0.1 The continuous and discrete problem

Let $\alpha > 0$, $\Omega \subset \mathbb{R}^n$ bounded polyhedral Lipschitz domain, and $f \in L^2(\Omega)$.

The continuous problem minimizes

$$E(v) := \frac{\alpha}{2}\|v\|^2_{L^2(\Omega)} + |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \int_\Omega f\, v\, \mathrm{d}x \tag{0.1}$$

amongst all $v \in V := \mathrm{BV}(\Omega) \cap L^2(\Omega)$ where the BVseminorm $|v|_{\mathrm{BV}(\Omega)}$ is equal to the $W^{1,1}$ seminorm for any $v \in W^{1,1}(\Omega)$.

The nonconforming problem minimizes

$$E_{\mathrm{NC}}(v_{\mathrm{CR}}) := \frac{\alpha}{2}\|v_{\mathrm{CR}}\|^2_{L^2(\Omega)} + |v_{\mathrm{CR}}|_{1,1,\mathrm{NC}} - \int_\Omega f\, v_{\mathrm{CR}}\, \mathrm{d}x \tag{0.2}$$

amongst all $v_{\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T})$ where $|\bullet|_{1,1,\mathrm{NC}} := \|\nabla_{\mathrm{NC}} \bullet\|_{L^1(\Omega)}$.

0.2 Refinement indicator and guaranteed lower energy bound

For some $n \in \mathbb{N}$ (here $n = 2$) and $0 < \gamma \leqslant 1$ define a refinement indicator $\eta := \sum_{T\in\mathcal{T}} \eta(T)$ with

$$\eta(T) := \underbrace{|T|^{2/n}\|f - \alpha u_{\mathrm{CR}}\|^2_{L^2(T)}}_{=:\eta_{\mathrm{Vol}}(T)} + \underbrace{|T|^{\gamma/n} \sum_{F\in\mathcal{F}(T)} \|[u_{\mathrm{CR}}]_F\|_{L^1(F)}}_{=:\eta_{\mathrm{Jumps}}(T)} \tag{0.3}$$

for any $T \in \mathcal{T}$.

For $f \in H^1_0(\Omega)$ and $u \in H^1_0(\Omega)$ $\big(u_{\mathrm{CR}} \in \mathrm{CR}^1_0(\Omega)\big)$ continuous (discrete) minimizer with minimal energy $E(u)$ ($E_{\mathrm{NC}}(u_{\mathrm{CR}})$) it holds

$$E_{\mathrm{NC}}(u_{\mathrm{CR}}) + \frac{\alpha}{2}\|u - u_{\mathrm{CR}}\|^2_{L^2(\Omega)} - \frac{\kappa_{\mathrm{CR}}}{\alpha}\|h_{\mathcal{T}}(f - \alpha u_{\mathrm{CR}})\|_{L^2(\Omega)}|f|_{1,2} \leqslant E(u) \tag{0.4}$$

where $|\bullet|_{1,2} = \|\nabla \bullet\|_{L^2(\Omega)}$.

Hence, for $\mathrm{GLEB} := E_{\mathrm{NC}}(u_{\mathrm{CR}}) - \frac{\kappa_{\mathrm{CR}}}{\alpha}\|h_{\mathcal{T}}(f - \alpha u_{\mathrm{CR}})\|_{L^2(\Omega)}|f|_{1,2}$, it holds $E_{\mathrm{NC}}(u_{\mathrm{CR}}) \geqslant \mathrm{GLEB}$ and $E(u) \geqslant \mathrm{GLEB}$.

0.3 Experiments

In the following sections on each level the termination criterion for the iteration was

$$\left\| \frac{\nabla_{\mathrm{NC}}(u_j - u_{j-1})}{\tau} \right\|_{L^2(\Omega)} < \varepsilon := 10^{-4}$$

for $\tau = \frac{1}{2}$.

0.4 Examples with exact solution

0.4.1 Example 1

For $\beta = 1$ define f as a function of the radius as

$$f(r) := \begin{cases} \alpha - 12(2 - 9r) & \text{if } 0 \leqslant r \leqslant \frac{1}{6}, \\ \alpha(1 + (6r-1)^{\beta}) - \frac{1}{r} & \text{if } \frac{1}{6} \leqslant r \leqslant \frac{1}{3}, \\ 2\alpha + 6\pi \sin(\pi(6r-2)) - \frac{1}{r}\cos(\pi(6r-2)) & \text{if } \frac{1}{3} \leqslant r \leqslant \frac{1}{2}, \\ 2\alpha(\frac{5}{2} - 3r)^{\beta} + \frac{1}{r} & \text{if } \frac{1}{2} \leqslant r \leqslant \frac{5}{6}, \\ -3\pi \sin(\pi(6r-5)) + \frac{1+\cos(\pi(6r-5))}{2r} & \text{if } \frac{5}{6} \leqslant r \leqslant 1, \end{cases} \tag{0.5}$$

with exact solution

$$u(r) := \begin{cases} 1 & \text{if } 0 \leqslant r \leqslant \frac{1}{6}, \\ 1 + (6r-1)^{\beta} & \text{if } \frac{1}{6} \leqslant r \leqslant \frac{1}{3}, \\ 2 & \text{if } \frac{1}{3} \leqslant r \leqslant \frac{1}{2}, \\ 2(\frac{5}{2} - 3r)^{\beta} & \text{if } \frac{1}{2} \leqslant r \leqslant \frac{5}{6}, \\ 0 & \text{if } \frac{5}{6} \leqslant r \leqslant 1. \end{cases} \tag{0.6}$$

For $\alpha = 1$ the exact energy $E(u)$ was computed before the experiment.

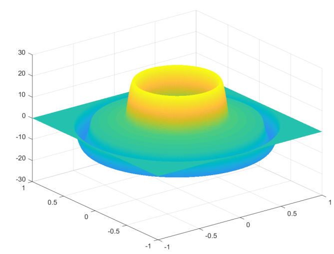

Figure 0.1: f for $\alpha = 1$

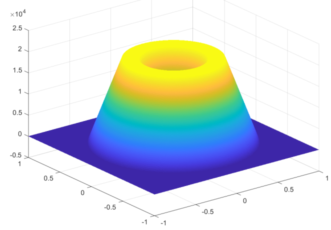

Figure 0.2: f for $\alpha = 10^4$

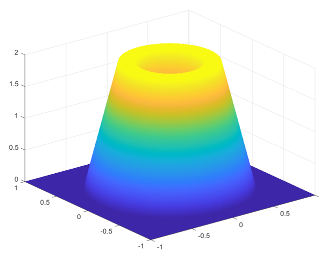

Figure 0.3: u

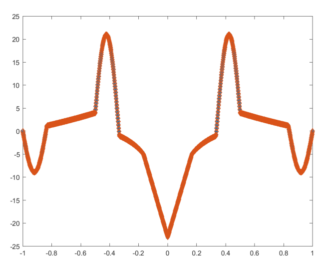

Figure 0.4: f along the axes for $\alpha = 1$

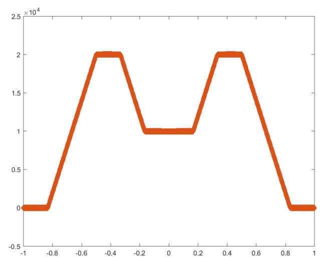

Figure 0.5: f along the axes for $\alpha = 10^4$

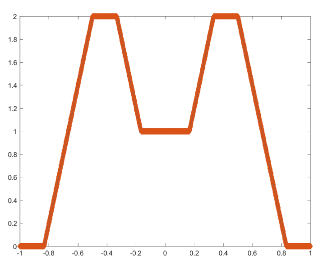

Figure 0.6: u along the axes

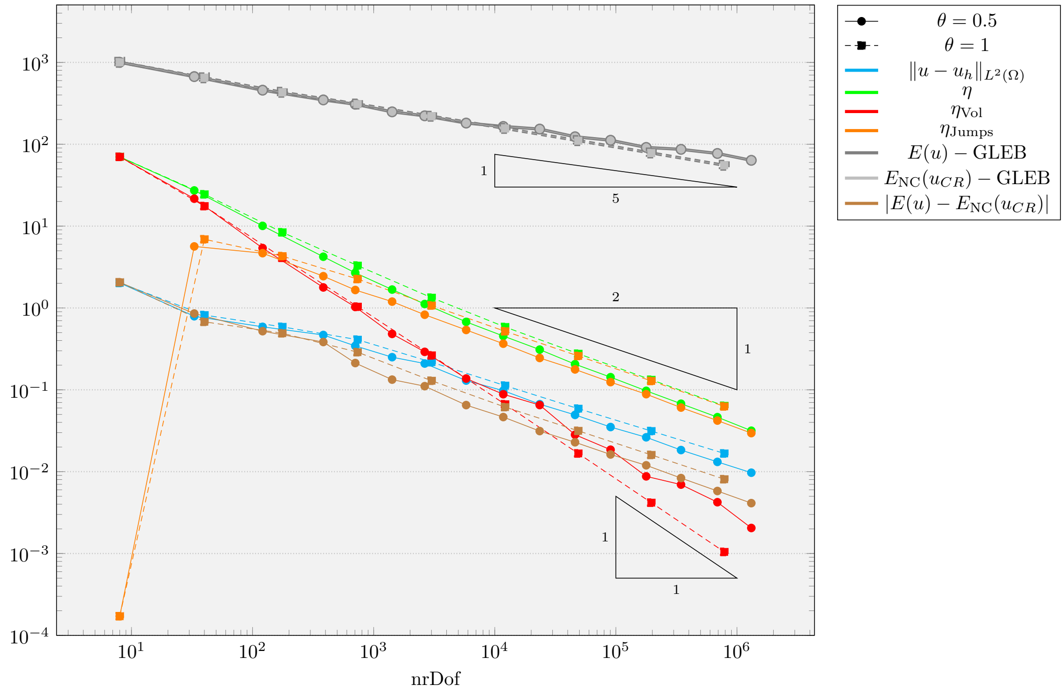

Figure 0.7: convergence history plot for $\alpha = 1$

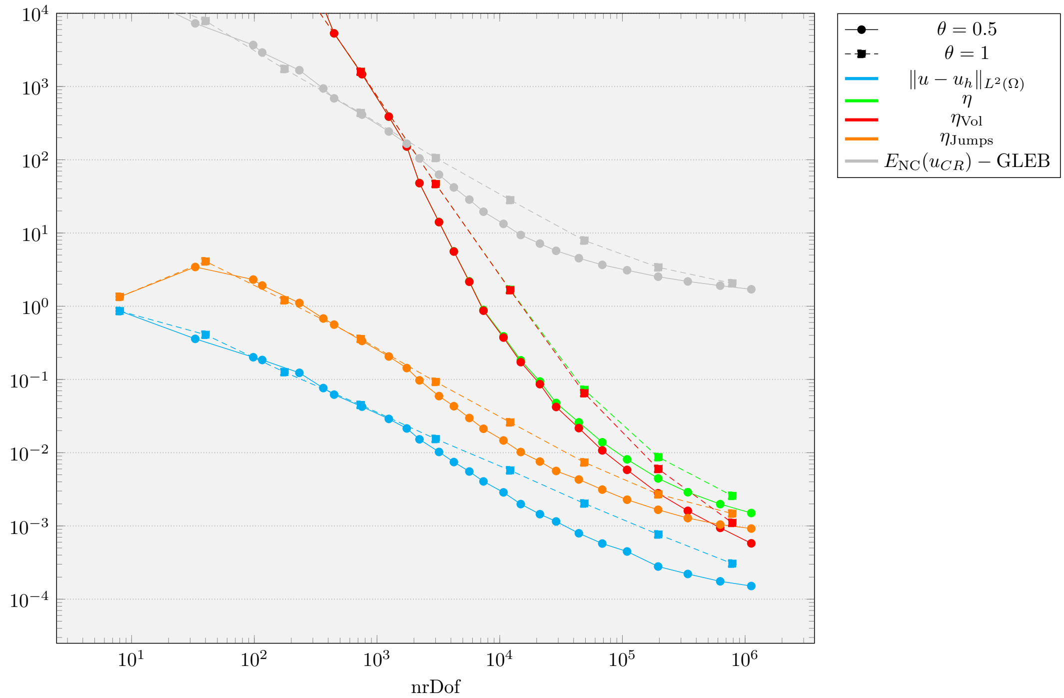

Figure 0.8: convergence history plot for $\alpha = 10^4$

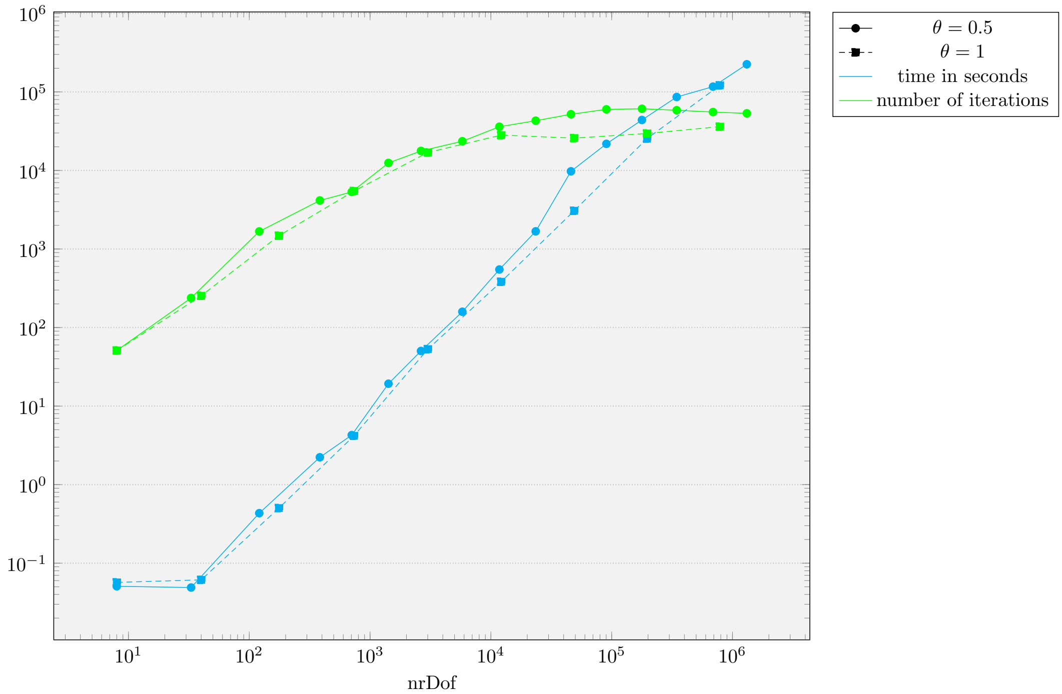

Figure 0.9: development of the number of iterations and the elapsed time for each iteration for $\alpha = 1$

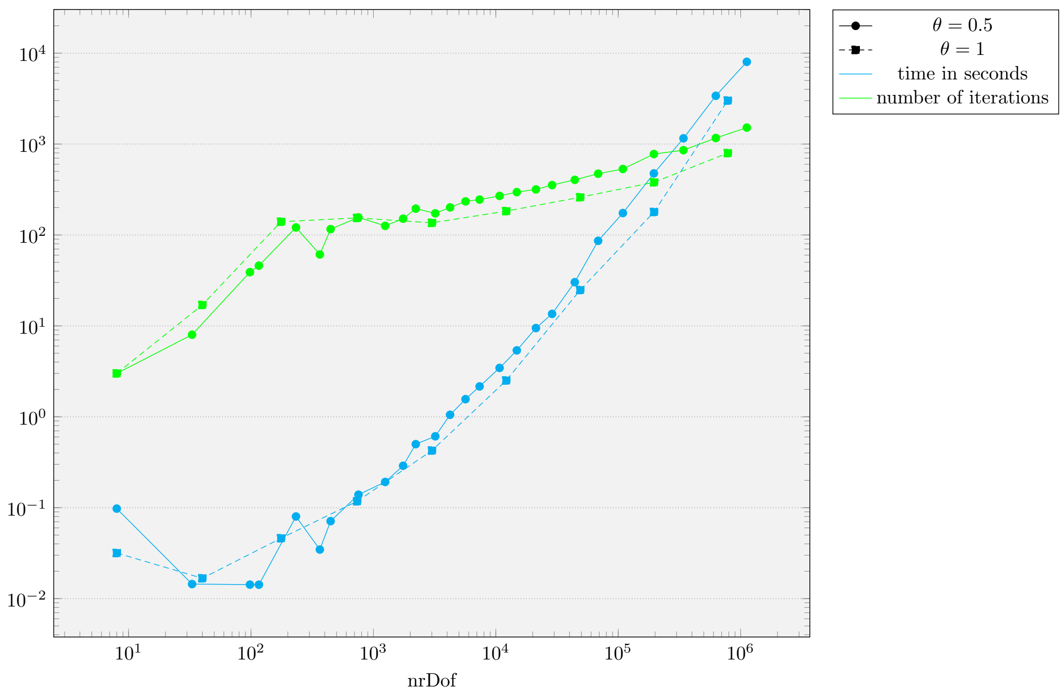

Figure 0.10: development of the number of iterations and the elapsed time for each iteration for $\alpha = 10^4$

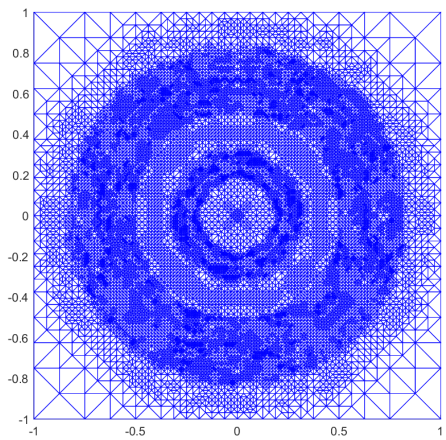

Figure 0.11: adaptive mesh for $\alpha = 1$ and $\theta = 0.5$ with 15393 nodes and 46016 degrees of freedom

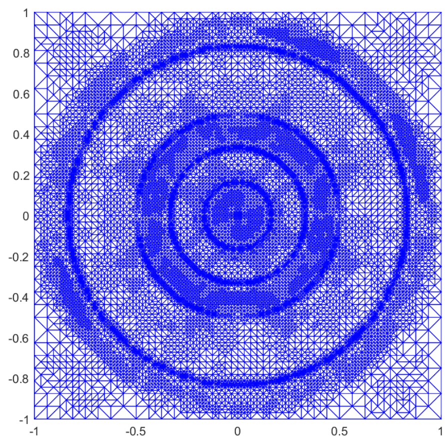

Figure 0.12: adaptive mesh for $\alpha = 10^4$ and $\theta = 0.5$ with 14808 nodes and 44157 degrees of freedom

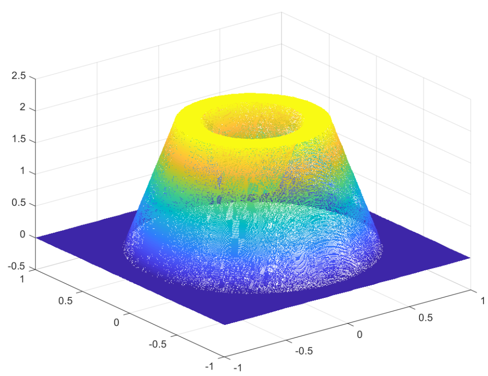

Figure 0.13: last iterate for $\alpha = 1$ and $\theta = 0.5$

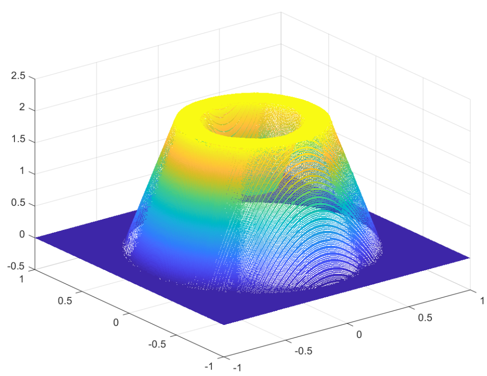

Figure 0.14: last iterate for $\alpha = 1$ and $\theta = 1$

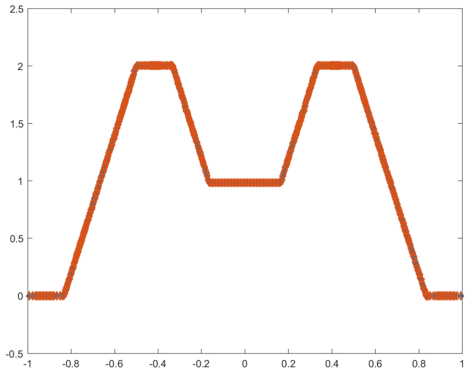

Figure 0.15: last iterate along the axes for $\alpha = 1$ and $\theta = 0.5$

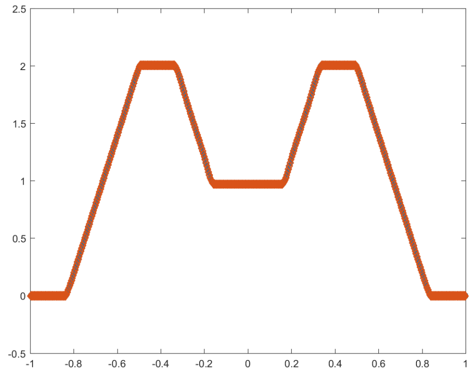

Figure 0.16: last iterate along the axes for $\alpha = 1$ and $\theta = 1$

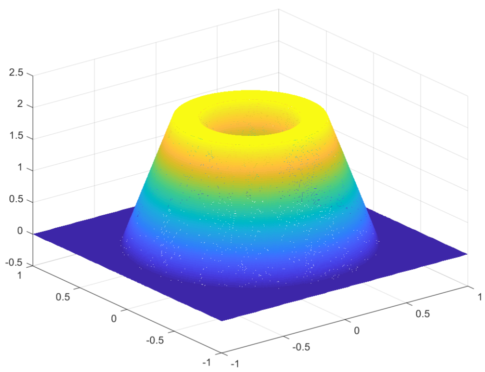

Figure 0.17: last iterate for $\alpha = 10^4$ and $\theta = 0.5$

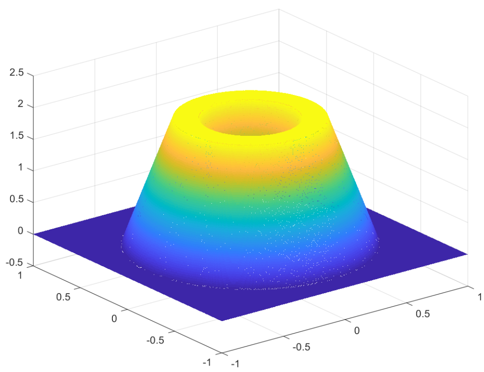

Figure 0.18: last iterate for $\alpha = 10^4$ and $\theta = 1$

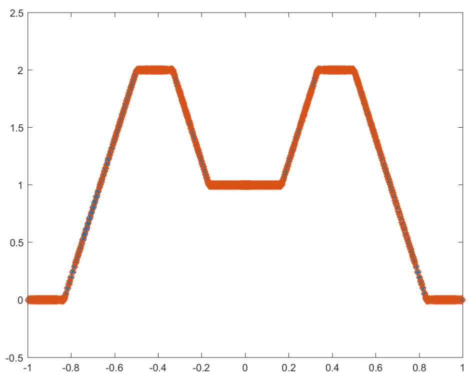

Figure 0.19: last iterate along the axes for $\alpha = 10^4$ and $\theta = 0.5$

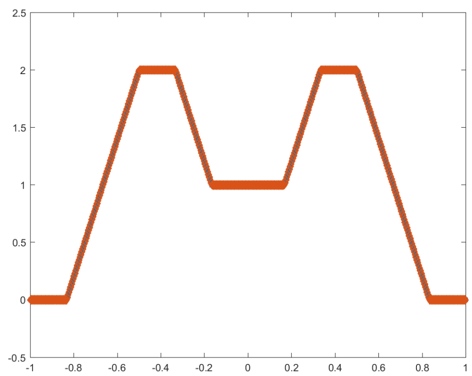

Figure 0.20: last iterate along the axes for $\alpha = 10^4$ and $\theta = 1$

0.4.2 Example 2

For $\alpha = 1$ and $\beta \in (0,1)$ define f as a function of the radius as

$$f(r) := \begin{cases} \alpha - \frac{4}{1-\beta}\left(\frac{3}{1-\beta}r - 2\right) & \text{if } 0 \leqslant r \leqslant \frac{1-\beta}{2}, \\ -\frac{\alpha}{\beta}\left(r - \frac{1+\beta}{2}\right) + \frac{1}{r} & \text{if } \frac{1-\beta}{2} \leqslant r \leqslant \frac{1+\beta}{2}, \\ \frac{-4}{(\beta-1)^3}\left(16r^2 - 9(\beta+3)r + 12(\beta+1) - \frac{3\beta+1}{r}\right) & \text{if } \frac{1+\beta}{2} \leqslant r \leqslant 1, \end{cases}$$

with exact solution

$$u_2(r) := \begin{cases} 1 & \text{if } 0 \leqslant r \leqslant \frac{1-\beta}{2}, \\ -\frac{1}{\beta}r + \frac{1+\beta}{2\beta} & \text{if } \frac{1-\beta}{2} \leqslant r \leqslant \frac{1+\beta}{2}, \\ 0 & \text{if } \frac{1+\beta}{2} \leqslant r. \end{cases}$$

The experiments were conducted for $\beta = \frac{1}{10}, \frac{1}{2}, \frac{9}{10}$ and the corresponding exact energies were computed beforehand.

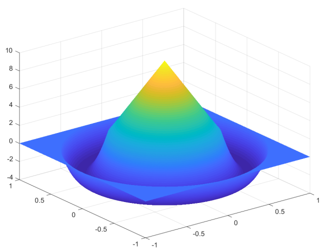

Figure 0.21: f for $\beta = \frac{1}{10}$

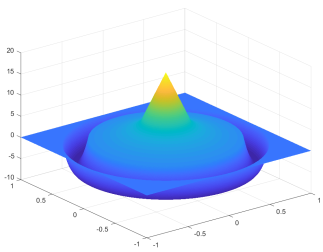

Figure 0.22: f for $\beta = \frac{1}{2}$

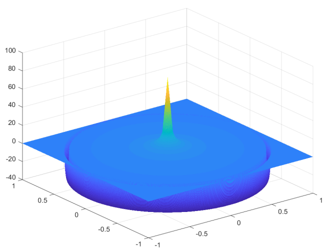

Figure 0.23: f for $\beta = \frac{9}{10}$

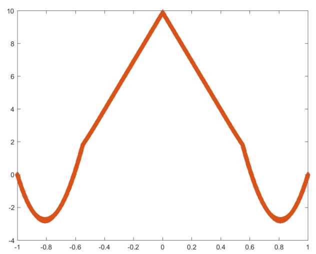

Figure 0.24: f along the axes for $\beta = \frac{1}{10}$

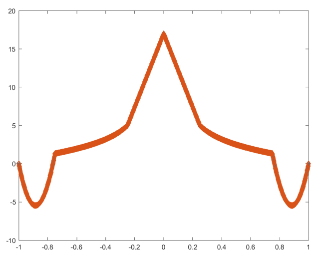

Figure 0.25: f along the axes for $\beta = \frac{1}{2}$

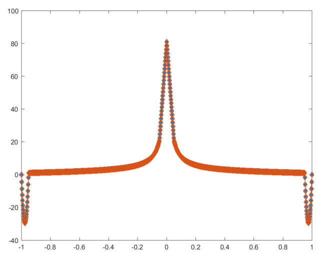

Figure 0.26: f along the axes for $\beta = \frac{9}{10}$

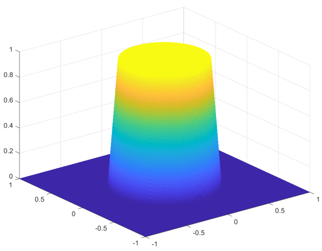

Figure 0.27: u for $\beta = \frac{1}{10}$

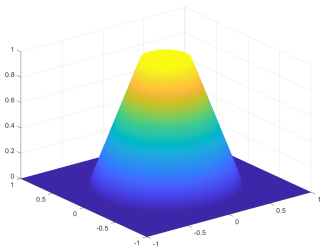

Figure 0.28: u for $\beta = \frac{1}{2}$

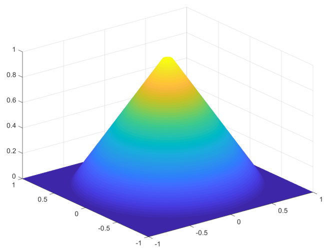

Figure 0.29: u for $\beta = \frac{9}{10}$

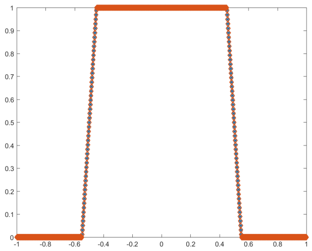

Figure 0.30: u along the axes for $\beta = \frac{1}{10}$

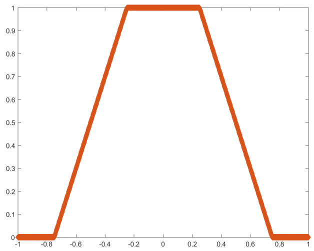

Figure 0.31: u along the axes for $\beta = \frac{1}{2}$

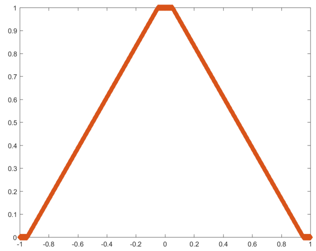

Figure 0.32: u along the axes for $\beta = \frac{9}{10}$

0.5 Application to image

For $\alpha = 10000$ let f represent the grayscale of an image in $[0,1]^{256\times 256}$ multiplied with α scaled to the domain $\Omega \in (0,1)^2$ as seen in fig. 0.33.

Figure 0.33: grayscale plot of the right-hand side f (view from above onto the x-y plane)

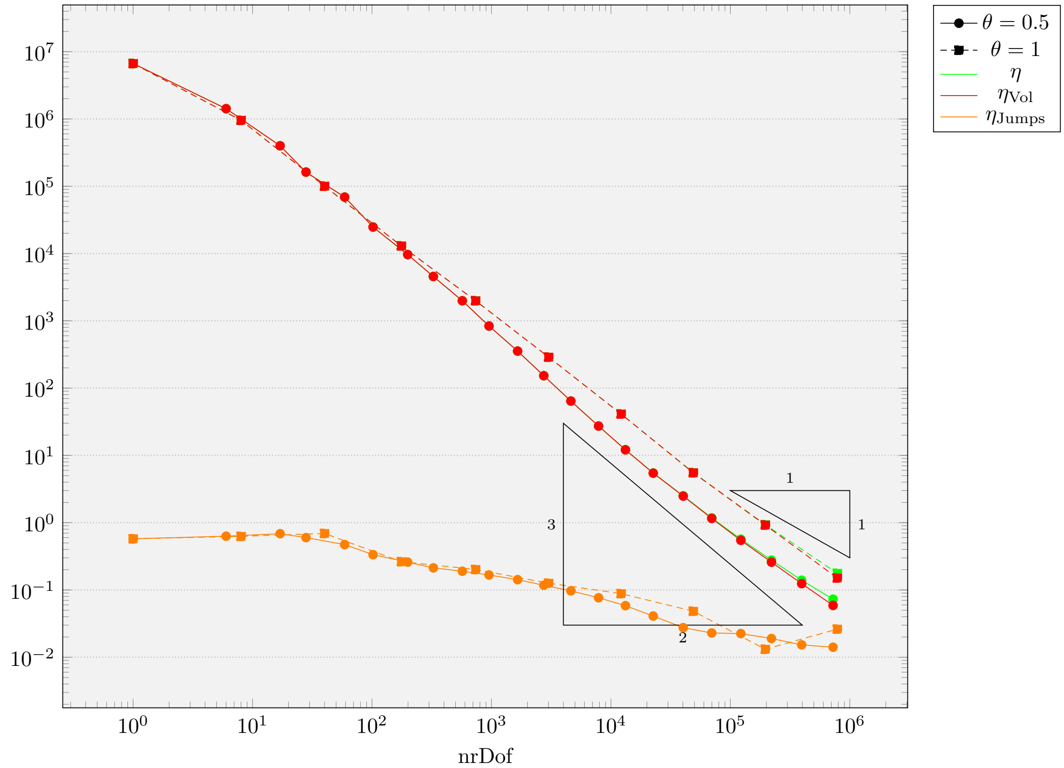

Figure 0.34: convergence history plot for η, η_{Vol}, and η_{Jumps}

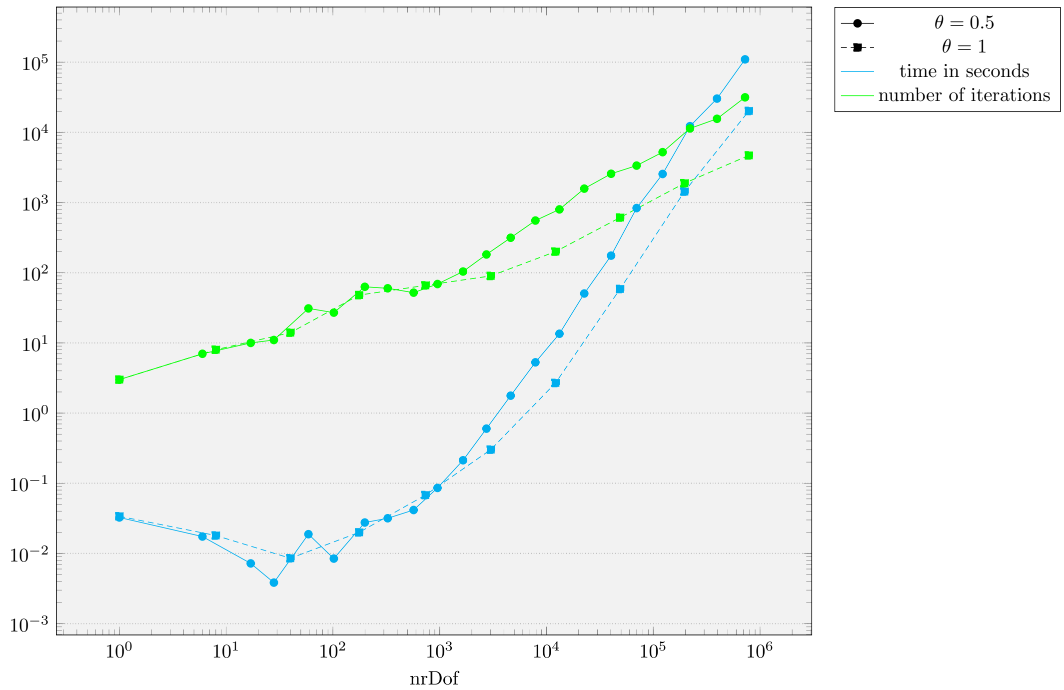

Figure 0.35: development of the number of iterations and the elapsed time for each iteration

Figure 0.36: grayscale plot of last iterate for $\theta = 0.5$

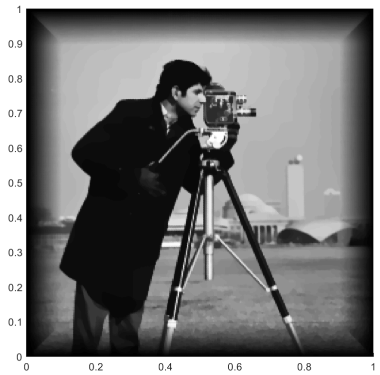

Figure 0.37: grayscale plot of last iterate for $\theta = 1$

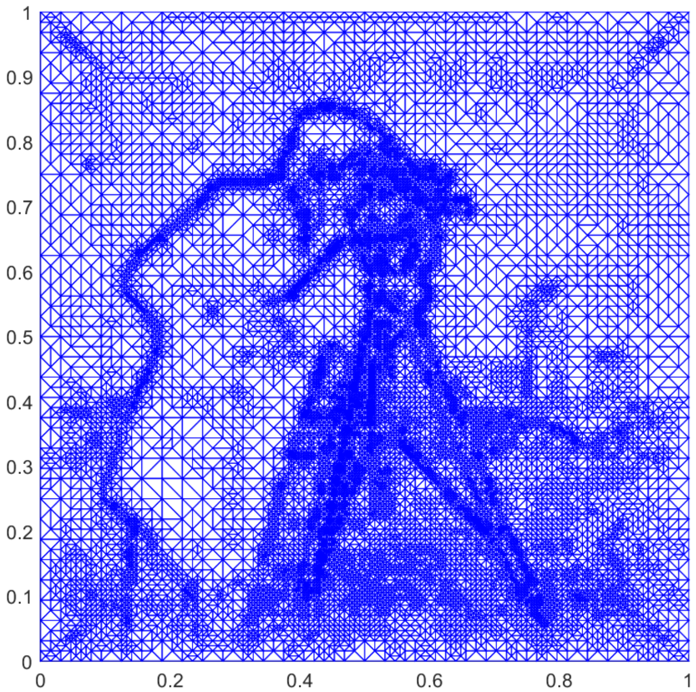

Figure 0.38: adaptive mesh for $\theta = 0.5$ with 13555 nodes and 40300 degrees of freedom

0.6 Application to a function with discontinuity set

For $\alpha = 100$ define

$$f(x) := \begin{cases} 100 & \text{if } \|x\|_\infty \leqslant \frac{1}{2}, \\ 0 & \text{else} \end{cases} \tag{0.7}$$

on $\Omega = (-1,1)^2$.

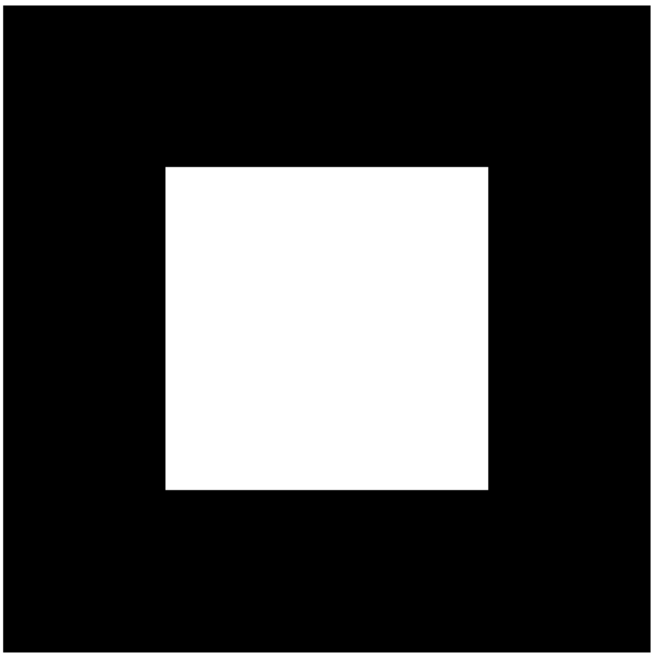

Figure 0.39: grayscale plot of the right-hand side f (view from above onto the x-y plane)

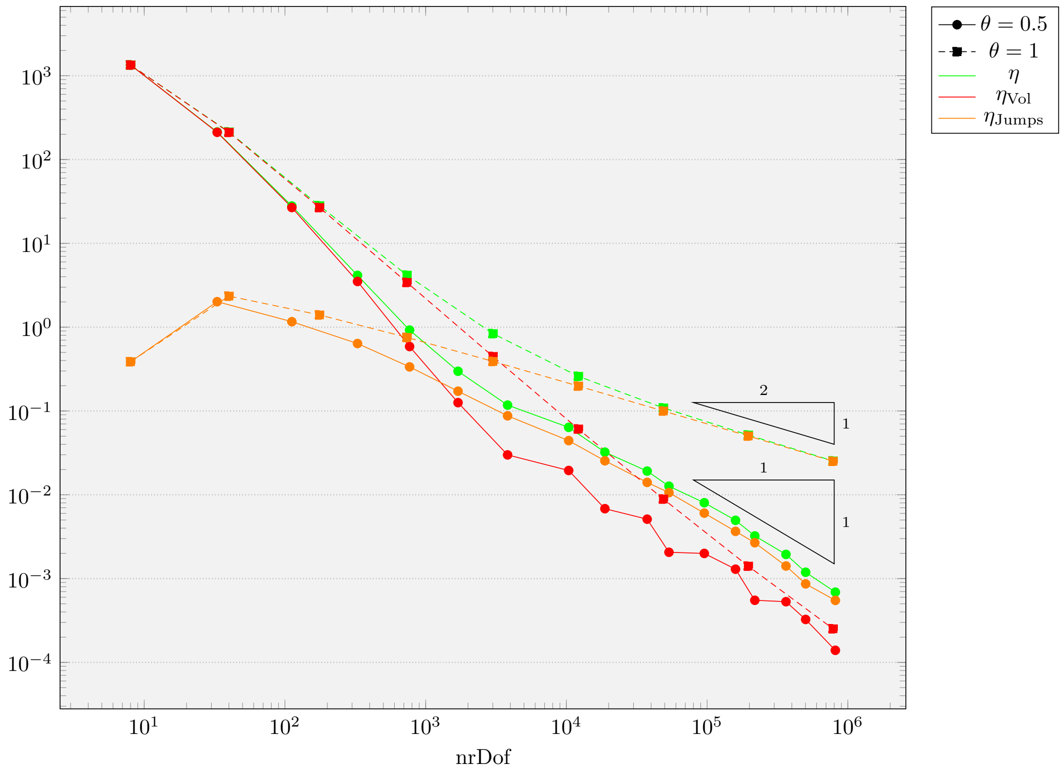

Figure 0.40: convergence history plot for η, η_{Vol}, and η_{Jumps}

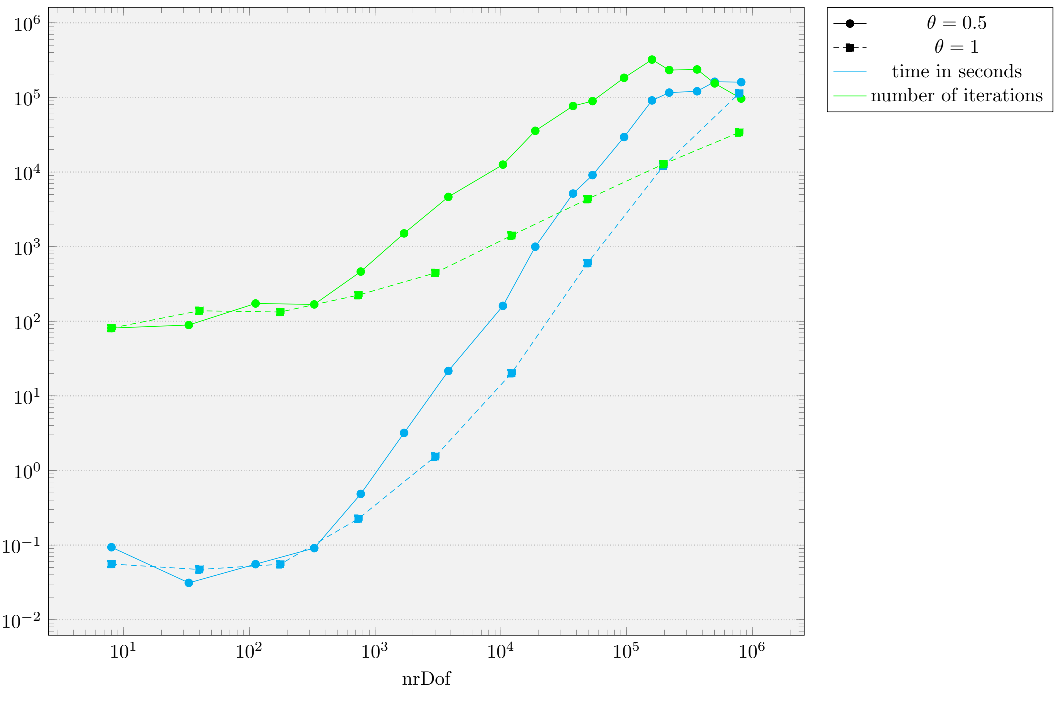

Figure 0.41: development of the number of iterations and the elapsed time for each iteration

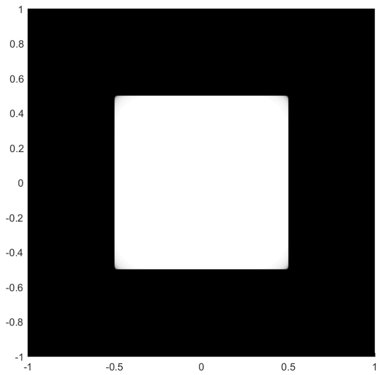

Figure 0.42: grayscale plot of last iterate for $\theta = 0.5$

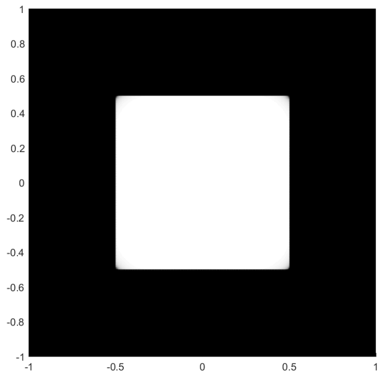

Figure 0.43: grayscale plot of last iterate for $\theta = 1$

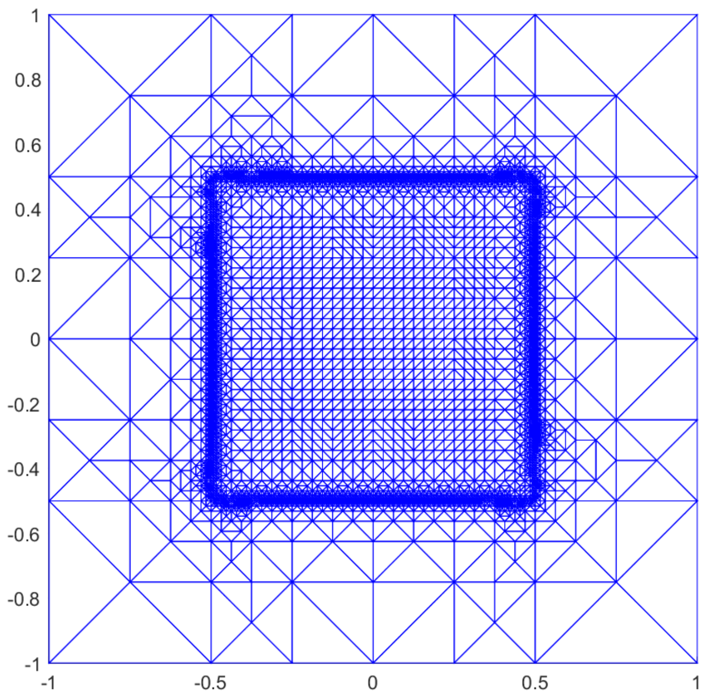

Figure 0.44: adaptive mesh for $\theta = 0.5$ with 6278 nodes and 18783 degrees of freedom