The Crouzeix-Raviart Finite Element Method for a Nonconforming Formulation of the Rudin-Osher-Fatemi Model Problem

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ROF Problem beschreiben mit kurzer Def von BV(Ω), α Bedeutung nachliefern, vielleicht auch hier schon die Bilder zeigen mit Entrauschen (s. Intro BA) nochmal Existenz und Eindeutigkeitsaussagen zeigen und gaaaanz grob beschreiben, wie die bewiesen werden nochmal Existenz und Eindeutigkeitsaussagen zeigen und gaaaanz grob beschreiben, wie die bewiesen werden vielleicht in eigener Section: noch die Aussagen aus der Arbeit verarbeiten, die über Existenz und Eindeutigkeit hinausgehen

Sören Bartels. **Numerical Methods for Nonlinear Partial Differential Equations**. Vol. 47. Springer Series in Computational Mathematics. Springer International Publishing, 2015. ISBN: 978-3-319-13796-4. DOI: 10.1007/978-3-319-13797-1, Chapter 10, p. 297-319

Sören Bartels. **Numerical Methods for Nonlinear Partial Differential Equations**. Vol. 47. Springer Series in Computational Mathematics. Springer International Publishing, 2015. ISBN: 978-3-319-13796-4. DOI: 10.1007/978-3-319-13797-1, Chapter 10, p. 297-319

Let $\Omega \subset \mathbb{R}^n$ be a bounded polyhedral Lipschitz domain.

For given $g \in L^2(\Omega)$ and $\alpha > 0$ minimize the functional

$$I(v) = |v|_{\mathrm{BV}(\Omega)} + \frac{\alpha}{2}\|v - g\|^2$$

amongst all $v \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$.

Functions of Bounded Variation

A function $v \in L^1(\Omega)$ with distributional derivative $Dv : C_C^\infty(\Omega;\mathbb{R}^n) \to \mathbb{R}$ is said to be of bounded variation if there exists $c > 0$ such that

$$\langle Dv, \phi \rangle := -\int_\Omega v \operatorname{div}(\phi)\,\mathrm{d}x \leqslant c\|\phi\|_{L^\infty(\Omega)}$$

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The minimal constant $c \geqslant 0$ satisfying this property is called total variation of Dv and is given by

$$|v|_{\mathrm{BV}(\Omega)} = \sup_{\substack{\phi \in C_C^1(\Omega;\mathbb{R}^n)\\ \|\phi\|_{L^\infty(\Omega)} \leqslant 1}} -\int_\Omega v \operatorname{div}(\phi)\,\mathrm{d}x.$$

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The space of all such functions is denoted by $\mathrm{BV}(\Omega)$.

Properties of $\mathrm{BV}(\Omega)$

$\mathrm{BV}(\Omega)$ is a Banach space equipped with the norm

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$W^{1,1}(\Omega) \subset \mathrm{BV}(\Omega)$ with $\|v\|_{\mathrm{BV}(\Omega)} = \|v\|_{W^{1,1}(\Omega)}$ for all $v \in W^{1,1}(\Omega)$.

Notions of convergence on $\mathrm{BV}(\Omega)$

Let $(v_n)_{n\in\mathbb{N}} \subset \mathrm{BV}(\Omega)$ and $v \in \mathrm{BV}(\Omega)$ such that $v_n \to v$ in $L^1(\Omega)$ as $n \to \infty$.

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(i) $(v_n)_{n\in\mathbb{N}}$ converges intermediately or strictly to v if $|v_n|_{\mathrm{BV}(\Omega)} \to |v|_{\mathrm{BV}(\Omega)}$ as $n \to \infty$.

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(ii) $(v_n)_{n\in\mathbb{N}}$ converges weakly to v if $\langle Dv_n, \phi\rangle \to \langle Dv, \phi\rangle$ for all $\phi \in C_0(\Omega;\mathbb{R}^n)$ as $n \to \infty$.

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There exists a linear operator $T : \mathrm{BV}(\Omega) \to L^1(\partial\Omega)$ such that $T(v) = v|_{\partial\Omega}$ for all $v \in \mathrm{BV}(\Omega) \cap C(\overline{\Omega})$.
T is continuous with respect to intermediate convergence in $\mathrm{BV}(\Omega)$ but not with respect to weak convergence in $\mathrm{BV}(\Omega)$.

Algorithmus und mglw LGS dazu, siehe Arbeit
wohl auch Konvergenztheorem aufführen, Bereich für τ kurz erläutern, vielleicht beim groben erläutern der Beweisidee

drüber nachdenken, was hier gezeigt werden soll. Idealerweise viele subsections mit Themenbereichen (f01, cam, termCrit, tau. . .)

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