HUMBOLDT-UNIVERSITÄT ZU BERLIN

Minimization of a Functional on the Space of BV Functions and Nonconforming Discretization of the Problem

I. Theoretical Basics and Characterization of Minimizers

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Sören Bartels. **Numerical Methods for Nonlinear Partial Differential Equations**. Vol. 47. Springer Series in Computational Mathematics. Springer International Publishing, 2015. ISBN: 978-3-319-13796-4. DOI: 10.1007/978-3-319-13797-1, Chapter 10, p. 297-319

Sören Bartels. **Numerical Methods for Nonlinear Partial Differential Equations**. Vol. 47. Springer Series in Computational Mathematics. Springer International Publishing, 2015. ISBN: 978-3-319-13796-4. DOI: 10.1007/978-3-319-13797-1, Chapter 10, p. 297-319

Let $\Omega \subset \mathbb{R}^n$ be a bounded polyhedral Lipschitz domain.

For given $g \in L^2(\Omega)$ and $\alpha > 0$ minimize the functional

$$I(v) = |v|_{\mathrm{BV}(\Omega)} + \frac{\alpha}{2}\|v - g\|^2$$

amongst all $v \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$.

Functions of Bounded Variation

A function $v \in L^1(\Omega)$ with distributional derivative $Dv : C_C^\infty(\Omega; \mathbb{R}^n) \to \mathbb{R}$ is said to be of bounded variation if there exists $c > 0$ such that

$$\langle Dv, \phi \rangle := -\int_\Omega v \operatorname{div}(\phi) \, \mathrm{d}x \leqslant c \|\phi\|_{L^\infty(\Omega)}$$

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The minimal constant $c \geqslant 0$ satisfying this property is called total variation of Dv and is given by

$$|v|_{\mathrm{BV}(\Omega)} = \sup_{\substack{\phi \in C_C^1(\Omega;\mathbb{R}^n) \\ \|\phi\|_{L^\infty(\Omega)} \leqslant 1}} -\int_\Omega v \operatorname{div}(\phi)\, \mathrm{d}x.$$

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The space of all such functions is denoted by $\mathrm{BV}(\Omega)$.

Properties of $\mathrm{BV}(\Omega)$

$\mathrm{BV}(\Omega)$ is a nonseparable Banach space equipped with the norm

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$W^{1,1}(\Omega) \subset \mathrm{BV}(\Omega)$ with $\|v\|_{\mathrm{BV}(\Omega)} = \|v\|_{W^{1,1}(\Omega)}$ for all $v \in W^{1,1}(\Omega)$.

Notions of convergence on $\mathrm{BV}(\Omega)$

Let $(v_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(\Omega)$ and $v \in \mathrm{BV}(\Omega)$ such that $v_n \to v$ in $L^1(\Omega)$ as $n \longrightarrow \infty$.

(i) $(v_n)_{n\in\mathbb{N}}$ converges intermediately or strictly to v if $|v_n|_{\mathrm{BV}(\Omega)} \longrightarrow |v|_{\mathrm{BV}(\Omega)}$ as $n \longrightarrow \infty$.

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(i) $(v_n)_{n\in\mathbb{N}}$ converges intermediately or strictly to v if $|v_n|_{\mathrm{BV}(\Omega)} \longrightarrow |v|_{\mathrm{BV}(\Omega)}$ as $n \longrightarrow \infty$.

(ii) $(v_n)_{n\in\mathbb{N}}$ converges weakly to v if $\langle Dv_n, \phi\rangle \to \langle Dv, \phi\rangle$ for all $\phi \in C_0(\Omega; \mathbb{R}^n)$ as $n \longrightarrow \infty$.

Further Properties of $\mathrm{BV}(\Omega)$

$C^\infty(\overline{\Omega})$ and $C^\infty(\Omega) \cap \mathrm{BV}(\Omega)$ are dense in $\mathrm{BV}(\Omega)$ with respect to intermediate convergence.

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The embedding $\mathrm{BV}(\Omega) \to L^p(\Omega)$ is continuous for $1 \leqslant p \leqslant n/(n-1)$ and compact for $1 \leqslant p < n/(n-1)$

There exists a linear operator $T : \mathrm{BV}(\Omega) \to L^1(\partial\Omega)$ such that $T(v) = v|_{\partial\Omega}$ for all $v \in \mathrm{BV}(\Omega) \cap C(\overline{\Omega})$.
T is continuous with respect to intermediate convergence in $\mathrm{BV}(\Omega)$ but not with respect to weak convergence in $\mathrm{BV}(\Omega)$.

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For given $f \in L^2(\Omega)$ and $\alpha > 0$ minimize the functional

$$E(v) := \frac{\alpha}{2}\|v\|^2_{L^2(\Omega)} + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \int_\Omega f\, v \,\mathrm{d}x$$

amongst all $v \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$.

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For $f = \alpha g$ we have

$$I(v) = |v|_{\mathrm{BV}(\Omega)} + \frac{\alpha}{2}\|v - g\|^2 = E(v) - \|v\|_{L^1(\partial\Omega)} + \frac{\alpha}{2}\|g\|^2_{L^2(\Omega)}$$

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I and E have the same minimizers in
$\left\{v \in \mathrm{BV}(\Omega) \cap L^2(\Omega) \mid \|v\|_{L^1(\partial\Omega)} = 0\right\}$.

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$$E(v) = \frac{\alpha}{2}\|v\|_{L^2(\Omega)}^2 + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \int_\Omega fv \,\mathrm{d}x$$

$$
\begin{aligned}
E(v) &= \frac{\alpha}{2}\|v\|_{L^2(\Omega)}^2 + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \int_\Omega fv \,\mathrm{d}x \\
&\geqslant \frac{\alpha}{2}\|v\|_{L^2(\Omega)}^2 + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}
\end{aligned}
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&\geqslant \frac{\alpha}{2}\|v\|^2_{L^2(\Omega)} + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \frac{1}{\alpha}\|f\|^2_{L^2(\Omega)} - \frac{\alpha}{4}\|v\|^2_{L^2(\Omega)}
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&\geqslant \frac{\alpha}{2}\|v\|^2_{L^2(\Omega)} + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \frac{1}{\alpha}\|f\|^2_{L^2(\Omega)} - \frac{\alpha}{4}\|v\|^2_{L^2(\Omega)} \\
&\geqslant \frac{\alpha}{4}\|v\|^2_{L^2(\Omega)} + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \frac{1}{\alpha}\|f\|^2_{L^2(\Omega)}
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&\geqslant \frac{\alpha}{4}\|v\|^2_{L^2(\Omega)} + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \frac{1}{\alpha}\|f\|^2_{L^2(\Omega)} \\
&\geqslant \frac{\alpha}{4|\Omega|}\|v\|^2_{L^1(\Omega)} + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \frac{1}{\alpha}\|f\|^2_{L^2(\Omega)}
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&\geqslant \frac{\alpha}{4|\Omega|}\|v\|^2_{L^1(\Omega)} + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \frac{1}{\alpha}\|f\|^2_{L^2(\Omega)} \\
&\geqslant -\frac{1}{\alpha}\|f\|^2_{L^2(\Omega)}.
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\begin{aligned}
E(v) &\geqslant \frac{\alpha}{4}\|v\|_{L^2(\Omega)}^2 + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \frac{1}{\alpha}\|f\|_{L^2(\Omega)}^2 \\
&\geqslant \frac{\alpha}{4|\Omega|}\|v\|_{L^1(\Omega)}^2 + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \frac{1}{\alpha}\|f\|_{L^2(\Omega)}^2 \\
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&\geqslant -\frac{1}{\alpha}\|f\|^2_{L^2(\Omega)}.
\end{aligned}
$$

- $\exists (u_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(\Omega) \cap L^2(\Omega)$ infimizing sequence of E
- $\|u_n\|_{\mathrm{BV}(\Omega)} \to \infty$ as $n \to \infty \Rightarrow E(u_n) \to \infty$ as $n \to \infty$
- $(u_n)_{n\in\mathbb{N}}$ bounded

Lawrence C. Evans and Ronald F. Gariepy. **Measure Theory and Fine Properties of Functions**. CRC Press, 1992. ISBN: 0-8493-7157-0, p. 183, Theorem 1

Let $v \in \mathrm{BV}(\Omega)$. For all $x \in \mathbb{R}^n$ define

$$\tilde{v}(x) := \begin{cases} v(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \backslash \Omega. \end{cases}$$

Then $\tilde{v} \in \mathrm{BV}\left(\mathbb{R}^n\right)$ and $|\tilde{v}|_{\mathrm{BV}(\mathbb{R}^n)} = |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)}$.

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