

0.1 The continuous and discrete problem

Let $\alpha > 0$, $\Omega \subset \mathbb{R}^n$ bounded polyhedral Lipschitz domain, and $f \in L^2(\Omega)$.

The continuous problem minimizes

$$E(v) := \frac{\alpha}{2} \|v\|_{L^2(\Omega)}^2 + |v|_{BV(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \int_{\Omega} f v \, dx \quad (0.1)$$

amongst all $v \in V := BV(\Omega) \cap L^2(\Omega)$ where the BVseminorm $|v|_{BV(\Omega)}$ is equal to the $W^{1,1}$ seminorm for any $v \in W^{1,1}(\Omega)$.

The nonconforming problem minimizes

$$E_{NC}(v_{CR}) := \frac{\alpha}{2} \|v_{CR}\|_{L^2(\Omega)}^2 + |v_{CR}|_{1,1,NC} - \int_{\Omega} f v_{CR} \, dx \quad (0.2)$$

amongst all $v_{CR} \in CR_0^1(\mathcal{T})$ where $|\bullet|_{1,1,NC} := \|\nabla_{NC} \bullet\|_{L^1(\Omega)}$.

0.2 Refinement indicator and guaranteed lower energy bound

For some $n \in \mathbb{N}$ (here $n = 2$) and $0 < \gamma \leq 1$ define a refinement indicator $\eta := \sum_{T \in \mathcal{T}} \eta(T)$ with

$$\eta(T) := \underbrace{|T|^{2/n} \|f - \alpha u_{CR}\|_{L^2(T)}^2}_{=: \eta_{Vol}(T)} + \underbrace{|T|^{\gamma/n} \sum_{F \in \mathcal{F}(T)} \|[u_{CR}]_F\|_{L^1(F)}}_{=: \eta_{Jumps}(T)} \quad (0.3)$$

for any $T \in \mathcal{T}$.

For $f \in H_0^1(\Omega)$ and $u \in H_0^1(\Omega)$ ($u_{CR} \in CR_0^1(\Omega)$) continuous (discrete) minimizer with minimal energy $E(u)$ ($E_{NC}(u_{CR})$) it holds

$$E_{NC}(u_{CR}) + \frac{\alpha}{2} \|u - u_{CR}\|_{L^2(\Omega)}^2 - \frac{\kappa_{CR}}{\alpha} \|h_{\mathcal{T}}(f - \alpha u_{CR})\|_{L^2(\Omega)} |f|_{1,2} \leq E(u) \quad (0.4)$$

where $|\bullet|_{1,2} = \|\nabla \bullet\|_{L^2(\Omega)}$.

Hence, for GLEB := $E_{NC}(u_{CR}) - \frac{\kappa_{CR}}{\alpha} \|h_{\mathcal{T}}(f - \alpha u_{CR})\|_{L^2(\Omega)} |f|_{1,2}$, it holds $E_{NC}(u_{CR}) \geq$ GLEB and $E(u) \geq$ GLEB.

0.3 Experiments

In the following sections on each level the termination criterion for the iteration was

$$\left\| \frac{\nabla_{NC}(u_j - u_{j-1})}{\tau} \right\|_{L^2(\Omega)} < \varepsilon := 10^{-4}$$

for $\tau = \frac{1}{2}$.

0.4 Examples with exact solution

0.4.1 Example 1

For $\beta = 1$ define f as a function of the radius as

$$f(r) := \begin{cases} \alpha - 12(2 - 9r) & \text{if } 0 \leq r \leq \frac{1}{6}, \\ \alpha(1 + (6r - 1)^\beta) - \frac{1}{r} & \text{if } \frac{1}{6} \leq r \leq \frac{1}{3}, \\ 2\alpha + 6\pi \sin(\pi(6r - 2)) - \frac{1}{r} \cos(\pi(6r - 2)) & \text{if } \frac{1}{3} \leq r \leq \frac{1}{2}, \\ 2\alpha(\frac{5}{2} - 3r)^\beta + \frac{1}{r} & \text{if } \frac{1}{2} \leq r \leq \frac{5}{6}, \\ -3\pi \sin(\pi(6r - 5)) + \frac{1+\cos(\pi(6r-5))}{2r} & \text{if } \frac{5}{6} \leq r \leq 1, \end{cases} \quad (0.5)$$

with exact solution

$$u(r) := \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{1}{6}, \\ 1 + (6r - 1)^\beta & \text{if } \frac{1}{6} \leq r \leq \frac{1}{3}, \\ 2 & \text{if } \frac{1}{3} \leq r \leq \frac{1}{2}, \\ 2(\frac{5}{2} - 3r)^\beta & \text{if } \frac{1}{2} \leq r \leq \frac{5}{6}, \\ 0 & \text{if } \frac{5}{6} \leq r \leq 1. \end{cases} \quad (0.6)$$

For $\alpha = 1$ the exact energy $E(u)$ was computed before the experiment.

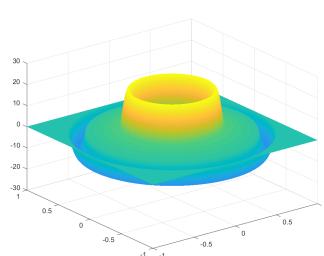


Figure 0.1: f for $\alpha = 1$

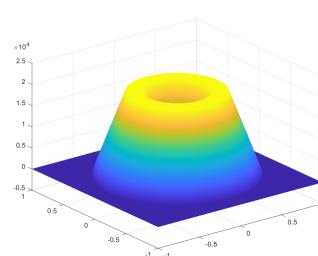


Figure 0.2: f for $\alpha = 10^4$

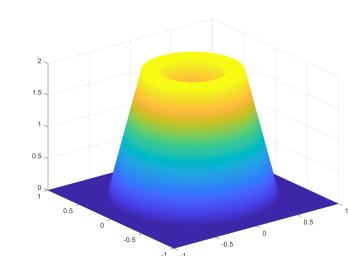


Figure 0.3: u

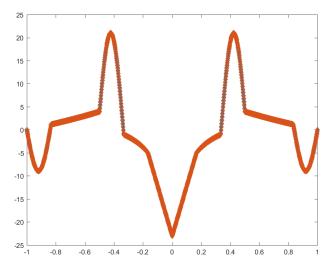


Figure 0.4: f along the axes for $\alpha = 1$

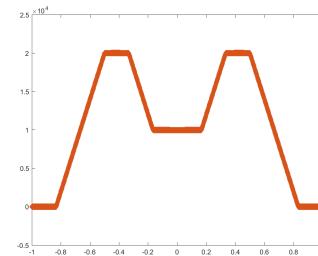


Figure 0.5: f along the axes for $\alpha = 10^4$

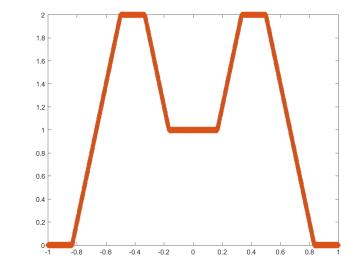


Figure 0.6: u along the axes

0.4 Examples with exact solution

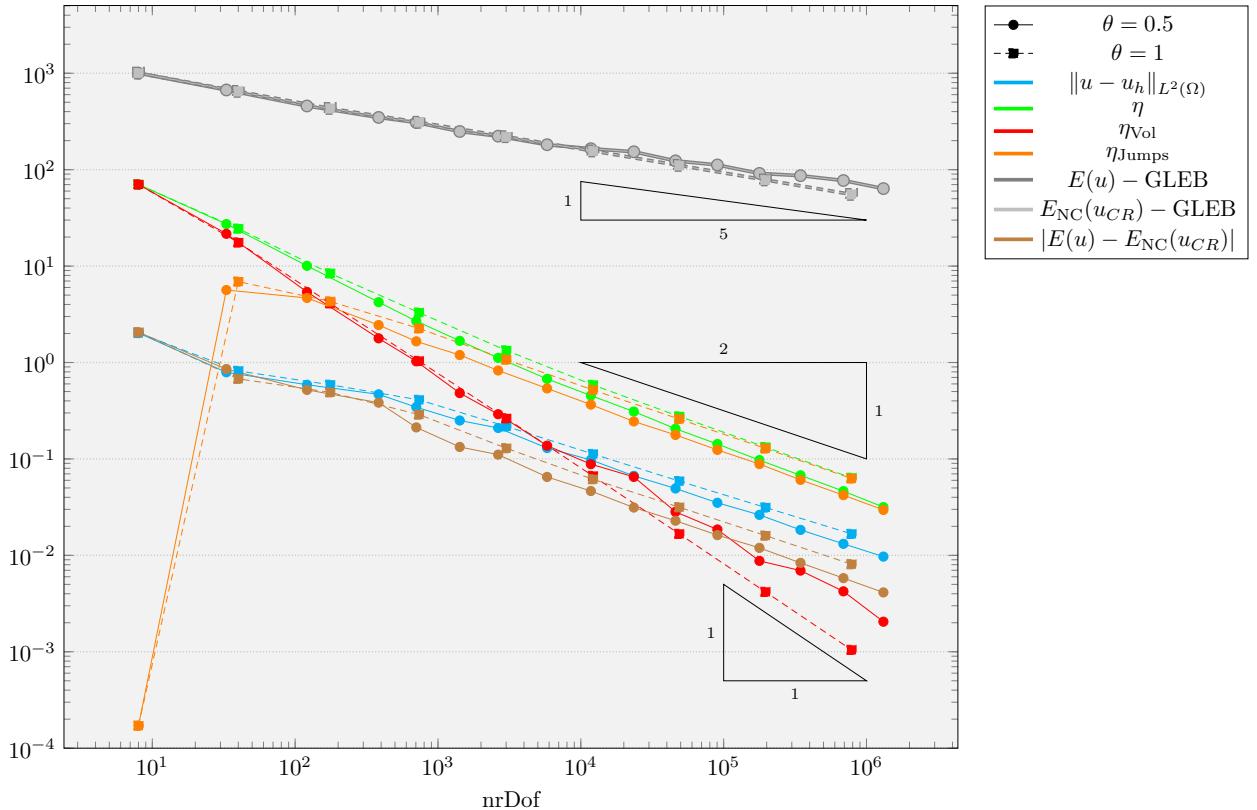


Figure 0.7: convergence history plot for $\alpha = 1$

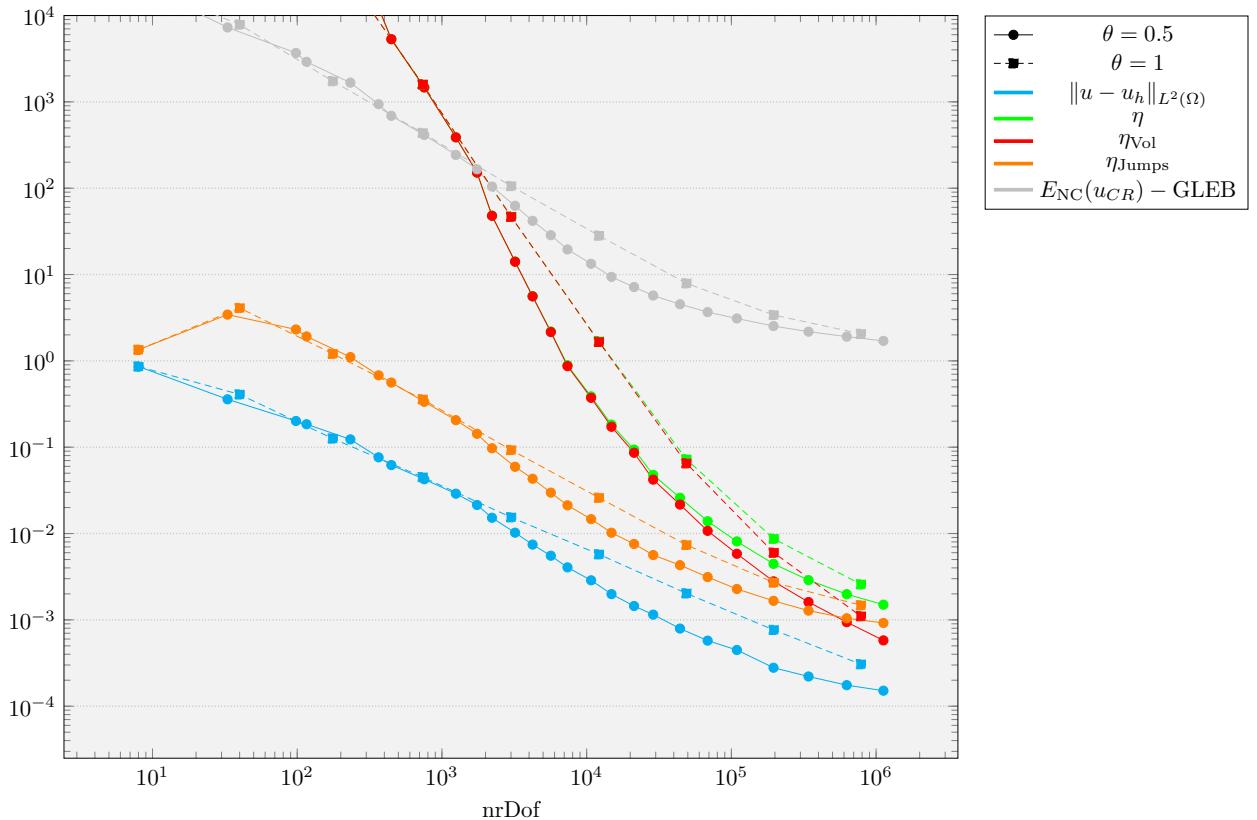


Figure 0.8: convergence history plot for $\alpha = 10^4$

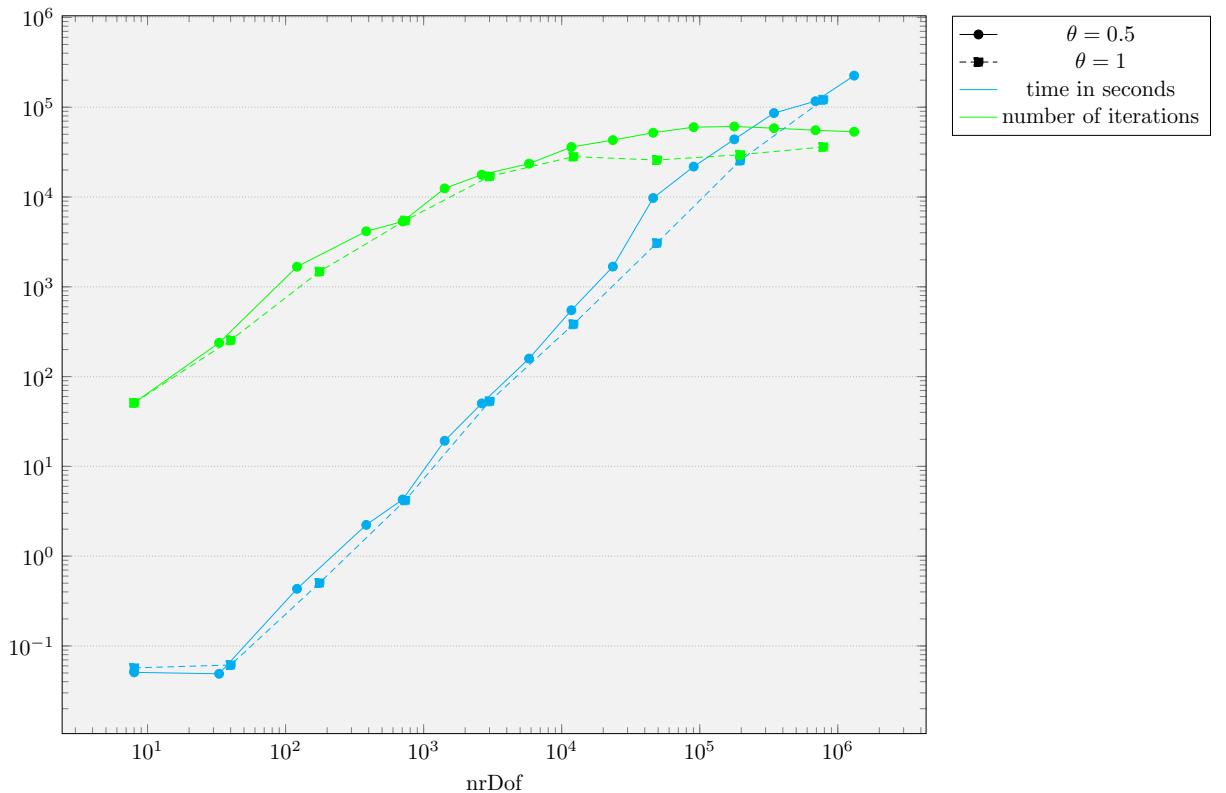


Figure 0.9: development of the number of iterations and the elapsed time for each iteration for $\alpha = 1$

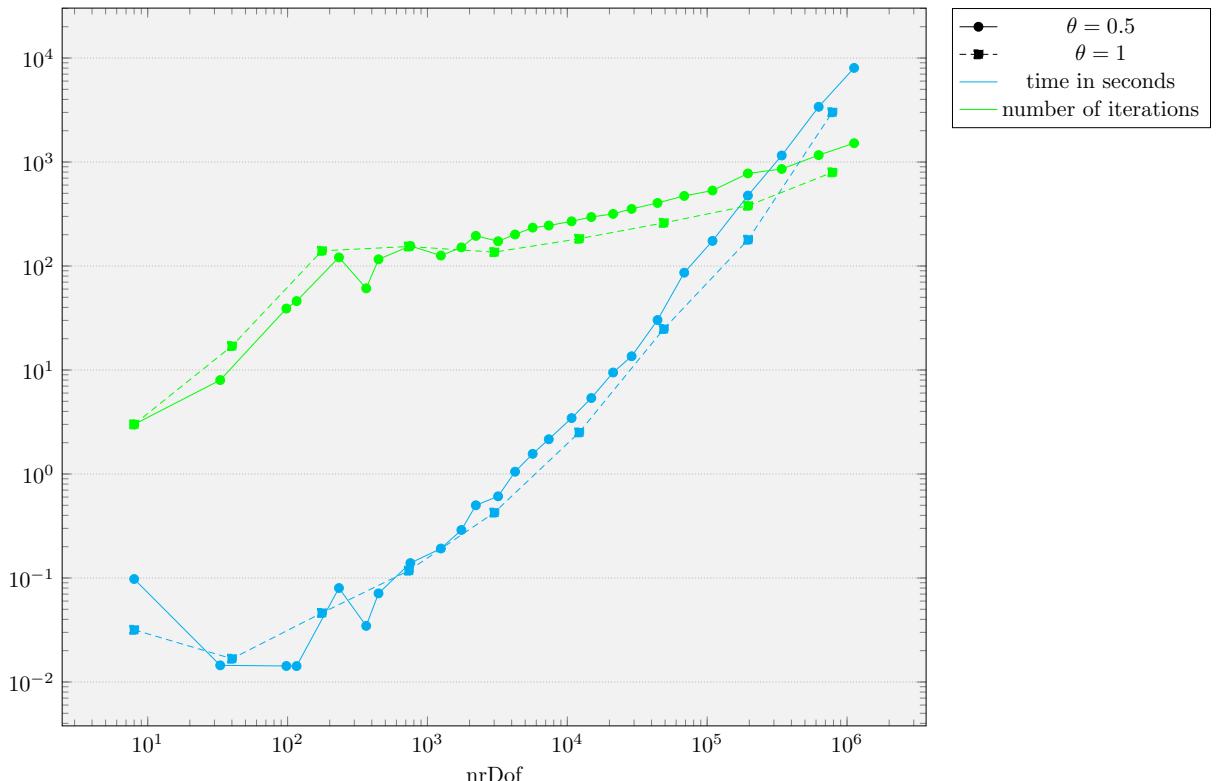


Figure 0.10: development of the number of iterations and the elapsed time for each iteration for $\alpha = 10^4$

0.4 Examples with exact solution

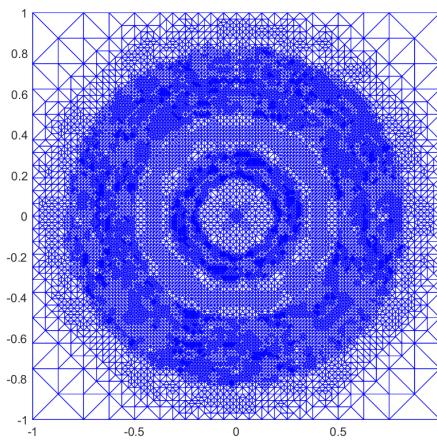


Figure 0.11: adaptive mesh for $\alpha = 1$ and $\theta = 0.5$ with 15393 nodes and 46016 degrees of freedom

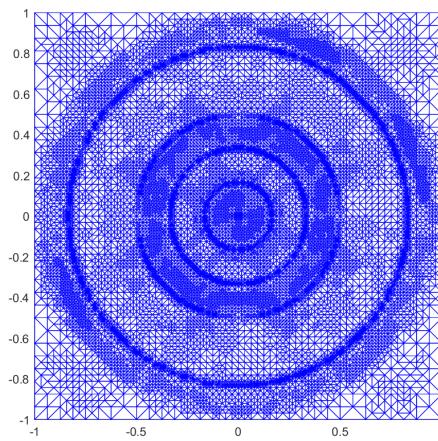


Figure 0.12: adaptive mesh for $\alpha = 10^4$ and $\theta = 0.5$ with 14808 nodes and 44157 degrees of freedom

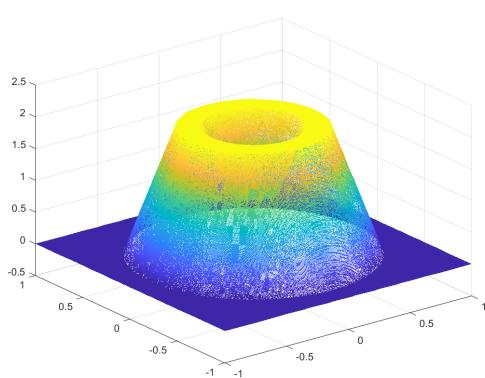


Figure 0.13: last iterate for $\alpha = 1$ and $\theta = 0.5$

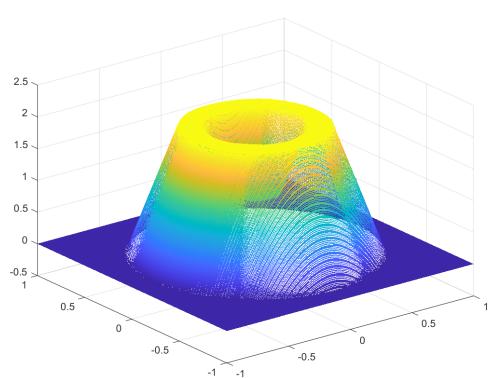


Figure 0.14: last iterate for $\alpha = 1$ and $\theta = 1$

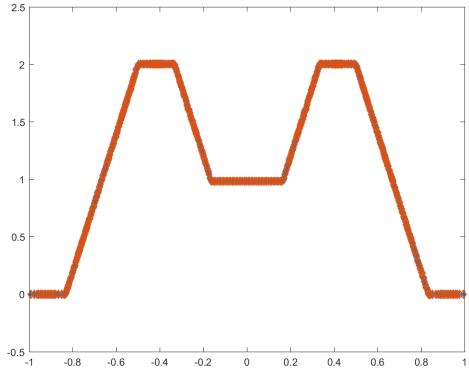


Figure 0.15: last iterate along the axes for $\alpha = 1$ and $\theta = 0.5$

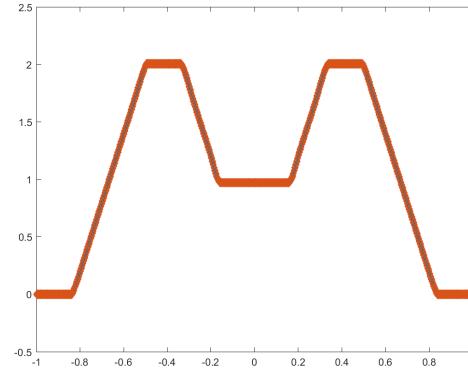


Figure 0.16: last iterate along the axes for $\alpha = 1$ and $\theta = 1$

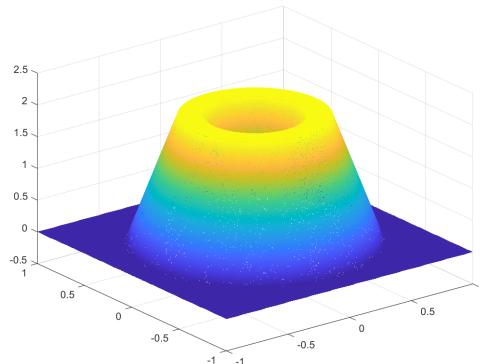


Figure 0.17: last iterate for $\alpha = 10^4$ and $\theta = 0.5$

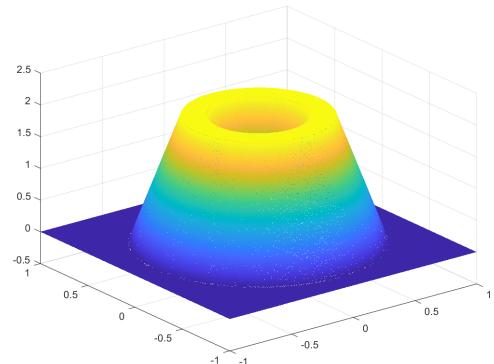


Figure 0.18: last iterate for $\alpha = 10^4$ and $\theta = 1$

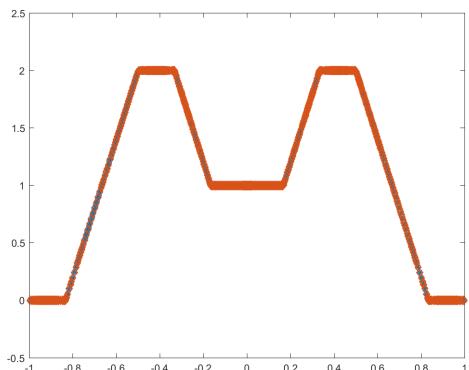


Figure 0.19: last iterate along the axes for $\alpha = 10^4$ and $\theta = 0.5$

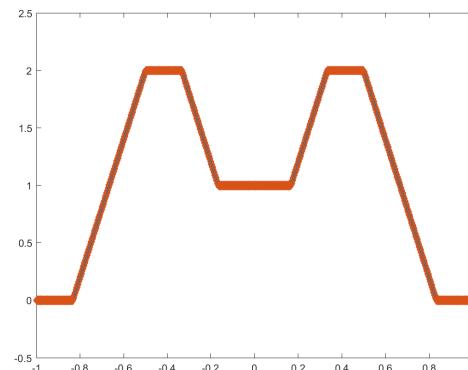


Figure 0.20: last iterate along the axes for $\alpha = 10^4$ and $\theta = 1$

0.4.2 Example 2

For $\alpha = 1$ and $\beta \in (0, 1)$ define f as a function of the radius as

$$f(r) := \begin{cases} \alpha - \frac{4}{1-\beta} \left(\frac{3}{1-\beta} r - 2 \right) & \text{if } 0 \leq r \leq \frac{1-\beta}{2}, \\ -\frac{\alpha}{\beta} \left(r - \frac{1+\beta}{2} \right) + \frac{1}{r} & \text{if } \frac{1-\beta}{2} \leq r \leq \frac{1+\beta}{2}, \\ \frac{-4}{(\beta-1)^3} (16r^2 - 9(\beta+3)r + 12(\beta+1) - \frac{3\beta+1}{r}) & \text{if } \frac{1+\beta}{2} \leq r \leq 1, \end{cases}$$

with exact solution

$$u_2(r) := \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{1-\beta}{2}, \\ -\frac{1}{\beta}r + \frac{1+\beta}{2\beta} & \text{if } \frac{1-\beta}{2} \leq r \leq \frac{1+\beta}{2}, \\ 0 & \text{if } \frac{1+\beta}{2} \leq r. \end{cases}$$

The experiments were conducted for $\theta = 0.5$ and $\beta = \frac{1}{10}, \frac{1}{2}, \frac{9}{10}$.
The corresponding exact energies were computed beforehand.

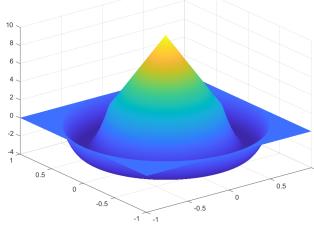


Figure 0.21: f for $\beta = \frac{1}{10}$

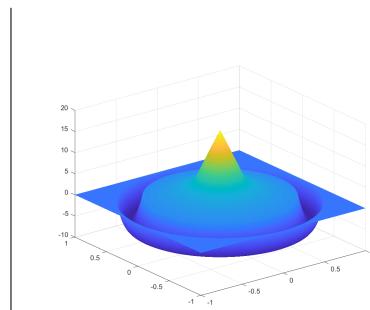


Figure 0.22: f for $\beta = \frac{1}{2}$

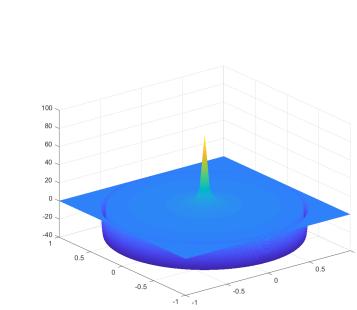


Figure 0.23: f for $\beta = \frac{9}{10}$

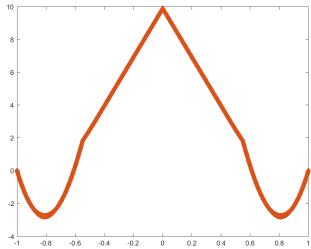


Figure 0.24: f along the axes for $\beta = \frac{1}{10}$

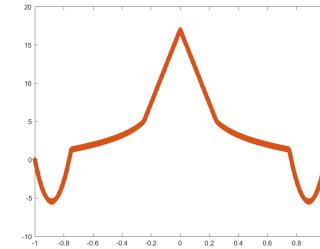


Figure 0.25: f along the axes for $\beta = \frac{1}{2}$

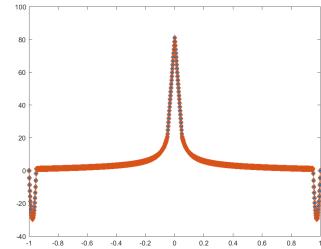


Figure 0.26: f along the axes for $\beta = \frac{9}{10}$

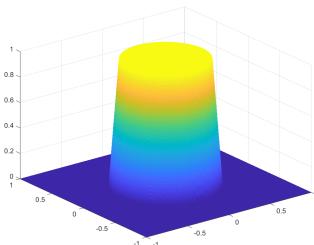


Figure 0.27: u for $\beta = \frac{1}{10}$

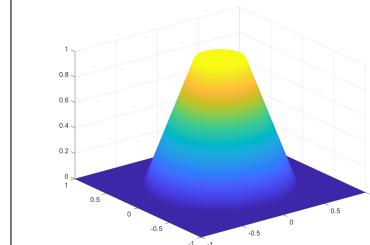


Figure 0.28: u for $\beta = \frac{1}{2}$

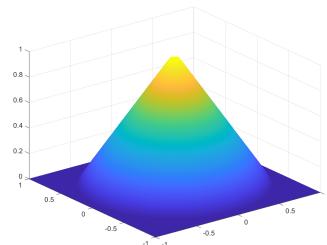


Figure 0.29: u for $\beta = \frac{9}{10}$

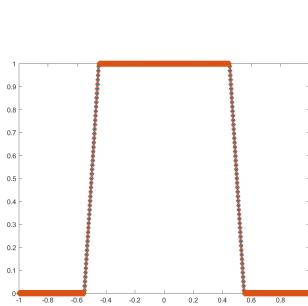


Figure 0.30: u along the axes for $\beta = \frac{1}{10}$

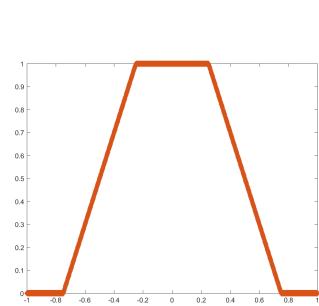


Figure 0.31: u along the axes for $\beta = \frac{1}{2}$

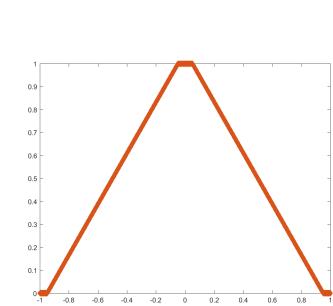


Figure 0.32: u along the axes for $\beta = \frac{9}{10}$

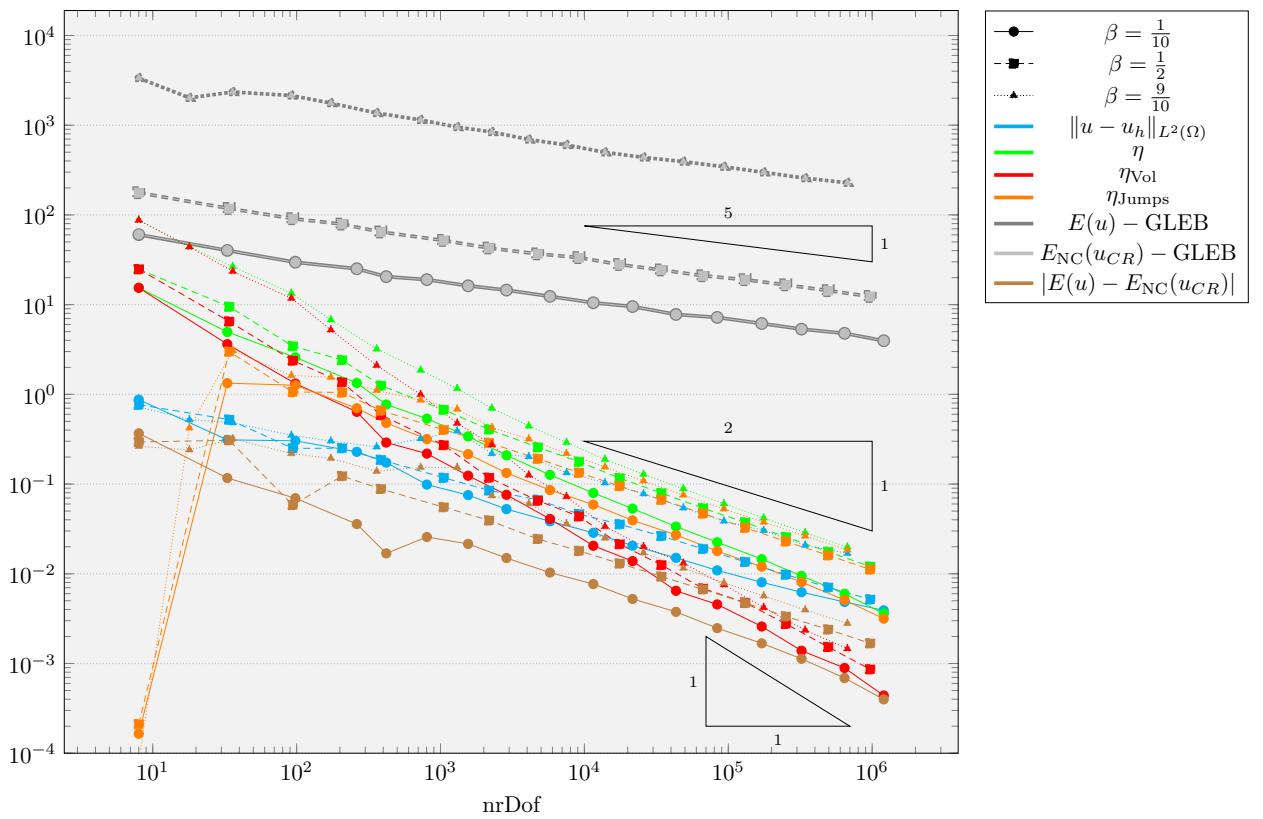


Figure 0.33: convergence history plot

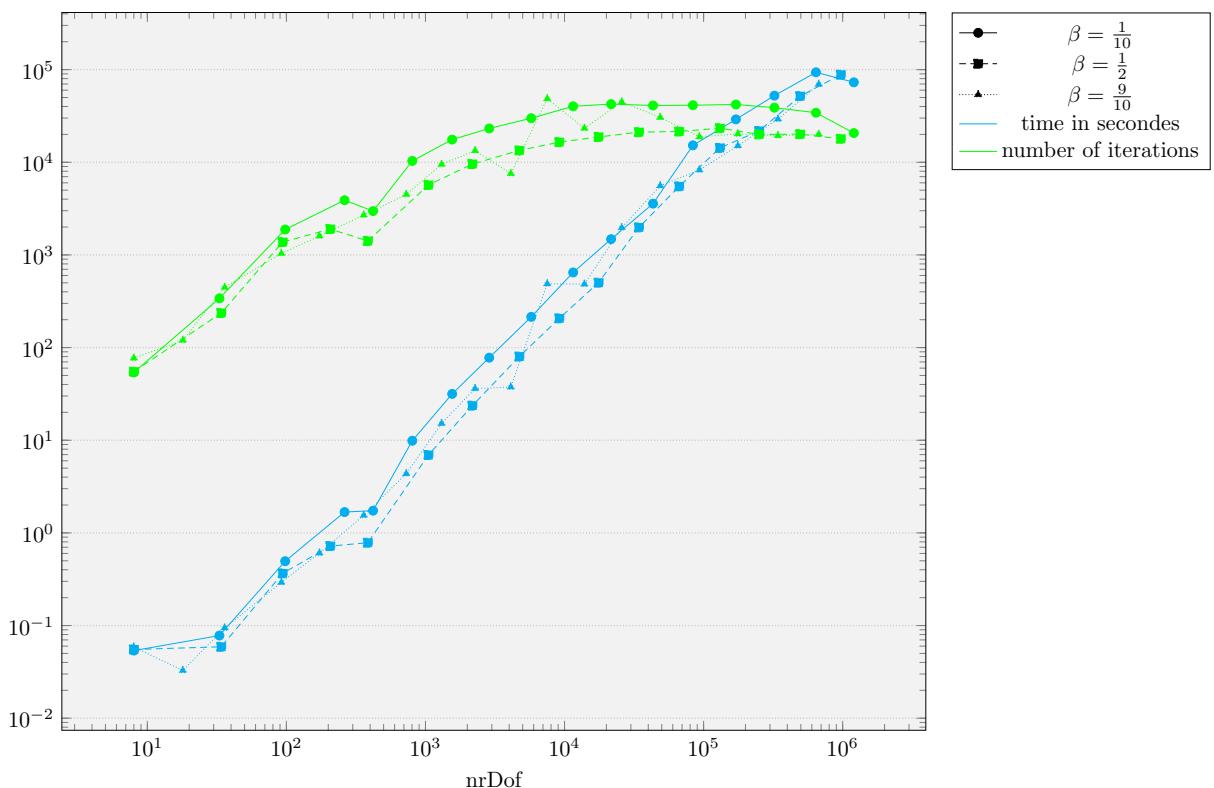


Figure 0.34: development of the number of iterations and the elapsed time for each iteration

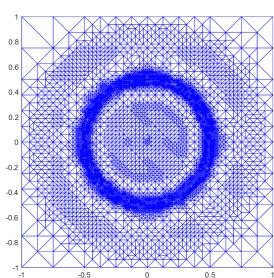


Figure 0.35: adaptive mesh for $\beta = \frac{1}{10}$ with 7237 nodes and 21612 degrees of freedom

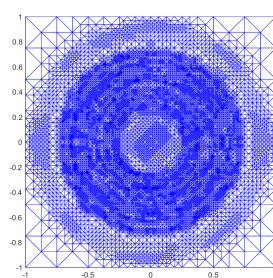


Figure 0.36: adaptive mesh for $\beta = \frac{1}{2}$ with 8675 nodes and 25782 degrees of freedom

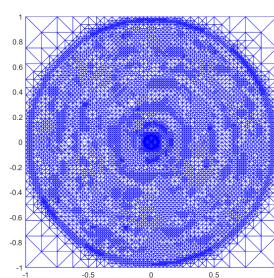


Figure 0.37: adaptive mesh for $\beta = \frac{9}{10}$ with 86752 nodes and 257822 degrees of freedom

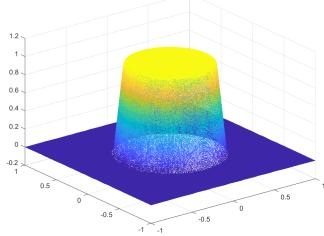


Figure 0.38: last iterate for $\beta = \frac{1}{10}$

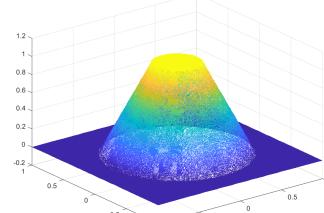


Figure 0.39: last iterate for $\beta = \frac{1}{2}$

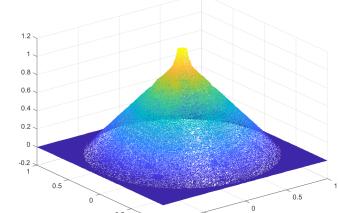


Figure 0.40: last iterate for $\beta = \frac{9}{10}$

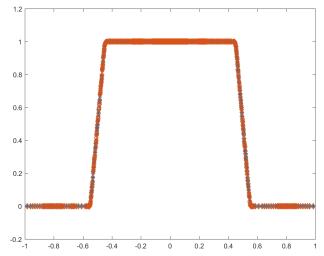


Figure 0.41: last iterate along the axes for $\beta = \frac{1}{10}$

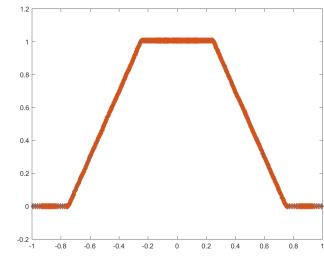


Figure 0.42: last iterate along the axes for $\beta = \frac{1}{2}$

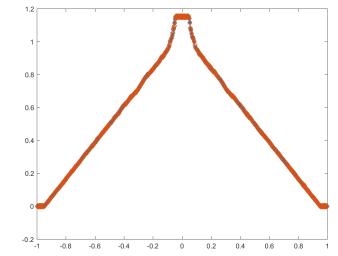


Figure 0.43: last iterate along the axes for $\beta = \frac{9}{10}$

0.5 Application to image

For $\alpha = 10000$ let f represent the grayscale of an image in $[0, 1]^{256 \times 256}$ multiplied with α scaled to the domain $\Omega \in (0, 1)^2$ as seen in fig. 0.44.



Figure 0.44: grayscale plot of the right-hand side f (view from above onto the x - y plane)

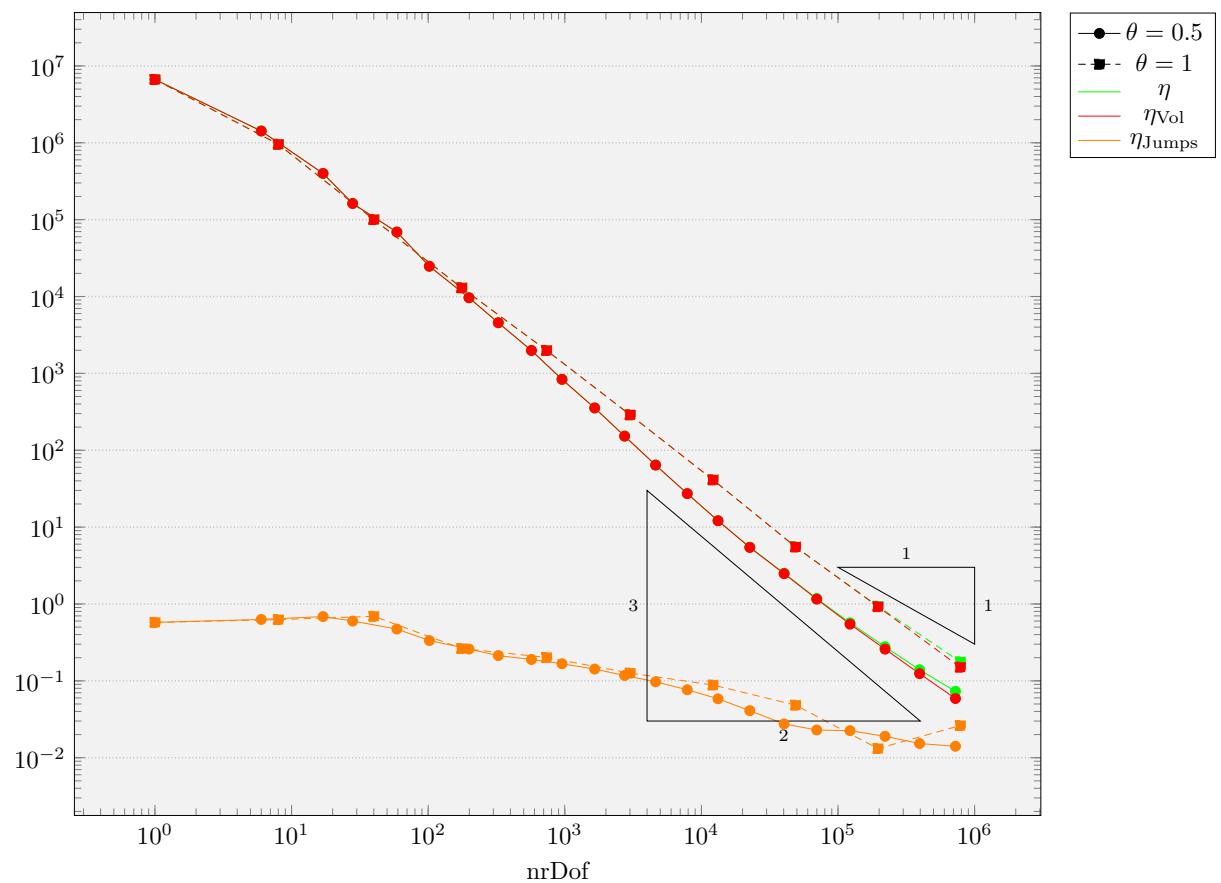


Figure 0.45: convergence history plot

0.5 Application to image

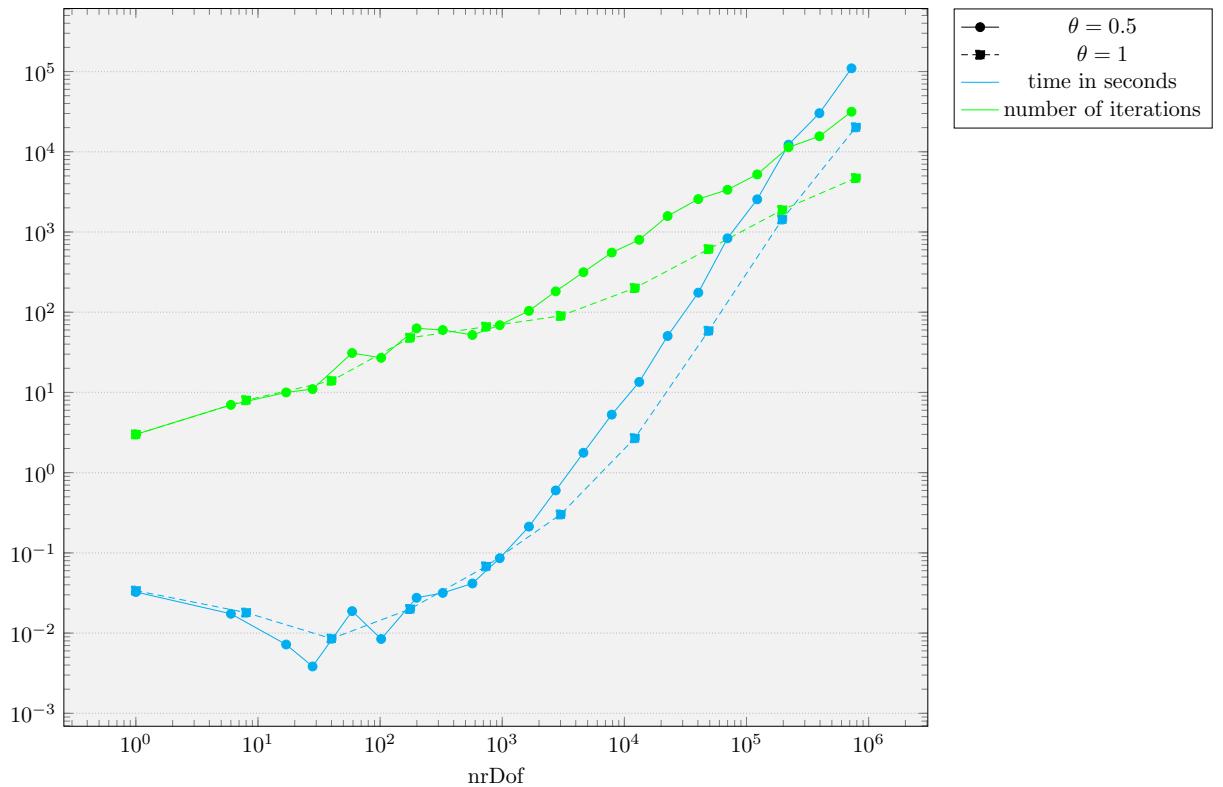


Figure 0.46: development of the number of iterations and the elapsed time for each iteration



Figure 0.47: grayscale plot of last iterate for $\theta = 0.5$



Figure 0.48: grayscale plot of last iterate for $\theta = 1$

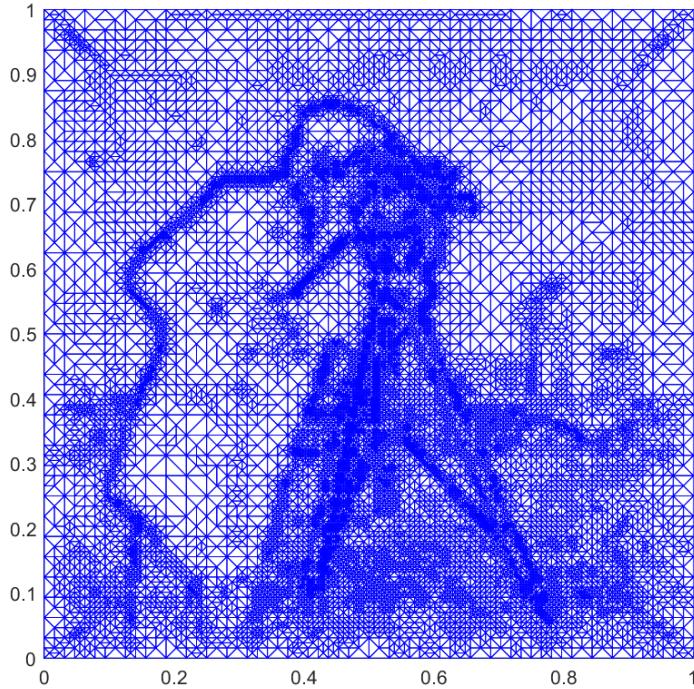


Figure 0.49: adaptive mesh for $\theta = 0.5$ with 13555 nodes and 40300 degrees of freedom

0.6 Application to a function with discontinuity set

For $\alpha = 100$ define

$$f(x) := \begin{cases} 100 & \text{if } \|x\|_\infty \leq \frac{1}{2}, \\ 0 & \text{else} \end{cases} \quad (0.7)$$

on $\Omega = (-1, 1)^2$.

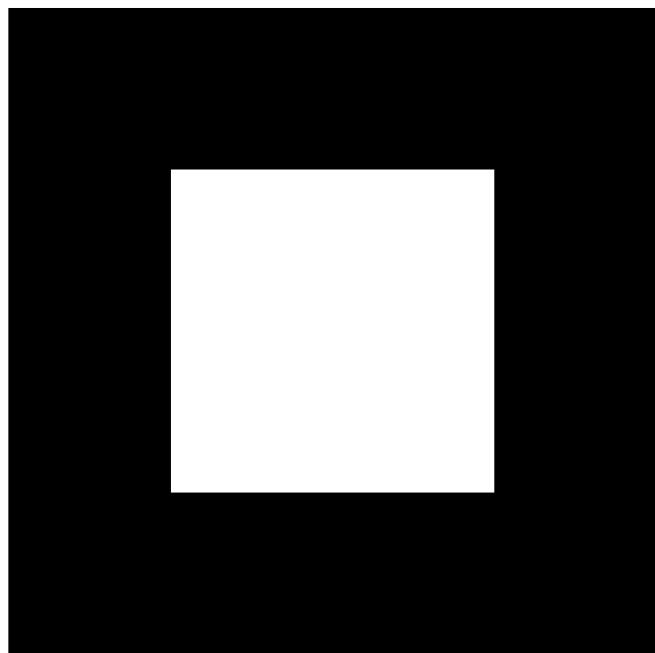


Figure 0.50: grayscale plot of the right-hand side f (view from above onto the x - y plane)

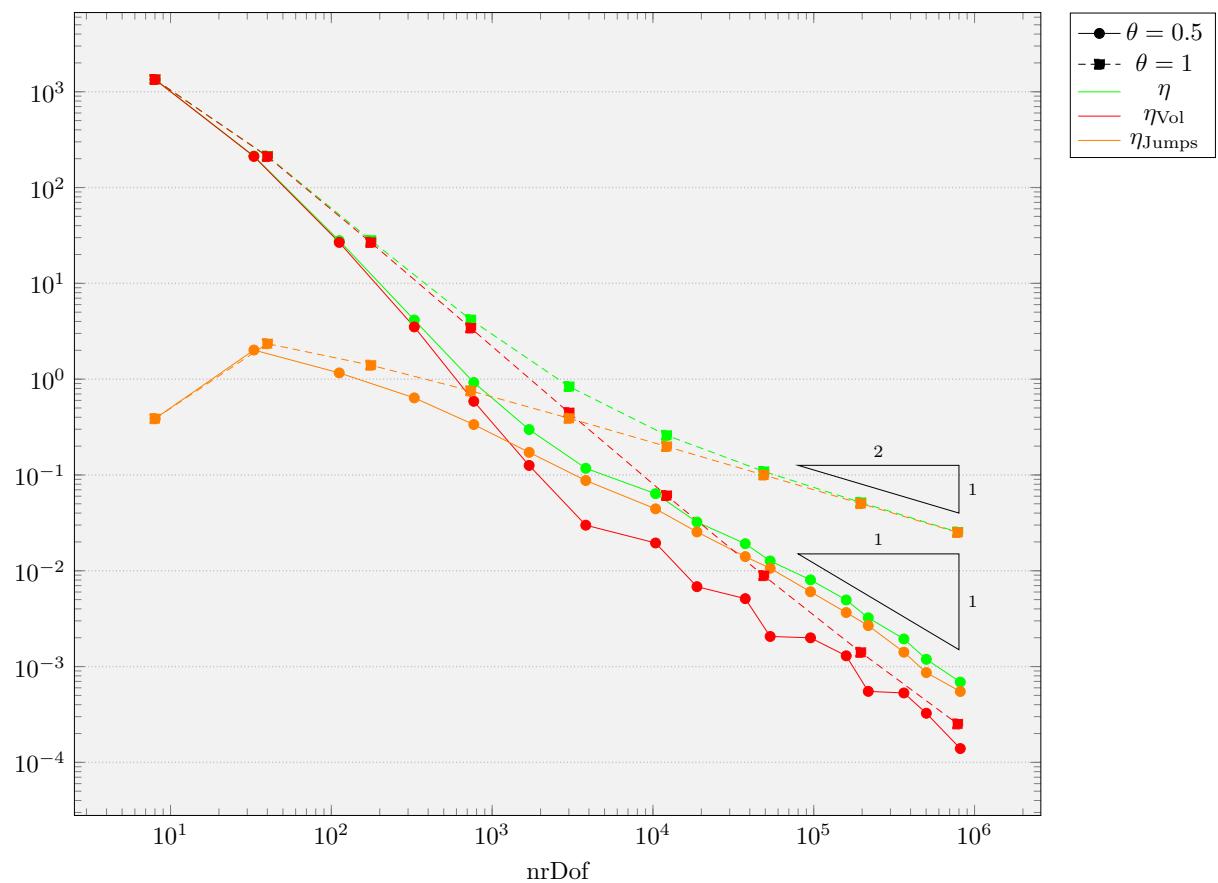


Figure 0.51: convergence history plot

0.6 Application to a function with discontinuity set

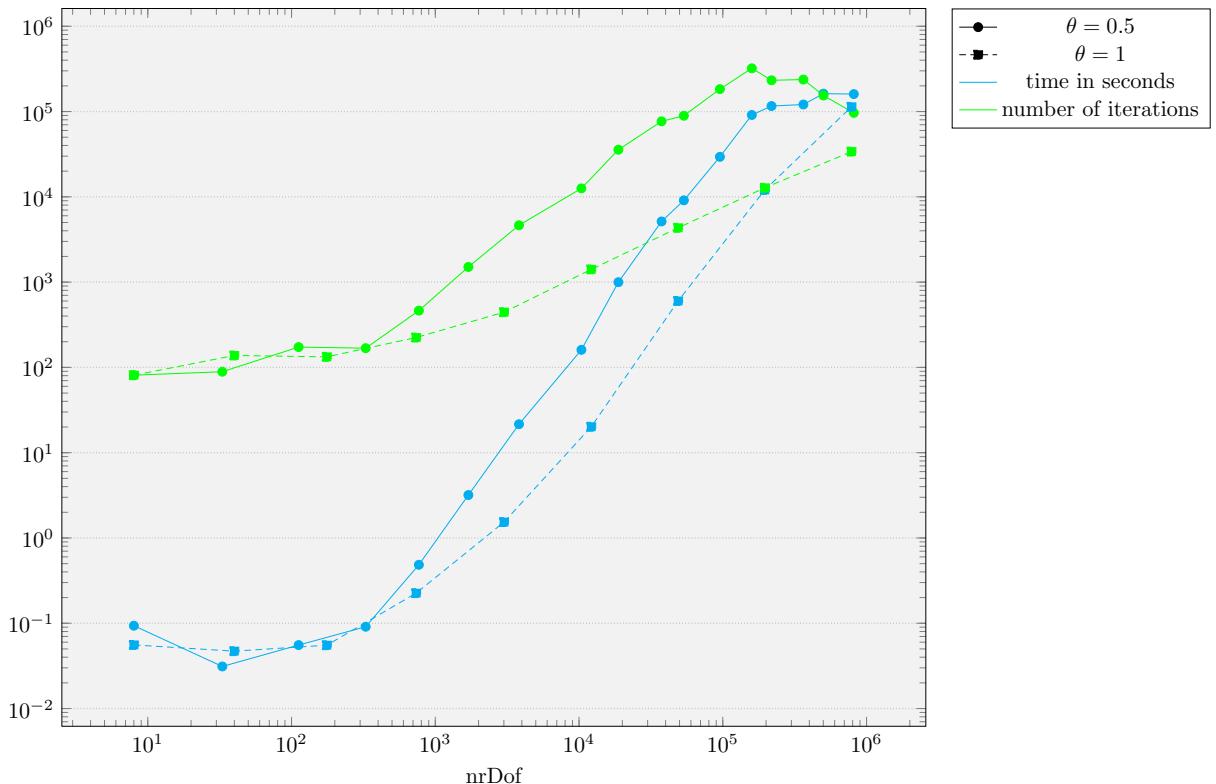


Figure 0.52: development of the number of iterations and the elapsed time for each iteration

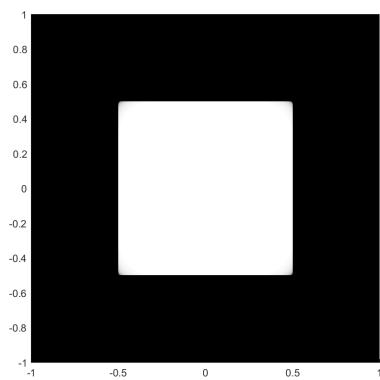


Figure 0.53: grayscale plot of last iterate for $\theta = 0.5$

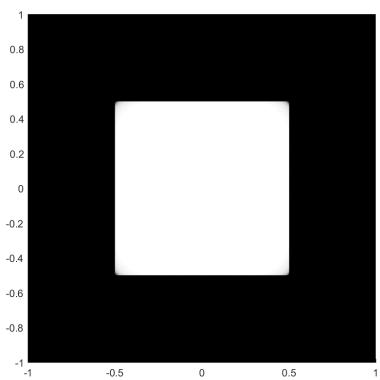


Figure 0.54: grayscale plot of last iterate for $\theta = 1$

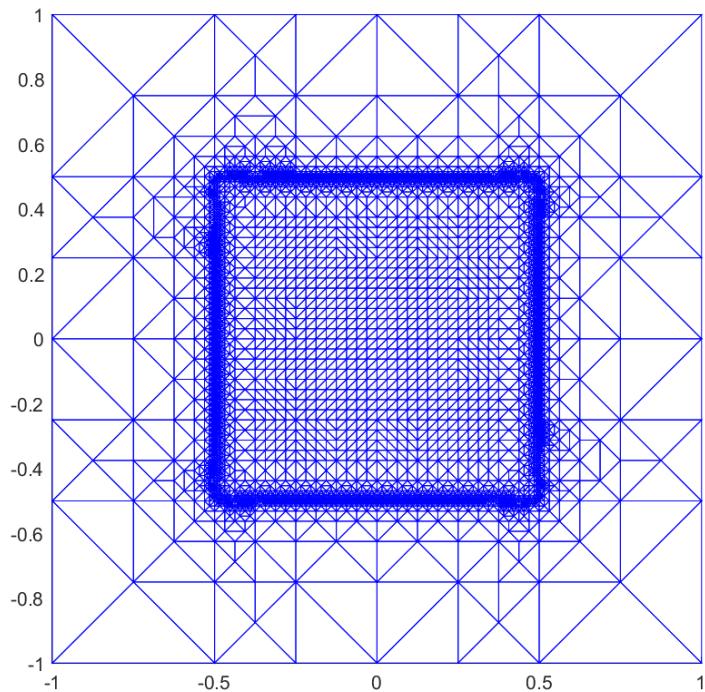


Figure 0.55: adaptive mesh for $\theta = 0.5$ with 6278 nodes and 18783 degrees of freedom

0.7 Another example without known exact solution

Define

$$f(x, y) := \sin(\pi x) \sin(\pi y) \quad (0.8)$$

on $\Omega = (-1, 1)^2$. Let $\alpha = 10^4$ and $\theta = 0.5$.

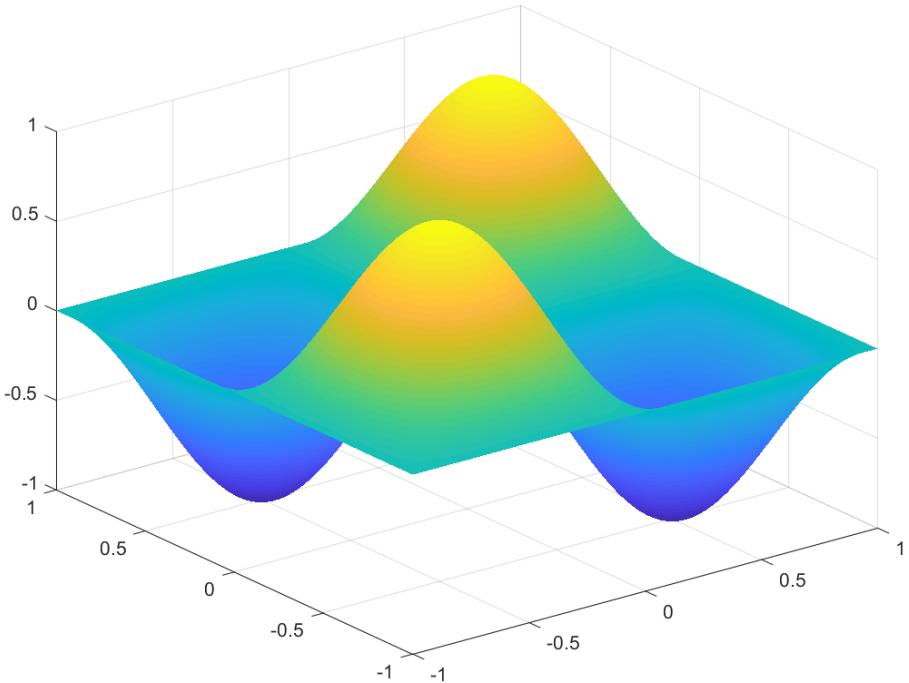


Figure 0.56: plot of the right-hand side f

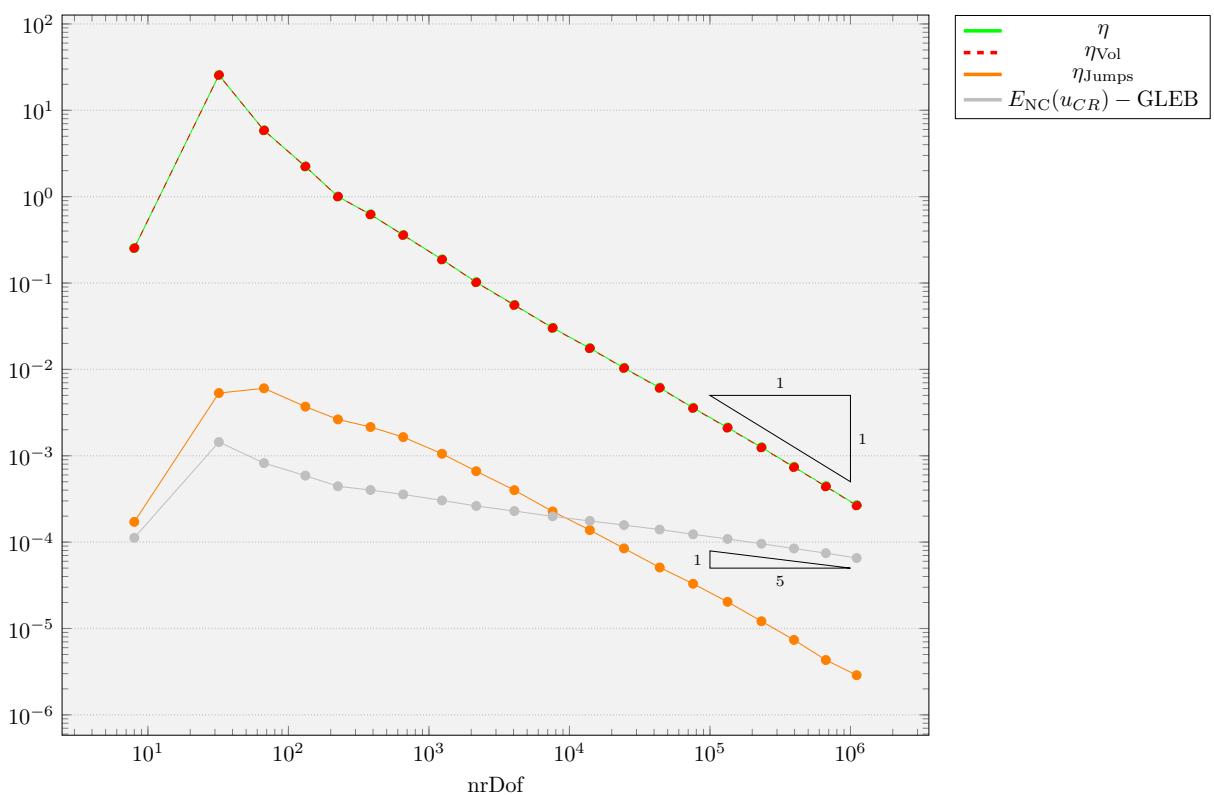


Figure 0.57: convergence history plot

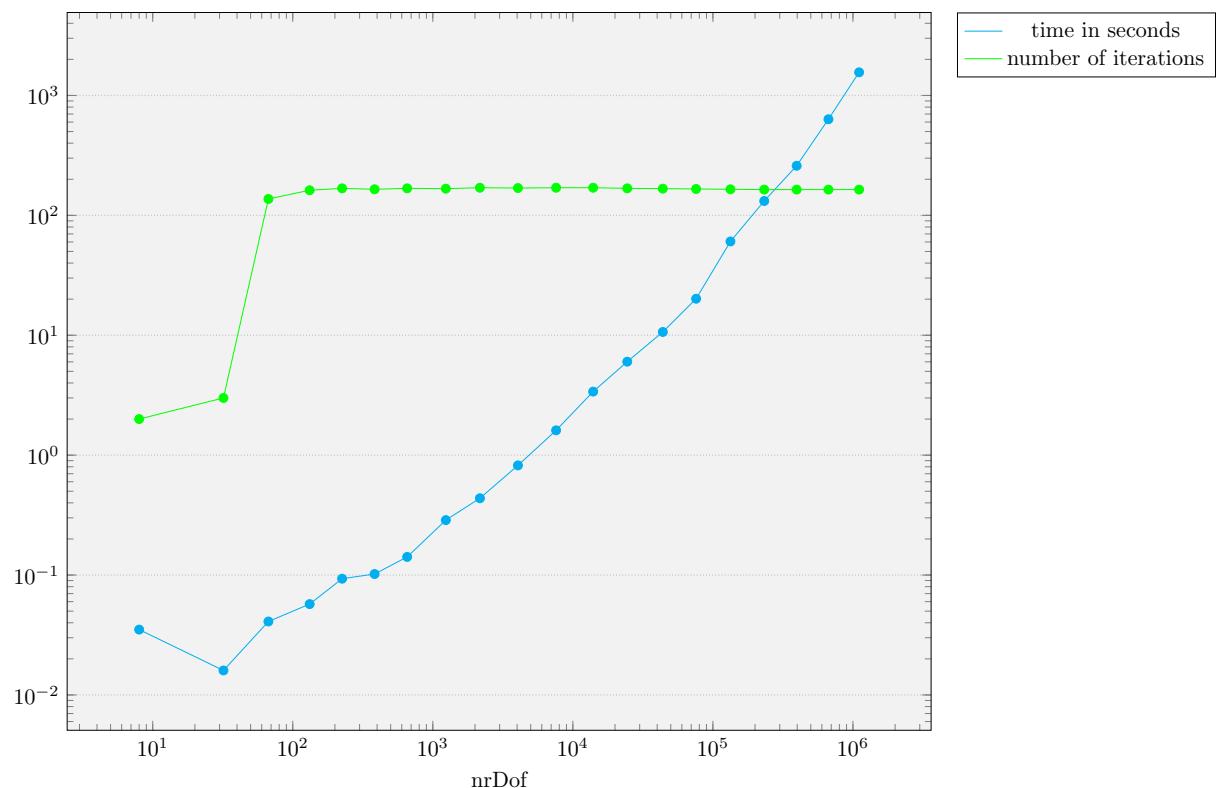


Figure 0.58: development of the number of iterations and the elapsed time for each iteration

0.7 Another example without known exact solution

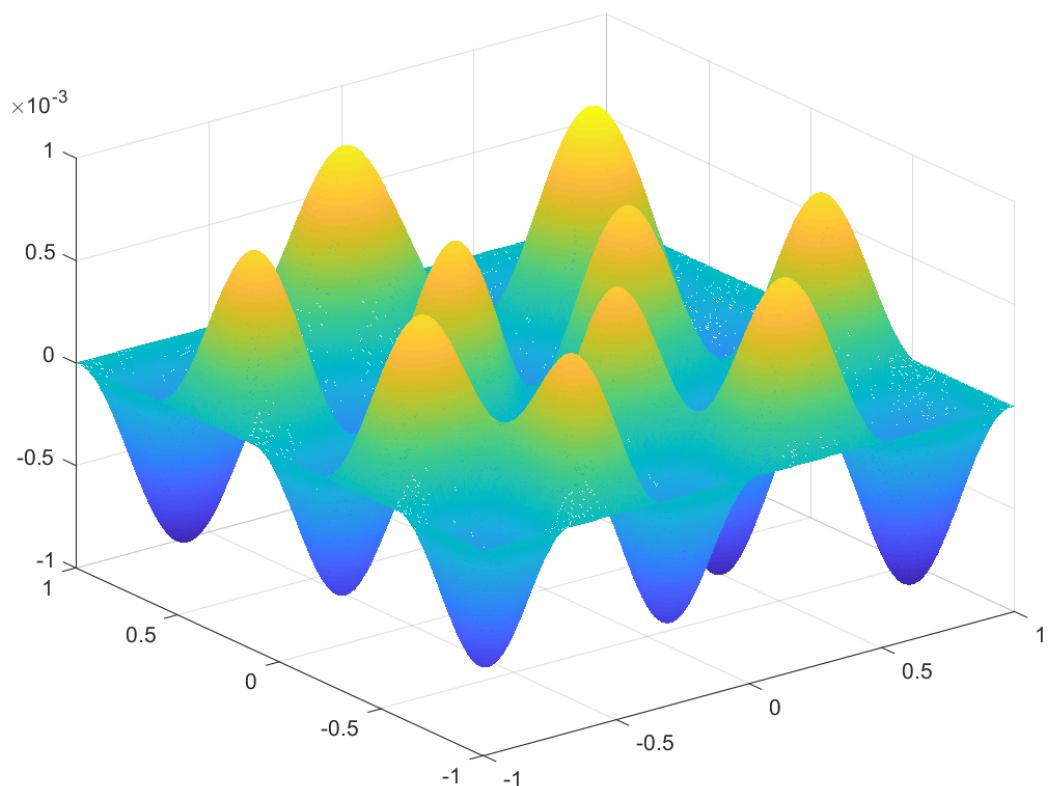


Figure 0.59: plot of last iterate

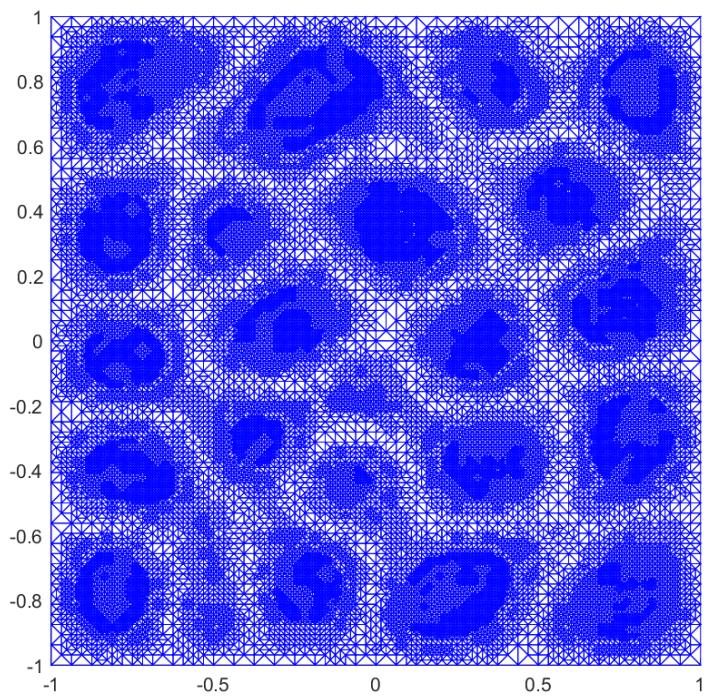


Figure 0.60: adaptive mesh with 25436 nodes and 75889 degrees of freedom