Chapter 13

Integral functionals of the calculus of variations

This chapter is devoted to the study of the sequential lower semicontinuity of certain types of functionals which occur in many variational problems. As noted in Chapter 11, lower semicontinuity is the key tool to apply the direct methods of the calculus of variations, and we deal in the sections below with some different cases, depending on the spaces the functionals are defined on. We will see that, due to the integral form of the functionals under consideration, the convexity or quasi-convexity conditions play a central role in all the results. We first complement Section 11.2 by establishing necessary and sufficient conditions for more general integral functionals to be lower semicontinuous. Then to complement Section 11.3, we deal with lower semicontinuity of functionals defined on the space of measures, on BV and SBV. We do not pretend to be exhaustive in this very widely studied field; we intend only to give here some principal results. For other cases and details see [26], [147], [153], [182], [225], [302].

13.1 ▪ Lower semicontinuity in the scalar case

In this section we consider integral functionals of the form

$$F(u)=\int_{\Omega} f(x,u,Du)\,dx, \tag{13.1}$$

where u varies on a Sobolev space $W^{1,p}(\Omega)$. We stress the fact that in this section we restrict our attention to the case of functions u which take their values in $\mathbf{R}$; some differences with the case of functionals defined on vector valued functions will be discussed in the next section.

To study the sufficient conditions for the lower semicontinuity of functionals of the form (13.1), it is convenient to consider first the case of functionals of the form

$$F(u,v)=\int_{\Omega} f\big(x,u(x),v(x)\big)\,d\mu(x), \tag{13.2}$$

where $(\Omega,\mathscr{A},\mu)$ is a measure space with the measure μ nonnegative and finite (i.e., $\mu\in\mathbf{M}^+(\Omega)$), $f:\Omega\times\mathbf{R}^m\times\mathbf{R}^n\to[0,+\infty]$ is an $\mathscr{A}\otimes\mathbf{B}_m\otimes\mathbf{B}_n$-measurable function ($\mathbf{B}_m$ and $\mathbf{B}_n$, respectively, denote the σ-algebras of Borel subsets of $\mathbf{R}^m$ and $\mathbf{R}^n$), and u, v, respectively, vary in the spaces $L^1_\mu(\Omega;\mathbf{R}^m)$, $L^1_\mu(\Omega;\mathbf{R}^n)$ of μ integrable $\mathbf{R}^m$, $\mathbf{R}^n$ valued functions.

The theorem below is a lower semicontinuity result for functionals of the form (13.2); the link with the case (13.1) will be discussed later.

Theorem 13.1.1. *Assume the function f satisfies the following conditions:*

(i) *for μ-a.e. $x \in \Omega$ the function $f(x,\cdot,\cdot)$ is lower semicontinuous on $\mathbf{R}^m \times \mathbf{R}^n$;*

(ii) *for μ-a.e. $x \in \Omega$ and for every $s \in \mathbf{R}^m$ the function $f(x,s,\cdot)$ is convex on $\mathbf{R}^n$.*

Then the functional F defined in (13.2) *is sequentially lower semicontinuous on the space $L^1_\mu(\Omega;\mathbf{R}^m) \times L^1_\mu(\Omega;\mathbf{R}^n)$ endowed with the strong topology on $L^1_\mu(\Omega;\mathbf{R}^m)$ and the weak topology on $L^1_\mu(\Omega;\mathbf{R}^n)$.*

PROOF. Let $u_h \to u$ strongly in $L^1_\mu(\Omega;\mathbf{R}^m)$ and $v_h \to v$ weakly in $L^1_\mu(\Omega;\mathbf{R}^n)$; we have to prove that

$$F(u,v) \le \liminf_{h\to+\infty} F(u_h,v_h). \tag{13.3}$$

Possibly passing to subsequences, we may assume without loss of generality that the lim inf in the right-hand side of (13.3) is a finite limit, that is,

$$\lim_{h\to+\infty} F(u_h,v_h) = c \in \mathbf{R}. \tag{13.4}$$

Since the sequence (v_h) is weakly compact in $L^1_\mu(\Omega;\mathbf{R}^n)$, we may use the Dunford–Pettis theorem, Theorem 2.4.5, and the De La Vallée–Poussin criterion, Theorem 2.4.4, to conclude that there exists a function $\vartheta : [0,+\infty[\to [0,+\infty[$, which can be taken convex and strictly increasing, with a superlinear growth, that is,

$$\lim_{t\to+\infty} \frac{\vartheta(t)}{t} = +\infty,$$

such that

$$\sup_{h\in\mathbf{N}} \int_\Omega \vartheta(|v_h|)\,d\mu \le 1. \tag{13.5}$$

Setting

$$\begin{cases} H(t) = \sqrt{t\vartheta(t)}, \\ \Phi(t) = \vartheta\big(H^{-1}(t)\big), \\ \xi_h(x) = H\big(|v_h(x)|\big), \end{cases}$$

it is easy to see that

(i) H is strictly increasing and $H(t)/t \to +\infty$ as $t \to +\infty$,

(ii) Φ is strictly increasing and $\Phi(t)/t \to +\infty$ as $t \to +\infty$,

(iii) $\vartheta(t)/H(t) \to +\infty$ as $t \to +\infty$,

(iv) $\Phi\big(\xi_h(x)\big) = \vartheta\big(|v_h(x)|\big)$.

Therefore, by (13.5) we have

$$\sup_{h\in\mathbf{N}} \int_\Omega \Phi(\xi_h)\,d\mu \le 1.$$

We can use now the Dunford–Pettis theorem again to deduce that the sequence (ξ_h) is weakly compact in $L^1_\mu(\Omega)$, hence (up to extracting a subsequence) we may assume that $\xi_h \to \eta$ weakly in $L^1_\mu(\Omega)$ for a suitable η. By the Mazur theorem, a suitable sequence of convex combinations of (ξ_h, v_h) is strongly convergent in $L^1_\mu(\Omega) \times L^1_\mu(\Omega;\mathbf{R}^n)$ to (η, v). More precisely, there exist $N_h \to +\infty$ and $\alpha_{i,h} \geq 0$ with

$$\sum_{i=N_h+1}^{N_{h+1}} \alpha_{i,h} = 1$$

such that the sequences

$$\eta_h = \sum_{i=N_h+1}^{N_{h+1}} \alpha_{i,h}\xi_i, \qquad \nu_h = \sum_{i=N_h+1}^{N_{h+1}} \alpha_{i,h} v_i$$

strongly converge to η in $L^1_\mu(\Omega)$ and to v $L^1_\mu(\Omega;\mathbf{R}^n)$, respectively. Possibly passing to subsequences we may also assume that $\eta_h \to \eta$, $\nu_h \to v$, and $u_h \to u$ pointwise μ-a.e. on Ω.

Consider now a point $x \in \Omega$ where all the convergences above occur, and set

$$\begin{cases} \varepsilon_h = \max\big\{|u(x) - u_i(x)| \,:\, N_h < i \leq N_{h+1}\big\}, \\ \lambda_h = \displaystyle\sum_{i=N_h+1}^{N_{h+1}} \alpha_{i,h} f\big(x, u_i(x), v_i(x)\big), \\ \mathscr{A}_h = \big\{(\nu, \eta, \lambda) \in \mathbf{R}^{n+2} \,:\, \eta = H(|\nu|),\ \exists s \in \mathbf{R}^m,\ |s - u(x)| \leq \varepsilon_h,\ \lambda \geq f(x, s, \nu)\big\}. \end{cases}$$

We have $\varepsilon_h \to 0$ and by definition of ν_h, η_h, λ_h we obtain that $\big(\nu_h(x), \eta_h(x), \lambda_h(x)\big)$ belongs to the convex hull $co\mathscr{A}_h$ of $\mathscr{A}_h$. Since $\mathscr{A}_h \subset \mathbf{R}^{n+2}$, by the Carathéodory theorem on convex hulls in Euclidean spaces the vector $\big(\nu_h(x), \eta_h(x), \lambda_h(x)\big)$ can be written as a convex combination of $n+3$ elements of $\mathscr{A}_h$, that is, there exist

$$\beta_{i,h} \geq 0, \quad \nu_{i,h} \in \mathbf{R}^n, \quad \eta_{i,h} \geq 0, \quad \lambda_{i,h} \geq 0 \qquad (i = 1, \dots, n+3)$$

such that $(\nu_{i,h}, \eta_{i,h}, \lambda_{i,h}) \in \mathscr{A}_h$ for every index i, and

$$\begin{cases} \displaystyle\sum_{i=1}^{n+3} \beta_{i,h} = 1, \qquad \sum_{i=1}^{n+3} \beta_{i,h}\nu_{i,h} = \nu_h(x), \\ \displaystyle\sum_{i=1}^{n+3} \beta_{i,h}\eta_{i,h} = \eta_h(x), \qquad \sum_{i=1}^{n+3} \beta_{i,h}\lambda_{i,h} = \lambda_h(x). \end{cases}$$

Therefore, for suitable $s_{i,h} \in \mathbf{R}^m$ with $|s_{i,h} - u(x)| \leq \varepsilon_h$ we have

$$\lambda_{i,h} \geq f(x, s_{i,h}, \nu_{i,h}).$$

By extracting subsequences, without loss of generality, we may assume that for every index i the sequence $|\nu_{i,h}|$ tends to a limit and, denoting by I the set of indices i such that this limit is finite, again by passing to subsequences, we may also assume that

$$\begin{cases} \nu_{i,h} \to \nu_i & \forall i \in I, \\ |\nu_{i,h}| \to +\infty & \forall i \notin I, \\ \beta_{i,h} \to \beta_i & \forall i = 1, \dots, n+3. \end{cases}$$

Since

$$\sum_{i=1}^{n+3} \beta_{i,h} H(|\nu_{i,h}|) = \eta_h(x) \to \eta(x),$$

the set I cannot be empty. From the relation

$$\sum_{i=1}^{n+3} \beta_{i,h} \nu_{i,h} = \nu_h(x) \to v(x),$$

we obtain that $\beta_i = 0$ for every $i \notin I$. Moreover, from

$$\eta_h(x) = \sum_{i=1}^{n+3} \beta_{i,h} \eta_{i,h} \geq \sum_{i \notin I} \beta_{i,h} \eta_{i,h} = \sum_{i \notin I} \beta_{i,h} |\nu_{i,h}| \frac{H(|\nu_{i,h}|)}{|\nu_{i,h}|}$$

we get

$$\beta_{i,h} |\nu_{i,h}| \to 0 \qquad \forall i \notin I$$

so that

$$\sum_{i \in I} \beta_i = 1, \qquad \sum_{i \in I} \beta_i \nu_i = v(x).$$

We now use the assumptions on the function f to obtain

$$\begin{aligned} f\big(x, u(x), v(x)\big) &\leq \sum_{i \in I} \beta_i f\big(x, u(x), \nu_i\big) \\ &\leq \liminf_{h \to +\infty} \sum_{i \in I} \beta_{i,h} f(x, s_{i,h}, \nu_{i,h}) \\ &\leq \liminf_{h \to +\infty} \sum_{i=1}^{n+3} \beta_{i,h} f(x, s_{i,h}, \nu_{i,h}) \\ &\leq \liminf_{h \to +\infty} \lambda_h(x) \end{aligned}$$

so that by Fatou's lemma,

$$\int_\Omega f(x, u, v)\, d\mu \leq \liminf_{h \to +\infty} \int_\Omega \lambda_h(x)\, d\mu \tag{13.6}$$

$$= \liminf_{h \to +\infty} \sum_{i=N_h+1}^{N_{h+1}} \alpha_{i,h} \int_\Omega f(x, u_i, v_i)\, d\mu. \tag{13.7}$$

Fix now $\varepsilon > 0$; by using (13.4) we obtain, for h large enough,

$$\int_\Omega f(x, u_i, v_i)\, d\mu \leq c + \varepsilon \qquad \forall i \in [N_h + 1, N_{h+1}]$$

and so by (13.7) $F(u, v) \leq c + \varepsilon$. The proof is then achieved by taking $\varepsilon \to 0^+$. ◻

Remark 13.1.1. It is easy to see that the result of Theorem 13.1.1 remains true if the measure μ is only assumed to be σ-finite.

The result above, under the slightly stronger assumption that $f(x, \cdot, \cdot)$ is continuous for μ-a.e. $x \in \Omega$, was first obtained by De Giorgi in 1968 in an unpublished paper. The original proof by De Giorgi is obtained by approximating from below the convex function $f(x, s, \cdot)$ by finite suprema of affine functions $f_k(x, s, \cdot)$ for which the proof is easier, and then passing to the limit as $k \to +\infty$; the interested reader may find further details about this type of proof in the book by Buttazzo [147]. The proof reported above follows on the contrary the scheme of the proof which was given in 1977 by Ioffe [246].

By using the result of Theorem 13.1.1 we can easily give some sufficient conditions for the sequential lower semicontinuity of functionals of the form (13.1) on the Sobolev space $W^{1,1}(\Omega)$. More precisely, the following result holds.

Theorem 13.1.2. *Let Ω be a Lipschitz domain of $\mathbf{R}^n$ and let $f : \Omega \times \mathbf{R}^m \times \mathbf{R}^{mn} \to [0,+\infty]$ be a function verifying the assumptions of Theorem* 13.1.1. *Then the functional F defined in* (13.1) *is sequentially weakly lower semicontinuous on the Sobolev space $W^{1,1}(\Omega;\mathbf{R}^m)$.*

PROOF. Let (u_h) be a sequence in $W^{1,1}(\Omega;\mathbf{R}^m)$ converging weakly to some function u. Setting $v_h = Du_h$ we have that v_h converges weakly to $v = Du$ in $L^1(\Omega;\mathbf{R}^{mn})$ and, by the Rellich theorem, u_h converge strongly to u in $L^1(\Omega;\mathbf{R}^m)$. By Theorem 13.1.1 we have

$$F(u) = \int_\Omega f(x,u,v)\,dx \le \liminf_{h\to+\infty} \int_\Omega f(x,u_h,v_h)\,dx = \liminf_{h\to+\infty} F(u_h),$$

which then proves the assertion. □

In the so-called scalar case (i.e., when $m = 1$), and in the case of ordinary integrals as well (i.e., when $n = 1$), we will prove below that the convexity of the integrand f with respect to the gradient is also a necessary condition for the semicontinuity. This is no longer true in the vector valued case $m > 1$ for multiple integrals, as we will discuss in the next section. For the sake of simplicity we here limit ourselves to consider only the case of functionals of the form

$$F(u) = \int_\Omega f(Du)\,dx; \tag{13.8}$$

the more general case (13.1) presents only technical differences in the proof, and we refer to one of the books mentioned at the beginning of this chapter for the details.

Theorem 13.1.3. *Assume that either $m = 1$ or $n = 1$ and that the functional F in* (13.8) *is sequentially weakly* lower semicontinuous in the Sobolev space $W^{1,\infty}$, in the sense that*

$$F(u) \le \liminf_{h\to+\infty} F(u_h) \tag{13.9}$$

for every sequence u_h converging to u uniformly in Ω and with Du_h uniformly bounded in Ω. Then the function f is convex and lower semicontinuous.

PROOF. We give the proof only in the case $m = 1$, the other one being similar. Let $z_1, z_2 \in \mathbf{R}^n$, let $t \in]0,1[$, and let $z = tz_1 + (1-t)z_2$. Denote the linear function $u_z(x) = z\cdot x$ by u_z and define

$$\begin{cases} z_0 = \dfrac{z_2 - z_1}{|z_2 - z_1|}, \\ \Omega^1_{hj} = \left\{x \in \Omega \,:\, \frac{j-1}{h} < z_0 \cdot x < \frac{j-1+t}{h}\right\}, \qquad j \in \mathbf{Z},\ h \in \mathbf{N}, \\ \Omega^2_{hj} = \left\{x \in \Omega \,:\, \frac{j-1+t}{h} < z_0 \cdot x < \frac{j}{h}\right\}, \qquad j \in \mathbf{Z},\ h \in \mathbf{N}, \\ \Omega^1_h = \bigcup\left\{\Omega^1_{hj} \,:\, j \in \mathbf{Z}\right\}, \\ \Omega^2_h = \bigcup\left\{\Omega^2_{hj} \,:\, j \in \mathbf{Z}\right\}, \\ u_h(x) = \begin{cases} c^1_{hj} + z_1 \cdot x & \text{if } x \in \Omega^1_{hj}, \\ c^2_{hj} + z_2 \cdot x & \text{if } x \in \Omega^2_{hj}, \end{cases} \end{cases}$$

where
$$c^1_{hj} = \frac{(j-1)(1-t)}{h}|z_2 - z_1|, \qquad c^2_{hj} = -\frac{jt}{h}|z_2 - z_1|.$$
It is easy to verify that, as $h \to +\infty$,
$$\frac{\text{meas}(\Omega^1_h)}{\text{meas}(\Omega)} \to t, \qquad \frac{\text{meas}(\Omega^2_h)}{\text{meas}(\Omega)} \to 1-t. \tag{13.10}$$
Moreover, the functions u_h are Lipschitz continuous and for every $x \in \Omega^1_{hj}$ we have
$$\begin{aligned}|u_h(x) - u_z(x)| = |c^1_{hj} + (z_1 - z)\cdot x| &= (1-t)\left|\frac{j-1}{h}|z_2 - z_1| + (z_1 - z_2)\cdot x\right| \\ &= (1-t)|z_2 - z_1|\left|\frac{j-1}{h} - z_0 \cdot x\right| \le \frac{t(1-t)}{h}|z_2 - z_1|.\end{aligned}$$
Analogously, a similar computation gives for every $x \in \Omega^2_{hj}$
$$|u_h(x) - u_z(x)| \le \frac{t(1-t)}{h}|z_2 - z_1|.$$
Therefore $u_h \to u_z$ uniformly on Ω. Moreover, it is immediate to see that the gradients Du_h are uniformly bounded on Ω, so that the sequence (u_h) converges to u_z weakly* in $W^{1,\infty}(\Omega)$. By the assumption (13.9), using also (13.10), we then obtain
$$\begin{aligned} f(z)\,\text{meas}(\Omega) = F(u_z) &\le \liminf_{h\to+\infty} F(u_h) \\ &= \liminf_{h\to+\infty}\Big(f(z_1)\,\text{meas}(\Omega^1_h) + f(z_2)\,\text{meas}(\Omega^2_h)\Big) \\ &= t f(z_1)\,\text{meas}(\Omega) + (1-t) f(z_2)\,\text{meas}(\Omega), \end{aligned}$$
which proves the convexity of f. The lower semicontinuity of f on $\mathbf{R}^n$ is a straightforward consequence of the sequential lower semicontinuity assumption (13.9) on F. □

13.2 ▪ Lower semicontinuity in the vectorial case

We have seen in the previous section that the convexity assumption on the integrand $f(x,s,\cdot)$ is necessary and sufficient for the sequential weak lower semicontinuity of the functional
$$F(u) = \int_\Omega f(x,u,Du)\,dx \tag{13.11}$$
in the case of scalar functions u. On the contrary, in the case of functions u with vector values, the convexity of the integrand with respect to the gradient variable describes only a small class of weakly lower semicontinuous functionals. For this reason, according to Morrey [302] we introduce the notion of quasi-convexity defined in Chapter 11 in the context of relaxation theory.

Definition 13.2.1. *A Borel function* $f : \mathbf{R}^{mn} \to [0,+\infty]$ *is said to be quasi-convex if*
$$f(z)\,\text{meas}(A) \le \int_A f\big(z + D\phi(x)\big)\,dx \tag{13.12}$$

for a suitable (hence for all) bounded open subset A of $\mathbf{R}^n$, every $m \times n$ matrix z, and every $\phi \in \mathbf{C}_0^1(A;\mathbf{R}^m)$.

Remark 13.2.1. It is possible to prove that when either $m = 1$ or $n = 1$ quasi-convexity reduces to the usual convexity; this will actually follow from Theorem 13.1.3 once the equivalence between lower semicontinuity and quasi-convexity will be proved. On the other hand, there are many examples of quasi-convex functions which are not convex, as, for instance, the function

$$z \mapsto f(z) = |\det z|.$$

If $f : \mathbf{R}^{mn} \to [0,+\infty]$ is quasi-convex, then for every $z \in \mathbf{R}^{mn}$ and $s \in \mathbf{R}^m$ the function $\phi_{s,z} : \mathbf{R}^n \to [0,+\infty]$ defined by

$$\phi_{s,z}(\xi) = f(z + s \otimes \xi) \tag{13.13}$$

is convex. This fact follows again from the results of the previous section on the scalar case, once we remark that by (13.12) we obtain

$$\phi_{s,z}(\xi)\,\mathrm{meas}(A) \le \int_A \phi_{s,z}\big(\xi + D\phi(x)\big)\,dx$$

for every vector $\xi \in \mathbf{R}^n$ and every scalar function $\phi \in \mathbf{C}_0^1(A)$. The property given by (13.13) is called rank-one convexity. From the convexity of the functions $\phi_{s,z}$ defined in (13.13), we obtain that every rank-one convex function f of class $\mathbf{C}^2(\mathbf{R}^{mn})$ satisfies the so-called Legendre–Hadamard condition:

$$\frac{\partial^2 f}{\partial z_{ij}\partial z_{hk}}(z_0)\alpha_i\alpha_h\beta_j\beta_k \ge 0$$

for all $z_0 \in \mathbf{R}^{mn}$ and for all $\alpha \in \mathbf{R}^m$, $\beta \in \mathbf{R}^n$. (The summation convention over repeated indices is adopted.) Moreover, we get that every rank-one convex finite-valued function is locally Lipschitz on $\mathbf{R}^{mn}$. This is made precise in Lemma 13.2.1.

Remark 13.2.2. A wide class of quasi-convex functions is the class of polyconvex functions, introduced by Ball in [79]. A function f is called polyconvex if it can be written in the form

$$f(z) = g\big(X(z)\big) \qquad \forall z \in \mathbf{R}^{m\times n},$$

where $X(z)$ denotes the vector of all subdeterminants of the matrix z and g is a convex function. For instance, if $m = n = 2$, every polyconvex function is of the form $g(z, \det z)$ with g convex on $\mathbf{R}^4 \times \mathbf{R}$; analogously, if $m = n = 3$, every polyconvex function is of the form $g(z, \operatorname{adj} z, \det z)$ with g convex on $\mathbf{R}^9 \times \mathbf{R}^9 \times \mathbf{R}$ and where $\operatorname{adj} z$ denotes the adjugate matrix of z, that is the transpose of the matrix of cofactors of z.

Lemma 13.2.1. *Let $f : \mathbf{R}^{mn} \to \mathbf{R}$ be a rank-one convex function such that*

$$0 \le f(z) \le c(1 + |z|^p) \quad \forall z \in \mathbf{R}^{mn}.$$

Then, for a suitable constant $k > 0$ we have

$$|f(z) - f(w)| \le k|z - w|\big(1 + |z|^{p-1} + |w|^{p-1}\big) \quad \forall z, w \in \mathbf{R}^{mn}.$$

PROOF. With f convex with respect to each column vector, it is enough to prove the inequality in the case f convex. Again, arguing component by component, we may assume that f is a convex function of only one variable t. These functions are differentiable almost everywhere, and we have

$$f'(t) \le \frac{f(t+h)-f(t)}{h} \quad \forall h>0, \tag{13.14}$$

$$f'(t) \ge \frac{f(t+h)-f(t)}{h} \quad \forall h<0. \tag{13.15}$$

Taking $h=1+|t|$ in (13.14) and $h=-1-|t|$ in (13.15) we obtain for a.e. $t \in \mathbf{R}$

$$|f'(t)| \le \frac{f(t+h)}{|h|} \le \frac{c}{1+|t|}\big(1+|t|^p+|h|^p\big) \le c\big(1+|t|^{p-1}\big),$$

from which the conclusion follows easily. ☐

For a systematic study of the properties of quasi-convex functions, see [303], [3], [286], [302], and [182]. The main interest of the notion of quasi-convexity consists in its relation with the lower semicontinuity of integral functionals, in the sense specified by the following theorem.

Theorem 13.2.1. *Let $p \ge 1$ and let $f : \Omega \times \mathbf{R}^m \times \mathbf{R}^{mn} \to \mathbf{R}$ be a Carathéodory integrand such that*

$$0 \le f(x,s,z) \le c\big(a(x)+|s|^p+|z|^p\big) \tag{13.16}$$

for all $(x,s,z) \in \Omega \times \mathbf{R}^m \times \mathbf{R}^{mn}$, where $c \ge 0$ is a constant, and $a \in L^1(\Omega)$. Then the following conditions are equivalent:

(i) *for a.e. $x \in \Omega$ and every $s \in \mathbf{R}^m$ the function $f(x,s,\cdot)$ is quasi-convex;*

(ii) *the functional F defined by* (13.11) *is sequentially lower semicontinuous on $W^{1,p}(\Omega;\mathbf{R}^m)$ with respect to its weak topology.*

PROOF. We give here the proof only in the basic case $f = f(z)$, where

$$0 \le f(z) \le c\big(1+|z|^p\big);$$

the proof of the general case can be found in the references mentioned above.

We start by proving that the quasi-convexity condition (i) implies the lower semicontinuity condition (ii). We have to prove that

$$F(u) \le \liminf_{h\to+\infty} F(u_h) \tag{13.17}$$

whenever $u_h \to u$ weakly in $W^{1,p}(\Omega;\mathbf{R}^m)$. This will be done in three steps.

First step. Inequality (13.17) *holds when u is affine and $u_h = u$ on $\partial\Omega$.* In fact, in this case Du is a constant matrix z and $u_h - u \in W_0^{1,p}(\Omega;\mathbf{R}^m)$ so that, by definition of quasi-convexity, we get

$$F(u) = |\Omega| f(z) \le \int_\Omega f\big(z + D(u_h-u)\big)\,dx = F(u_h),$$

hence (13.17).

Second step. Inequality (13.17) *holds when u is affine.* We use a slicing method near the boundary of Ω (see also the proof of Proposition 11.2.3). Let Ω_0 be a compact subset of Ω, let $R = \frac{1}{2}\operatorname{dist}(\Omega_0, \partial\Omega)$, and let $N \in \mathbf{N}$. For every integer $i = 1, \dots, N$ define

$$\Omega_i = \left\{ x \in \Omega \,:\, \operatorname{dist}(x, \Omega_0) < \frac{iR}{N} \right\}$$

and let φ_i be a smooth function such that

$$0 \le \varphi_i \le 1, \quad \varphi_i = 0 \text{ on } \Omega \setminus \Omega_i, \quad \varphi_i = 1 \text{ on } \Omega_{i-1}, \quad |D\varphi_i| \le \frac{2N}{R}.$$

Finally take

$$v_{i,h} = u + \varphi_i \cdot (u_h - u).$$

For every $i = 1, \dots, N$ we have $v_{i,h} \to u$ weakly in $W^{1,p}(\Omega; \mathbf{R}^m)$ as $h \to +\infty$, and $v_{i,h} = u$ on $\partial\Omega$ so that, by the first step,

$$\begin{aligned}
F(u) &\le F(v_{i,h}) \\
&= \int_{\Omega_{i-1}} f(Du_h)\,dx + \int_{\Omega_i \setminus \Omega_{i-1}} f(Dv_{i,h})\,dx + \int_{\Omega \setminus \Omega_i} f(Du)\,dx \\
&\le F(u_h) + c \int_{\Omega_i \setminus \Omega_{i-1}} \left(1 + |Dv_{i,h}|^p\right) dx + c \int_{\Omega \setminus \Omega_i} \left(1 + |Du|^p\right) dx \\
&\le F(u_h) + c \int_{\Omega_i \setminus \Omega_{i-1}} \left(1 + |Du_h|^p + |Du|^p + |u_h - u|^p \frac{N^p}{R^p}\right) dx \\
&\qquad + c \int_{\Omega \setminus \Omega_i} \left(1 + |Du|^p\right) dx \\
&\le F(u_h) + c \int_{\Omega_i \setminus \Omega_{i-1}} \left(|Du_h|^p + |u_h - u|^p \frac{N^p}{R^p}\right) dx + c \int_{\Omega \setminus \Omega_0} \left(1 + |Du|^p\right) dx.
\end{aligned}$$

Summing for $i = 1, \dots, N$ and dividing by N gives

$$F(u) \le F(u_h) + \frac{c}{N} \int_{\Omega \setminus \Omega_0} \left(|Du_h|^p + |u_h - u|^p \frac{N^p}{R^p}\right) dx + c \int_{\Omega \setminus \Omega_0} \left(1 + |Du|^p\right) dx.$$

Passing to the limit as $h \to +\infty$ and taking into account that (u_h) is bounded in $W^{1,p}(\Omega; \mathbf{R}^m)$ yields

$$F(u) \le \liminf_{h \to +\infty} F(u_h) + \frac{c}{N} + c \int_{\Omega \setminus \Omega_0} \left(1 + |Du|^p\right) dx.$$

Now, to achieve the proof of the second step, it is enough to pass to the limit as $N \to +\infty$ and as $\Omega_0 \uparrow \Omega$.

Third step. Inequality (13.17) *holds in the general case $u \in W^{1,p}(\Omega; \mathbf{R}^m)$.* Let us fix $\varepsilon > 0$ and let w be a piecewise affine function such that $\|u - w\|_{W^{1,p}} < \varepsilon$. In particular, there exist open sets Ω_i and constant matrices z_i such that $Dw = z_i$ on Ω_i. Setting

$$w_{i,h}(x) = u_h(x) - u(x) + z_i x \quad \text{on } \Omega_i,$$

we have $w_{i,h} \to z_i x$ weakly in $W^{1,p}(\Omega_i;\mathbf{R}^m)$ so that, by the second step,

$$\int_{\Omega_i} f(z_i)\,dx \le \liminf_{h\to+\infty} \int_{\Omega_i} f(Dw_{i,h})\,dx.$$

By Lemma 13.2.1 we get

$$\begin{aligned}
&\left| F(u_h) - \sum_i \int_{\Omega_i} f(Dw_{i,h})\,dx \right| \le \sum_i \int_{\Omega_i} |f(Du_h) - f\big(Dw_{i,h}\big)|\,dx \\
&\le c \sum_i \int_{\Omega_i} |Du_h - Dw_{i,h}|\big(1 + |Du_h|^{p-1} + |Dw_{i,h}|^{p-1}\big)\,dx \\
&\le c \sum_i \int_{\Omega_i} |Du - Dw|\big(1 + |Du_h|^{p-1} + |Du - Dw|^{p-1}\big)\,dx \\
&\le c \left(\int_\Omega |Du - Dw|^p\,dx \right)^{1/p} \left(\int_\Omega \big[1 + |Du_h|^p + |Du - Dw|^p\big]\,dx \right)^{1-1/p} \\
&\le c\varepsilon.
\end{aligned}$$

Analogously,

$$\left| F(u) - \sum_i \int_{\Omega_i} f(z_i)\,dx \right| \le c\varepsilon.$$

Therefore,

$$\begin{aligned}
F(u) &\le c\varepsilon + \sum_i \int_{\Omega_i} f(z_i)\,dx \\
&\le c\varepsilon + \sum_i \liminf_{h\to+\infty} \int_{\Omega_i} f(Dw_{i,h})\,dx \\
&\le c\varepsilon + \liminf_{h\to+\infty} \sum_i \int_{\Omega_i} f(Dw_{i,h})\,dx \\
&\le 2c\varepsilon + \liminf_{h\to+\infty} F(u_h),
\end{aligned}$$

and the conclusion follows by taking the limit as $\varepsilon \to 0^+$.

We prove now that lower semicontinuity condition (ii) implies quasi-convexity of f. The proof is based on the following result, which is, as said in Example 2.4.2 about oscillation phenomena, a straightforward consequence of a general ergodic theorem. (See also Lemma 12.3.1.)

Lemma 13.2.2. *Let Q be any open cube in $\mathbf{R}^n$ of size $L > 0$, v a function in $L^p(Q,\mathbf{R}^m)$, and $\tilde{v}$ its Q-periodic extension, i.e., the function of $L^p_{loc}(\mathbf{R}^n,\mathbf{R}^m)$ defined by*

$$\tilde{v}(x) = v(x - z) \text{ if } x \in Q + z,\ z \in L\mathbf{Z}^n.$$

Then the sequence $(v_h)_{h\in\mathbf{N}}$ defined by $v_h(x) = \tilde{v}(hx)$ weakly converges in $L^p_{loc}(\mathbf{R}^n,\mathbf{R}^m)$ (weak if $p = +\infty$) to its mean value $\frac{1}{\mathrm{meas}(Q)} \int_Q v(x)\,dx$.*

The proof is a straightforward consequence of Proposition 13.2.1 that we establish below because of its own interest.

Let now z be an $m \times n$ matrix, l_z the linear function defined by $l_z(x) = zx$, $u \in \mathbf{C}_0^1(Q, \mathbf{R}^m)$, $\tilde{u}$ its Q-periodic extension, and set for every $x \in \mathbf{R}^n$, $u_h(x) = \tilde{u}(hx)/h$. It is easily seen that u_h strongly converges to 0 in $L^p_{loc}(\mathbf{R}^n, \mathbf{R}^m)$. On the other hand, according to Lemma 13.2.2, Du_h weakly converges to

$$\frac{1}{\mathrm{meas}(Q)} \int_Q Du(x)\, dx = 0$$

in $L^p_{loc}(\mathbf{R}^m, \mathbf{R}^{mn})$. Therefore $u_h + l_z$ weakly converges to l_z in $W^{1,p}(Q, \mathbf{R}^m)$ and, by hypothesis (ii),

$$\liminf_{h \to +\infty} \int_Q f(Du_h + z)\, dx \geq \int_Q f(z)\, dz = \mathrm{meas}(Q) f(z). \tag{13.18}$$

A change of scale and the periodicity assumption on $\tilde{u}$ gives

$$\begin{aligned} \frac{1}{\mathrm{meas}(Q)} \int_Q f(Du_h + z)\, dx &= \frac{1}{\mathrm{meas}(hQ)} \int_{hQ} f(D\tilde{u} + z)\, dx \\ &= \frac{1}{\mathrm{meas}(Q)} \int_Q f(Du + z)\, dx \end{aligned}$$

so that (13.18) yields

$$\frac{1}{\mathrm{meas}(Q)} \int_Q f(Du + z)\, dx \geq f(z),$$

which completes the proof. □

Remark 13.2.3. When $p = +\infty$ the result of Theorem 13.2.1 still holds if we substitute the weak topology with the weak* topology of $W^{1,\infty}(\Omega; \mathbf{R}^m)$, and condition (13.16) with

$$0 \leq f(x, s, z) \leq \alpha(x, |s|, |z|) \tag{13.19}$$

for all $(x, s, z) \in \Omega \times \mathbf{R}^m \times \mathbf{R}^{mn}$, where $\alpha(x, t, \tau)$ is a function which is summable in x and increasing in t and τ.

Remark 13.2.4. From what was presented above, it follows that the implications

$$f \text{ convex} \quad \Rightarrow \quad f \text{ polyconvex} \quad \Rightarrow \quad f \text{ quasi-convex} \quad \Rightarrow \quad f \text{ rank-one convex}$$

hold true. We stress the fact that, as shown in Theorem 13.2.1, quasi-convexity is the right property to use when dealing with the lower semicontinuity of integral functionals of the calculus of variations. However, due to its intrinsic definition, it is not easy to work with quasi-convex functions, while polyconvexity and rank-one convexity conditions are much more explicit: in many cases, the sufficiency of the first and the necessity of the second are of great help.

None of the implications above can be reversed; this can be seen very easily for the first one (indeed $|\det z|$ is a polyconvex function which is not convex), whereas more delicate counterexamples are needed for the remaining ones. See the book by Dacorogna [182] and the paper by Sverak [342] for details concerning these topics. The only problem that remains open in the study of the implications above is the equivalence between quasi-convexity and rank-one convexity in the case $m = n = 2$: neither counterexamples nor proofs of the equivalence are known.

We state now the ergodic theorem stated in Lemma 13.2.2, which is a generalization of the convergence result described in Example 2.4.2 about oscillations phenomena, and a particular case of Lemma 12.3.1.

Proposition 13.2.1. *With the notation of Lemma* 12.3.1, *let Q be a cube in $\mathbf{R}^n$ of the form $\Pi_{i=1}^n(a_i, a_i + L)$ for some $L > 0$, and $\mathscr{A} : \mathscr{B}_b(\mathbf{R}^n) \to \mathbf{R}$ satisfying*

(i) $\mathscr{A}_{A\cup B} = \mathscr{A}_A + \mathscr{A}_B$ *for every disjoint set of $\mathscr{B}_b(\mathbf{R}^n)$,*

(ii) $\mathscr{A}_{A+z} = \mathscr{A}_A$ *for every set of $\mathscr{B}_b(\mathbf{R}^n)$ and every z in $L\mathbf{Z}^n$.*

Then, for any bounded Borel convex subset B of $\mathbf{R}^n$,

$$\lim_{h\to+\infty} \frac{\mathscr{A}_{hB}}{\text{meas}(hB)} = \frac{1}{\text{meas}(Q)} \mathscr{A}_Q.$$

Consequently, if v belongs to $L^p(Q, \mathbf{R}^m)$ and $\tilde{v}$ denotes its Q-periodic extension, then v_h defined by $v_h(x) = \tilde{v}(hx)$, $h \in \mathbf{N}$, weakly converges in $L^p_{loc}(\mathbf{R}^n, \mathbf{R}^m)$ (weakly if $p = +\infty$) to its mean value $\frac{1}{\text{meas}(Q)} \int_Q v(x)\, dx$.*

PROOF. When $B = Q$, the first assertion is a straightforward consequence of the following decomposition:

$$hQ = \bigcup_{z\in L\mathbf{Z}^n\cap hQ} (z + Q).$$

For the proof of the general case, see [167]. We are going to prove the second assertion. Let U be an open ball of $\mathbf{R}^n$; then v_h is bounded in $L^p(U, \mathbf{R}^m)$. Indeed

$$\frac{1}{\text{meas}(U)} \int_U |v_h|^p \, dx = \frac{1}{\text{meas}(hU)} \int_{hU} |\tilde{v}|^p \, dx,$$

which converges thanks to the first assertion. Therefore, in the case when $p > 1$, there exists a subsequence of $(v_h)_{h\in\mathbf{N}}$ (not relabeled) and $u \in L^p(U, \mathbf{R}^m)$ such that $v_h \rightharpoonup u$ weakly in $L^p(U, \mathbf{R}^m)$. To identify the weak limit u, we work in the space of measures $\mathbf{M}(U, \mathbf{R}^m)$. Obviously $v_h \rightharpoonup u\, \mathscr{L}^n\lfloor U$ weakly in $\mathbf{M}(U, \mathbf{R}^m)$. For a.e. x_0 in U, ρ in $(0, 1) \setminus N$ where N is a countable subset of $\mathbf{R}$ (see Lemma 4.2.2), according to the first assertion, we have

$$\begin{aligned}
u(x_0) &= \lim_{\rho\to 0} \frac{1}{\text{meas}(B_\rho(x_0))} \int_{B_\rho(x_0)} u(x)\, dx \\
&= \lim_{\rho\to 0} \lim_{h\to+\infty} \frac{1}{\text{meas}(B_\rho(x_0))} \int_{B_\rho(x_0)} v_h(x)\, dx \\
&= \lim_{\rho\to 0} \lim_{h\to+\infty} \frac{1}{\text{meas}(hB_\rho(x_0))} \int_{hB_\rho(x_0)} \tilde{v}(x)\, dx \\
&= \frac{1}{\text{meas}(Q)} \int_Q v(x)\, dx.
\end{aligned}$$

It remains to treat the case $p = 1$. We establish the uniform integrability of the sequence $(v_h)_{h\in\mathbf{N}}$ in $L^1(U, \mathbf{R}^m)$; the conclusion will follow by the Dunford–Pettis theorem,

Theorem 2.4.5, and by the above procedure for identifying the weak limit. We now use a truncation argument. Let $\delta > 0$ intended to go to $+\infty$ and set

$$\begin{cases} v_{h,\delta} = v_h \wedge \delta \vee (-\delta), \\ v_\delta = v \wedge \delta \vee (-\delta), \\ w_{h,\delta} = v_h - v_{h,\delta}, \\ w_\delta = v - v_\delta. \end{cases}$$

Note that $v_{h,\delta} = (v_\delta)_h$. For every Borel subset A of U, we have

$$\int_A |v_h|\, dx \leq \int_U |w_{h,\delta}|\, dx + \int_A |v_{h,\delta}|\, dx. \tag{13.20}$$

On the other hand, according to the first assertion,

$$\lim_{h\to+\infty} \int_U |w_{h,\delta}|\, dx = \frac{1}{\mathrm{meas}(Q)} \int_Q |w_\delta|\, dx.$$

Then, given $\varepsilon > 0$, since there exists δ large enough such that

$$\frac{1}{\mathrm{meas}(Q)} \int_Q |w_\delta|\, dx < \frac{\varepsilon}{4},$$

there exists $h(\varepsilon) \in \mathbf{N}$ such that

$$\sup_{h\geq h(\varepsilon)} \int_U |w_{h,\delta}|\, dx < \frac{\varepsilon}{2}. \tag{13.21}$$

Collecting (13.20) and (13.21), we deduce

$$\sup_{h\geq h(\varepsilon)} \int_A |v_h|\, dx < \frac{\varepsilon}{2} + \delta\ \mathrm{meas}(A)$$

and the conclusion follows by taking $\mathrm{meas}(A) < \frac{\varepsilon}{2\delta}$. □

Remark 13.2.5. Proposition 13.2.1 is a particular case, in a deterministic setting, of the ergodic theorem for spatial processes established by Nguyen and Zessin which generalizes the famous Birkhoff ergodic theorem. For a proof, see [311]. For an application to homogenization, see [167]. For a systematic study of ergodic theorems, see [262], and for applications to homogenization, see [185], [121], [291], [1], and references therein. For an application to one-dimensional models in fracture mechanics, see Theorem 14.2.2.

13.3 ▪ Lower semicontinuity for functionals defined on the space of measures

In this subsection, we establish a semicontinuity result for integral functionals defined on the space $\mathbf{M}(\Omega, \mathbf{R}^m)$. The set Ω is a separable locally compact, metrizable, σ-compact topological space, for example, $\mathbf{R}^N$ or an open subset of $\mathbf{R}^N$. As a consequence, we recover, in a convex setting, some lower semicontinuity results concerning integral functionals defined on $BV(\Omega)$ obtained in Section 11.3.

Let $f:\mathbf{R}^m \to [0,+\infty]$ be a lower semicontinuous convex function. The recession function of a quasi-convex function was defined in Theorem 11.3.1. For the convex function f, it is easily seen that the limit

$$\lim_{t\to+\infty} \frac{f(w+ta)}{t}$$

exists and does not depend on the choice of w. This limit then coincides with the recession function defined in Theorem 11.3.1. More precisely, we define

$$f^\infty(a) := \lim_{t\to+\infty} \frac{f(w+ta)}{t}$$

(when $f \equiv +\infty$ we set $f^\infty \equiv +\infty$). It is straightforward to show that f^∞ is convex, lower semicontinuous, and positively homogeneous of degree 1, that is, it satisfies $f^\infty(ta) = tf^\infty(a)$ for all $t\in\mathbf{R}^+$ and all a in $\mathbf{R}^m$.

Theorem 13.3.1. *Let $f:\mathbf{R}^m \to [0,+\infty]$ be a lower semicontinuous convex function and μ be a Borel measure in $\mathbf{M}^+(\Omega)$. Then the integral functional defined on $\mathbf{M}(\Omega,\mathbf{R}^m)$ by*

$$F(\lambda) = \int_\Omega f\left(\frac{d\lambda}{d\mu}\right) d\mu + \int_\Omega f^\infty\left(\frac{d\lambda^s}{d|\lambda^s|}\right) d|\lambda^s|,$$

where $\lambda = d\lambda/d\mu\cdot\mu + d\lambda^s$ is the Radon–Nikodým decomposition of λ with respect to μ, is lower semicontinuous for the weak convergence in $\mathbf{M}(\Omega,\mathbf{R}^m)$.

PROOF. Our strategy is to apply localization lemma, Lemma 4.2.2. Since f is lower semicontinuous and convex, there exist $a_h\in\mathbf{R}$ and $b_h\in\mathbf{R}^m$ such that (see Theorem 9.3.1)

$$f(s) = \sup\{a_h + b_h\cdot s \ : \ h\in\mathbf{N}\} \qquad \forall s\in\mathbf{R}^m.$$

Let us set, for all $h\in\mathbf{N}$ and all Borel subsets B of Ω,

$$\begin{cases} f_h(s) = [a_h + b_h\cdot s]^+, \\ F_h(\lambda,B) = \displaystyle\int_B f_h\left(\frac{d\lambda}{d\mu}\right) d\mu + \int_B f_h^\infty\left(\frac{d\lambda^s}{d|\lambda^s|}\right) d|\lambda^s|. \end{cases}$$

Clearly $f = \sup\{f_h \ : \ h\in\mathbf{N}\}$. Let $\lambda\in\mathbf{M}(\Omega,\mathbf{R}^m)$ and let $\mathscr{N}$ be a μ-measurable subset of Ω where λ^s is concentrated. We moreover set $\nu = \mu + |\lambda^s|$ and

$$\tilde f(x) = \begin{cases} f\left(\dfrac{d\lambda}{d\mu}(x)\right) & \text{if } x\in\Omega\setminus\mathscr{N}, \\ f^\infty\left(\dfrac{d\lambda^s}{d|\lambda^s|}(x)\right) & \text{if } x\in\mathscr{N}, \end{cases}$$

$$\tilde f_h(x) = \begin{cases} f_h\left(\dfrac{d\lambda}{d\mu}(x)\right) & \text{if } x\in\Omega\setminus\mathscr{N}, \\ f_h^\infty\left(\dfrac{d\lambda^s}{d|\lambda^s|}(x)\right) & \text{if } x\in\mathscr{N}. \end{cases}$$

We have $F(\lambda) = \int_\Omega \tilde f\, d\nu$, $F_h(\lambda) = \int_\Omega \tilde f_h\, d\nu$ and, according to Lemma 4.2.2,

$$F(\lambda) = \sup\left\{\sum_{i\in I} F_i(\lambda, A_i) \ : \ A_i \text{ disjoint open subsets of } \Omega\right\}.$$

To conclude, it suffices to notice that for all $h \in \mathbf{N}$ and all open subsets A of Ω, $\lambda \mapsto F_h(\lambda, A)$ is (sequentially) lower semicontinuous on $\mathbf{M}(\Omega, \mathbf{R}^m)$. □

Remark 13.3.1. A more general semicontinuity result has been established in [110] for functionals of the form

$$F(\lambda) = \int_\Omega f\left(\frac{d\lambda}{d\mu}\right) d\mu + \int_{\Omega \setminus A_\lambda} f^\infty(\lambda^s) + \int_\Omega g(\lambda(x))\, d\mathscr{H}^0, \tag{13.22}$$

where $f : \mathbf{R}^m \to [0, +\infty]$ is assumed to be convex and lower semicontinuous, $g : \mathbf{R}^m \to [0, +\infty]$ is subadditive and lower semicontinuous, $f(0) = g(0) = 0$, and $f^\infty = g^0$, where

$$g^0(s) = \lim_{t \to 0^+} \frac{g(ts)}{t}.$$

The set A_λ is the subset of the atoms of λ and $\lambda(x)$ is the value $\lambda(\{x\})$.

The result of Remark 13.3.1 is complete in the following sense: every functional lower semicontinuous on $\mathbf{M}(\Omega, \mathbf{R}^m)$ for the weak convergence has an integral representation of the form (13.22) provided that it satisfies a local property: a functional F defined on $\mathbf{M}(\Omega, \mathbf{R}^m)$ is said to be local iff

$$\lambda_1 \perp \lambda_2 \quad \Rightarrow \quad F(\lambda_1 + \lambda_2) = F(\lambda_1) + F(\lambda_2).$$

The following theorem is established in Bouchitté and Buttazzo [111].

Theorem 13.3.2. *Let $F : \mathbf{M}(\Omega, \mathbf{R}^m) \to [0, +\infty]$ be a local lower semicontinuous functional for the weak convergence of $\mathbf{M}(\Omega, \mathbf{R}^m)$. Then F is of the form*

$$F(\lambda) = \int_\Omega f\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_{\Omega \setminus A_\lambda} f^\infty\left(x, \frac{d\lambda^s}{d|\lambda^s|}\right) d|\lambda^s| + \int_{A_\lambda} g(x, \lambda(x))\, d\mathscr{H}^0,$$

where $f, g : \Omega \times \mathbf{R}^m \to [0, +\infty]$ are two Borel functions satisfying

(a) *$f(x, \cdot)$ is convex, lower semicontinuous on $\mathbf{R}^m$ and $f(x, 0) = 0$ for μ-a.e. $x \in \Omega$;*

(b) *g is lower semicontinuous on $\Omega \times \mathbf{R}^m$, $g(x, \cdot)$ is subadditive on $\mathbf{R}^m$, and $g(x, 0) = 0$ for all $x \in \Omega$;*

(c) *$f^\infty = g^0$ on $(\Omega \setminus N) \times \mathbf{R}^m$ where N is a countable subset of Ω.*

Note that the lower semicontinuity hypothesis does not generally imply the convexity of F as it is the case, however, for functionals defined on $L^p(\Omega)$. Indeed, consider the functional

$$F(\lambda) = \int_\Omega \left|\frac{d\lambda}{d\mu}\right|^2 d\mu + \mathscr{H}^0(A_\lambda) + \chi_{\{\lambda^s = 0 \text{ on } \Omega \setminus A_\lambda\}}(\lambda).$$

It is easily seen that F is local and lower semicontinuous but nonconvex because of the term $\mathscr{H}^0(A_\lambda)$ and that

$$\chi_{\{\lambda^s = 0 \text{ on } \Omega \setminus A_\lambda\}}(\lambda) = \int_{\Omega \setminus A_\lambda} f^\infty\left(x, \frac{d\lambda^s}{d|\lambda^s|}\right) d|\lambda^s|,$$

where $f(x,s) = |s|^2$ (see Bouchitté and Buttazzo [112] and Ambrosio and De Giorgi [195]).

When F is convex, its integral representation is as earlier established in Ambrosio and Buttazzo [23].

Theorem 13.3.3. *Let $F : \mathbf{M}(\Omega, \mathbf{R}^m) \to [0, +\infty]$ be a convex, local, and lower semicontinuous functional for the weak convergence of $\mathbf{M}(\Omega, \mathbf{R}^m)$. Then F is of the form*

$$F(\lambda) = \int_\Omega f\left(x, \frac{d\lambda}{d\mu}\right) d\mu + \int_\Omega f^\infty\left(x, \frac{d\lambda^s}{d|\lambda^s|}\right) d|\lambda^s|,$$

where $f : \Omega \times \mathbf{R}^n \to \mathbf{R}$ is a Borel function such that $f(x,\cdot)$ is convex in $\mathbf{R}^m$ for μ-a.e. $x \in \Omega$ and $f^\infty(x,\cdot)$ is the recession function of $f(x,\cdot)$.

13.4 ▪ Functionals with linear growth: Lower semicontinuity in BV and SBV

13.4.1 ▪ Lower semicontinuity and relaxation in BV

The main objective of this section is, in a convex situation, to provide an alternative proof of the general relaxation result established in Theorem 11.3.1 for integral functionals defined in $L^1(\Omega)$ by

$$F(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\, dx & \text{if } u \in W^{1,1}(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \qquad G(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\, dx & \text{if } u \in W_0^{1,1}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

when Ω is an open bounded subset of $\mathbf{R}^N$ with a Lipschitz boundary Γ, and f is convex and satisfies suitable linear growth conditions. Let us first state the following corollary of Theorem 13.3.1 where we do not assume growth conditions on f.

Proposition 13.4.1. *Let $Du = \nabla u\, \mathscr{L}^N \lfloor \Omega + D^s u$ be the Lebesgue–Nikodým decomposition of the measure Du and $f : \mathbf{R}^N \to [0, +\infty]$ a lower semicontinuous convex function. Then, the integral functional defined on $BV(\Omega)$ by*

$$F(u) = \int_\Omega f(\nabla u)\, dx + \int_\Omega f^\infty\left(\frac{D^s u}{|D^s u|}\right) |D^s u|$$

is lower semicontinuous for the weak convergence of $BV(\Omega)$.

PROOF. Take $\mu = \mathscr{L}^N \lfloor \Omega$, $\lambda = Du$, and apply Theorem 13.3.1 to the functional defined by

$$\tilde{F}(Du) := \int_\Omega f(\nabla u)\, dx + \int_\Omega f^\infty\left(\frac{D^s u}{|D^s u|}\right) |D^s u|.$$

Indeed, $u_n \rightharpoonup u$ weakly in $BV(\Omega)$ implies $Du_n \rightharpoonup Du$ weakly in $\mathbf{M}(\Omega, \mathbf{R}^N)$. □

In what follows, we denote the integral

$$\int_\Omega f^\infty\left(\frac{D^s u}{|D^s u|}\right) |D^s u|$$

by $\int_\Omega f^\infty(D^s u)$ and the functional F by

$$F(u) := \int_\Omega f(Du).$$

We consider a convex function $f : \mathbf{R}^N \to \mathbf{R}^+$ satisfying for all a in $\mathbf{R}^N$ the growth condition

$$0 \le f(a) \le \beta(1 + |a|) \tag{13.23}$$

for some positive constant β. We recall that the Fenchel transform of f is the function $f^* : \mathbf{R}^N \to [-\infty, +\infty]$ defined for all b in $\mathbf{R}^N$ by

$$f^*(b) = \sup_{a \in \mathbf{R}^N} \{b.a - f(a)\}.$$

It is well known (see Section 9.3) that f^* is a $\mathbf{R} \cup \{+\infty\}$-value convex function, proper and lower semicontinuous. According to the growth condition (13.23), a straightforward calculation shows that its domain K is contained in the closed ball $\overline{\mathbf{B}(0,\beta)}$ of $\mathbf{R}^N$ with radius β and centered at 0 and that for all $b \in K$, $f^*(b) \ge -\beta$. We assume moreover that f^* is bounded from above on its domain: there exists $C > 0$ such that

$$\forall b \in K \quad f^*(b) \le C. \tag{13.24}$$

Approximating Theorem 10.1.2 may be generalized as follows.

Theorem 13.4.1. *Let f be a nonnegative convex function satisfying* (13.23) *and* (13.24); *then the space $\mathbf{C}^\infty(\Omega) \cap BV(\Omega)$ is dense in $BV(\Omega)$ equipped with the intermediate convergence associated with f. Namely, for all u in $BV(\Omega)$, there exists u_n in $\mathbf{C}^\infty(\Omega) \cap BV(\Omega)$ such that*

$$u_n \to u \quad \textit{strongly in } L^1(\Omega);$$
$$\int_\Omega |Du_n|\,dx \to \int_\Omega |Du|;$$
$$\int_\Omega f(Du_n)\,dx \to \int_\Omega f(Du).$$

PROOF. Proceed exactly as in the proof of Theorem 10.1.2. For a proof explained in detail, consult Temam [348]. □

With the intention of obtaining a complete description of the two lower semicontinuous envelopes of the functionals F and G, we assume a coerciveness condition on f: there exists a positive constant α such that

$$\forall a \in \mathbf{R}^N \quad \alpha(|a| - 1) \le f(a). \tag{13.25}$$

Theorem 13.4.2. *Let Ω be a Lipschitz bounded open subset of $\mathbf{R}^N$ with boundary Γ and f a nonnegative and convex function satisfying* (13.23), (13.24), *and* (13.25). *Then the lsc envelopes $\overline{F}$ and $\overline{G}$ of the functionals F and G in $L^1(\Omega)$ equipped with its strong topology are* *defined by*

$$\overline{F}(u) = \begin{cases} \int_\Omega f(\nabla u)\,dx + \int_\Omega f^\infty(D^s u) & \text{if } u \in BV(\Omega), \\ +\infty & \text{otherwise;} \end{cases}$$

$$\overline{G}(u) = \begin{cases} \int_\Omega f(\nabla u)\,dx + \int_\Omega f^\infty(D^s u) + \int_\Gamma f^\infty(\gamma_0(u)\nu)\,d\mathcal{H}^{N-1} & \text{if } u \in BV(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

PROOF. We argue as in the proof of Proposition 11.3.2. For the function F, we must establish that for all u in $L^1(\Omega)$,

$$\text{if } u_n \to u \text{ in } L^1(\Omega), \text{ then } \overline{F}(u) \le \liminf_{n\to+\infty} F(u_n),$$
$$\text{there exists } u_n \text{ converging to } u \text{ in } L^1(\Omega), \text{ such that } \overline{F}(u) \ge \limsup_{n\to+\infty} F(u_n).$$

These two assertions are straightforward consequences of Proposition 13.4.1 and Theorem 13.4.1. Indeed, let $u_n \to u$ strongly in $L^1(\Omega)$, such that $\liminf_{n\to+\infty} F(u_n) < +\infty$. From (13.25), for a nonrelabeled subsequence, $u_n \rightharpoonup u$ in $BV(\Omega)$ and, according to Proposition 13.4.1,

$$\overline{F}(u) \le \liminf_{n\to+\infty} F(u_n).$$

On the other hand, for $u \in BV(\Omega)$, Theorem 13.4.1 provides a sequence of functions u_n in $\mathbf{C}^\infty(\Omega) \cap BV(\Omega)$ such that $\overline{F}(u) \ge \limsup_{n\to+\infty} F(u_n)$.

For the function G, the proof is exactly that of Proposition 11.3.2. Let us recall that it suffices to apply the previous result related to F after enlarging the set Ω to obtain the first assertion and to approach Ω from below to derive the second. □

13.4.2 ▪ Compactness and lower semicontinuity in SBV

The following result, due to Ambrosio, is a key tool in the so-called direct method in the calculus of variations when working with integral functionals defined in $SBV(\Omega)$ equipped with the strong topology of $L^1(\Omega)$ (see Chapter 14). In this section, Ω denotes an open bounded subset of $\mathbf{R}^N$.

Theorem 13.4.3. *Let $(u_n)_{n\in\mathbf{N}}$ be a sequence in $SBV(\Omega)$ satisfying for $p > 1$,*

$$\sup_{n\in\mathbf{N}} \left\{ \|u_n\|_\infty + \int_\Omega |\nabla u_n|^p\,dx + \mathcal{H}^{N-1}(S_{u_n}) \right\} < +\infty.$$

Then there exists a subsequence $(u_{n_k})_{k\in\mathbf{N}}$ and a function u in $SBV(\Omega)$, such that

$$u_{n_k} \to u \quad \text{strongly in } L^1_{loc}(\Omega);$$

$$\nabla u_{n_k} \rightharpoonup \nabla u \quad \text{weakly in } L^p(\Omega, \mathbf{R}^N);$$

$$\mathcal{H}^{N-1}(S_u) \le \liminf_{k\longrightarrow+\infty} \mathcal{H}^{N-1}(S_{u_{n_k}}).$$

Moreover, the Lebesgue part and the jump part of the derivatives converge separately. More precisely, $\nabla u_{n_k} \rightharpoonup \nabla u$ weakly in $L^1(\Omega)$ and $J u_{n_k} \rightharpoonup J u$ weakly in $\mathbf{M}(\Omega, \mathbf{R}^N)$.

PROOF. In what follows C denotes various constants which do not depend on n. Thanks to inequalities

$$\int_\Omega |\nabla u_n|^p \, dx \le C \text{ and } |u_n|_\infty \le C,$$

we have $\|u_n\|_{BV(\Omega)} \le C$. Indeed, according to Remark 10.3.4, for $\mathscr{H}^{N-1}$ a.e. x in S_{u_n}, $|u_n^+(x)| \le \|u_n\|_{L^\infty(\Omega)} \le C$ and $|u_n^-(x)| \le \|u_n\|_{L^\infty(\Omega)} \le C$. From the compactness of the embedding of $BV(U)$ into $L^1(U)$, for each regular open subset of Ω, there exists a subsequence $(u_{n_k})_{k\in\mathbf{N}}$ and u in $BV(\Omega)$ such that

$$\begin{aligned} u_{n_k} &\rightharpoonup u \quad \text{weakly in } BV(\Omega), \\ u_{n_k} &\to u \quad \text{strongly in } L^1_{loc}(\Omega). \end{aligned}$$

Let us show that $u \in SBV(\Omega)$. Since $u_{n_k} \in SBV(\Omega)$, according to Theorem 10.5.1, there exists a Borel measure μ_{n_k} in $\mathbf{M}(\Omega \times \mathbf{R}, \mathbf{R}^N)$ such that for all $\Phi \in \mathbf{C}^1_c(\Omega, \mathbf{R}^N)$ and all $\varphi \in \mathbf{C}^1_0(\mathbf{R})$,

$$\int_{\Omega\times\mathbf{R}} \varphi(s)\Phi(x)\, \mu_{n_k}(dx, ds) = -\int_\Omega \big(\varphi'(u_{n_k})\, \nabla u_{n_k} \,.\, \Phi(x) + \varphi(u_{n_k}) \operatorname{div} \Phi(x)\big)\, dx,$$
$$|\mu_{n_k}|(\Omega \times \mathbf{R}) = 2\, H^{N-1}(S_{u_{n_k}}).$$

The second equality and the hypothesis $\mathscr{H}^{N-1}(S_{u_{n_k}}) \le C$ yield the existence of a Borel measure μ in $\mathbf{M}(\Omega \times \mathbf{R}, \mathbf{R}^N)$ and a subsequence (not relabeled) of $(\mu_{n_k})_{k\in\mathbf{N}}$, weakly converging to μ in $\mathbf{M}(\Omega\times\mathbf{R}, \mathbf{R}^N)$. On the other hand, for a further nonrelabeled subsequence, there exists $a \in L^1(\Omega, \mathbf{R}^N)$ such that ∇u_{n_k} weakly converges to a in $L^1(\Omega, \mathbf{R}^N)$. Going to the limit in the first equality, we obtain

$$\int_{\Omega\times\mathbf{R}} \varphi(s)\Phi(x)\, \mu(dx, ds) = -\int_\Omega \big(\varphi'(u)\, a \,.\, \Phi(x) + \varphi(u) \operatorname{div} \Phi(x)\big)\, dx,$$

which, from Theorem 10.5.1, yields $u \in SBV(\Omega)$ and $a = \nabla u$.

On the other hand, according to the lower semicontinuity of the total variation, one has

$$\begin{aligned} 2\mathscr{H}^{N-1}(S_u) &= |\mu|(\Omega \times \mathbf{R}) \\ &\le \liminf_{k\longrightarrow+\infty} |\mu_{n_k}|(\Omega \times \mathbf{R}) \\ &= \liminf_{k\longrightarrow+\infty} 2\mathscr{H}^{N-1}(S_{u_{n_k}}). \end{aligned}$$

Finally, since Du_{n_k} weakly converges to Du in $\mathbf{M}(\Omega, \mathbf{R}^N)$ and ∇u_{n_k} weakly converges to ∇u in $L^1(\Omega, \mathbf{R}^N)$, one has $Ju_{n_k} = Du_{n_k} - \nabla u_{n_k} \mathscr{L}^N$ weakly converges to $Ju = Du - \nabla u \mathscr{L}^N$ in $\mathbf{M}(\Omega, \mathbf{R}^N)$. □

When $(u_n)_{n\in\mathbf{N}}$ is not bounded in $L^\infty(\Omega)$, we obtain the same result provided that $(u_n)_{n\in\mathbf{N}}$ be bounded in $BV(\Omega)$. Note also that the boundedness of $(\nabla u_n)_{n\in\mathbf{N}}$ in $L^p(\Omega)$ with $p > 1$, implies the equi-integrability of $(\nabla u_n)_{n\in\mathbf{N}}$. In this sense, the next theorem generalizes Theorem 13.4.3.

Theorem 13.4.4. *Let $(u_n)_n$ be a sequence in $SBV(\Omega)$ satisfying*

(i) $\sup_{n\in\mathbf{N}}\{|u_n|_{BV(\Omega)}\} < +\infty$;

(ii) *the approximate gradients ∇u_n are equi-integrable (i.e., $(\nabla u_n)_{n\in\mathbf{N}}$ is relatively compact with respect to the weak topology of $L^1(\Omega,\mathbf{R}^N)$);*

(iii) *the sequence $(\mathscr{H}^{N-1}(S_{u_n}))_{n\in\mathbf{N}}$ is bounded.*

Then there exists a subsequence $(u_{n_k})_{k\in\mathbf{N}}$ weakly converging to some $u \in SBV(\Omega)$ such that

$$u_{n_k} \to u \quad \text{strongly in } L^1_{loc}(\Omega);$$

$$\nabla u_{n_k} \rightharpoonup \nabla u \quad \text{weakly in } L^1(\Omega,\mathbf{R}^N);$$

$$J u_{n_k} \rightharpoonup J u \quad \text{weakly in } \mathbf{M}(\Omega,\mathbf{R}^N);$$

$$\mathscr{H}^{N-1}(S_u) \le \liminf_{k\longrightarrow+\infty} \mathscr{H}^{N-1}(S_{u_{n_k}}).$$

PROOF. According to Theorem 2.4.5, equi-integrability condition (ii) yields the existence of a subsequence of $(\nabla u_n)_{n\in\mathbf{N}}$ and a in $L^1(\Omega,\mathbf{R}^N)$ such that $\nabla u_n \rightharpoonup a$ in $L^1(\Omega,\mathbf{R}^N)$. Then we argue as in the proof of Theorem 13.4.3 and adopt the same notation. We only have to justify the convergence of

$$\int_\Omega \varphi'(u_{n_k})\, \nabla u_{n_k} \,.\, \Phi(x)\, dx$$

to

$$\int_\Omega \varphi'(u)\, \nabla u \,.\, \Phi(x)\, dx.$$

According to Egorov's theorem, since $\varphi'(u_{n_k})$ converges a.e. to $\varphi'(u)$, for all $\varepsilon > 0$ there exists a Borel subset Ω_ε of Ω such that $\mathscr{L}^N(\Omega\setminus\Omega_\varepsilon) < \varepsilon$ and $\lim_{k\to+\infty}\sup_{x\in\Omega_\varepsilon}|\varphi(u_{n_k}) - \varphi(u)| = 0$. Let us write

$$\int_\Omega \varphi'(u_{n_k})\, \nabla u_{n_k} \,.\, \Phi(x)\, dx = \int_{\Omega_\varepsilon} \varphi'(u_{n_k})\, \nabla u_{n_k} \,.\, \Phi(x)\, dx + \int_{\Omega\setminus\Omega_\varepsilon} \varphi'(u_{n_k})\, \nabla u_{n_k} \,.\, \Phi(x)\, dx.$$

Since moreover $\sup_{k\in\mathbf{N}}\int_\Omega |\nabla u_{n_k}|\, dx < +\infty$, one easily obtains that the first term in the right-hand side tends to

$$\int_{\Omega_\varepsilon} \varphi'(u)\, \nabla u \,.\, \Phi(x)\, dx.$$

Letting $\varepsilon \to 0$, the conclusion then follows, provided that we establish

$$\lim_{\varepsilon\to 0}\lim_{k\to+\infty}\int_{\Omega\setminus\Omega_\varepsilon} \varphi'(u_{n_k})\, \nabla u_{n_k} \,.\, \Phi(x)\, dx = 0,$$

which is a straightforward consequence of

$$\left|\int_{\Omega\setminus\Omega_\varepsilon} \varphi'(u_{n_k})\, \nabla u_{n_k} \,.\, \Phi(x)\, dx\right| \le C\int_{\Omega\setminus\Omega_\varepsilon} |\nabla u_{n_k}|\, dx$$

and equi-integrability of $(\nabla u_n)_{n\in\mathbf{N}}$. $\square$

Remark 13.4.1. If one of the two conditions (ii) and (iii) is not satisfied, the conclusion may fail. Indeed, in the Cantor–Vitali example of Section 10.4, u_n belongs to $SBV(0,1)$, weakly converges to the Cantor–Vitali function u in $BV(0,1)$ and $\mathcal{H}^0(S_{u_n}) = 0$. Nevertheless $(\nabla u_n)_{n\in\mathbf{N}}$ is not equi-integrable since $\int_{(0,1)} \nabla u_n \, dx = 1$ does not converge to $\int_{(0,1)} \nabla u \, dx = 0$. With the notation of this example, consider now v_n in $SBV(0,1)$ defined by $v_n = u_n 1_{(0,1)\setminus C_n}$. It is easily seen that v_n weakly converges to u in $BV(0,1)$ and that $\nabla v_n = 0$ is obviously equi-integrable. But $\mathcal{H}^0(S_{v_n}) = 2(2^n - 1)$ is not uniformly bounded.

Condition (iii) of Theorem 13.4.4 can be weakened by the following condition (iii′): there exists a function $\psi : [0,+\infty) \to [0,+\infty]$ such that $\psi(t)/t \to +\infty$ as $t \to 0$ and

$$\sup_{n\in\mathbf{N}} \int_{S_{u_n}} \psi\big(|u_n^+ - u_n^-|\big)\, d\mathcal{H}^{N-1} < +\infty.$$

For a proof, consult Braides [122].

Remark 13.4.2. Theorem 13.4.4 obviously holds in the vectorial case, i.e., when the considered sequences belong to $SBV(\Omega, \mathbf{R}^m)$.

We now deal with the lower semicontinuity property of functionals of the form

$$\int_\Omega f(\nabla u)\, dx + \int_\Omega g(u^+, u^-) h(\nu_u)\, d\mathcal{H}^{N-1} \lfloor S_u,$$

where f, g, and h verify suitable conditions.

Theorem 13.4.5. *Let us consider a function $f : \mathbf{R}^N \to \mathbf{R}^+$ satisfying the De La Vallée–Poussin criterion: f is convex and*

$$\lim_{a\to+\infty} \frac{f(a)}{|a|} = +\infty.$$

Let, moreover, $g : \mathbf{R}\times\mathbf{R} \to \mathbf{R}^+$ be a lower semicontinuous symmetric and subadditive function, i.e.,

$$g(a,b) = g(b,a) \le g(b,c) + g(c,a) \quad \forall\, a,\ b,\ c \in \mathbf{R},$$

and assume that $g(a,b) \ge \max(\psi(|a-b|), \delta|a-b|)$ for all a, $b \in \mathbf{R}$ where the function $\psi : [0,+\infty) \to [0,+\infty]$ satisfies the condition $\psi(t)/t \to +\infty$ as $t \to 0$, and δ is some positive constant.

Let finally $h : \mathbf{R}^N \to [0,+\infty)$ be a convex, even function, positively homogeneous of degree 1 and satisfying, for all $\nu \in \mathbf{R}^N$, $h(\nu) \ge c|\nu|$ for some positive constant c. Then the functional defined in $L^1(\Omega)$ by

$$F(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\, dx + \int_\Omega g(u^+, u^-) h(\nu_u)\, d\mathcal{H}^{N-1} \lfloor S_u & \text{if } u \in SBV(\Omega), \\ +\infty & \text{otherwise} \end{cases}$$

is lower semicontinuous for the strong topology of $L^1(\Omega)$.

If the lower semicontinuous, symmetric, and subadditive function g only satisfies the condition $g(a,b) \ge \psi(|a-b|)$, then, given a nonempty compact subset K of $\mathbf{R}$, the functional $\tilde{F}$

defined in $L^1(\Omega)$ *by*

$$\tilde{F}(u) = \begin{cases} \displaystyle\int_\Omega f(\nabla u)\,dx + \int_\Omega g(u^+, u^-) h(\nu_u)\, d\mathcal{H}^{N-1} \lfloor S_u & \textit{if } u \in SBV(\Omega) \textit{ and } u(x) \in K, \\ +\infty \quad \textit{otherwise} & \end{cases}$$

is lower semicontinuous for the strong topology of $L^1(\Omega)$.

SKETCH OF THE PROOF. The proof is based on the following lemma.

Lemma 13.4.1. *Let* $g : \mathbf{R} \times \mathbf{R} \to \mathbf{R}^+$ *be a lower semicontinuous, symmetric, and subadditive function and* $h : \mathbf{R}^N \to [0, +\infty)$ *be a convex, even function, positively homogeneous of degree* 1*. Then*

$$\int_\Omega g(u^+, u^-) h(\nu)\, d\mathcal{H}^{N-1} \lfloor S_u \leq \liminf_{n \to +\infty} \int_\Omega g(u_n^+, u_n^-) h(\nu_{u_n})\, d\mathcal{H}^{N-1} \lfloor S_{u_n}$$

whenever u_n, u *satisfy the thesis of Theorem* 13.4.4 *(with condition* (iii) *or* (iii′)*).*

For the proof of Lemma 13.4.1, consult Braides [122, Theorem 2.12]. Let $(u_n)_{n \in \mathbf{N}}$ be a sequence strongly converging to some u in $L^1(\Omega)$ and such that $\liminf_{n \to +\infty} F(u_n) < +\infty$. From the De La Vallée–Poussin criterion, there exists a subsequence of $(u_n)_{n \in \mathbf{N}}$ (not relabeled) such that

$$\nabla u_{n_k} \text{ weakly converges in } L^1(\Omega, \mathbf{R}^N).$$

On the other hand, according to the coercivity assumption on g,

$$\sup_{n \in \mathbf{N}} \int_\Omega |u_n^+ - u_n^-|\, d\mathcal{H}^{N-1} \lfloor S_{u_n} < +\infty.$$

The sequence $(u_n)_{n \in \mathbf{N}}$ is then bounded in $BV(\Omega)$. Moreover, from the assumption made on ψ, condition (iii′) of Remark 13.4.1 is satisfied. Thus conditions (i), (ii), and (iii′) of Theorem 13.4.4 are fulfilled. The conclusion then follows from the convexity of f and Lemma 13.4.1. The proof of the lower semicontinuity property of the functional $\tilde{F}$ follows the same scheme. The boundedness

$$\sup_{n \in \mathbf{N}} \int_\Omega |u_n^+ - u_n^-|\, d\mathcal{H}^{N-1} \lfloor S_{u_n} < +\infty$$

is now satisfied thanks to $|u_n^+(x)| \leq \|u_n\|_{L^\infty(\Omega)} \leq C$ and $|u_n^-(x)| \leq \|u_n\|_{L^\infty(\Omega)} \leq C$ for $\mathcal{H}^{N-1}$ a.e. x in S_{u_n} (cf. Remark 10.3.4). □

Remark 13.4.3. In the vectorial case, Theorem 13.4.5 holds with the same conditions on g and h (now $g : \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^+$), and when $f : \mathbf{R} \times \mathbf{M}^{m \times N} \to \mathbf{R}$ is quasi-convex and satisfies the growth conditions of order $p > 1$: for all $A \in \mathbf{M}^{m \times N}$,

$$|A|^p \leq f(x, A) \leq \beta(1 + |A|^p)$$

for some positive constants α and β. For a proof, consult Ambrosio [18]. This result will be essential in Section 14.2 to establish the existence of a solution for the weak Griffith model in the framework of fracture mechanics.