

## 0.1 The continuous and discrete problem

Let  $\alpha > 0$ ,  $\Omega \subset \mathbb{R}^n$  bounded polyhedral Lipschitz domain, and  $f \in L^2(\Omega)$ .

The continuous problem minimizes

$$E(v) := \frac{\alpha}{2} \|v\|_{L^2(\Omega)}^2 + |v|_{\text{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} - \int_{\Omega} f v \, dx \quad (0.1)$$

amongst all  $v \in V := \text{BV}(\Omega) \cap L^2(\Omega)$  where the BVseminorm  $|v|_{\text{BV}(\Omega)}$  is equal to the  $W^{1,1}$  seminorm for any  $v \in W^{1,1}(\Omega)$ .

The nonconforming problem minimizes

$$E_{\text{NC}}(v_{\text{CR}}) := \frac{\alpha}{2} \|v_{\text{CR}}\|_{L^2(\Omega)}^2 + |v_{\text{CR}}|_{1,1,\text{NC}} - \int_{\Omega} f v_{\text{CR}} \, dx \quad (0.2)$$

amongst all  $v_{\text{CR}} \in \text{CR}_0^1(\mathcal{T})$  where  $|\cdot|_{1,1,\text{NC}} := \|\nabla_{\text{NC}} \cdot\|_{L^1(\Omega)}$ .

## 0.2 Refinement indicator and guaranteed lower energy bound

For some  $n \in \mathbb{N}$  (here  $n = 2$ ) and  $0 < \beta \leq 1$  define a refinement indicator  $\eta := \sum_{T \in \mathcal{T}} \eta(T)$  with

$$\eta(T) := \underbrace{|T|^{2/n} \|f - \alpha u_{\text{CR}}\|_{L^2(T)}^2}_{=: \eta_{\text{Vol}}(T)} + \underbrace{|T|^{\beta/n} \sum_{F \in \mathcal{F}(T)} \|[u_{\text{CR}}]_F\|_{L^1(F)}}_{=: \eta_{\text{Jumps}}(T)} \quad (0.3)$$

for any  $T \in \mathcal{T}$ .

For  $f \in H_0^1(\Omega)$  and  $u \in H_0^1(\Omega)$  ( $u_{\text{CR}} \in \text{CR}_0^1(\Omega)$ ) continuous (discrete) minimizer with minimal energy  $E(u)$  ( $E_{\text{NC}}(u_{\text{CR}})$ ) it holds

$$E_{\text{NC}}(u_{\text{CR}}) + \frac{\alpha}{2} \|u - u_{\text{CR}}\|_{L^2(\Omega)}^2 - \frac{\kappa_{\text{CR}}}{\alpha} \|h_{\mathcal{T}}(f - \alpha u_{\text{CR}})\|_{L^2(\Omega)} |f|_{1,2} \leq E(u) \quad (0.4)$$

where  $|\cdot|_{1,2} = \|\nabla \cdot\|_{L^2(\Omega)}$ .

Hence, for  $\text{GLEB} := E_{\text{NC}}(u_{\text{CR}}) - \frac{\kappa_{\text{CR}}}{\alpha} \|h_{\mathcal{T}}(f - \alpha u_{\text{CR}})\|_{L^2(\Omega)} |f|_{1,2}$ , it holds  $E_{\text{NC}}(u_{\text{CR}}) \geq \text{GLEB}$  and  $E(u) \geq \text{GLEB}$ .

## 0.3 Experiments

In the following sections it holds  $\beta = 1$  and  $\varepsilon = 10^{-5}$ .

## 0.4 Example with exact solution

For  $\alpha = 1$  define  $f$  as a function of the radius as

$$f(r) := \begin{cases} \alpha - 12(2 - 9r) & \text{if } 0 \leq r \leq \frac{1}{6}, \\ \alpha(1 + (6r - 1)^\beta) - \frac{1}{r} & \text{if } \frac{1}{6} \leq r \leq \frac{1}{3}, \\ 2\alpha + 6\pi \sin(\pi(6r - 2)) - \frac{1}{r} \cos(\pi(6r - 2)) & \text{if } \frac{1}{3} \leq r \leq \frac{1}{2}, \\ 2\alpha(\frac{5}{2} - 3r)^\beta + \frac{1}{r} & \text{if } \frac{1}{2} \leq r \leq \frac{5}{6}, \\ -3\pi \sin(\pi(6r - 5)) + \frac{1 + \cos(\pi(6r - 5))}{2r} & \text{if } \frac{5}{6} \leq r \leq 1, \end{cases} \quad (0.5)$$

with exact solution

$$u(r) := \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{1}{6}, \\ 1 + (6r - 1)^\beta & \text{if } \frac{1}{6} \leq r \leq \frac{1}{3}, \\ 2 & \text{if } \frac{1}{3} \leq r \leq \frac{1}{2}, \\ 2(\frac{5}{2} - 3r)^\beta & \text{if } \frac{1}{2} \leq r \leq \frac{5}{6}, \\ 0 & \text{if } \frac{5}{6} \leq r \leq 1. \end{cases} \quad (0.6)$$

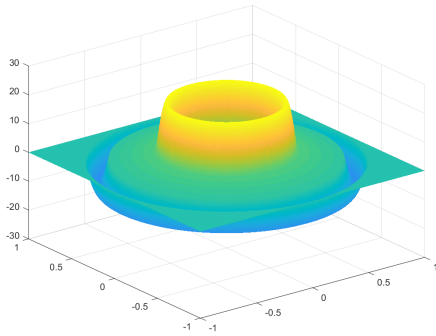


Figure 0.1: right-hand side  $f$

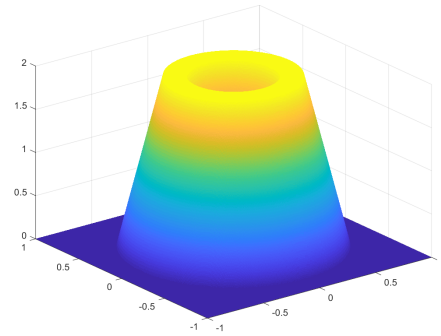


Figure 0.2: exact solution  $u$

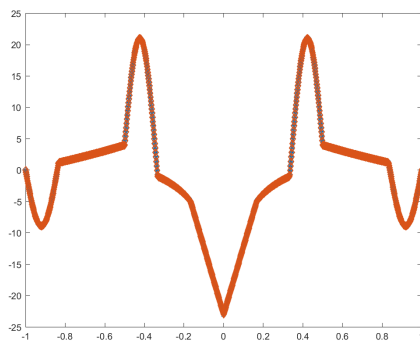


Figure 0.3: right-hand side  $f$  along the axes

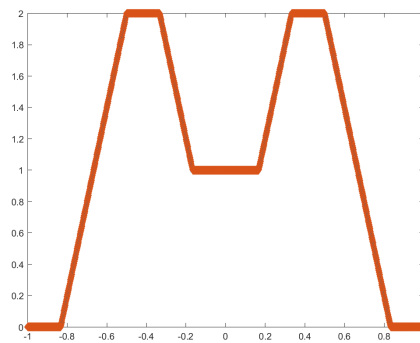


Figure 0.4: exact solution  $u$  along the axes

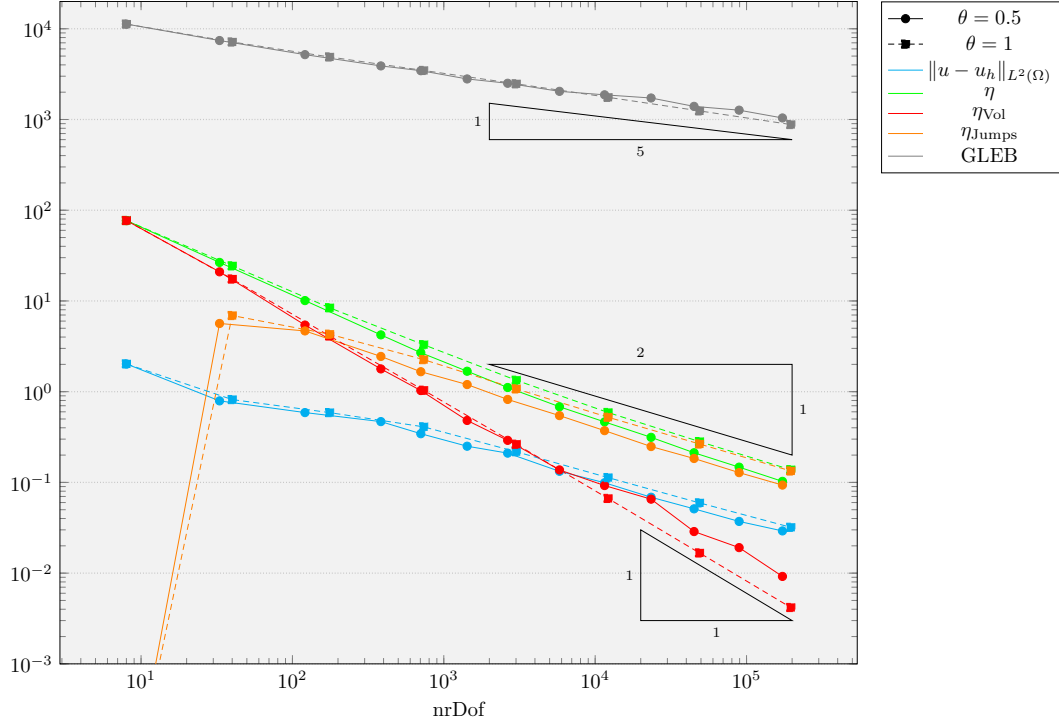


Figure 0.5: convergence history plot for the  $L^2$  error,  $\eta$ ,  $\eta_{\text{Vol}}$ ,  $\eta_{\text{Jumps}}$ , and the guaranteed lower energy bound

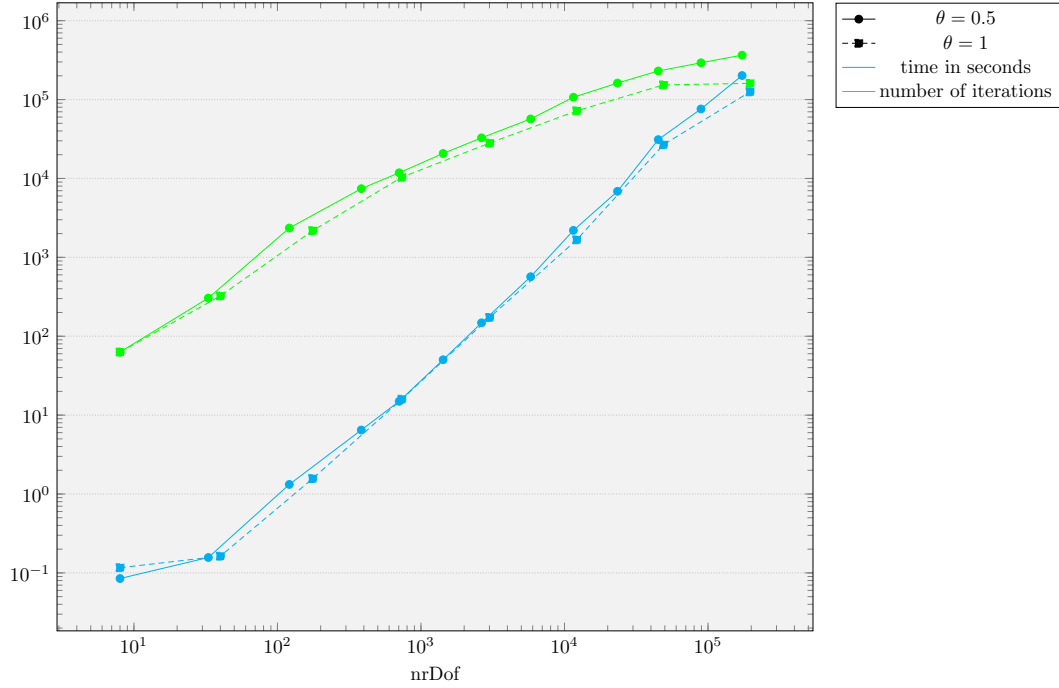


Figure 0.6: development of the number of iterations and the elapsed time for each iteration

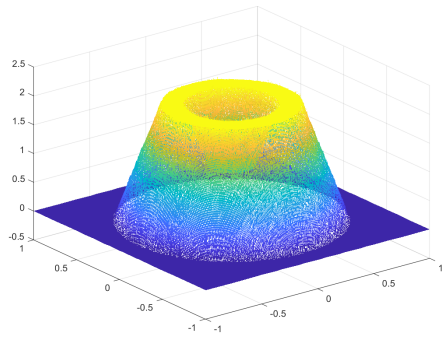


Figure 0.7: last iterate for  $\theta = 0.5$

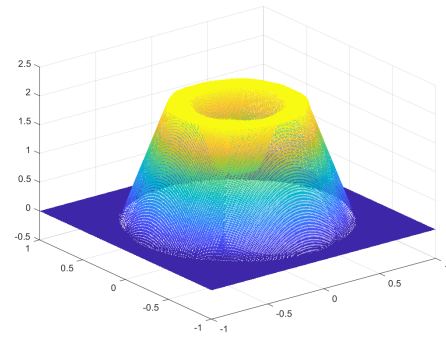


Figure 0.8: last iterate for  $\theta = 1$

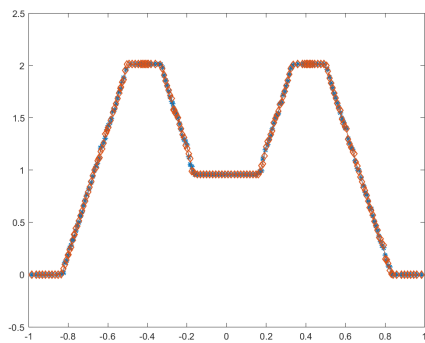


Figure 0.9: last iterate along the axes  
for  $\theta = 0.5$

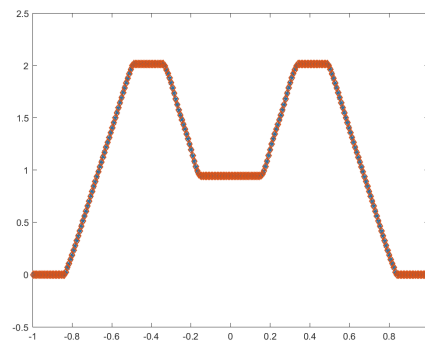


Figure 0.10: last iterate along the  
axes for  $\theta = 1$

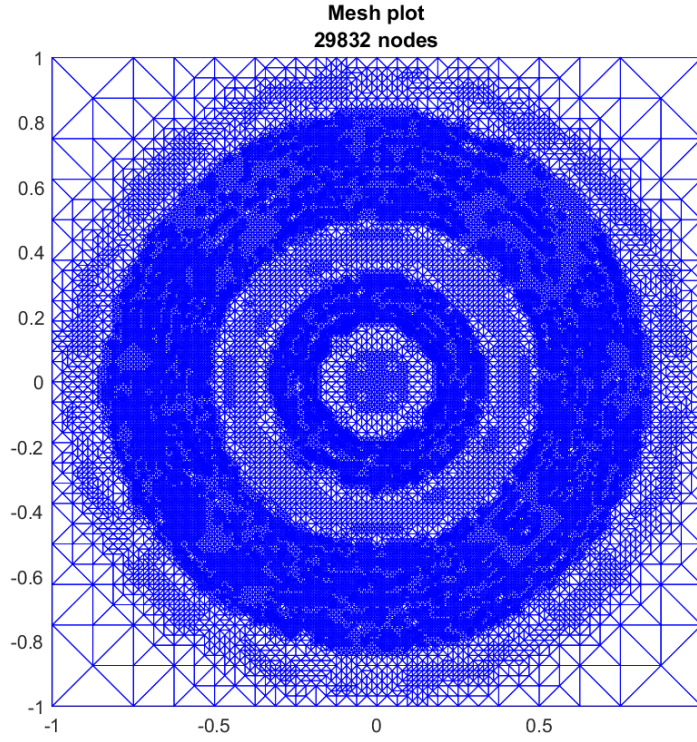


Figure 0.11: adaptive mesh for  $\theta = 0.5$  with 29832 nodes and 89317 degrees of freedom

## 0.5 Application to an image

For  $\alpha = 10000$  let  $f$  represent the grayscale of an image in  $[0, 1]^{256 \times 256}$  scaled to the domain  $\Omega \in (0, 1)^2$  as seen in fig. 0.12.



Figure 0.12: grayscale plot of the right-hand side  $f$  (view from above onto the  $x$ - $y$  plane)

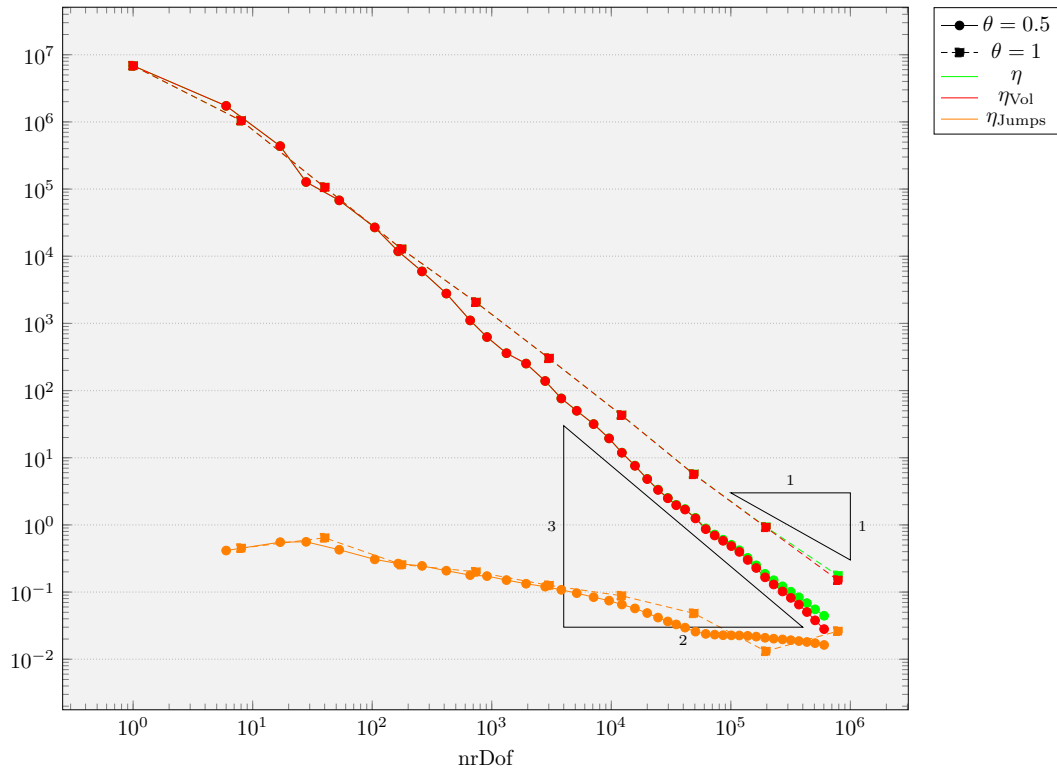


Figure 0.13: convergence history plot for  $\eta$ ,  $\eta_{Vol}$ , and  $\eta_{Jumps}$

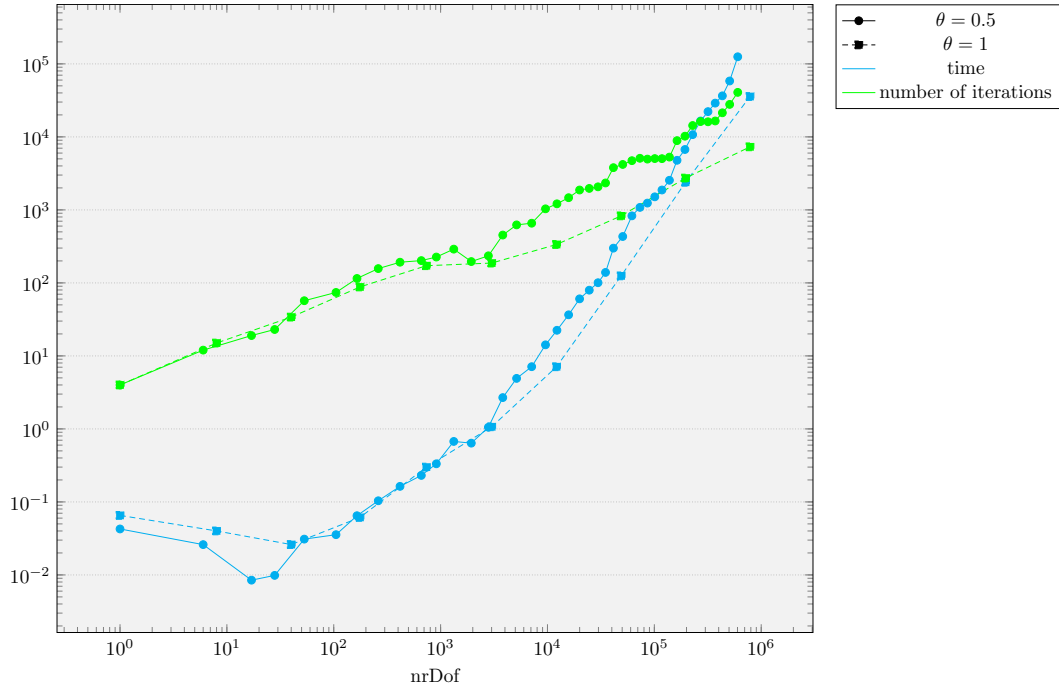


Figure 0.14: development of the number of iterations and the elapsed time for each iteration



Figure 0.15: grayscale plot of last iterate for  $\theta = 0.5$  Figure 0.16: grayscale plot of last iterate for  $\theta = 1$

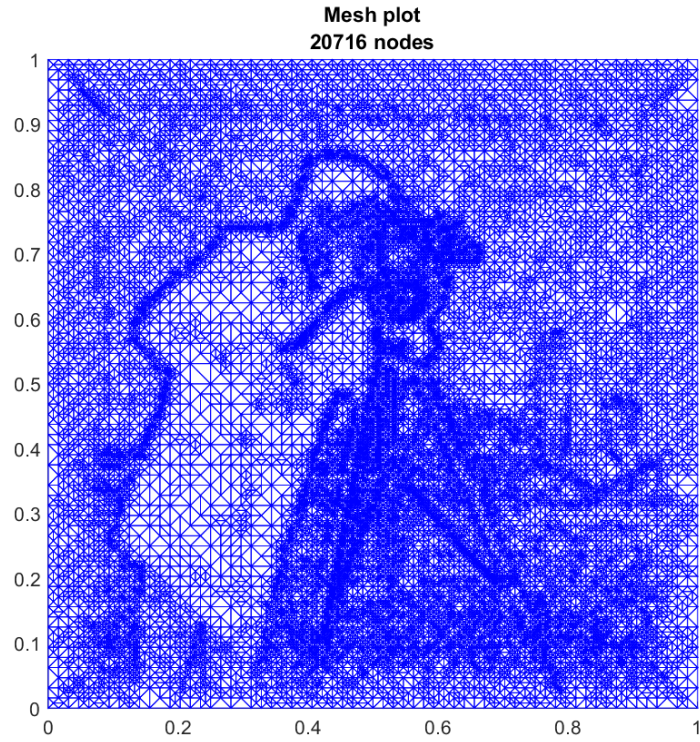


Figure 0.17: adaptive mesh for  $\theta = 0.5$  with 20716 nodes and 61657 degrees of freedom

## 0.6 Application to a function with discontinuity set

For  $\alpha = 100$  define

$$f(x) := \begin{cases} 100 & \text{if } \|x\|_{\infty} \leq \frac{1}{2}, \\ 0 & \text{else} \end{cases} \quad (0.7)$$

on  $\Omega = (-1, 1)^2$ .



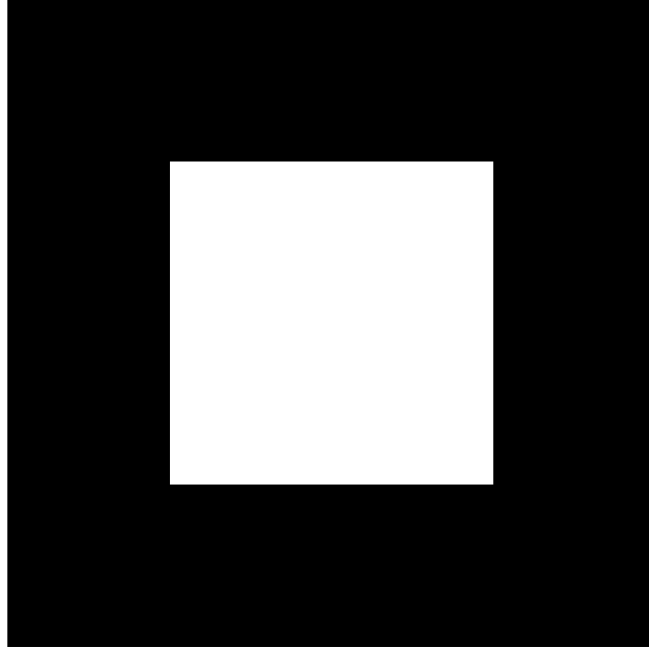


Figure 0.18: grayscale plot of the right-hand side  $f$  (view from above onto the  $x$ - $y$  plane)

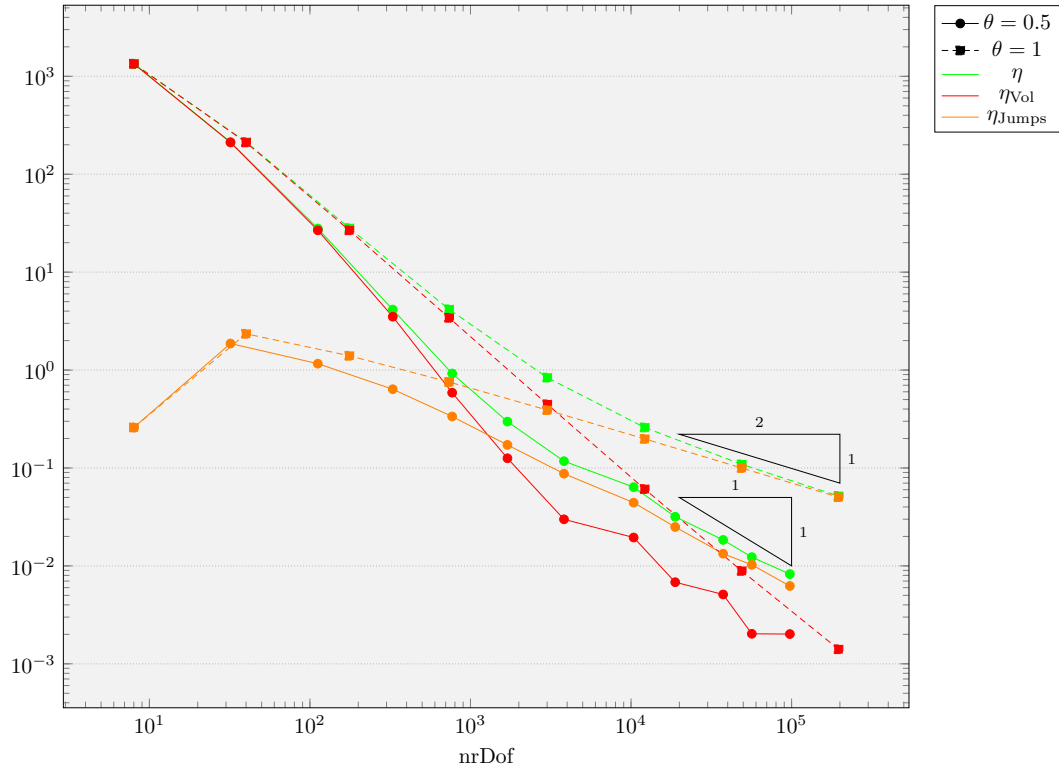


Figure 0.19: convergence history plot for  $\eta$ ,  $\eta_{\text{Vol}}$ , and  $\eta_{\text{Jumps}}$

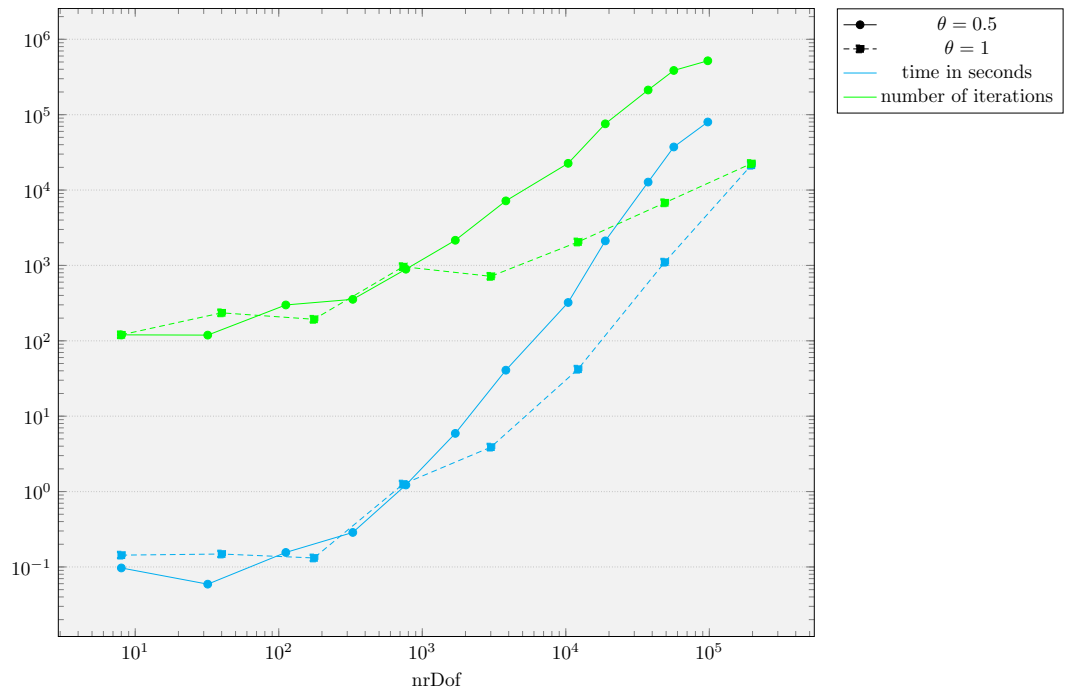


Figure 0.20: development of the number of iterations and the elapsed time for each iteration

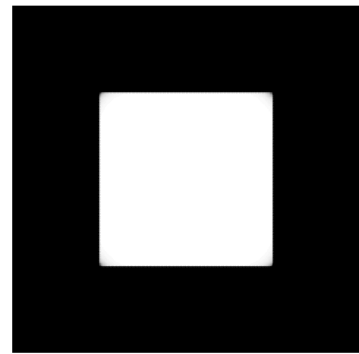
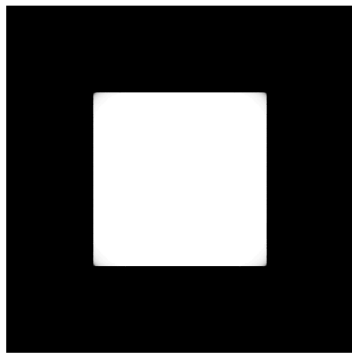


Figure 0.21: grayscale plot of last iterate for  $\theta = 0.5$  Figure 0.22: grayscale plot of last iterate for  $\theta = 1$

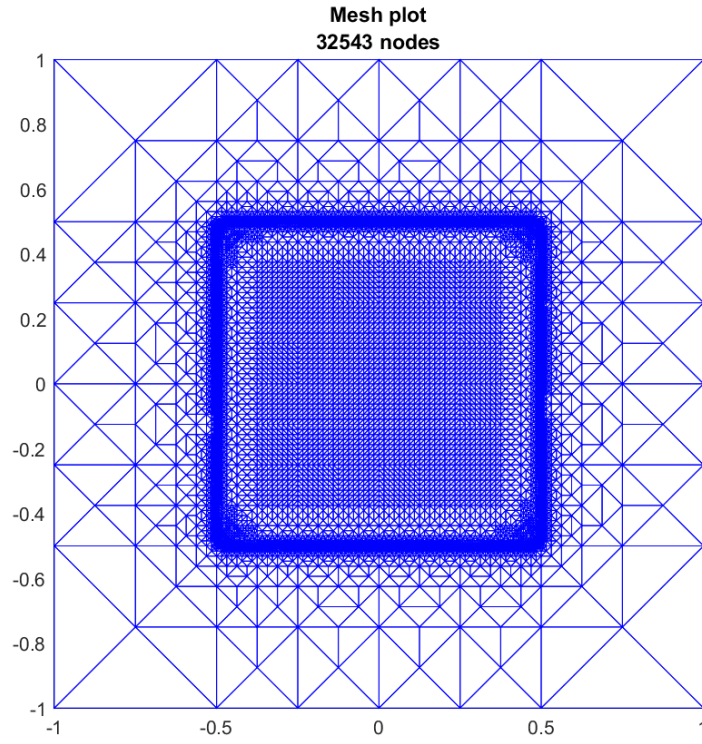


Figure 0.23: adaptive mesh for  $\theta = 0.5$  with 32543 nodes and 97578 degrees of freedom