The Crouzeix-Raviart Finite Element Method for a Nonconforming Formulation of the Rudin-Osher-Fatemi Model Problem

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Let U be an open subset of $\mathbb{R}^d$. A function $v \in L^1(U)$ is a function of bounded variation iff

$$|v|_{\mathrm{BV}(U)} := \sup_{\substack{\phi \in C_c^1(U;\mathbb{R}^d) \\ \|\phi\|_{L^\infty(U)} \leqslant 1}} \int_U v \operatorname{div}(\phi)\,\mathrm{d}x < \infty.$$

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We have $W^{1,1}(\Omega) \subset \mathrm{BV}(\Omega)$ with $\|v\|_{\mathrm{BV}(\Omega)} = \|v\|_{W^{1,1}(\Omega)}$ for all $v \in W^{1,1}(\Omega)$.

Hedy Attouch, Giuseppe Buttazzo, and Gérard Michaille. **Variational Analysis in Sobolev and BV Spaces. Applications to PDEs and Optimization**. Second Edition. Vol. 17. MOS-SIAM Series on Optimization. Philadelphia: Society for Industrial and Applied Mathematics, Mathematical Optimization Society, 2014. ISBN: 978-1-611973-47-1

Lawrence C. Evans and Ronald F. Gariepy. **Measure Theory and Fine Properties of Functions**. CRC Press, 1992. ISBN: 0-8493-7157-0

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Rudin-Osher-Fatemi (ROF) model problem

For a parameter $\alpha \in \mathbb{R}_+$ and an input signal $g \in L^2(\Omega)$ minimize the functional

$$I(v) := |v|_{\mathrm{BV}(\Omega)} + \frac{\alpha}{2}\|v - g\|^2_{L^2(\Omega)}$$

amongst all $v \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$.

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Original picture[0]

[0] https://homepages.cae.wisc.edu/~ece533/images/cameraman.tif

Original picture[0]

Input signal

The input signal was created by adding AWGN with a SNR of 20 to the original picture.

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$\alpha = 10^5$

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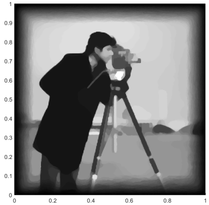

$\alpha = 10^3$

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Original picture

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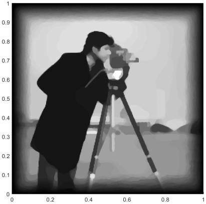

$\alpha = 10^3$

$\alpha = 10^4$

$\alpha = 10^5$

Pascal Getreuer. “Rudin-Osher-Fatemi Total Variation Denoising using Split Bregman”. In: **Image Processing On Line** 2 (2012), pp. 74–95. URL: https://doi.org/10.5201/ipol.2012.g-tvd

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Continuous problem

For a parameter $\alpha \in \mathbb{R}_+$ and an input signal $f \in L^2(\Omega)$ minimize the functional

$$E(v) := \frac{\alpha}{2}\|v\|^2 + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \int_\Omega fv \,\mathrm{d}x$$

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For $f = \alpha g$ the functional E has the same minimizers as

$$I(v) = |v|_{\mathrm{BV}(\Omega)} + \frac{\alpha}{2}\|v - g\|^2_{L^2(\Omega)}$$

in $\left\{v \in \mathrm{BV}(\Omega) \cap L^2(\Omega) \mid \|v\|_{L^1(\partial\Omega)} = 0\right\}$.

Theorem (Existence and uniqueness of a minimizer)

There exists a unique minimizer $u \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$ *for* $E(v) = \frac{\alpha}{2}\|v\|^2 + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \int_\Omega fv \,\mathrm{d}x$ *amongst all* $v \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$.

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Lemma

Let $v \in \mathrm{BV}(\Omega)$. *For all* $x \in \mathbb{R}^d$, *define*

$$\tilde{v}(x) := \begin{cases} v(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^d \setminus \overline{\Omega}. \end{cases}$$

Then $\tilde{v} \in \mathrm{BV}(\mathbb{R}^d)$ *and* $|\tilde{v}|_{\mathrm{BV}(\mathbb{R}^d)} = |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)}$.

Let U be an open subset of $\mathbb{R}^d$.

Definition (Weak convergence in $\mathrm{BV}(U)$)

Let $(v_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(U)$ and $v \in \mathrm{BV}(U)$ with $v_n \to v$ in $L^1(U)$ as $n \to \infty$. Then $(v_n)_{n\in\mathbb{N}}$ converges weakly to v in $\mathrm{BV}(U)$ iff, for all $\phi \in C_0(U;\mathbb{R}^d)$, it holds

$$\int_U v_n \operatorname{div}(\phi)\,\mathrm{d}x \to \int_U v \operatorname{div}(\phi)\,\mathrm{d}x \quad \text{as } n \to \infty.$$

We write $v_n \rightharpoonup v$ as $n \to \infty$.

Theorem

Let $v \in L^1(U)$ *and* $(v_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(U)$ *with* $\sup_{n\in\mathbb{N}} |v_n|_{\mathrm{BV}(U)} < \infty$ *and* $v_n \to v$ *in* $L^1(U)$ *as* $n \to \infty$*. Then* $v \in \mathrm{BV}(U)$ *and* $|v|_{\mathrm{BV}(U)} \leqslant \liminf_{n\to\infty} |v_n|_{\mathrm{BV}(U)}$*. Furthermore,* $v_n \rightharpoonup v$ *in* $\mathrm{BV}(U)$*.*

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Let U be a bounded Lipschitz domain.

Theorem

Let $(v_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(U)$ *be bounded. Then there exists some subsequence* $(v_{n_k})_{k\in\mathbb{N}}$ *of* $(v_n)_{n\in\mathbb{N}}$ *and* $v \in \mathrm{BV}(U)$ *such that* $v_{n_k} \to v$ *in* $L^1(U)$ *as* $k \to \infty$.

Theorem (Stability)

Let $f_1, f_2 \in L^2(\Omega)$. *For* $\ell \in \{1,2\}$, *let* $u_\ell \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$ *minimize*

$$E_\ell(v) := \frac{\alpha}{2}\|v\|^2 + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \int_\Omega f_\ell v \,\mathrm{d}x$$

amongst all $v \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$. *Then*

$$\|u_1 - u_2\| \leqslant \frac{1}{\alpha}\|f_1 - f_2\|.$$

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Sören Bartels. **Numerical Methods for Nonlinear Partial Differential Equations**. Vol. 47. Springer Series in Computational Mathematics. Springer International Publishing, 2015. ISBN: 978-3-319-13796-4. DOI: 10.1007/978-3-319-13797-1, Chapter 10, p. 297-319.

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Let $\mathcal{T}$ be a regular triangulation of Ω.

For all $v_{\mathrm{CR}} \in \mathrm{CR}^1(\mathcal{T})$,

$$|v_{\mathrm{CR}}|_{\mathrm{BV}(\Omega)} = \|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^1(\Omega)} + \sum_{F \in \mathcal{E}(\Omega)} \|[v_{\mathrm{CR}}]_F\|_{L^1(F)}.$$

In particular, $\mathrm{CR}^1(\mathcal{T}) \subset \mathrm{BV}(\Omega)$.

$$E(v_{\mathrm{CR}}) = \frac{\alpha}{2}\|v_{\mathrm{CR}}\|^2 + |v_{\mathrm{CR}}|_{\mathrm{BV}(\Omega)} + \|v_{\mathrm{CR}}\|_{L^1(\partial\Omega)} - \int_\Omega f v_{\mathrm{CR}}\,\mathrm{d}x$$

$$|v_{\mathrm{CR}}|_{\mathrm{BV}(\Omega)} + \|v_{\mathrm{CR}}\|_{L^1(\partial\Omega)} = \|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^1(\Omega)} + \sum_{F\in\mathcal{E}} \|[v_{\mathrm{CR}}]_F\|_{L^1(F)}$$

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Discrete problem

For a parameter $\alpha \in \mathbb{R}_+$ and an input signal $f \in L^2(\Omega)$ minimize the functional

$$E_{\mathrm{NC}}(v_{\mathrm{CR}}) := \frac{\alpha}{2}\|v_{\mathrm{CR}}\|^2 + \|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^1(\Omega)} - \int_\Omega f v_{\mathrm{CR}}\,\mathrm{d}x$$

amongst all $v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$.

$$E(v_{CR}) = \frac{\alpha}{2}\|v_{CR}\|^2 + |v_{CR}|_{BV(\Omega)} + \|v_{CR}\|_{L^1(\partial\Omega)} - \int_\Omega f v_{CR}\,\mathrm{d}x$$

$$|v_{CR}|_{BV(\Omega)} + \|v_{CR}\|_{L^1(\partial\Omega)} = \|\nabla_{NC} v_{CR}\|_{L^1(\Omega)} + \sum_{F\in\mathcal{E}} \|[v_{CR}]_F\|_{L^1(F)}$$

Discrete problem

For a parameter $\alpha \in \mathbb{R}_+$ and an input signal $f \in L^2(\Omega)$ minimize the functional

$$E_{NC}(v_{CR}) := \frac{\alpha}{2}\|v_{CR}\|^2 + \|\nabla_{NC} v_{CR}\|_{L^1(\Omega)} - \int_\Omega f v_{CR}\,\mathrm{d}x$$

amongst all $v_{CR} \in CR_0^1(\mathcal{T})$.

Theorem (Existence and uniqueness of a minimizer)

There exists a unique minimizer $u_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ *for* $E_{\mathrm{NC}}(v_{\mathrm{CR}}) := \frac{\alpha}{2}\|v_{\mathrm{CR}}\|^2 + \|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^1(\Omega)} - \int_\Omega f v_{\mathrm{CR}}\,\mathrm{d}x$ *amongst all* $v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$.

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Let $K := \left\{\Lambda \in L^\infty(\Omega;\mathbb{R}^2) \,\middle|\, |\Lambda(\bullet)| \leqslant 1 \text{ a.e. in } \Omega\right\}$ and, for all $(v_{\mathrm{CR}}, \Lambda_0) \in \mathrm{CR}^1_0(\mathcal{T}) \times P_0(\mathcal{T};\mathbb{R}^2)$,

$$L(v_{\mathrm{CR}}, \Lambda_0) := \int_\Omega \Lambda_0 \cdot \nabla_{\mathrm{NC}} v_{\mathrm{CR}}\,\mathrm{d}x + \frac{\alpha}{2}\|v_{\mathrm{CR}}\|^2 - \int_\Omega f v_{\mathrm{CR}}\,\mathrm{d}x - I_K(\Lambda_0).$$

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Minimax problem

Find $\left(\tilde{u}_{\mathrm{CR}}, \bar{\Lambda}_0\right) \in \mathrm{CR}_0^1(\mathcal{T}) \times P_0\left(\mathcal{T};\mathbb{R}^2\right)$ such that

$$L(\tilde{u}_{\mathrm{CR}}, \bar{\Lambda}_0) = \inf_{v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})} \sup_{\Lambda_0 \in P_0(\mathcal{T};\mathbb{R}^2)} L(v_{\mathrm{CR}}, \Lambda_0).$$

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This problem has a solution
$(\tilde{u}_{\mathrm{CR}}, \bar{\Lambda}_0) \in \mathrm{CR}_0^1(\mathcal{T}) \times \big(P_0(\mathcal{T}; \mathbb{R}^2) \cap K\big)$.

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$$L(\tilde{u}_{\mathrm{CR}},\bar{\Lambda}_0) = \inf_{v_{\mathrm{CR}}\in\mathrm{CR}^1_0(\mathcal{T})}\ \sup_{\Lambda_0\in P_0(\mathcal{T};\mathbb{R}^2)} L(v_{\mathrm{CR}},\Lambda_0).$$

This problem has a solution
$\left(\tilde{u}_{\mathrm{CR}},\bar{\Lambda}_0\right) \in \mathrm{CR}^1_0(\mathcal{T}) \times \left(P_0\left(\mathcal{T};\mathbb{R}^2\right)\cap K\right)$.

R. Tyrrell Rockafellar. **Convex Analysis**. New Jersey: Princeton University Press, 1970. ISBN: 0-691-08069-0

Theorem (Equivalent characterizations)

For a function $\tilde{u}_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ *the following statements are equivalent.*

(i) $\tilde{u}_{\mathrm{CR}}$ *solves the discrete problem.*

(ii) *There exists* $\bar{\Lambda}_0 \in P_0\big(\mathcal{T};\mathbb{R}^2\big)$ *with* $\big|\bar{\Lambda}_0(\bullet)\big| \leqslant 1$ *a.e. in* Ω *s.t.*

$$\bar{\Lambda}_0(\bullet) \cdot \nabla_{\mathrm{NC}} \tilde{u}_{\mathrm{CR}}(\bullet) = |\nabla_{\mathrm{NC}} \tilde{u}_{\mathrm{CR}}(\bullet)| \quad \text{a.e. in } \Omega$$

and

$$\big(\bar{\Lambda}_0, \nabla_{\mathrm{NC}} v_{\mathrm{CR}}\big) = (f - \alpha \tilde{u}_{\mathrm{CR}}, v_{\mathrm{CR}}) \quad \text{for all } v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T}).$$

(iii) *For all* $v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$,

$$(f - \alpha \tilde{u}_{\mathrm{CR}}, v_{\mathrm{CR}} - \tilde{u}_{\mathrm{CR}}) \leqslant \|\nabla_{\mathrm{NC}} v_{\mathrm{CR}}\|_{L^1(\Omega)} - \|\nabla_{\mathrm{NC}} \tilde{u}_{\mathrm{CR}}\|_{L^1(\Omega)}\,.$$

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Algorithm (Primal-dual iteration)

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$$v_j := \frac{u_j - u_{j-1}}{\tau}.$$ *Terminate iteration if* $|||v_j||| < \varepsilon_{\text{stop}}$.

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Theorem (Convergence of the primal-dual iteration)

Let $u_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ *solve the discrete problem,* $\bar{\Lambda}_0 \in P_0\big(\mathcal{T};\mathbb{R}^2\big)$ *satisfy* $\big|\bar{\Lambda}_0(\bullet)\big| \leqslant 1$ *a.e. in* Ω *as well as*

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$$\sum_{j=1}^{\infty}\|u_{\mathrm{CR}}-u_j\|^2 \leqslant \frac{1}{2\alpha\tau}\left(\|\!|u_{\mathrm{CR}}-u_0|\!\|_{\mathrm{NC}}^2 + \|\bar{\Lambda}_0-\Lambda_0\|^2\right).$$

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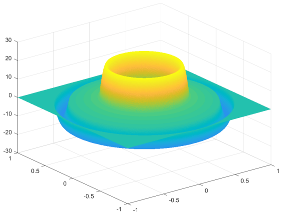
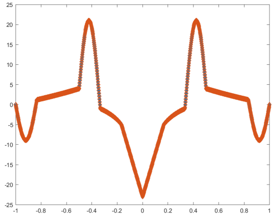
Let $\Omega = (-1,1)^2$. Define $f \in H_0^1(\Omega)$ by $f(x) = \tilde{f}(|x|)$ for all $x \in \Omega$ with

$$\tilde{f}(r) := \begin{cases} \alpha - 12(2-9r) & \text{if } 0 \leqslant r \leqslant \frac{1}{6}, \\ 6r\alpha - \frac{1}{r} & \text{if } \frac{1}{6} \leqslant r \leqslant \frac{1}{3} \\ 2\alpha + 6\pi \sin(\pi(6r-2)) - \frac{1}{r}\cos(\pi(6r-2)) & \text{if } \frac{1}{3} \leqslant r \leqslant \frac{1}{2}, \\ \alpha(5-6r) + \frac{1}{r} & \text{if } \frac{1}{2} \leqslant r \leqslant \frac{5}{6}, \\ -3\pi \sin(\pi(6r-5)) + \frac{1+\cos(\pi(6r-5))}{2r} & \text{if } \frac{5}{6} \leqslant r \leqslant 1. \end{cases}$$



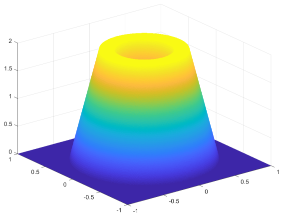
f for $\alpha = 1$



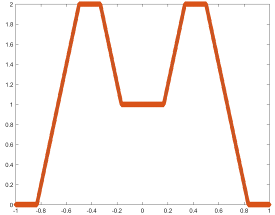
f for $\alpha = 1$ along the axes

Then the solution to the continuous problem with input signal f is given by $u \in H_0^1(\Omega)$ defined by $u(x) = \tilde{u}(|x|)$ for all $x \in \Omega$ with

$$\tilde{u}(r) := \begin{cases} 1 & \text{if } 0 \leqslant r \leqslant \frac{1}{6}, \\ 6r & \text{if } \frac{1}{6} \leqslant r \leqslant \frac{1}{3}, \\ 2 & \text{if } \frac{1}{3} \leqslant r \leqslant \frac{1}{2}, \\ 5 - 6r & \text{if } \frac{1}{2} \leqslant r \leqslant \frac{5}{6}, \\ 0 & \text{if } \frac{5}{6} \leqslant r. \end{cases}$$



u



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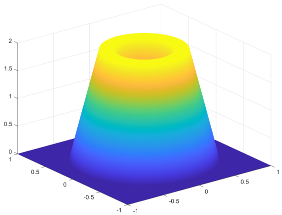

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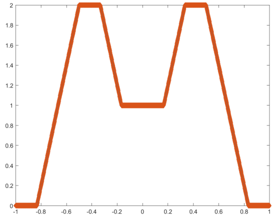

u along the axes

It holds $E(u) \approx -2.058034062391$.

For $\alpha = 10000$ let the input signal represent the grayscale of an image in $[0,1]^{256\times 256}$ multiplied with α scaled to the domain $\Omega = (0,1)^2$.

Image cameraman

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TODO

show that the energy converges from above to the exact energy (not necessarily monotonly, i.e. choose some example where it simply converges and one example where there are oscillations, i.e. two pictures here

also mention that it converges to something slightly below the exact energy, i.e. choose the pictures accordingly, maybe even plot the error between the energies of the iterates and the exact energy

THAT MEANS choose one picture from f where the exact energy can be seen and maybe one from the cameraman

mention that during the afem loop we will see, that this undershooting decrease (as one would expect, the discrete energy converges to the exact energy)

definitely mention degrees of freedom if only one level is shown!!!

show the result of an iteration (plot of uApprox next to uExact) (along the axes as well) and show cameraman grayscale plot definitely mention degrees of freedom if only one level is shown!!!

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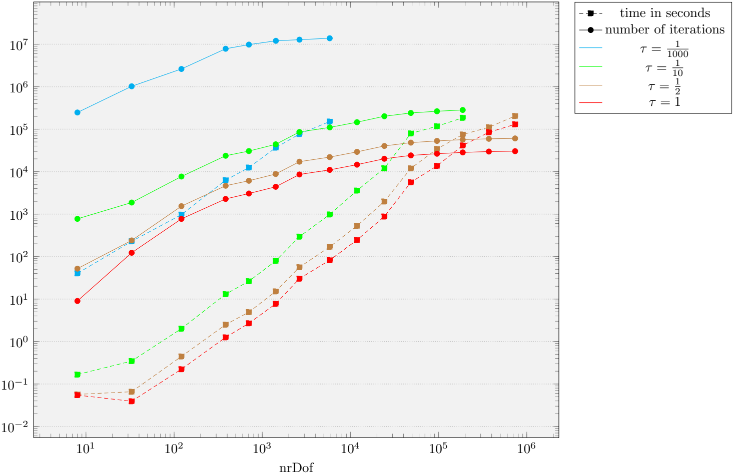
Choice of τ

For the rest of the presentation (unless otherwise specified) let the bulk parameter be $\theta = 0.5$, and $\varepsilon_{\text{stop}} = 10^{-4}$.



Input signal f

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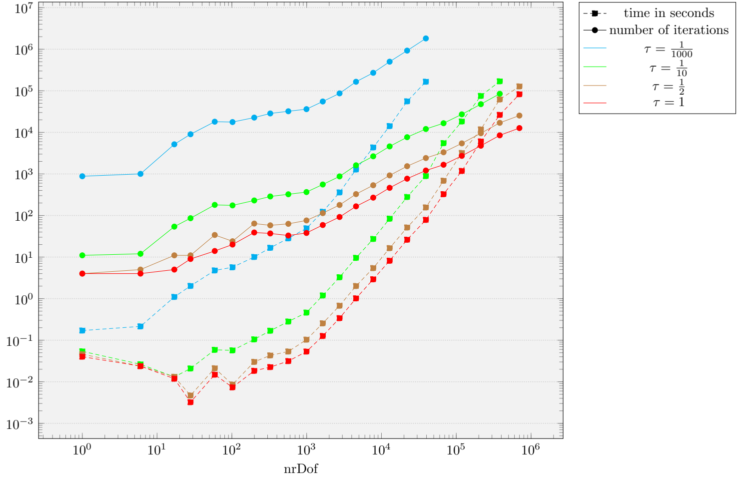

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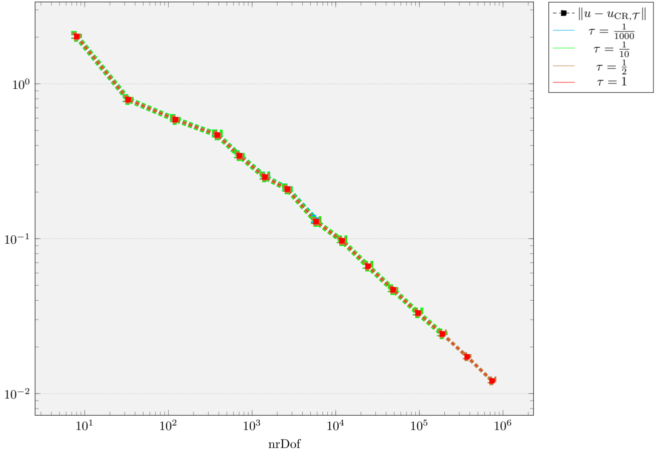

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Conclusion and Hypothesis

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$$\sum_{j=1}^{\infty} \|u_{\mathrm{CR}} - u_j\|^2 \leqslant \frac{1}{2\alpha\tau} \left(|||u_{\mathrm{CR}} - u_0|||_{\mathrm{NC}}^2 + \|\bar{\Lambda}_0 - \Lambda_0\|^2 \right).$$

Settings with $\tau = 1.2$ and no convergence were observed.

For $v_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$, define $J_1 : \mathrm{CR}_0^1(\mathcal{T}) \to P_1(\mathcal{T}) \cap C_0(\Omega)$ by

$$J_1 v_{\mathrm{CR}}(z) := |\mathcal{T}(z)|^{-1} \sum_{T \in \mathcal{T}(z)} v_{\mathrm{CR}}|_T(z) \quad \text{for all } z \in \mathcal{N}(\Omega).$$

Conclusion and Hypothesis

For the rest of the presentation $\tau = 1$.

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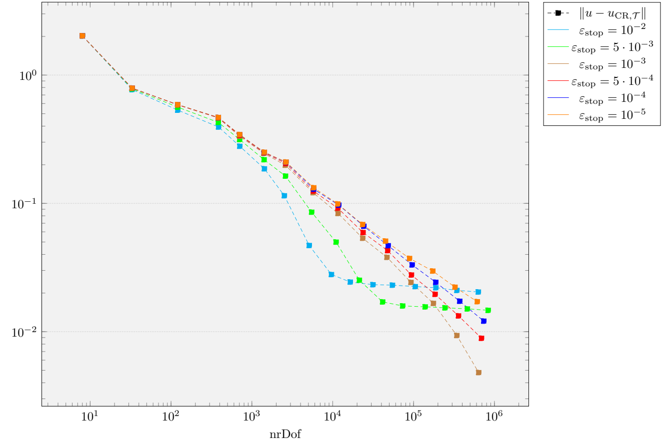
Use $\hat{u}_0 := J_1 u_{\mathrm{CR},\mathcal{T}} \in P_1(\mathcal{T}) \cap C_0(\Omega) \subseteq P_1(\hat{\mathcal{T}}) \cap C_0(\Omega) \subseteq \mathrm{CR}_0^1(\hat{\mathcal{T}})$ as input for the iteration on the refined triangulation $\hat{\mathcal{T}}$.

Choice of $\varepsilon_{\text{stop}}$

With $\tau = 1$ the stopping criterion reads $|||u_j - u_{j-1}|||_{\text{NC}} < \varepsilon_{\text{stop}}$.



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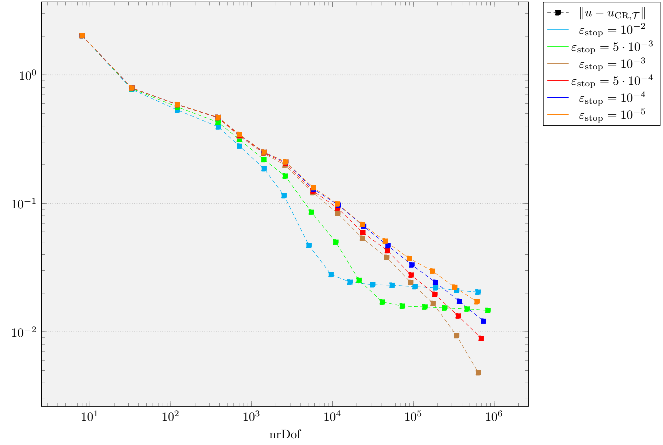

For the rest of the presentation $\varepsilon_{\text{stop}} = 10^{-4}$.

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TODO translate and rewrite

Theorem

Sei Ω konvex, $f \in H^1_0(\Omega)$ das Eingangssignal für mit Lösung $u \in H^1_0(\Omega)$ und minimaler Energie $E(u)$ sowie für mit Lösung $u_{\mathrm{CR}} \in \mathrm{CR}^1_0(\Omega)$ und minimaler Energie $E_{\mathrm{NC}}(u_{\mathrm{CR}})$. Dann gilt

$$E_{\mathrm{NC}}(u_{\mathrm{CR}}) + \frac{\alpha}{2}\|u - u_{\mathrm{CR}}\|^2 - \frac{\kappa_{\mathrm{CR}}}{\alpha}\|h_{\mathcal{T}}(f - \alpha u_{\mathrm{CR}})\|\|\nabla f\| \leqslant E(u).$$

Dabei ist die Konstante $\kappa_{\mathrm{CR}} := \sqrt{1/48 + 1/j_{1,1}^2}$ mit der kleinsten positiven Nullstelle $j_{1,1}$ der Bessel-Funktion erster Art. Insbesondere gilt dann für

$$E_{\mathrm{GLEB}} := E_{\mathrm{NC}}(u_{\mathrm{CR}}) - \frac{\kappa_{\mathrm{CR}}}{\alpha}\|h_{\mathcal{T}}(f - \alpha u_{\mathrm{CR}})\|\|\nabla f\|, \tag{1}$$

dass $E_{\mathrm{NC}}(u_{\mathrm{CR}}) \geqslant E_{\mathrm{GLEB}}$ und $E(u) \geqslant E_{\mathrm{GLEB}}$.

TODO translate and rewrite, leave out the d and make it 2. Say we choose $\gamma = 1$ because we want to refine towards the discontinuities

Definition (Verfeinerungsindikator)

Für $d \in \mathbb{N}$ (in dieser Arbeit stets $d = 2$) und $0 < \gamma \leqslant 1$ definieren wir für alle $T \in \mathcal{T}$ und $u_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ die Funktionen

$$\eta_{\mathrm{V}}(T) := |T|^{2/d} \|f - \alpha u_{\mathrm{CR}}\|_{L^2(T)}^2 \quad \text{und}$$
$$\eta_{\mathrm{J}}(T) := |T|^{\gamma/d} \sum_{F \in \mathcal{E}(T)} \|[u_{\mathrm{CR}}]_F\|_{L^1(F)} .$$

Damit definieren wir den Verfeinerungsindikator $\eta := \sum_{T \in \mathcal{T}} \eta(T)$, wobei

$$\eta(T) := \eta_{\mathrm{V}}(T) + \eta_{\mathrm{J}}(T) \quad \text{für alle } T \in \mathcal{T}.$$

cameraman triangulation figures to show the effect of the refinement indicator

convergence graphs error and refinement indicator and probably its volume and jump contributions
adaptive and uniform
maybe also plot refinement indicator and its contributions for cameraman
say expected rates from bartels and compare to them

plot differences between energies and gleb
plot the differences between the discrete energies and the exact energy energy in the graph with all gleb differences
note that Enc -Egleb and E - Egleb dont differ much obviously because Enc - E tends to zero (as seen in the plots)
show strict convecity theorem here to show that it holds (even 'better' than the theorem guarantees)

show error, eta (without contributions) together with the differences of the energies and EGLEB

End slide with some picture maybe or just end on the last slide in experiments (with might be a plot with everything in it, which would be nice

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Appendix

tien schicken spätestens am Wochenende vor der Präsi, CC vor der Präsi die fertige Präsi + akuteller Stand der Arbeit schicken

L2 Sprünge vielleicht auswerten (bleiben sie konstant. . . , if we consider them, it becomes conforming
die L2 Sprung entwicklung einiger experiment (iteration auswerten, iteration selbst und Afem loop insgesamt). bleiben sicherlich konstant oder sowas

differenct norm for termination criteria comparison (energy difference not good because oscillations, everything else (dont show L2 error squared) is similar, just different height

Let $u_P : [0, \infty) \to \mathbb{R}$ with $u_P(r) = 0$ for $r \geqslant 1$, and, for all $x \in \Omega$, $u(x) = u_P\big(|x|\big)$.

Let $u_P : [0,\infty) \to \mathbb{R}$ with $u_P(r) = 0$ for $r \geqslant 1$, and, for all $x \in \Omega$, $u(x) = u_P\big(|x|\big)$. Furthermore, assume the existence of $\partial_r u_P$ a.e. in $[0,\infty)$, the existence of the derivative of

$$\operatorname{sgn}\big(\partial_r u_P(r)\big) := \begin{cases} -1 & \text{für } \partial_r u_P(r) < 0, \\ x \in [0,1] & \text{für } \partial_r u_P(r) = 0, \\ 1 & \text{für } \partial_r u_P(r) > 0. \end{cases}$$

a.e. in $[0,\infty)$, and that $\operatorname{sgn}\big(\partial_r u_P(r)\big)/r \to 0$ as $r \to 0$.

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$$f_P(r) := \alpha u_P(r) - \partial_r\left(\operatorname{sgn}\big(\partial_r u_P(r)\big)\right) - \frac{\operatorname{sgn}\big(\partial_r u_P(r)\big)}{r}$$

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Then u solves the continuous problem on $\Omega \supseteq \big\{w \in \mathbb{R}^2 \,\big|\, |w| \leqslant 1\big\}$ if the input signal is $f(x) := f_P\big(|x|\big)$.