FUNCTIONS OF BOUNDED VARIATION AND FREE DISCONTINUITY PROBLEMS
(Oxford Mathematical Monographs)

By Luigi Ambrosio, Nicolo Fucso and Diego Pallara: 434 pp., £55.00, isbn 0-19-850254-1 (Clarendon Press, Oxford, 2000).

Around 1936, L. Tonelli defined, in analogy with the one-dimensional case, a continuous function of two variables to be *of bounded variation* if the surface areas of the projections of the graph of the function onto the vertical coordinate planes (counting multiplicities) are finite. This definition was soon extended by L. Cesari [**1**, **4**] to require only that there exists a Lebesgue representative of the function for which the areas of the projections are finite. This extended definition applies to discontinuous functions, and is independent of coordinates. Note that the variation of a function is related to the area of its graph.

Since then, other variants of the notion of bounded variation have been considered, motivated by various points of view: the theory of distributions, generalized surfaces, variational problems involving surface area, and so forth.

Today, there is an extensive and fascinating theory of functions of bounded variation (BV functions). In the book under review, a BV function is defined as follows. A function $u \in L^1(\Omega)$ is BV in the open set $\Omega \subset \mathbb{R}^n$ if the distributional partial derivatives of u are representable by finite Radon measures; that is, if there exist finite Radon measures $D_i u$ such that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega \varphi \, dD_i u$$

for each $\varphi \in C_c^\infty(\Omega)$ and $1 \leqslant i \leqslant n$. The set of all functions of bounded variation on Ω is denoted by $BV(\Omega)$.

It may not be obvious, but the above definition is equivalent to that of Tonelli–Cesari. Moreover, if u is continuous and Ω has finite n-dimensional Lebesgue measure ($\mathscr{L}^n(\Omega) < \infty$), then u is BV if, and only if, the n-dimensional measure of the graph of u is finite. Furthermore, the Lebesgue surface area of the nonparametric surface corresponding to u is equal to the total variation of the $\mathbb{R}^{n+1}$-valued measure $(\mathscr{L}^n, D_1 u, \ldots, D_n u)$; see [**2**].

The first half (Chapters 1–3) of the book under review provides a systematic and largely self-contained presentation (for example, no prior knowledge of Sobolev spaces is required) of the theory of BV functions (and sets of finite perimeter), and a gentle introduction to geometric measure theory. Among many other things presented in Chapter 3 is a precise description of the structure of the distributional derivative of a BV function. Let $\mathscr{H}^m$ denote m-dimensional Hausdorff measure. For each $u \in BV(\Omega)$, we see that S_u, the set of approximate discontinuities of u, is $\mathscr{H}^{n-1}$-rectifiable and $\mathscr{H}^{n-1}$-almost every point $x \in S_u$ is an approximate jump point of u with direction of jump $v_u(x)$ perpendicular to the approximate tangent space to S_u at x. Denote the set of jump points by J_u. Moreover, u is approximately differentiable $\mathscr{L}^n$ almost everywhere. The distributional derivative of u can be written as

$$Du = \nabla u \mathscr{L}^n + Du \llcorner J_u + D^s u \llcorner (\Omega - S_u),$$

where ∇u is the approximate derivative of u and D^s is the singular part of the measure Du with respect to $\mathscr{L}^n$. The three terms on the right-hand side above are called the *absolutely continuous part*, the *jump part* and the *Cantor part*, respectively, of Du. The jump set J_u is countably rectifiable, and is oriented by the jump vector $\nu_u(x)$. Thus

$$Du \llcorner J_u(B) = \int_{B \cap J_u} \big(u^+(x) - u^-(x)\big) \otimes \nu_u(x)\, d\mathscr{H}^{n-1}$$

whenever B is a Borel subset of Ω. Here $u^\pm(x)$ denote the one-sided limits at a point $x \in J_u$ relative to the normal $\nu_u(x)$.

The second half of the book is directed towards the study of certain variational problems modeled on the Mumford–Shah image-segmentation problem. Suppose that Ω is a bounded domain in $\mathbb{R}^n$, that $g \in L^\infty(\Omega)$, and that α and β are positive parameters. Let

$$J(K, u) := \int_{\omega - K} |\nabla u|^2 + \alpha(u - g)\, dx + \beta \mathscr{H}^{n-1}(K \cap \Omega).$$

The Mumford–Shah problem is to minimize J among all admissible pairs

$$\mathscr{A} := \left\{(K, u) \;:\; K \subset \overline{\Omega} \text{ closed},\; u \in W^{1,2}_{\mathrm{loc}}(\Omega - K)\right\}.$$

In two dimensions, this problem was proposed by D. Mumford and J. Shah [**3**] as a variational formulation of the image-segmentation problem in computer vision.

Chapter 4 studies the subspace of BV called SBV (for *special* BV), consisting of those BV functions whose distributional derivatives have zero Cantor part. These functions provide an appropriate setting for certain types of variational problems that involve both volume and surface energies. In particular, important closure and compactness properties hold for minimizing sequences. Chapter 5 establishes lower semicontinuity with respect to weak* convergence in BV for various integral functionals. Chapter 6 surveys in detail the state of the current knowledge regarding the Mumford–Shah image-segmentation problem. The last two chapters present existence and regularity results for the minimization problems modeled on the Mumford–Shah problem. The presentation in the second half of the book benefits in its clarity from the fact that the authors have contributed significantly to the subject.

This book provides an excellent account of the theory of BV functions (apparently for the first time in book form), and a nice introduction to geometric measure theory, as well as a rigorous survey of results for 'free discontinuity' problems modeled on the Mumford–Shah problem. It should serve as a standard reference, especially for the BV theory, for years to come.

References

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