Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Chapter 8

Spectral analysis of the Laplacian

8.1 ▪ Introduction

From the very beginning of the 20th century, the study of the eigenvalue problem for the Laplace equation emerged as a fundamental topic in the theory of partial differential equations. In 1922, J. B. J. Fourier was faced with this question to develop the so-called separation of variables method. Let us illustrate this method in the case of the wave equation with Dirichlet boundary data, which, for example, gives a model for the vibrations of an elastic membrane which is clamped on its boundary.

Given initial data $u_0, u_1 : \Omega \to \mathbf{R}$, one looks for a solution $u : Q = \Omega \times (0, +\infty) \to \mathbf{R}$ of the boundary value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0 & \text{on } Q, \\ u = 0 & \text{on } \Sigma = \partial\Omega \times (0, +\infty), \\ u(x,0) = u_0(x) & \text{on } \Omega, \\ \dfrac{\partial u}{\partial t}(x,0) = u_1(x) & \text{on } \Omega. \end{cases}$$

The idea is to look for a solution u of the form

$$u(x,t) = w(x)\varphi(t), \tag{8.1}$$

where the dependence of u with respect to (x,t) has been separated. The wave equation then becomes

$$w(x)\varphi''(t) - \varphi(t)\Delta w(x) = 0$$

or, equivalently,

$$\frac{\varphi''(t)}{\varphi(t)} = \frac{\Delta w(x)}{w(x)}. \tag{8.2}$$

Since the left-hand side of (8.2) is a function only of t and the right-hand side only of x, this forces these two expressions to be constant, that is,

$$\frac{\varphi''(t)}{\varphi(t)} = \frac{\Delta w(x)}{w(x)} = -\lambda$$

for some constant λ.

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

This method leads to the study of the spectral problem for the so-called Laplace–Dirichlet operator,

$$\begin{cases} -\Delta w = \lambda w & \text{on } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.3}$$

and the resolution of the ordinary differential equation

$$\varphi''(t) + \lambda\varphi(t) = 0. \tag{8.4}$$

One can easily verify that the eigenvalues of the spectral problem (8.3) are positive. (It is enough to multiply by w and integrate by parts on Ω.) Therefore, the solutions of (8.4) are of the following form:

$$\varphi(t) = A\cos\sqrt{\lambda}t + B\sin\sqrt{\lambda}t.$$

The question is now, can one, by linear combinations of such separate solutions, obtain a solution

$$u(x,t) = \sum_i w_i(x)\Big(A\cos\sqrt{\lambda_i}\,t + B\sin\sqrt{\lambda_i}\,t\Big) \tag{8.5}$$

which satisfies the initial data $u(x,0) = u_0(x)$ and $(\partial u/\partial t)(x,t) = u_1(x)$?

Indeed, this question is intimately related to the possibility of generating any given function by linear combinations (indeed, series!) of eigenvectors of the Laplace–Dirichlet operator.

We shall give a positive answer to this question and prove the following theorem (which is the main result of this chapter). Assume Ω is a bounded open set in $\mathbf{R}^N$. Then, there exists a complete orthonormal system of eigenvectors of the Laplace–Dirichlet operator in the space $L^2(\Omega)$. A complete orthonormal system is also called a Hilbertian basis. This is a deep result of Rellich which is precisely based on the Rellich–Kondrakov theorem (compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$; see Theorem 5.3.3) and on the abstract spectral decomposition theorem for compact self-adjoint operators. Indeed, variational methods play a central role in the theory of eigenvalues of elliptic partial differential equations. Another striking result in this direction is the variational characterization of eigenvalues of the Laplace–Dirichlet operator. One of the formulas provided by the Courant–Fisher min-max principle is the following: the first eigenvalue of the Laplace–Dirichlet operator is given by the variational formula

$$\lambda_1(-\Delta) = \min\left\{ \int_\Omega |\nabla v(x)|^2\,dx \ :\ v \in H_0^1(\Omega), \int_\Omega v(x)^2\,dx = 1 \right\}.$$

This formula and its companions provide powerful tools for studying the properties of the eigenvalues and eigenvectors of the Laplace–Dirichlet operator. In 1911, Weyl used this principle to solve the problem on the asymptotic distribution of the eigenvalues of the Laplace–Dirichlet operator. We shall briefly describe such a result in Section 8.5.

In the last two decades, spectral methods, just like finite element methods, have proved to be very efficient in the numerical treatment of some partial differential operators. They give raise to approximation methods where the finite dimensional approximating spaces V_n are based on (orthogonal) polynomials of high degree (by contrast with finite element methods, where the degree is fixed, one for the P_1 method, two for the P_2 method, for example). Here, the degree of polynomials of V_n increases with n. These methods provide accurate approximations of the solution, which are limited only by the regularity of the solution. But on the counterpart, there are some restrictions on the geometry of Ω.

For pedagogical reasons and simplicity of exposition, we restrict our attention to the spectral analysis of the Laplace equation with Dirichlet boundary condition. In Section 8.6, we shall briefly survey some straight extensions of these results.

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

8.2 ▪ The Laplace–Dirichlet operator: Functional setting

Our objective is to study the eigenvalue problem for the Laplace equation on Ω with Dirichlet boundary conditions on $\partial\Omega$. We seek for $\lambda \in \mathbf{R}$ and $u \neq 0$ such that

$$\begin{cases} -\Delta u = \lambda u & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.6}$$

Throughout this chapter, Ω is assumed to be a bounded open set in $\mathbf{R}^N$.

Let us give a precise meaning to this definition and write its variational formulation.

Definition 8.2.1. *We say that $\lambda \in \mathbf{R}$ is an eigenvalue of the Laplace–Dirichlet operator if there exists some $u \in H_0^1(\Omega)$, $u \neq 0$, such that*

$$\begin{cases} \displaystyle\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx = \lambda \int_\Omega u(x) v(x)\, dx \quad \forall v \in H_0^1(\Omega), \\ u \in H_0^1(\Omega). \end{cases} \tag{8.7}$$

When such u exists it is called an eigenvector related to the eigenvalue λ.

Remark 8.2.1. Let us make some comments to the definition above:

(a) If (8.7) is satisfied, then

$$\begin{cases} -\Delta u = \lambda u & \text{in the distribution sense,} \\ u = 0 & \text{in the trace sense,} \end{cases}$$

i.e., (8.6) is satisfied in a weak sense. The next step consists in proving that u is regular and hence it is a classical solution of (8.6).

(b) One may wonder whether "$\lambda \in \mathbf{R}$" is not too restrictive, and take instead $\lambda \in \mathbf{C}$. Indeed, by taking $v = u$ in (8.7) one obtains

$$\lambda = \frac{\displaystyle\int_\Omega |\nabla u|^2\, dx}{\displaystyle\int_\Omega u^2(x)\, dx},$$

which implies that all the eigenvalues of the Laplace–Dirichlet operator are positive real numbers.

Let us now come to the central idea which will permit us to formulate the above problem in terms of classical operator theory.

The classical theory for spectral analysis of operators in infinite dimensional spaces works with operators

$$T : H \longrightarrow H,$$

where H is a Hilbert space and $T \in L(H)$ is a linear, continuous, and compact operator from H into H.

One cannot write $-\Delta$ in such a setting because, as for any differential operator, there is a loss of regularity when passing from u to $-\Delta u$. Indeed, $-\Delta$ is a linear continuous operator

$$-\Delta : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega).$$

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Another way to treat $-\Delta$ is to consider it as an operator from $L^2(\Omega)$ into $L^2(\Omega)$, but with a domain, i.e.,

$$\mathrm{dom}(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega).$$

By contrast, and that is the central idea, the inverse operator $T = (-\Delta)^{-1}$ is a nice operator which fits well with the classical theory. The straight relation which connects the spectrum of an operator and the spectrum of its inverse operator will permit us to conclude our analysis.

Let us now define the operator T as the inverse of the Laplace–Dirichlet operator.

Definition 8.2.2. *The inverse of the Laplace–Dirichlet operator is the operator $T : L^2(\Omega) \longrightarrow L^2(\Omega)$ which is defined for every $h \in L^2(\Omega)$ by the following: $Th \in H_0^1(\Omega) \subset L^2(\Omega)$ is the unique solution of the variational problem*

$$\begin{cases} \displaystyle\int_\Omega \nabla(Th)(x) \cdot \nabla v(x)\, dx = \int_\Omega h(x) v(x)\, dx \quad \forall v \in H_0^1(\Omega), \\ Th \in H_0^1(\Omega). \end{cases}$$

Equivalently, Th is the variational solution of the Dirichlet problem (see Theorem 5.1.1)

$$\begin{cases} -\Delta(Th) = h \quad \text{on } \Omega, \\ Th = 0 \quad \text{on } \partial\Omega. \end{cases}$$

At this point let us notice that

$$T : L^2(\Omega) \longrightarrow H_0^1(\Omega).$$

To consider T as a linear continuous operator from a space H into itself we have two possibilities: either

$$T : L^2(\Omega) \longrightarrow L^2(\Omega)$$

or

$$T : H_0^1(\Omega) \longrightarrow H_0^1(\Omega).$$

These two approaches lead to similar parallel developments. We choose to consider T as acting from $L^2(\Omega)$ into $L^2(\Omega)$. We then have

$$(-\Delta) \circ T = id_H, \qquad H = L^2(\Omega),$$

i.e., T is the right inverse of $-\Delta$.

The introduction of $T = (-\Delta)^{-1}$ is justified in the context of the spectral analysis of the Laplace–Dirichlet operator by the following result.

Lemma 8.2.1. *The real number λ is an eigenvalue of the Laplace–Dirichlet operator iff $1/\lambda$ is an eigenvalue of $T = (-\Delta)^{-1}$.*

PROOF. Let us assume that λ is an eigenvalue of the Laplace–Dirichlet operator, i.e., there exists some $u \in H_0^1(\Omega)$, $u \neq 0$, such that

$$\int_\Omega \nabla u \cdot \nabla v\, dx = \lambda \int_\Omega uv\, dx \quad \forall v \in H_0^1(\Omega).$$

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

By definition of $T = (-\Delta)^{-1}$ this is equivalent to

$$u = T(\lambda u).$$

By linearity of T (this is proved in the next proposition), and using the fact that $\lambda \neq 0$, we deduce

$$T(u) = \frac{1}{\lambda} u,$$

i.e., $1/\lambda$ is an eigenvalue of T. □

The following properties of the operator T will play a central role in its spectral analysis.

Proposition 8.2.1. *The operator T satisfies the following properties:*

(i) $T : L^2(\Omega) \longrightarrow L^2(\Omega)$ *is a linear continuous operator,*

(ii) T *is self-adjoint in* $L^2(\Omega)$*,*

(iii) T *is compact from* $L^2(\Omega)$ *into* $L^2(\Omega)$*,*

(iv) T *is positive definite.*

PROOF. $(\mathrm{i})_1$ Take $h_1,\ h_2 \in L^2(\Omega)$ and $\alpha_1, \alpha_2 \in \mathbf{R}$. By definition of T

$$\int_\Omega \nabla(Th_1) \cdot \nabla v \, dx = \int_\Omega h_1 v \, dx \quad \forall v \in H_0^1(\Omega),$$

$$\int_\Omega \nabla(Th_2) \cdot \nabla v \, dx = \int_\Omega h_2 v \, dx \quad \forall v \in H_0^1(\Omega).$$

By taking a linear combination of these two equalities we obtain

$$\begin{cases} \displaystyle\int_\Omega \nabla(\alpha_1 Th_1 + \alpha_2 Th_2) \cdot \nabla v \, dx = \int_\Omega (\alpha_1 h_1 + \alpha_2 h_2) v \, dx \quad \forall v \in H_0^1(\Omega), \\ \alpha_1 Th_1 + \alpha_2 Th_2 \in H_0^1(\Omega). \end{cases}$$

By uniqueness of the solution of the Dirichlet problem, we get

$$T(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 Th_1 + \alpha_2 Th_2,$$

which is the linearity of T.

$(\mathrm{i})_2$ Let us now prove that $T : L^2(\Omega) \longrightarrow L^2(\Omega)$ is continuous. By definition of T, the equality

$$\int_\Omega \nabla(Th) \cdot \nabla v \, dx = \int_\Omega hv \, dx$$

holds true for any $v \in H_0^1(\Omega)$. In particular, it is satisfied by $v = Th \in H_0^1(\Omega)$, which gives

$$\int_\Omega |\nabla(Th)|^2 \, dx = \int_\Omega hT(h) \, dx. \tag{8.8}$$

Using the Cauchy–Schwarz inequality, we obtain

$$\int_\Omega |\nabla(Th)|^2 \, dx \le \left(\int_\Omega h^2 \, dx \right)^{1/2} \left(\int_\Omega (Th)^2 \, dx \right)^{1/2}. \tag{8.9}$$

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Let us now use the Poincaré inequality and the fact that Ω is bounded (Theorem 5.3.1). There exists a constant $C > 0$ which depends only on Ω such that

$$\forall v \in H_0^1(\Omega) \quad \left(\int_\Omega v(x)^2\,dx\right)^{1/2} \le C\left(\int_\Omega |\nabla v(x)|^2\,dx\right)^{1/2}.$$

In particular, since for every $h \in L^2(\Omega)$ we have $Th \in H_0^1(\Omega)$, we can write the Poincaré inequality with $v = Th$, which gives

$$\left(\int_\Omega |Th(x)|^2\,dx\right)^{1/2} \le C\left(\int_\Omega |\nabla(Th(x))|^2\,dx\right)^{1/2}. \tag{8.10}$$

Let us combine inequalities (8.9) and (8.10) to obtain

$$\int_\Omega |\nabla(Th)|^2\,dx \le C\left(\int_\Omega h^2\,dx\right)^{1/2}\left(\int_\Omega |\nabla(Th)|^2\,dx\right)^{1/2}.$$

Equivalently

$$\left(\int_\Omega |\nabla(Th)|^2\,dx\right)^{1/2} \le C\left(\int_\Omega h^2\,dx\right)^{1/2}, \tag{8.11}$$

which, together with (8.10), gives

$$\left(\int_\Omega |Th(x)|^2\,dx\right)^{1/2} \le C^2\left(\int_\Omega h^2\,dx\right)^{1/2}. \tag{8.12}$$

Thus, we have obtained

$$\forall h \in L^2(\Omega) \quad ||Th||_{L^2(\Omega)} \le C^2||h||_{L^2(\Omega)}, \tag{8.13}$$

i.e., T is a linear continuous operator from $L^2(\Omega)$ into $L^2(\Omega)$. Indeed, we have obtained a sharper result: from (8.11) and (8.12) we deduce that

$$\forall h \in L^2(\Omega) \qquad ||Th||_{H_0^1(\Omega)} \le C\sqrt{1+C^2}||h||_{L^2(\Omega)}, \tag{8.14}$$

i.e., $T : L^2(\Omega) \longrightarrow H_0^1(\Omega)$ is a linear continuous operator. This proves that one can also treat T as a linear continuous operator from $H_0^1(\Omega)$ into $H_0^1(\Omega)$.

(ii) Let us now prove that T is a self-adjoint operator in $L^2(\Omega)$, i.e.,

$$\forall g, h \in L^2(\Omega) \quad \langle Th, g\rangle_{L^2(\Omega)} = \langle h, Tg\rangle_{L^2(\Omega)},$$

which means

$$\int_\Omega (Th)(x)g(x)\,dx = \int_\Omega h(x)(Tg)(x)\,dx.$$

By definition of Th and Tg we have

$$\int_\Omega \nabla(Th)\cdot\nabla v\,dx = \int_\Omega hv\,dx \quad \forall v \in H_0^1(\Omega),$$

$$\int_\Omega \nabla(Tg)\cdot\nabla v\,dx = \int_\Omega gv\,dx \quad \forall v \in H_0^1(\Omega).$$

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Take $v = Tg \in H_0^1(\Omega)$ in the first equality and $v = Th \in H_0^1(\Omega)$ in the second equality. We obtain

$$\int_\Omega \nabla(Th)\cdot\nabla(Tg)\,dx = \int_\Omega hT(g)\,dx = \langle h, Tg\rangle_{L^2(\Omega)},$$
$$\int_\Omega \nabla(Tg)\cdot\nabla(Th)\,dx = \int_\Omega gT(h)\,dx = \langle g, Th\rangle_{L^2(\Omega)}.$$

Hence

$$\langle h, Tg\rangle_{L^2(\Omega)} = \langle g, Th\rangle_{L^2(\Omega)} = \int_\Omega \nabla(Th)\cdot\nabla(Tg)\,dx,$$

which shows that T is self-adjoint in $L^2(\Omega)$.

(iii) T is compact from $L^2(\Omega)$ into $L^2(\Omega)$. Take B a bounded set in $L^2(\Omega)$. By (8.14), since T is linear and continuous from $L^2(\Omega)$ into $H_0^1(\Omega)$, the set $T(B)$ is bounded in $H_0^1(\Omega)$. We now use the fact that Ω is bounded and the Rellich–Kondrakov theorem, Theorem 5.3.3, to conclude that $T(B)$ is relatively compact in $L^2(\Omega)$.

(iv) T is positive definite. By (8.8) we have

$$\forall v \in L^2(\Omega) \quad \langle Th, h\rangle = \int_\Omega |\nabla(Th)|^2\,dx \ge 0,$$

that is, T is positive. Moreover, if $\langle Th, h\rangle = 0$, then $\nabla(Th) = 0$, that is, Th is locally constant. Since $Th \in H_0^1(\Omega)$, this forces Th to be equal to zero. Coming back to the definition of Th, $Th = 0$ means that $\int_\Omega hv\,dx = 0$ for all $v \in H_0^1(\Omega)$. By the density of $H_0^1(\Omega)$ into $L^2(\Omega)$, we conclude that $h = 0$. □

We can summarize the results of this section and say that the operator $T = (-\Delta)^{-1}$ is a linear continuous, self-adjoint, compact, positive operator from $L^2(\Omega)$ into $L^2(\Omega)$. This will allow us to obtain in the next section the spectral decomposition of the Laplace–Dirichlet operator.

8.3 ▪ Existence of a Hilbertian basis of eigenvectors of the Laplace–Dirichlet operator

Let us first recall the well-known (see, for instance, Brezis [137]) abstract “diagonalization” theorem for compact self-adjoint positive definite operators.

Theorem 8.3.1. *Let us assume that H is a separable Hilbert space with* $\dim H = +\infty$. *Let $T : H \longrightarrow H$ be a linear continuous self-adjoint compact and positive definite operator. Then we have the following:*

(i) *T is diagonalizable: there exists a Hilbertian basis of eigenvectors of T.*

(ii) *The set $\Lambda(T)$ of eigenvalues of T is countable. It can be written as a sequence $(\mu_n)_{n\in\mathbf{N}}$ of positive distinct real numbers that decreases to zero as $n \to +\infty$*

$$0 \leftarrow \mu_n < \cdots < \mu_3 < \mu_2 < \mu_1.$$

(iii) *For each $\mu_n \in \Lambda(T)$, $E_{\mu_n} = \ker(T - \mu_n I)$ is a finite dimensional subspace of H: it is the eigensubspace relative to the eigenvalue μ_n. Its dimension is called the multiplicity of μ_n.*

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

(iv) *For all* $\mu_i \neq \mu_j$, $\mu_i, \mu_j \in \Lambda(T)$, $E_{\mu_i} \perp E_{\mu_j}$ *(orthogonal subspaces).*

(v) $H = \oplus_{n\in\mathbf{N}} E_{\mu_n}$, *i.e.,*

$$\forall x \in H \qquad x = \sum_{n\in\mathbf{N}} proj_{E_{\mu_n}}(x)$$

and

$$\forall x \in H \qquad Tx = \sum_{n\in\mathbf{N}} \mu_n proj_{E_{\mu_n}}(x).$$

Remark 8.3.1. The situation described in the above statement is simplified by the fact that here we have assumed T be positive definite, i.e., $\ker T = 0$, which allows us to avoid considering $\mu = 0$ in the spectral decomposition. (The null space $\ker T$ may be infinite dimensional.)

SKETCH OF THE PROOF OF THEOREM 8.3.1. It is worthwhile to recall some of the basic ingredients of the proof of Theorem 8.3.1.

(a) First notice that $\Lambda(T)$ is a bounded subset of $(0,+\infty)$: if $\mu \in \Lambda(T)$, then there exists some $u \in H$, $u \neq 0$ such that

$$Tu = \mu u.$$

Hence

$$\langle Tu, u\rangle = \mu |u|^2.$$

Since $u \neq 0$ we have $\langle Tu, u\rangle > 0$ (T is positive definite), which forces μ to be positive. On the other hand, since T is linear continuous,

$$\mu|u|^2 \leq |Tu|_H |u|_H \leq ||T||_{L(H,H)} |u|^2,$$

which gives

$$0 < \mu \leq ||T||_{L(H,H)}.$$

(b) Now take $\mu \neq \nu$ with $\mu, \nu \in \Lambda(T)$. By definition of $\Lambda(T)$ and E_μ, E_ν we have

$$\begin{aligned} &\forall h \in E_\mu \quad Th = \mu h,\\ &\forall k \in E_\nu \quad Tk = \nu k. \end{aligned}$$

We deduce

$$\begin{aligned} \langle Th, k\rangle &= \mu\langle h, k\rangle,\\ \langle Tk, h\rangle &= \nu\langle k, h\rangle. \end{aligned}$$

Since T is self-adjoint, we have $\langle Th, k\rangle = \langle h, Tk\rangle$. Hence

$$(\mu - \nu)\langle h, k\rangle = 0.$$

We have assumed $\mu \neq \nu$. This forces h and k to satisfy $\langle h, k\rangle = 0$, that is, $E_\mu \perp E_\nu$.

(c) It is interesting to see where the compactness on T comes into play. Let us notice that for any $\mu \in \Lambda(T)$ the subspace E_μ is closed and invariant by T; if $h \in E_\mu$, i.e., $Th = \mu h$, then $T(Th) = \mu(Th)$, i.e., $Th \in E_\mu$. Hence $T : E_\mu \longrightarrow E_\mu$ and E_μ is a Hilbert space for the induced structure of H. Moreover, if B_{E_μ} denotes the unit ball in E_μ, we have

$$T(B_{E_\mu}) = \mu B_{E_\mu}.$$

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Since $\mu \neq 0$ and T is compact, this forces B_{E_μ} to be relatively compact, that is, $\dim E_\mu < +\infty$. □

We have obtained the following decomposition of H as a Hilbertian sum of eigenspaces:

$$H = \bigoplus_{n \in \mathbf{N}} E_{\mu_n}. \tag{8.15}$$

To derive from this formula a Hilbertian basis of H we have to pick up in each E_{μ_n} an orthonormal basis whose cardinal is equal to the (finite) multiplicity of μ_n.

To keep a quite simple notation we adopt the following convention.

Definition 8.3.1. *We now decide to count the eigenvalues of T according to their multiplicity, that is,*

μ_1 is repeated k_1 times where k_1 is the multiplicity of μ_1

...

μ_n is repeated k_n times where k_n is the multiplicity of μ_n

...

and so on. Clearly, in this way, we obtain a sequence of positive real numbers, which we still denote by $(\mu_n)_{n\in\mathbf{N}}$, such that

$$0 \leftarrow \mu_n \leq \cdots \mu_3 \leq \mu_2 \leq \mu_1.$$

Note that now the μ_i are not necessarily distinct.

The convention above allows us to pick up an orthonormal basis in each finite dimensional eigensubspace to obtain a Hilbertian basis $(h_n)_{n\in\mathbf{N}}$ in H which satisfies

$$Th_n = \mu_n h_n$$

for every $n \in \mathbf{N}$.

Let us now come back to our model example. Clearly, by Proposition 8.2.1, the operator $T = (-\Delta)^{-1}$ which is considered as acting from $H = L^2(\Omega)$ into $L^2(\Omega)$ satisfies all the conditions of the abstract diagonalization theorem, Theorem 8.3.1. Thus, there exists a Hilbertian basis $(e_n)_{n\in\mathbf{N}}$ of $L^2(\Omega)$ such that for each $n \in \mathbf{N}$ e_n is an eigenvector of T. More precisely,

$$Te_n = \mu_n e_n,$$

and $(\mu_n)_{n\in\mathbf{N}}$ is a sequence of positive numbers which decreases to zero. By Lemma 8.2.1, we deduce that $1/\mu_n$ is an eigenvalue of the Laplace–Dirichlet operator and that e_n is a corresponding eigenvector. This means

$$\begin{cases} -\Delta e_n = \frac{1}{\mu_n} e_n & \text{on } \Omega, \\ e_n = 0 \quad \text{on } \partial\Omega, \end{cases}$$

the solution e_n being taken in the variational sense, i.e., $e_n \in H_0^1(\Omega)$ and

$$\int_\Omega \nabla e_n \cdot \nabla v \, dx = \frac{1}{\mu_n} \int_\Omega e_n v \, dx \quad \forall v \in H_0^1(\Omega).$$

Indeed, it is immediate to verify that the above equality is equivalent to $T(e_n/\mu_n) = e_n$, that is, $T(e_n) = \mu_n e_n$.

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Let us now forget the operator $T = (-\Delta)^{-1}$ which was just a technical ingredient in our study and convert the previous results directly in terms of $-\Delta$, the Laplace–Dirichlet operator. Noticing that the sequence $(\lambda_n)_{n\in\mathbf{N}}$ with $\lambda_n = 1/\mu_n$ is now an increasing sequence of positive numbers which tends to $+\infty$ as $n \to +\infty$, we obtain the following theorem.

Theorem 8.3.2. *The Laplace–Dirichlet operator has a countable family of eigenvalues $(\lambda_n)_{n\in\mathbf{N}}$ which can be written as an increasing sequence of positive numbers which tends to $+\infty$ as $n \to +\infty$:*

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots.$$

Each eigenvalue is repeated a number of times equal to its multiplicity (which is finite).

There exists a Hilbertian basis $(e_n)_{n\in\mathbf{N}}$ of $L^2(\Omega)$ such that for each $n \in \mathbf{N}$, e_n is an eigenvector of the Laplace–Dirichlet operator relatively to the eigenvalue λ_n:

$$\begin{cases} -\Delta e_n = \lambda_n e_n & \text{on } \Omega, \\ e_n = 0 & \text{on } \partial\Omega. \end{cases}$$

We already mentioned that the spectral analysis of the Laplace–Dirichlet operator could have been, as well, developed in the space $H_0^1(\Omega)$. Indeed, there is a direct link between these two approaches, which is described below.

Proposition 8.3.1. *The family $\left(e_n/\sqrt{\lambda_n}\right)_{n\in\mathbf{N}}$ is a Hilbertian basis of the space $H_0^1(\Omega)$ equipped with the scalar product $\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v\, dx$.*

PROOF. (a) Let us start from the definition of e_n:

$$\int_\Omega \nabla e_n \cdot \nabla v\, dx = \lambda_n \int_\Omega e_n v\, dx \quad \forall v \in H_0^1(\Omega).$$

By taking $v = e_n \in H_0^1(\Omega)$, we obtain

$$\int_\Omega |\nabla e_n|^2\, dx = \lambda_n \int_\Omega e_n^2\, dx = \lambda_n.$$

Hence

$$\|e_n\|^2_{H_0^1(\Omega)} = \lambda_n$$

and

$$\left\| \frac{e_n}{\sqrt{\lambda_n}} \right\|_{H_0^1(\Omega)} = 1.$$

(b) Let us verify the orthogonality property in $H_0^1(\Omega)$: take $n \ne m$,

$$\begin{aligned}\left\langle \frac{e_n}{\sqrt{\lambda_n}}, \frac{e_m}{\sqrt{\lambda_m}} \right\rangle_{H_0^1(\Omega)} &= \frac{1}{\sqrt{\lambda_n\lambda_m}} \int_\Omega \nabla e_n \cdot \nabla e_m\, dx \\ &= \frac{1}{\sqrt{\lambda_n\lambda_m}} \lambda_n \int_\Omega e_n e_m\, dx,\end{aligned}$$

which is equal to zero because $(e_n)_{n\in\mathbf{N}}$ is an orthogonal system in $L^2(\Omega)$.

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

(c) Let us verify that $\left(e_n/\sqrt{\lambda_n}\right)_{n\in\mathbf{N}}$ generates a vector space which is dense in $H_0^1(\Omega)$. Equivalently, we have to verify that if $f \in H_0^1(\Omega)$ is such that for all $n \in \mathbf{N}$,

$$\left\langle f, \frac{e_n}{\sqrt{\lambda_n}} \right\rangle_{H_0^1(\Omega)} = 0,$$

then $f = 0$. Let us notice that

$$\begin{aligned} \left\langle f, \frac{e_n}{\sqrt{\lambda_n}} \right\rangle_{H_0^1(\Omega)} &= \frac{1}{\sqrt{\lambda_n}} \int_\Omega \nabla f \cdot \nabla e_n \, dx \\ &= \sqrt{\lambda_n} \int_\Omega f e_n \, dx. \end{aligned}$$

Since $\lambda_n \neq 0$, our assumption becomes $\int_\Omega f e_n \, dx = 0$ for all $n \in \mathbf{N}$, which clearly implies $f = 0$ because $(e_n)_{n\in\mathbf{N}}$ is an orthonormal basis in $L^2(\Omega)$. □

8.4 ▪ The Courant–Fisher min-max and max-min formulas

Let us start with the variational characterization of the first eigenvalue $\lambda_1(-\Delta)$ of the Laplace–Dirichlet operator. Without ambiguity, we write λ_1. To introduce this result, let us notice that if λ is an eigenvalue of $-\Delta$, then there exists some $u \in H_0^1(\Omega)$, $u \neq 0$, such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \lambda \int_\Omega u v \, dx \quad \forall v \in H_0^1(\Omega).$$

By taking $v = u$ and noticing that $u \neq 0$ we obtain

$$\lambda = \frac{\displaystyle\int_\Omega |\nabla u(x)|^2 \, dx}{\displaystyle\int_\Omega u(x)^2 \, dx}. \tag{8.16}$$

The above expression plays a central role in the variational approach to eigenvalue problems for the Laplace equation.

Definition 8.4.1. *For any $v \in H_0^1(\Omega)$, $v \neq 0$, let us write*

$$\mathscr{R}(v) = \frac{\displaystyle\int_\Omega |\nabla v(x)|^2 \, dx}{\displaystyle\int_\Omega v(x)^2 \, dx}.$$

$\mathscr{R} : H_0^1(\Omega) \longrightarrow \mathbf{R}^+$ *is called the Rayleigh quotient.*

From (8.16) we immediately obtain that for any eigenvalue λ of the Laplace–Dirichlet operator,

$$\lambda \geq \inf\left\{\mathscr{R}(v) \,:\, v \in H_0^1(\Omega),\ v \neq 0\right\},$$

which is equivalent to saying

$$\lambda_1 \geq \inf\left\{\mathscr{R}(v) \,:\, v \in H_0^1(\Omega),\ v \neq 0\right\}. \tag{8.17}$$

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Indeed, there is equality between these two expressions. That is the object of the following theorem.

Theorem 8.4.1 (Courant–Fisher formula). *Assume Ω is a bounded open subset of $\mathbf{R}^N$. The first eigenvalue λ_1 of the Laplace–Dirichlet operator on Ω is given by the following variational formula:*

$$\lambda_1 = \min \left\{ \frac{\displaystyle\int_\Omega |\nabla v(x)|^2 \, dx}{\displaystyle\int_\Omega v^2(x) \, dx} \; : \; v \in H_0^1(\Omega), \; v \neq 0 \right\}.$$

Moreover, the infimum above is achieved and the solutions of this variational problem are the eigenvectors relative to the first eigenvalue λ_1.

PROOF. By (8.17) we need only to prove the inequality

$$\inf\left\{\mathscr{R}(v) \; : \; v \in H_0^1(\Omega), \; v \neq 0\right\} \geq \lambda_1,$$

that is,

$$\forall v \in H_0^1(\Omega), \; v \neq 0, \quad \mathscr{R}(v) \geq \lambda_1. \tag{8.18}$$

The idea is to express, for any $v \in H_0^1(\Omega)$, $\mathscr{R}(v)$ in a Hilbertian basis of eigenvectors of $(-\Delta)$. Indeed, we know by Theorem 8.3.2 and Proposition 8.3.1 that there exists a Hilbertian basis $(e_n)_{n \in \mathbf{N}}$ of $L^2(\Omega)$ such that e_n is an eigenvector of $-\Delta$ relatively to the eigenvalue λ_n and that $\left(e_n/\sqrt{\lambda_n}\right)$ is a corresponding Hilbertian basis in $H_0^1(\Omega)$.

By using the Bessel–Parseval inequality respectively in $H_0^1(\Omega)$ and $L^2(\Omega)$, we have

$$||v||^2_{H_0^1(\Omega)} = \int_\Omega |\nabla v(x)|^2 \, dx = \sum_{n=1}^{+\infty} \left\langle v, \frac{e_n}{\sqrt{\lambda_n}} \right\rangle^2_{H_0^1(\Omega)}, \tag{8.19}$$

$$||v||^2_{L^2(\Omega)} = \int_\Omega v(x)^2 \, dx = \sum_{n=1}^{+\infty} \langle v, e_n \rangle^2_{L^2(\Omega)}. \tag{8.20}$$

One can easily compare these two quantities by using that e_n is an eigenvalue of $-\Delta$ relatively to λ_n. Indeed, (8.19) gives

$$\begin{aligned} ||v||^2_{H_0^1(\Omega)} &= \sum_{n=1}^{+\infty} \frac{1}{\lambda_n} \Big(\int_\Omega \nabla v \cdot \nabla e_n \, dx \Big)^2 \\ &= \sum_{n=1}^{+\infty} \frac{\lambda_n^2}{\lambda_n} \Big(\int_\Omega e_n v \, dx \Big)^2 = \sum_{n=1}^{+\infty} \lambda_n \langle v, e_n \rangle^2_{L^2(\Omega)} \\ &\geq \lambda_1 \sum_{n=1}^{+\infty} \langle v, e_n \rangle^2_{L^2(\Omega)} = \lambda_1 ||v||^2_{L^2(\Omega)}, \end{aligned}$$

where in the last equality we use (8.20). Hence, $\mathscr{R}(v) \geq \lambda_1$ for any $v \in H_0^1(\Omega)$, $v \neq 0$, which concludes the first part of the theorem.

Let us now notice that by (8.16), any eigenvector v relative to the eigenvalue λ_1 is a solution of the Courant–Fisher minimization problem. Conversely, if v is such a solution, by the above Parseval equalities (8.19), (8.20) we must have

$$\mathscr{R}(v) = \frac{||v||^2_{H_0^1(\Omega)}}{||v||^2_{L^2(\Omega)}} = \frac{\sum_{n=1}^{+\infty} \lambda_n \langle v, e_n \rangle^2}{\sum_{n=1}^{+\infty} \langle v, e_n \rangle^2} = \lambda_1, \tag{8.21}$$

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

i.e.,

$$\sum_{n=1}^{+\infty}(\lambda_n-\lambda_1)\langle v,e_n\rangle^2=0.$$

This forces $\langle v,e_n\rangle$ to be equal to zero for all indices $n\in\mathbf{N}^*$ such that $\lambda_n\neq\lambda_1$. Equivalently, v must belong to the eigenspace relative to λ_1. $\square$

Another elegant and direct approach to the Courant–Fisher formula relies on the theory of Lagrange multipliers. Let us start with the following equivalent formulation of the Courant–Fisher variational formula for λ_1:

$$\inf\left\{\int_\Omega|\nabla v(x)|^2\,dx\ :\ v\in H_0^1(\Omega),\ \int_\Omega v(x)^2\,dx=1\right\}. \tag{$\mathscr{P}$}$$

Problem $(\mathscr{P})$ is now seen as a constrained minimization problem, namely, the minimization of the Dirichlet energy functional on the unit sphere of $L^2(\Omega)$. That is where Lagrange multipliers naturally come into play!

Let us first notice that by using the direct methods of the calculus of variations and the Rellich–Kondrakov theorem, one can easily prove that problem $(\mathscr{P})$ admits a solution u. To write some optimality condition satisfied by such solution u we take $V=H_0^1(\Omega)$ with $\langle u,v\rangle=\int_\Omega\nabla u\cdot\nabla v\,dx$ and $||v||^2=\langle v,v\rangle$, and we consider the functionals

$$F(v)=\int_\Omega|\nabla v(x)|^2\,dx=||v||^2,$$
$$G(v)=\int_\Omega v^2(x)\,dx.$$

Then $(\mathscr{P})$ is equivalent to

$$\min\left\{F(v)\ :\ G(v)=1,\ v\in V\right\}.$$

The sphere $S=\left\{v\in V\ :\ G(v)=1\right\}$ is a submanifold of class $\mathbf{C}^1$ and codimension 1 in V. Indeed, for any u,v we have, as $t\to 0$,

$$\frac{1}{t}\big[G(u+tv)-G(u)\big]\longrightarrow 2\int_\Omega uv\,dx.$$

Let us interpret this limit as a linear continuous form on $H_0^1(\Omega)$:

$$\int_\Omega uv\,dx=\langle h,v\rangle_{H_0^1(\Omega)}=\int_\Omega\nabla h\cdot\nabla v\,dx\qquad\forall v\in H_0^1(\Omega)$$

means

$$\begin{cases}-\Delta h=u & \text{on }\Omega,\\ h=0 & \text{on }\partial\Omega,\end{cases}$$

i.e., $h=Tu$, where $T=(-\Delta)^{-1}$. Hence G is Fréchet differentiable on V and $\nabla G(u)=2Tu$.

The theory of Lagrange multipliers applies in our situation and we have that there exists $\mu\in\mathbf{R}$ such that

$$\nabla F(u)=\mu\nabla G(u),$$

that is,

$$u=\mu Tu.$$

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Equivalently,

$$\begin{cases} -\Delta u = \mu u & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which implies

$$\mu = \mathscr{R}(u) = \frac{\displaystyle\int_\Omega |\nabla u(x)|^2\,dx}{\displaystyle\int_\Omega u(x)^2\,dx} = \lambda_1.$$

Let us summarize the previous results in the following statement.

Proposition 8.4.1. *The first eigenvalue $\lambda_1(-\Delta)$ of the Laplace–Dirichlet operator is a Lagrange multiplier of the constrained minimization problem*

$$\min\left\{ \int_\Omega |\nabla v(x)|^2\,dx \;:\; v \in H_0^1(\Omega),\ \int_\Omega v^2(x)\,dx = 1 \right\}.$$

The above approach provides a direct variational proof of the existence of an eigenvalue (indeed, the first $\lambda_1(-\Delta)$) of the Laplace–Dirichlet operator. The whole theory can then be developed in this way, by using a recursive argument: in the next step we can apply the same argument in the orthogonal subspace of V_1 (which is the eigenspace relative to λ_1) and so on. Let us make this precise in the following statement.

Proposition 8.4.2. *Let $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$ be the sequence of the eigenvalues of the Laplace–Dirichlet operator (repeated in accordance with their multiplicities) and $(e_n)_{n\in\mathbf{N}}$ a corresponding Hilbertian basis of eigenvectors in $L^2(\Omega)$:*

$$\begin{cases} -\Delta e_n = \lambda_n e_n & \text{on } \Omega, \\ e_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us denote by V_n the subspace of $H_0^1(\Omega)$ generated by the first eigenvectors $e_1, \dots, e_n$,

$$V_n = \operatorname{span}\{e_1, e_2, \dots, e_n\},$$

and by $V_n^\perp$ the orthogonal of V_n in $H_0^1(\Omega)$ with respect to the scalar product of $H_0^1(\Omega)$:

$$\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v\,dx.$$

Then the following variational formulas hold:

$$\begin{aligned}
&\lambda_1 = \min\left\{\mathscr{R}(v) \;:\; v \in H_0^1(\Omega),\ v \neq 0\right\},\\
&\lambda_2 = \min\left\{\mathscr{R}(v) \;:\; v \in V_1^\perp,\ v \neq 0\right\},\\
&\dots\\
&\lambda_n = \min\left\{\mathscr{R}(v) \;:\; v \in V_{n-1}^\perp,\ v \neq 0\right\},\\
&\dots .
\end{aligned}$$

PROOF. By definition of λ_n and e_n

$$\int_\Omega \nabla e_n \cdot \nabla v\,dx = \lambda_n \int_\Omega e_n v\,dx \quad \forall v \in H_0^1(\Omega).$$

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

By taking $v = e_n$ we obtain

$$\mathscr{R}(e_n) = \lambda_n, \quad n = 1, 2, \ldots.$$

Hence, by noticing that $e_n \in V_{n-1}^{\perp}$, we first obtain

$$\lambda_n \geq \inf\left\{\mathscr{R}(v) \,:\, v \in V_{n-1}^{\perp},\ v \neq 0\right\}. \tag{8.22}$$

To obtain the reverse inequality, we use an argument similar to the one used in the proof of Theorem 8.4.1, which relies on the Bessel–Parseval equality. Recall that (see (8.21)) for any $v \in H_0^1(\Omega)$

$$\mathscr{R}(v) = \frac{\sum_{i=1}^{+\infty} \lambda_i \langle v, e_i \rangle_{L^2(\Omega)}^2}{\sum_{i=1}^{+\infty} \langle v, e_i \rangle_{L^2(\Omega)}^2}$$

and that $\langle v, e_i \rangle_{H_0^1(\Omega)} = \lambda_i \langle v, e_i \rangle_{L^2(\Omega)}^2$, which allows passing from orthogonality in $H_0^1(\Omega)$ to orthogonality in $L^2(\Omega)$ and vice versa.

Hence, for any $v \in V_{n-1}^{\perp}$ we have

$$\mathscr{R}(v) = \frac{\sum_{i \geq n} \lambda_i \langle v, e_i \rangle_{L^2(\Omega)}^2}{\sum_{i \geq n} \langle v, e_i \rangle_{L^2(\Omega)}^2} \geq \lambda_n.$$

Therefore

$$\inf\left\{\mathscr{R}(v) \,:\, v \in V_{n-1}^{\perp},\ v \neq 0\right\} \geq \lambda_n. \tag{8.23}$$

Combining inequalities (8.22) and (8.23) yields the result. The fact that the infimum is actually attained follows from the direct methods of the calculus of variations, being the functional $v \mapsto \mathscr{R}(v)$ coercive and weakly lower semicontinuous. □

We can now state the Courant–Fisher min-max and max-min formulas.

Theorem 8.4.2 (Courant–Fisher min-max and max-min formulas). *With the same notation as in Proposition* 8.4.2, *we have*

$$\begin{aligned}\lambda_n &= \min_{M \in \mathscr{L}_n} \max_{v \in M,\ v \neq 0} \mathscr{R}(v) \\ &= \max_{M \in \mathscr{L}_{n-1}} \min_{v \in M^{\perp},\ v \neq 0} \mathscr{R}(v),\end{aligned}$$

where $\mathscr{L}_n$ is the class of all n-dimensional linear subspaces in $H_0^1(\Omega)$ and $M^{\perp}$ stands for the orthogonal subspace of M in $H_0^1(\Omega)$.

PROOF. (a) Take first $M = V_n = \operatorname{span}\{e_1, \ldots, e_n\}$. For any $v \in V_n$, we have $\langle v, e_i \rangle_{H_0^1(\Omega)} = \langle v, e_i \rangle_{L^2(\Omega)}$ for all $i > n$. By expressing $\mathscr{R}(v)$ as (see (8.21))

$$\mathscr{R}(v) = \frac{\sum_{i=1}^{+\infty} \lambda_i \langle v, e_i \rangle_{L^2(\Omega)}^2}{\sum_{i=1}^{+\infty} \langle v, e_i \rangle_{L^2(\Omega)}^2},$$

we obtain

$$\mathscr{R}(v) = \frac{\sum_{i=1}^{n} \lambda_i \langle v, e_i \rangle_{L^2(\Omega)}^2}{\sum_{i=1}^{n} \langle v, e_i \rangle_{L^2(\Omega)}^2} \leq \lambda_n.$$

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Hence

$$\max\left\{\mathcal{R}(v) \,:\, v \in V_n,\ v \neq 0\right\} \leq \lambda_n \tag{8.24}$$

(indeed equality holds taking $v = e_n$) and consequently

$$\min_{M \in \mathcal{L}_n} \max_{v \in M, v \neq 0} \mathcal{R}(v) \leq \lambda_n.$$

Let us now prove the reverse inequality. Equivalently, we have to prove that for any n-dimensional subspace M of $H_0^1(\Omega)$,

$$\lambda_n \leq \max\left\{\mathcal{R}(v) \,:\, v \in M,\ v \neq 0\right\}. \tag{8.25}$$

Take a subspace M of $H_0^1(\Omega)$ with $\dim M = n$. We claim that

$$M \cap V_{n-1}^{\perp} \neq \{0\}.$$

This is a consequence of the classical relation linking the dimension of the image and the kernel of a linear mapping: take $P : M \longrightarrow V_{n-1}$ to be the linear mapping which, to any $v \in M$, associates $P(v) = \mathrm{proj}_{V_{n-1}} v$, the projection of v on V_{n-1}. We have

$$\dim M = n = \dim(\ker P) + \dim(P(M)).$$

Since $P(M)$, the image of M per P, is contained in V_{n-1}, we have

$$\dim(P(M)) \leq n - 1.$$

Hence

$$\dim(\ker P) \geq n - (n-1) = 1.$$

Equivalently, there exists some $v \in M$, $v \neq 0$ such that $\mathrm{proj}_{V_{n-1}} v = 0$, that is, $v \in M \cap V_{n-1}^{\perp}$, which proves the claim.

Take now any $\overline{v} \in M \cap V_{n-1}^{\perp}$, $\overline{v} \neq 0$. We know by Proposition 8.4.2 that

$$\lambda_n = \min\left\{\mathcal{R}(v) \,:\, v \in V_{n-1}^{\perp}\right\}.$$

Hence

$$\begin{aligned}\lambda_n &\leq \mathcal{R}(\overline{v}) \\ &\leq \max\left\{\mathcal{R}(v) \,:\, v \in M,\ v \neq 0\right\},\end{aligned}$$

which proves (8.25) and completes the proof of the min-max formula.

(b) The proof of the max-min formula is very similar to the proof of the min-max formula.

First note that by Proposition 8.4.2,

$$\lambda_n = \min\left\{\mathcal{R}(v) \,:\, v \in V_{n-1}^{\perp},\ v \neq 0\right\}. \tag{8.26}$$

Hence

$$\lambda_n \leq \sup_{M \in \mathcal{L}_{n-1}} \inf_{v \in M^{\perp},\ v \neq 0} \mathcal{R}(v). \tag{8.27}$$

To prove the reverse inequality, we need to show that for any $M \in \mathcal{L}_{n-1}$ we have

$$\lambda_n \geq \inf_{v \in M^{\perp},\ v \neq 0} \mathcal{R}(v). \tag{8.28}$$

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

We claim that there exists some $\overline{v} \in M^{\perp} \cap V_n$ with $\overline{v} \neq 0$. This can be obtained by considering the linear mapping $Q : V_n \longrightarrow M$ which, to any $v \in V_n$, associates $Q(v) = \mathrm{proj}_M v$. We have

$$n = \dim V_n = \dim(\ker Q) + \dim\big(Q(V_n)\big).$$

Since $\dim Q(V_n) \leq \dim M = n-1$, we have $\dim(\ker Q) \geq 1$. Equivalently, there exists some $\overline{v} \in V_n$, $\overline{v} \neq 0$ such that $\mathrm{proj}_M \overline{v} = 0$, that is $\overline{v} \in M^{\perp} \cap V_n$.

We now use (8.24) to obtain

$$\begin{aligned} \lambda_n &\geq \mathscr{R}(\overline{v}) \\ &\geq \inf_{v \in M^{\perp},\ v \neq 0} \mathscr{R}(v), \end{aligned}$$

that is, (8.28). Hence $\lambda_n = \sup_{M \in \mathscr{L}_{n-1}} \inf_{v \in M^{\perp},\ v \neq 0} \mathscr{R}(v)$.

Moreover, by (8.26) we have that the sup is a max (it is precisely attained by taking $M = V_{n-1}$) and the inf is a min (take $v = e_n$). Finally,

$$\lambda_n = \max_{M \in \mathscr{L}_{n-1}} \min_{v \in M^{\perp}\ v \neq 0} \mathscr{R}(v),$$

which ends the proof. □

Remark 8.4.1. (1) It is worth pointing out that the Courant–Fisher min-max and max-min principles, which give a variational characterization of the eigenvalues of the Laplace–Dirichlet operator, hold for the sequence $(\lambda_n)_{n \in \mathbf{N}}$ of eigenvalues which is expressed according to the multiplicity condition. This is another justification of this convention which gives the information on the values of the eigenvalues and on their multiplicities.

For these reasons, we call the $(\lambda_n)_{n \in \mathbf{N}}$ with the multiplicity condition (i.e., λ_n is repeated a number of times equal to its multiplicity) *the sequence of eigenvalues of the Laplace–Dirichlet operator*.

(2) In Proposition 8.4.2, the $(\lambda_n)_{n \in \mathbf{N}}$ are obtained by a recursive formula: one has first to know V_{n-1} to obtain λ_n. By contrast, the Courant–Fisher min-max and max-min principles provide a direct variational formulation of the eigenvalues of the Laplace–Dirichlet operator.

(3) The Courant–Fisher min-max principle is the point of departure for the Ljusternik–Schnirelman theory of critical points. Indeed, in 1930, Ljusternik wrote, "The theory of eigenvalues of quadratic form developed by R. Courant enables one to discern their existence and reality without calculations. We shall generalize their theory to arbitrary functions having continuous second partial derivatives."

Typically, the Ljusternik theory deals with the variational approach to nonlinear eigenvalue problems of the type

$$F'(u) = \lambda u, \quad u \in X,\ \lambda \in \mathbf{R},\ ||u|| = 1,$$

where X is a separable Hilbert space, $\dim X = +\infty$, and $F : X \longrightarrow \mathbf{R}$ is even, of class $\mathbf{C}^1$, with F' compact.

Let us conclude this section by making a direct connection between the first eigenvalue $\lambda_1(-\Delta)$ of the Laplace–Dirichlet operator and the Poincaré constant (cf. Definition 5.3.1). Let us recall that the Poincaré constant is the smallest constant C such that for any $v \in H_0^1(\Omega)$

$$\left(\int_\Omega v(x)^2\, dx\right)^{1/2} \leq C\left(\int_\Omega |\nabla v(x)|^2\, dx\right)^{1/2}.$$

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

This is equivalent to saying that

$$\frac{1}{C^2} = \inf\left\{\mathscr{R}(v) \;:\; v \in H_0^1(\Omega),\ v \neq 0\right\},$$

i.e., $1/C^2 = \lambda_1$. In other words, we have obtained the following result.

Proposition 8.4.3. *The Poincaré constant C and the first eigenvalue λ_1 of the Laplace–Dirichlet operator are related by the following formula:*

$$\frac{1}{C^2} = \lambda_1.$$

8.5 ▪ Multiplicity and asymptotic properties of the eigenvalues of the Laplace–Dirichlet operator

The first eigenvalue $\lambda_1(-\Delta)$ plays a fundamental role, for example, in the analysis of the resonance phenomena for vibrating structures and in some related shape optimization problems. Indeed, the first eigenvalue $\lambda_1(-\Delta)$ enjoys remarkable properties as stated in the following result.

Theorem 8.5.1. *Let Ω be a bounded connected regular open set in $\mathbf{R}^N$. The first eigenvalue λ_1 of the Laplace–Dirichlet operator has multiplicity equal to one. Its eigenspace is generated by a vector $e_1 \in H_0^1(\Omega)$ such that $e_1 > 0$ on Ω.*

PROOF. Let us denote by E_1 the eigenspace relative to the first eigenvalue λ_1. We recall that the Courant–Fisher theorem, Theorem 8.4.1, asserts that the elements $v \in E_1$, $v \neq 0$, are the solutions of the minimization problem

$$\min\left\{\frac{\displaystyle\int_\Omega |\nabla v(x)|^2\,dx}{\displaystyle\int_\Omega v(x)^2\,dx} \;:\; v \in H_0^1(\Omega),\ v \neq 0\right\}. \qquad (\mathscr{P})$$

An important consequence of this formula is that if $v \in E_1$, then automatically $|v| \in E_1$. This follows from the fact that the truncations operate on the space $H_0^1(\Omega)$. In particular, see Corollary 5.8.1, for any $v \in H_0^1(\Omega)$, $|v| \in H_0^1(\Omega)$, and $\mathscr{R}(|v|) = \mathscr{R}(v)$.

The following argument proceeds by contradiction and makes use of the strong maximum principle. Suppose that one can find two elements v_1 and v_2 in the eigensubspace E_1 which are not proportional.

Because of the regularity assumption on Ω, v_1 and v_2 are smooth functions and it makes sense to consider their values at any point $x \in \Omega$. Thus we can find x_0 and $x_1 \in \Omega$ such that

$$\begin{aligned}\alpha_1 v_1(x_0) + \alpha_2 v_2(x_0) &= 0,\\ \alpha_1 v_1(x_1) + \alpha_2 v_2(x_1) &\neq 0\end{aligned}$$

for some $\alpha_1,\ \alpha_2 \in \mathbf{R}$.

Now take $w = |\alpha_1 v_1 + \alpha_2 v_2|$. Since $v_1,\ v_2 \in E_1$, we have $\alpha_1 v_1 + \alpha_2 v_2 \in E_1$ and $w = |\alpha_1 v_1 + \alpha_2 v_2|$ still belongs to E_1 (as shown just above, as a consequence of the Courant–Rayleigh variational formula for λ_1). Let us summarize the properties of w:

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

$$\begin{cases} w \in E_1, \\ w \geq 0, \\ w(x_0) = 0, \\ w(x_1) \neq 0. \end{cases}$$

Since $-\Delta w = \lambda_1 w$, from $\lambda_1 > 0$ and $w \geq 0$, we deduce that

$$\begin{cases} -\Delta w \geq 0, \\ w = 0 \quad \text{on } \partial\Omega, \\ w(x_0) = 0, \quad x_0 \in \Omega. \end{cases}$$

The strong maximum principle property of Hopf now implies that $w = 0$ on Ω, a clear contradiction to the fact that $w(x_1) \neq 0$.

Thus E_1 has dimension one. By taking any vector $w \in E_1 \setminus \{0\}$ and $e_1 = |w|$ we obtain a vector in E_1 which satisfies, by using again the strong maximum principle, $e_1 > 0$ on Ω. □

Corollary 8.5.1. *Let Ω be as in Theorem 8.5.1 and take any eigenvector e_i of the Laplace–Dirichlet operator corresponding to an eigenvalue $\lambda_i > \lambda_1$. Then the sign of e_i is not constant on Ω.*

PROOF. Since $\lambda_i \neq \lambda_1$ we have $E(\lambda_i) \perp E(\lambda_1)$ and $\int_\Omega e_i(x)e_1(x)\,dx = 0$. Since $e_1 > 0$ on Ω, this forces e_i to change sign on Ω. □

This means that the status of the first eigenvalue is very particular. It is the only eigenvalue which possesses an eigenvector with constant sign. Indeed, in the analysis of the second eigenvalue problem $\lambda_2(-\Delta)$, the nodal set of a second eigenvector (the set where it is equal to zero) plays a central role.

The explicit computation of the spectrum of the Laplace–Dirichlet operator is possible only in very particular situations. Nevertheless, even in situations where such a computation is not possible (or is too complicated), one can get rather precise information on the spectrum by using comparison arguments. To stress the dependence of eigenvalues with respect to Ω, let us denote by $(\lambda_n(\Omega))_{n\in\mathbf{N}}$ the sequence of eigenvalues of the Laplace–Dirichlet operator on Ω (with the multiplicity convention). Then, as a direct consequence of the Courant–Fisher min-max principle, we have the following comparison result.

Proposition 8.5.1. *Let Ω and $\tilde{\Omega}$ be two open bounded subsets of $\mathbf{R}^N$ with $\Omega \subset \tilde{\Omega}$. Then, for any $n \geq 1$,*

$$\lambda_n(\tilde{\Omega}) \leq \lambda_n(\Omega),$$

i.e., $\lambda_n(\Omega)$ is a decreasing function of Ω.

PROOF. For any $v \in H_0^1(\Omega)$, let us denote by $\tilde{v}$ the function which is equal to v on Ω and zero on $\tilde{\Omega} \setminus \Omega$. By Proposition 5.1.1, we have $\tilde{v} \in H_0^1(\tilde{\Omega})$. Moreover,

$$\int_\Omega |v(x)|^2\,dx = \int_{\tilde{\Omega}} |\tilde{v}(x)|^2\,dx,$$
$$\int_\Omega |\nabla v(x)|^2\,dx = \int_{\tilde{\Omega}} |\nabla \tilde{v}(x)|^2\,dx.$$

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Hence, $H_0^1(\Omega)$ can be isometrically identified with a subspace of $H_0^1(\tilde{\Omega})$ by the mapping

$$v \in H_0^1(\Omega) \xrightarrow{\tilde{i}} \tilde{v} = \tilde{i}(v) \in H_0^1(\tilde{\Omega}).$$

If $M \in \mathscr{L}_n(\Omega)$ is an n-dimensional subspace of $H_0^1(\Omega)$, then $\tilde{i}(M) \in \mathscr{L}_n(\tilde{\Omega})$ is an n-dimensional subspace of $H_0^1(\tilde{\Omega})$. We can now apply the Courant–Fisher min-max formula (Theorem 8.4.2) to obtain

$$\begin{aligned}
\lambda_n(\Omega) &= \min_{M\in\mathscr{L}_n(\Omega)} \max_{v\in M,\ v\neq 0} \mathscr{R}(v,\Omega) \\
&= \min_{M\in\mathscr{L}_n(\Omega)} \max_{v\in M,\ v\neq 0} \mathscr{R}(\tilde{v},\tilde{\Omega}) \\
&= \min_{W=\tilde{i}(M),\ M\in\mathscr{L}_n(\Omega)} \max_{w\in W\setminus\{0\}} \mathscr{R}(w,\tilde{\Omega}) \\
&\geq \min_{M\in\mathscr{L}_n(\tilde{\Omega})} \max_{w\in W\setminus\{0\}} \mathscr{R}(w,\tilde{\Omega}) = \lambda_n(\tilde{\Omega}),
\end{aligned}$$

which ends the proof. □

To go further and use this comparison result we need to know some particular situations where the spectrum of $-\Delta$ can be explicitly computed. Let us start with the simplest situation, that is, $N=1$ and $\Omega=(0,1)$.

Proposition 8.5.2. *Let $N=1$ and $\Omega=(0,1)$. Then the eigenvalues $(\lambda_n)_{n\in\mathbf{N}}$ of the Laplace–Dirichlet operator are given by*

$$\lambda_n = n^2\pi^2, \quad n = 1,2\dots,$$

and the corresponding orthonormal basis $(e_n)_{n\in\mathbf{N}}$ of eigenvectors in $L^2(\Omega)$ is given by

$$e_n(x) = \sqrt{2}\sin(n\pi x).$$

PROOF. The proof is elementary. When solving the ordinary differential equation

$$u'' + \lambda u = 0,$$

one obtains

$$u(x) = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x).$$

The boundary condition $u(0)=0$ gives $B=0$ and the boundary condition $u(1)=0$ gives $\sin\sqrt{\lambda}=0$, that is, $\lambda = n^2\pi^2$, for some $n\geq 1$. The corresponding solution is $u(x)=A\sin(n\pi x)$. After L^2-normalization one obtains $A=\sqrt{2}$. □

In this very simple situation, each eigenvalue has multiplicity one. Let us now study the Laplace equation with the Dirichlet boundary condition on the N-cube $\Omega=(0,1)^N$ and the corresponding eigenvalue problem.

Proposition 8.5.3. *Let $\Omega=(0,1)^N$. For each $p=(p_1,p_2,\dots,p_N)$ with $p_i\in\mathbf{N}\setminus\{0\}$, $i=1,2,\dots,N$ (i.e., $p\in(\mathbf{N}^*)^N$), the positive real number*

$$\lambda_p := \pi^2(p_1^2+p_2^2+\cdots+p_N^2)$$

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

is an eigenvalue of the Laplace–Dirichlet operator on $\Omega = (0,1)^N$ *and the function*

$$u_p(x) = 2^{N/2} \prod_{i=1}^{N} \sin(\pi p_i x_i)$$

is an eigenfunction corresponding to the eigenvalue λ_p*. Indeed,*

$$\Lambda(-\Delta, \Omega) = \left\{ \lambda_p \,:\, p \in (\mathbf{N}^*)^N \right\},$$

i.e., all the eigenvalues of $-\Delta$ *on* $\Omega = (0,1)^N$ *can be expressed in this way, and the family* $\left\{ u_p \,:\, p \in (\mathbf{N}^*)^N \right\}$ *is an orthonormal basis of* $L^2(\Omega)$*.*

PROOF. Take $p = (p_1, p_2, \dots, p_N)$, $v \in \mathscr{D}(\Omega)$ and compute

$$\int_\Omega \nabla u_p(x) \cdot \nabla v(x)\, dx = \int_\Omega \sum_{i=1}^{N} \frac{\partial u_p}{\partial x_i}(x) \frac{\partial v}{\partial x_i}(x)\, dx.$$

Let us notice that

$$u_p(x) = \prod_{i=1}^{N} e_{p_i}(x_i) \quad \text{with } e_{p_i}(x_i) = \sqrt{2} \sin(\pi p_i x_i).$$

Consequently,

$$\frac{\partial}{\partial x_i} u_p(x) = \left[\prod_{j \neq i} e_{p_j}(x_j) \right] e'_{p_i}(x_i).$$

(e'_{p_i} stands for the derivative of the function of one variable $e_{p_i}(\cdot)$.) Hence

$$\int_\Omega \nabla u_p(x) \cdot \nabla v(x)\, dx = \sum_{i=1}^{N} \left(\int_0^1 e'_{p_i}(x_i) \frac{\partial v}{\partial x_i}(x)\, dx_i \right) \cdot \int_{(0,1)^{N-1}} \prod_{j \neq i} e_{p_j}(x_j)\, dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_N.$$

An integration by parts yields

$$\begin{aligned} \int_0^1 e'_{p_i}(x_i) \frac{\partial v}{\partial x_i}(x)\, dx_i &= -\int_0^1 e''_{p_i}(x_i) v(x)\, dx_i \\ &= \pi^2 p_i^2 \int_0^1 e_{p_i}(x_i) v(x)\, dx_i. \end{aligned}$$

(The last equality follows from the fact that $e''_{p_i} + \pi^2 p_i^2 e_{p_i} = 0$, since e_{p_i} is an eigenvector relative to the eigenvalue $\pi^2 p_i^2$ of the one-dimensional Laplace–Dirichlet problem.) Thus

$$\int_\Omega \nabla u_p(x) \cdot \nabla v(x)\, dx = \pi^2 \left(\sum_{i=1}^{N} p_i^2 \right) \int_\Omega u_p(x) v(x)\, dx.$$

By a classical density and extension by continuity argument, this equality can be extended to an arbitrary $v \in H_0^1(\Omega)$, and

$$\begin{cases} \displaystyle\int_\Omega \nabla u_p(x) \cdot \nabla v(x)\, dx = \lambda_p \int_\Omega u_p v\, dx \quad \forall v \in H_0^1(\Omega), \\ u_p \in H_0^1(\Omega), \end{cases}$$

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

which precisely means that $\lambda_p = \pi^2 \sum_{i=1}^N p_i^2$ is an eigenvalue of the Laplace–Dirichlet operator on $(0,1)^N$, and $u_p(x) = \prod_{i=1}^N e_{p_i}(x_i)$ is a corresponding eigenvector.

Let us now show that the family of eigenfunctions $\{u_p \ : \ p \in (\mathbf{N}^*)^N\}$ is an orthonormal basis of $L^2(\Omega)$.

First let us notice that if $p \neq q$, then there exists at least one $i \in \{1,2,\dots,N\}$ such that $p_i \neq q_i$. From the orthogonality in $L^2(0,1)$ of the two functions $\sin(p_i \pi x)$ and $\cos(q_i \pi x)$, we immediately obtain that the family $\{u_p \ : \ p \in (\mathbf{N}^*)^N\}$ is orthogonal in $L^2(\Omega)$.

The point that is more delicate is to prove that the family $\{u_p \ : \ p \in (\mathbf{N}^*)^N\}$ generates $L^2(\Omega)$ in the topological sense, that is, the vector space generated by this family of vectors is dense in $L^2(\Omega)$. Indeed, by a careful application of the Fubini theorem, one can prove the following result (which is quite classical in integration theory and we omit its proof).

Lemma 8.5.1. *Let $(v_p)_{p \in \mathbf{N}^*}$ and $(w_q)_{q \in \mathbf{N}^*}$ be two Hilbertian bases of $L^2(0,1)$. Then the family of functions*

$$(x,y) \mapsto v_p(x) w_q(y)$$

is a Hilbertian basis of $L^2((0,1)^2)$.

Thus, by iterating this result a finite number of times we obtain that the family $\{u_p \ : \ p \in (\mathbf{N}^*)^N\}$ is an orthonormal basis of $L^2(\Omega)$. This clearly implies that by taking $\Lambda = \{\lambda_p \ : \ p \in (\mathbf{N}^*)^N\}$ we have obtained all the eigenvalues; otherwise, there would exist some $v \in H_0^1(\Omega)$, $v \neq 0$, which is an eigenvector corresponding to some eigenvalue $\lambda \notin \Lambda$. By the orthogonality property this would imply that v is orthogonal in $L^2(\Omega)$ to all the $\{u_p \ : \ p \in (\mathbf{N}^*)^N\}$ which forms a basis, and hence $v = 0$, a clear contradiction. This completes the proof of the spectral analysis of the Laplace–Dirichlet operator in the case $\Omega = (0,1)^N$. $\square$

Propositions 8.5.3 and 8.5.1 (comparison principle) permit us to obtain a sharp estimation of the asymptotic behavior of the sequence $(\lambda_n(\Omega))_{n \in \mathbf{N}^*}$ of the eigenvalues of the Laplace–Dirichlet operator in a bounded open set Ω in $\mathbf{R}^N$. Indeed, we can prove the following result.

Theorem 8.5.2. *Let $(\lambda_n(\Omega))_{n \in \mathbf{N}}$ be the sequence of the eigenvalues of the Laplace–Dirichlet operator in a bounded open set Ω in $\mathbf{R}^N$ (with the multiplicity convention). Then, there exist two positive constants c_Ω and d_Ω, which depend only on Ω, such that for all $n \geq 1$,*

$$c_\Omega n^{2/N} \leq \lambda_n(\Omega) \leq d_\Omega n^{2/N}.$$

SKETCH OF THE PROOF. The proof is quite technical but the idea is very simple. The idea consists in the comparison of Ω with two N-cubes Q_a and Q_b such that $Q_a \subset \Omega \subset Q_b$, $Q_a = (-a/2, a/2)^N$, $Q_b = (-b/2, b/2)^N$. Then Proposition 8.5.1 applies and one obtains

$$\lambda_n(Q_a) \leq \lambda_n(\Omega) \leq \lambda_n(Q_b).$$

Then the problem has been reduced to the evaluation of $\lambda_n(Q_a)$ and $\lambda_n(Q_b)$. Clearly $\lambda_n(Q_a) = \lambda_n(Q)/a^2$ and $\lambda_n(Q_b) = \lambda_n(Q)/b^2$, where $Q = (0,1)^N$. By Proposition 8.5.3, the numbers $\lambda_n(Q)/\pi^2$ are precisely the positive integers of the form $\sum_{i=1}^N p_i^2$ with $p_i \in \mathbf{N} \setminus \{0\}$. Thus, one has to arrange the numbers $\{\sum_{i=1}^N p_i^2 \ : \ p_i \in \mathbf{N}^*\}$ as an increasing sequence to obtain the sequence $\{\lambda_n(Q)/\pi^2 \ : \ n \in \mathbf{N}^*\}$: this is just a combinatorial

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

problem! To that end, it is convenient to introduce for any $t>0$ the quantity $\nu_N(t)$ which is the cardinal of all the elements $p\in(\mathbf{N}^*)^N$ such that $p=(p_1,\dots,p_N)$ with $\sum_{i=1}^N p_i^2\le t$. Then, the key of the proof consists in showing the following estimate: $\nu_N(t)\sim C_N t^{N/2}$ for some constant $C_N>0$. □

Remark 8.5.1. From Proposition 8.5.3, we can obtain, as indicated before, after some combinatorial argument, a complete description of the sequence $(\lambda_n)_{n\in\mathbf{N}}$ of the eigenvalues of the Laplace–Dirichlet operator in $\Omega=(0,1)^N$. For example,

(a) $N=2$. Then

$$\begin{array}{ll} \lambda_1(\Omega)=2\pi^2, & \text{multiplicity}=1 \text{ (no surprise!)},\\ \lambda_2(\Omega)=5\pi^2, & \text{multiplicity}=2,\\ \lambda_3(\Omega)=8\pi^2, & \text{multiplicity}=1,\\ \lambda_4(\Omega)=10\pi^2, & \text{multiplicity}=2,\\ \dots. & \end{array}$$

(b) $N=3$. Then

$$\begin{array}{ll} \lambda_1(\Omega)=3\pi^2, & \text{multiplicity}=1 \text{ (no surprise!)},\\ \lambda_2(\Omega)=6\pi^2, & \text{multiplicity}=3,\\ \lambda_3(\Omega)=9\pi^2, & \text{multiplicity}=3,\\ \dots. & \end{array}$$

Indeed, in the various examples we have encountered, the multiplicity of the second eigenvalue of the Laplace–Dirichlet operator does not obey a simple rule: it may be equal to one, two, three, ...

This makes a sharp contrast to the first eigenvalue $\lambda_1(-\Delta)$, which always has multiplicity one, and makes the second eigenvalue more delicate to work with.

Remark 8.5.2. The previous analysis of the spectrum of the Laplace–Dirichlet operator on a general bounded open set Ω relies on the comparison of Ω with a reference set $\tilde{\Omega}$ such that either $\Omega\subset\tilde{\Omega}$ or $\tilde{\Omega}\subset\Omega$. There is another way to obtain comparison results, which consists in the use of rearrangement results (Steiner symmetrization). The idea is that this type of transformation preserves the measure and the L^2 norm of the functions and makes smaller the $(L^2)^N$ norm of the gradient of a function (Dirichlet integral). In this way it is possible to compare the corresponding Rayleigh quotients. This device is very useful in shape optimization; it permits us, for example, to solve an old problem from Rayleigh which consists in proving that the ball minimizes the first eigenvalue among all open sets of given volume.

8.6 ▪ A general abstract theory for spectral analysis of elliptic boundary value problems

So far, we have considered the spectral analysis of the Laplacian with Dirichlet boundary conditions. To be able to develop a similar analysis for more general linear elliptic operators and for different types of boundary conditions (like Neumann, mixed,...), let us introduce the following abstract setting:

(i) Let V and H be two real Hilbert spaces (infinite dimensional spaces) such that $V\xrightarrow{i}H$; we assume that

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

• V can be embedded in H by i which is linear continuous and one to one,
• V is dense in H (i.e., $\overline{i(V)}^H = H$),
• V is compactly embedded in H (i.e., i is compact)

(as a typical example, take $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$ with their usual Hilbertian structures).

(ii) Let

$$a : V \times V \longrightarrow \mathbf{R},$$
$$(u,v) \mapsto a(u,v),$$

be a bilinear form on $V \times V$ which is symmetric continuous and coercive:

$$\exists \alpha > 0 \text{ such that } \forall\, v \in V \quad a(v,v) \geq \alpha ||v||^2.$$

Here $||\cdot||$ stands for the norm in V.

The norm and the scalar product in H are respectively denoted by $|\cdot|_H$ ($|\cdot|$ without ambiguity) and $\langle \cdot,\cdot \rangle_H$ ($\langle \cdot,\cdot \rangle$ without ambiguity). Note that we are in the situation of the Lax–Milgram theorem, Theorem 3.1.2, and for any $L \in V^*$ there exists a unique $u \in V$ which satisfies

$$a(u,v) = L(v) \quad \forall v \in V.$$

Noticing that for any $h \in H$ the linear form

$$v \in V \mapsto \langle h, v\rangle_H$$

is continuous on V, we deduce from the Lax–Milgram theorem the existence of a unique solution $u = Th$ of the following problem:

$$\begin{cases} a(Th, v) = \langle h, v\rangle_H, \\ Th \in V. \end{cases}$$

By using the same device as in Proposition 8.2.1, we can easily prove that $T : H \longrightarrow H$ is a linear continuous, self-adjoint, compact, and positive definite operator. Thus, one can apply to T the abstract diagonalization theorem, Theorem 8.3.1, for compact, self-adjoint, positive definite operators and conclude that there exists a Hilbertian basis $(e_n)_{n\in\mathbf{N}}$ in H of eigenvectors,

$$Te_n = \mu_n e_n,$$

with $(\mu_n)_{n\in\mathbf{N}}$, the decreasing sequence of positive eigenvalues (with multiplicity condition), which tends to zero as $n \to +\infty$. Note that now the family $(e_n)_{n\in\mathbf{N}}$ is a Hilbertian basis of V, when V is equipped with the scalar product

$$\langle\langle u, v\rangle\rangle := a(u,v)$$

(which is equivalent to the initial one).

We can now give a precise description of the solutions of the abstract spectral problem: find $\lambda \in \mathbf{R}$ such that there exists $u \in V$, $u \neq 0$, which satisfies

$$a(u,v) = \lambda \langle u, v\rangle_H \quad \forall v \in V.$$

(When such $u \neq 0$ exists, it is called an eigenvector relative to the eigenvalue λ.)

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Theorem 8.6.1. *Assume that the canonical injection of V into H is dense and compact and that the continuous bilinear form $a : V \times V \longrightarrow \mathbf{R}$ is symmetric and coercive on $V \times V$ (V-elliptic). Then the eigenvalues λ of the abstract variational problem*

$$\begin{cases} \text{find } \lambda \in \mathbf{R} \text{ such that there exists } u \in V,\ u \neq 0, \\ a(u,v) = \lambda \langle u, v\rangle_H \ \forall\, v \in V, \end{cases}$$

can be written as an increasing sequence of positive numbers $(\lambda_n)_{n\in\mathbf{N}}$ which tends to $+\infty$ as $n \to +\infty$

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \cdots.$$

(We again adopt the multiplicity convention: each eigenvalue is repeated a number of times equal to its multiplicity, which is finite.)

There exists an orthonormal basis (Hilbertian basis) $(e_n)_{n\in\mathbf{N}}$ of H such that for each $n \in \mathbf{N}$, e_n is an eigenvector relative to the eigenvalue λ_n:

$$\begin{cases} a(e_n, v) = \lambda_n \langle e_n, v\rangle_H \quad \forall v \in V, \\ e_n \in V. \end{cases}$$

Moreover, the sequence $\left(e_n/\sqrt{\lambda_n}\right)_{n\in\mathbf{N}}$ is a Hilbertian basis of V when this space is equipped with the (equivalent) scalar product $a(\cdot,\cdot)$.

We consider now some applications of the results above.

(1) Neumann problem. Take in this case

$$V = \left\{ v \in H^1(\Omega) : \int_\Omega v(x)\, dx = 0 \right\} \quad \text{and} \quad a(u,v) = \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx.$$

The coercivity of a on $V \times V$ follows from the Poincaré–Wirtinger inequality (Corollary 5.4.1) and the compact embedding $V \longrightarrow H = L^2(\Omega)$ from the Rellich–Kondrakov theorem, Theorem 5.4.2 (Ω is assumed smooth and bounded). From Theorem 8.6.1, we deduce the existence of a Hilbertian basis $(e_n)_{n\in\mathbf{N}}$ in $L^2(\Omega)$ such that

$$\begin{cases} -\Delta e_n = \lambda_n e_n \quad \text{on } \Omega, \\ \dfrac{\partial e_n}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\frac{\partial}{\partial \nu}$ stands for the normal outward derivative on the boundary $\partial\Omega$.

(2) Mixed Dirichlet–Neumann problem (see Section 6.3). By taking $V = \{v \in H^1(\Omega) : \gamma_0(v) = 0 \text{ on } \Gamma_0\}$ with $\mathscr{H}^{N-1}(\Gamma_0) > 0$, one obtains the existence of a Hilbertian basis in $L^2(\Omega)$, $(e_n)_{n\in\mathbf{N}}$ such that

$$\begin{cases} -\Delta e_n = \lambda_n e_n \quad \text{on } \Omega, \\ e_n = 0 \quad \text{on } \Gamma_0, \\ \dfrac{\partial e_n}{\partial \nu} = 0 \quad \text{on } \Gamma_1 = \Gamma \setminus \Gamma_0. \end{cases}$$

(3) One can obtain similar results by replacing $-\Delta$ by an elliptic linear operator A of the form

$$Av = -\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{i,j} \frac{\partial v}{\partial x_j} \right) + a_0 u$$

or more generally when considering elliptic systems (elasticity, Stokes, ...).

Downloaded 01/09/19 to 141.20.217.242. Redistribution subject to SIAM license or copyright; see http://www.siam.org/journals/ojsa.php

Remark 8.6.1. The variational approach of Courant–Fisher works without any particular difficulty in such a general setting. One introduces the abstract Rayleigh quotient $\mathscr{R}(v) = a(v,v)/|v|_H^2$ and thus

$$\begin{aligned}
\lambda_1 &= \min\left\{\mathscr{R}(v) \,:\, v \in V,\ v \neq 0\right\},\\
\lambda_n &= \min_{M \in \mathscr{L}_n} \max_{v \in M,\ v \neq 0} \mathscr{R}(v)\\
&= \max_{M \in \mathscr{L}_{n-1}} \min_{v \in M^\perp,\ v \neq 0} \mathscr{R}(v).
\end{aligned}$$

Let us end this chapter and return to the situation which was our first motivation for this study, the method of separation of variables of Fourier, applied to the wave equation

$$\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0 \quad \text{on } Q = \Omega \times (0, +\infty),\\
u = 0 \quad \text{on } \Sigma = \partial\Omega \times (0, +\infty),\\
u(x,0) = u_0(x) \quad \text{on } \Omega,\\
\dfrac{\partial u}{\partial t}(x,0) = u_1(x) \quad \text{on } \Omega.
\end{cases}$$

Denote by $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \cdots$ the eigenvalues of the Laplace–Dirichlet operator and by $(e_n)_{n \in \mathbf{N}}$ a corresponding Hilbertian basis in $L^2(\Omega)$ of eigenvectors. Set $\omega_n = \sqrt{\lambda_n}$. Then the unique (variational) solution of the above problem is given by the following formula (for $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$ given):

$$u(t) = \sum_{n=1}^{+\infty} \left[\langle u_0, e_n \rangle_{L^2(\Omega)} \cos(\omega_n t) + \frac{1}{\omega_n} \langle u_1, e_n \rangle_{L^2(\Omega)} \sin(\omega_n t) \right] e_n.$$

Here the variational solution is taken in the following sense: for any $0 < T < +\infty$

$$u \in \mathbf{C}\big((0,T); H_0^1(\Omega)\big) \cap \mathbf{C}^1\big((0,T); L^2(\Omega)\big)$$

and

$$\begin{cases}
\dfrac{d^2}{dt^2} \langle u(t), v \rangle_{L^2(\Omega)} + \displaystyle\int_\Omega \nabla u(t) \cdot \nabla v \, dx = 0 \quad \text{in the distributional sense on } (0,T)\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall v \in H_0^1(\Omega),\\
u(0) = u_0, \quad \dfrac{du}{dt}(0) = u_1.
\end{cases}$$

(See, for example, Raviart–Thomas [323, Section 8.2] for further details.)