

0.1 The continuous and discrete problem

Let $\alpha > 0$, $\Omega \subset \mathbb{R}^n$ bounded polyhedral Lipschitz domain, and $f \in L^2(\Omega)$.

The continuous problem minimizes

$$E(v) := \frac{\alpha}{2} \|v\|_{L^2(\Omega)}^2 + |v|_{\text{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \int_{\Omega} f v \, dx \quad (0.1)$$

amongst all $v \in V := \text{BV}(\Omega) \cap L^2(\Omega)$ where the BVseminorm $|v|_{\text{BV}(\Omega)}$ is equals to the $W^{1,1}$ seminorm for any $v \in W^{1,1}(\Omega)$.

The nonconforming problem minimizes

$$E_{\text{NC}}(v_{\text{CR}}) := \frac{\alpha}{2} \|v_{\text{CR}}\|_{L^2(\Omega)}^2 + |v_{\text{CR}}|_{1,1,\text{NC}} - \int_{\Omega} f v_{\text{CR}} \, dx \quad (0.2)$$

amongst all $v_{\text{CR}} \in \text{CR}_0^1(\mathcal{T})$ where $|\cdot|_{1,1,\text{NC}} := \|\nabla_{\text{NC}} \cdot\|_{L^1(\Omega)}$.

0.2 Refinement indicator and guaranteed lower energy bound

For some $n \in \mathbb{N}$ (here $n = 2$) and $0 < \beta \leq 1$ define a refinement indicator $\eta := \sum_{T \in \mathcal{T}} \eta(T)$ with

$$\eta(T) := \underbrace{|T|^{2/n} \|f - \alpha u_{\text{CR}}\|_{L^2(T)}^2}_{=: \eta_{\text{Vol}}(T)} + \underbrace{|T|^{\beta/n} \sum_{F \in \mathcal{F}(T)} \|[u_{\text{CR}}]_F\|_{L^1(F)}}_{=: \eta_{\text{Jumps}}(T)} \quad (0.3)$$

for any $T \in \mathcal{T}$.

For $f \in H_0^1(\Omega)$ and $u \in H_0^1(\Omega)$ ($u_{\text{CR}} \in \text{CR}_0^1(\Omega)$) continuous (discrete) minimizer with minimal energy $E(u)$ ($E_{\text{NC}}(u_{\text{CR}})$) it holds

$$E_{\text{NC}}(u_{\text{CR}}) + \frac{\alpha}{2} \|u - u_{\text{CR}}\|_{L^2(\Omega)}^2 - \frac{\kappa_{\text{CR}}}{\alpha} \|h_{\mathcal{T}}(f - \alpha u_{\text{CR}})\|_{L^2(\Omega)} |f|_{1,2} \leq E(u) \quad (0.4)$$

where $|\cdot|_{1,2} = \|\nabla \cdot\|_{L^2(\Omega)}$.

0.3 Example with exact solution

For $\alpha = \beta = 1$ define f as a function of the radius as

$$f(r) := \begin{cases} \alpha - 12(2 - 9r) & \text{if } 0 \leq r \leq \frac{1}{6}, \\ \alpha(1 + (6r - 1)^{\beta}) - \frac{1}{r} & \text{if } \frac{1}{6} \leq r \leq \frac{1}{3}, \\ 2\alpha + 6\pi \sin(\pi(6r - 2)) - \frac{1}{r} \cos(\pi(6r - 2)) & \text{if } \frac{1}{3} \leq r \leq \frac{1}{2}, \\ 2\alpha(\frac{5}{2} - 3r)^{\beta} + \frac{1}{r} & \text{if } \frac{1}{2} \leq r \leq \frac{5}{6}, \\ -3\pi \sin(\pi(6r - 5)) + \frac{1 + \cos(\pi(6r - 5))}{2r} & \text{if } \frac{5}{6} \leq r \leq 1, \end{cases} \quad (0.5)$$

with exact solution

$$u(r) := \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{1}{6}, \\ 1 + (6r - 1)^\beta & \text{if } \frac{1}{6} \leq r \leq \frac{1}{3}, \\ 2 & \text{if } \frac{1}{3} \leq r \leq \frac{1}{2}, \\ 2(\frac{5}{2} - 3r)^\beta & \text{if } \frac{1}{2} \leq r \leq \frac{5}{6}, \\ 0 & \text{if } \frac{5}{6} \leq r \leq 1. \end{cases} \quad (0.6)$$

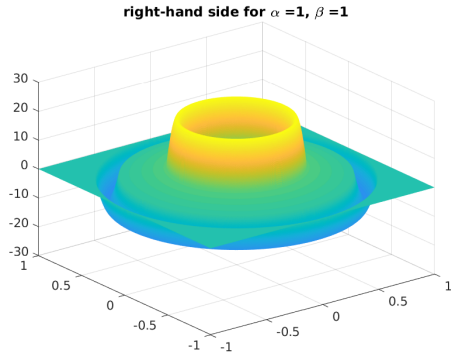


Figure 0.1: right-hand side f

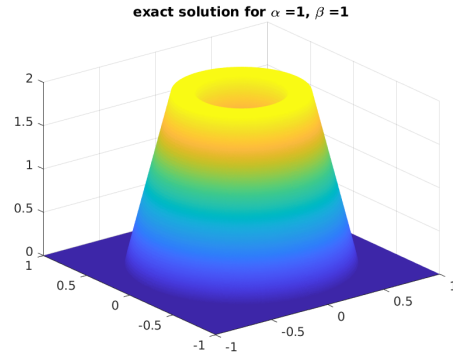


Figure 0.2: exact solution u

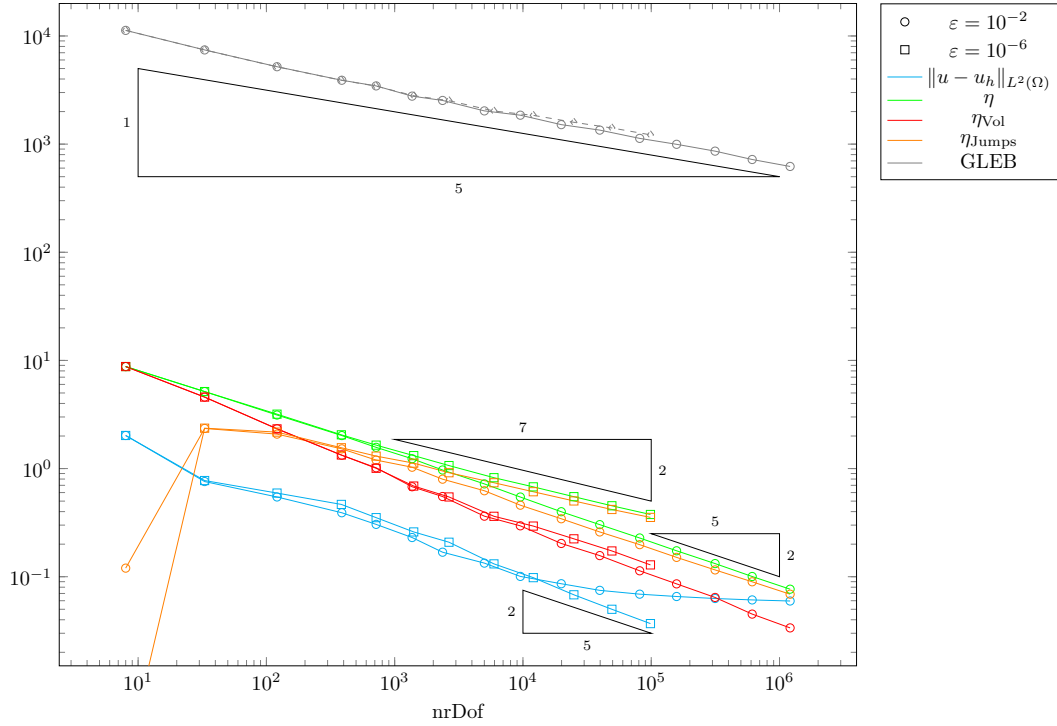


Figure 0.3: convergence history plot for the L^2 error, η , η_{Vol} , η_{Jumps} , and the guaranteed lower energy bound

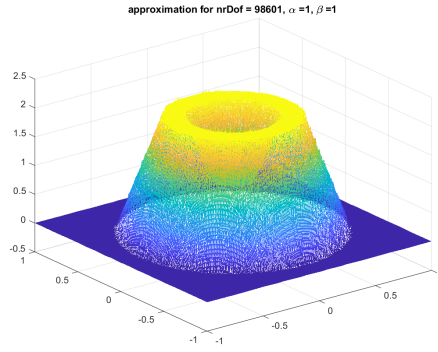
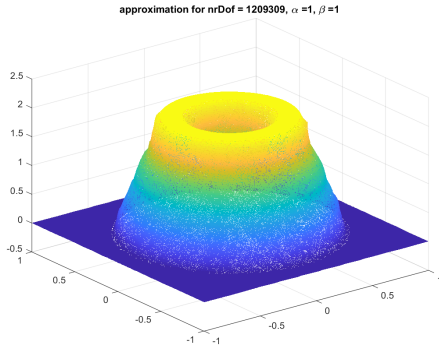


Figure 0.4: last iterate for $\varepsilon = 10^{-2}$ Figure 0.5: last iterate for $\varepsilon = 10^{-6}$

0.4 Application to an image

For $\alpha = 10000$ and $\beta = 1$ let f represent the grayscale of an image in $[0, 1]^{256 \times 256}$ scaled to the domain $\Omega \in (0, 1)^2$ as seen in fig. 0.6.

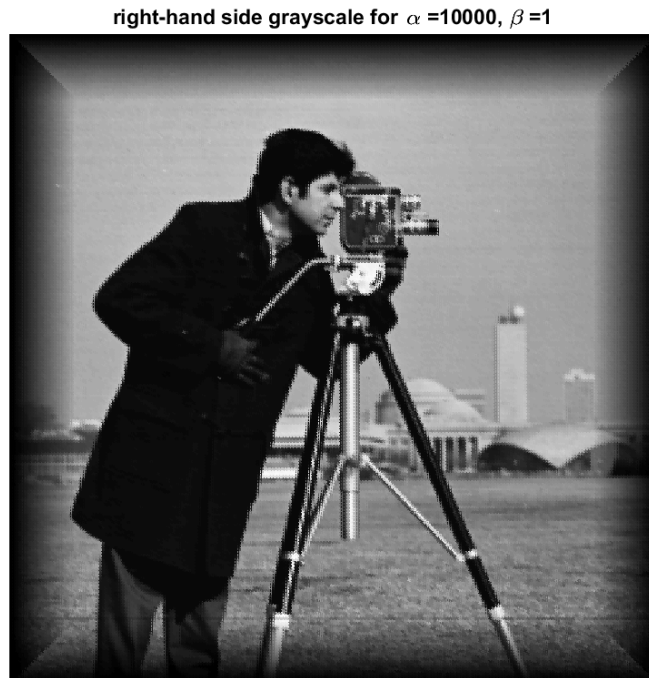


Figure 0.6: grayscale plot of the right-hand side f (view from above onto the x - y plane)

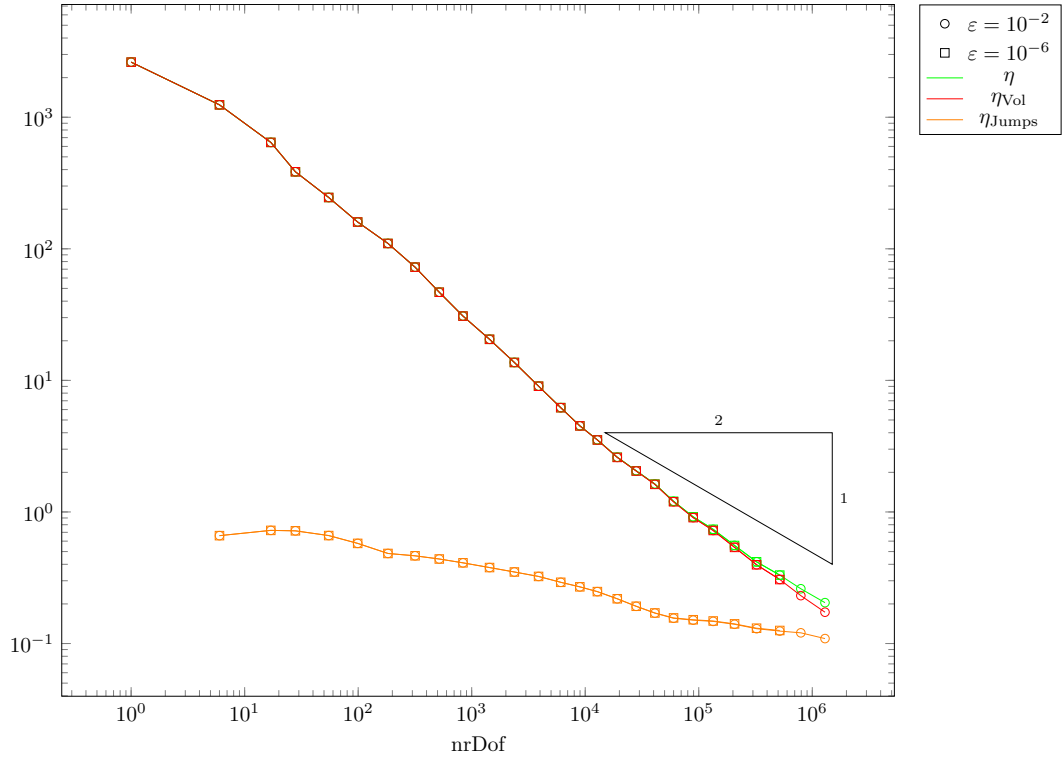


Figure 0.7: convergence history plot for η , η_{Vol} , and η_{Jumps}



Figure 0.8: grayscale plot of last iterate for $\varepsilon = 10^{-2}$ Figure 0.9: grayscale plot of last iterate for $\varepsilon = 10^{-6}$

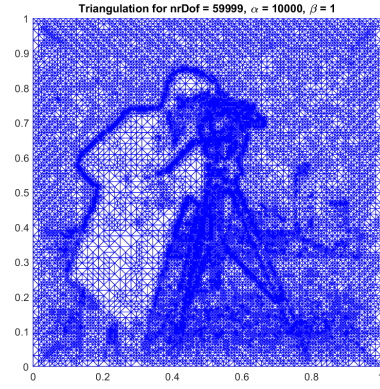
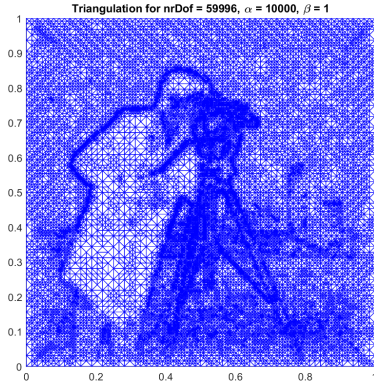


Figure 0.10: adaptive triangulation for 59996 degrees of freedom for $\varepsilon = 10^{-2}$ Figure 0.11: adaptive triangulation for 59999 degrees of freedom for $\varepsilon = 10^{-6}$