Chapter 5
Sobolev spaces

In this chapter we introduce the key notion of Sobolev spaces, which can be considered as one of the main tools that made possible the wide development of the theory of PDEs in the last several decades.

Motivations. In Chapter 2, Section 2.3.1, it was shown that the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

can be formulated in a weak sense as

$$\begin{cases} \text{find } u \in V \text{ such that} \\ \displaystyle\int_\Omega Du \cdot Dv \, dx = \int_\Omega f v \, dx \quad \forall v \in V, \end{cases} \tag{5.1}$$

where V is a functional space, containing $\mathscr{D}(\Omega)$, which has to be adequately chosen. Indeed, one can observe that problem (5.1) can be naturally attacked thanks to the Lax–Milgram theorem. One has to find a Hilbert space V such that the bilinear form

$$a(u,v) = \int_\Omega Du \cdot Dv \, dx$$

is continuous and coercive and the linear form

$$L(v) = \int_\Omega f v dx$$

is continuous. Noticing that

$$\begin{aligned} a(v,v) &= \int_\Omega |Dv|^2 \, dx, \\ |a(u,v)| &\le \left(\int_\Omega |Du|^2 \, dx \right)^{1/2} \left(\int_\Omega |Dv|^2 \, dx \right)^{1/2}, \\ |L(v)| &\le \left(\int_\Omega f^2 \, dx \right)^{1/2} \left(\int_\Omega v^2 \, dx \right)^{1/2}, \end{aligned}$$

it is natural to take V equal to the completion of the space $D(\Omega)$ with respect to the norm

$$||v|| = \left(\int_\Omega v^2 + |Dv|^2 \, dx \right)^{1/2}.$$

That is precisely the Sobolev space $V = H_0^1(\Omega)$. For pedagogical reasons, we prefer to introduce these spaces in a direct analytical way, by using the concept of distributional derivative. Then, we prove that regular functions are dense in the Sobolev spaces with respect to the corresponding Sobolev norms, which involve the L^p-norms of the functions and of their derivatives. This, combined with the completeness of the Sobolev spaces, establishes that Sobolev spaces are precisely the spaces provided by the above completion approach.

The key for a simple direct definition of Sobolev spaces is the notion of derivative in the sense of distributions developed in Section 2.2.

5.1 ▪ Sobolev spaces: Definition, density results

Unless specified, in the following Ω is a general open subset of $\mathbf{R}^N$.

Definition 5.1.1. *The Sobolev space $H^1(\Omega)$ is defined by*

$$H^1(\Omega) = \left\{ v \in L^2(\Omega) \, : \, \frac{\partial v}{\partial x_i} \in L^2(\Omega), \quad i = 1, \dots, N \right\},$$

where $\frac{\partial v}{\partial x_i}$ is taken in the distributional sense. The space $H^1(\Omega)$ is equipped with the scalar product

$$\langle u, v \rangle = \int_\Omega \left(uv + \sum_{i=1}^N \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right) dx$$

and the corresponding norm

$$||v||_{H^1(\Omega)} = \left[\int_\Omega \left(v^2 + \sum_{i=1}^N \left(\frac{\partial v}{\partial x_i} \right)^2 \right) dx \right]^{1/2}.$$

Remark 5.1.1. By definition of distributional derivative, the following conditions are equivalent:

(a) $v \in H^1(\Omega)$;

(b) $v \in L^2(\Omega)$ and there exist $g_1, g_2, \dots, g_N \in L^2(\Omega)$ such that

$$\forall \varphi \in \mathcal{D}(\Omega) \qquad \int_\Omega v \, \frac{\partial \varphi}{\partial x_i} \, dx = -\int_\Omega g_i \varphi \, dx.$$

Then, by definition, $\frac{\partial v}{\partial x_i} = g_i$ in distribution sense.

The above definition can be naturally extended when replacing the $L^2(\Omega)$ space by a general $L^p(\Omega)$ space, $1 \le p \le +\infty$.

Definition 5.1.2. *For any $1 \le p \le +\infty$, the Sobolev space $W^{1,p}(\Omega)$ is defined by*

$$W^{1,p}(\Omega) = \left\{ v \in L^p(\Omega) \, : \, \frac{\partial v}{\partial x_i} \in L^p(\Omega), \quad i = 1, 2, \dots, N \right\},$$

where $\frac{\partial v}{\partial x_i}$ is taken in the distribution sense. The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|v\|_{W^{1,p}(\Omega)} = \left[\int_\Omega \left(|v|^p + \sum_{i=1}^N \left| \frac{\partial v}{\partial x_i} \right|^p \right) dx \right]^{1/p} \quad \text{for } 1 \le p < +\infty,$$

$$\|v\|_{W^{1,\infty}(\Omega)} = \max \left\{ \|v\|_\infty ; \left\| \frac{\partial v}{\partial x_1} \right\|_\infty ; \dots ; \left\| \frac{\partial v}{\partial x_N} \right\|_\infty \right\} \quad \text{for } p = +\infty.$$

When $p = 2$, the space $W^{1,2}(\Omega)$ is often denoted by $H^1(\Omega)$. Both notations are commonly used, the notation H^1 recalling the Hilbert structure which is so obtained when $p = 2$ (H is the initial of Hilbert). The next extension of the above notions is obtained when considering higher-order derivatives.

Definition 5.1.3. *Take m a nonnegative integer and $1 \le p \le +\infty$. The Sobolev space $W^{m,p}(\Omega)$ is defined by*

$$W^{m,p}(\Omega) = \{ v \in L^p(\Omega) \, : \, D^\alpha v \in L^p(\Omega) \ \forall \ \alpha \text{ with } |\alpha| \le m \},$$

where $D^\alpha v$ is the distribution derivative of v of symbol α. We recall that for $\alpha = (\alpha_1, \dots, \alpha_N)$,

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_N^{\alpha_N}}$$

with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$. The space $W^{m,p}(\Omega)$ is equipped with the norm

$$\|v|_{W^{m,p}(\Omega)} = \left[\sum_{0 \le |\alpha| \le m} \int_\Omega |D^\alpha v|^p \, dx \right]^{1/p} \quad \text{for } 1 \le p < +\infty,$$

$$\|v\|_{W^{m,\infty}(\Omega)} = \max_{0 \le |\alpha| \le m} \|D^\alpha v\|_\infty \quad \text{for } p = +\infty.$$

When $p = 2$, we also use the notation $H^m(\Omega)$ for $W^{m,2}(\Omega)$ to enhance the Hilbertian structure of the space $W^{m,2}(\Omega)$.

The space $W^{1,p}(\Omega)$ may be also equipped with the equivalent norm

$$\|v\|_{L^p} + \sum_{i=1}^N \left\| \frac{\partial v}{\partial x_i} \right\|_{L^p},$$

but we prefer the choice of the norm made in Definition 5.1.2, because when $p = 2$ it yields a Hilbertian norm (which is not the case of the above one).

Clearly, $\mathcal{D}(\Omega)$ is always a subspace of $W^{m,p}(\Omega)$ for any $m \in \mathbf{N}$, $1 \le p \le +\infty$, so one can consider its closure in $W^{m,p}(\Omega)$.

Definition 5.1.4. *By definition,*

$H_0^1(\Omega) =$ closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$,

$W_0^{1,p}(\Omega) =$ closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$,

$W_0^{m,p}(\Omega) =$ closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$.

As we will see, the elements of $H_0^1(\Omega)$ are precisely the elements of $H^1(\Omega)$ whose trace on $\partial\Omega$ is equal to zero. This, obviously, requires introduction of a notion of trace for functions of $H^1(\Omega)$ which extends the classical restriction operation for regular functions.

Example 5.1.1. Let us first examine the case $N = 1$, Ω being an open interval of $\mathbf{R}$.

(a) Take $\Omega =]-1,1[$ an open interval in $\mathbf{R}$ and $v(x) = |x|$. Clearly, v is not differentiable in the classical sense at $x = 0$. Let us compute its distribution derivative. Given $\varphi \in \mathcal{D}(]-1,1[)$ we have

$$\begin{aligned}\int_{-1}^{1} v(x)\varphi'(x)dx &= \int_{-1}^{0} (-x)\varphi'(x)dx + \int_{0}^{1} x\varphi'(x)dx\\ &= \int_{-1}^{0} \varphi(x)dx - \int_{0}^{1} \varphi(x)dx\\ &= -\int_{-1}^{1} \operatorname{sign}(x)\varphi(x)dx,\end{aligned}$$

where

$$\operatorname{sign}(x) = \begin{cases} -1 & \text{if} \quad -1 \le x < 0,\\ 1 & \text{if} \quad 0 < x \le 1.\end{cases}$$

So, $v'(x) = \operatorname{sign}(x)$ belongs to $L^\infty(\Omega)$ and v belongs to $W^{1,\infty}(-1,1)$ and thus to any $W^{1,p}(-1,1)$, $1 \le p \le +\infty$. Clearly, the above argument can be easily iterated and any continuous piecewise affine (and in fact piecewise $\mathbf{C}^1$) belongs to $W^{1,\infty}(a,b)$ for any bounded interval (a,b).

(b) Let us now exhibit a continuous function on a bounded interval of $\mathbf{R}$ which does not belong to H^1. Take $\Omega =]-1,1[$ and consider for $0 < \alpha \le 1$ $v_\alpha(x) = |x|^\alpha$. A similar computation as above yields $v'_\alpha(x) = \alpha \operatorname{sign} x |x|^{\alpha-1}$.

Hence

$$\int_{-1}^{1} v'_\alpha(x)^2\, dx = \alpha^2 \int_{-1}^{1} x^{2\alpha-2}\, dx,$$

which is finite iff $2 - 2\alpha < 1$, i.e., $\alpha > \frac{1}{2}$. Thus, $|x|^\alpha$ belongs to $H^1(-1,1)$ iff $\alpha > \frac{1}{2}$. As a consequence, the function $u(x) = \sqrt{|x|}$ does not belong to $H^1(-1,1)$!

(c) We are going to prove that any element $v \in H^1(a,b)$ has a continuous representative. Indeed, this is a very special property of the one-dimensional case. It is no more true as soon as $N \ge 2$. We will further carefully study the regularity properties of elements of Sobolev spaces. Before proving this result, let us observe that a function v which has a jump at one point in $\mathbf{R}$ does not belong to a Sobolev space $W^{1,p}$. For simplicity, take $\Omega =]-1,1[$ and

$$v(x) = \begin{cases} 1 & \text{if } x \ge 0,\\ 0 & \text{if } x < 0.\end{cases}$$

An elementary computation yields that the distributional derivative of v is equal to the Dirac mass at the origin, indeed,

$$v' = \delta_0,$$

which, as we already observed, is a distribution which is not attached to a function $f \in L^1(\Omega)$.

Theorem 5.1.1. *Take $1 \le p \le +\infty$. Let $\Omega = (a,b)$ be an open interval of* $\mathbf{R}$.

(i) *Let $v \in W^{1,p}(a,b)$ and denote by $v' \in L^p(a,b)$ its first distributional derivative. Then there exists a continuous function $\tilde{v} \in \mathbf{C}([a,b])$ such that*

$$\begin{cases} v(x) = \tilde{v}(x) \text{ for a.e. } x \in (a,b), \\ \tilde{v}(x) - \tilde{v}(y) = \displaystyle\int_y^x v'(t)dt \ \forall\, x,y \in [a,b]. \end{cases}$$

We say in this case that v admits a continuous representative $\tilde{v}$. Indeed $\tilde{v}$ is unique, and, when $p > 1$, it belongs to $\mathbf{C}^{0,\alpha}([a,b])$ with $\alpha = \frac{1}{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$, i.e.,

$$|\tilde{v}(x) - \tilde{v}(y)| \le C\,|x-y|^{\alpha} \quad \forall x,y \in [a,b].$$

As a consequence,

$$v(x) - v(y) = \int_y^x v'(t)dt \quad \text{for a.e. } x,y \in (a,b).$$

(ii) *Conversely, let us assume that $v \in L^p(\Omega)$ and there exists a function $g \in L^p(\Omega)$ such that*

$$v(x) - v(y) = \int_y^x g(t)dt \quad \text{for a.e. } x,y \in (a,b).$$

Then, $v \in W^{1,p}(a,b)$ and $v' = g$ in the distributional sense.

Before proving Theorem 5.1.1, let us recall that the fundamental theorem of calculus states that for any $\mathbf{C}^1$ function v we have

$$v(x) = v(a) + \int_a^x v'(t)\,dt.$$

The above theorem allows us to extend such formula to the class of functions $v \in W^{1,p}$, the derivative v' being taken in the distributional sense and the integral in the Lebesgue sense.

PROOF OF THEOREM 5.1.1. Let $v \in W^{1,p}(a,b)$ and $v' \in L^p(a,b)$ its distributional derivative. Let us denote by $w(\cdot)$ the function

$$w(x) = \int_a^x v'(t)\,dt.$$

When $1 < p \le +\infty$, $w(\cdot)$ is Hölder continuous: indeed, for all $x,y \in (a,b)$,

$$\begin{aligned} |w(y) - w(x)| &= \left| \int_x^y v'(t)\,dt \right| \\ &\le \int_x^y |v'(t)|\,dt \\ &\le |y-x|^{1/p'} \left(\int_x^y |v'(t)|^p\,dt \right)^{1/p} \\ &\le \|v'\|_{L^p(a,b)} |y-x|^{1/p'}. \end{aligned}$$

Let us prove that there exists a constant $C \in \mathbf{R}$ such that

$$v(x) - w(x) = C \quad \text{for a.e. } x \in (a,b).$$

To that end, let us compute w' in a distributional sense and prove that $w' = v'$. Taking $\varphi \in \mathcal{D}(a,b)$ we have

$$\begin{aligned}
\langle w', \varphi \rangle_{(D',D)} &= -\int_a^b w(x)\varphi'(x)\,dx \\
&= -\int_a^b \left(\int_a^x v'(t)\,dt \right) \varphi'(x)\,dx \\
&= -\int_a^b \left(\int_a^b 1_{[a,x]}(t) v'(t)\,dt \right) \varphi'(x)\,dx.
\end{aligned}$$

The function $(t,x) \longmapsto 1_{[a,x]}(t)v'(t)\varphi'(x)$ belongs to $L^1((a,b)\times(a,b))$, which allows us to apply the Fubini theorem to obtain

$$\begin{aligned}
\langle w', \varphi \rangle_{(D',D)} &= -\int_a^b v'(t) \left(\int_a^b 1_{[a,x]}(t) \varphi'(x)\,dx \right) dt \\
&= -\int_a^b v'(t) \left(\int_t^b \varphi'(x)\,dx \right) dt \\
&= \int_a^b v'(t)\varphi(t)\,dt \\
&= \langle v', \varphi \rangle_{(\mathcal{D}',\mathcal{D})}.
\end{aligned}$$

Hence, $w' = v'$ in $\mathcal{D}'(a,b)$, and as a consequence $(w - v)' = 0$. We conclude the proof of assertion (i) by the help of the lemma below; the proof of assertion (ii) is easy and thus is left to the reader. $\square$

Lemma 5.1.1. *Let $f \in L^1_{loc}(a,b)$ such that $f' = 0$ in distributional sense. Then, there exists a constant $C \in \mathbf{R}$ such that $f(x) = C$ for a.e. $x \in (a,b)$.*

PROOF. By assumption $f' = 0$ in $\mathcal{D}'(a,b)$, which is equivalent to saying that

$$\int_a^b f(x)\varphi'(x)dx = 0 \quad \forall\, \varphi \in \mathcal{D}(a,b).$$

To exploit this information we need to understand what is the space

$$W = \{\varphi' \,:\, \varphi \in \mathcal{D}(a,b)\}.$$

Clearly, for any $\varphi \in \mathcal{D}(a,b)$ we have that φ' still belongs to $\mathcal{D}(a,b)$ and

$$\int_a^b \varphi'(x)dx = \varphi(b) - \varphi(a) = 0.$$

Conversely, take $\psi \in \mathcal{D}(a,b)$ such that $\int_a^b \psi(x)dx = 0$ and prove that $\psi = \varphi'$ for some $\varphi \in \mathcal{D}(a,b)$. Indeed

$$\varphi(x) = \int_a^x \psi(t)dt$$

belongs to $\mathbf{C}^\infty(a,b)$ and satisfies $\varphi' = \psi$. Assuming that $\psi \equiv 0$ outside of $[c,d]$ with $a<c<d<b$, we have $\varphi(x)=0$ for all $x \le c$ and, for all $x > d$,

$$\varphi(x) = \int_a^x \psi(t)dt = \int_a^b \psi(t)dt = 0.$$

Hence $\varphi \in \mathscr{D}(a,b)$. We have obtained that

$$\begin{aligned} W &= \{\varphi' \,:\, \varphi \in \mathscr{D}(a,b)\} \\ &= \left\{\psi \in \mathscr{D}(a,b) : \int_a^b \psi(x)dx = 0\right\}. \end{aligned}$$

To construct functions $\psi \in W$, we first take some function $\theta \in D(a,b)$ such that $\int_a^b \theta(x)\, dx = 1$. Then, observe that for any $\chi \in D(a,b)$ the function $h(\cdot)$ which is defined by

$$h(x) := \chi(x) - \left(\int_a^b \chi(t)\,dt\right)\theta(x)$$

satisfies $h \in \mathscr{D}(a,b)$ and $\int_a^b h(x)dx = 0$. Hence $h \in W$ and

$$\int_a^b f(x)h(x)\,dx = 0.$$

Equivalently,

$$\int_a^b f(x)\chi(x)dx = \left(\int_a^b \chi(t)dt\right)\int_a^b f(x)\theta(x)\,dx.$$

Denoting $C := \int_a^b f(x)\theta(x)dx$, we obtain

$$\forall \chi \in \mathscr{D}(a,b) \quad \int_a^b (f(x)-C)\chi(x)\,dx = 0.$$

We conclude thanks to Theorem 2.2.1. $\square$

As soon as the dimension N is greater than or equal to 2, the situation is more complex, and, in general, elements of $H^1(\Omega)$ have no continuous representative. Let us illustrate that fact with the help of the following example.

Example 5.1.2. Take $N=2$ and $\Omega = B(0,R)$ with $R<1$, that is, $\Omega = \{x=(x_1,x_2) \in \mathbf{R}^2 : |x| = \sqrt{x_1^2+x_2^2} < R\}$. On Ω, we consider the function $v(x) = |\ln|x||^k$, where k is a real parameter.

Let us examine, depending on the value of the parameter k, whether the function v belongs to $H^1(\Omega)$. To do so, it is natural to use radial coordinates and take $r=|x|$. As a classical result, we have $|\nabla v|^2 = (\frac{\partial \tilde{v}}{\partial r})^2$, where $v(x) = \tilde{v}(r)$. Here $\tilde{v}(r) = |\ln r|^k$. It follows

$$\int_{B(0,R)} v^2(x)dx = 2\pi \int_0^R (\ln r)^{2k}\, r\,dr,$$

$$\int_{B(0,R)} |\nabla v(x)|^2 dx = 2\pi \int_0^R \frac{k^2}{r^2}(\ln r)^{2k-2}\, r\,dr.$$

Let us make the change of variable $t = -\ln r$ to obtain

$$\int_\Omega (v^2 + |\nabla v|^2)dx = 2\pi \int_{-\ln R}^{+\infty} t^{2k} e^{-2t}\, dt + 2\pi k^2 \int_{-\ln R}^{+\infty} \frac{dt}{t^{2-2k}}.$$

It follows that v belongs to $H^1(\Omega)$ iff $2-2k > 1$, i.e., $k < \frac{1}{2}$. By taking $0 < k < \frac{1}{2}$, we obtain a function belonging to $H^1(\Omega)$ which blows up to $+\infty$ at the origin and which does not have a continuous representative.

Remark 5.1.2. It is a central question to know the best regularity results on the elements of the Sobolev spaces. As a general rule, for a function $v \in L^p(\Omega)$, which is a priori defined only almost everywhere on Ω, to know that some of its distribution derivatives belong to $L^p(\Omega)$ allows us, even if the function v has no continuous representative, to treat it more precisely than almost everywhere with respect to x. We will describe further three distinct approaches to this question: the Sobolev embedding theorem, the trace theory, and the capacity theory.

Let us now examine the general properties of the Sobolev spaces.

Theorem 5.1.2. *Let Ω be an open subset of $\mathbf{R}^N$. For any nonnegative integer m and any real number p with $1 \le p \le +\infty$, $W^{m,p}(\Omega)$ is a Banach space. When $p = 2$, $W^{m,p}(\Omega) = H^m(\Omega)$ is a Hilbert space.*

PROOF. Let $(v_n)_{n\in\mathbf{N}}$ be a Cauchy sequence in $W^{m,p}(\Omega)$. For any multi-index

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$$

with $|\alpha| \le m$, we have

$$\|D^\alpha v_n - D^\alpha v_n\|_{L^p} \le \|v_n - v_m\|_{W^{m,p}}.$$

Hence, the sequence $(D^\alpha v_n)_{n\in\mathbf{N}}$ is a Cauchy sequence in $L^p(\Omega)$, which is a Banach space. Let

$$\begin{aligned} v_n &\to v \quad && \text{in } L^p(\Omega) \quad \text{as } n \to +\infty,\\ D^\alpha v_n &\to g_\alpha \quad && \text{in } L^p(\Omega) \quad \text{as } n \to +\infty. \end{aligned}$$

Convergence in $L^p(\Omega)$ clearly implies convergence in the distribution sense.

By continuity of the derivation operation with respect to the convergence in the distribution sense (see Proposition 2.2.7), we obtain

$$D^\alpha v_n \to g_\alpha = D^\alpha v \quad \text{in } L^p(\Omega) \text{ as } n \to +\infty.$$

Hence $v \in W^{m,p}(\Omega)$ and the sequence $(v_n)_{n\in\mathbf{N}}$ converges to v in $W^{m,p}(\Omega)$. □

Remark 5.1.3. Notice that the above result relies essentially on the fact that the Sobolev spaces are built by means of the Lebesgue spaces $L^p(\Omega)$ (which are Banach spaces) and of the generalized notion of derivation, namely, the distributional derivative.

The following important result makes the link between the definition of Sobolev spaces relying on the notion of distributional derivative and the completion approach.

Theorem 5.1.3. *Let $\Omega = \mathbf{R}^N$. For any $1 \le p < +\infty$ and $m \in \mathbf{N}$, $\mathscr{D}(\mathbf{R}^N)$ is a dense subspace of $W^{m,p}(\mathbf{R}^N)$. Equivalently,*

$W_0^{m,p}(\mathbf{R}^N) = W^{m,p}(\mathbf{R}^N)$,
$W^{m,p}(\mathbf{R}^N)$ *is the completion of* $\mathscr{D}(\mathbf{R}^N)$ *with respect to the* $\|\cdot\|_{W^{m,p}}$ *norm.*

PROOF. For simplicity of notation, let us consider the case $m = 1$ and prove that $\mathscr{D}(\mathbf{R}^N)$ is dense in $W^{1,p}(\mathbf{R}^N)$, $(1 \le p < +\infty)$. This is a two-step approximation procedure:

(a) *Truncation (of the domain).* Let $M \in \mathscr{D}(\mathbf{R}^N)$ such that $M(0) = 1$. For any $n \in \mathbf{N}^*$, let us define $M_n(\xi) = M(\frac{\xi}{n})$. Clearly, $M_n \in \mathscr{D}(\mathbf{R}^N)$ and, for any $\xi \in \mathbf{R}^N$, $\lim_{n\to+\infty} M_n(\xi) = M(0) = 1$.

The most commonly used truncation function is a function $M : \mathbf{R}^N \to \mathbf{R}^+$ such that $0 \le M \le 1$, $M(x) = 1$ on $B(0, \frac{1}{2})$, and $M(x) = 0$ for $\|x\| \ge 1$. Such a function exists by using the Tietze–Urysohn lemma, or it can be explicitly described if needed. But notice that for our purpose, we will only exploit the fact that $M \in \mathscr{D}(\mathbf{R}^N)$ and $M(0) = 1$.

For any $v \in W^{1,p}(\mathbf{R}^N)$, let us define $v_n(x) = M_n(x)v(x)$. Clearly, v_n has a compact support, since

$$\operatorname{spt} v_n \subset \operatorname{spt} M_n \subset n \operatorname{spt} M.$$

Let us verify that v_n still belongs to $W^{1,p}(\mathbf{R}^N)$:

$$\int_{\mathbf{R}^N} |v_n|^p\,dx \le \|M\|_\infty^p \int_{\mathbf{R}^N} |v|^p\,dx < +\infty.$$

On the other hand, the classical differentiation rule is still valid in this context. It is worthwhile to state it as a lemma.

Lemma 5.1.2. *Let* $v \in W^{1,p}(\Omega)$ *and* $M \in \mathscr{D}(\Omega)$. *Then* $Mv \in W^{1,p}(\Omega)$ *and*

$$\frac{\partial}{\partial x_i}(Mv) = M\frac{\partial v}{\partial x_i} + \frac{\partial M}{\partial x_i}v.$$

PROOF. Take $\varphi \in \mathscr{D}(\Omega)$ as a test function. By definition,

$$\begin{aligned}\left\langle \frac{\partial}{\partial x_i}(Mv), \varphi \right\rangle_{(\mathscr{D}'(\Omega),\mathscr{D}(\Omega))} &= -\left\langle Mv, \frac{\partial \varphi}{\partial x_i} \right\rangle_{(\mathscr{D}',\mathscr{D})} \\ &= -\int_\Omega M(x)v(x)\frac{\partial \varphi}{\partial x_i}\,dx.\end{aligned}$$

Since $M \in \mathscr{D}(\Omega)$, we can use the classical differentiation rule

$$\frac{\partial}{\partial x_i}(M\varphi) = M\frac{\partial \varphi}{\partial x_i} + \frac{\partial M}{\partial x_i}\varphi.$$

Hence

$$\left\langle \frac{\partial}{\partial x_i}(Mv), \varphi \right\rangle_{(D',D)} = -\int_\Omega v.\frac{\partial}{\partial x_i}(M\varphi)dx + \int_\Omega v\frac{\partial M}{\partial x_i}\varphi\,dx.$$

Noticing that $M\varphi \in \mathscr{D}(\Omega)$ we obtain

$$\left\langle \frac{\partial}{\partial x_i}(Mv), \varphi \right\rangle_{(D',D)} = \left\langle \frac{\partial v}{\partial x_i}, M\varphi \right\rangle_{(D',D)} + \int v\frac{\partial M}{\partial x_i}\varphi\,dx.$$

Since $v \in W^{1,p}(\Omega)$,

$$\left\langle \frac{\partial}{\partial x_i}(Mv), \varphi \right\rangle_{(D',D)} = \int_\Omega \left(\frac{\partial v}{\partial x_i} M + v \frac{\partial M}{\partial x_i} \right) \varphi \, dx = \left\langle \frac{\partial v}{\partial x_i} M + v \frac{\partial M}{\partial x_i}, \varphi \right\rangle_{(D',D)}.$$

We conclude by noticing that

$$\frac{\partial v}{\partial x_i} M + v \frac{\partial M}{\partial x_i} \in L^p(\Omega). \qquad \square$$

PROOF OF THEOREM 5.1.3 CONTINUED. Applying Lemma 5.1.2 we have

$$\frac{\partial v_n}{\partial x_i} = M_n \frac{\partial v}{\partial x_i} + \frac{\partial M_n}{\partial x_i} v.$$

Since M_n, $\frac{\partial M_n}{\partial x_i}$ belong to $\mathcal{D}(\mathbf{R}^N)$, we have $\frac{\partial v_n}{\partial x_i} \in L^p(\mathbf{R}^N)$.

Let us prove that the sequence $(v_n)_{n\in\mathbf{N}}$ norm converges to v in $W^{1,p}(\mathbf{R}^N)$ as $n \to +\infty$:

$$\int_{\mathbf{R}^N} |v_n - v|^p \, dx = \int_{\mathbf{R}^N} |1 - M_n|^p \; |v|^p \, dx.$$

Then notice that

$$|1 - M_n|^p |v|^p \to 0 \quad \text{pointwise as } n \to +\infty,$$
$$|1 - M_n|^p |v|^p \leq (1 + \|M\|_\infty)^p |v|^p \quad \forall n \in \mathbf{N},$$

and apply the Lebesgue dominated convergence theorem to obtain that $v_n \to v$ in $L^p(\mathbf{R}^N)$. Similarly,

$$M_n \frac{\partial v}{\partial x_i} \to \frac{\partial v}{\partial x_i} \quad \text{in } L^p(\mathbf{R}^N) \text{ as } n \to +\infty.$$

To prove that $\frac{\partial v_n}{\partial x_i} \to \frac{\partial v}{\partial x_i}$ in $L^p(\mathbf{R}^N)$, we just need to prove that

$$\frac{\partial M_n}{\partial x_i} v \to 0 \quad \text{in } L^p(\mathbf{R}^N) \text{ as } n \to +\infty.$$

This follows immediately from

$$\begin{aligned} \left| \frac{\partial M_n}{\partial x_i}(x) v(x) \right| &= \frac{1}{n} \left| \frac{\partial M}{\partial x_i}\left(\frac{x}{n} \right) \right| \, |v(x)| \\ &\leq \frac{1}{n} \left\| \frac{\partial M}{\partial x_i} \right\|_\infty |v(x)|, \end{aligned}$$

and hence

$$\left\| \frac{\partial M_n}{\partial x_i} v \right\|_{L^p} \leq \frac{1}{n} \left\| \frac{\partial M}{\partial x_i} \right\|_\infty \|v\|_{L^p}.$$

(b) *Regularization*. The second step consists in proving that any element $u \in W^{1,p}(\mathbf{R}^N)$ with compact support can be approximated in $W^{1,p}(\mathbf{R}^N)$ by a sequence $(u_n)_{n\in\mathbf{N}}$ of elements in $\mathcal{D}(\mathbf{R}^N)$. To that end, we use the regularization by convolution method. With the same notation as in Section 2.2.2, we introduce $\rho \in \mathscr{D}(\mathbf{R}^N)$, $\rho \geq 0$, with $\operatorname{spt}\rho \subset B(0,1)$, and $\int_{\mathbf{R}^N} \rho(x)\,dx = 1$ and define for each $n \in \mathbf{N}^*$

$$\rho_n(x) := n^N \rho(nx),$$

which satisfies

$$\begin{cases} \rho_n \in \mathscr{D}(\mathbf{R}^N), \rho_n \geq 0, \\ \operatorname{spt}\rho_n \subset \mathbf{B}(0, 1/n), \\ \displaystyle\int_{\mathbf{R}^N} \rho_n(x)\,dx = 1. \end{cases}$$

Given $u \in W^{1,p}(\mathbf{R}^N)$, let us define for all $n \in \mathbf{N}^*$, $u_n := u * \rho_n$, that is,

$$u_n(x) = \int_{\mathbf{R}^N} u(x-y)\rho_n(y)\,dy \tag{5.2}$$

$$= \int_{\mathbf{R}^N} u(y)\rho_n(x-y)\,dy. \tag{5.3}$$

To pass from one equality to the other in the formula above, one just has to make a change of variable and use that the Lebesgue measure on $\mathbf{R}^N$ is invariant by translation. Depending on the situation, we will use one formula or the other: note that the x variable appears in (5.2) in u, while in (5.3) it appears in ρ_n.

Let us verify that for each $n \in \mathbf{N}$, u_n belongs to $\mathscr{D}(\mathbf{R}^N)$. This means that u_n has a compact support and that u_n belongs to $\mathbf{C}^\infty(\mathbf{R}^N)$. Take $R > 0$ such that $u(x) = 0$ for $|x| > R$. By construction, for each $n \in \mathbf{N}^*$, $\rho_n(x) = 0$ for $|x| > \frac{1}{n}$. Let us verify that $u_n(x) = u * \rho_n(x)$ is equal to zero when $|x| > R + \frac{1}{n}$. This follows from the fact that the function $y \longmapsto u(x-y)\rho_n(y)$ is identically zero when $|x| > R + \frac{1}{n}$. Indeed, either $|y| > \frac{1}{n}$, in which case $\rho_n(y) = 0$, or $|y| \leq \frac{1}{n}$, in which case

$$|x-y| \geq |x| - |y| > R + \frac{1}{n} - \frac{1}{n} = R$$

and $u(x-y) = 0$.

(One should notice that the above argument can be easily generalized to obtain that for any two functions f and g, $\operatorname{spt}(f \star g) \subset \operatorname{spt} f + \operatorname{spt} g$.)

Let us now verify that $u_n = u * \rho_n$ belongs to $\mathbf{C}^\infty(\mathbf{R}^N)$. At this point, one has to use precisely formula (5.3), where the x variable, with respect to which one wants to derive, appears in ρ_n. It is the differentiability property of ρ_n, which makes $u_n = u * \rho_n$ differentiable too!

One has to derive under the sum sign, which is a direct consequence of the differentiability properties of ρ_n and of the Lebesgue dominated convergence theorem. As an illustration, let us prove that for each $i = 1, 2, \ldots, N$, for each $x \in \mathbf{R}^N$, $\frac{\partial u_n}{\partial x_i}(x)$ exists and

$$\frac{\partial u_n}{\partial x_i}(x) = \left(\frac{\partial u}{\partial x_i} * \rho_n\right)(x).$$

To avoid any confusion, we prefer to write $\frac{\partial u_n}{\partial e_i}$ instead of $\frac{\partial u_n}{\partial x_i}$, where e_i is the ith vector of the canonical basis of $\mathbf{R}^N$. For any $t \neq 0$ let us consider the differential quotient

$$\frac{1}{t}[u_n(x+te_i) - u_n(x)] = \int_{\mathbf{R}^N} u(y)\frac{\rho_n(x-y+te_i) - \rho_n(x-y)}{t}\,dy.$$

Clearly

$$u(y)\frac{\rho_n(x-y+te_i)-\rho_n(x-y)}{t} \to u(y)\left(\frac{\partial \rho_n}{\partial e_i}\right)(x-y) \quad \text{as } t \to 0,$$

$$\left|u(y)\frac{\rho_n(x-y+te_i)-\rho_n(x-y)}{t}\right| \leq |u(y)| \sup_{z\in\mathbf{R}^N}\left|\frac{\partial \rho_n}{\partial e_i}(z)\right| \leq M|u(y)|,$$

which belongs to $L^p(\mathbf{R}^N)$. Note that in the above inequality, we have used that for each $n \in \mathbf{N}$, since ρ_n belongs to $\mathscr{D}(\mathbf{R}^N)$, ρ_n and its partial derivatives are bounded functions on $\mathbf{R}^N$. (M here is a constant which depends on n.) By using the Lebesgue dominated convergence theorem, we obtain

$$\frac{\partial u_n}{\partial e_i}(x) = \int_{\mathbf{R}^N} u(y)\left(\frac{\partial \rho_n}{\partial e_i}\right)(x-y)\,dy.$$

Let us now notice that for x fixed, the function $y \longmapsto (\frac{\partial \rho_n}{\partial e_i})(x-y)$ can be written as a partial derivative of a function of $\mathscr{D}(\mathbf{R}^N)$. Take $\xi_n(y) = -\rho_n(x-y)$. Then

$$\begin{aligned}\frac{\partial \xi_n}{\partial e_i}(y) &= \lim_{t\to 0}\frac{1}{t}[\xi_n(y+te_i)-\xi_n(y)]\\ &= \lim_{t\to 0}\frac{\rho_n(x-y-te_i)-\rho_n(x-y)}{-t}\\ &= \left(\frac{\partial \rho_n}{\partial e_i}\right)(x-y).\end{aligned}$$

Thus

$$\frac{\partial u_n}{\partial e_i}(x) = \int_{\mathbf{R}^N} u(y)\frac{\partial \xi_n}{\partial e_i}(y)\,dy.$$

Since $u \in W^{1,p}(\mathbf{R}^N)$, and by definition of the distribution derivative

$$\begin{aligned}\frac{\partial u_n}{\partial e_i}(x) &= -\int_{\mathbf{R}^N}\frac{\partial u}{\partial e_i}(y)\xi_n(y)dy\\ &= \int_{\mathbf{R}^N}\frac{\partial u}{\partial e_i}(y)\,\rho_n(x-y)dy,\end{aligned}$$

that is,

$$\frac{\partial u_n}{\partial e_i} = \frac{\partial u}{\partial e_i} * \rho_n.$$

Returning to the usual notation with $\frac{\partial}{\partial x_i}$ instead of $\frac{\partial}{\partial e_i}$ we have

$$\frac{\partial u_n}{\partial x_i} = \frac{\partial u}{\partial x_i} * \rho_n.$$

Let us apply Proposition 2.2.4(iii) to obtain

$$\frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i} \quad \text{in } L^p(\mathbf{R}^N) \text{ as } n \to +\infty,$$

that is, $u_n \to u$ in $W^{1,p}(\mathbf{R}^N)$. □

In general, when $\Omega \neq \mathbf{R}^N$, the two spaces $W_0^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$ do not coincide. The space $W_0^{1,p}(\Omega)$ is strictly included in $W^{1,p}(\Omega)$. We will justify this fact a little further by proving that $W_0^{1,p}(\Omega)$ consists of functions whose trace on $\partial\Omega$ is equal to zero.

The strict inclusion $W_0^{1,p}(\Omega) \subsetneq W^{1,p}(\Omega)$ for $\Omega \neq \mathbf{R}^N$ (at least for Ω with a smooth boundary) also can be seen as a consequence of the following result of independent interest.

Proposition 5.1.1. *Let Ω be an open set in $\mathbf{R}^N$ and let $u \in W_0^{m,p}(\Omega)$. Then the function $\tilde{u}$ which is equal to u on Ω and zero on $\mathbf{R}^N \setminus \Omega$ belongs to $W^{m,p}(\mathbf{R}^N)$. The linear mapping p defined by $p(u) = \tilde{u}$ is an isometry from $W_0^{m,p}(\Omega)$ into $W^{m,p}(\mathbf{R}^N)$.*

PROOF. Let us consider the linear mapping

$$p \;:\; \mathscr{D}(\Omega) \longrightarrow \mathscr{D}(\mathbf{R}^N), \qquad u \longmapsto p(u) = \tilde{u}.$$

It is important to notice that since $u \in \mathscr{D}(\Omega)$, for each $x \in \partial\Omega$ there exists a neighborhood of x on which u is equal to zero. This clearly implies that $\tilde{u} \in \mathscr{D}(\mathbf{R}^N)$. Moreover, p is an isometry for the $W^{m,p}$ norms:

$$\forall u \in \mathscr{D}(\Omega) \quad ||p(u)||_{W^{m,p}(\mathbf{R}^N)} = ||u||_{W^{m,p}(\Omega)}.$$

Thus, p is a uniformly continuous mapping

$$p : \mathscr{D}(\Omega) \to W^{m,p}(\mathbf{R}^N),$$

and moreover $W^{m,p}(\mathbf{R}^N)$ is a Banach space. It can be continuously extended into a mapping (that we still denote by p)

$$p : \overline{\mathscr{D}(\Omega)}^{W^{m,p}(\Omega)} = W_0^{m,p}(\Omega) \to W^{m,p}(\mathbf{R}^N).$$

For $u \in W_0^{m,p}(\Omega)$, there exists a sequence $u_n \in \mathscr{D}(\Omega)$ such that $u_n \to u$ in $W_0^{m,p}(\Omega)$, and so

$$p(u_n) = \tilde{u}_n \to p(u) \quad \text{in } W^{m,p}(\mathbf{R}^N).$$

The convergence in $W^{m,p}$ implies the convergence in L^p, which implies the convergence almost everywhere of a subsequence. It follows that

$$p(u) = \tilde{u}, \quad \text{that is, } \tilde{u} \in W^{m,p}(\mathbf{R}^N).$$

Moreover, for any $u \in W_0^{m,p}(\Omega)$, $||p(u)||_{W^{m,p}(\mathbf{R}^N)} = ||u||_{W^{m,p}(\Omega)}$. □

Example 5.1.3. Take Ω a smooth, bounded domain in $\mathbf{R}^N$, for example, $\Omega = \mathbf{B}(0,1)$ the open ball centered at the origin with radius one. The constant function $u \equiv 1$ on Ω belongs to $W^{1,p}(\Omega)$ but does not belong to $W_0^{1,p}(\Omega)$. Otherwise, its extension $\tilde{u}$ by zero outside of Ω would belong to $W^{1,p}(\mathbf{R}^N)$, which is not true since its first partial derivatives $\frac{\partial \tilde{u}}{\partial x_i}$ are measures supported by the sphere $\partial\Omega$. When $N = 1$, $\frac{du}{dx} = \delta_{\{-1\}} - \delta_{\{1\}}$, a distribution which cannot be represented by an integrable function!

When $\Omega \subset \mathbf{R}^N$, one can adapt the previous argument, that is, truncation on the domain and regularization by convolution.

Theorem 5.1.4 (Meyers–Serrin). *If $1 \leq p < \infty$, then $\mathbf{C}^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$.*

PROOF. For each integer $k \geq 1$ set

$$\Omega_k = \left\{ x \in \Omega \,:\, ||x|| \leq k \text{ and } \mathrm{dist}(x, \partial\Omega) > \frac{1}{k} \right\}$$

and $\Omega_0 = \emptyset$ the empty set. Define for each $k = 1, 2, \ldots$

$$G_k := \Omega_{k+1} \cap (\overline{\Omega}_{k-1})^c.$$

Then $(G_k)_{k\geq 1}$ is an open covering of Ω. Let $(\alpha_k)_{k\geq 1}$ be a $\mathbf{C}^\infty$ partition of unity for Ω subordinate to the open cover $(G_k)_{k\geq 1}$, that is,

$$\begin{cases} \mathrm{spt}\,\alpha_k \subset G_k, \alpha_k \in \mathscr{D}(G_k), \alpha_k \geq 0, \\ \displaystyle\sum_{k=1}^{+\infty} \alpha_k = 1. \end{cases}$$

Let $u \in W^{m,p}(\Omega)$. Given $\varepsilon > 0$, we are going to construct an element $\varphi \in \mathbf{C}^\infty(\Omega)$ such that $||u - \varphi||_{W^{m,p}(\Omega)} < \varepsilon$.

Consider the truncated function $\alpha_k u$. The same argument as in Theorem 5.1.3 applies and $\alpha_k u \in W_0^{m,p}(G_k)$. It can be extended by zero outside of G_k into a function belonging to $W^{m,p}(\mathbf{R}^N)$. We now use a regularization by convolution method with a kernel $(\rho_n)_{n\in\mathbf{N}}$. By taking $n = n(k)$ sufficiently large, we have that

$$||\rho_{n(k)} * (\alpha_k u) - \alpha_k u||_{W^{m,p}(\Omega)} \leq \frac{\varepsilon}{2^k}$$

and

$$\mathrm{spt}[\rho_{n(k)} * (\alpha_k u)] \subset G_k.$$

Define $\varphi := \sum_{k=1}^{+\infty} \rho_{n(k)} * (\alpha_k u)$ and note that for $x \in G_k$ we have

$$\varphi(x) = \sum_{j=-1}^{+1} [\rho_{n(k+j)} * (\alpha_{k+j} u)](x),$$

i.e., there are at most three terms which are nonzero in this sum. Hence $\varphi \in \mathbf{C}^\infty(\Omega)$ and

$$\begin{aligned} ||u - \varphi||_{W^{m,p}(\Omega)} &= \left\| \sum_{k=1}^{\infty} \big(\alpha_k u - (\alpha_k u) * \rho_{n(k)}\big) \right\|_{W^{m,p}} \\ &\leq \sum_{k=1}^{\infty} ||\alpha_k u - (\alpha_k u) * \rho_{n(k)}||_{W^{m,p}} \\ &\leq \varepsilon \end{aligned}$$

and the proof is complete. □

Remark 5.1.4. The above result shows that, equivalently, the space $W^{m,p}(\Omega)$ can be defined as the completion with respect to the $||\cdot||_{W^{m,p}(\Omega)}$ norm of the space $\mathscr{V} = \{v \in \mathbf{C}^\infty(\Omega) \,:\, ||v||_{W^{m,p}(\Omega)} < +\infty\}$. This result was obtained in 1964 by Meyers and Serrin.

5.2 ▪ The topological dual of $H_0^1(\Omega)$. The space $H^{-1}(\Omega)$.

When studying a functional space, a fundamental question which arises is the description of its topological dual space. Let us first consider the Sobolev space $H_0^1(\Omega)$, in which case this question can be easily solved thanks to the Hilbert structure of $H_0^1(\Omega)$ and the density of $\mathscr{D}(\Omega)$.

Let us take $T \in H_0^1(\Omega)^*$ an element of the topological dual of $H_0^1(\Omega)$. By using the Riesz representation theorem of the elements of the topological dual of a Hilbert space, we obtain the existence of a unique element $g \in H_0^1(\Omega)$ such that

$$\forall v \in H_0^1(\Omega) \quad T(v) = \langle v, g\rangle = \int_\Omega \left(vg + \sum_{i=1}^N \frac{\partial v}{\partial x_i}\frac{\partial g}{\partial x_i}\right) dx. \tag{5.4}$$

Moreover,

$$\|T\|_{H_0^1(\Omega)^*} = \|g\|_{H_0^1(\Omega)} = \left[\int_\Omega \left(g^2 + \sum_{i=1}^N \left(\frac{\partial g}{\partial x_i}\right)^2\right) dx\right]^{1/2}. \tag{5.5}$$

This is a first description of $H_0^1(\Omega)^*$. Indeed, one can give another description of the elements of $H_0^1(\Omega)^*$ as distributions. To that end, let us introduce the identity mapping

$$i : \mathscr{D}(\Omega) \hookrightarrow H_0^1(\Omega).$$

To each element $T \in H_0^1(\Omega)^*$ one can associate its restriction $T \circ i$ to $\mathscr{D}(\Omega)$

$$T \circ i : \mathscr{D}(\Omega) \xrightarrow{i} H_0^1(\Omega) \xrightarrow{T} \mathbf{R}.$$

Noticing that $\mathscr{D}(\Omega)$ is dense in $H_0^1(\Omega)$ for the $\|\cdot\|_{H_0^1}$ norm, it is equivalent to know T or its restriction $T \circ i$ to $\mathscr{D}(\Omega)$. Moreover, the convergence for the topology of $\mathscr{D}(\Omega)$ (Definition 2.2.1) is stronger than the norm convergence in $H_0^1(\Omega)$. Therefore, $T \circ i$ is a continuous linear form on $\mathscr{D}(\Omega)$ and $T \circ i$ is a distribution. One can summarize the above results by saying that the mapping

$$\begin{aligned} H_0^1(\Omega)^* &\hookrightarrow \mathscr{D}'(\Omega), \\ T &\longmapsto T \circ i = T|_{\mathscr{D}(\Omega)} \end{aligned}$$

is a continuous embedding of $H_0^1(\Omega)^*$ into the space of distributions $\mathscr{D}'(\Omega)$.

Let us now give a precise description of the distributions which are so obtained: for any $\varphi \in \mathscr{D}(\Omega)$

$$\langle T \circ i, \varphi\rangle = T(\varphi) = \int_\Omega \left(g\varphi + \sum_{i=1}^N \frac{\partial \varphi}{\partial x_i}\frac{\partial g}{\partial x_i}\right) dx.$$

Let us write $g_0 = g$ and $g_i = -\frac{\partial g}{\partial x_i}$ for $i = 1, \ldots, N$. Then,

$$\begin{aligned} \langle T \circ i, \varphi\rangle &= \int_\Omega g_0\varphi - \sum_{i=1}^N g_i \frac{\partial \varphi}{\partial x_i}\, dx \\ &= \left\langle g_0 + \sum_{i=1}^N \frac{\partial g_i}{\partial x_i}, \varphi \right\rangle_{(\mathscr{D}'(\Omega), \mathscr{D}(\Omega))}. \end{aligned}$$

Therefore, we can identify T with a distribution of the form $g_0+\sum_{i=1}^N \frac{\partial g_i}{\partial x_i}$ with $g_0, g_1, \dots, g_N \in L^2(\Omega)$. Moreover,

$$\|T\|_{H_0^1(\Omega)^*} = \|g\|_{H_0^1(\Omega)} = \left(\sum_{i=0}^N \int_\Omega g_i(x)^2\,dx\right)^{1/2}. \tag{5.6}$$

Conversely, any distribution $T \in \mathscr{D}'(\Omega)$ which can be written as $T = g_0 + \sum_{i=1}^N \frac{\partial g_i}{\partial x_i}$ with $g_0, g_1 \dots g_N \in L^2(\Omega)$ clearly satisfies

$$\begin{aligned}\forall \varphi \in \mathscr{D}(\Omega) \quad \langle T, \varphi\rangle_{(D',D)} &= \int_\Omega \left(g_0\varphi - \sum_{i=1}^N g_i \frac{\partial \varphi}{\partial x_i}\right) dx \\ &\le \left(\sum_{i=0}^N \int_\Omega g_i^2(x)\,dx\right)^{1/2} \|\varphi\|_{H_0^1(\Omega)}.\end{aligned}$$

As a consequence, T can be uniquely extended by density to a continuous linear form on $\overline{\mathscr{D}(\Omega)}^{H_0^1(\Omega)} = H_0^1(\Omega)$ which we still denote by T. Moreover,

$$\|T\|_{H_0^1(\Omega)^*} \le \left(\sum_{i=0}^N \int_\Omega g_i(x)^2\,dx\right)^{1/2}. \tag{5.7}$$

It is important to notice that there is not a unique way to write T as a sum $T = g_0+\sum \frac{\partial g_i}{\partial x_i}$. Indeed, take any g such that $\Delta g = 0$, then $\sum \frac{\partial}{\partial x_i}(\frac{\partial g}{\partial x_i}) = 0$, and T can, as well, be written as $T = g_0 + \sum \frac{\partial}{\partial x_i}(g_i + \frac{\partial g}{\partial x_i})$. By using (5.6) and (5.7), we obtain that

$$\|T\|_{H_0^1(\Omega)^*} = \min\left\{\left(\sum_{i=0}^N \int_\Omega g_i(x)^2\,dx\right)^{1/2} : T = g_0 + \sum_{i=1}^N \frac{\partial g_i}{\partial x_i}\right\}. \tag{5.8}$$

Note that the minimum is achieved precisely by taking the $(g_0, \dots, g_N)$ provided by the Riesz representation (5.4), (5.5).

Let us summarize the above results in the following statement.

Theorem 5.2.1. *Let us define $H^{-1}(\Omega)$ as the topological dual space of $H_0^1(\Omega)$, i.e., $H^{-1}(\Omega) = H_0^1(\Omega)^*$. Then $H^{-1}(\Omega)$ is isometrically isomorphic to the Hilbert space of distributions $T \in \mathscr{D}'(\Omega)$ satisfying*

$$T = g_0 + \sum_{i=1}^N \frac{\partial g_i}{\partial x_i} \quad \textit{for some } g_0, g_1, \dots, g_N \in L^2(\Omega)$$

with

$$\|T\|_{H^{-1}(\Omega)} = \inf\left\{\left(\sum_{i=0}^N \|g_i\|_{L^2(\Omega)}^2\right)^{1/2} : \ T = g_0 + \sum_{i=1}^N \frac{\partial g_i}{\partial x_i}\right\}.$$

The above results can be easily extended to higher-order Sobolev spaces and to Sobolev spaces built on $L^p(\Omega)$ spaces instead of $L^2(\Omega)$. In that case, one uses the duality between $L^p(\Omega)$ and $L^{p'}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ to obtain the following result.

Theorem 5.2.2. *Let $1 \leq p < +\infty$. The topological dual space of $W_0^{m,p}(\Omega)$ is denoted by $W^{-m,p'}(\Omega)$, where $\frac{1}{p}+\frac{1}{p'}=1$. It is isometrically isomorphic to the Banach space consisting of those distributions $T \in \mathscr{D}'(\Omega)$ satisfying*

$$T = \sum_{0\leq|\alpha|\leq m} D^\alpha g_\alpha \quad \text{with each } g_\alpha \in L^{p'}(\Omega)$$

with

$$\|T\|_{W^{-m,p'}(\Omega)} = \inf\left\{ \left(\sum_{0\leq|\alpha|\leq m} \int_\Omega |g_\alpha|^{p'}(x)\,dx \right)^{1/p'} : T = \sum_{0\leq|\alpha|\leq m} D^\alpha g_\alpha \right\}.$$

5.3 ▪ Poincaré inequality and Rellich–Kondrakov theorem in $W_0^{1,p}(\Omega)$

The Poincaré inequality is a basic ingredient of the variational approach to the Dirichlet problem. It provides the coercivity of the Dirichlet integral $\int_\Omega |Dv|^2\,dx$ on the space $H_0^1(\Omega)$.

Theorem 5.3.1 (Poincaré inequality). *Let Ω be an open subset of $\mathbf{R}^N$ which is bounded in one direction. Then, for each $1 \leq p < +\infty$, there exists a constant $C_{p,N}(\Omega)$ which depends only on Ω, p, and N such that*

$$\left(\int_\Omega |v(x)|^p\,dx \right)^{1/p} \leq C_{p,N}(\Omega) \left(\int_\Omega \sum_{i=1}^N \left| \frac{\partial v}{\partial x_i} \right|^p dx \right)^{1/p} \quad \forall\, v \in W_0^{1,p}(\Omega). \tag{5.9}$$

PROOF. Since Ω is bounded in a direction, we can find a system of coordinates, which for simplicity we still denote $(x_1, x_2, \ldots, x_N)$, such that Ω is contained in the strip $a \leq x_N \leq b$. Let us write $x = (x', x_N)$ with $x' = (x_1, x_2, \ldots, x_{N-1})$.

Since $\mathscr{D}(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ (by definition of $W_0^{1,p}(\Omega)$) let us first take $v \in \mathscr{D}(\Omega)$. We will then extend the result by a density and continuity argument. Let us define

$$\tilde{v} = \begin{cases} v & \text{on } \Omega, \\ 0 & \text{on } \mathbf{R}^N \setminus \Omega. \end{cases}$$

We have, for any $x \in \mathbf{R}^N$, with $a \leq x_N \leq b$, $x' \in \mathbf{R}^{N-1}$

$$\tilde{v}(x) = \tilde{v}(x', x_N) = \tilde{v}(x', a) + \int_a^{x_N} \frac{\partial \tilde{v}}{\partial x_N}(x', t)\,dt.$$

Since $\tilde{v}(x', a) = 0$

$$\tilde{v}(x', x_N) = \int_a^{x_N} \frac{\partial \tilde{v}}{\partial x_N}(x', t)\,dt.$$

Applying the Hölder inequality we obtain ($\frac{1}{p}+\frac{1}{p'}=1$)

$$\begin{aligned} |\tilde{v}(x', x_N)|^p &\leq (x_N - a)^{p/p'} \int_a^{x_N} \left| \frac{\partial \tilde{v}}{\partial x_N}(x', t) \right|^p dt \\ &\leq (x_N - a)^{p/p'} \int_{\mathbf{R}} \left| \frac{\partial \tilde{v}}{\partial x_N}(x', t) \right|^p dt. \end{aligned}$$

Let us first integrate with respect to $x' \in \mathbf{R}^{N-1}$. We obtain

$$\int_{\mathbf{R}^{N-1}} |\tilde{v}(x', x_N)|^p \, dx' \leq (x_N - a)^{p/p'} \int_{\mathbf{R}^N} \left| \frac{\partial \tilde{v}}{\partial x_N}(x) \right|^p dx.$$

Let us now integrate with respect to x_N ($a \leq x_N \leq b$) to obtain

$$\int_{\mathbf{R}^N} |\tilde{v}(x)|^p \, dx \leq \frac{(b-a)^{1+p/p'}}{1+p/p'} \int_{\mathbf{R}^N} \left| \frac{\partial \tilde{v}}{\partial x_N}(x) \right|^p dx.$$

Since $\frac{\partial \tilde{v}}{\partial x_N} = Dv \cdot n$, where n is a unit normal vector to the strip containing Ω, we have

$$\begin{aligned} \left| \frac{\partial \tilde{v}}{\partial x_N} \right|^p = \left| \sum_{i=1}^N \frac{\partial \tilde{v}}{\partial x_i} n_i \right|^p &\leq \left(\sum_{i=1}^N \left| \frac{\partial \tilde{v}}{\partial x_i} \right|^p \right) \left(\sum_{i=1}^N n_i^{p'} \right)^{p/p'} \\ &\leq N^{p/p'} \sum_{i=1}^N \left| \frac{\partial \tilde{v}}{\partial x_i} \right|^p. \end{aligned}$$

Hence, noticing that $1 + \frac{p}{p'} = p$

$$\int_{\mathbf{R}^N} |\tilde{v}(x)|^p \, dx \leq \frac{(b-a)^p}{p} N^{p/p'} \int_{\mathbf{R}^N} \sum_{i=1}^N \left| \frac{\partial \tilde{v}}{\partial x_i} \right|^p dx.$$

Since $\tilde{v}$ and all the $\frac{\partial \tilde{v}}{\partial x_i}$ ($i = 1, \ldots, N$) are equal to zero outside of Ω and are equal, respectively, to v and $\frac{\partial v}{\partial x_i}$ on Ω, we obtain

$$\left(\int_\Omega |v(x)|^p \, dx \right)^{1/p} \leq C_{p,N}(\Omega) \left(\int_\Omega \sum_{i=1}^N \left| \frac{\partial v}{\partial x_i} \right|^p dx \right)^{1/p} \quad \forall\, v \in \mathscr{D}(\Omega)$$

with $C_{p,N}(\Omega) = (b-a) \frac{N^{1/p'}}{p^{1/p}}$.

By the density of $\mathscr{D}(\Omega)$ in $W_0^{1,p}(\Omega)$, this inequality can be directly extended to $v \in W_0^{1,p}(\Omega)$. □

Definition 5.3.1. *The Poincaré constant is the smallest constant C for which the inequality* (5.9) *holds for all $v \in W_0^{1,p}(\Omega)$. We denote it by $\bar{C}_{p,N}(\Omega)$:*

$$\bar{C}_{p,N}(\Omega) = \inf \left\{ C : \left(\int_\Omega |v|^p \, dx \right)^{1/p} \leq C \left(\int_\Omega \sum_{i=1}^N \left| \frac{\partial v}{\partial x_i} \right|^p dx \right)^{1/p} \; \forall v \in W_0^{1,p}(\Omega) \right\}.$$

Equivalently,

$$\frac{1}{\bar{C}_{p,N}(\Omega)} = \inf \left\{ \left(\int_\Omega \sum_{i=1}^N \left| \frac{\partial v}{\partial x_i} \right|^p dx \right)^{1/p} : \int_\Omega |v|^p \, dx = 1,\; v \in W_0^{1,p}(\Omega) \right\}.$$

In some instances (see the "cloud of ice" in Attouch [37], for example), it is useful to know precisely how this constant depends on the size of Ω.

Proposition 5.3.1. *For any $R > 0$, for any Ω in $\mathbf{R}^N$,*

$$\bar{C}_{p,N}(R\Omega) = R\bar{C}_{p,N}(\Omega).$$

PROOF. Let $v \in W_0^{1,p}(R\Omega)$. Let us define $v_R(x) = v(Rx)$. Clearly $v_R \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega |v_R(x)|^p\,dx = \int_\Omega |v(Rx)|^p\,dx = R^{-N}\int_{R\Omega} |v(y)|^p\,dy.$$

$$\begin{aligned}\int_\Omega |Dv_R(x)|^p\,dx &= \int_\Omega \sum_{i=1}^N \left|\frac{\partial v_R}{\partial x_i}(x)\right|^p dx = \int_\Omega \sum_{i=1}^N \left|\frac{\partial}{\partial x_i} v(Rx)\right|^p dx \\ &= R^p \int_\Omega \sum_{i=1}^N \left|\frac{\partial v}{\partial x_i}(Rx)\right|^p dx \\ &= R^{p-N}\int_{R\Omega} \sum_{i=1}^N \left|\frac{\partial v}{\partial x_i}(\xi)\right|^p d\xi.\end{aligned}$$

Hence

$$\begin{aligned}\int_{R\Omega} |v(y)|^p\,dy &= R^N \int_\Omega |v_R(x)|^p\,dx \\ &\le R^N \bar{C}_p(\Omega)^p \int_\Omega |Dv_R|^p\,dx \\ &\le R^N R^{p-N} \bar{C}_p(\Omega)^p \int_{R\Omega} |Dv|^p\,dx \\ &= (R\,\bar{C}_p(\Omega))^p \int_{R\Omega} |Dv|^p\,dx.\end{aligned}$$

This being true for any $v \in W_0^{1,p}(R\Omega)$, it follows that

$$\bar{C}_{p,N}(R\Omega) \le R\bar{C}_{p,N}(\Omega).$$

Conversely, since $\Omega = \frac{1}{R}(R\Omega)$, we have $\bar{C}_{p,N}(\Omega) \le \frac{1}{R}\bar{C}_{p,N}(R\Omega)$, and thus the equality $\bar{C}_{p,N}(R\Omega) = R\bar{C}_{p,N}(\Omega)$ holds. □

We will see later when $p = 2$ in Chapter 8, Theorem 8.4.1, how the Poincaré constant can be related to the first eigenvalue of the Laplacian operator with Dirichlet boundary condition.

Another basic result is the compact embedding theorem of Rellich–Kondrakov. For that purpose, we need to recall the Kolmogorov compactness criteria in $L^p(\mathbf{R}^N)$.

Theorem 5.3.2 (Kolmogorov). *Let $p \in [1, +\infty[$ and let $\mathcal{F}$ be a subset of $L^p(\mathbf{R}^N)$. Then $\mathcal{F}$ is relatively compact in $L^p(\mathbf{R}^N)$ iff the following three conditions are satisfied:*

(i) *$\mathcal{F}$ is bounded in $L^p(\mathbf{R}^N)$;*

(ii) $\lim_{R\to+\infty} \int_{\{|x|>R\}} |v(x)|^p\,dx = 0$ *uniformly with respect to $v \in \mathcal{F}$;*

(iii) $\lim_{h\to 0} \|\tau_h v - v\|_{L^p(\mathbf{R}^N)} = 0$ *uniformly with respect to $v \in \mathcal{F}$, where $\tau_h v$ is the translated function $(\tau_h v)(x) := v(x-h)$.*

PROOF. Let us prove the implication which is useful for applications, that is, (i), (ii), (iii) $\Longrightarrow \mathscr{F}$ relatively compact in $L^p(\mathbf{R}^N)$.

Equivalently, we have to prove that $\mathscr{F}$ is precompact, which means that for any $\varepsilon > 0$, there exists a finite number of balls $B(v_1,\varepsilon),\dots,B(v_k,\varepsilon)$ which cover $\mathscr{F}$. So, let us give $\varepsilon > 0$. By (ii) there exists some $R > 0$ such that

$$\forall v \in \mathscr{F} \quad \int_{|x|>R} |v(x)|^p \, dx < \varepsilon.$$

Let $(\rho_n)_{n\in\mathbf{N}}$ be a mollifier. It follows from Proposition 2.2.4 that

$$\forall n \geq 1, \ \forall v \in L^p(\mathbf{R}^N) \quad \|v - v*\rho_n\|_{L^p}^p \leq \int_{\mathbf{R}^N} \rho_n(y)\|v - \tau_y v\|_{L^p}^p dy.$$

Hence

$$\|v - v*\rho_n\|_{L^p} \leq \sup_{|y|\leq 1/n} \|v - \tau_y v\|_{L^p}.$$

By (iii), there exists some integer $N(\varepsilon) \in \mathbf{N}$ such that

$$\forall v \in \mathscr{F} \quad \|v - v*\rho_{N(\varepsilon)}\|_{L^p} < \varepsilon.$$

On the other hand, for any $x, x' \in \mathbf{R}^N$, for any $v \in L^p(\mathbf{R}^N)$ and $n \in \mathbf{N}$,

$$\begin{aligned} |(v*\rho_n)(x) - (v*\rho_n)(x')| &\leq \int |v(x-y) - v(x'-y)|\rho_n(y)dy \\ &\leq \|\tau_x \check{v} - \tau_{x'}\check{v}\|_{L^p} \, \|\rho_n\|_{L^{p'}} \\ &\leq \|\tau_{x-x'} v - v\|_{L^p} \, \|\rho_n\|_{L^{p'}}. \end{aligned}$$

(Note that this last property follows from the invariance property of the Lebesgue measure.) Moreover,

$$|(v*\rho_n)(x)| \leq \|v\|_{L^p} \, \|\rho_n\|_{L^{p'}}.$$

Let us consider the family $\mathscr{H} = \{v*\rho_{N(\varepsilon)} \, : \, B(0,R) \to \mathbf{R}, \ v \in \mathscr{F}\}$. By using (i) and (iii), we have that it satisfies the conditions of the Ascoli theorem. Hence, it is precompact for the topology of the uniform convergence on $B(0,R)$, and we have the existence of a finite set $\{v_1,\dots,v_k\}$ of elements of $\mathscr{F}$ such that

$$\bigcup_{i=1}^{k} \mathbf{B}(v_i*\rho_{N(\varepsilon)}, \varepsilon R^{-N/p}) \supset \mathscr{H}.$$

So, for all $v \in \mathscr{F}$, there exists some $j \in \{1,2,\dots,k\}$ such that

$$\forall x \in B(0,R) \quad |v*\rho_{N(\varepsilon)}(x) - v_j*\rho_{N(\varepsilon)}(x)| \leq \varepsilon |B(0,R)|^{-1/p}.$$

Hence,

$$\begin{aligned} \|v - v_j\|_{L^p(\mathbf{R}^N)} \leq &\left(\int_{|x|>R} |v|^p \, dx\right)^{1/p} + \left(\int_{|x|>R} |v_j|^p \, dx\right)^{1/p} \\ &+ \|v - v*\rho_{N(\varepsilon)}\|_{L^p} + \|v_j - v_j*\rho_{N(\varepsilon)}\|_{L^p} \\ &+ \|v*\rho_{N(\varepsilon)} - v_j*\rho_{N(\varepsilon)}\|_{L^p(B(0,R))}. \end{aligned}$$

The last term can be majorized as follows:

$$\begin{aligned}\|v*\rho_{N(\varepsilon)}-v_j*\rho_{N(\varepsilon)}\|_{L^p(B(0,R))} &= \left(\int_{B(0,R)} |v*\rho_{N(\varepsilon)}(x)-v_j*\rho_{N(\varepsilon)}(x)|^p\,dx\right)^{1/p}\\ &\le \varepsilon|B(0,R)|^{-1/p}|B(0,R)|^{1/p}=\varepsilon.\end{aligned}$$

Finally,

$$\|v-v_j\|_{L^p(\mathbf{R}^N)} \le 5\varepsilon,$$

which proves that $\mathscr{F}$ is precompact in $L^p(\mathbf{R}^N)$. □

The rate of convergence in $L^p(\mathbf{R}^N)$ of $\tau_h v$ to v as $|h|\to 0$ can be made precise for functions v belonging to $W^{1,p}(\mathbf{R}^N)$.

Proposition 5.3.2. *For all $1\le p\le +\infty$ and all $v\in W^{1,p}(\mathbf{R}^N)$ the following inequality holds:*

$$\forall h\in\mathbf{R}^N \quad \|\tau_h v - v\|_{L^p(\mathbf{R}^N)} \le \|Dv\|_{L^p(\mathbf{R}^N)}|h|,$$

where $\|Dv\|_{L^p(\mathbf{R}^N)}=(\int_{\mathbf{R}^N}|Dv(x)|^p\,dx)^{1/p}$ and $|Dv(x)|$ is the Euclidean norm of $Dv(x)$.

PROOF. Since $\mathscr{D}(\mathbf{R}^N)$ is dense in $W^{1,p}(\mathbf{R}^N)$, we just need to prove this inequality for $v\in\mathscr{D}(\mathbf{R}^N)$. We have

$$\begin{aligned}(\tau_h v)(x)-v(x) &= v(x-h)-v(x)\\ &= -\int_0^1 Dv(x-th)\,h\,dt.\end{aligned}$$

Hence, by the Cauchy–Schwarz inequality

$$|(\tau_h v)(x)-v(x)| \le \int_0^1 |Dv(x-th)||h|\,dt,$$

and then, by Hölder's inequality,

$$|(\tau_h v)(x)-v(x)|^p \le |h|^p\int_0^1 |Dv(x-th)|^p\,dt.$$

Integrating over $\mathbf{R}^N$ we have

$$\int_{\mathbf{R}^N}|(\tau_h v)(x)-v(x)|^p\,dx \le |h|^p\int_{\mathbf{R}^N}\left(\int_0^1|Dv(x-th)|^p\,dt\right)dx.$$

Let us use the Fubini–Tonelli theorem and the invariance by translation of the Lebesgue measure in $\mathbf{R}^N$ to obtain

$$\int_{\mathbf{R}^N}|\tau_h v - v|^p\,dx \le |h|^p\int_{\mathbf{R}^N}|Dv(x)|^p\,dx,$$

which ends the proof. □

We can now state the main compactness theorem in Sobolev spaces.

Theorem 5.3.3 (Rellich–Kondrakov). *Let Ω be a bounded open subset of $\mathbf{R}^N$. Then the canonical embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact. In other words, every bounded subset of $W_0^{1,p}(\Omega)$ is relatively compact in $L^p(\Omega)$.*

PROOF. Let us denote by p the natural extension operator by zero outside of Ω, which is a continuous operator from $W_0^{1,p}(\Omega)$ into $W^{1,p}(\mathbf{R}^N)$. Indeed, it follows from Proposition 5.1.1 that p is a linear isometry.

The restriction operator $r : L^p(\mathbf{R}^N) \to L^p(\Omega)$ defined by $r(v) = v|_\Omega$ is clearly linear continuous (with norm less than or equal to one). So, the embedding

$$i : W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$$

can be written as the composition $i = r \circ j \circ p$,

$$W_0^{1,p}(\Omega) \xrightarrow{p} W^{1,p}(\mathbf{R}^N) \xrightarrow{j} L^p(\mathbf{R}^N) \xrightarrow{r} L^p(\Omega),$$

where j is the canonical embedding from $W^{1,p}(\mathbf{R}^N)$ into $L^p(\mathbf{R}^N)$.

Let $\mathbf{B}$ be the unit ball in $W_0^{1,p}(\Omega)$. Since r is continuous and the image of a compact set by a continuous mapping is still compact, we need to show that $(j \circ p)(\mathbf{B})$ is relatively compact in $L^p(\mathbf{R}^N)$. To that end, we use the Kolmogorov compactness criteria in $L^p(\mathbf{R}^N)$; cf. Theorem 5.3.2.

(i) Since j and p are linear continuous, $(j \circ p)(\mathbf{B})$ is bounded in $L^p(\mathbf{R}^N)$.

(ii) Since Ω is bounded, there exists $R > 0$ such that $\Omega \subset B(0,R)$. Hence for all $v \in \mathbf{B}$, $p(v) = 0$ on $\mathbf{R}^N \setminus B(0,R)$ and $\int_{\{|x|>R\}} |p(v)|^p \, dx = 0$.

(iii) Since $p(\mathbf{B})$ is contained in the unit ball of $W^{1,p}(\mathbf{R}^N)$, it follows from Proposition 5.3.2 that there exists a constant $C > 0$ such that

$$\forall v \in \mathbf{B} \quad ||\tau_h p(v) - p(v)||_{L^p(\mathbf{R}^N)} \leq C|h|,$$

which proves that $\tau_h p(v)$ tends to $p(v)$ in $L^p(\mathbf{R}^N)$ as $h \to 0$, uniformly with respect to $v \in \mathbf{B}$.

This completes the proof. □

By a proof similar to the one above we obtain the following useful result.

Corollary 5.3.1. *Let $\mathscr{F}$ be a subset of $W^{1,p}(\mathbf{R}^N)$, $1 \leq p < +\infty$, which satisfies the following two conditions:*

(i) *$\mathscr{F}$ is bounded in $W^{1,p}(\mathbf{R}^N)$, i.e., $\sup_{v\in\mathscr{F}} ||v||_{W^{1,p}(\mathbf{R}^N)} < +\infty$.*

(ii) *$\mathscr{F}$ is L^p-equi-integrable at infinity, i.e., $\lim_{R\to+\infty} \int_{\{|x|>R\}} |v(x)|^p \, dx = 0$ uniformly with respect to $v \in \mathscr{F}$.*

Then $\mathscr{F}$ is relatively compact in $L^p(\mathbf{R}^N)$.

5.4 ▪ Extension operators from $W^{1,p}(\Omega)$ into $W^{1,p}(\mathbf{R}^N)$. Poincaré inequalities and the Rellich–Kondrakov theorem in $W^{1,p}(\Omega)$

We have been able to prove some important properties of the space $W_0^{1,p}(\Omega)$, without any regularity assumptions on the boundary of the open set Ω, because there always exists a continuous extension operator from $W_0^{1,p}(\Omega)$ into $W^{1,p}(\mathbf{R}^N)$, namely, the extension by zero (Proposition 5.1.1). When working with the space $W^{1,p}(\Omega)$, the situation is more delicate and the regularity of the boundary of Ω will play a crucial role in the proofs and the statements of the results. Once more, a basic ingredient will be the obtainment of an extension operator. So doing, we will be able to use the previous results in $W^{1,p}(\mathbf{R}^N)$, where techniques like convolution and translation naturally apply.

Notation. Given $x\in\mathbf{R}^N$ we write $x=(x',x_N)$ with $x'\in\mathbf{R}^{N-1}$, $x'=(x_1,x_2,\dots,x_{N-1})$. We write

$\mathbf{R}^N_+=\{x=(x',x_N)\ :\ x_N>0\}$ the open upper half-space;

$B=B(0,1)=\{x\in\mathbf{R}^N\ :\ |x|=(\sum_{i=1}^N x_i^2)^{1/2}<1\}$ the open unit ball in $\mathbf{R}^N$;

$B_+=B(0,1)\cap\mathbf{R}^N_+$;

$B_0=B\cap\mathbf{R}^{N-1}=\{x=(x',x_N)\in\mathbf{R}^N\ :\ |x'|\le 1 \text{ and } x_N=0\}$.

A $\mathbf{C}^1$-diffeomorphism from an open set $U\subset X$ into an open set $V\subset Y$, where X and Y are normed linear spaces, is a one-to-one mapping φ from U into V which is continuously differentiable and such that its inverse φ^{-1} is continuously differentiable from V onto U.

Definition 5.4.1. *Let Ω be an open subset of $\mathbf{R}^N$. We say that Ω is of class $\mathbf{C}^1$ if for all $x\in\Gamma=\partial\Omega$ (the topological boundary of Ω), there exists an open neighborhood G of x and a $\mathbf{C}^1$-diffeomorphism φ from $B(0,1)$ onto G such that*

$$\varphi(B_+)=G\cap\Omega,$$
$$\varphi(B_0)=G\cap\Gamma.$$

Theorem 5.4.1. *Let Ω be an open subset of class $\mathbf{C}^1$ which is bounded (or $\Omega=\mathbf{R}^N_+$). Then there exists an extension operator $\mathbf{P}:W^{1,p}(\Omega)\to W^{1,p}(\mathbf{R}^N)$ which is linear and continuous. More precisely, for all $v\in W^{1,p}(\Omega)$,*

(i) $\mathbf{P}v|_\Omega=v$;

(ii) $||\mathbf{P}v||_{L^p(\mathbf{R}^N)}\le C\,||v||_{L^p(\Omega)}$;

(iii) $||\mathbf{P}v||_{W^{1,p}(\mathbf{R}^N)}\le C||v||_{W^{1,p}(\Omega)}$.

The constant C above depends only on Ω and p.

The proof of Theorem 5.4.1 first proves the result in the case where $\Omega=\mathbf{R}^N_+$ is a half-space then passes to the general situation by using partition of unity and local coordinates.

Lemma 5.4.1. *There exists an extension operator $\mathbf{P}:W^{1,p}(\mathbf{R}^N_+)\to W^{1,p}(\mathbf{R}^N)$ which is obtained by reflection:*

$$(\mathbf{P}u)(x',x_N)=\begin{cases} u(x',x_N) & \text{if } x_N>0,\\ u(x',-x_N) & \text{if } x_N<0.\end{cases}$$

Moreover, $\mathbf{P}$ *is linear and continuous, with*

$$||\mathbf{P}u||_{L^p(\mathbf{R}^N)} \le 2||u||_{L^p(\mathbf{R}^N_+)};$$
$$||\mathbf{P}u||_{W^{1,p}(\mathbf{R}^N)} \le 2||u||_{W^{1,p}(\mathbf{R}^N_+)}.$$

PROOF. Clearly $\mathbf{P}u \in L^p(\mathbf{R}^N)$ and

$$\left(\int_{\mathbf{R}^N} |\mathbf{P}u(x)|^p\,dx\right)^{1/p} = \left(2\int_{\mathbf{R}^N_+} |u(x)|^p\,dx\right)^{1/p} = 2^{1/p}||u||_{L^p} \le 2||u||_{L^p}.$$

Let us prove that

$$\frac{\partial}{\partial x_i}(\mathbf{P}u) = \mathbf{P}\left(\frac{\partial u}{\partial x_i}\right) \quad \text{for } i = 1,2,\dots,N-1,$$
$$\frac{\partial}{\partial x_N}(\mathbf{P}u) = \mathbf{S}\left(\frac{\partial u}{\partial x_N}\right),$$

where we have set

$$(\mathbf{S}v)(x',x_N) = \begin{cases} v(x',x_N) & \text{if } x_N > 0, \\ -v(x',-x_N) & \text{if } x_N \le 0. \end{cases}$$

We need to introduce a truncation (on the domain) function. Let $\eta \in \mathbf{C}^\infty(\mathbf{R})$ be such that

$$\eta(t) = \begin{cases} 0 & \text{if } t < \frac{1}{2}, \\ 1 & \text{if } t > 1 \end{cases}$$

and define $\eta_k(t) = \eta(kt)$ for $k \in \mathbf{N}^*$. Given $\varphi \in \mathscr{D}(\mathbf{R}^N)$ let us compute $\langle \frac{\partial}{\partial x_i}\mathbf{P}u, \varphi\rangle_{D',D}$:

(a) Take first $1 \le i \le N-1$. By definition

$$\left\langle \frac{\partial}{\partial x_i}(\mathbf{P}u), \varphi \right\rangle_{(\mathscr{D}'(\mathbf{R}^N),\mathscr{D}(\mathbf{R}^N))} = -\int_{\mathbf{R}^N} \mathbf{P}u.\frac{\partial \varphi}{\partial x_i}\,dx. \tag{5.10}$$

By definition of $\mathbf{P}$,

$$\begin{aligned} \int_{\mathbf{R}^N} \mathbf{P}u\frac{\partial \varphi}{\partial x_i}\,dx &= \int_{\mathbf{R}^{N-1}} dx' \int_{\mathbf{R}^+} u(x',x_N)\frac{\partial \varphi}{\partial x_i}(x',x_N)\,dx_N \\ &\quad + \int_{\mathbf{R}^{N-1}} dx' \int_{\mathbf{R}^-} u(x',-x_N)\frac{\partial \varphi}{\partial x_i}(x',x_N)\,dx_N \\ &= \int_{\mathbf{R}^N_+} u(x)\frac{\partial \psi}{\partial x_i}\,dx, \end{aligned} \tag{5.11}$$

where $\psi(x',x_N) = \varphi(x',x_N) + \varphi(x',-x_N)$.

But ψ does not belong to $\mathscr{D}(\mathbf{R}^N_+)$; this is where we need to use a truncation method.

Take as a test function $(\eta_k\psi)(x',x_N) = \eta_k(x_N)\psi(x',x_N)$. Note that $\eta_k\psi$ belongs to $\mathscr{D}(\mathbf{R}^N_+)$ and since $u \in W^{1,p}(\mathbf{R}^N_+)$

$$\int_{\mathbf{R}^N_+} u\frac{\partial}{\partial x_i}(\eta_k\psi)\,dx = -\int_{\mathbf{R}^N_+} \frac{\partial u}{\partial x_i}\eta_k\psi\,dx.$$

Noticing that $\frac{\partial}{\partial x_i}(\eta_k \psi) = \eta_k \frac{\partial \psi}{\partial x_i}$ (since $1 \le i \le N-1$), we obtain

$$\int_{\mathbf{R}^N_+} u\,\eta_k \frac{\partial \psi}{\partial x_i}\,dx = -\int_{\mathbf{R}^N_+} \frac{\partial u}{\partial x_i}\eta_k \psi\,dx.$$

Then, pass to the limit as $k \to +\infty$. By using the Lebesgue dominated convergence theorem,

$$\int_{\mathbf{R}^N_+} u \frac{\partial \psi}{\partial x_i}\,dx = -\int_{\mathbf{R}^N_+} \frac{\partial u}{\partial x_i}\psi\,dx. \tag{5.12}$$

Then combine (5.10), (5.11), (5.12) to obtain

$$\begin{aligned}\left\langle \frac{\partial}{\partial x_i}(\mathbf{P}u), \varphi \right\rangle &= \int_{\mathbf{R}^N_+} \frac{\partial u}{\partial x_i}\psi\,dx \\ &= \int_{\mathbf{R}^N_+} \frac{\partial u}{\partial x_i}\varphi(x', x_N)\,dx + \int_{\mathbf{R}^N_+} \frac{\partial u}{\partial x_i}\varphi(x', -x_N)\,dx \\ &= \int_{\mathbf{R}^N} \mathbf{P}\left(\frac{\partial u}{\partial x_i}\right)\varphi\,dx.\end{aligned}$$

Hence $\frac{\partial}{\partial x_i}(\mathbf{P}u) = \mathbf{P}(\frac{\partial u}{\partial x_I})$ for $i = 1, 2, \ldots, N-1$.

(b) Take now $i = N$ and compute

$$\begin{aligned}\int_{\mathbf{R}^N} \mathbf{P}u.\frac{\partial \varphi}{\partial x_N}\,dx = &\int_{\mathbf{R}^{N-1}} dx' \int_{\mathbf{R}^+} u(x', x_N)\frac{\partial \varphi}{\partial x_N}(x', x_N)\,dx_N \\ &+ \int_{\mathbf{R}^{N-1}} dx' \int_{\mathbf{R}^-} u(x', -x_N)\frac{\partial \varphi}{\partial x_N}(x', x_N)\,dx_N.\end{aligned}$$

Then note that

$$\int_{\mathbf{R}^-} u(x', -x_N)\frac{\partial \varphi}{\partial x_N}(x', x_N)\,dx_N = -\int_{\mathbf{R}^+} u(x', x_N)\frac{\partial}{\partial x_N}(\varphi(x', -x_N))\,dx_N.$$

Let us introduce $\chi(x', x_N) := \varphi(x', x_N) - \varphi(x', -x_N)$. We have

$$\int_{\mathbf{R}^N} \mathbf{P}u.\frac{\partial \varphi}{\partial x_N}\,dx = \int_{\mathbf{R}^N_+} u\frac{\partial \chi}{\partial x_N}\,dx. \tag{5.13}$$

Since $\chi(x', 0) = 0$, there exists some constant $M > 0$ such that

$$|\chi(x', x_N)| \le M|x_N| \quad \text{for } |x_N| \le R, \quad \text{where } \operatorname{spt}\varphi \subset B(0, R).$$

The same argument as before yields, with η_k as a truncation function (note that $\eta_k \chi \in \mathscr{D}(\mathbf{R}^N_+)$),

$$\int_{\mathbf{R}^N_+} u\frac{\partial}{\partial x_N}(\eta_k \chi)\,dx = -\int_{\mathbf{R}^N_+} \frac{\partial u}{\partial x_N}\eta_k \chi\,dx.$$

We have

$$\frac{\partial}{\partial x_N}(\eta_k \chi) = \eta_k \frac{\partial \chi}{\partial x_N} + \eta'_k \chi$$

and

$$\eta_k'(x_N) = k\eta'(k\,x_N).$$

Hence

$$\int_{\mathbf{R}_+^N} \frac{\partial u}{\partial x_N} \eta_k \chi\, dx = -\int_{\mathbf{R}_+^N} u\eta_k \frac{\partial \chi}{\partial x_N}\, dx - \int_{\mathbf{R}_+^N} u\; k\eta'(k\,x_N)\chi(x', x_N)\, dx. \tag{5.14}$$

Let now pass to the limit as $k \to +\infty$. By the Lebesgue dominated convergence theorem,

$$\int_{\mathbf{R}_+^N} \frac{\partial u}{\partial x_N} \eta_k \chi\, dx \to \int_{\mathbf{R}_+^N} \frac{\partial u}{\partial x_N} \chi\, dx,$$
$$\int_{\mathbf{R}_+^N} u\eta_k \frac{\partial \chi}{\partial x_N}\, dx \to \int_{\mathbf{R}_+^N} u \frac{\partial \chi}{\partial x_N}\, dx,$$

and the last integral in (5.14) can be majorized as

$$\begin{aligned} \left| \int_{\mathbf{R}_+^N} k u \eta'(k\,x_N)\chi(x', x_N)\, dx \right| &\le k M C \int_{0<x_N<1/k} |u| x_N\, dx \\ &\le MC \int_{0<x_N<1/k} |u(x)|\, dx, \end{aligned}$$

where $C = \sup_{t\in[0,1]} \eta'(t)$. Hence

$$\int_{\mathbf{R}_+^N} \frac{\partial u}{\partial x_N} \chi\, dx = -\int_{\mathbf{R}_+^N} u \frac{\partial \chi}{\partial x_N}\, dx,$$

and, returning to (5.13),

$$\begin{aligned} \int_{\mathbf{R}^N} \mathbf{P}u . \frac{\partial \varphi}{\partial x_N}\, dx &= -\int_{\mathbf{R}_+^N} \frac{\partial u}{\partial x_N} \chi\, dx \\ &= -\int_{\mathbf{R}_+^N} \frac{\partial u}{\partial x_N} (\varphi(x', x_N) - \varphi(x', -x_N))\, dx \\ &= -\int_{\mathbf{R}_+^N} \frac{\partial u}{\partial x_N} \varphi + \int_{\mathbf{R}_-^N} \left(\frac{\partial u}{\partial x_N} \right)(x', -x_N) \varphi\, dx \\ &= -\int_{\mathbf{R}^N} \mathbf{S}\left(\frac{\partial u}{\partial x_N} \right) \varphi\, dx, \end{aligned}$$

i.e.,

$$\frac{\partial}{\partial x_N}(\mathbf{P}u) = \mathbf{S}\left(\frac{\partial u}{\partial x_N} \right). \qquad \square$$

Let us now consider a general open set. We will use a partition of unity. Since Ω is an open set of class $\mathbf{C}^1$ which is bounded, there exists a finite number of open sets $G_0, G_1, \dots, G_k$ such that $\bar{\Omega} \subset \bigcup_{i=0}^k G_i$ with $\bar{G}_0 \subset \Omega$ and for each $i = 1, \dots, k$ a system of local coordinates $\varphi_i : B(0,1) \to G_i$.

Let us introduce a partition of the unity relatively to the open covering $\{G_0, G_1, \dots, G_k\}$ of the compact set $\bar{\Omega}$: there exists $\{\alpha_0, \alpha_1, \dots, \alpha_k\}$ with $\alpha_i \in \mathscr{D}(G_i)$, $0 \le i \le k$, and $\sum_{i=0}^k \alpha_i = 1$ on $\bar{\Omega}$. We have now all the elements to prove Theorem 5.4.1.

PROOF OF THEOREM 5.4.1. Given $v \in W^{1,p}(\Omega)$, since $1 = \sum_{i=0}^k \alpha_i$ on Ω we have

$$v = \left(\sum_{i=0}^k \alpha_i\right) v = \sum_{i=0}^k \alpha_i v = \sum_{i=0}^k v_i, \quad \text{where } v_i = \alpha_i v.$$

We now extend each of the functions v_i to the whole of $\mathbf{R}^N$. We have to distinguish the case $i = 0$ from the case $i \ge 1$.

(a) *Extension of* v_0. The natural extension consists in taking

$$\tilde{v}_0(x) = \begin{cases} v_0(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbf{R}^N \setminus \Omega. \end{cases}$$

One can easily verify that

$$\frac{\partial}{\partial x_i}\tilde{v}_0 = \alpha_0 \widetilde{\frac{\partial v}{\partial x_i}} + \frac{\partial \alpha_0}{\partial x_i}\tilde{v}_0,$$

which belongs to $L^p(\mathbf{R}^N)$. Hence $\tilde{v}_0 \in W^{1,p}(\mathbf{R}^N)$.

(b) *Extension of* v_i, $1 \le i \le k$. We use the local coordinates φ_i on G_i to define, for $1 \le i \le k$,

$$w_i = \begin{cases} (\alpha_i v) \circ \varphi_i & \text{on } B_+, \\ 0 & \text{on } \mathbf{R}^N_+ \setminus B_+. \end{cases}$$

This function belongs to $W^{1,p}(\mathbf{R}^N_+)$. We can apply the extension by reflection operator $\mathbf{P}$ (cf. Lemma 5.4.1) to obtain

$$\mathbf{P}w_i \in W^{1,p}(\mathbf{R}^N) \quad \text{and} \quad \operatorname{spt} \mathbf{P}w_i \subset B(0,1).$$

We then return to G_i by considering the function $\mathbf{P}w_i \circ \varphi_i^{-1}$ and define

$$\hat{v}_i = \begin{cases} \mathbf{P}w_i \circ \varphi_i^{-1} & \text{on } G_i, \\ 0 & \text{on } \mathbf{R}^N \setminus G_i. \end{cases}$$

Since $v = \Sigma v_i$, and since we look for a linear extension operator, we set

$$\mathbf{P}v = \sum_{i=0}^k \hat{v}_i.$$

It is easy to verify that Pv coincides with v on Ω and since all the operations involved in this construction are continuous on $W^{1,p}$, so is $\mathbf{P}$. □

As a direct consequence of the extension Theorem 5.4.1, we have the following result, where we denote $\mathscr{D}(\bar{\Omega}) = \{v|_\Omega \ : \ v \in \mathscr{D}(\mathbf{R}^N)\}$.

Proposition 5.4.1. *Let Ω be a bounded open subset of class $\mathbf{C}^1$ in $\mathbf{R}^N$, or $\Omega = \mathbf{R}^N_+$. Then $\mathscr{D}(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$ $(1 \le p < +\infty)$.*

PROOF. Let $v \in W^{1,p}(\Omega)$. By Theorem 5.4.1, $\mathbf{P}v \in W^{1,p}(\mathbf{R}^N)$. By the density of $\mathscr{D}(\mathbf{R}^N)$ in $W^{1,p}(\mathbf{R}^N)$ (see Theorem 5.1.3), there exists a sequence $(v_n)_{n\in\mathbf{N}}$ in $\mathscr{D}(\mathbf{R}^N)$ with $v_n \underset{n\to+\infty}{\longrightarrow} \mathbf{P}v$ in $W^{1,p}(\mathbf{R}^N)$. Then $v_n|_\Omega \in \mathscr{D}(\bar{\Omega})$ and $v_n|_\Omega \to \mathbf{P}v|_\Omega = v$ in $W^{1,p}(\Omega)$.

When $\Omega = \mathbf{R}^N_+$ the conclusion can be achieved by a truncation argument (see [137, Corollary IX.8]). □

Another important consequence of Theorem 5.4.1 is the Rellich–Kondrakov compact embedding theorem in $W^{1,p}(\Omega)$.

Theorem 5.4.2. *Let Ω be a bounded open set in $\mathbf{R}^N$ of class $\mathbf{C}^1$. For $1 \le p \le +\infty$, the canonical embedding $W^{1,p}(\Omega) \to L^p(\Omega)$ is compact.*

PROOF. Let $(v_n)_{n\in\mathbf{N}}$ be a sequence which is bounded in $W^{1,p}(\Omega)$. By Theorem 5.4.1, the sequence $(\mathbf{P}(v_n))$ is bounded in $W^{1,p}(\mathbf{R}^N)$. Since Ω is bounded, there exists some $R > 0$ such that $\Omega \subset B(0,R)$. Take $\alpha \in \mathscr{D}(\mathbf{R}^N)$ such that $\alpha = 1$ on $B(0,R)$ and $\alpha = 0$ on $\mathbf{R}^N \setminus B(0,2R)$. The sequence $(\alpha\, \mathbf{P}(v_n))_{n\in\mathbf{N}}$ is bounded in $W^{1,p}(\mathbf{R}^N)$ and is identically equal to zero on $\mathbf{R}^N \setminus B(0,2R)$. It follows from the Rellich–Kondrakov theorem in $W^{1,p}(\mathbf{R}^N)$ (cf. Corollary 5.3.1) that the sequence $(\alpha\mathbf{P}(v_n))_{n\in\mathbf{N}}$ is relatively compact in $L^p(\mathbf{R}^N)$. Let $\alpha\mathbf{P}(v_{n_k}) \to v$ in $L^p(\mathbf{R}^N)$. Then

$$\alpha\mathbf{P}(v_{n_k})|_\Omega \to v|_\Omega \quad \text{in } L^p(\Omega).$$

Since $\alpha\mathbf{P}(v_{n_k})|_\Omega = v_{n_k}$, we have $v_{n_k} \to v|_\Omega$ in $L^p(\Omega)$. □

As an application of the Rellich–Kondrakov theorem in $W^{1,p}(\Omega)$ we have the following result.

Theorem 5.4.3. *Let $\Omega \subset \mathbf{R}^N$ be an open connected set which is bounded and of class $\mathbf{C}^1$. Let $V \subset W^{1,p}(\Omega)$ be a linear subspace of $W^{1,p}(\Omega)$ which is closed and such that the only constant function belonging to V is the function which is identically zero. Then, there exists a constant $C > 0$ such that*

$$\|v\|_{L^p(\Omega)} \le C \left(\int_\Omega \sum_{i=1}^N \left| \frac{\partial v}{\partial x_i} \right|^p dx \right)^{1/p} \qquad \forall\, v \in V.$$

PROOF. Let us argue by contradiction. There exists a sequence $(v_n)_{n\in\mathbf{N}}$ in V, $v_n \not\equiv 0$, such that

$$\|v_n\|_{L^p} > n\, \|Dv_n\|_{L^p},$$

where we briefly denote $\|Dv\|_{L^p} = (\Sigma|\frac{\partial v}{\partial x_i}|^p)^{1/p}$.

Take $u_n := v_n/\|v_n\|_{L^p}$. We have $u_n \in V$ (since V is a linear subspace), $\|u_n\|_{L^p} = 1$, and $\|Du_n\|_{L^p} < 1/n$, that is, $\lim_{n\to+\infty} \|Du_n\|_{L^p} = 0$. The sequence $(u_n)_{n\in\mathbf{N}}$ is bounded in $W^{1,p}(\Omega)$. By the Rellich–Kondrakov theorem, it is relatively compact in $L^p(\Omega)$. So, we can extract a subsequence $(u_{n_k})_{k\in\mathbf{N}}$ such that

$$u_{n_k} \to u \quad \text{in } L^p(\Omega).$$

Since

$$Du_{n_k} \to 0 \quad \text{in } L^p(\Omega)$$

we conclude that $Du = 0$. Since Ω is connected we have $u \equiv C$ and

$$u_{n_k} \to C \quad \text{in } W^{1,p}(\Omega).$$

Since V is closed in $W^{1,p}(\Omega)$, C belongs to V, which by assumption forces C to be equal to zero. On the other hand, since $\|u_{n_k}\|_{L^p} = 1$ and $u_{n_k} \to C$ in L^p we have $|C||\Omega|^{1/p} = 1$, a contradiction with the fact that $C = 0$. □

Corollary 5.4.1 (Poincaré–Wirtinger inequality). *Let Ω satisfy the assumptions of Theorem 5.4.3. Then there exists a constant $C_p > 0$ such that*

$$\forall v \in W^{1,p}(\Omega) \qquad \left\| v - \frac{1}{|\Omega|}\int_\Omega v(x)\,dx \right\|_{L^p(\Omega)} \le C_p \|Dv\|_{L^p(\Omega)}.$$

Proof. Take $V = \{v \in W^{1,p}(\Omega) \ : \ \int_\Omega v(x)\,dx = 0\}$. Clearly, V is a closed linear subspace of $W^{1,p}(\Omega)$ and the only constant function which is in V is the function which vanishes identically. Then the proof is concluded by noticing that $v - \frac{1}{|\Omega|}\int_\Omega v\,dx$ belongs to V for every $v \in W^{1,p}(\Omega)$. □

5.5 ▪ The Fourier approach to Sobolev spaces. The space $H^s(\Omega)$, $s \in \mathbf{R}$

We give here a description of the space $H^1(\mathbf{R}^N)$ by using the Fourier transform. We recall that for any $v \in L^1(\mathbf{R}^N)$, its Fourier transform $\hat{v}$ is defined by

$$\hat{v}(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbf{R}^N} e^{-i\xi\cdot x} v(x)\,dx. \tag{5.15}$$

It can be shown that $\hat{v}$ belongs to $\mathbf{C}_0(\mathbf{R}^N)$. When $\hat{v} \in L^1(\mathbf{R}^N)$, one can invert the Fourier transform and obtain v from $\hat{v}$ by using the following formula:

$$v(x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbf{R}^N} e^{i\xi\cdot x} \hat{v}(\xi)\,d\xi. \tag{5.16}$$

Indeed, the condition $\hat{v} \in L^1(\mathbf{R}^N)$ is not always easy to handle. One often prefers to work with the Fourier–Plancherel transform, which is defined as follows.

The basic property is that if $v \in L^1(\mathbf{R}^N) \cap L^2(\mathbf{R}^N)$, then $\hat{v} \in L^2(\mathbf{R}^N)$ and, in fact,

$$\|\hat{v}\|_{L^2(\mathbf{R}^N)} = \|v\|_{L^2(\mathbf{R}^N)}. \tag{5.17}$$

Note that, since the Lebesgue measure of $\mathbf{R}^N$ is infinite, $L^2(\mathbf{R}^N)$ is not a subspace of $L^1(\mathbf{R}^N)$ and one cannot define directly $\hat{v}$ for $v \in L^2(\mathbf{R}^N)$ by using formula (5.15). However, if $v \in L^1 \cap L^2$, formula (5.15) makes sense and (5.17) tells us that $v \to \hat{v}$ is an isometry for the L^2 norm.

By using the density of $L^1 \cap L^2$ into L^2 (note, for example, that $L^1 \cap L^2$ contains the continuous functions with compact support), one can extend the Fourier transform to L^2.

The so-obtained mapping is called the Fourier–Plancherel transform and we denote it by $\mathscr{F}$. The basic properties of $\mathscr{F}$ are summarized in the following proposition.

Proposition 5.5.1. *One can associate to each function $v \in L^2(\mathbf{R}^N)$ a function $\mathscr{F}v \in L^2(\mathbf{R}^N)$, called the Fourier–Plancherel transform of v, so that the following properties hold:*

(i) *For all $v \in L^1(\mathbf{R}^N) \cap L^2(\mathbf{R}^N)$, $\mathscr{F}v = \hat{v}$.*

(ii) *For all $v \in L^2(\mathbf{R}^N)$, $\|\mathscr{F}v\|_{L^2(\mathbf{R}^N)} = \|v\|_{L^2(\mathbf{R}^N)}$.*

(iii) *The mapping $\mathscr{F}$ is an isometric isomorphism from $L^2(\mathbf{R}^N)$ onto $L^2(\mathbf{R}^N)$.*

(iv) *For $v \in L^2(\mathbf{R}^N)$, setting for each $n \in \mathbf{N}$*

$$w_n(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{|x|\le n} e^{-i\xi\cdot x} v(x)\,dx,$$

we have $w_n \to \mathscr{F}v$ in $L^2(\mathbf{R}^N)$ as $n \to +\infty$.

(v) *Conversely, for $v \in L^2(\mathbf{R}^N)$, setting for each $n \in \mathbf{N}$*

$$v_n(x) = \frac{1}{(2\pi)^{N/2}} \int_{\{|\xi|\le n\}} e^{i\xi\cdot x} \mathscr{F}(v)(\xi)\,d\xi,$$

we have $v_n \to v$ in $L^2(\mathbf{R}^N)$ as $n \to +\infty$.

We can now give the following characterization of the Sobolev space $H^1(\mathbf{R}^N)$.

Theorem 5.5.1. *We have*

$$H^1(\mathbf{R}^N) = \{v \in L^2(\mathbf{R}^N) : (1+|\xi|^2)^{1/2}\mathscr{F}(v) \in L^2(\mathbf{R}^N)\}$$

and, for any $v \in H^1(\mathbf{R}^N)$,

$$\|v\|_{H^1(\mathbf{R}^N)} = \|(1+|\xi|^2)^{1/2}\mathscr{F}(v)\|_{L^2(\mathbf{R}^N)}.$$

PROOF. Take first $v \in \mathscr{D}(\mathbf{R}^N)$. By definition,

$$\widehat{\frac{\partial v}{\partial x_k}}(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbf{R}^N} e^{-i\xi\cdot x} \frac{\partial v}{\partial x_k}(x)\,dx.$$

Let us integrate by parts the above formula. Since v has compact support, we obtain

$$\widehat{\frac{\partial v}{\partial x_k}}(\xi) = (i\,\xi_k)\hat{v}(\xi). \tag{5.18}$$

We now use (see Theorem 5.1.3) the density of $\mathscr{D}(\mathbf{R}^N)$ in $H^1(\mathbf{R}^N)$. For any $v \in H^1(\mathbf{R}^N)$ there exists a sequence $(v_n)_{n\in\mathbf{N}}$ in $\mathscr{D}(\mathbf{R}^N)$ such that $v_n \to v$ with respect to the norm topology of $H^1(\mathbf{R}^N)$. So for each $n \in \mathbf{N}$,

$$\widehat{\frac{\partial v_n}{\partial x_k}}(\xi) = (i\,\xi_k)\hat{v}_n(\xi). \tag{5.19}$$

By definition of the Fourier–Plancherel transform (which coincides with the classical Fourier transform for functions in $\mathscr{D}(\mathbf{R}^N)$) we have

$$\mathscr{F}\left(\frac{\partial v_n}{\partial x_k}\right)(\xi) = (i\,\xi_k)\mathscr{F}(v_n)(\xi). \tag{5.20}$$

Since $v_n \to v$ for the $\|\cdot\|_{H^1(\mathbf{R}^N)}$ norm, we have that $v_n \to v$ in $L^2(\mathbf{R}^N)$ and $\frac{\partial v_n}{\partial x_k} \longrightarrow \frac{\partial v}{\partial x_k}$ in $L^2(\mathbf{R}^N)$. We now use the continuity property of $\mathscr{F}$ for the $L^2(\mathbf{R}^N)$ norm and pass to the limit in (5.20): one can extract a subsequence $n(p)$ such that

$$\begin{aligned} &\mathscr{F}(v_{n(p)})(\xi) \longrightarrow \mathscr{F}(v)(\xi) && \text{for a.e. } \xi \in \mathbf{R}^N,\\ &\mathscr{F}\left(\frac{\partial v_{n(p)}}{\partial x_k}\right)(\xi) \longrightarrow \mathscr{F}\left(\tfrac{\partial v}{\partial x_k}\right)(\xi) && \text{for a.e. } \xi \in \mathbf{R}^N \end{aligned}$$

and so obtain

$$\mathscr{F}\left(\frac{\partial v}{\partial x_k}\right)(\xi) = i\,\xi_k\,\mathscr{F}(v)(\xi) \quad \text{for a.e. } \xi \in \mathbf{R}^N. \tag{5.21}$$

It follows from the above argument and the isometry property of $\mathscr{F}$ in $L^2(\mathbf{R}^N)$ that

$$\begin{aligned} v \in H^1(\mathbf{R}^N) &\Longleftrightarrow \mathscr{F}(v) \in L^2(\mathbf{R}^N) \text{ and } i\,\xi_k\,\mathscr{F}(v) \in L^2(\mathbf{R}^N)\\ &\qquad \forall\, k = 1,2,\dots,N,\\ &\Longleftrightarrow (1+|\xi|^2)^{1/2}\mathscr{F}(v) \in L^2(\mathbf{R}^N). \end{aligned}$$

Moreover,

$$\begin{aligned} \|v\|^2_{H^1(\mathbf{R}^N)} &= \int_{\mathbf{R}^N} v^2(x) + \sum_{k=1}^N \left(\frac{\partial v}{\partial x_k}\right)^2(x)\,dx\\ &= \int_{\mathbf{R}^N} \left(|\mathscr{F}(v)(\xi)|^2 + \sum_{k=1}^N \left|\mathscr{F}\left(\frac{\partial v}{\partial x_k}\right)(\xi)\right|^2\right) d\xi\\ &= \int_{\mathbf{R}^N} (1+|\xi|^2)\,|\mathscr{F}(v)(\xi)|^2\,d\xi\\ &= \|(1+|\xi|^2)^{1/2}\mathscr{F}(v)\|^2_{L^2(\mathbf{R}^N)} \end{aligned}$$

and the proof is complete. $\square$

The above characterization of the space $H^1(\mathbf{R}^N)$ via the Fourier–Plancherel transform is quite useful. It permits us to obtain in an elegant way a number of important properties of Sobolev spaces. Let us first show how to obtain by this way the Rellich–Kondrakov theorem.

Theorem 5.5.2 (Rellich–Kondrakov, second proof). *Let Ω be a bounded open subset of $\mathbf{R}^N$ with a boundary $\partial\Omega$ of class $\mathbf{C}^1$. Then the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ is compact.*

PROOF. Let $(v_n)_{n\in\mathbf{N}}$ be a sequence in $H^1(\Omega)$ which is bounded for the $\|\cdot\|_{H^1(\Omega)}$ norm, i.e., $\sup_{n\in\mathbf{N}} \|v_n\|_{H^1(\Omega)} < +\infty$. By using the extension operator $\mathbf{P}$ defined in Theorem 5.4.1, which is linear and continuous from $H^1(\Omega)$ into $H^1(\mathbf{R}^N)$, and by using a truncation argument (Ω is assumed to be bounded), we can assume that the sequence v_n is bounded in $H^1(\mathbf{R}^N)$ and $v_n \equiv 0$ outside a ball $B(0,R)$ (with R independent of $n \in \mathbf{N}$).

Since $(v_n)_{n\in\mathbf{N}}$ is bounded in $L^2(\mathbf{R}^N)$, we can extract a weakly convergent subsequence

$$v_{n(k)} \rightharpoonup v \quad \text{in } w-L^2(\mathbf{R}^N).$$

We are going to prove that, because of the uniform bound on the $H^1(\mathbf{R}^N)$ norm of the $v_{n(k)}$, the convergence is actually strong in $L^2(\mathbf{R}^N)$. Without loss of generality we can assume $v=0$ (replacing $v_{n(k)}$ by $v_{n(k)} - v$!).

Let us simplify the notation and write v_k instead of $v_{n(k)}$. We have

$$v_k \to 0 \quad \text{in } w-L^2(\mathbf{R}^N), \tag{5.22}$$

$$v_k \equiv 0 \quad \text{outside of } B(0,R), \tag{5.23}$$

$$\sup_{k\in\mathbf{N}} ||v_k||_{H^1(\mathbf{R}^N)} < +\infty, \tag{5.24}$$

and we want to prove that $v_k \to 0$ in $s-L^2(\mathbf{R}^N)$. To that end, we use the Fourier–Plancherel transformation and its isometrical property from $L^2(\mathbf{R}^N)$ onto $L^2(\mathbf{R}^N)$. We need to prove that

$$||\mathscr{F}(v_k)||_{L^2(\mathbf{R}^N)} \longrightarrow 0 \quad \text{as } k \to +\infty,$$

and we know (see Theorem 5.5.1) that

$$\sup_{k\in\mathbf{N}} ||(1+|\xi|^2)^{1/2}\mathscr{F}(v_k)||_{L^2(\mathbf{R}^N)} := C < +\infty.$$

Let us write

$$\begin{aligned}
\int_{\mathbf{R}^N} |\mathscr{F}(v_k)(\xi)|^2\, d\xi &= \int_{|\xi|\le M} |\mathscr{F}(v_k)(\xi)|^2\, d\xi + \int_{|\xi|\ge M} |\mathscr{F}(v_k)(\xi)|^2\, d\xi \\
&\le \int_{|\xi|\le M} |\mathscr{F}(v_k)(\xi)|^2\, d\xi \\
&\quad + \frac{1}{1+M^2}\int_{\mathbf{R}^N} (1+|\xi|^2)|\mathscr{F}(v_k)(\xi)|^2\, d\xi \\
&\le \int_{|\xi|\le M} |\mathscr{F}(v_k)(\xi)|^2\, d\xi + \frac{C^2}{1+M^2}.
\end{aligned} \tag{5.25}$$

On the other hand, since $v_k \equiv 0$ outside of $B(0,R)$, we have $v_k \in L^2(\mathbf{R}^N) \cap L^1(\mathbf{R}^N)$ and $\mathscr{F}(v_k) = \hat{v}_k$, i.e.,

$$\begin{aligned}
\mathscr{F}(v_k)(\xi) &= \frac{1}{(2\pi)^{N/2}} \int_{\mathbf{R}^N} e^{-i\,\xi\cdot x} v_k(x)\, dx \\
&= \frac{1}{(2\pi)^{N/2}} \int_{\mathbf{R}^n} \mathbf{1}_{B(0,R)}(x) e^{-i\,\xi\cdot x} v_k(x)\, dx.
\end{aligned}$$

For any $\xi \in \mathbf{R}^N$, the function $x \mapsto \mathbf{1}_{B(0,R)}(x)e^{-i\,\xi\cdot x}$ belongs to $L^2(\mathbf{R}^N)$.

By using (5.22), we obtain

$$\forall \xi \in \mathbf{R}^N \quad \mathscr{F}(v_k)(\xi) \to 0 \quad \text{as } k \to +\infty. \tag{5.26}$$

To apply the Lebesgue dominated convergence theorem, we notice that by the Cauchy–Schwarz inequality,

$$|\mathscr{F}(v_k)(\xi)|^2 \le \frac{1}{(2\pi)^N} \|v_k\|^2_{L^2(\mathbf{R}^N)} \, |B(0,R)|, \tag{5.27}$$

where $|B(0,R)|$ is the Lebesgue measure of the ball $B(0,R)$.

By using (5.26) and (5.27), we obtain

$$\int_{|\xi|\le M} |\mathscr{F}(v_k)(\xi)|^2 \, d\xi \to 0 \quad \text{as } k \to +\infty. \tag{5.28}$$

Returning to (5.25), by using (5.28), we obtain

$$\limsup_{k\to+\infty} \int_{\mathbf{R}^N} |\mathscr{F}(v_k)(\xi)|^2 \, d\xi \le \frac{C^2}{1+M^2}.$$

This being true for arbitrarily large M, by letting $M \to +\infty$, we finally obtain

$$\lim_{k\to+\infty} \|\mathscr{F}(v_k)\|_{L^2(\mathbf{R}^N)} = 0,$$

which completes the proof. $\square$

Clearly, the characterization of the space $H^1(\mathbf{R}^N)$ by means of the Fourier–Plancherel transform can be easily extended to higher-order Sobolev spaces $H^m(\Omega)$, $m \in \mathbf{N}$.

Theorem 5.5.3. *For any* $m \in \mathbf{N}$

$$H^m(\mathbf{R}^N) = \{v \in L^2(\mathbf{R}^N) : (1+|\xi|^2)^{m/2} \mathscr{F}(v) \in L^2(\mathbf{R}^N)\}$$

and, for any $v \in H^m(\mathbf{R}^N)$,

$$\|v\|_{H^m(\mathbf{R}^n)} = \|(1+|\xi|^2)^{m/2} \mathscr{F}(v)\|_{L^2(\mathbf{R}^N)}.$$

A major interest of the above approach is that it suggests a natural definition of the space $H^s(\mathbf{R}^N)$ for $s \in \mathbf{R}$, s being not necessarily an integer (s being possibly a fraction or more generally any real number), and s being not necessarily positive. The central point is that the property for the integral

$$\int_{\mathbf{R}^N} (1+|\xi|^2)^m \, |\mathscr{F}(v)(\xi)|^2 \, d\xi$$

to be finite makes sense for any exponent $m \in \mathbf{R}$, since the quantity a^x makes sense for any $x \in \mathbf{R}$ as soon as $a > 0$. (Here $a = 1+|\xi|^2$, which is clearly positive.)

We are led to the following definition.

Definition 5.5.1. *Let* $s \ge 0$ *be a nonnegative real number. Let us define*

$$H^s(\mathbf{R}^N) = \{v \in L^2(\mathbf{R}^N) : (1+|\xi|^2)^{s/2} \mathscr{F}(v) \in L^2(\mathbf{R}^N)\},$$

which is equipped with the scalar product, for any $u, v \in H^s(\mathbf{R}^N)$

$$\langle u, v\rangle_{H^s(\mathbf{R}^N)} := \int_{\mathbf{R}^N} (1+|\xi|^2)^s \mathscr{F}(u)(\xi) \overline{\mathscr{F}(v)}(\xi) \, d\xi,$$

and the corresponding norm,

$$\|v\|_{H^s(\mathbf{R}^N)} := \left(\int_{\mathbf{R}^N} (1+|\xi|^2)^s |\mathscr{F}(v)(\xi)|^2 \, d\xi \right)^{1/2}.$$

Let us summarize in the following statement some of the basic properties of the Sobolev spaces $H^s(\mathbf{R}^N)$.

Proposition 5.5.2. *For any $s \in \mathbf{R}^+$, $H^s(\mathbf{R}^N)$ is a Hilbert space. When $s = m \in \mathbf{N}$ we have that $H^s(\mathbf{R}^N) = H^m(\mathbf{R}^N) = W^{m,2}(\Omega)$ is the classical Sobolev space.*

PROOF. Clearly $\mathscr{F}$ is an isomorphism from $H^s(\mathbf{R}^N)$ onto the Lebesgue space $L^2(a\,dm)$, where $a\,dm$ is the measure with density $a(\xi) = (1+|\xi|^2)^s$ with respect to the Lebesgue measure dm on $\mathbf{R}^N$. Moreover, $\mathscr{F}$ is an isometry and the Hilbert structure of the weighted Lebesgue space $L^2(a\,dm)$ is transferred to $H^s(\mathbf{R}^N)$ by the Fourier–Plancherel mapping $\mathscr{F}$. So

$$H^s(\mathbf{R}^N) \cong L^2((1+|\xi|^2)^s dm) \quad \text{(isomorphism)}$$

and the proof is complete. □

In accordance with the definition of $H^{-1}(\Omega)$ as the topological dual space of $H_0^1(\Omega)$, let us give the following definition of the Sobolev space $H^{-s}(\mathbf{R}^N)$ when $s \geq 0$.

Definition 5.5.2. *Let s be a nonnegative real number. By definition*

$$H^{-s}(\mathbf{R}^N) = H^s(\mathbf{R}^N)^*$$

is the topological dual of $H^s(\mathbf{R}^N)$.

5.6 ▪ Trace theory for $W^{1,p}(\Omega)$ spaces

To study the Dirichlet problem by variational techniques one needs to solve the following problem: "For an arbitrary $v \in H^1(\Omega)$, is it possible to give a meaning to the boundary condition $v = g$ on $\partial\Omega$?"

Clearly, considering v only as an element of $L^2(\Omega)$ is not sufficient information to talk about v on $\partial\Omega$, because the Lebesgue measure of $\partial\Omega$ is zero (for Ω smooth enough). Therefore, one has to rely on the additional information on v, namely, "$\frac{\partial v}{\partial x_i}$ belongs to $L^2(\Omega)$ for any $i = 1, \ldots, N$," to give meaning to v on $\partial\Omega$.

In this section and the next two, we give different answers to this question by using very different techniques. Note that except when $N = 1$, the space $H^1(\Omega)$ is not embedded in the space of continuous functions $\mathbf{C}(\bar{\Omega})$. In this section, we use the geometrical properties of $\partial\Omega$ (the fact that it is locally an $(N-1)$-dimensional manifold) to prove that for a general $v \in W^{1,p}(\Omega)$, $1 \leq p \leq +\infty$, it is possible to give meaning to v on $\partial\Omega$, that is, the trace of v on $\partial\Omega$. Note that for a regular function v, this notion has to reduce to the restriction of v on $\partial\Omega$. This naturally suggests defining the notion of trace by using a density and extension by continuity argument.

Theorem 5.6.1. *Let Ω be a bounded open set whose boundary $\partial\Omega$ is of class $\mathbf{C}^1$. Then, for any $1 \leq p < +\infty$, $\mathscr{D}(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$ and the restriction mapping*

$$\begin{aligned} \gamma_0 :\ & \mathscr{D}(\bar{\Omega}) \longrightarrow L^p(\partial\Omega), \\ & v \longmapsto \gamma_0(v) = v|_{\partial\Omega}, \end{aligned}$$

which to each element $v \in \mathscr{D}(\bar{\Omega})$ associates its restriction to $\partial\Omega$, can be extended by continuity into a linear continuous mapping from $W^{1,p}(\Omega)$ into $L^p(\partial\Omega)$, which we still denote by γ_0. (Without ambiguity, one can still use the simpler notation $v|_{\partial\Omega}$.)

The so-defined mapping

$$\gamma_0 : W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega)$$

is called the trace operator of order zero.

PROOF. By Proposition 5.4.1, and the regularity property of the boundary $\partial\Omega$, we know that $\mathscr{D}(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$. For any $v \in \mathscr{D}(\bar{\Omega})$, without any ambiguity, we can define the restriction of v to $\partial\Omega$, setting $\gamma_0(v) := v|_{\partial\Omega}$.

Assume for a moment that we have been able to prove that

$$\gamma_0 : (\mathscr{D}(\bar{\Omega}), \|\cdot\|_{W^{1,p}(\Omega)}) \longrightarrow (L^p(\partial\Omega), \|\cdot\|_{L^p(\partial\Omega)})$$

is continuous. Since γ_0 is linear, it is uniformly continuous. The space $L^p(\partial\Omega)$ is a Banach space (it is complete). Therefore, all the conditions of the extension by continuity theorem are fulfilled, which provides the existence of $\hat{\gamma}_0$, the unique linear and continuous extension of γ_0

$$\hat{\gamma}_0 : W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega).$$

For simplicity of notation, we still denote by γ_0 the so-defined operator, which is the trace operator.

Thus, we just need to prove that γ_0 is continuous. To do so, we first consider the case of a half-space $\Omega = \mathbf{R}^N_+$ and then use local coordinates.

Lemma 5.6.1. *Take $\Omega = \mathbf{R}^N_+ = \{x = (x', x_N) \in \mathbf{R}^{N-1} \times \mathbf{R} : x_N > 0\}$. Then, for any $1 \le p < +\infty$, the following inequality holds:*

$$\forall v \in \mathscr{D}(\bar{\mathbf{R}}^N_+) \quad \|\gamma_0(v)\|_{L^p(\mathbf{R}^{N-1})} \le p^{1/p} \|v\|_{W^{1,p}(\mathbf{R}^N_+)}.$$

PROOF. Let $v \in \mathscr{D}(\bar{\mathbf{R}}^N_+)$. For any $x' \in \mathbf{R}^{N-1}$ we have

$$\begin{aligned} |v(x',0)|^p &= -\int_0^{+\infty} \frac{\partial}{\partial x_N} |v(x', x_N)|^p \, dx_N \\ &\le p \int_0^{+\infty} |v(x', x_N)|^{p-1} \left| \frac{\partial v}{\partial x_N}(x', x_N) \right| dx_N. \end{aligned}$$

Let us apply the Young convexity inequality

$$ab \le \frac{1}{p} a^p + \frac{1}{p'} b^{p'}$$

with $\frac{1}{p} + \frac{1}{p'} = 1$ to the following situation:

$$a = \left| \frac{\partial v}{\partial x_N}(x', x_N) \right| \quad \text{and} \quad b = |v(x', x_N)|^{p-1}.$$

We obtain

$$|v(x',0)|^p \le p \left[\int_0^{+\infty} \left(\frac{1}{p} \left| \frac{\partial v}{\partial x_N}(x', x_N) \right|^p + \frac{1}{p'} |v(x', x_N)|^{(p-1)p'} \right) dx_N \right].$$

By using the relation $(p-1)p' = p$ we obtain

$$|v(x',0)|^p \le (p-1)\int_0^{+\infty} |v(x',x_N)|^p \, dx_N + \int_0^{+\infty} \left|\frac{\partial v}{\partial x_N}(x',x_N)\right|^p dx_N.$$

Integrating over $\mathbf{R}^{N-1}$ yields

$$\begin{aligned}\int_{\mathbf{R}^{N-1}} |v(x',0)|^p \, dx' &\le (p-1)\int_{\mathbf{R}^N_+} |v(x)|^p \, dx + \int_{\mathbf{R}^N_+} \left|\frac{\partial v}{\partial x_N}(x)\right|^p dx \\ &\le (p-1)\int_{\mathbf{R}^N_+} |v(x)|^p dx + \int_{\mathbf{R}^N_+} \sum_{i=1}^N \left|\frac{\partial v}{\partial x_i}\right|^p dx \\ &\le p \int_{\mathbf{R}^N_+} \left(|v(x)|^p + \sum_{i=1}^N \left|\frac{\partial v}{\partial x_i}\right|^p \right) dx.\end{aligned}$$

Hence

$$||v(\cdot,0)||_{L^p(\mathbf{R}^{N-1})} \le p^{1/p} ||v||_{W^{1,p}(\mathbf{R}^N_+)}$$

and the proof is complete. $\square$

Proof of Theorem 5.6.1 continued. We use a system of local coordinates. With the same notation as in Section 5.4, we set

$$\bar{\Omega} \subset \bigcup_{i=0}^{k} G_i \qquad \text{with } \bar{G}_0 \subset \Omega,\ G_i \text{ open } \forall i = 0,\dots,k,$$

while $\varphi_i : B(0,1) \to G_i$, $i = 1,2,\dots,k$, are the local coordinates, and $\{\alpha_0,\dots,\alpha_i\}$ is an associated partition of unity, i.e., $\alpha_i \in \mathscr{D}(G_i)$, $\alpha_i \ge 0$, $\sum_{i=0}^k \alpha_i = 1$ on $\bar{\Omega}$.

Take $v \in \mathscr{D}(\bar{\Omega})$. For any $1 \le i \le k$, let us define

$$w_i = \begin{cases} (\alpha_i v)\circ \varphi_i & \text{on } B_+, \\ 0 & \text{on } \mathbf{R}^N_+ \setminus B_+. \end{cases}$$

Clearly w_i belongs to $\mathscr{D}(\mathbf{R}^N_+)$. By Lemma 5.6.1, we have

$$||w_i(\cdot,0)||_{L^p(\mathbf{R}^{N-1})} \le p^{1/p} ||w_i||_{W^{1,p}(\mathbf{R}^N_+)}. \tag{5.29}$$

By using classical differential calculus rules (note that all the functions α_i, v, φ_i are continuously differentiable), one obtains the existence, for any $i = 1,\dots,k$, of a constant C_i such that

$$||w_i||_{W^{1,p}(\mathbf{R}^N_+)} \le C_i ||v||_{W^{1,p}(\Omega)}. \tag{5.30}$$

Combining the two inequalities (5.29) and (5.30), we obtain

$$||w_i(\cdot,0)||_{L^p(\mathbf{R}^{N-1})} \le C_i p^{1/p} ||v||_{W^{1,p}(\Omega)}. \tag{5.31}$$

We now use the definition of the $L^p(\partial\Omega)$ norm which is based on the use of local coordinates. One can show that an equivalent norm to the $L^p(\partial\Omega)$ norm can be obtained by using local coordinates: denoting by $\sim$ the extension by zero outside of $\mathbf{R}^{N-1} \setminus \{y \in \mathbf{R}^{N-1} : |y| < 1\}$, we have that

$$L^p(\partial\Omega) = \{v : \partial\Omega \to \mathbf{R} \,:\, \widetilde{(\alpha_i v) \circ \varphi_i}(\cdot, 0) \in L^p(\mathbf{R}^{N-1}), 1 \le i \le k\}$$

and

$$v \longmapsto \left(\sum_{i=1}^{k} \|\widetilde{(\alpha_i v) \circ \varphi_i}\|^p_{L^p(\mathbf{R}^{N-1})} \right)^{1/p} \tag{5.32}$$

is an equivalent norm to the $L^p(\partial\Omega)$ norm.

This definition of the $L^p(\partial\Omega)$ norm and the inequality (5.31) (note that $w_i = \widetilde{(\alpha_i v) \circ \varphi_i)}$ yield

$$\|v\|_{L^p(\partial\Omega)} \le C(p, N, \Omega) \|v\|_{W^{1,p}(\Omega)}$$

for some constant $C(p, N, \Omega)$. Thus, γ_0 is continuous, which ends the proof of Theorem 5.6.1. $\square$

Let us now give some of the most important properties of the trace operator γ_0.

Proposition 5.6.1. *Let us assume that Ω is an open bounded subset of $\mathbf{R}^N$ whose boundary $\partial\Omega$ is $\mathbf{C}^1$. Then, for any $1 \le p < \infty$, $W_0^{1,p}(\Omega)$ is equal to the kernel of γ_0, i.e.,*

$$W_0^{1,p}(\Omega) = \{v \in W^{1,p}(\Omega) : \gamma_0(v) = 0\}.$$

PROOF. We first show the inclusion $W_0^{1,p}(\Omega) \subset \ker \gamma_0$.

Take $v \in W_0^{1,p}(\Omega)$. By definition of $W_0^{1,p}(\Omega)$, there exists a sequence of functions $(v_n)_{n\in\mathbf{N}}$, $v_n \in \mathscr{D}(\Omega)$ such that $v_n \to v$ in $W^{1,p}(\Omega)$. Since $\gamma_0(v_n) = v_n|_{\partial\Omega} = 0$, by continuity of γ_0 we obtain that $\gamma_0(v) = 0$, i.e., $v \in \ker \gamma_0$.

The other inclusion is a bit more involved. We just sketch the main lines of its proof. Using local coordinates, we prove the following result.

Take $v \in W^{1,p}(\mathbf{R}^N_+)$ such that $\gamma_0(v) = 0$. Prove that $v \in W_0^{1,p}(\mathbf{R}^N_+)$. Let us first extend v by zero outside of $\mathbf{R}^N_+$. By using the information $\gamma_0(v) = 0$ one can verify that the so-obtained extension $\tilde{v}$ belongs to $W^{1,p}(\mathbf{R}^N)$. Then let us translate $\tilde{v}$ and consider for any $h > 0$

$$\tau_h \tilde{v}(x', x_N) = \tilde{v}(x', x_N - h).$$

Finally, one regularizes by convolution the function $\tau_h \tilde{v}$. We have that for ε sufficiently small, $\rho_\varepsilon \star (\tau_h \tilde{v})$ belongs to $\mathscr{D}(\mathbf{R}^N_+)$ and $\rho_\varepsilon \star (\tau_h \tilde{v})$ tends to v in $W^{1,p}(\mathbf{R}^N_+)$ as $h \to 0$ and $\varepsilon \to 0$. Hence $v \in W_0^{1,p}(\mathbf{R}^N_+)$. $\square$

Proposition 5.6.2 (Green's formula). *Let Ω be an open bounded set in $\mathbf{R}^N$ whose boundary $\partial\Omega$ is of class $\mathbf{C}^1$. Then, for any $u, v \in H^1(\Omega)$ and for any $1 \le i \le N$, we have*

$$\int_\Omega \frac{\partial u}{\partial x_i} v \, dx = -\int_\Omega u \frac{\partial v}{\partial x_i} dx + \int_{\partial\Omega} \gamma_0(u)\gamma_0(v)(\vec{n}.\vec{e}_i) \, d\sigma.$$

PROOF. Let us first establish the Green's formula for smooth functions $u, v \in \mathscr{D}(\bar{\Omega})$. Let us start from the divergence theorem, which states that for all $\mathbf{C}^1$ real-valued function u and vector-valued function $\vec{V}$,

$$\int_\Omega \operatorname{div}(u\vec{V})dx = \int_{\partial\Omega} u(\vec{V}\cdot\vec{n})\,d\sigma.$$

Take now $\vec{V} = (0,\dots,v,\dots,0) = v\vec{e}_i$, all components being equal to zero, except the component of rank i. We obtain

$$\int_\Omega \left(u\frac{\partial v}{\partial x_i} + v\frac{\partial u}{\partial x_i} \right) dx = \int_{\partial\Omega} uv(\vec{n}\cdot\vec{e}_i)\,d\sigma.$$

Let us now consider arbitrary elements $u, v \in H^1(\Omega)$ and use a density argument.

By Proposition 5.4.1, there exist approximating sequences $(u_k)_{k\in\mathbf{N}}$ (respectively, $(v_k)_{k\in\mathbf{N}}$) of elements of $\mathscr{D}(\bar{\Omega})$ such that u_k converges to u in $H^1(\Omega)$ (respectively, v_k converges to v in $H^1(\Omega)$). For each $k \in \mathbf{N}$, we have

$$\int_\Omega \left(u_k\frac{\partial v_k}{\partial x_i} + v_k\frac{\partial u_k}{\partial x_i} \right) dx = \int_{\partial\Omega} u_k v_k(\vec{n}\cdot\vec{e}_i)\,d\sigma. \tag{5.33}$$

Let us now apply Theorem 5.6.1. By definition of γ_0, and the continuity property of γ_0 from $H^1(\Omega)$ into $L^2(\partial\Omega)$, we have

$$\begin{aligned} u_k|_{\partial\Omega} &\longrightarrow \gamma_0(u) \quad \text{in } L^2(\partial\Omega),\\ v_k|_{\partial\Omega} &\longrightarrow \gamma_0(v) \quad \text{in } L^2(\partial\Omega). \end{aligned}$$

Hence

$$\int_{\partial\Omega} u_k v_k(\vec{n}\cdot\vec{e}_i)\,d\sigma \longrightarrow \int_{\partial\Omega} \gamma_0(u)\gamma_0(v)(\vec{n}\cdot\vec{e}_i)\,d\sigma.$$

One can pass to the limit, without any difficulty, on the left-hand side of (5.33). We finally obtain

$$\int_\Omega \left(u\frac{\partial v}{\partial x_i} + v\frac{\partial u}{\partial x_i} \right) dx = \int_{\partial\Omega} \gamma_0(u)\gamma_0(v)(\vec{n}\cdot\vec{e}_i)\,d\sigma,$$

which ends the proof. □

Remark 5.6.1. (a) One should retain from the above argument the general method of the proof of Green's formulas: first establish it for smooth functions, then pass to the limit by using the continuity properties of the trace operators.

(b) The same formula holds for $u \in W^{1,p}(\Omega)$ and $v \in W^{1,q}(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Let us now examine the space of traces $\{\gamma_0(v) : v \in W^{1,p}(\Omega)\}$. By Theorem 3.6.1 we know that the space trace is included in $L^p(\partial\Omega)$. Indeed, we are going to show that the range of γ_0 is a strict subspace of $L^p(\partial\Omega)$, namely, $\gamma_0(v) \in W^{1-1/p,p}(\partial\Omega)$. This result requires the use of fractional Sobolev spaces on a manifold, which is a quite involved subject. When $p = 2$, one can give a simpler proof of it thanks to the Fourier approach to Sobolev spaces.

Proposition 5.6.3 (range of γ_0). *Let Ω be an open bounded set in $\mathbf{R}^N$ whose boundary $\partial\Omega$ is of class $\mathbf{C}^1$. Then the trace operator γ_0 is linear continuous and onto from $H^1(\Omega) \to H^{1/2}(\partial\Omega)$.*

PROOF. The definition of $H^{1/2}(\partial\Omega)$ is obtained by using local coordinates. Thus, we just need to prove Proposition 5.6.3 when $\Omega=\mathbf{R}^N_+$ and $\partial\Omega=\mathbf{R}^{N-1}$. By using the continuity of the reflection operator $\mathbf{P}: H^1(\mathbf{R}^N_+)\to H^1(\mathbf{R}^N)$ (see Lemma 5.4.1), we just need to prove that the trace operator

$$H^1(\mathbf{R}^N)\xrightarrow{\gamma_0} H^{1/2}(\mathbf{R}^{N-1})$$

is continuous. By using the density of $\mathscr{D}(\mathbf{R}^N)$ in $H^1(\mathbf{R}^N)$ and the definition of γ_0, we can finally reduce our study to the following situation.

Prove the existence of some constant $C\geq 0$ such that for any $v\in\mathscr{D}(\mathbf{R}^N)$

$$\|v(\cdot,0)\|_{H^{1/2}(\mathbf{R}^{N-1})}\leq C\,\|v\|_{H^1(\mathbf{R}^N)}. \tag{5.34}$$

To prove (5.34), we use the Fourier transform as defined in Section 5.5.

Denoting $m_N=(1/(2\pi)^{N/2})dx$, we have

$$\mathscr{F}_N(v)(\xi)=\int_{\mathbf{R}^N} e^{-i\,\xi.x}v(x)dm_N,$$

where $\mathscr{F}_N$ refers to the Fourier transform in $\mathbf{R}^N$. Let us use the classical notation $x=(x',x_N)\in\mathbf{R}^{N-1}\times\mathbf{R}$. For any $v\in\mathscr{D}(\mathbf{R}^N)$, by using the Fubini theorem, we obtain

$$\begin{aligned}\mathscr{F}_N(v)(x',x_N)&=\int_{\mathbf{R}} e^{-i\,x_N\,t}\left(\int_{\mathbf{R}^{N-1}} e^{-i\,x'\cdot y}v(y,t)dm_{N-1}(y)\right)dm_1(t)\\&=\mathscr{F}_1(\mathscr{F}_{N-1}v(x',\cdot))(x_N)\\&=\mathscr{F}_{N-1}(\mathscr{F}_1 v(\cdot,x_N))(x').\end{aligned} \tag{5.35}$$

Let us compute the $H^{1/2}(\mathbf{R}^{N-1})$ norm of $\gamma_0(v)$. We have

$$\gamma_0(v)(x')=v(x',0)=\bar{\mathscr{F}}_1\mathscr{F}_1(v(x',\cdot))(0),$$

where we used the Fourier inversion formula (Proposition 5.5.1). Hence

$$\gamma_0(v)(x')=\int_{-\infty}^{+\infty}\mathscr{F}_1(v(x',\cdot))(t)\,dt. \tag{5.36}$$

Let us apply $\mathscr{F}_{N-1}$ to the two sides of (5.36) and use (5.35) to obtain

$$\mathscr{F}_{N-1}(\gamma_0(v))(x')=\int_{-\infty}^{+\infty}(\mathscr{F}_N v)(x',t)\,dt. \tag{5.37}$$

From Theorem 5.6.1 we have

$$\begin{aligned}\|v\|_{H^1(\mathbf{R}^N)}&=\int_{\mathbf{R}^N}(1+|x|^2)|(\mathscr{F}_N v)(x)|^2dx\\&=\int_{\mathbf{R}^N}(1+|x'|^2+t^2)|(\mathscr{F}_N v)(x',t)|^2dx'dt.\end{aligned}$$

We may rewrite (5.37) as

$$\mathscr{F}_{N-1}(\gamma_0(v))(x')=\int_{-\infty}^{+\infty}(\mathscr{F}_N v)(x',t).(1+|x'|^2+t^2)^{1/2}\frac{dt}{(1+|x'|^2+t^2)^{1/2}}$$

and use the Cauchy–Schwarz inequality to obtain

$$|\mathscr{F}_{N-1}(\gamma_0(v))(x')|^2 \leq \int_{-\infty}^{+\infty} |(\mathscr{F}_N v)(x',t)|^2(1+|x'|^2+t^2)\,dt \int_{-\infty}^{+\infty} \frac{dt}{1+|x'|^2+t^2}. \tag{5.38}$$

An elementary computation yields (after the change of variable $t=(1+|x'|^2)^{1/2}s$)

$$\int_{-\infty}^{+\infty} \frac{dt}{1+|x'|^2+t^2} = \frac{\pi}{(1+|x'|^2)^{1/2}}. \tag{5.39}$$

Combining (5.38) and (5.39), we obtain

$$|(1+|x'|^2)^{1/4}\mathscr{F}_{N-1}(\gamma_0(v))(x')|^2 \leq \pi \int_{-\infty}^{+\infty} |(\mathscr{F}_N v)(x',t)|^2(1+|x'|^2+t^2)dt. \tag{5.40}$$

Let us integrate (5.40) over $\mathbf{R}^{N-1}$ to obtain

$$\begin{aligned}&\int_{\mathbf{R}^{N-1}} |(1+|x'|^2)^{1/4}\mathscr{F}_{N-1}(\gamma_0(v))(x')|^2 dx' \\ &\leq \pi \int_{\mathbf{R}^N} |(\mathscr{F}_N v)(x',t)|^2(1+|x'|^2+t^2)dx'dt.\end{aligned} \tag{5.41}$$

By using again Theorem 5.5.1 and Definition 5.5.1 of $H^{1/2}$, we thus get

$$||\gamma_0(v)||_{H^{1/2}(\mathbf{R}^{N-1})} \leq \sqrt{\pi}||v||_{H^1(\mathbf{R}^N)},$$

which expresses that γ_0 is continuous from $H^1(\mathbf{R}^N)$ into $H^{1/2}(\mathbf{R}^{N-1})$. □

Remark 5.6.2. When v belongs to $W^{2,p}(\Omega)$, by a similar argument one can give a meaning to $\frac{\partial v}{\partial n}$. Just notice that $\nabla v \in W^{1,p}(\Omega)^N$, and hence the trace of ∇v on $\partial\Omega$ belongs to $L^p(\partial\Omega)^N$. One defines

$$\frac{\partial v}{\partial n} := \gamma_0(\nabla v)\cdot \vec{n},$$

which belongs to $L^p(\partial\Omega)$. Indeed, one can show that

$$\frac{\partial v}{\partial n} \in W^{1-1/p,p}(\partial\Omega).$$

When $p=2$, for $v\in H^2(\Omega)$ we have $\frac{\partial v}{\partial n} \in H^{1/2}(\partial\Omega)$.

One can also show that the operator $v \mapsto \{v|_{\partial\Omega}, \frac{\partial v}{\partial n}\}$ is linear continuous and onto from $W^{2,p}(\Omega)$ onto $W^{2-1/p,p}(\partial\Omega)\times W^{1-1/p,p}(\partial\Omega)$.

5.7 ▪ Sobolev embedding theorems

Let Ω be an open set in $\mathbf{R}^N$. We have seen in Section 5.1 (Theorem 5.1.1) that each element of the Sobolev space $W^{1,p}(a,b)$, $1\leq p\leq +\infty$, has a continuous representative. This is no longer true for the elements of the space $W^{1,2}(\Omega)$ as soon as the dimension of the space $N\geq 2$. This raises a natural question: Is there a general relation between the numbers m, p, N which allows us to conclude that $W^{m,p}(\Omega) \hookrightarrow \mathbf{C}(\bar{\Omega})$?

Indeed, the answer is yes, and the Sobolev embedding Theorem 5.7.2 establishes that this is true as soon as $mp > N$. Another important aspect of the Sobolev embedding theorem is that, even if $v \in W^{m,p}(\Omega)$ and $mp < N$, one can say better than $v \in L^p(\Omega)$: indeed $v \in L^q(\Omega)$ with $\frac{1}{q} = \frac{1}{p} - \frac{m}{N}$.

We stress the fact that the Sobolev embedding theorem plays a crucial role in the variational approach to partial differential equations. It allows us to make the link between the two scales of spaces: $W^{m,p}(\Omega)$ and $\mathbf{C}^{k,\alpha}(\Omega)$.

When $p = 2$, an incisive approach to this question consists in using the Fourier–Plancherel transformation, as developed in Section 5.5.

Theorem 5.7.1. *Let $s > 0$ and assume that $2s > N$. Then $H^s(\mathbf{R}^N)$ is continuously embedded in $\mathbf{C}(\mathbf{R}^N)$.*

PROOF. We recall (see Theorem 5.5.1 and Definition 5.5.1) that

$$v \in H^s(\mathbf{R}^N) \Longleftrightarrow (1+|\xi|^2)^{s/2}\mathscr{F}(v) \in L^2(\mathbf{R}^N).$$

Let us set $g = (1+|\xi|^2)^{s/2}\mathscr{F}(v)$, which, by assumption, belongs to $L^2(\mathbf{R}^N)$. We have $\mathscr{F}(v) = g(1+|\xi|^2)^{-s/2}$ and $v = \bar{\mathscr{F}}(\mathscr{F}v)$. When $\mathscr{F}(v) = g(1+|\xi|^2)^{-s/2}$ belongs to $L^1(\mathbf{R}^N)$, then $\bar{\mathscr{F}}$ coincides with the classical inverse Fourier transform, and consequently $v \in \mathbf{C}(\mathbf{R}^N)$. Since g belongs to $L^2(\mathbf{R}^N)$, the function $g(1+|\xi|^2)^{-s/2}$ belongs to $L^1(\mathbf{R}^N)$ as soon as $(1+|\xi|^2)^{-s/2}$ belongs to $L^2(\mathbf{R}^N)$, i.e., $\int_{\mathbf{R}^N}(1+|\xi|^2)^{-s}\, d\xi < +\infty$. We have

$$\int_{\mathbf{R}^N} \frac{d\xi}{(1+|\xi|^2)^s} = c\int_0^{+\infty} \frac{r^{N-1}}{(1+r^2)^s}\, dr$$

and this last integral is finite iff $2s-(N-1) > 1$, i.e., $2s > N$. □

Let us now examine the general case $1 \le p \le +\infty$ and first consider the space $W^{1,p}(\Omega)$. By induction on m, we will then derive the result for $W^{m,p}(\Omega)$.

Theorem 5.7.2 (Sobolev). *Let Ω be an open bounded subset of $\mathbf{R}^N$ with a $\mathbf{C}^1$ boundary $\partial\Omega$. Let $1 \le p \le +\infty$ and consider the Sobolev space $W^{1,p}(\Omega)$. Then, the following continuous embedding results hold:*

(i) *If $1 \le p < N$, then $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$. More precisely, any element $v \in W^{1,p}(\Omega)$ belongs to $L^{p^*}(\Omega)$ and there exists a constant C, depending only on p, N, and Ω such that for all $v \in W^{1,p}(\Omega)$,*

$$||v||_{L^{p^*}(\Omega)} \le C||v||_{W^{1,p}(\Omega)}.$$

(ii) *If $p = N$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \le q < +\infty$.*

(iii) *If $p > N$, then $W^{1,p}(\Omega) \hookrightarrow \mathbf{C}(\bar{\Omega})$. More precisely, we have $W^{1,p}(\Omega) \hookrightarrow \mathbf{C}^{0,\alpha}(\Omega)$ with $\alpha = 1 - \frac{N}{p}$, i.e., each element $v \in W^{1,p}(\Omega)$ is Hölder continuous with exponent α, and there exists a constant C depending only on p, N, and Ω such that for all $v \in W^{1,p}(\Omega)$,*

$$|v(x) - v(y)| \le C||v||_{W^{1,p}(\Omega)}|x-y|^\alpha \quad \textit{for a.e. } x, y \in \Omega.$$

PROOF. The proof method is similar to the one used in Section 5.4. We first prove the Sobolev continuous embeddings (i), (ii), (iii) in the case $\Omega = \mathbf{R}^N$: indeed, one obtains in this case slightly more precise results as stated in Theorem 5.7.3 (Sobolev–Gagliardo–Nirenberg for the case $1 \le p < N$), Theorem 5.7.4 (Morrey for the case $p > N$), and Theorem 5.7.5 for the critical case $p = N$.

Let us introduce the linear continuous extension operator

$$\mathbf{P} : W^{1,p}(\Omega) \hookrightarrow W^{1,p}(\mathbf{R}^N),$$

whose properties are described in Theorem 5.4.1.

Let us consider, for example, the case $1 \le p < N$. The composition of the continuous operators

$$W^{1,p}(\Omega) \overset{\mathbf{P}}{\hookrightarrow} W^{1,p}(\mathbf{R}^N) \overset{Thm.\ 4.7.3}{\hookrightarrow} L^{p^*}(\mathbf{R}^N) \overset{r}{\hookrightarrow} L^{p^*}(\Omega)$$

(where r is the restriction operator to Ω) is still continuous, and it is the canonical embedding from $W^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$, which is the identity. The same argument works for the cases $p > N$ and $p = N$.

Note that to use the extension operator technique, which is developed in Theorem 5.4.1, we need to make some regularity assumptions on Ω, namely, $\partial\Omega$ is assumed to be of class $\mathbf{C}^1$. □

From now on in this section, we work on the whole space $\mathbf{R}^N$.

5.7.1 ▪ Case $1 \le p < N$

Theorem 5.7.3 (Sobolev–Gagliardo–Nirenberg). *Let $1 \le p < N$. Then $W^{1,p}(\mathbf{R}^N) \hookrightarrow L^{p^*}(\mathbf{R}^N)$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$. More precisely, there exists a constant $C = C(p,N)$ such that*

$$\forall v \in W^{1,p}(\mathbf{R}^N) \quad \|v\|_{L^{p^*}(\mathbf{R}^N)} \le C\|\nabla v\|_{L^p(\mathbf{R}^N)}.$$

Remark 5.7.1. Before proving Theorem 5.7.3, it is worth pointing out that by an elementary homogeneity argument, one can obtain the precise value of $p^* = pN/(N-p)$. Let us assume that there exists a constant C and some $1 \le q < +\infty$ such that for all $v \in W^{1,p}(\mathbf{R}^N)$,

$$\|v\|_{L^q(\mathbf{R}^N)} \le C\|\nabla v\|_{L^p(\mathbf{R}^N)}.$$

Let us prove that, necessarily, $q = p^*$. In the above inequality, instead of v, let us take $v_\lambda(x) = v(\lambda x)$ with $\lambda > 0$. We have

$$\left(\int_{\mathbf{R}^N} |v(\lambda x)|^q dx\right)^{1/q} \le C\left(\int_{\mathbf{R}^N} \lambda^p |\nabla v(\lambda x)|^p dx\right)^{1/p}.$$

Let us make the change of variable $y = \lambda x$. We obtain

$$\frac{1}{\lambda^{N/q}}\|v\|_{L^q(\mathbf{R}^N)} \le C\lambda \frac{1}{\lambda^{N/p}}\|\nabla v\|_{L^p(\mathbf{R}^N)},$$

which, after simplification, yields

$$\|v\|_{L^q(\mathbf{R}^N)} \le C\,\lambda^{(1-N/p+N/q)}\|\nabla v\|_{L^p(\mathbf{R}^N)}.$$

Clearly, this formula makes sense only when $1-\frac{N}{p}+\frac{N}{q}=0$, i.e., $q=p^*$. Otherwise, if $1-\frac{N}{p}+\frac{N}{q}>0$, let $\lambda\to 0$ in the above inequality, and if $1-\frac{N}{p}+\frac{N}{q}<0$, let $\lambda\to+\infty$. In both cases, one obtains that for any $v\in W^{1,p}(\mathbf{R}^N)$, $v\equiv 0$, which is an absurd statement!

To prove Theorem 5.7.3 we will use the following lemma, which is due to Gagliardo.

Lemma 5.7.1. *Let $N\geq 2$ and $g_1,g_2,\dots,g_N\in L^{N-1}(\mathbf{R}^{N-1})$. Then the function g defined by $g(x)=g_1(\tilde{x}_1)g_2(\tilde{x}_2)\dots g_N(\tilde{x}_N)$ belongs to $L^1(\mathbf{R}^N)$ and*

$$||g||_{L^1(\mathbf{R}^N)}\leq\prod_{i=1}^N||g_i||_{L^{N-1}(\mathbf{R}^{N-1})}.$$

PROOF. The inequality is obvious when $N=2$. So let us argue by induction, assume that the result has been proved until N, and prove it for $N+1$. Let us give $N+1$ functions $g_1,g_2,\dots,g_{N+1}$ belonging to $L^N(\mathbf{R}^N)$ and consider the function g defined for any $x=(x_1,x_2,\dots,x_{N+1})\in\mathbf{R}^{N+1}$ by

$$\begin{aligned}g(x)&=g_1(\tilde{x}_1)g_2(\tilde{x}_2)\dots g_{N+1}(\tilde{x}_{N+1})\\&=[g_1(\tilde{x}_1)\dots g_N(\tilde{x}_N)]g_{N+1}(\tilde{x}_{N+1}).\end{aligned}$$

Let us fix x_{N+1} and apply the Hölder inequality

$$\int_{\mathbf{R}^N}|g(x)|dx_1dx_2\dots dx_N\leq||g_{N+1}||_{L^N(\mathbf{R}^N)}\left(\int_{\mathbf{R}^N}|g_1(\tilde{x}_1)\dots g_N(\tilde{x}_N)|^{N'}dx_1\dots dx_N\right)^{1/N'}\tag{5.42}$$

with $N'=N/(N-1)$. Then, note that the functions $|g_1(\tilde{x}_1)|^{N'},\dots,|g_N(\tilde{x}_N)|^{N'}$ belong to $L^{N-1}(\mathbf{R}^{N-1})$. By the induction hypothesis,

$$\int_{\mathbf{R}^N}|g_1(\tilde{x}_1)\dots g_N(\tilde{x}_N)|^{N'}dx_1\dots dx_N\leq\prod_{i=1}^N||g_i(.,x_{N+1})||^{N'}_{L^N(\mathbf{R}^{N-1})}.\tag{5.43}$$

Combining (5.42) and (5.43) we obtain

$$\int_{\mathbf{R}^N}|g(x)|dx_1\dots dx_N\leq||g_{N+1}||_{L^N(\mathbf{R}^N)}\prod_{i=1}^N||g_i(\cdot,x_{N+1})||_{L^N(\mathbf{R}^{N-1})}.\tag{5.44}$$

Let us now make x_{N+1} vary and integrate (5.44) over $\mathbf{R}$. We have

$$\int_{\mathbf{R}^{N+1}}|g(x)|dx_1\dots dx_{N+1}\leq||g_{N+1}||_{L^N(\mathbf{R}^N)}\int_{\mathbf{R}}\prod_{i=1}^N||g_i(\cdot,x_{N+1})||_{L^N(\mathbf{R}^{N-1})}dx_{N+1}.\tag{5.45}$$

Let us notice that for all $i=1,\dots,N$ the function $||g_i(\cdot,x_{N+1})||_{L^N(\mathbf{R}^{N-1})}$ belongs to $L^N(\mathbf{R})$. Applying Hölder's inequality to (5.45) with $\frac{1}{N}+\dots+\frac{1}{N}=1$ (N times) we obtain

$$\int_{\mathbf{R}^{N+1}}|g(x)|dx_1\dots dx_{N+1}\leq||g_{N+1}||_{L^N(\mathbf{R}^N)}\prod_{i=1}^N||g_i||_{L^N(\mathbf{R}^N)},$$

i.e.,

$$||g||_{L^1(\mathbf{R}^N)} \leq \prod_{i=1}^{N+1} ||g_i||_{L^N(\mathbf{R}^N)},$$

which completes the induction and the proof. □

PROOF OF THEOREM 5.7.3. (a) Let us use a density argument and prove that it is equivalent to know that the inequality

$$||v||_{L^{p^*}(\mathbf{R}^N)} \leq C||\nabla v||_{L^p(\mathbf{R}^N)} \tag{5.46}$$

holds for any $v \in \mathscr{D}(\mathbf{R}^N)$ or for any $v \in W^{1,p}(\mathbf{R}^N)$. Given $v \in W^{1,p}(\mathbf{R}^N)$, by using Theorem 5.1.3, we can find a sequence $(v_n)_{n\in\mathbf{N}}$ of elements $v_n \in \mathscr{D}(\mathbf{R}^N)$ such that

$$v_n \to v \text{ in } W^{1,p}(\mathbf{R}^N) \text{ and } v_n(x) \to v(x) \text{ for almost every } x \in \mathbf{R}^N.$$

Let us write

$$||v_n||_{L^{p^*}(\mathbf{R}^N)} \leq C||\nabla v_n||_{L^p(\mathbf{R}^N)}.$$

Hence,

$$\begin{aligned} ||v||_{L^{p^*}(\mathbf{R}^N)} &\leq \liminf_n ||v_n||_{L^{p^*}(\mathbf{R}^N)} \\ &\leq C \lim_n ||\nabla v_n||_{L^p(\mathbf{R}^N)} = C||\nabla v||_{L^p(\mathbf{R}^N)}, \end{aligned}$$

where the first above inequality is obtained by using Fatou's lemma.

(b) Let us now verify that it is enough to prove (5.46) when $p = 1$. To do so, let us assume that it is true for $p = 1$, i.e.,

$$\forall v \in W^{1,1}(\mathbf{R}^N) \qquad ||v||_{L^{1^*}(\mathbf{R}^N)} \leq C_1(N)||\nabla v||_{L^1(\mathbf{R}^N)}, \tag{5.47}$$

and prove that it is true for all $1 \leq p < N$.

Let us observe that if $v \in \mathscr{D}(\mathbf{R}^N)$, then $|v|^{p^*/1^*}$ belongs to $W^{1,1}(\mathbf{R}^N)$. Indeed, since $p > 1$ we have $p^* > 1^*$ and the function $|v|^{p^*/1^*}$ is continuously differentiable with compact support. Hence $|v|^{p^*/1^*} \in W^{1,1}(\mathbf{R}^N)$ and

$$\nabla\left(|v|^{p^*/1^*}\right) = \frac{p^*}{1^*} \operatorname{sign} v |v|^{(p^*/1^*)-1} \nabla v. \tag{5.48}$$

Let us replace v by $|v|^{p^*/1^*}$ in (5.47). Applying (5.48) we obtain

$$\left(\int_{\mathbf{R}^N} |v|^{p^*} dx\right)^{1/1^*} \leq C_1(N) \frac{p^*}{1^*} \int_{\mathbf{R}^N} |v|^{(p^*/1^*)-1} |\nabla v| dx. \tag{5.49}$$

Let us apply Hölder's inequality with $\frac{1}{p} + \frac{1}{p'} = 1$ to the right-hand side of (5.49):

$$\left(\int_{\mathbf{R}^N} |v|^{p^*} dx\right)^{1/1^*} \leq C_1(N) \frac{p^*}{1^*} \left(\int_{\mathbf{R}^N} |v|^{(\frac{p^*}{1^*}-1)p'} dx\right)^{1/p'} \left(\int_{\mathbf{R}^N} |\nabla v|^p dx\right)^{1/p}. \tag{5.50}$$

An elementary computation yields the equality $\frac{1}{1^*} - \frac{1}{p^*} = \frac{1}{p'}$, which is equivalent to $(\frac{p^*}{1^*} - 1)p' = p^*$, and allows us to simplify (5.50),

$$||v||_{L^{p^*}(\mathbf{R}^N)} \leq C_1(N) \frac{p^*}{1^*} ||\nabla v||_{L^p(\mathbf{R}^N)}, \tag{5.51}$$

which is precisely (5.46) for an arbitrary $1 \leq p < N$. Note that we have obtained that one can take

$$C(p,N) = C_1(N)\frac{p^*}{1^*}. \tag{5.52}$$

(c) Thus, we just need to prove that for any $v \in \mathscr{D}(\mathbf{R}^N)$,

$$\|v\|_{L^{1^*}(\mathbf{R}^N)} \leq C_1(N)\|\nabla v\|_{L^1(\mathbf{R}^N)}. \tag{5.53}$$

For any $x = (x_1, x_2, \ldots, x_N) \in \mathbf{R}^N$ we have

$$\begin{aligned} |v(x)| &= \left| \int_{-\infty}^{x_i} \frac{\partial v}{\partial x_i}(x_1, x_2, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_N)\, dt \right| \\ &\leq \int_{-\infty}^{x_i} \left| \frac{\partial v}{\partial x_i}(x_1, x_2, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_N) \right| dt. \end{aligned}$$

Symmetrically,

$$|v(x)| \leq \int_{x_i}^{+\infty} \left| \frac{\partial v}{\partial x_i}(x_1, x_2, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_N) \right| dt.$$

Adding these two inequalities we obtain

$$|v(x)| \leq \frac{1}{2}\int_{-\infty}^{+\infty} \left| \frac{\partial v}{\partial x_i}(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_N) \right| dt. \tag{5.54}$$

Let us adopt the following notation. For any $x \in \mathbf{R}^N$ and $i = 1, 2, \ldots, N$

$$\tilde{x}_i := (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N),$$

$$f_i(\tilde{x}_i) := \int_{-\infty}^{+\infty} \left| \frac{\partial v}{\partial x_i}(x_1, x_2, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_N) \right| dt.$$

We can rewrite (5.54) as $|v(x)| \leq \frac{1}{2} f_i(\tilde{x}_i)$ for all $i = 1, \ldots, N$, which implies, as a more symmetric expression,

$$|v(x)|^N \leq \frac{1}{2^N} \prod_{i=1}^{N} f_i(\tilde{x}_i). \tag{5.55}$$

Indeed, in (5.53) we need to majorize $\|v\|_{L^{1^*}}$. Noticing that $1^* = \frac{N}{N-1}$, let us write (5.55) as

$$|v(x)|^{1^*} \leq \frac{1}{2^{N/N-1}} \prod_{i=1}^{N} f_i(\tilde{x}_i)^{\frac{1}{N-1}}. \tag{5.56}$$

We note that for each $i = 1, \ldots, N$ the function $g_i(\tilde{x}_i) := f_i(\tilde{x}_i)^{\frac{1}{N-1}}$ belongs to $L^{N-1}(\mathbf{R}^{N-1})$ and

$$\|g_i\|_{L^{N-1}(\mathbf{R}^{N-1})} = \left\| \frac{\partial v}{\partial x_i} \right\|_{L^1(\mathbf{R}^N)}^{\frac{1}{N-1}}. \tag{5.57}$$

Applying Lemma 5.7.1 to (5.56) and using (5.57) we obtain

$$\begin{aligned}\int_{\mathbf{R}^N} |v(x)|^{1^*} dx &\leq \frac{1}{2^{N/N-1}} \left\| \prod_{i=1}^{N} g_i(\tilde{x}_i) \right\|_{L^1(\mathbf{R}^N)} \\ &\leq \frac{1}{2^{N/N-1}} \prod_{i=1}^{N} ||g_i||_{L^{N-1}(\mathbf{R}^{N-1})} \\ &\leq \frac{1}{2^{N/N-1}} \prod_{i=1}^{N} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^1(\mathbf{R}^N)}^{\frac{1}{N-1}}.\end{aligned}$$

Since $1^* = \frac{N}{N-1}$, it follows that

$$||v||_{L^{1^*}(\mathbf{R}^N)} \leq \frac{1}{2} \prod_{i=1}^{N} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^1(\mathbf{R}^N)}^{1/N}. \tag{5.58}$$

From the convexity inequality $(\prod_{i=1}^{N} a_i)^{1/N} \leq \frac{1}{N} \sum_{i=1}^{N} a_i$, we finally obtain

$$||v||_{L^{1^*}(\mathbf{R}^N)} \leq \frac{1}{2N} \sum_{i=1}^{N} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^1(\mathbf{R}^N)} = \frac{1}{2N} ||\nabla v||_{L^1(\mathbf{R}^N)}, \tag{5.59}$$

which ends the proof. □

Remark 5.7.2. Combining (5.47), (5.52), and (5.59), we obtain that one can take

$$C(p,N) = \frac{1}{2N} \frac{p^*}{1^*} = \frac{p(N-1)}{2N(N-p)}.$$

But this is not the best estimate. The best constant is strictly less than this one; it is known and quite involved (cf. Aubin [63], Talenti [345], and Lieb [275]).

As a direct consequence of the Sobolev–Gagliardo–Nirenberg theorem, one obtains the following Poincaré–Sobolev inequality.

Proposition 5.7.1. *There exists a constant $C(p,N)$ such that for any open subset Ω in $\mathbf{R}^N$ and for any $1 \leq p < N$, the following inequality holds:*

$$\forall v \in W_0^{1,p}(\Omega) \quad ||v||_{L^{p^*}(\Omega)} \leq C(p,N) ||\nabla v||_{L^p(\Omega)}.$$

PROOF. Given $v \in W_0^{1,p}(\Omega)$, let $\tilde{v}$ be the extension of v by zero outside of Ω. By Proposition 5.1.1, $\tilde{v}$ belongs to $W^{1,p}(\mathbf{R}^N)$. Applying Theorem 5.7.3 to $\tilde{v}$ and noticing that $||\tilde{v}||_{L^{p^*}(\mathbf{R}^N)} = ||v||_{L^{p^*}(\Omega)}$ and $||\nabla \tilde{v}||_{L^p(\mathbf{R}^N)} = ||\nabla v||_{L^p(\Omega)}$, we obtain the desired conclusion. □

Remark 5.7.3. (a) A striking feature of the above Poincaré–Sobolev inequality is that it is valid for an arbitrary open set Ω (not necessarily bounded) and the constant $C(p,N)$ is independent of Ω. Indeed, one can take as a value of $C(p,N)$ the number $\frac{p(N-1)}{2N(N-p)}$.

These properties rely on the fact that one estimates $||v||_{L^{p^*}}$ from above by $||\nabla v||_{L^p}$. This makes a great contrast with the classical Poincaré inequality, where one estimates from above $||v||_{L^p}$ by $||\nabla v||_{L^p}$: in the classical Poincaré inequality (Theorem 5.3.1) one has to assume that Ω is bounded (at least in one direction) and the Poincaré constant does depend on Ω.

(b) When Ω is bounded, one can recover the classical Poincaré inequality from Proposition 5.7.1 just by using Hölder's inequality. Indeed, assuming that $1 \le p < N$,

$$\int_\Omega |v(x)|^p dx \le |\Omega|^{1-p/p^*} \left(\int_\Omega |v(x)|^{p^*} dx \right)^{p/p^*}.$$

Hence

$$\begin{aligned} ||v||_{L^p(\Omega)} &\le |\Omega|^{1/p-1/p^*} ||v||_{L^{p^*}(\Omega)} \\ &\le |\Omega|^{1/p-1/p^*} C(p,N) ||\nabla v||_{L^p(\Omega)}. \end{aligned}$$

Using the equality $\frac{1}{p} - \frac{1}{p^*} = \frac{1}{N}$, we obtain

$$||v||_{L^p(\Omega)} \le |\Omega|^{1/N} C(p,N) ||\nabla v||_{L^p(\Omega)}.$$

This is another way to see how the Poincaré constant $\bar{C}_p(\Omega)$ depends on Ω. Observing that $|R\Omega|^{1/N} = R|\Omega|$, we find again the conclusion of Proposition 5.3.1, which is $\bar{C}_p(R\Omega) = R\bar{C}_p(\Omega)$.

Let us summarize the above results in the following statement.

Corollary 5.7.1. *Let Ω be a bounded open subset of $\mathbf{R}^N$ and take $1 \le p < N$. Then, for any $v \in W_0^{1,p}(\Omega)$, the following inequality holds:*

$$||v||_{L^p(\Omega)} \le |\Omega|^{1/N} C(p,N) ||\nabla v||_{L^p(\Omega)}.$$

For example, when $\Omega = B(0,r)$, we obtain

$$\int_{B_r} v(x)^2 dx \le C r^2 \int_{B_r} |\nabla v(x)|^2 dx \qquad \forall v \in H_0^1(B_r).$$

We also have a Poincaré–Wirtinger–Sobolev inequality.

Proposition 5.7.2. *Let Ω be a bounded, connected, open set in $\mathbf{R}^N$ whose boundary $\partial\Omega$ is of class $\mathbf{C}^1$. Then there exists a constant $C(p,N,\Omega)$ such that for any $1 \le p < +\infty$, the following inequality holds:*

$$\forall v \in W^{1,p}(\Omega) \qquad \left\| v - \frac{1}{|\Omega|} \int_\Omega v(x)\, dx \right\|_{L^{p*}(\Omega)} \le C(p,N,\Omega) ||\nabla v||_{L^p(\Omega)}.$$

PROOF. Let us denote $M(v) = \frac{1}{|\Omega|} \int_\Omega v(x)\, dx$ and apply the Sobolev embedding Theorem 5.7.2(i) to the function $v - M(v)$. We obtain

$$\begin{aligned} ||v - M(v)||_{L^{p^*}(\Omega)} &\le C_1(p,N,\Omega) ||v - M(v)||_{W^{1,p}(\Omega)} \\ &\le C_1(p,N,\Omega) \big[||v - M(v)||_{L^p(\Omega)} + ||\nabla v||_{L^p(\Omega)} \big]. \end{aligned}$$

Let us now apply the classical Poincaré–Wirtinger inequality (Corollary 5.4.1) to v:

$$\|v - M(v)\|_{L^p(\Omega)} \le C_2(p,N,\Omega)\|\nabla v\|_{L^p(\Omega)}.$$

Combining the two last inequalities, we obtain

$$\|v - M(v)\|_{L^{p^*}(\Omega)} \le C(p,N,\Omega)\|\nabla v\|_{L^p(\Omega)},$$

which ends the proof. □

5.7.2 ▪ Case $p > N$

We now consider the space $W^{1,p}(\mathbf{R}^N)$ with $p > N$. The following theorem is due to Morrey [302].

Theorem 5.7.4 (Morrey). *Assume that $p > N$. Then there exists a continuous embedding $W^{1,p}(\mathbf{R}^N) \hookrightarrow \mathbf{C}^{0,\alpha}(\mathbf{R}^N)$ with $\alpha = 1 - \frac{N}{p}$. More precisely, there exists a constant $C(p,N)$ such that for all $v \in W^{1,p}(\mathbf{R}^N)$,*

$$|v(y) - v(x)| \le C(p,N)\|\nabla v\|_{L^p(\mathbf{R}^N)}|y-x|^\alpha \quad \textit{for a.e. } x,y \in \mathbf{R}^N.$$

PROOF. Let us first take $v \in \mathscr{D}(\mathbf{R}^N)$. The proof is then completed by a density argument. Let Q be a cube containing the origin and whose edges are parallel to the coordinate axes in $\mathbf{R}^N$ and have a common length equal to $r > 0$. For each $x \in Q$ we have

$$v(x) - v(0) = \int_0^1 \frac{d}{dt} v(tx)\,dt.$$

From this, we infer

$$\begin{aligned} |v(x) - v(0)| &\le \int_0^1 \sum_{i=1}^N |x_i| \left| \frac{\partial v}{\partial x_i}(tx) \right| dt \\ &\le r \sum_{i=1}^N \int_0^1 \left| \frac{\partial v}{\partial x_i}(tx) \right| dt. \end{aligned} \tag{5.60}$$

Let $\bar{v} := \frac{1}{|Q|}\int_Q v(x)dx$ denote the mean value of v on Q. Integrating (5.60) on Q we obtain

$$|\bar{v} - v(0)| \le \frac{r}{|Q|} \int_Q dx \sum_{i=1}^N \int_0^1 \left| \frac{\partial v}{\partial x_i}(tx) \right| dt.$$

Let us exchange the order of integration (Fubini's theorem):

$$|\bar{v} - v(0)| \le \frac{1}{r^{N-1}} \int_0^1 dt \int_Q \sum_{i=1}^N \left| \frac{\partial v}{\partial x_i}(tx) \right| dx.$$

Making the change of variable $y = tx$, we obtain

$$|\bar{v} - v(0)| \le \frac{1}{r^{N-1}} \int_0^1 dt \int_{tQ} \sum_{i=1}^N \left| \frac{\partial v}{\partial x_i}(y) \right| \frac{dy}{t^N}. \tag{5.61}$$

We now use Hölder's inequality and majorize this last integral as follows:

$$\int_{tQ} \left| \frac{\partial v}{\partial x_i}(y) \right| dy \le |tQ|^{1/p'} \left(\int_{tQ} \left| \frac{\partial v}{\partial x_i}(y) \right|^p dy \right)^{1/p}. \tag{5.62}$$

Since $0 \in Q$, we have $tQ \subset Q$ for all $0 \le t \le 1$. From the above inequalities (5.61) and (5.62) it follows that

$$\begin{aligned} |\bar{v} - v(0)| &\le \frac{r^{N/p'}}{r^{N-1}} \int_0^1 \frac{t^{N/p'}}{t^N} \, dt \sum_{i=1}^N \left(\int_Q \left| \frac{\partial v}{\partial x_i}(y) \right|^p dy \right)^{1/p} \\ &\le \frac{r^{1-N/p}}{1-N/p} \|\nabla v\|_{L^p(Q)}. \end{aligned}$$

By translation, this inequality remains true for any cube Q whose edges are parallel to the coordinate axes and have common length equal to r. Hence, for any $x \in Q$,

$$|\bar{v} - v(x)| \le \frac{r^{1-N/p}}{1-N/p} \|\nabla v\|_{L^p(Q)}.$$

We use the triangle inequality

$$|v(y) - v(x)| \le |v(y) - \bar{v}| + |\bar{v} - v(x)|$$

to obtain

$$|v(y) - v(x)| \le \frac{2r^{1-N/p}}{1-N/p} \|\nabla v\|_{L^p(Q)} \qquad \forall x, y \in Q.$$

Then we observe that for any two points $x, y \in \mathbf{R}^N$, there exists an open cube Q which is constructed as above and with $r = 2|y - x|$. It follows that for any $x, y \in \mathbf{R}^N$,

$$|v(y) - v(x)| \le C(p,N) \|\nabla v\|_{L^p(Q)} |y - x|^{1-N/p},$$

where $C(p,N)$ depends only on p and N. Here we have obtained $C(p,N) = \frac{2^{2-N/p}}{1-N/p}$. The proof is then completed by a standard density argument. □

5.7.3 ▪ Case $p = N$

Let us first show that for any $1 \le q < +\infty$, $W^{1,N}(\mathbf{R}^N)$ is continuously embedded in $L^q_{loc}(\mathbf{R}^N)$. This result follows easily from the Sobolev–Gagliardo–Nirenberg theorem in the case $1 \le p < N$ and the fact that $p^* = \frac{pN}{N-p}$ tends to $+\infty$ as p goes to N.

We recall that $v \in L^q_{loc}(\mathbf{R}^N)$ means that for any $R > 0$, $\int_{B(0,R)} |v(x)|^q \, dx < +\infty$. The topology on $L^q_{loc}(\mathbf{R}^N)$ is generated by the family of seminorms $\{\|\cdot\|_k, k = 1, 2, \dots\}$

$$\|v\|_k = \left(\int_{B(0,R_k)} |v(x)|^q \, dx \right)^{1/q},$$

where R_k is an arbitrary sequence tending to $+\infty$ with k. We obtain in this way a Fréchet topology (metrizable and complete), which does not depend on the choice of the sequence $(R_k)_{k \in \mathbf{N}}$, $R_k \to +\infty$. It is equivalent to say that $v_n \to v$ in $L^q_{loc}(\mathbf{R}^N)$ and

$$\forall R < +\infty \qquad \int_{B(0,R)} |v_n(x) - v(x)|^q \, dx \to 0.$$

We can now state the following result.

Theorem 5.7.5 (the limiting case $p = N$). *We have*

$$W^{1,p}(\mathbf{R}^N) \hookrightarrow L^q_{loc}(\mathbf{R}^N) \quad \forall\, 1 \le q < +\infty$$

with a continuous injection.

PROOF. Let $v \in W^{1,N}(\mathbf{R}^N)$. Let us use a truncation (on the domain) argument. Lemma 5.1.2 shows that for any $M \in \mathscr{D}(\mathbf{R}^N)$ the function Mv belongs to $W^{1,N}(\mathbf{R}^N)$ and has a compact support. As a consequence

$$Mv \in W^{1,N-\varepsilon}(\mathbf{R}^N) \quad \forall\, \varepsilon > 0 \text{ arbitrarily small.}$$

(To obtain this result, we use that for any $R > 0$, for any $\varepsilon > 0$, $L^N(B(0,R)) \subset L^{N-\varepsilon}(B(0,R))$. Note that this is false on the whole of $\mathbf{R}^N$!) Let us apply Theorem 5.7.3 with $p = N - \varepsilon < N$. We obtain

$$Mv \in L^q(\mathbf{R}^N) \quad \text{with} \quad \frac{1}{q} = \frac{1}{N-\varepsilon} - \frac{1}{N} = \frac{\varepsilon}{N(N-\varepsilon)}.$$

Hence, for any $M \in \mathscr{D}(\mathbf{R}^N)$, for any $\varepsilon > 0$, $Mv \in L^{\frac{N(N-\varepsilon)}{\varepsilon}}(\mathbf{R}^N)$.

Noticing that $\frac{N(N-\varepsilon)}{\varepsilon} \to +\infty$ as $\varepsilon \to 0$, and taking $M \equiv 1$ on $B(0,R)$, we obtain that for any $1 \le q < +\infty$, for any $R > 0$,

$$v \in L^q(B(0,R)).$$

One can easily verify that the above operations are continuous and so is the embedding $W^{1,N}(\mathbf{R}^N) \hookrightarrow L^q_{loc}(\mathbf{R}^N)$ for all $1 \le q < +\infty$.

Remark 5.7.4. 1. Note that $W^{1,N}(\mathbf{R}^N) \hookrightarrow L^q_{loc}(\mathbf{R}^N)$ for all q finite. In general $v \in W^{1,N}(\mathbf{R}^N)$ does not imply that v is bounded on the bounded sets, as the example $v(x) = |\ln(x)|^k$ described in Section 5.1 shows.

2. One can show (see, for example, [137, Corollary IX.11] that $W^{1,N}(\mathbf{R}^N) \hookrightarrow L^q(\mathbf{R}^N)$ for all $1 \le q < +\infty$. This result clearly implies the conclusion of Theorem 5.7.5.

3. Indeed, functions $v \in W^{1,N}(\mathbf{R}^N)$ are not in $L^\infty_{loc}(\mathbf{R}^N)$, but one can say something better than $v \in L^q_{loc}(\mathbf{R}^N)$ for each $1 \le q < +\infty$. To do so, we need to use a sharper scaling of spaces than the classical Lebesgue $\{L^p; 1 \le p \le +\infty\}$ scaling. We use Orlicz spaces scaling as shown in the following proposition, which is again an easy consequence of Theorem 5.7.3.

Proposition 5.7.3. *There exist two constants $K > 0$ and $L > 0$ such that for any $R > 0$, for any function $v \in W^{1,N}(\mathbf{R}^N)$ which satisfies* $\operatorname{spt} v \subset B(0,R)$ *and* $\|\nabla v\|_{L^N(\mathbf{R}^N)} \le 1$, *we have*

$$\int_{B(0,R)} e^{Kv(x)}dx \le L|B(0,R)|.$$

PROOF. Let us return to Theorem 5.7.3. We have seen that for any $1 \le p < N$, for any $v \in W^{1,p}(\mathbf{R}^N)$,

$$\|v\|_{L^{p^*}(\mathbf{R}^N)} \le \frac{(N-1)p}{2N(N-p)} \|\nabla v\|_{L^p(\mathbf{R}^N)}.$$

We have

$$\frac{(N-1)p}{2N(N-p)} = \frac{(N-1)}{2N^2} \cdot \frac{Np}{N-p}.$$

Noticing that $p^* = \frac{Np}{N-p}$ and using the inequality $\frac{N-1}{2N^2} \leq 1$, we obtain that $\frac{(N-1)p}{2N(N-p)} \leq p^*$. Hence

$$||v||_{L^{p^*}(\mathbf{R}^N)} \leq p^* ||\nabla v||_{L^p(\mathbf{R}^N)}.$$

Let us now suppose that $\operatorname{spt} v \subset B(0,R)$. Using Hölder's inequality with $\frac{1}{p^*/p} + \frac{1}{N/p} = 1$, we obtain

$$\begin{aligned} ||\nabla v||_{L^p(\mathbf{R}^N)} &= \left(\int_{B(0,R)} |\nabla v|^p \, dx \right)^{1/p} \\ &\leq |B_R|^{1/p^*} \left(\int_{B(0,R)} |\nabla v|^N dx \right)^{1/N}. \end{aligned}$$

Combining the two last inequalities, we obtain

$$||v||_{L^{p^*}(B_R)} \leq p^* |B_R|^{1/p^*} ||\nabla v||_{L^N(B_R)}.$$

Then note that when $1 \leq p < N$, the corresponding $p^* = \frac{Np}{N-p}$ varies from $\frac{N}{N-1}$ to $+\infty$. Replacing p^* by q, we obtain the following intermediate result: for any q such that $\frac{N}{N-1} \leq q < +\infty$,

$$||v||_{L^q(B_R)} \leq q |B_R|^{1/q} ||\nabla v||_{L^N(B_R)}. \tag{5.63}$$

We now use the asymptotic development

$$e^{K|v|} = 1 + K|v| + \frac{K^2}{2!}|v|^2 + \cdots + \frac{K^q}{q!}|v|^q + \cdots$$

to obtain

$$\int_{B_R} e^{K|v|} dx = |B_R| + K \int_{B_R} |v| \, dx + \sum_{q=2}^{+\infty} \frac{K^q}{q!} \int_{B_R} |v|^q \, dx. \tag{5.64}$$

Let us assume that $N \geq 2$. This implies $\frac{N}{N-1} \leq 2$, and (5.63) is valid for any $q \geq 2$. Since $||\nabla v||_{L^N(B_R)} \leq 1$, we have

$$\forall q \geq 2 \quad \frac{1}{|B_R|} \int_{B_R} |v|^q dx \leq q^q. \tag{5.65}$$

When $q = 1$, we use the Hölder inequality

$$\int_{B_R} |v(x)| dx \leq |B_R|^{1/2} \left(\int_{B_R} |v(x)|^2 dx \right)^{1/2}$$

and the inequality (5.65) with $q = 2$ to obtain

$$\frac{1}{|B_R|} \int_{B_R} |v(x)| dx \leq \left(\frac{1}{|B_R|} \int_{B_R} |v(x)|^2 dx \right)^{1/2} \leq 2. \tag{5.66}$$

Combining (5.64), (5.65), and (5.66), we obtain

$$\frac{1}{|B_R|} \int_{B_R} e^{K|v(x)|} dx \leq 1 + 2K + \sum_{q=2}^{+\infty} \frac{K^q q^q}{q!}. \tag{5.67}$$

Set $u_q := K^q q^q / q!$. We have

$$\lim_{q\to+\infty} \frac{u_{q+1}}{u_q} = K \lim_{q\to+\infty} \left(1+\frac{1}{q}\right)^q = Ke.$$

Hence, the series $\sum K^q q^q/q!$ is convergent when $K < 1/e$. Choosing $K < 1/e$, we take $L = 1 + 2K + \sum_{q=2}^{+\infty} K^q q^q / q!$ and so we obtain for any $R > 0$

$$\frac{1}{|B_R|}\int_{B_R} e^{Kv(x)}dx \leq L,$$

which ends the proof. □

One can improve the conclusion of Proposition 5.7.3. Instead of (5.63), one can prove the sharper estimation

$$||v||_{L^q} \leq C_N q^{1-\frac{1}{N}} |B_R|^{1/q} ||\nabla v||_{L^N}.$$

At this point, note the importance of getting the best constant $C(p,N)$ in the Sobolev embedding theorem. Then, by using the same argument as above, one obtains the following inequality, which is due to Trudinger [352] and Moser [306].

Proposition 5.7.4. *Under the assumptions of Proposition* 5.7.3, *there exist some constants* $\sigma > 0$ *and* $K(N) > 0$ *such that for any* v *in* $W^{1,N}(\mathbf{R}^N)$ *with* $\operatorname{spt} v \subset B_R$ *and* $||\nabla v||_{L^N(B_R)} \leq 1$, *we have*

$$\int_{B_R} e^{\sigma |v(x)|^{N/N-1}} dx \leq K(N)|B_R|.$$

Note that the power $\frac{N}{N-1}$ is the best possible power.

Let us now return to the general situation and complete the study of the Sobolev embedding theorem by the following results. Repeated applications of Theorem 5.7.2 enable us to obtain the statement below.

Theorem 5.7.6. *Let* Ω *be an open bounded subset of* $\mathbf{R}^N$ *with a* $\mathbf{C}^1$ *boundary* $\partial\Omega$. *Let* $1 \leq p < +\infty$ *and let* $m \geq 0$ *be an integer.*

(i) *If* $mp < N$, *then* $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ *with* $\frac{1}{q} = \frac{1}{p} - \frac{m}{N}$.

(ii) *If* $mp = N$, *then* $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ *for all* $1 \leq q < +\infty$.

(iii) *If* $mp > N$, *let us set* $k = [m - \frac{N}{p}]$ *and* $\alpha = m - \frac{N}{p} - k$ $(0 \leq \alpha < 1)$. *Then* $W^{m,p}(\Omega) \hookrightarrow \mathbf{C}^{k,\alpha}(\Omega)$, *where* $v \in C^{k,\alpha}(\Omega)$ *means that* $v \in \mathbf{C}^k(\Omega)$ *and* $D^l v \in \mathbf{C}^{0,\alpha}(\Omega)$ *for any* l *with* $|l| = k$.

Remark 5.7.5. As an illustration of the above theorem, we notice that $H^2(\Omega)$, where $\Omega \subset \mathbf{R}^N$, is continuously embedded in $\mathbf{C}(\bar{\Omega})$ as soon as $N < 4$, i.e., for $N = 1,2,3$.

Let us end this section by the following compactness embedding theorem. Using the Sobolev embedding theorem, one can improve the Rellich–Kondrakov theorem, Theorem 5.4.2.

Theorem 5.7.7. *Let* Ω *be an open bounded subset of* $\mathbf{R}^N$ *which has a* $\mathbf{C}^1$ *boundary* $\partial\Omega$. *Then we have the following compact injections:*

(i) *If* $p < N$, $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ *for any* $q < p^*$ *with* $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$.

(ii) *If* $p = N$, $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ *for any* $1 \le q < +\infty$.

(iii) *If* $p > N$, $W^{1,p}(\Omega) \hookrightarrow \mathbf{C}(\bar{\Omega})$.

PROOF. The case (iii) follows from Theorem 5.7.2(iii) and Ascoli's theorem: any bounded subset of $W^{1,p}(\Omega)$ is bounded in $\mathbf{C}^{0,\alpha}(\Omega)$ with $\alpha = 1 - \frac{N}{p}$ and hence equicontinuous.

Let us consider the case (i). Take a bounded sequence $(v_n)_{n\in\mathbf{N}}$ in $W^{1,p}(\Omega)$. By the Sobolev embedding Theorem 5.7.2(i), the sequence $(v_n)_{n\in\mathbf{N}}$ is bounded in $L^{p^*}(\Omega)$. By the classical Rellich–Kondrakov theorem, Theorem 5.4.2, we can extract a convergent subsequence $v_{n_k} \to v$ in $L^p(\Omega)$. Let us prove that, indeed, the convergence of v_{n_k} to v holds in every $L^q(\Omega)$, with $1 \le q < p^*$.

We use the following generalized version of Hölder's inequality: If $f_1, f_2, \dots, f_k$ satisfy $f_i \in L^{p_i}(\Omega)$ for $1 \le i \le k$ and

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} \le 1,$$

then the product $f = \prod_{i=1}^k f_i$ belongs to $L^p(\Omega)$ and

$$\|f\|_{L^p} \le \|f_1\|_{L^{p_1}} \cdots \|f_k\|_{L^{p_k}}.$$

In particular, if $f \in L^p(\Omega) \cap L^q(\Omega)$ with $1 \le p \le q \le +\infty$, then $f \in L^r(\Omega)$ for all $p \le r \le q$ and the following interpolation formula holds:

$$\|f\|_{L^r} \le \|f\|_{L^p}^{\alpha} \|f\|_{L^q}^{1-\alpha} \quad \text{with} \quad \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q} \quad (0 \le \alpha \le 1), \tag{5.68}$$

where we just applied Hölder's inequality to $|f|^\alpha \in L^{p/\alpha}$ and $|f|^{1-\alpha} \in L^{q/1-\alpha}$ noticing that

$$\frac{1}{r} = \frac{1}{p/\alpha} + \frac{1}{q/1-\alpha}.$$

So, let us take $p \le q < p^*$ and apply the above interpolation formula (5.68) to $|v_{n_k} - v|$ with $\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{p^*}$ and $\alpha > 0$ (because $q < p^*$). We have

$$\begin{aligned} \|v_{n_k} - v\|_{L^q(\Omega)} &\le \|v_{n_k} - v\|_{L^p}^{\alpha} \|v_{n_k} - v\|_{L^{p^*}}^{1-\alpha} \\ &\le C \|v_{n_k} - v\|_{L^p}^{\alpha} \end{aligned}$$

since $(v_{n_k})_{k\in\mathbf{N}}$ is bounded in $L^{p^*}(\Omega)$. Note that by Fatou's lemma, for example, one has also that v belongs to $L^{p^*}(\Omega)$. Since $\alpha > 0$ and $v_{n_k} \to v$ in $L^p(\Omega)$ we obtain that $v_{n_k} \to v$ in $L^q(\Omega)$ for any $p \le q < p^*$ and hence for any $1 \le q < p^*$ (Ω is bounded). □

Remark 5.7.6. Another proof of Theorem 5.7.7 relies on the use of Vitali's theorem: one can notice that for any $1 < q < p^*$ the sequence $\{|v_{n_k}|^q \ : \ k \in \mathbf{N}\}$ is equi-integrable (by a direct application of the classical Hölder's inequality) and converges in measure (after extraction of a further subsequence).

5.8 ▪ Capacity theory and elements of potential theory

In this section, we consider the Sobolev spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$, with $1 \le p < +\infty$, and pay particular attention to the case $p = 2$. We introduce the notion of capacity with respect to the energy functional $\Phi(v) = \int_\Omega |\nabla v(x)|^p \, dx$. As a key tool, we use that the contractions operate on the space $W^{1,p}(\Omega)$, as explained below.

5.8.1 ▪ Contractions operate on $W^{1,p}(\Omega)$

The central idea, which goes back to Deny and Beurling [95], is that the contractions operate on the spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$. Let us make this more precise. We say that $T : \mathbf{R} \to \mathbf{R}$ is a contraction if $T(0) = 0$ and $|T(r) - T(s)| \le |r - s|$ for all $r, s \in \mathbf{R}$. To say that "the contractions operate on $W^{1,p}(\Omega)$" means that, for any $v \in W^{1,p}(\Omega)$, $T \circ v \in W^{1,p}(\Omega)$ and $\|T \circ v\|_{W^{1,p}(\Omega)} \le \|v\|_{W^{1,p}(\Omega)}$.

We denote by $T \circ v$ the composition of the two functions $(T \circ v)(x) = T(v(x))$. When $p = 2$, this property is the basis of the so-called theory of Dirichlet spaces and Dirichlet forms.

The most commonly used contraction functions are $T(r) = r^+$, $T(r) = |r|$, and $T(r) = 1 \wedge r^+$ (fundamental contraction).

To state the results for a general open set Ω, we need to use the following density result, which is closely related to Theorem 5.1.3.

Theorem 5.8.1 (Friedrichs). *Let Ω be an arbitrary open set in $\mathbf{R}^N$. Then, for any $v \in W^{1,p}(\Omega)$, $1 \le p < +\infty$, there exists a sequence $v_n \in \mathcal{D}(\mathbf{R}^N)$ such that*

$$v_n|_\Omega \to v \quad \text{in } L^p(\Omega) \text{ and a.e.},$$

$$\left.\frac{\partial v_n}{\partial x_i}\right|_\omega \to \left.\frac{\partial v}{\partial x_i}\right|_\omega \quad \text{in } L^p(\omega) \quad \forall\, \omega \subset\subset \Omega,\ i = 1, \dots, N.$$

PROOF. Let $\bar{v}$ be the extension by zero of v outside of Ω, i.e., $\bar{v}(x) = v(x)$ if $x \in \Omega$, $\bar{v}(x) = 0$ if $x \in \mathbf{R}^N \setminus \Omega$. Note that, except when $v \in W_0^{1,p}(\Omega)$, $\bar{v}$ does not belong to $W^{1,p}(\mathbf{R}^N)$!

We proceed analogously to the proof of Theorem 5.1.3. We regularize $\bar{v}$ by taking $v_n = M_n(\bar{v}(x) \star \rho_n)$, where M_n is a truncation function (on the domain) and ρ_n is a smoothing kernel. Clearly, v_n belongs to $\mathcal{D}(\mathbf{R}^N)$. Then note that for any $\omega \subset\subset \Omega$ one can find a function $\alpha \in \mathcal{D}(\Omega)$ such that $\alpha = 1$ on ω and $0 \le \alpha \le 1$. The point is that for n sufficiently large,

$$\bar{v} \star \rho_n = \alpha \bar{v} \star \rho_n \text{ on } \omega.$$

Since $\alpha\bar{v}$ belongs to $W^{1,p}(\mathbf{R}^N)$ one can apply the same argument as in the proof of Theorem 5.1.3 to obtain the result. After extraction of a subsequence one can also obtain the convergence almost everywhere. □

Let us first consider the case of smooth truncations.

Proposition 5.8.1. *Let $T \in \mathbf{C}^1(\mathbf{R})$ be a smooth truncation, i.e., $T : \mathbf{R} \to \mathbf{R}$ is a $\mathbf{C}^1$ function which satisfies $T(0) = 0$ and $|T'(r)| \le 1$ for all $r \in \mathbf{R}$. Let Ω be an arbitrary open set in $\mathbf{R}^N$, and let $1 \le p < +\infty$. Then the following properties hold:*

(a) *for all $v \in W^{1,p}(\Omega)$, $T \circ v \in W^{1,p}(\Omega)$, $\frac{\partial}{\partial x_i}(T \circ v) = T'(v)\frac{\partial v}{\partial x_i}$ for all $1 \le i \le N$, and $\|T \circ v\|_{W^{1,p}(\Omega)} \le \|v\|_{W^{1,p}(\Omega)}$;*

(b) *when $v \in W_0^{1,p}(\Omega)$ we have $T \circ v \in W_0^{1,p}(\Omega)$.*

PROOF. We have that for all $r, s \in \mathbf{R}$, $|T(r) - T(s)| \le |r - s|$. This combined with $T(0) = 0$ yields $|T(r)| \le |r|$ for all $r \in \mathbf{R}$. Hence

$$|T \circ v(x)| \le |v(x)|$$

and $T \circ v \in L^p(\Omega)$. In a similar way,

$$\left| T'(v) \frac{\partial v}{\partial x_i} \right| \le \left| \frac{\partial v}{\partial x_i} \right|$$

and $T'(v)\frac{\partial v}{\partial x_i}$ belongs to $L^p(\Omega)$. Thus, we just need to prove that $\frac{\partial}{\partial x_i}(T \circ v) = T'(v)\frac{\partial v}{\partial x_i}$ in the distribution sense, which means that

$$\forall \varphi \in \mathscr{D}(\Omega) \quad \int_\Omega (T \circ v) \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega T'(v) \frac{\partial v}{\partial x_i} \varphi \, dx. \tag{5.69}$$

To prove (5.69) we use Theorem 5.8.1, which provides us a sequence $(v_n)_{n \in \mathbf{N}}$ in $\mathscr{D}(\mathbf{R}^N)$ such that $v_n \to v$ in $L^p(\Omega)$ and a.e., and $\frac{\partial v_n}{\partial x_i} \to \frac{\partial v}{\partial x_i}$ in $L^p(\omega)$ for all $\omega \subset\subset \Omega$.

Since $T \circ v_n$ belongs to $\mathbf{C}^1(\Omega)$ and $\varphi \in \mathscr{D}(\Omega)$, we have by using classical differential calculus that for all $n \in \mathbf{N}$

$$\int_\Omega (T \circ v_n) \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega T'(v_n) \frac{\partial v_n}{\partial x_i} \varphi \, dx. \tag{5.70}$$

We have

$$|T \circ v_n - T \circ v| \le |v_n - v|,$$

which implies that $T \circ v_n \to T \circ v$ in $L^p(\Omega)$.

On the other hand, let $\varphi \equiv 0$ outside of $\omega \subset\subset \Omega$. We have

$$\begin{aligned} \left| T'(v_n)\frac{\partial v_n}{\partial x_i} - T'(v)\frac{\partial v}{\partial x_i} \right| &\le \left| T'(v_n)\left(\frac{\partial v_n}{\partial x_i} - \frac{\partial v}{\partial x_i} \right) \right| + \left| (T'(v_n) - T'(v)) \frac{\partial v}{\partial x_i} \right| \\ &\le \left| \frac{\partial v_n}{\partial x_i} - \frac{\partial v}{\partial x_i} \right| + |T'(v_n) - T'(v)| \left| \frac{\partial v}{\partial x_i} \right|. \end{aligned}$$

Since $v_n \to v$ a.e. and T' is continuous, we have

$$T'(v_n) \to T'(v) \quad \text{a.e.}$$

and

$$|T'(v_n) - T'(v)|^p \left| \frac{\partial v}{\partial x_i} \right|^p \le 2^p \left| \frac{\partial v}{\partial x_i} \right|^p.$$

We obtain, thanks to the Lebesgue dominated convergence theorem,

$$T'(v_n)\frac{\partial v_n}{\partial x_i} \to T'(v)\frac{\partial v}{\partial x_i} \text{ in } \quad L^p(\omega).$$

We can now pass to the limit in (5.70) to obtain (5.69).

Now take $v \in W_0^{1,p}(\Omega)$. The proof is even simpler, since one can take as an approximating sequence $v_n \in \mathscr{D}(\Omega)$ with $v_n \to v$ in $W^{1,p}(\Omega)$. We have $T \circ v_n \to T \circ v$ in $W^{1,p}(\Omega)$ and $T \circ v_n|_{\partial\Omega} = 0$. By continuity of the trace operator, we obtain $T \circ v \in W_0^{1,p}(\Omega)$. $\square$

Let us now consider nonsmooth contractions. We start with the important case $T(r) = r^+ = r \vee 0$. The idea is to regularize the function $T(r)$ to reduce ourselves to the previous situation. This elementary construction is described below.

Lemma 5.8.1. *Let $T_n : \mathbf{R} \to \mathbf{R}$ be defined by*

$$T_n(r) = \begin{cases} r & \text{if } r \geq 0, \\ r + n\frac{r^2}{2} & \text{if } -\frac{1}{n} \leq r \leq 0, \\ -\frac{1}{2n} & \text{if } r \leq -\frac{1}{n}. \end{cases}$$

Then $T_n \in \mathbf{C}^1(\mathbf{R})$, $T_n(0) = 0$, $|T_n'(r)| \leq 1$ for all $r \in \mathbf{R}$ and, for all $r \in \mathbf{R}$, $T_n(r) \to r^+$ as $n \to +\infty$.

PROOF. The function $r \to r^+$ is not a $\mathbf{C}^1$ function. Its distribution derivative is not continuous. It is indeed the Heaviside function $\theta(r) = 1$ if $r \geq 0$, and $\theta(r) = 0$ elsewhere. Let us approximate θ by a sequence of continuous functions, taking for $n \geq 1$

$$\theta_n(r) = \begin{cases} 1 & \text{if } r \geq 0, \\ nr + 1 & \text{if } -\frac{1}{n} \leq r \leq 0, \\ 0 & \text{if } r \leq -\frac{1}{n}. \end{cases}$$

Set

$$T_n(r) = \int_0^r \theta_n(s)ds.$$

Since $0 \leq \theta_n \leq 1$ we obtain

$$|T_n'(r)| = |\theta_n(r)| \leq 1,$$

and all the properties of T_n are easily verified. □

We can now state the following result.

Theorem 5.8.2. *Let Ω be an arbitrary open set in $\mathbf{R}^N$ and let $1 \leq p < +\infty$. Then, for any $v \in W^{1,p}(\Omega)$, $v^+ \in W^{1,p}(\Omega)$ and*

$$\frac{\partial}{\partial x_i} v^+ = \mathbf{1}_{\{v \geq 0\}} \frac{\partial v}{\partial x_i}, \quad i = 1, \ldots, N,$$

$$\|v^+\|_{W^{1,p}(\Omega)} \leq \|v\|_{W^{1,p}(\Omega)}.$$

Moreover, when $v \in W_0^{1,p}(\Omega)$, one still has $v^+ \in W_0^{1,p}(\Omega)$.

PROOF. Let T_n be the approximation of $T(r) = r^+$ which is defined in Lemma 5.8.1. Since T_n belongs to $\mathbf{C}^1(\mathbf{R})$, $T_n(0) = 0$ and $|T_n'(r)| \leq 1$ for all $r \in \mathbf{R}$, we can apply Proposition 5.8.1 to obtain

$$\frac{\partial}{\partial x_i} T_n \circ v = T_n'(v) \frac{\partial v}{\partial x_i}. \tag{5.71}$$

Let us pass to the limit on (5.71) as $n \to +\infty$. By Lemma 5.8.1, we have $T_n(r) \to r^+$ for all $r \in \mathbf{R}$. Hence

$$T_n \circ v \to v^+ \quad \text{a.e.}$$

and

$$|T_n \circ v| \leq |v|.$$

By the Lebesgue dominated convergence theorem, we have

$$T_n \circ v \to v^+ \quad \text{in } L^p(\Omega). \tag{5.72}$$

We now examine the right-hand side of (5.71). We have

$$T_n'(r) = 1 \quad \forall\, r \geq 0,$$

$$T_n'(r) \to 0 \text{ as } n \to +\infty \quad \forall\, r < 0.$$

Hence

$$T_n'(v) \to \mathbf{1}_{\{v \geq 0\}} \quad \text{a.e.}$$

and

$$T_n'(v)\frac{\partial v}{\partial x_i} \to \mathbf{1}_{\{v \geq 0\}}\frac{\partial v}{\partial x_i} \quad \text{a.e.}$$

Moreover,

$$\left| T_n'(v)\frac{\partial v}{\partial x_i} \right| \leq \left| \frac{\partial v}{\partial x_i} \right|.$$

We apply again the Lebesgue dominated convergence theorem and obtain

$$T_n'(v)\frac{\partial v}{\partial x_i} \to \mathbf{1}_{\{v \geq 0\}}\frac{\partial v}{\partial x_i} \quad \text{in } L^p(\Omega). \tag{5.73}$$

Combining (5.71), (5.72), and (5.73) we conclude that

$$\frac{\partial v^+}{\partial x_i} = \mathbf{1}_{\{v \geq 0\}}\frac{\partial v}{\partial x_i}. \tag{5.74}$$

In other words, the equality above follows from the continuity of the derivation with respect to the convergence in distribution and from the fact that the $L^p(\Omega)$-convergence implies the convergence in distribution.

From (5.74) we obtain

$$\left| \frac{\partial v^+}{\partial x_i} \right| \leq \left| \frac{\partial v}{\partial x_i} \right|.$$

This combined with $|v^+| \leq |v|$ yields

$$\|v^+\|_{W^{1,p}(\Omega)} \leq \|v\|_{W^{1,p}(\Omega)}.$$

Finally, note that (5.72) and (5.73) imply that $T_n \circ v \to v^+$ in $W^{1,p}(\Omega)$. If $v \in W_0^{1,p}(\Omega)$, we know by Proposition 5.8.1 that $T_n \circ v \in W_0^{1,p}(\Omega)$. Hence $v^+ \in W_0^{1,p}(\Omega)$. □

Let us derive from Theorem 5.8.2 some useful results.

Proposition 5.8.2. *Let $v \in W^{1,p}(\Omega)$, $1 \leq p < +\infty$. Then, for each $i = 1, \ldots, N$,*

$$\mathbf{1}_{\{v=0\}}\frac{\partial v}{\partial x_i} = 0.$$

In other words,

$$\frac{\partial v}{\partial x_i} = 0 \quad \text{a.e. in } E = \{x \in \Omega : v(x) = 0\},\ i = 1, \ldots, N.$$

PROOF. Let us consider the truncation

$$T(r) = r^- = (-r) \vee 0.$$

An argument similar to the one of the proof of Theorem 5.8.2 shows that $v^- \in W^{1,p}(\Omega)$ and

$$\frac{\partial v^-}{\partial x_i} = -\mathbf{1}_{\{v \le 0\}} \frac{\partial v}{\partial x_i}.$$

From $v = v^+ - v^-$ we obtain

$$\begin{aligned}
\frac{\partial v}{\partial x_i} &= \frac{\partial v^+}{\partial x_i} - \frac{\partial v^-}{\partial x_i} \\
&= \left(\mathbf{1}_{\{v \ge 0\}} + \mathbf{1}_{\{v \le 0\}}\right) \frac{\partial v}{\partial x_i} \\
&= \left(\mathbf{1}_{\{v > 0\}} + \mathbf{1}_{\{v \le 0\}}\right) \frac{\partial v}{\partial x_i} + \mathbf{1}_{\{v = 0\}} \frac{\partial v}{\partial x_i} \\
&= \frac{\partial v}{\partial x_i} + \mathbf{1}_{\{v = 0\}} \frac{\partial v}{\partial x_i}.
\end{aligned}$$

Hence,

$$\mathbf{1}_{\{v = 0\}} \frac{\partial v}{\partial x_i} = 0,$$

which completes the proof. □

Corollary 5.8.1. *Let $v \in W^{1,p}(\Omega)$, $1 \le p < +\infty$. Then $|v| \in W^{1,p}(\Omega)$ and*

$$\frac{\partial}{\partial x_i} |v| = \mathbf{1}_{\{v \ge 0\}} \frac{\partial v}{\partial x_i} - \mathbf{1}_{\{v < 0\}} \frac{\partial v}{\partial x_i}.$$

Moreover, $\| |v| \|_{W^{1,p}(\Omega)} = \|v\|_{W^{1,p}(\Omega)}$. Furthermore, if $v \in W_0^{1,p}(\Omega)$, then $|v| \in W_0^{1,p}(\Omega)$.

PROOF. We have $|v| = v^+ + v^-$. Hence

$$\begin{aligned}
\frac{\partial}{\partial x_i} |v| &= \frac{\partial v^+}{\partial x_i} + \frac{\partial v^-}{\partial x_i} \\
&= \mathbf{1}_{\{v \ge 0\}} \frac{\partial v}{\partial x_i} - \mathbf{1}_{\{v \le 0\}} \frac{\partial v}{\partial x_i}.
\end{aligned}$$

By using Proposition 5.8.2, we obtain

$$\frac{\partial}{\partial x_i} |v| = \mathbf{1}_{\{v \ge 0\}} \frac{\partial v}{\partial x_i} - \mathbf{1}_{\{v < 0\}} \frac{\partial v}{\partial x_i},$$

which ends the proof. □

Corollary 5.8.2. *The following facts hold:*

(a) *Let $u, v \in W^{1,p}(\Omega)$. Then $u \wedge v$ and $u \vee v$ still belong to $W^{1,p}(\Omega)$, and*

$$\frac{\partial}{\partial x_i} u \wedge v = \mathbf{1}_{\{u<v\}} \frac{\partial u}{\partial x_i} + \mathbf{1}_{\{u \geq v\}} \frac{\partial v}{\partial x_i},$$
$$\frac{\partial}{\partial x_i} u \vee v = \mathbf{1}_{\{u \geq v\}} \frac{\partial u}{\partial x_i} + \mathbf{1}_{\{v>u\}} \frac{\partial v}{\partial x_i}.$$

(b) *The same holds true for $u, v \in W_0^{1,p}(\Omega)$.*

In particular, if $v \in W_0^{1,p}(\Omega)$, then $T_1 \circ v \in W_0^{1,p}(\Omega)$, where $T_1(r) = 1 \wedge r^+$, and

$$\int_\Omega |\nabla(T_1 \circ v)|^p \, dx \leq \int_\Omega |\nabla v|^p \, dx.$$

PROOF. Just notice that

$$u \wedge v = u - (u-v)^+.$$

By Theorem 5.8.2, $u \wedge v \in W^{1,p}(\Omega)$ and

$$\begin{aligned}
\frac{\partial}{\partial x_i} u \wedge v &= \frac{\partial u}{\partial x_i} - \frac{\partial}{\partial x_i}(u-v)^+ \\
&= \frac{\partial u}{\partial x_i} - \mathbf{1}_{\{u-v \geq 0\}} \left(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i} \right) \\
&= \mathbf{1}_{\{u-v<0\}} \frac{\partial u}{\partial x_i} + \mathbf{1}_{\{u-v \geq 0\}} \frac{\partial v}{\partial x_i} \\
&= \mathbf{1}_{\{u<v\}} \frac{\partial u}{\partial x_i} + \mathbf{1}_{\{u \geq v\}} \frac{\partial v}{\partial x_i},
\end{aligned}$$

which completes the proof for the function $u \wedge v$. The proof for $u \vee v$ is similar. □

5.8.2 ▪ Capacity

Let us introduce the capacity $\mathrm{Cap}_p(\cdot)$ which is associated to the energy functional defined by $\Phi(v) = \int_\Omega |\nabla v(x)|^p \, dx$. The capacity $\mathrm{Cap}_p(E)$ of a subset E of Ω is a nonnegative real number. The capacity theory allows us to study small sets in $\mathbf{R}^N$. Indeed, one can show that there are sets in $\mathbf{R}^N$ which are negligible with respect to the Lebesgue measure and whose capacity is strictly greater than zero. It makes sense to speak of the values of an arbitrary function $v \in W^{1,p}(\Omega)$ on sets $E \subset \Omega$ such that $\mathrm{Cap}_p(E) > 0$.

For example, a one-codimensional manifold in $\mathbf{R}^N$ (the boundary of a smooth open set, for example) has a strictly positive capacity, which highlights the fact that one can develop a trace theory on such sets (see Section 5.6). One can make these considerations precise and prove that each element of $W^{1,p}(\Omega)$ admits a quasi-continuous representative (see Section 5.8.3, Evans and Gariepy [211], and Ziemer [366]). This is a finer representation than the Lebesgue representation, where the functions are defined only a.e.

When $p > N$, the Morrey theorem, Theorem 5.7.4 says that each element $v \in W^{1,p}(\Omega)$ admits a continuous representative. In that case, the capacity theory is not useful because only the empty set will be Cap_p-negligible. So we assume that $1 \leq p \leq N$. For simplicity of the exposition, we assume that Ω is bounded. We denote by $|\nabla v(x)|$ the Euclidean norm of $\nabla v(x)$, so that

$$\int_\Omega |\nabla v(x)|^p\,dx = \int_\Omega \left(\sum_{i=1}^N \left|\frac{\partial v}{\partial x_i}(x)\right|^2\right)^{p/2} dx.$$

We may as well work with equivalent norms (like $|\nabla v|^p = \Sigma|\frac{\partial v}{\partial x_i}|^p$). This does not make any difference in the development of the capacity theory since the truly important notions for a set, such as "to have zero capacity" or "to have a capacity greater than zero," are invariant by using equivalent norms on $W_0^{1,p}(\Omega)$.

Definition 5.8.1. *Let Ω be an open bounded set in $\mathbf{R}^N$ and let $1 \le p \le N$.*

(a) *For any open subset G of Ω, the p-capacity of G with respect to Ω is defined by*

$$\mathrm{Cap}_p(G,\Omega) := \inf\left\{ \int_\Omega |\nabla v(x)|^p\,dx : v \in W_0^{1,p}(\Omega), v(x) \ge 1 \text{ a.e. on } G \right\}.$$

(b) *This definition is extended to any subset E of Ω in the following way:*

$$\mathrm{Cap}_p(E,\Omega) := \inf\left\{ \mathrm{Cap}_p(G,\Omega) : G \text{ open},\ G \supset E \right\}.$$

Remark 5.8.1. (a) Let us notice that the above definition of the capacity Cap_p is a two-step procedure. For G open, one takes the infimum of the energy $\Phi(v)$ over all $v \in W_0^{1,p}(\Omega)$ which satisfy $v(x) \ge 1$ a.e. on G. Indeed, when G is open, the constraint $v \ge 1$ on G is easy to describe; it just has to be taken in the sense a.e. Then for an arbitrary set E, $\mathrm{Cap}_p(E)$ is the infimum of the capacity of the open sets G which contain E. We stress the fact that this definition is coherent since, clearly, for G open

$$\mathrm{Cap}_p(G,\Omega) = \inf\{\mathrm{Cap}_p(G',\Omega) : G' \text{ open},\ G' \supset G\}.$$

(b) When $p = 2$, $\mathrm{Cap}_2(E,\Omega)$ is the harmonic capacity of E with respect to Ω. This notion comes from physics. In electrostatics, take a condenser whose internal part E has a potential equal to one and whose external part $\partial\Omega$ has a potential equal to zero. Then, $\mathrm{Cap}_2(E,\Omega)$ is, up to a constant factor, the total amount of charge (the energy) of the condenser.

Let us give equivalent formulations of $\mathrm{Cap}_p(E)$ for an arbitrary set E.

Proposition 5.8.3. *For any set $E \subset \Omega$, the following hold:*

(a) $\mathrm{Cap}_p(E,\Omega) = \inf\left\{ \int_\Omega |\nabla v(x)|^p\,dx : v \in W_0^{1,p}(\Omega), v \ge 1 \text{ a.e. on a neighborhood of } E \right\}$, *where "$v \ge 1$ on a neighborhood of E" means that there exists an open set G which contains E and such that $v(x) \ge 1$ on G for almost every $x \in \Omega$.*

(b) $\mathrm{Cap}_p(E,\Omega) = \inf\left\{ \int_\Omega |\nabla v(x)|^p\,dx : v \in W_0^{1,p}(\Omega), v \ge 0, v = 1 \text{ a.e. on a neighborhood of } E \right\}$.

PROOF. (a) Take $v \in W_0^{1,p}(\Omega)$, $v \geq 1$ a.e. on a neighborhood of E. This means that there exists an open set $G \supset E$ such that $v \geq 1$ a.e. on G. By definition of $\mathrm{Cap}_p(G,\Omega)$, we have

$$\mathrm{Cap}_p(G,\Omega) \leq \int_\Omega |\nabla v(x)|^p \, dx.$$

Since $G \supset E$, G open, by definition of $\mathrm{Cap}_p(E,\Omega)$

$$\mathrm{Cap}_p(E,\Omega) \leq \mathrm{Cap}_p(G,\Omega),$$

which combined with the previous inequality yields

$$\mathrm{Cap}_p(E,\Omega) \leq \int_\Omega |\nabla v(x)|^p \, dx.$$

This being true for any such v, we infer that

$$\mathrm{Cap}_p(E,\Omega) \leq \inf\left\{ \int_\Omega |\nabla v(x)|^p \, dx : v \in W_0^{1,p}(\Omega), v \geq 1 \right.$$
$$\left. \text{a.e. on a neighborhood of } E \right\}.$$

Let us now prove the opposite inequality. If $\mathrm{Cap}_p(E,\Omega) = +\infty$, it is obvious. So, we assume $\mathrm{Cap}_p(E,\Omega) < +\infty$. For any $\varepsilon > 0$, there exists G_ε open, $G_\varepsilon \supset E$ such that

$$\mathrm{Cap}_p(G_\varepsilon,\Omega) \leq \mathrm{Cap}_p(E,\Omega) + \varepsilon.$$

By definition of $\mathrm{Cap}_p(G_\varepsilon,\Omega)$, there exists $v_\varepsilon \in W_0^{1,p}(\Omega)$ such that $v_\varepsilon \geq 1$ a.e. on G_ε and

$$\int_\Omega |\nabla v_\varepsilon(x)|^p \, dx \leq \mathrm{Cap}_p(G_\varepsilon,\Omega) + \varepsilon.$$

Adding the two last inequalities, we obtain

$$\int_\Omega |\nabla v_\varepsilon(x)|^p \, dx \leq \mathrm{Cap}_p(E,\Omega) + 2\varepsilon$$

with $v_\varepsilon \geq 1$ a.e. on a neighborhood of E. Hence

$$\inf\left\{ \int_\Omega |\nabla v|^p \, dx : v \in W_0^{1,p}(\Omega), v \geq 1 \quad \text{a.e. on a neighborhood of } E \right\}$$
$$\leq \mathrm{Cap}_p(E,\Omega) + 2\varepsilon.$$

Then let $\varepsilon \to 0$ to obtain the result.

(b) Let $T(r) = 1 \wedge r^+$ be the fundamental contraction of the potential theory. Notice that if $v \in W_0^{1,p}(\Omega)$ satisfies "$v \geq 1$ a.e. on a neighborhood of E," then $T \circ v \geq 0$, $T \circ v = 1$ a.e. on a neighborhood of E.

Moreover, by Theorem 5.8.2 and Corollary 5.8.2 we have that $T \circ v$ belongs to $W_0^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla (T \circ v)|^p \, dx \leq \int_\Omega |\nabla v|^p \, dx.$$

Hence,

$$\inf\left\{\int_\Omega |\nabla w|^p\,dx \;:\; w\in W_0^{1,p}(\Omega), w\ge 0, w=1 \text{ a.e. on a neighborhood of } E\right\}$$
$$\le \int_\Omega |\nabla(T\circ v)|^p\,dx$$
$$\le \int_\Omega |\nabla v|^p\,dx.$$

Then pass to the infimum on v to obtain

$$\inf\left\{\int_\Omega |\nabla w|^p\,dx \;:\; w\in W_0^{1,p}(\Omega), w\ge 0, w=1 \right.$$
$$\left.\text{a.e. on a neighborhood of } E\right\}$$
$$\le \inf\left\{\int_\Omega |\nabla v|^p\,dx \;:\; v\in W_0^{1,p}(\Omega), v\ge 1 \right.$$
$$\left.\text{a.e. on a neighborhood of } E\right\}.$$

The opposite inequality is clearly satisfied, since in the last infimum we can easily reduce ourselves to consider only nonnegative functions v. □

Let us now make the connection with the Choquet definition of capacities.

Proposition 5.8.4. (a) *For any compact set* $K\subset\Omega$

$$\mathrm{Cap}_p(K)=\inf\left\{\int_\Omega |\nabla v|^p dx : v\in\mathscr{D}(\Omega), v(x)\ge 1\ \forall\ x\in K\right\}.$$

(b) *For all* G *open sets in* Ω

$$\mathrm{Cap}_p(G)=\sup\{\mathrm{Cap}_p(K) : K \textit{ compact, } K\subset G\}.$$

PROOF. (a) The key point is that if G is an open set containing a compact K, then G contains an ε-enlargement K_ε of K with $\varepsilon>0$, where

$$K_\varepsilon=\{x\in\Omega : \mathrm{dist}(x,K)<\varepsilon\}.$$

To see this, just notice that $x\mapsto \mathrm{dist}(x,\Omega^c)$ is a continuous function on a compact set K which takes only positive values. Hence, its minimal value ε satisfies $\varepsilon>0$.

As a consequence, if $v\in W_0^{1,p}(\Omega)$ satisfies "$v\ge 1$ a.e. on a neighborhood of K," there exists some open set $G\supset K$ such that $v\ge 1$ a.e. on G and some $\varepsilon>0$ such that $G\supset K_\varepsilon$. Hence, $v\ge 1$ a.e. on K_ε.

Then use standard regularization techniques as developed in Theorem 5.1.3 to approximate v by a sequence $v_n\in\mathscr{D}(\Omega)$ which satisfies

$$\text{“}v_n\ge 1 \quad \text{a.e. on a neighborhood of } K\text{”}$$

and

$$\int_\Omega |\nabla v_n|^p dx \longrightarrow \int_\Omega |\nabla v|^p dx.$$

To that end one should notice that, given $(\rho_n)_{n\in\mathbf{N}}$ a regularization kernel with $\rho_n \equiv 0$ outside of $\mathbf{B}(0,1/n)$, we have

$$(v \star \rho_n)(x) = \int_{B(0,1/n)} v(x-y)\rho_n(y)\,dy.$$

When $1/n < \varepsilon$, $x \in K$, we have that for all $y \in \mathbf{B}(0,1/n)$, $x-y \in K_\varepsilon$ and hence $v(x-y) \geq 1$. It follows that for $1/n < 1/\varepsilon$ and $x \in K$,

$$(v \star \rho_n)(x) \geq \int_{\mathbf{B}(0,1/n)} \rho_n(y)\,dy = 1.$$

Then combine this regularization by the convolution method with a truncation on the domain (to preserve the Dirichlet boundary condition) to obtain v_n.

As a consequence,

$$\begin{aligned}\mathrm{Cap}_p(K) = \inf\Big\{ \int_\Omega |\nabla v|^p\,dx : v \in \mathscr{D}(\Omega), v(x) \geq 1 \\ \forall\, x \text{ in a neighborhood of } K\Big\} \\ \geq \inf\Big\{ \int_\Omega |\nabla v|^p\,dx : v \in \mathscr{D}(\Omega),\ v(x) \geq 1\ \forall\, x \in K\Big\},\end{aligned}$$

the last inequality being a consequence of the fact that one minimizes on a larger set.

(b) Since $\mathrm{Cap}_p(\cdot)$ is a monotone set function, we have

$$\mathrm{Cap}_p(G) \geq \sup\{\mathrm{Cap}_p(K) \,:\, K \text{ compact}, K \subset G\}.$$

To prove the opposite inequality, we consider only the nontrivial case when the right-hand side is finite. Take a sequence (K_n) of compact subsets of G such that $\cup_n K_n = G$ and for any integer n, by using statement (a) above, let $u_n \in \mathscr{D}(\Omega)$ such that

$$u_n \geq 1 \text{ on } K_n, \quad \int_\Omega |\nabla u_n|^p\,dx \leq \frac{1}{n} + \mathrm{Cap}_p(K_n).$$

The sequence (u_n) is bounded in $W_0^{1,p}(\Omega)$ and we may extract a subsequence (that we still denote by (u_n)) weakly converging to some $u \in W_0^{1,p}(\Omega)$. Then $u \geq 1$ a.e. on G and

$$\begin{aligned}\mathrm{Cap}_p(G) &\leq \int_\Omega |\nabla u|^p\,dx \\ &\leq \liminf_{n\to+\infty} \int_\Omega |\nabla u_n|^p\,dx \\ &\leq \liminf_{n\to+\infty} \mathrm{Cap}_p(K_n) \\ &\leq \sup\{\mathrm{Cap}_p(K) \,:\, K \text{ compact}, K \subset G\},\end{aligned}$$

which concludes the proof. □

The capacity of a set $E \subset \mathbf{R}^N$ can be defined independently of the domain Ω; to do that it is enough to take $\Omega = \mathbf{R}^N$, and in this case the L^p norm of the competing functions u has to be taken into account:

$$\mathrm{cap}_p(E) = \inf\Big\{ \int_{\mathbf{R}^N} \big(|\nabla u|^p + |u|^p\big)\,dx \,:\, u \in \mathscr{U}_{p,E}\Big\}, \tag{5.75}$$

where $\mathscr{U}_{p,E}$ is the set of all functions $u \in W^{1,p}(\mathbf{R}^N)$ such that $u \geq 1$ a.e. (in the Lebesgue sense) in a neighborhood of E. We stress the fact that for a set, what is important is not the precise value of its capacity but whether its capacity is zero.

The term $\int_{\mathbf{R}^N} |u|^p\,dx$ in the definition of $\mathrm{cap}(E)$ is essential when the relative set Ω is unbounded or is the entire space $\mathbf{R}^N$. In particular, if $N=2$, without this term, every bounded set would have capacity zero. Indeed, for a disk B_R centered at the origin, taking $t > R$, the function

$$u_t(x) = \frac{\log(|x|/t)}{\log(R/t)} \qquad \text{with } u(x) = 1 \text{ on } B_R$$

would give

$$\int_{\mathbf{R}^2} |\nabla u_t|^2\,dx = 2\pi \int_R^t \frac{1}{\log^2(R/t)} \frac{1}{r}\,dr = 2\pi \log(t/R),$$

so that

$$\mathrm{cap}(B_R) \leq 2\pi \log(t/R).$$

Letting $t \to +\infty$ would give $\mathrm{cap}(B_R) = 0$. Since every bounded set E is contained in a ball B_R, this would give $\mathrm{cap}(E) = 0$ for every bounded set $E \subset \mathbf{R}^2$.

From the definition (5.75) above we obtain immediately $\mathrm{cap}_p(E) \geq |E|$ for every E, and thus every set with capacity zero is also Lebesgue negligible. The opposite is not true: for instance a smooth $N-1$ dimensional surface in $\mathbf{R}^N$ has a positive 2-capacity but zero Lebesgue measure. More precisely, the following result holds.

Proposition 5.8.5. *Let S be a smooth k-dimensional surface in $\mathbf{R}^N$ and let $p > 1$. Then*

(i) $\mathrm{cap}_p(S) = 0$ *for every* $1 < p \leq N-k$*;*

(ii) $\mathrm{cap}_p(S) > 0$ *for every* $p > N-k$*.*

PROOF. By a smooth change of variables and by using the countable subadditivity of the capacity we may reduce ourselves to the case of S a k-dimensional plane, and in this case assertions (i) and (ii) are equivalent to showing that a point in $\mathbf{R}^d$ (with $d = N-k$)

(i′) has zero p-capacity for every $1 < p \leq d$;

(ii′) has a positive p-capacity if $p > d$.

To prove assertion (i′) in the case $p = d$ it is enough to take in polar coordinates

$$u(r) = \frac{\log r}{\log \varepsilon} \quad \text{if } \varepsilon < r < 1, \qquad u(r) = 1 \quad \text{if } r \leq \varepsilon. \tag{5.76}$$

This gives

$$\mathrm{cap}_p(\{0\}) \leq C_d \int_\varepsilon^1 \frac{r^{d-1}}{(\log \varepsilon)^p} \left(\frac{1}{r^d} + (\log r)^p \right) dr + |B(0,\varepsilon)|,$$

where C_d is a constant depending only on d, and an easy calculation shows that as $\varepsilon \to 0$ we obtain

$$\mathrm{cap}_p(\{0\}) = 0.$$

In the case $p \neq d$ a similar calculation can be done with

$$u(r) = \frac{1 - r^{(p-d)/(p-1)}}{1 - \varepsilon^{(p-d)/(p-1)}} \quad \text{if } \varepsilon < r < 1, \qquad u(r) = 1 \quad \text{if } r \leq \varepsilon, \tag{5.77}$$

and we obtain that

$$\operatorname{cap}_p(\{0\}) = 0 \qquad \text{for every } 1 < p < d.$$

To prove (ii′) we take as a lower bound of $\operatorname{cap}_p(\{0\})$ the quantity, in polar coordinates,

$$\lim_{\varepsilon \to 0} \min \left\{ C_d \int_\varepsilon^1 r^{d-1} |u'(r)|^p \, dr \ : \ u(\varepsilon) = 1, \ u(1) = 0 \right\}.$$

The Euler–Lagrange equations of the minimum problems above give as solutions the functions in (5.76) and (5.77) and, by a simple integration, we obtain the conclusion. □

In the following, unless specified differently, we will use the capacity with $p = 2$, then omit the index p. If a property $P(x)$ holds for all $x \in E$ except for the elements of a set $Z \subset E$ with $\operatorname{cap}(Z) = 0$, we say that the property $P(x)$ holds *quasi-everywhere* (*q.e.*) on E. The expression *almost everywhere* (*a.e.*) refers, as usual, to the Lebesgue measure.

We summarize here the main properties of the capacity; the interested reader may find all the details and the related proofs in one of the classical books [198], [200], [222], [366].

- The capacity $\operatorname{cap}(E)$ is a monotone set function, that is,

$$\operatorname{cap}(E_1) \leq \operatorname{cap}(E_2) \qquad \text{whenever } E_1 \subset E_2.$$

- The set function $\operatorname{cap}(E)$ is continuous for increasing sequences, that is,

$$\operatorname{cap}(E_n) \uparrow \operatorname{cap}(E) \qquad \text{whenever } E_n \uparrow E.$$

- The set function $\operatorname{cap}(E)$ is countably subadditive, that is,

$$\operatorname{cap}(E) \leq \sum_{n \in \mathbf{N}} \operatorname{cap}(E_n) \qquad \text{whenever } E = \bigcup_{n \in \mathbf{N}} E_n.$$

- The set function $\operatorname{cap}(E)$ is not additive, that is, for $E_1 \cap E_2 = \emptyset$ the inequality

$$\operatorname{cap}(E_1 \cup E_2) \leq \operatorname{cap}(E_1) + \operatorname{cap}(E_2)$$

  may be, in general, strict.

5.8.3 ▪ Quasi-open sets, quasi-continuity

A subset A of $\mathbf{R}^N$ is said to be *quasi-open* (respectively, *quasi-closed*) if for every $\varepsilon > 0$ there exists an open (respectively, closed) subset A_ε of $\mathbf{R}^N$, such that $\operatorname{cap}(A_\varepsilon \Delta A) < \varepsilon$, where Δ denotes the symmetric difference of sets. In the definition of a quasi-open set we can additionally require that $A \subset A_\varepsilon$.

In a similar way, if Ω is an open domain, a function $f : \Omega \to \mathbf{R}$ is said to be *quasi-continuous* (respectively, *quasi-lower semicontinuous*) if for every $\varepsilon > 0$ there exists a continuous (respectively, lower semicontinuous) function $f_\varepsilon : \Omega \to \mathbf{R}$ such that $\operatorname{cap}(\{f \neq f_\varepsilon\}) < \varepsilon$, where $\{f \neq f_\varepsilon\} = \{x \in \Omega \ : \ f(x) \neq f_\varepsilon(x)\}$. It is well known (see, e.g., Ziemer

[366]) that every function u of the Sobolev space $H^1(\Omega)$ has a quasi-continuous representative, which is uniquely defined up to a set of capacity zero. It is convenient for our purposes to identify the function u with its quasi-continuous representative, so that a pointwise condition can be imposed on $u(x)$ for quasi-every $x \in \Omega$. Notice that with this convention we have for every subset E of Ω

$$\mathrm{cap}(E,\Omega) = \min\left\{ \int_\Omega |\nabla u|^2\,dx \ : \ u \in H_0^1(\Omega),\ u \geq 1 \text{ q.e. on } E \right\}.$$

We recall the following theorems from [5].

Theorem 5.8.3. *Let $u \in H^1(\mathbf{R}^N)$. Then a quasi-continuous representative $\tilde{u}$ of u is given, for q.e. $x \in \mathbf{R}^N$, by*

$$\tilde{u}(x) = \lim_{\varepsilon \to 0} \frac{1}{|B_{x,\varepsilon}|} \int_{B_{x,\varepsilon}} u(y)\,dy = \tilde{u}(x).$$

Theorem 5.8.4. *Every strongly convergent sequence in $H^1(\mathbf{R}^N)$ has a subsequence converging q.e. in $\mathbf{R}^N$.*

It is important to notice that the Sobolev space $H_0^1(A)$ can be defined for every quasi-open set A as the space of all functions $u \in H_0^1(\mathbf{R}^N)$ such that $u = 0$ q.e. on $\mathbf{R}^N \setminus A$. The Hilbert space structure of $H_0^1(A)$ is inherited from $H_0^1(\mathbf{R}^N)$. Note that if $A \subset \Omega$, $H_0^1(A)$ is a closed subspace of $H_0^1(\Omega)$ as a consequence of the properties above of quasi-continuous representatives of Sobolev functions. If A is an open set, then the previous definition of $H_0^1(A)$ is equivalent to the usual one (see Adams and Hedberg [5]). Indeed, we recall the following result.

Theorem 5.8.5. *Let $A \subset \mathbf{R}^N$ be an open set. A function $u \in H^1(\mathbf{R}^N)$ belongs to $H_0^1(A)$ iff $u = 0$ q.e. on $\mathbf{R}^N \setminus A$.*

In the statement above, the assertion u *belongs to* $H_0^1(A)$ has to be understood in the sense that u is the strong limit in $H^1(\mathbf{R}^N)$ of a sequence of $C_c^\infty(\mathbf{R}^N)$ functions with support in A.

Most of the properties that hold for Sobolev spaces over open sets can be extended to this larger framework; for instance, the following result holds (see [146]).

Lemma 5.8.2. *Let A_1, A_2 be two quasi-open sets that are quasi-disjoint, that is, with* $\mathrm{cap}(A_1 \cap A_2) = 0$. *Then*

$$H_0^1(A_1 \cup A_2) = H_0^1(A_1) \cap H_0^1(A_2)$$

in the sense that for every $u \in H_0^1(A_1 \cup A_2)$ we have $u_{|A_1} \in H_0^1(A_1)$ and $u_{|A_2} \in H_0^1(A_2)$.

Since the family of quasi-open sets of $\mathbf{R}^N$ is not a topology (only countable unions of quasi-open sets are quasi-open), when dealing with arbitrary unions of quasi-open sets sometimes it is more interesting to work with the so-called *finely open sets*, that is, open sets with respect to the *fine topology* defined below.

The fine topology on Ω is the coarsest topology making all super-harmonic functions continuous. The relation between quasi-open sets and the fine topology is studied in [5], [222], [255]. We recall the following theorem from [255].

Theorem 5.8.6. *Suppose $A \subset \mathbf{R}^N$. Then the following assertions are equivalent:*

(i) *A is quasi-open;*

(ii) *A is the union of a finely open set and a set of zero capacity;*

(iii) *$A = \{u > 0\}$ for some nonnegative quasi-continuous function $u \in H^1(\mathbf{R}^N)$.*

In addition, if A is a quasi-open subset of $\mathbf{R}^N$ and u is a function on A, then the following assertions are equivalent:

(i) *u is quasi-lower semicontinuous;*

(ii) *the sets $\{u > c\}$ are quasi-open for all $c \in \mathbf{R}$;*

(iii) *u is finely lower semicontinuous up to a set of zero capacity.*

Remark 5.8.2. All the definitions and results presented in this section have natural extension to the Sobolev spaces $W_0^{1,p}(\Omega)$ with $1 < p < +\infty$. We refer to [237] for a review of the main definitions and properties of the p-capacity. From the shape optimization point of view, the most interesting case is when $1 < p \leq N$, since for $p > N$ the p-capacity of a point is strictly positive and every $W^{1,p}$-function has a continuous representative. For this reason, a property which holds p-quasi-everywhere, with $p > N$, holds in fact everywhere, and this explains why for shape optimization problems the most interesting case is when $p \leq N$.

5.8.4 ▪ Capacitary measures

The class of quasi-open sets is considerably larger than that of classical domains; nevertheless several shape optimization problems,

$$\min\big\{F(\Omega)\ :\ \Omega \subset D,\ \Omega \text{ quasi-open}\big\},$$

do not admit a solution. In Section 16.3 we show an example in which this nonexistence phenomenon occurs.

On the other hand, minimizing sequences $(\Omega_n)_{n\in\mathbf{N}}$ always exist, and it is interesting to study the asymptotic behavior of them as $n \to \infty$. As a general philosophy of relaxation problems (see Section 3.2.4), to do that we have to endow the class

$$\mathscr{A} = \big\{\Omega \subset D,\ \Omega \text{ quasi-open}\big\} \tag{5.78}$$

with a metric convergence γ, to consider the completion $\overline{\mathscr{A}}$ with respect to this metric, and to define the relaxed functional $\overline{F}$ on $\overline{\mathscr{A}}$ by setting

$$\overline{F}(\overline{\Omega}) = \inf\Big\{\liminf_{n\to\infty} F(\Omega_n)\ :\ \Omega_n \to_\gamma \overline{\Omega}\Big\}$$

for every $\overline{\Omega} \in \overline{\mathscr{A}}$.

In this way, under a coercivity assumption on the functional F (i.e., assuming that the sublevel sets $\{F \leq t\}$ are relatively γ-compact), the minimizing sequences $(\Omega_n)_{n\in\mathbf{N}}$ will γ-converge, up to extraction of subsequences, to minimum points of the relaxed problem

$$\min\big\{\overline{F}(\overline{\Omega})\ :\ \overline{\Omega} \in \overline{\mathscr{A}}\big\}$$

that, by the coercivity assumption above, always admits a solution.

We refer to [147] for a complete discussion about relaxation theory on general spaces and in particular for the integral functional on function spaces as Sobolev spaces, Lebesgue spaces, BV spaces, and spaces of measures. Here we highlight the case when the space is the class $\mathscr{A}$ of admissible domains, endowed with a suitable γ-convergence, and the cost functional is a shape functional $F(\Omega)$ defined for every $\Omega \in \mathscr{A}$.

Definition 5.8.2. *Let D be a bounded open set of $\mathbf{R}^N$ and let $\mathscr{A}$ be the class of its quasi-open subdomains defined in* (5.78)*. We say that a sequence $(\Omega_n)_{n\in\mathbf{N}}$ in $\mathscr{A}$ γ-converges to $\Omega \in \mathscr{A}$ if for every right-hand side $f \in L^2(D)$ the solutions of the elliptic boundary value problems*

$$-\Delta u_n = f \ \text{in } \Omega_n, \qquad u \in H_0^1(\Omega_n),$$

extended by zero to $D \setminus \Omega_n$, converge in $L^2(\mathscr{D})$ to the solution u of

$$-\Delta u = f \ \text{in } \Omega, \qquad u \in H_0^1(\Omega).$$

Proposition 5.8.6. *The convergence $u_n \to u$ of Definition* 5.8.2 *is actually strong in $H_0^1(D)$.*

PROOF. From the equations $-\Delta u_n = f$, multiplying by u_n and integrating by parts, we obtain

$$\int_D |\nabla u_n|^2\,dx = \int_D f\,u_n\,dx.$$

Similarly, we obtain for the limit solution u

$$\int_D |\nabla u|^2\,dx = \int_D f\,u\,dx.$$

Since $u_n \to u$ in $L^2(D)$, we have

$$\lim_{n\to\infty} \int_D |\nabla u_n|^2\,dx = \lim_{n\to\infty} \int_D f\,u_n\,dx = \int_D f\,u\,dx = \int_D |\nabla u|^2\,dx,$$

which gives strong convergence of u_n to u in $H_0^1(D)$. $\square$

Remark 5.8.3. In [145] it is shown that in Definition 5.8.2 it is equivalent to require the convergence of u_n to u only for the right-hand side $f = 1$. Moreover, if for every $\Omega \in \mathscr{A}$ we denote by $R_\Omega : L^2(D) \to L^2(D)$ the *resolvent operator* which associates to every right-hand side $f \in L^2(D)$ the solution $u \in H_0^1(\Omega) \subset L^2(D)$ of $-\Delta u = f$ in Ω, the convergence $\Omega_n \to_\gamma \Omega$ can be restated as

$$\forall f \in L^2(D) \qquad R_{\Omega_n}(f) \to R_\Omega(f) \quad \text{in } L^2(D).$$

Some conditions equivalent to γ-convergence are listed in the proposition below (see proposition 4.5.3 of [145]).

Proposition 5.8.7. *Let $(\Omega_n)_{n\in\mathbf{N}}$ and Ω be quasi-open subsets of a given bounded open set D. The following assertions are equivalent.*

1. $\Omega_n \to_\gamma \Omega$.
2. *For every $f \in L^2(D)$ we have $R_{\Omega_n}(f) \to R_\Omega(f)$ strongly in $H_0^1(D)$.*

3. *We have $R_{\Omega_n}(1) \to R_\Omega(1)$ strongly in $H^1_0(D)$.*

4. *The resolvent operators R_{Ω_n} converge in the operator norm of $\mathscr{L}(L^2(D))$ to R_Ω.*

Remark 5.8.4. Denoting by $G(\Omega, u)$ the Dirichlet energy functional

$$G(\Omega,u) = \begin{cases} \displaystyle\int_D |\nabla u|^2\,dx & \text{if } u \in H^1_0(\Omega),\\ +\infty & \text{if } u \in L^2(D)\setminus H^1_0(\Omega),\end{cases}$$

a further equivalence can be stated in terms of Γ-convergence (see Chapter 12): $\Omega_n \to_\gamma \Omega$ iff

$$G(\Omega_n,\cdot) \to_\Gamma G(\Omega,\cdot),$$

where the Γ-convergence is intended with respect to the $L^2(D)$ topology.

The γ-convergence is very strong; if $\Omega_n \to_\gamma \Omega$, not only do the solutions $R_{\Omega_n}(f)$ of the corresponding boundary value problems converge strongly in $H^1_0(D)$ for every right-hand side f, but, as a consequence of the norm convergence of the resolvent operators (see Proposition 16.5.1), we also have the convergence of the entire spectrum $\Lambda(\Omega_n)$ to $\Lambda(\Omega)$. By the ellipticity of the operator $-\Delta$ these spectra consist of discrete eigenvalues (see Chapter 8) which then converge in the sense that

$$\lambda_k(\Omega_n) \to \lambda_k(\Omega) \qquad \text{for every } k \in \mathbf{N}.$$

In this way, many shape functionals turn out to be γ-lower semicontinuous or even γ-continuous. Two important classes are the following ones.

Integral functionals. Given a right-hand side $f \in L^2(D)$ we consider the solution $R_\Omega(f)$ of the PDE

$$-\Delta u = f \text{ in } \Omega, \qquad u \in H^1_0(\Omega),$$

that we assume extended by zero outside of Ω. The integral shape cost functionals we may consider are of the form

$$F(\Omega) = \int_D j\big(x, R_\Omega(f), \nabla R_\Omega(f)\big)\,dx,$$

where j is a suitable integrand. Since the γ-convergence implies the strong H^1_0 convergence of the corresponding solutions, as a consequence of Fatou's lemma and of the Sobolev embedding theorem (see Section 5.7), we obtain that the functional F is γ-lower semicontinuous provided the integrand j satisfies the following properties:

(1) $j(x,s,z)$ is measurable in x and lower semicontinuous in (s,z);

(2) $j(x,s,z) \geq -a(x) - c\big(|s|^p + |z|^2\big)$, where $a \in L^1(D)$, $p = 2N/(N-2)$ ($p < +\infty$ if $N = 2$), $c \in \mathbf{R}$.

The shape functional F is γ-continuous if assumptions (1) and (2) above are replaced by

(1′) $j(x,s,z)$ is measurable in x and continuous in (s,z);

(2′) $|j(x,s,z)| \leq a(x) + c\big(|s|^p + |z|^2\big)$, where $a \in L^1(D)$, $p = 2N/(N-2)$ ($p < +\infty$ if $N = 2$), $c \in \mathbf{R}$.

Spectral functionals. For every domain Ω of the class $\mathscr{A}$ we consider the spectrum $\Lambda(\Omega)$ of the Dirichlet Laplacian $-\Delta$ on $H_0^1(\Omega)$. Since our domains Ω are bounded, the Dirichlet Laplacian $-\Delta$ has a compact resolvent and so its spectrum $\Lambda(\Omega)$ is discrete:

$$\Lambda(\Omega) = \big(\lambda_1(\Omega), \lambda_2(\Omega), \dots\big),$$

where $\lambda_k(\Omega)$ are the eigenvalues counted with their multiplicity. The spectral shape cost functionals we may consider are of the form

$$F(\Omega) = \Phi\big(\Lambda(\Omega)\big)$$

for a suitable function $\Phi : \mathbf{R}^{\mathbf{N}} \to \overline{\mathbf{R}}$. For instance, taking $\Phi(\Lambda) = \lambda_k$ we obtain

$$F(\Omega) = \lambda_k(\Omega).$$

The functional F is then γ-lower semicontinuous provided the function Φ is lower semicontinuous, that is,

$$\lambda_k^n \to \lambda_k \ \forall k \quad \Rightarrow \quad \Phi(\Lambda) \le \liminf_{n\to+\infty} \Phi(\Lambda^n).$$

Analogously, F is γ-continuous if the function Φ is continuous, that is,

$$\lambda_k^n \to \lambda_k \ \forall k \quad \Rightarrow \quad \Phi(\Lambda^n) \to \Phi(\Lambda).$$

The γ-convergence, defined on the class $\mathscr{A}$ of (5.78), is actually a metric convergence; from Definition 5.8.2 and from Remark 5.8.3 we have that the distance

$$d_\gamma(\Omega_1, \Omega_2) = ||R_{\Omega_1}(1) - R_{\Omega_2}(1)||$$

generates the γ-convergence. As a consequence, the space $(\mathscr{A}, d_\gamma)$ is a metric space.

The following density result was proved in [186].

Proposition 5.8.8. *The class of smooth domains $A \subset D$ is d_γ-dense in $\mathscr{A}$.*

The metric space $(\mathscr{A}, d_\gamma)$ is then separable. However, it is not compact, as the following example shows.

Example 5.8.1. This example is due to Cioranescu and Murat (see [175]). Let $D =]0,1[\times]0,1[$ be the unit square of $\mathbf{R}^2$ and let $f \in L^2(D)$. We construct a sequence (Ω_n) of open subsets of D such that the solutions $u_n = R_{\Omega_n}(f)$ of

$$\begin{cases} -\Delta u = f & \text{in } \Omega_n, \\ u \in H_0^1(\Omega_n), \end{cases} \tag{5.79}$$

extended by zero on $D \setminus \Omega_n$, converge weakly in $H_0^1(D)$ to a function u which is not of the form $u = R_\Omega(f)$, then proving the noncompactness of the metric space $(\mathscr{A}, d_\gamma)$.

Consider, for n large enough, the sequence of sets

$$C_n = \bigcup_{i,j=0}^{n} \overline{B}_{(i/n, j/n), r_n}, \qquad \Omega_n = D \setminus C_n,$$

where $r_n = e^{-cn^2}$, being $c > 0$ a fixed positive constant. It is easy to see that $u_n = R_{\Omega_n}(f)$ are bounded in $H_0^1(D)$; then we may assume for a subsequence (that we denote by the same indices) that u_n converges to some function u weakly in $H_0^1(D)$.

It is convenient to introduce the functions $z_n \in H^1(D)$ defined by

$$z_n = \begin{cases} 0 & \text{on } C_n, \\ \dfrac{\ln\sqrt{(x-i/n)^2+(y-j/n)^2}+cn^2}{cn^2-\ln(2n)} & \text{on } \overline{B}_{(i/n,j/n),1/(2n)} \setminus C_n, \\ 1 & \text{on } D \setminus \bigcup_{i,j=0}^n \overline{B}_{(i/n,j/n),1/(2n)}. \end{cases}$$

We notice that $0 \le z_n \le 1$ and that ∇z_n converges to zero weakly in $L^2(D)$; hence z_n converges weakly in $H^1(D)$ to a constant function. Computing the limit of $\int_D z_n\,dx$ we find that this constant is equal to 1.

For every $\varphi \in C_0^\infty(D)$ the function $z_n\varphi$ belongs to $H_0^1(\Omega_n)$, and thus we can take $z_n\varphi$ as a test function for (5.79) on Ω_n:

$$\int_D \varphi \nabla u_n \nabla z_n\,dx + \int_D z_n \nabla u_n \nabla \varphi\,dx = \int_D f\varphi z_n\,dx.$$

The second and third terms of this equality converge to $\int_D \nabla u \nabla\varphi\,dx$ and $\int_D f\varphi\,dx$, respectively. For the first term, the Gauss–Green formula gives

$$\int_D \varphi \nabla u_n \nabla z_n\,dx = \sum_{i,j=0}^n \int_{\partial B_{(i/n,j/n),1/(2n)}} u_n \frac{\partial z_n}{\partial \nu}\varphi\,d\sigma - \int_D u_n \nabla z_n \nabla\varphi\,dx.$$

The term with Δz_n does not appear since z_n is harmonic on $B_{(i/n,j/n),1/(2n)} \setminus C_n$; similarly, the boundary term on $\partial B_{(i/n,j/n),r_n}$ vanishes since u_n vanishes on it. The last term of the identity above tends to 0 as $n \to \infty$.

We now compute the boundary integral. We have

$$\begin{aligned} \sum_{i,j=0}^n \int_{\partial B_{(i/n,j/n),1/(2n)}} u_n \frac{\partial z_n}{\partial \nu}\varphi\,d\sigma &= \sum_{i,j=0}^n \int_{\partial B_{(i/n,j/n),1/(2n)}} \frac{2n}{cn^2-\ln(2n)} u_n\varphi\,d\sigma \\ &= \frac{2n^2}{cn^2-\ln(2n)} \sum_{i,j=0}^n \int_{\partial B_{(i/n,j/n),1/(2n)}} \frac{1}{n} u_n\varphi\,d\sigma. \end{aligned}$$

Let us denote by $\mu_n \in H^{-1}(D)$ the distribution defined by

$$\langle \mu_n, \psi \rangle_{H^{-1}(D)\times H_0^1(D)} = \sum_{i,j=0}^n \int_{\partial B_{(i/n,j/n),1/(2n)}} \frac{1}{n}\psi\,d\sigma.$$

We prove that μ_n converges strongly in $H^{-1}(D)$ to $\pi\,dx$. Indeed, we introduce the functions $v_n \in H^1(D)$ defined by

$$\begin{cases} \Delta v_n = 4 & \text{in } \bigcup_{i,j} B_{(i/n,j/n),1/(2n)}, \\ v_n = 0 & \text{on } D \setminus \bigcup_{i,j} B_{(i/n,j/n),1/(2n)}. \end{cases}$$

Therefore

$$\frac{\partial v_n}{\partial \nu} = \frac{1}{n} \quad \text{on } \bigcup_{i,j} \partial B_{(i/n,j/n),1/(2n)}.$$

We notice that v_n converges to zero strongly in $H^1(D)$, and therefore Δv_n converges to zero strongly in $H^{-1}(D)$. Moreover,

$$\begin{aligned}\langle -\Delta v_n, \psi\rangle_{H^{-1}(D)\times H_0^1(D)} &= \sum_{i,j=0}^{n} \int_{B_{(i/n,j/n),1/(2n)}} \nabla v_n \nabla \psi \, dx \\ &= \sum_{i,j=0}^{n} \int_{\partial B_{(i/n,j/n),1/(2n)}} \frac{1}{n}\psi \, d\sigma - \sum_{i,j=0}^{n} \int_{B_{(i/n,j/n),1/(2n)}} 4\psi \, dx.\end{aligned}$$

Passing to the limit as $n \to \infty$ and using the fact that $1_{\bigcup_{i,j} B_{(i/n,j/n),1/(2n)}}$ tends to $\pi/4$ weakly in $L^2(D)$, we get that μ_n tends to $\pi\, dx$ strongly in $H^{-1}(D)$.

Summarizing, the equation satisfied by $u \in H_0^1(D)$ is

$$\int_D \nabla u \nabla \varphi \, dx + \frac{2\pi}{c}\int_D u\varphi \, dx = \int_D f\varphi \, dx \qquad \forall \varphi \in C_0^\infty(D),$$

that is,

$$-\Delta u + \frac{2\pi}{c} u = f.$$

Remark 5.8.5. The two-dimensional example above can be repeated, with similar calculations, for any dimension $N > 2$; it is enough to replace the critical radius $r_n = e^{-cn^2}$ by

$$r_n = cn^{-N/(N-2)}.$$

An important question is now to characterize the completion $\overline{\mathscr{A}}$ of $\mathscr{A}$ with respect to d_γ, that is, all possible γ-limits of sequences of domains of $\mathscr{A}$. The conclusion of Example 5.8.1 can be rephrased by saying that all constant nonnegative functions belong to $\overline{\mathscr{A}}$. The characterization of $\overline{\mathscr{A}}$ was achieved in [186], where the following result was proved.

Theorem 5.8.7. *The completion $\overline{\mathscr{A}}$ of $\mathscr{A}$ with respect to d_γ is the class $\mathbf{M}_0(D)$ of all nonnegative regular Borel measures μ on D (not necessarily finite) such that*

$$\mu(E) = 0 \quad \textit{whenever } \operatorname{cap}(E) = 0.$$

The regularity in Theorem 5.8.7 is intended in the usual sense of measures, that is,

$$\mu(E) = \inf\{\mu(A) \,:\, A \text{ open}\,,\ E \subset A\}.$$

The measures of $\mathbf{M}_0(D)$ will be called *capacitary measures*. Notice that not all regular Borel measures belong to $\mathbf{M}_0(D)$; if $N \geq 2$, a point has zero capacity (see Proposition 5.8.5), and hence the Dirac measures δ_{x_0} do not belong to $\mathbf{M}_0(D)$.

An element of $\mathscr{A}$, that is, a quasi-open set $\Omega \subset D$, can be identified with the measure

$$\infty_{D\setminus\Omega}(E) = \begin{cases} 0 & \text{if } \operatorname{cap}(E\setminus\Omega) = 0, \\ +\infty & \text{otherwise.}\end{cases}$$

For every $\mu \in \mathbf{M}_0(D)$ and $f \in L^2(D)$ we may consider the PDE formally written as

$$\begin{cases} -\Delta u + \mu u = f, \\ u \in H_0^1(D), \end{cases} \tag{5.80}$$

whose precise meaning has to be given in the weak form

$$\int_D \nabla u \nabla \varphi \, dx + \int_D u\varphi \, d\mu = \int_D f\varphi \qquad \forall \varphi \in H_0^1(D) \cap L_\mu^2(D).$$

It is possible to show (see [149]) that the space $X = H_0^1(D) \cap L_\mu^2(D)$ is a Hilbert space with the norm

$$\|u\|_X^2 = \int_D |\nabla u|^2 \, dx + \int_D |u|^2 \, d\mu;$$

then by the Lax–Milgram theorem, Theorem 3.1.2, we obtain that the PDE (5.80) admits a unique solution. Notice that, when $\mu = \infty_{D\setminus\Omega}$, (5.80) becomes the PDE

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases}$$

For every measure $\mu \in \mathbf{M}_0(D)$ we may define the resolvent operator $R_\mu : L^2(D) \to L^2(D)$ which associates to every $f \in L^2(D)$ the solution u of (5.80). The definition of γ-convergence can be now extended to $\mathbf{M}_0(D)$ by setting

$$\mu_n \to_\gamma \mu \iff R_{\mu_n}(f) \to R_\mu(f) \quad \text{for every } f \in L^2(D).$$

Again, a result similar to the one of Remark 5.8.3 holds, showing that $\mu_n \to_\gamma \mu$ iff $R_{\mu_n}(1) \to R_\mu(1)$ in $L^2(D)$. In this way, the distance d_γ can be extended to $\mathbf{M}_0(D)$ by setting

$$d_\gamma(\mu_1, \mu_2) = \|R_{\mu_1}(1) - R_{\mu_2}(1)\|_{L^2(D)}.$$

As shown by Example 5.8.1, finely perforated domains may tend, in the γ-convergence, to capacitary measures (a multiple of the Lebesgue measure in the example); see Figure 5.1.

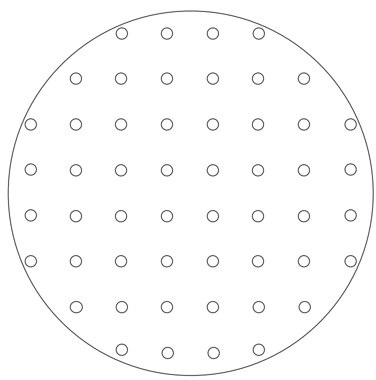

Figure 5.1. *A finely perforated set as in the Cioranescu–Murat example.*

Let us summarize the properties of the γ-convergence on the space $\mathbf{M}_0(D)$ (we refer the reader to [145] for the proofs and all the details):

1. The space $\mathbf{M}_0(D)$ endowed with the distance d_γ is a compact metric space.

2. The class $\mathscr{A}$ is included in $\mathbf{M}_0(D)$ via the identification $\Omega \mapsto \infty_{D\setminus\Omega}$ and $\mathscr{A}$ is d_γ-dense in $\mathbf{M}_0(D)$. In fact, also the class of all smooth domains Ω is d_γ-dense in $\mathbf{M}_0(D)$.

3. The measures of the form $a(x)\,dx$ with $a \in L^1(D)$ belong to $\mathbf{M}_0(D)$ and are d_γ-dense in $\mathbf{M}_0(D)$. In fact, also the class of measures $a(x)\,dx$ with a smooth is d_γ-dense in $\mathbf{M}_0(D)$.

4. If $\mu_n \to \mu$ for the γ-convergence, then the spectrum of the compact resolvent operator $\mathbf{R}_{\mu_n}$ converges to the spectrum of $\mathbf{R}_\mu$; in other words, the eigenvalues of the Schrödinger-like operator $-\Delta + \mu_n$ defined on $H_0^1(D)$ converge to the corresponding eigenvalues of the operator $-\Delta + \mu$.