HUMBOLDT-UNIVERSITÄT ZU BERLIN

The Crouzeix-Raviart Finite Element Method for a Nonconforming Formulation of the Rudin-Osher-Fatemi Model Problem

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Let U be an open subset of $\mathbb{R}^d$. A function $v \in L^1(U)$ is a function of bounded variation iff

$$|v|_{\mathrm{BV}(U)} := \sup_{\substack{\phi \in C_C^1(U;\mathbb{R}^d) \\ \|\phi\|_{L^\infty(U)} \leqslant 1}} \int_U v \operatorname{div}(\phi)\, \mathrm{d}x < \infty.$$

The space of all such functions is denoted by $\mathrm{BV}(U)$.

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We have $W^{1,1}(\Omega) \subset \mathrm{BV}(\Omega)$ with $\|v\|_{\mathrm{BV}(\Omega)} = \|v\|_{W^{1,1}(\Omega)}$ for all $v \in W^{1,1}(\Omega)$.

Hedy Attouch, Giuseppe Buttazzo, and Gérard Michaille. **Variational Analysis in Sobolev and BV Spaces. Applications to PDEs and Optimization**. Second Edition. Vol. 17. MOS-SIAM Series on Optimization. Philadelphia: Society for Industrial and Applied Mathematics, Mathematical Optimization Society, 2014. ISBN: 978-1-611973-47-1

Lawrence C. Evans and Ronald F. Gariepy. **Measure Theory and Fine Properties of Functions**. CRC Press, 1992. ISBN: 0-8493-7157-0

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Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal Lipschitz domain.

Rudin-Osher-Fatemi (ROF) model problem

For a parameter $\alpha \in \mathbb{R}_+$ and an input signal $g \in L^2(\Omega)$ minimize the functional

$$I(v) := |v|_{\mathrm{BV}(\Omega)} + \frac{\alpha}{2}\|v - g\|^2_{L^2(\Omega)}$$

amongst all $v \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$.

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Original picture[0]

[0] https://homepages.cae.wisc.edu/~ece533/images/cameraman.tif

Original picture[0]

Input signal

The input signal was created by adding AWGN with a SNR of 20 to the original picture.

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$\alpha = 10^5$

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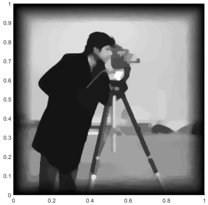

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Original picture

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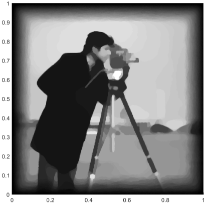

$\alpha = 10^3$

$\alpha = 10^4$

$\alpha = 10^5$

Pascal Getreuer. “Rudin-Osher-Fatemi Total Variation Denoising using Split Bregman”. In: **Image Processing On Line** 2 (2012), pp. 74–95. URL: https://doi.org/10.5201/ipol.2012.g-tvd

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Continuous problem

For a parameter $\alpha \in \mathbb{R}_+$ and an input signal $f \in L^2(\Omega)$ minimize the functional

$$E(v) := \frac{\alpha}{2}\|v\|^2 + |v|_{BV(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \int_\Omega fv \, \mathrm{d}x$$

amongst all $v \in BV(\Omega) \cap L^2(\Omega)$.

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amongst all $v \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$.

For $f = \alpha g$ the functional E has the same minimizers as

$$I(v) = |v|_{\mathrm{BV}(\Omega)} + \frac{\alpha}{2}\|v - g\|^2_{L^2(\Omega)}$$

in $\left\{v \in \mathrm{BV}(\Omega) \cap L^2(\Omega) \mid \|v\|_{L^1(\partial\Omega)} = 0\right\}$.

Theorem (Existence and uniqueness)

There exists a unique minimizer $u \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$ *for* $E = \frac{\alpha}{2}\|v\|^2 + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \int_\Omega fv \, \mathrm{d}x$ *in* $v \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$.

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Lemma

Let $v \in \mathrm{BV}(\Omega)$. *For all* $x \in \mathbb{R}^d$, *define*

$$\tilde{v}(x) := \begin{cases} v(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^d \backslash \overline{\Omega}. \end{cases}$$

Then $\tilde{v} \in \mathrm{BV}(\mathbb{R}^d)$ *and* $|\tilde{v}|_{\mathrm{BV}(\mathbb{R}^d)} = |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)}$.

Let U be an open subset of $\mathbb{R}^d$.

Definition (Weak convergence in $\mathrm{BV}(U)$)

Let $(v_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(U)$ and $v \in \mathrm{BV}(U)$ with $v_n \to v$ in $L^1(U)$ as $n \to \infty$. Then $(v_n)_{n\in\mathbb{N}}$ converges weakly to v in $\mathrm{BV}(U)$ iff, for all $\phi \in C_0(U;\mathbb{R}^d)$, it holds

$$\int_U v_n \operatorname{div}(\phi)\,\mathrm{d}x \to \int_U v \operatorname{div}(\phi)\,\mathrm{d}x \quad \text{as } n \to \infty.$$

We write $v_n \rightharpoonup v$ as $n \to \infty$.

Theorem

Let $v \in L^1(U)$ *and* $(v_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(U)$ *with* $\sup_{n\in\mathbb{N}} |v_n|_{\mathrm{BV}(U)} < \infty$ *and* $v_n \to v$ *in* $L^1(U)$ *as* $n \to \infty$. *Then* $v \in \mathrm{BV}(U)$ *and* $|v|_{\mathrm{BV}(U)} \leqslant \liminf_{n\to\infty} |v_n|_{\mathrm{BV}(U)}$. *Furthermore,* $v_n \rightharpoonup v$ *in* $\mathrm{BV}(U)$.

Theorem

Let $v \in L^1(U)$ and $(v_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(U)$ with $\sup_{n\in\mathbb{N}} |v_n|_{\mathrm{BV}(U)} < \infty$ and $v_n \to v$ in $L^1(U)$ as $n \to \infty$. Then $v \in \mathrm{BV}(U)$ and $|v|_{\mathrm{BV}(U)} \leqslant \liminf_{n\to\infty} |v_n|_{\mathrm{BV}(U)}$. Furthermore, $v_n \rightharpoonup v$ in $\mathrm{BV}(U)$.

Let U be a bounded Lipschitz domain.

Theorem ([EG92, S. 176, Theorem 4])

Let $(v_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(U)$ be bounded. Then there exists some subsequence $(v_{n_k})_{k\in\mathbb{N}}$ of $(v_n)_{n\in\mathbb{N}}$ and $v \in \mathrm{BV}(U)$ such that $v_{n_k} \to v$ in $L^1(U)$ as $k \to \infty$.

Stability

Let $f_1, f_2 \in L^2(\Omega)$. For $\ell \in \{1, 2\}$, let $u_\ell \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$ minimize

$$E_\ell := \frac{\alpha}{2}\|v\|^2 + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \int_\Omega f_\ell v \,\mathrm{d}x$$

amongst all $v \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$. Then

$$\|u_1 - u_2\| \leqslant \frac{1}{\alpha}\|f_1 - f_2\|.$$

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Sören Bartels. **Numerical Methods for Nonlinear Partial Differential Equations**. Vol. 47. Springer Series in Computational Mathematics. Springer International Publishing, 2015. ISBN: 978-3-319-13796-4. DOI: 10.1007/978-3-319-13797-1, Chapter 10.

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nochmal Existenz und Eindeutigkeitsaussagen zeigen und gaaaanz grob beschreiben, wie die bewiesen werden

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vielleicht in eigener Section: noch die Aussagen aus der Arbeit verarbeiten, die über Existenz und Eindeutigkeit hinausgehen

Sören Bartels. **Numerical Methods for Nonlinear Partial Differential Equations**. Vol. 47. Springer Series in Computational Mathematics. Springer International Publishing, 2015. ISBN: 978-3-319-13796-4. DOI: 10.1007/978-3-319-13797-1, Chapter 10, p. 297-319

Properties of BV(Ω)

Notions of convergence on $\mathrm{BV}(\Omega)$

Let $(v_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(\Omega)$ and $v \in \mathrm{BV}(\Omega)$ such that $v_n \to v$ in $L^1(\Omega)$ as $n \to \infty$.

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(i) $(v_n)_{n\in\mathbb{N}}$ converges intermediately or strictly to v if $|v_n|_{\mathrm{BV}(\Omega)} \to |v|_{\mathrm{BV}(\Omega)}$ as $n \to \infty$.

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(i) $(v_n)_{n\in\mathbb{N}}$ converges intermediately or strictly to v if $|v_n|_{\mathrm{BV}(\Omega)} \to |v|_{\mathrm{BV}(\Omega)}$ as $n \to \infty$.

(ii) $(v_n)_{n\in\mathbb{N}}$ converges weakly to v if $\langle Dv_n, \phi\rangle \to \langle Dv, \phi\rangle$ for all $\phi \in C_0(\Omega; \mathbb{R}^n)$ as $n \to \infty$.

Further Properties of $\mathrm{BV}(\Omega)$

$C^\infty(\overline{\Omega})$ and $C^\infty(\Omega) \cap \mathrm{BV}(\Omega)$ are dense in $\mathrm{BV}(\Omega)$ with respect to intermediate convergence.

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The embedding $\mathrm{BV}(\Omega) \to L^p(\Omega)$ is continuous for $1 \leqslant p \leqslant n/(n-1)$ and compact for $1 \leqslant p < n/(n-1)$.

There exists a linear operator $T : \mathrm{BV}(\Omega) \to L^1(\partial\Omega)$ such that $T(v) = v|_{\partial\Omega}$ for all $v \in \mathrm{BV}(\Omega) \cap C(\overline{\Omega})$.
T is continuous with respect to intermediate convergence in $\mathrm{BV}(\Omega)$ but not with respect to weak convergence in $\mathrm{BV}(\Omega)$.

Algorithmus und mglw LGS dazu, siehe Arbeit
wohl auch Konvergenztheorem aufführen, Bereich für τ kurz erläutern, vielleicht beim groben erläutern der Beweisidee

drüber nachdenken, was hier gezeigt werden soll. Idealerweise viele subsections mit Themenbereichen (f01, cam, termCrit, tau. . .)
termination criteria experiments only in the end if questions arise, only mention the possible termination criteria and that they seem equally valid (except for energy difference)
show tau experiments
energy during a iteration (convergence of subsequences from above, i.e. also choose one exampe with osscilating convergence)
find good alpha for denoising
show adaptive mesh for camerman and maybe for square to show the working of the refinement indicator

tien schicken spätestens am Wochenende vor der Präsi, CC vor der Präsi die fertige Präsi + akuteller Stand der Arbeit schicken