Primer on MCMC for Actuaries

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Learning Objectives

- Understand the core algorithm behind Monte Carlo Markov Chains (MCMC)
  This will allow actuaries who oversee work being done with MCMC models to be able to ask questions to the team doing the work; and also to present the findings to less technical audiences.

- "Peel back the curtain" and understand how the Metropolis Hastings Algorithm samples from the target stationary distribution

- See how MCMC can be implemented in MS Excel to solve a demonstrative reserving problem, which aims to provide educational value unavailable through modeling with pre-built programing packages

Overview

- Chocolate Chips – MLE & Bootstrap
- Chocolate Chips – a Bayesian Analysis
- More Complex Model – Bayesian Analysis
- Metropolis Hastings – Theory and Excel
- Gibbs Sampling – Theory and Excel

Chocolate Chips – MLE & Bootstrap

- Curious to know how many chocolate chips are in a cookie
- Estimate of the Mean (# of Chips in a Cookie)
- Distribution of the Mean
- Distribution of Chips in Cookies
- Draw 6 cookies

5, 5, 7, 10, 10, 11

Chocolate Chips – MLE & Bootstrap

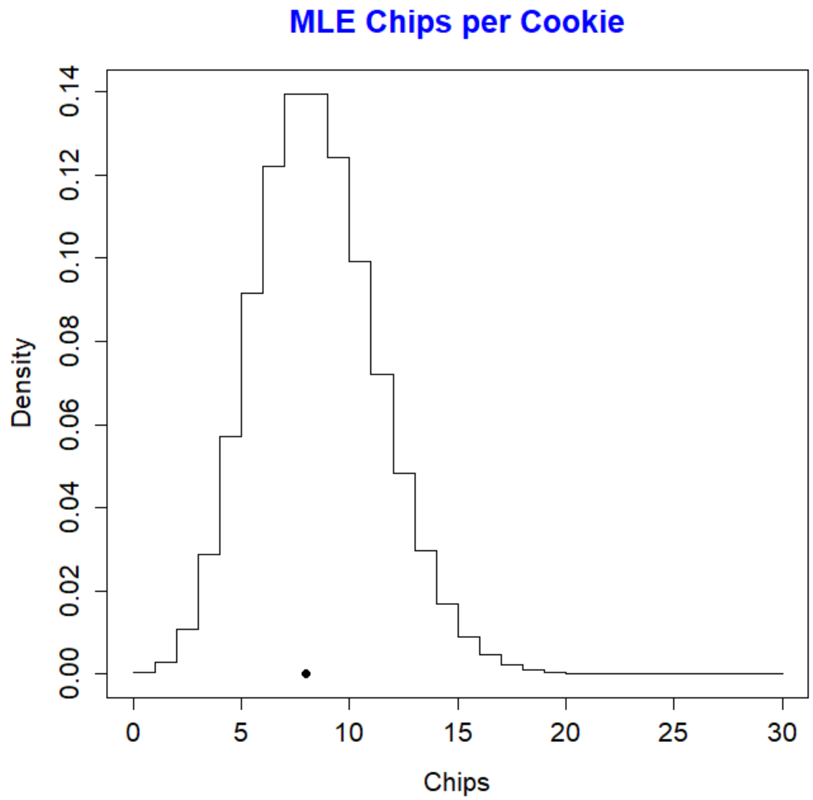
5, 5, 7, 10, 10, 11

- Assume they follow a Poisson Distribution
- The Maximum Likelihood Estimator is the Sample Mean

$$\mu = \bar{x} = \frac{5 + 5 + 7 + 10 + 10 + 11}{6} = 8.00$$

Chocolate Chips – MLE & Bootstrap

5, 5, 7, 10, 10, 11



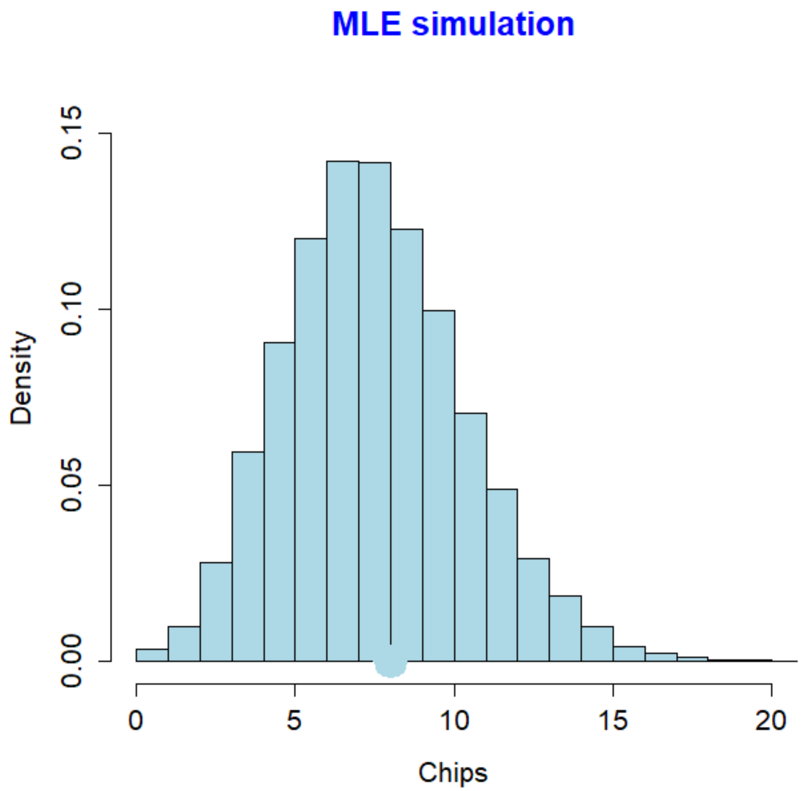
- Does not consider Parameter Risk for λ
- Assumes we estimated λ perfectly
- The Distribution here only has Process Risk

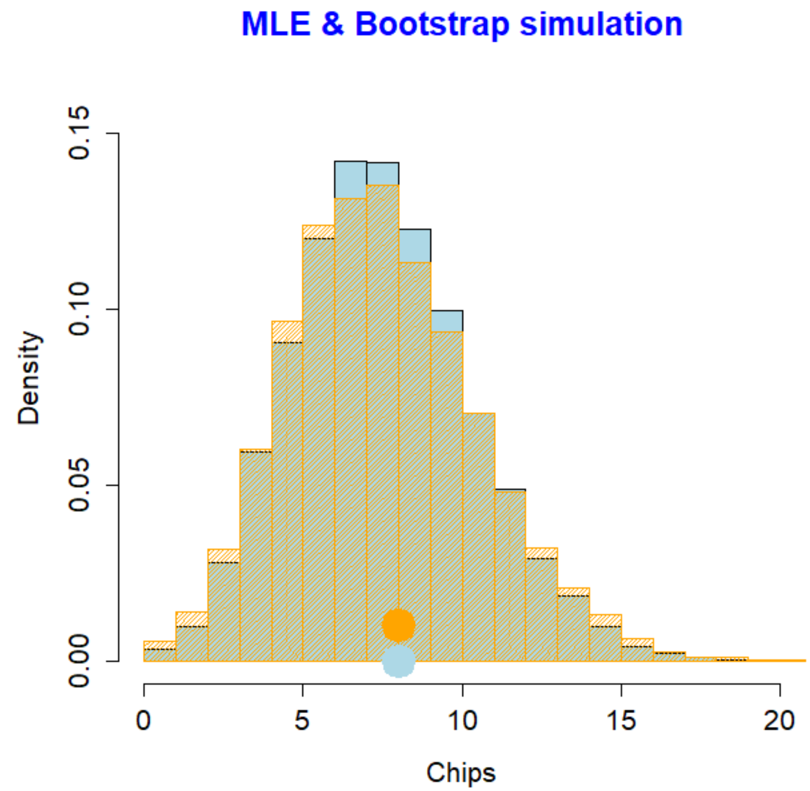
Chocolate Chips – MLE & Bootstrap

5, 5, 7, 10, 10, 11

- To add Parameter Risk, we can **Bootstrap**
- We draw 6 "cookies", from our set above, with replacement
- Calculate a sample mean
- Draw one cookie from Poisson with this sample mean
- Repeat 20,000 times

Chocolate Chips – MLE & Bootstrap





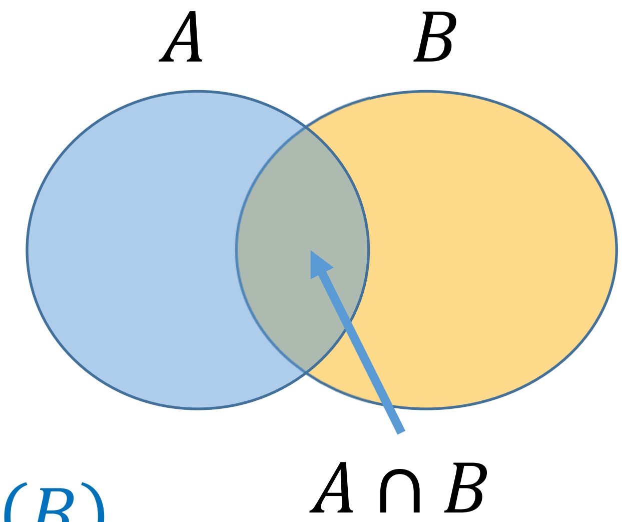
Chocolate Chips – a Bayesian Analysis

- Bootstrap was useful in adding Parameter Risk
- Bayesian Analysis provides another way to do this
- Also allows us to consider Expert Opinion

Chocolate Chips – a Bayesian Analysis

- Bayes Theorem

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

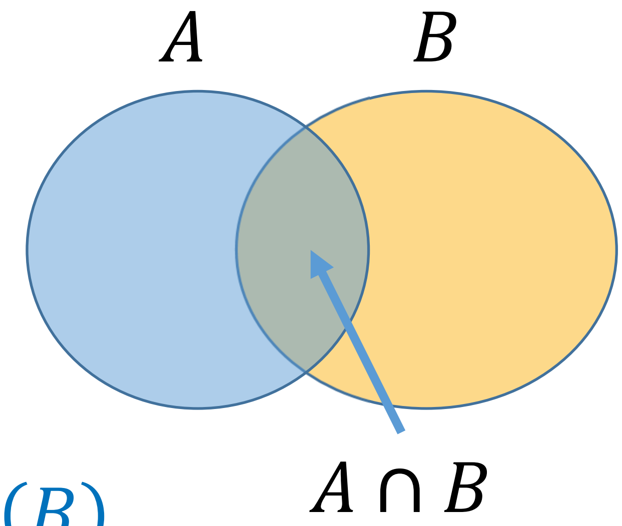


$$P(A \cap B) = P(A|B) \cdot P(B)$$
$$= P(B|A) \cdot P(A)$$

Chocolate Chips – a Bayesian Analysis

- Bayes Theorem

$$P(A|B) = \frac{\boldsymbol{P(A \cap B)}}{P(B)}$$



$$P(A \cap B) = P(A|B) \cdot P(B)$$
$$= \boldsymbol{P(B|A) \cdot P(A)}$$

$$\boldsymbol{P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}}$$

Chocolate Chips – a Bayesian Analysis

$$\boldsymbol{P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}}$$

- Using Continuous Density notation $f(\cdot)$
- Let x be a set of data
- Let y be a collection of parameters

$$\boldsymbol{f(y|x) = \frac{f(x|y) \cdot f(y)}{f(x)}}$$

Chocolate Chips – a Bayesian Analysis

$$f(y|x) = \frac{\boldsymbol{f(x|y)} \cdot f(y)}{f(x)}$$

- $f(x|y)$ – is the Likelihood (LL)
  - This is the density of our data, given a set of parameters
  - The set of parameters, $\hat{y}$ that maximizes the Likelihood are called the Maximum Likelihood Estimators - MLE
  - The MLE is used often to find a “best” set of parameters

Chocolate Chips – a Bayesian Analysis

$$f(y|x) = \frac{f(x|y) \cdot \boldsymbol{f(y)}}{f(x)}$$

- $f(y)$ – is the Prior Distribution of y
  - This is our Opinion on y **before** we have collected data
  - This can be an **informed** opinion, or an **uninformed** opinion
  - In the latter case, we typically select a large variance

Chocolate Chips – a Bayesian Analysis

$$f(y|x) = \frac{f(x|y) \cdot f(y)}{\boldsymbol{f(x)}}$$

- $f(x)$ – is the integral of the numerator

$$f(x) = \int \boldsymbol{f(x|y)} \cdot \boldsymbol{f(y)}\, dy$$

- Often – Calculate the numerator, and then integrate to get the denominator
- Allows us to drop constants in the numerator

Chocolate Chips – a Bayesian Analysis

$$\boldsymbol{f(y|x) = \frac{f(x|y) \cdot f(y)}{f(x)}}$$

- $f(y|x)$ – is the **Posterior Distribution** of y
- $f(x|y)$ – is the **Likelihood**
- $f(y)$ – is the **Prior Distribution** of y
- $f(x)$ – is the **Normalizing Constant**

Chocolate Chips – a Bayesian Analysis

$$f(y|x) = \frac{f(x|y) \cdot f(y)}{f(x)}$$

- Let's return to our Chocolate Chip problem
- x are the data points (the 6 cookies)
- $x = \{5,5,7,10,10,11\}$
- y is the parameter of the Poisson distribution
- We will now estimate a distribution for $y|x$
- Then, we can estimate a distribution for x

Chocolate Chips – a Bayesian Analysis

$$f(y|x) = \frac{f(x|y) \cdot \boldsymbol{f(y)}}{f(x)}$$

- We need a **Prior** Distribution for $\boldsymbol{y}$
- We are not confident in our prior, so select a wide variance
- Select **Gamma**, with mean $\mu = 10$, and $\sigma = 4$
- $\alpha = \frac{\mu^2}{\sigma^2} = 6.25\,; \beta = \frac{\mu}{\sigma^2} = 0.625$

$$f(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot y^{\alpha-1} \cdot e^{-\beta y} \qquad f(y) \propto y^{5.25} \cdot e^{-0.625y}$$

Chocolate Chips – a Bayesian Analysis

$$f(y|x) = \frac{\boldsymbol{f(x|y)} \cdot f(y)}{f(x)}$$

5, 5, 7, 10, 10, 11

- We need the Poisson Likelihood:

$$f(x_i|y) = \frac{e^{-y} y^{x_i}}{x_i!}$$

$$f(x|y) = \prod_{i=1}^{6} f(x_i|y) = \prod_{i=1}^{6} \frac{e^{-y} y^{x_i}}{x_i!}$$

$$\propto e^{-6y} y^{\Sigma x_i} = \boldsymbol{e^{-6y} y^{48}}$$

Chocolate Chips – a Bayesian Analysis

$$f(y|x) = \frac{\boldsymbol{f(x|y) \cdot f(y)}}{f(x)}$$

$$f(x|y) \cdot f(y) \propto [e^{-6y}y^{48}] \times [y^{5.25} \cdot e^{-0.625y}]$$

$$= y^{\mathbf{53.25}} \cdot e^{\mathbf{-6.625}y}$$

- This is the Gamma Distribution:

$$f(y) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot y^{\boldsymbol{\alpha - 1}} \cdot e^{\boldsymbol{-\beta} y}$$

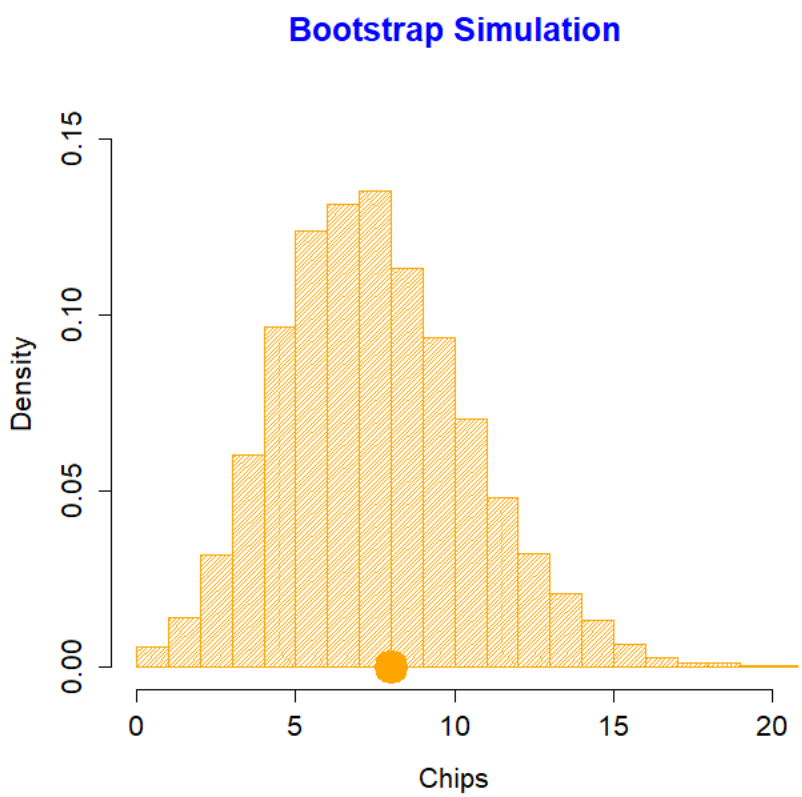
$$\alpha = 54.25$$
$$\beta = 6.625$$

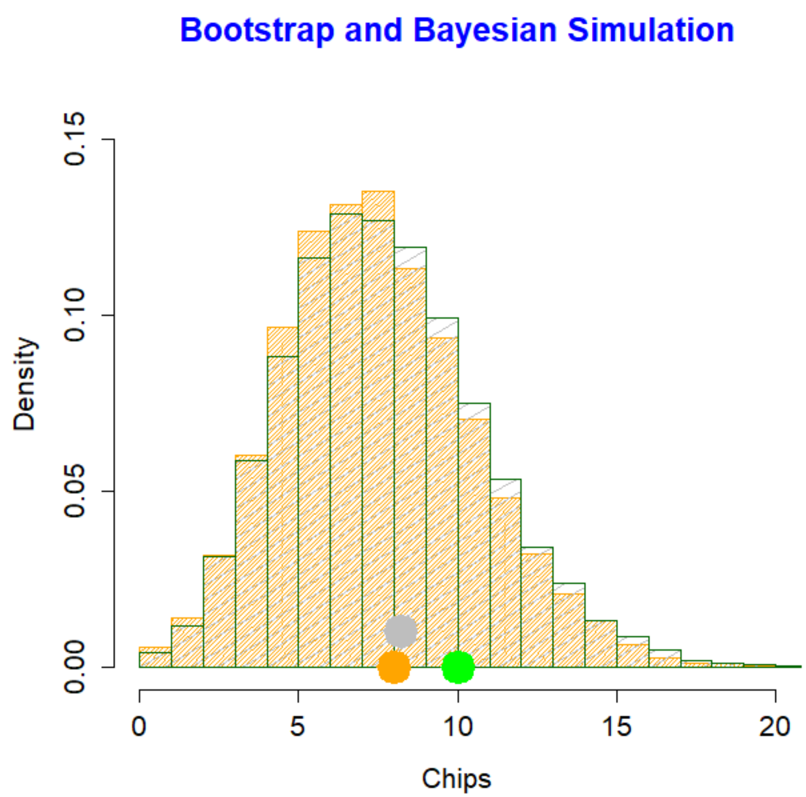
$$\text{Mean} = \frac{\alpha}{\beta} = 8.19 \quad \text{Std. Dev} = \sqrt{\frac{\alpha}{\beta^2}} = 1.11$$

Chocolate Chips – a Bayesian Analysis

- We Calculated the Likelihood Formula, $f(x|y)$
- Selected a Prior, $f(y)$
- Calculated the Posterior, $f(y|x)$
- Draw 20,000 samples from the posterior for y
- For each y, draw 1 cookie

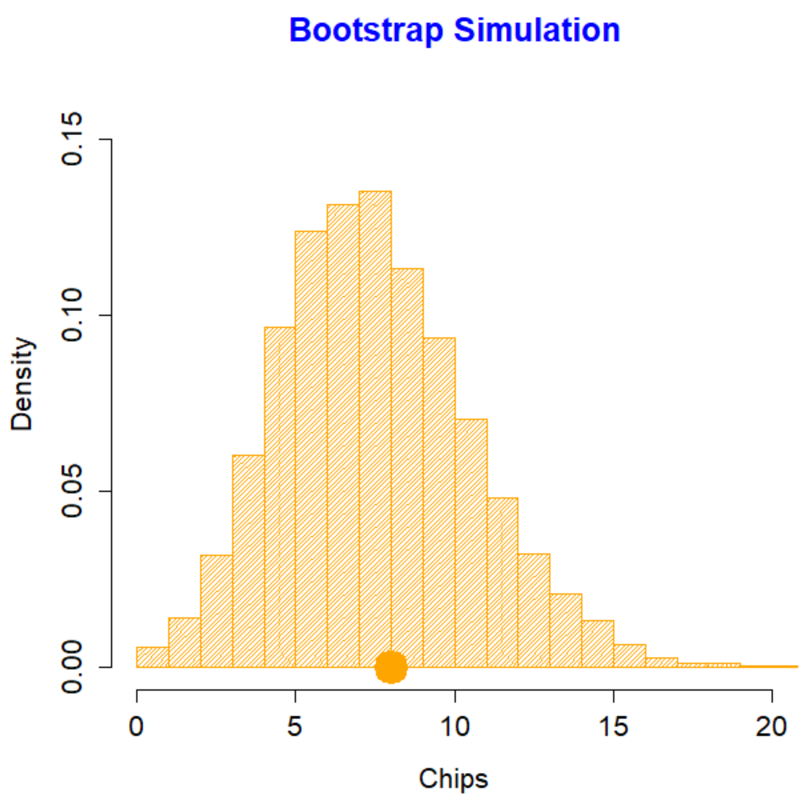
Chocolate Chips – a Bayesian Analysis

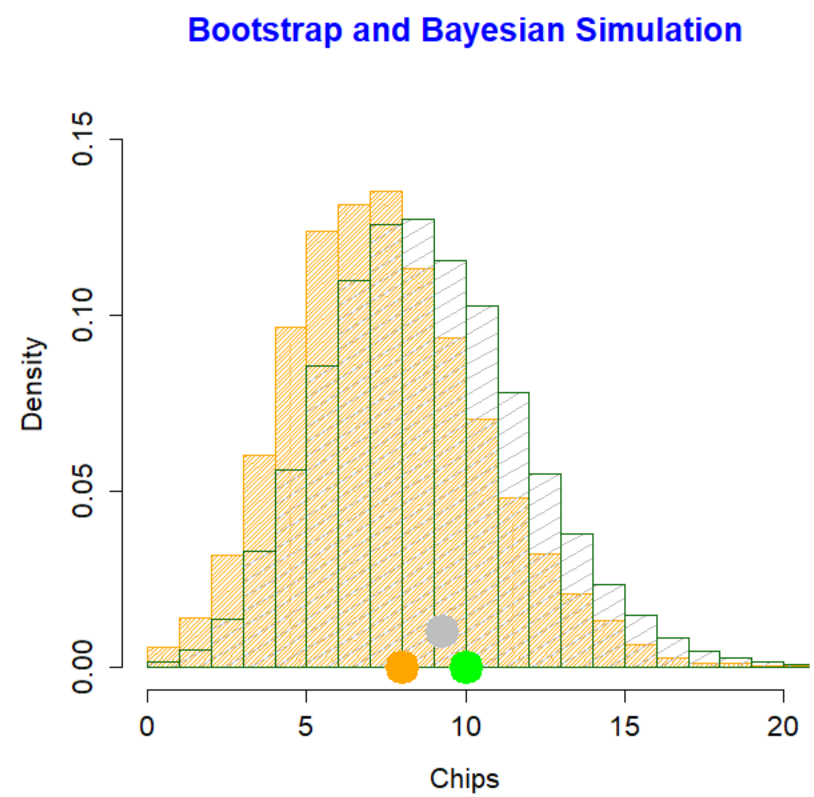




Chocolate Chips – a Bayesian Analysis

- Prior Distribution of Mean of Chips to have:
- Mean= 10, Stdev= 1





Bootstrap vs Bayesian

- Both Bootstrap and Bayesian gave us predictive distribution of the # of Chips in a Cookie
- Bayesian – allowed us to consider our Prior, Expert Opinion

Bayesian Model

- The cookie problem had a relatively simple Posterior Distribution – we lucked out, and it was the Gamma Distribution
- In insurance modeling, we'll have many more parameters, and solving the integral to determine the normalizing constant is intractable

Bayesian Model

- Take a 4x4 Triangle
- Incremental Losses, X_{ij}, are Over Dispersed Poisson, with fixed (but unknown) dispersion parameter ϕ
- 4 Row parameters: α_i ; $i \in 1..4$
- 3 Column Parameters β_j ; $j \in 2..4$
- $\beta_1 = 1$; fixed
- Mean of each cell is: $\alpha_i \cdot \beta_j$
- Variance $= \alpha_i \cdot \beta_j \cdot \phi$
- α_i, β_j have Gamma Priors

Bayesian Model

$$f(X|\alpha,\beta,\phi) = \prod_{(i,j)\in\Delta} \frac{e^{-(\alpha_i\beta_j)/\phi}\left(\frac{\alpha_i\beta_j}{\phi}\right)^{\frac{X_{ij}}{\phi}}}{(X_{ij}/\phi)!}$$

$$f(\alpha,\beta) \propto \prod_{i=1}^{4} \alpha_i^{a_i-1} e^{-b_i\alpha_i} \times \prod_{j=2}^{4} \beta_j^{c_j-1} e^{-d_j\beta_j}$$

$$\alpha_i \sim Gamma(a_i,b_i)\,;\beta_j \sim Gamma(c_j,d_j)$$

Bayesian Model

- We would be able to calculate the Posterior Distribution to within a Normalizing Constant
- We would **not** be able to integrate it – it's intractable
- $g(\alpha, \beta, \phi) \propto f(\alpha, \beta | \phi, X)$
- We can find the **ratio** of the density for any set of parameters to any other set of parameters; but we **don't** have the actual density
- There is an algorithm that allows us to **sample** from this distribution – Metropolis Hastings

Metropolis-Hastings

- Markov Chain is a mapping where the probability of the next state is dependent only on the current state
- A continuous version, with a single parameter could be written as a probability density

$$J(y_i|y_{i-1})$$

Metropolis-Hastings

- If we have, posterior to within a constant:

$$g(y) \propto f(y|x)$$

- Proposal Distribution

$$J(y_i|y_{i-1})$$

- The following algorithm will (in the limit) be samples from the distribution with density $f(y|x)$

Metropolis-Hastings

- 1. Select an initial y_0
- 2. Draw y^* from $J(y^*|y_{i-1})$
- 3. Calculate R

$$R_1 = \frac{g(y^*)}{g(y_{i-1})} \qquad R_2 = \frac{J(y_{i-1}|y^*)}{J(y^*|y_{i-1})}$$

$$R = R_1 \cdot R_2$$

- 4. Draw $u \in Unif(0,1)$
- 5. If $u < R$ then $y_i = y^*$; otherwise $y_i = y_{i-1}$

Repeat Steps 2-5 many times

Metropolis-Hastings

- Numerical Example with Cookies

Metropolis-Hastings

- In the cookie example we calculated a Posterior:

$$f(y|x) \propto g(y) = e^{-6.625y} \cdot y^{53.25}$$

- Proposal Function is Uniform with width 2
- $y_0 = 10$

$$J(y^*|y_0 = 10) = \left\{ \begin{matrix} 0.5 & : y^* \in [9,11] \\ 0 & : \text{else} \end{matrix} \right\}$$

Metropolis-Hastings

$$g(y) = e^{-6.625y} \cdot y^{53.25}$$

- $y_0 = 10$

$$J(y^*|y_0 = 10) = \begin{Bmatrix} 0.5 & : y^* \in [9,11] \\ 0 & : \text{else} \end{Bmatrix}$$

$$y^* = 9.72$$

$$g(y^* = 9.72) = 4.235 \cdot 10^{24}$$

$$g(y_0 = 10) = 3.006 \cdot 10^{24}$$

$$J(y^*|y_0) = J(y_0|y^*) = 0.5$$

$$R = R_1 \cdot R_2 = \frac{g(y^*)}{g(y_0)} \cdot \frac{J(y_0|y^*)}{J(y^*|y_0)} = \frac{g(y^*)}{g(y_0)} = \frac{4.235}{3.006} = 1.409$$

Since $R > 1$, $\boldsymbol{y_1 = y^* = 9.72}$

Metropolis-Hastings

$$g(y) = e^{-6.625y} \cdot y^{53.25}$$

- $y = \{10, 9.72\}$

$$J(y^*|\boldsymbol{y_1} = \mathbf{9.72}) = \begin{Bmatrix} 0.5 & : y^* \in [8.72, 10.72] \\ 0 & : \text{else} \end{Bmatrix}$$

$$y^* = 10.18$$

$$g(y^* = 10.18) = 2.359 \cdot 10^{24}$$

$$g(y_1 = 9.72) = 4.235 \cdot 10^{24}$$

$$R = R_1 \cdot R_2 = \frac{g(y^*)}{g(y_0)} \cdot \frac{J(y_0|y^*)}{J(y^*|y_0)} = \frac{2.359}{4.235} = 0.557$$

$$u \sim Unif(0,1); u = 0.779$$

$$u < R: \ 0.779 < 0.557 \ \ FALSE, \text{do not move}$$

$$\boldsymbol{y_2} = \boldsymbol{y_1} = \mathbf{9.72}$$

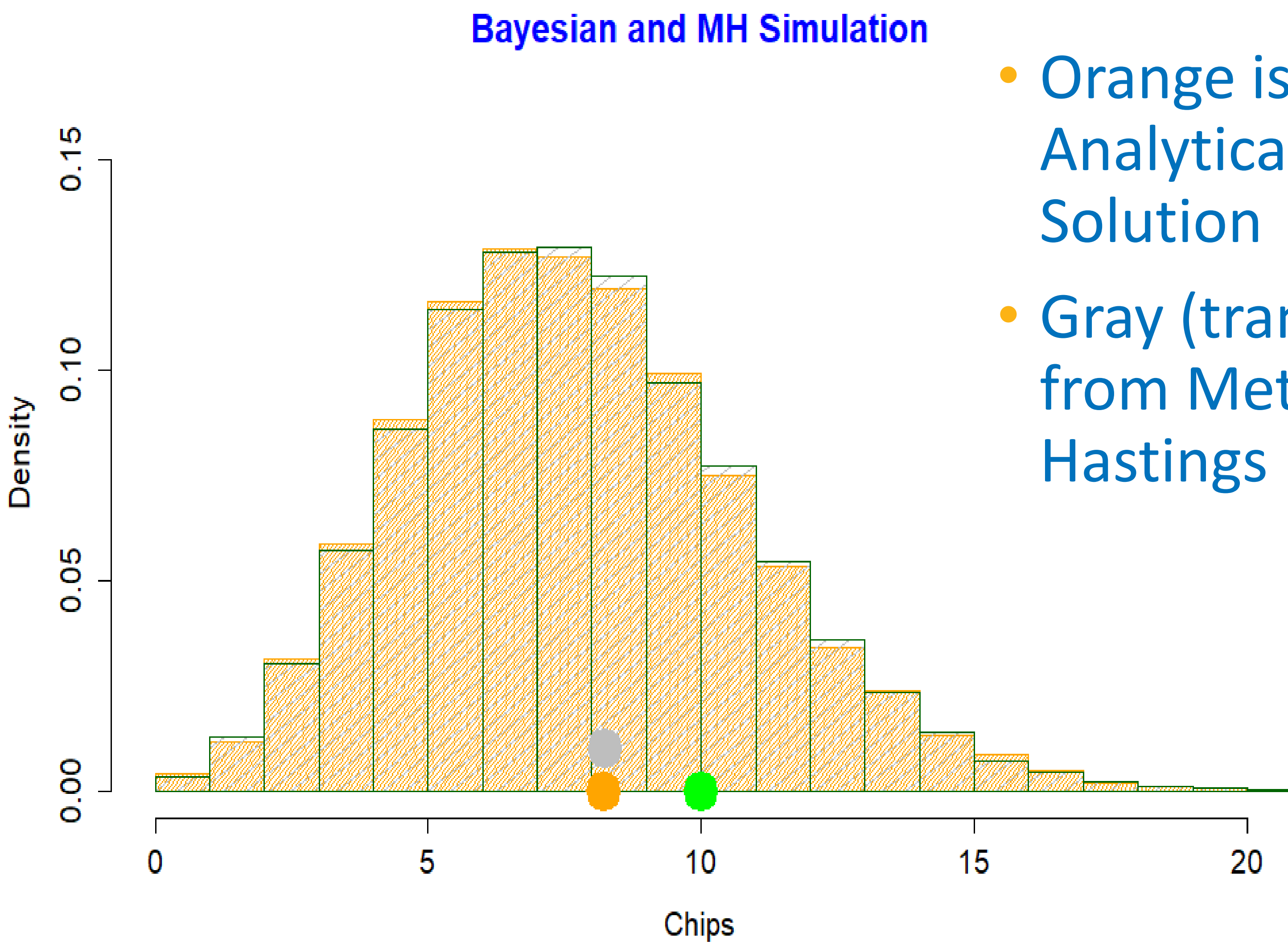
Excel Example

- Do this example of Metropolis Hastings in Excel

Metropolis-Hastings

- Want Acceptance Rate between 23% and 50%
- A High Acceptance Rate will result in the Proposal Distribution
- To Decrease Acceptance – Increase Variance of Proposal Distribution
- Switch from $Uniform(y_{i-1} \pm 1)$
- To $Uniform(y_{i-1} \pm 3)$

Metropolis-Hastings



- Orange is using the Analytical Bayesian Solution
- Gray (transparent) is from Metropolis Hastings

Metropolis-Hastings

- 1. Select an initial y_0
- 2. Draw y^* from $J(y^*|y_{i-1})$
- 3. Calculate R

$$R_1 = \frac{g(y^*)}{g(y_{i-1})} \qquad R_2 = \frac{J(y_{i-1}|y^*)}{J(y^*|y_{i-1})}$$

$$R = R_1 \cdot R_2$$

- 4. Draw $u \in Unif(0,1)$
- 5. If $u < R$ then $y_i = y^*$; otherwise $y_i = y_{i-1}$

Repeat Steps 2-5 many times

Gibbs Sampling

- $g(\alpha, \beta\ ) \propto f(\alpha, \beta|x)$
- Calculate
- $g(\alpha|\beta) \propto f(\alpha|\beta, x)$
- Treat β as a constant; and drop constant terms
- $g(\beta|\alpha)$
- When sampling, do the MH algorithm, assuming β is known; and draw a sample from α
- Then use, this α as a constant, and use MH to draw a sample from β
- Repeat

Gibbs Sampling

- A two-parameter example from CAS Monograph 1, implemented in Excel (for learning) and R (for real)

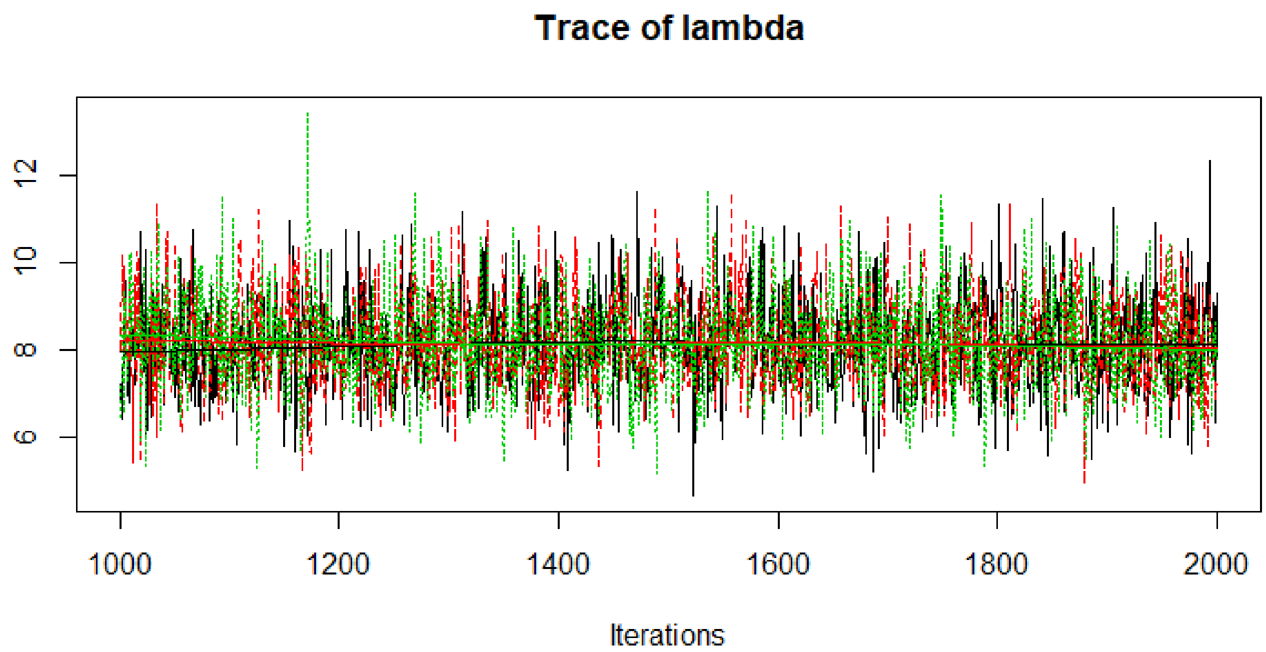
Diagnostics

- Gelman Diagnostic
- Auto Correlation
- Effective Size

Diagnostics

- Gelman Diagnostic
- Run multiple Markov Chains
- Compare the variance within each chain to the variance in other chains

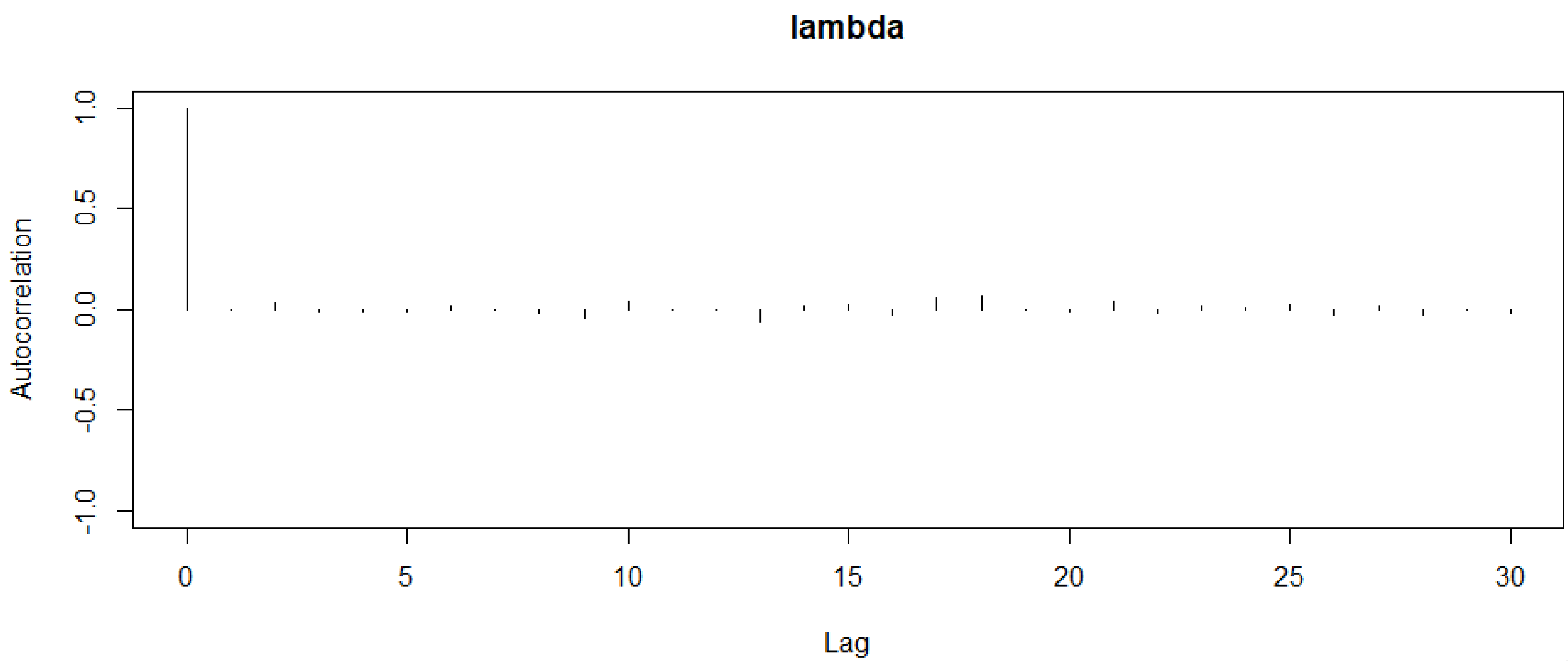
Diagnostics – Gelman



- 3 Chains each of length 1,000
- Gelman Diagnostic is 1.000

Diagnostics - Autocorrelation

- High Autocorrelation, reduces the information in the Markov Chain
- This dataset has low Autocorrelation



Diagnostics - Autocorrelation

- High Autocorrelation, reduces the information in the Markov Chain
- Data can be "thinned"
- Eg. Only use every 20 points from the Markov Chain if the Autocorrelation is small at lag 20 and beyond

Diagnostics – Effective Size

- Effective Size
- If there is Autocorrelation in the Markov Chain, it does not have the same amount of information as the same number of points drawn from the target distribution
- Effective Size tells you how much information is in your markov chain in terms of if it was actual pulls from the target distribution

Summary

- Bayesian Analysis – allows for a **prior** opinion on parameters
- Create a $g(\tilde{y}) \propto f(\tilde{y}|\tilde{x})$, **posterior** distribution
- Use Metropolis-Hastings to **sample** from the posterior distribution
- Use Gibbs Sampling if you have more than one parameter
- Use Diagnostics to test convergence
- You have a sample of the posterior of **all** the parameters
- Simulate Data, using draws from the sampled parameters
- Walk with Excel, run with VBA, fly with R

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