathbf{C}_c^1(\Omega, \mathbf{R}^N)$

$$
\begin{aligned}
\int_\Omega \varphi(u)\ \mathrm{div}\ \Phi\, dx &= -\langle D(\varphi(u)), \Phi\rangle \\
&= -\int_\Omega \varphi'(u)\nabla u.\Phi\, dx - \int_\Omega \big(\varphi(u^+) - \varphi(u^-)\big)\Phi.\nu_u\ d\mathscr{H}^{N-1}\lfloor S_u,
\end{aligned}
$$

which we write

$$
\int_\Omega \big(\varphi(u^+) - \varphi(u^-)\big)\Phi.\nu_u\ d\mathscr{H}^{N-1}\lfloor S_u = -\int_\Omega \Big(\varphi'(u)\,\nabla u\,.\,\Phi + \varphi(u)\ \mathrm{div}\ \Phi \Big)\, dx.
$$

According to the definition of the image of a measure, the left-hand side above is nothing but

$$\int_{\Omega\times\mathbf{R}} \varphi(s)\Phi(x)\,d\mu(x,s).$$

Finally, since $\Lambda^+(\Omega)$ and $\Lambda^-(\Omega)$ are disjoint sets, and by injectivity of Λ^+ and Λ^-, one has

$$\begin{aligned}|\mu| &= |\Lambda^+_\#(\nu_u\,\mathcal{H}^{N-1}\lfloor S_u) - \Lambda^-_\#(\nu_u\,\mathcal{H}^{N-1}\lfloor S_u)|\\ &= |\Lambda^+_\#(\nu_u\,\mathcal{H}^{N-1}\lfloor S_u)| + |\Lambda^-_\#(\nu_u\,\mathcal{H}^{N-1}\lfloor S_u)|\\ &= \Lambda^+_\#(\mathcal{H}^{N-1}\lfloor S_u) + \Lambda^-_\#(\mathcal{H}^{N-1}\lfloor S_u).\end{aligned}$$

Hence $|\mu|(\Omega\times\mathbf{R}) = 2\mathcal{H}^{N-1}(S_u)$. The proof of the converse condition proceeds in three steps.

First step. We establish $a(x_0)=\nabla u(x_0)$ for a.e. x_0 in Ω. Let us fix $x_0\in\Omega$ such that

$$\begin{cases} a_\rho(y) := a(x_0+\rho y)\to a(x_0) \quad \text{strongly in } L^1(B),\\ u_\rho \to \nabla u(x_0).y \quad \text{strongly in } L^1(B),\\ \lim\limits_{\rho\longrightarrow 0} \dfrac{1}{\rho^{N-1}}|\mu|(B_\rho(x_0)\times\mathbf{R}) = 0,\end{cases}$$

where u_ρ denotes the rescaled function

$$u_\rho(y) := \frac{1}{\rho}\big(u(x_0+\rho y) - u(x_0)\big),$$

and $B_\rho(x_0)$, B, respectively, the open ball in $\mathbf{R}^N$ with radius ρ, centered at x_0, and the unit open ball in $\mathbf{R}^N$ centered at 0. Let us justify the possible choice of such x_0. Actually x_0 is chosen to be a Lebesgue point of $y\mapsto a_\rho(y) := a(x_0+\rho y)$. On the other hand, according to Proposition 10.4.1, the second property is satisfied a.e. in Ω. Finally, denoting by π the projection from $\Omega\times\mathbf{R}$ onto Ω and by $\pi\#|\mu|$ the image of the measure $|\mu|$ by π, the limit

$$\lim_{\rho\longrightarrow 0}\frac{1}{\rho^N}|\mu|(B_\rho(x_0)\times\mathbf{R}) = \lim_{\rho\longrightarrow 0}\frac{1}{\rho^N}\pi_\#|\mu|(B_\rho(x_0))$$

exists for almost every x_0 in Ω and, up to a positive multiplicative constant, is equal to the density of the regular part of the measure $\pi_\#|\mu|$ in its Lebesgue–Nikodým decomposition.

In what follows, x_0 is a fixed element in Ω where these three properties are satisfied. Applying condition (10.43) to the function Φ defined by $\Phi(x) := \tilde{\Phi}(\frac{x-x_0}{\rho})$, where $\tilde{\Phi}\in \mathbf{C}^1_c(B,\mathbf{R}^N)$, one obtains

$$-\frac{1}{\rho}\int_{B_\rho(x_0)} \operatorname{div}\,\tilde{\Phi}\left(\frac{x-x_0}{\rho}\right)\varphi(u)\,dx - \int_{B_\rho(x_0)} \tilde{\Phi}\left(\frac{x-x_0}{\rho}\right)\varphi'(u).a\,dx = \int_{\Omega\times\mathbf{R}}\varphi(s)\Phi(x)\,d\mu,$$

and the change of scale $y=\frac{x-x_0}{\rho}$ gives

$$-\int_B\Big(\operatorname{div}\,\tilde{\Phi}\;\varphi(u(x_0+\rho y)) + \rho\varphi'(u(x_0+\rho y))\tilde{\Phi}(y).a_\rho(y)\Big)dy = \frac{1}{\rho^{N-1}}\int_{\Omega\times\mathbf{R}}\varphi(s)\Phi(x)\,d\mu.$$

Testing this equality with the function φ defined by $\varphi(s) := \gamma(\frac{s-u(x_0)}{\rho})$, one obtains

$$-\int_B \Big(\operatorname{div} \tilde{\Phi}\, \gamma(u_\rho) + \gamma'(u_\rho)\tilde{\Phi}(y).a_\rho(y) \Big) dy = \frac{1}{\rho^{N-1}} \int_{\Omega\times\mathbf{R}} \varphi(s)\Phi(x)\, d\mu$$

and letting $\rho \to 0$,

$$\int_B \operatorname{div} \tilde{\Phi}\, \gamma(\nabla u(x_0).y) + \gamma'(\nabla u(x_0).y)\tilde{\Phi}(y).a(x_0)\, dy = 0.$$

Since

$$\begin{aligned}\int_B \Big(\operatorname{div} \tilde{\Phi}\, \gamma(\nabla u(x_0).y) + \gamma'(\nabla u(x_0).y)\tilde{\Phi}(y).\nabla u(x_0) \Big) dy &= \int_B \operatorname{div} \big(\gamma(\nabla u(x_0).y)\tilde{\Phi}\big) dy \\ &= 0,\end{aligned}$$

we deduce

$$\big(a(x_0) - \nabla u(x_0)\big). \int_B \gamma'(\nabla u(x_0).y)\tilde{\Phi}(y)\, dy = 0.$$

The choice of $\tilde{\Phi}$ and γ being arbitrary, we deduce $a(x_0) = \nabla u(x_0)$ and the proof of the first step is complete.

Second step. We establish $Cu = 0$. From Proposition 17.2.5, for all φ in $\mathbf{C}^1_0(\mathbf{R})$, $\varphi \circ u \in BV(\Omega)$ so that for all Φ in $\mathbf{C}^1_c(\Omega, \mathbf{R}^N)$, one has

$$\int_\Omega \Big(\operatorname{div} \Phi\, \varphi(u) + \varphi'(u)\Phi.\nabla u \Big) dx = -\int_{S_u} \big(\varphi(u^+) - \varphi(u^-)\big)\Phi.\nu_u\, d\mathcal{H}^{N-1} - \int_\Omega \varphi'(u)\Phi\, dCu,$$

and condition (10.43) yields

$$\int_\Omega \varphi'(u)\Phi\, dCu = \int_{\Omega\times\mathbf{R}} \varphi(s)\Phi(x)\, \mu(dx, ds) - \int_{S_u} \big(\varphi(u^+) - \varphi(u^-)\big)\Phi.\nu_u\, d\mathcal{H}^{N-1}. \quad (10.44)$$

We now focus on a careful analysis of the measure $\varphi'(u)\, Cu$. Let us apply the slicing Theorem 4.2.4 for the measure μ. Let τ_i denote the density of μ_i with respect to $|\mu_i|$, where $(\mu_i)_{1\le i\le N}$ is the family of components of the measure μ. According to Theorem 4.2.4 one has

$$\int_{\Omega\times\mathbf{R}} \varphi(s)\Phi_i(x)\, \mu_i(dx, ds) = \int_\Omega \Phi_i(x) \Big(\int_\mathbf{R} \varphi(s)\tau_i(x,s)\, \theta_x(ds) \Big)\, \sigma_i(dx),$$

where σ_i is the image of the measure $|\mu_i|$ by the projection of $\Omega \times \mathbf{R}$ on Ω and $(\theta_x)_{x\in\mathbf{R}}$ is a family of probability measures on $\mathbf{R}$. Then (10.44) yields, for $i = 1, \dots, N$,

$$\varphi'(u)C_i u = \Big(\int_\mathbf{R} \varphi(s)\tau_i(x,s)\, \theta_x(ds) \Big)\sigma_i - \big(\varphi(u^+) - \varphi(u^-)\big)\nu_{u,i}\, \mathcal{H}^{N-1}\lfloor S_u, \quad (10.45)$$

and finally, since Cu and $\mathcal{H}^{N-1}\lfloor S_u$ are mutually singular,

$$\varphi'(u)C_i u = \Big(\int_\mathbf{R} \varphi(s)\tau_i(.,s)\, \theta_.(ds) \Big)\lambda_i \quad (10.46)$$

with $\lambda_i := \sigma_i \lfloor_{\Omega \setminus S_u}$. To complete the proof, the idea is to express (10.46) in terms of functional identity. Consider the densities b and c of, respectively, $C_i u$ and λ_i with respect to the measure $\alpha := |C_i u| + \lambda_i$, equality (10.46) yields, for α a.e. x in Ω,

$$b(x)(\varphi' \circ u)(x) = c(x) \int_{\mathbf{R}} \varphi(s)\tau_i(x,s)\, \theta_x(ds).$$

Indeed, there exists a Borel set N_φ with $\mathscr{L}^N(N_\varphi) = 0$ such that above equality holds true for all x in $\Omega \setminus N_\varphi$. Since $\mathbf{C}_0^1(\Omega)$ possesses a dense countable subset D, it also holds for all x in $\Omega' = \Omega \setminus \bigcup_{\varphi \in D} N_\varphi$. Let then x in Ω' and assume that $b(x) \neq 0$. We deduce

$$\varphi' \circ u(x) = \frac{c(x)}{b(x)} \int_{\mathbf{R}} \varphi(s)\tau_i(x,s)\, \theta_x(ds) \quad \forall \varphi \in \mathbf{C}_0^1(\mathbf{R}).$$

Equality between the two linear forms

$$\varphi \mapsto (\varphi' \circ u)(x) \text{ and } \varphi \mapsto \frac{c(x)}{b(x)} \int_{\mathbf{R}} \varphi(s)\tau_i(x,s)\, \theta_x(ds)$$

brings a contradiction. (The first is not continuous in $\mathbf{C}_0^1(\mathbf{R})$.) Consequently, $b(x) = 0$ and $C_i u = b\alpha = 0$, which ends the proof of the second step.

Last step. It remains to establish $\mu = \Lambda_\#^+(\nu_u\, \mathscr{H}^{N-1}\lfloor S_u) - \Lambda_\#^-(\nu_u\, \mathscr{H}^{N-1}\lfloor S_u)$. According to the previous step, equality (10.45) now becomes

$$\left(\int_{\mathbf{R}} \varphi(s)\tau_i(x,s)\, \theta_x(ds)\right)\sigma_i = \big(\varphi(u^+) - \varphi(u^-)\big)\nu_{u,i}\, \mathscr{H}^{N-1}\lfloor S_u.$$

Let $\beta := \sigma_i + \mathscr{H}^{N-1}\lfloor S_u$ and let b, c denote now the densities of $\mathscr{H}^{N-1}\lfloor S_u$ and σ_i with respect to the measure β. We have for all $x \in \Omega \setminus N$ satisfying $\beta(N) = 0$, and for all $\varphi \in \mathbf{C}_0^1(\mathbf{R})$,

$$c(x) \int_{\mathbf{R}} \varphi(s)\tau_i(x,s)\, \theta_x(ds) = b(x)\big(\varphi(u^+) - \varphi(u^-)\big)\nu_{u,i}.$$

Thus, for all x in $\Omega \setminus N$

$$c(x)\tau_i(x,.)\, \theta_x = b(x)\nu_{u,i}(x)\big(\delta_{u^+(x)} - \delta_{u^-(x)}\big).$$

For all bounded Borel function $f : \Omega \times \mathbf{R} \longrightarrow \mathbf{R}$, we now obtain

$$\begin{aligned}
\int_{\Omega\times\mathbf{R}} f(x,s)\, d\mu_i(x,s) &= \int_\Omega \left(\int_{\mathbf{R}} f(x,s)\tau_i(x,s)\, \theta_x(ds)\right) \sigma_i(dx) \\
&= \int_\Omega \left(\int_{\mathbf{R}} f(x,s)\tau_i(x,s)\, \theta_x(ds)\right) c(x)\beta(dx) \\
&= \int_{S_u} \big(f(x,u^+(x)) - f(x,u^-(x))\big)\nu_{u,i}(x)\, d\mathscr{H}^{N-1}(x)
\end{aligned}$$

and the proof is complete. □

We now state, without proof, another criterion for a function u in $L^\infty(\Omega)$ to belong to $SBV(\Omega)$. This criterion concerns the restrictions of u to the one-dimensional slices of

Ω. Let us define for all $\nu \in S^{N-1}$

$$\begin{cases} \pi_\nu = \{x \in \mathbf{R}^N : x.\nu = 0\}, \\ \Omega_x = \{t \in \mathbf{R} : x + t\nu \in \Omega\}, \quad x \in \pi_\nu, \\ \Omega_\nu = \{x \in \pi_\nu : \Omega_x \neq \emptyset\}. \end{cases}$$

On the other hand, for all Borel functions $u : \Omega \to \mathbf{R}$ and x in Ω_ν, we define the Borel function u_x for all t in Ω_x by : $u_x(t) = u(x + t\nu)$. For a proof of the theorem below, consult Braides [122].

Theorem 10.5.2. *Let u be a given function in $L^\infty(\Omega)$ such that for all $\nu \in S^{N-1}$*

(i) $u_x \in SBV(\Omega_x)$ *for* $\mathscr{H}^{N-1}$ *a.e.* $x \in \Omega_\nu$;

(ii) $\displaystyle\int_{\Omega_\nu} \Big(\int_{\Omega_x} |\nabla u_x| \, dt + \mathscr{H}^0(S_{u_x}) \Big) \mathscr{H}^{N-1}(dx) < +\infty.$

Then u belongs to $SBV(\Omega)$. Conversely, if u belongs to $SBV(\Omega) \cap L^\infty(\Omega)$, conditions (i) *and* (ii) *are satisfied for all ν in S^{N-1}. Moreover, for $\mathscr{H}^{N-1}$ a.e. x in Ω_ν,*

$$\nabla u(x + t\nu).\nu = \nabla u_x(t)$$

and

$$\int_{\Omega_\nu} H_0(S_{u_x}) \, \mathscr{H}^{N-1}(dx) = \int_{S_u} |\nu_u \cdot \nu| \, d\mathscr{H}^{N-1}.$$