VARIATIONAL ANALYSIS IN SOBOLEV AND *BV* SPACES

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Variational Analysis in Sobolev and *BV* Spaces

Applications to PDEs and Optimization

Second Edition

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Preface to the Second Edition

This second edition takes advantage of several comments received by colleagues and students. With respect to the first edition (published by SIAM in 2006) several new sections have been added and the organization of the material is now slightly different. The section on capacity theory and elements of potential theory has been completed by the notions of quasi-open sets and quasi-continuity. We also have increased the number of examples in Section 6 (the linearized elasticity system, obstacles problems, convection-diffusions and semilinear equations, ...). Section 11, devoted to the relaxation theory, has been completed by a section on mass transportation problems and the Kantorovich relaxed formulation of the Monge problem. We have added a subsection on stochastic homogenization to the section devoted to the Gamma-convergence: we establish the mathematical tools coming from ergodic theory and illustrate them in the scope of statistically homogeneous materials. Section 16 has been augmented by two examples illustrating the shape optimization procedure. The main novelty of this second edition is the new and very comprehensive section devoted to gradient flows, as well as the dynamical approach to equilibria.

Preface to the First Edition

Most of the material in this book comes from graduate-level courses on variational analysis, PDEs, and optimization which have been given during the last decades by the authors, H. Attouch and G. Michaille at the University of Montpellier, France, and G. Buttazzo at the University of Pisa, Italy. Our objective is twofold.

The first objective is to provide to students the basic tools and methods of variational analysis and optimization in infinite dimensional spaces together with applications to classical PDE problems. This corresponds to the first part of the book, Chapters 1 through 9, and takes place in classical Sobolev spaces. We have made an effort to provide, as much as possible, a self-contained exposition, and we try to introduce each new development from various perspectives (historical, numerical, ...).

The second objective, which is oriented more toward research, is to present new trends in variational analysis and some of the most recent developments and applications. This corresponds to the second part of the book, Chapters 10 through 16, where in particular are introduced the $BV(\Omega)$ spaces.

This organization is intended to make the book accessible to a large audience, from students to researchers, with various backgrounds in mathematics, as well as physicists, engineers, and others. As a guideline, we try to portray direct methods in modern variational analysis—one century after D. Hilbert delineated them in his famous lecture at Collège de France, Paris, 1900. The extraordinary success of these methods is intimately linked with the development, throughout the 20th century, of new branches in mathematics: functional analysis, measure theory, numerical analysis, (nonlinear) PDEs, and optimization.

We try to show in this book the interplay among all these theories and also between theory and applications. Variational methods have proved to be very flexible. In recent years, they have been developed to study a number of advanced problems of modern technology, like composite materials, phase transitions, thin structures, large deformations, fissures, and shape optimization.

To grasp these often involved phenomena, the classical framework of variational analysis, which is presented in the first part, must be enlarged. This is the motivation for the introduction in the second part of the book of some advanced techniques, like BV and SBV spaces, Young measures, Γ-convergence, recession analysis, and relaxation methods.

Finally, we wish to stress that variational analysis is a remarkable example of international collaboration. All mathematical schools have contributed to its success, and it is a modest symbol that this book has been written in collaboration between mathematicians of two schools, French and Italian. This book owes much to the support of the Universities of Montpellier and Pisa and of their mathematical departments, and the convention of cooperation that connects them.

Acknowledgments. We would like to express our sincere thanks to all the students and colleagues whose comments and encouragement helped us in writing the final manuscript. Year by year, the redaction of the book profited much from their comments.

We are grateful to our colleagues in the continuous optimization community, who strongly influenced the contents of the book and encouraged us from the very beginning in writing this book. The chapter on convex analysis benefited much from the careful reading of L. Thibault and M. Valadier.

We would like to thank SIAM and the editorial board of the MPS-SIAM Series on Optimization for the quality of the editing process. We address special thanks to B. Lacan in Montpellier, who helped us when we started the project.

Finally, we take this opportunity to express our consideration and gratitude to H. Brezis and E. De Giorgi, who were our first guides in the discovery of this fascinating world of variational methods and their applications.