First-Order Differential Equations

CHAPTER 2

CHAPTER CONTENTS

2.1 Solution Curve Without a Solution

- **Introduction**

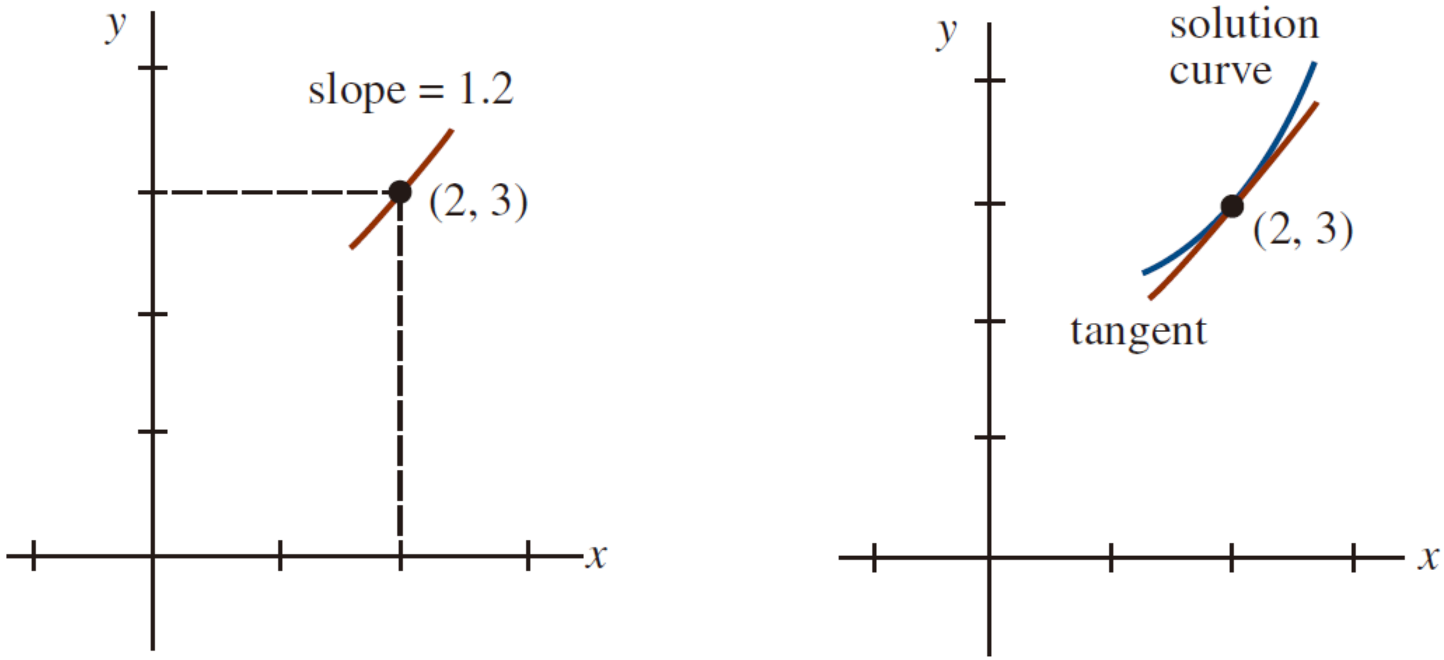
  Begin our study of first-order DE with analyzing a DE **qualitatively**.

- **Slope**

  A derivative dy/dx of $y = y(x)$ gives slopes of tangent lines at points.

- **Lineal Element**

  Assume $dy/dx = f(x, y(x))$. The value $f(x, y)$ represents the slope of a line, or a line element is called a **lineal element**. See Fig 2.1.1.



(a) $f(2, 3) = 1.2$ is slope of lineal element at (2, 3)

(b) A solution curve passing through (2, 3)

$dy/dx = 0.2xy$

then $f(x, y) = 0.2xy$

FIGURE 2.1.1 Solution curve is tangent to lineal element at (2, 3)

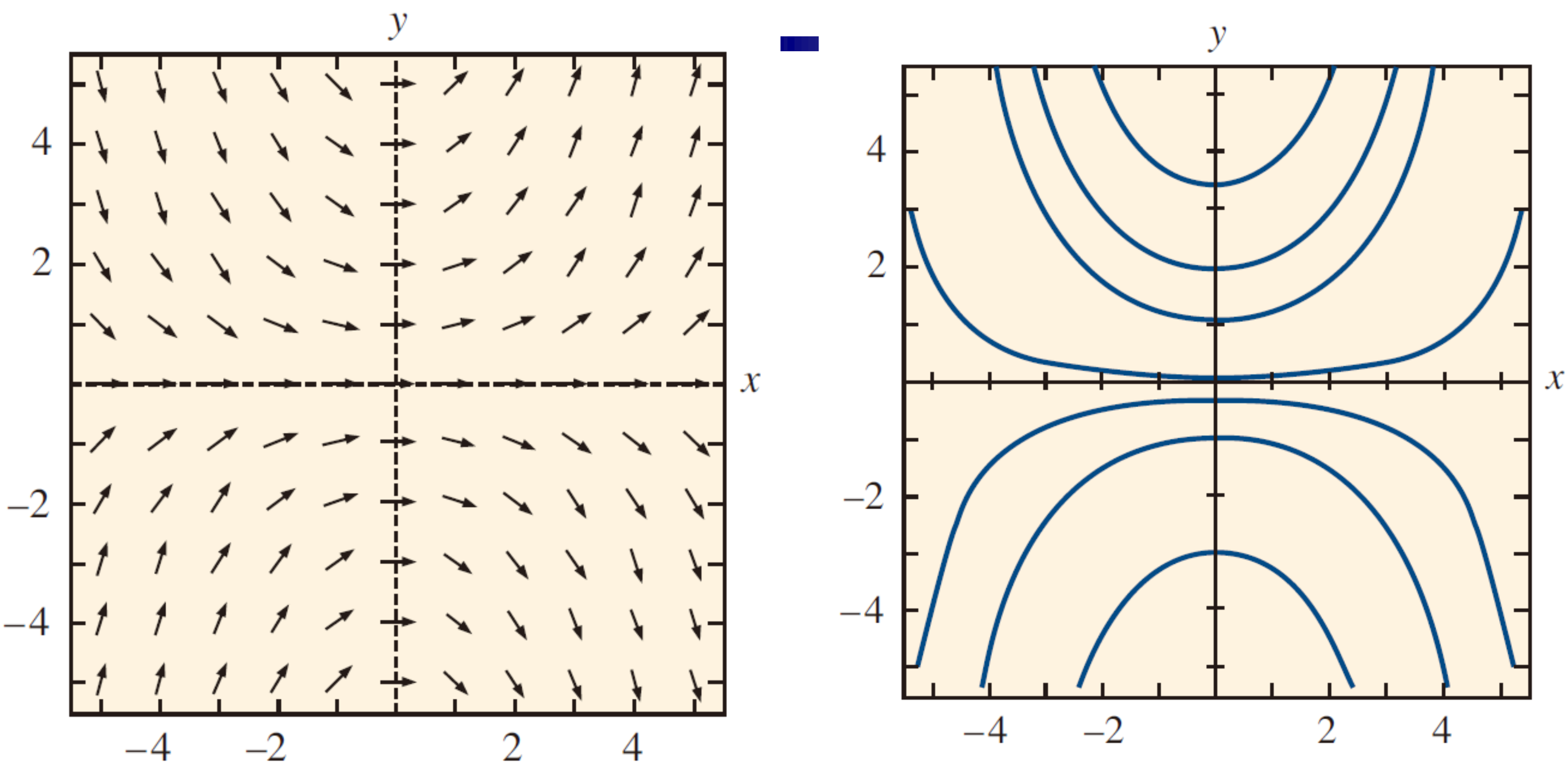
- **Direction Field**

  If we evaluate f over a rectangular grid of points, and draw a lineal element at each point (x, y) of the grid with slope $f(x, y)$, then the collection is called a **direction field** or a **slope field** of the following DE

$$dy/dx = f(x, y)$$

Example 1 Direction Field

- For the DE $dy/dx = 0.2xy$,
- The direction field of $dy/dx = 0.2xy$ is shown in Fig 2.1.2(a) and for comparison with Fig 2.1.2(a), some representative graphs of this family are shown in Fig 2.1.2(b).



(a) Direction field for $dy/dx = 0.2xy$

(b) Some solution curves in the family $y = ce^{0.1x^2}$

FIGURE 2.1.2 Direction field and solution curves in Example 1

Example 2 Direction Field

Use a direction field to draw an approximate solution curve for $dy/dx = \sin y$, $y(0) = -3/2$.

Solution:

Recall from the continuity of $f(x, y)$ and $\partial f/\partial y = \cos y$. Theorem 1.2.1 guarantees the existence of a unique solution curve passing any specified points in the plane. Now split the region containing (0, -3/2) into grids. We calculate the lineal element of each grid to obtain Fig 2.1.3.

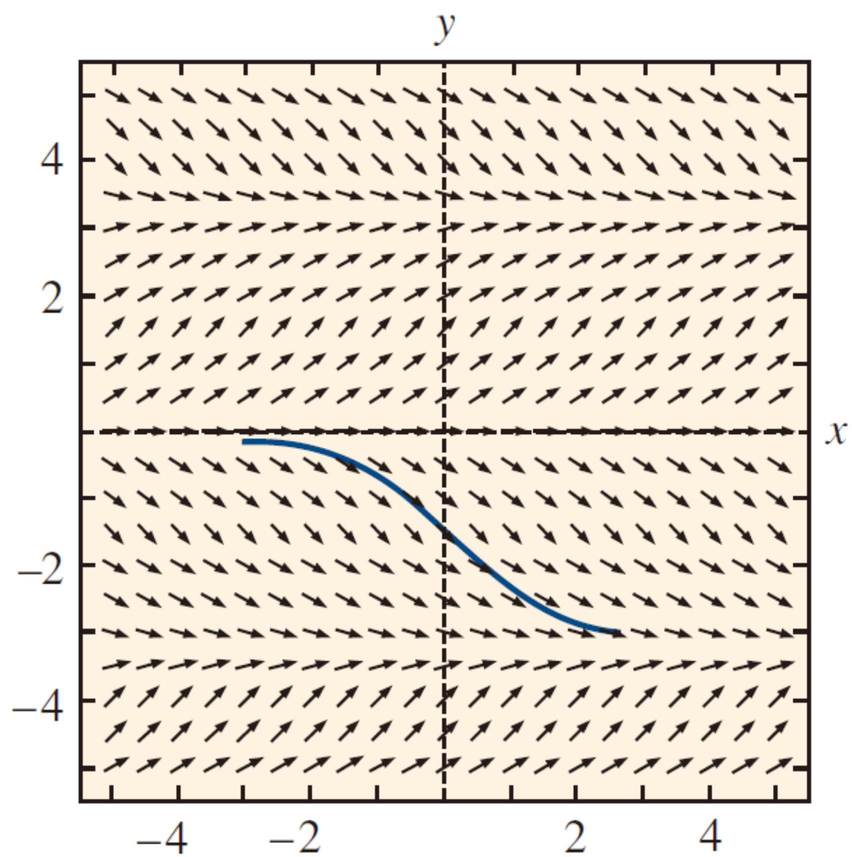


FIGURE 2.1.3 Direction field for $dy/dx = \sin y$ in Example 2

- **Increasing/Decreasing**

  If $dy/dx > 0$ for all x in I, then $y(x)$ is increasing in I.

  If $dy/dx < 0$ for all x in I, then $y(x)$ is decreasing in I.

- **DEs Free of the Independent variable**

$$dy/dx = f(y) \tag{1}$$

  is called **autonomous**. We shall assume f and f' are continuous on some I.

- **Critical Points** $dy/dx = f(y)$ (1)

  The zeros of f in (1) are important. If $f(c) = 0$, then c is a **critical point**, **equilibrium point** or **stationary point**.

- Substitute $y(x) = c$ into (1), then we have $0 = f(c) = 0$.

  If c is a critical point, then $y(x) = c$, is a solution of (1).

- A constant solution $y(x) = c$ of (1) is called an **equilibrium solution**.

Example 3 An Autonomous DE

The following DE

$$dP/dt = P(a - bP),$$

where a and b are positive constants, is autonomous.

From $f(P) = P(a - bP) = 0$, the equilibrium solutions are $P(t) = 0$ and $P(t) = a/b$.

Put the critical points on a vertical line. The arrows in Fig 2.1.4 indicate the algebraic sign of $f(P) = P(a - bP)$. If the sign is positive or negative, then P is increasing or decreasing on that interval.

$dy/dx = f(y)$ (1)
$dP/dt = f(P)$ (1)

$f(P) = P(a - bP) = 0$, the equilibrium solutions are $P(t) = 0$ and $P(t) = a/b$.

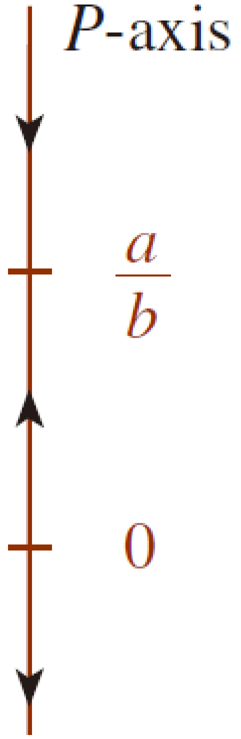


FIGURE 2.1.4 Phase portrait for Example 3

- **Solution Curves**
  If we guarantee the existence and uniqueness of solution of (1), through any point (x_0, y_0) in R, there is only one solution curve. See Fig 2.1.5(a).

- Suppose (1) possesses exactly two critical points, c_1, and c_2, where $c_1 < c_2$. The graph of the equilibrium solution $y(x) = c_1$, $y(x) = c_2$ are horizontal lines and split R into three regions, say R_1, R_2 and R_3 as in Fig 2.1.5(b).

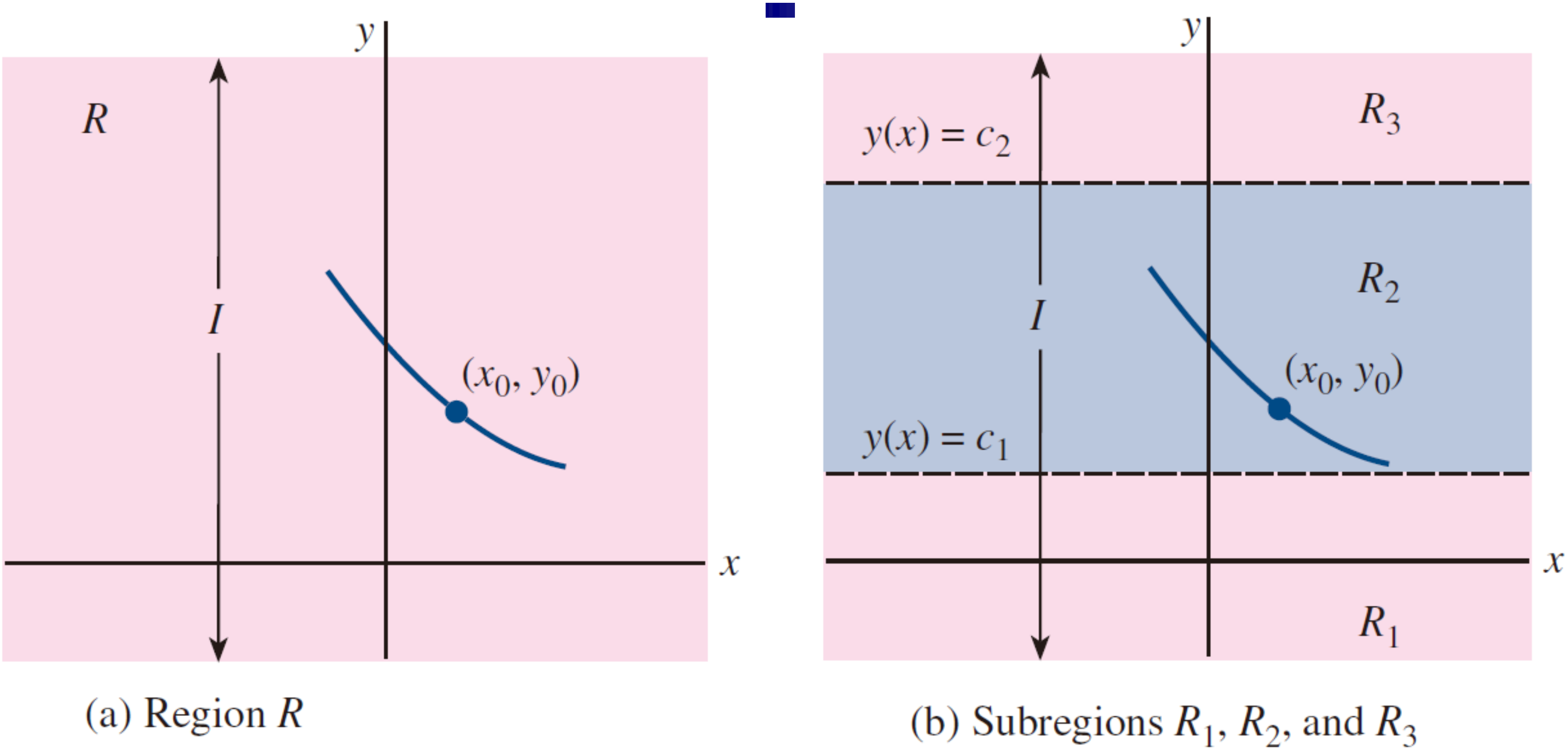


FIGURE 2.1.5 Lines $y(x) = c_1$ and $y(x) = c_2$ partition R into three horizontal subregions

- Some discussions without proof:

  (1) If (x_0, y_0) in R_i, $i = 1, 2, 3$, when $y(x)$ passes through (x_0, y_0), will remain in the same subregion. See Fig 2.1.5(b).

  (2) By continuity of f, $f(y)$ can not change signs in a subregion.

  (3) Since $dy/dx = f(y(x))$ is either positive or negative in R_i, a solution $y(x)$ is **monotonic**.

(4) If $y(x)$ is **bounded above** by c_1, ($y(x) < c_1$), the graph of $y(x)$ will approach $y(x) = c_1$;

If $c_1 < y(x) < c_2$, it will approach $y(x) = c_1$ and $y(x) = c_2$;

If $c_2 < y(x)$, it will approach $y(x) = c_2$;

Example 4 Example 3 Revisited

Referring to example 3, $P = 0$ and $P = a/b$ are two critical points, so we have three intervals for P:

$$R_1 : (-\infty, 0),\ \ R_2 : (0, a/b),\ \ R_3 : (a/b, \infty)$$

Let $P(0) = P_0$ and when a solution pass through P_0, we have three kind of graph according to the interval where P_0 lies on. See Fig 2.1.6.

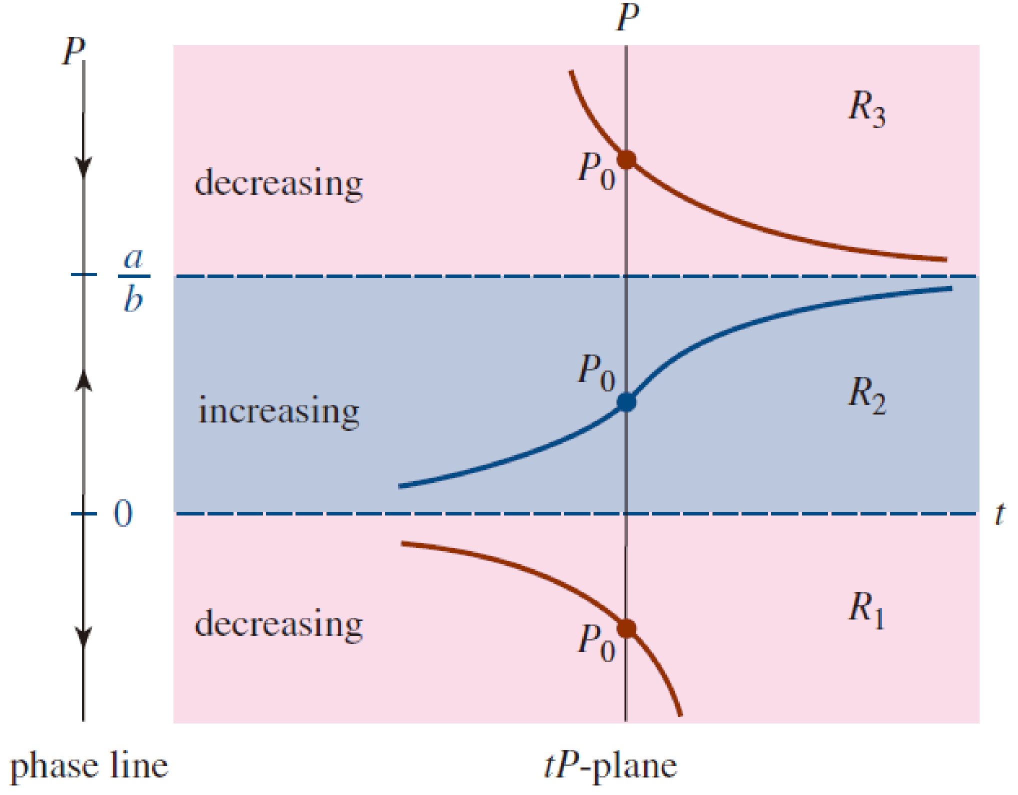


FIGURE 2.1.6 Phase portrait and solution curves in each of the three subregions in Example 4

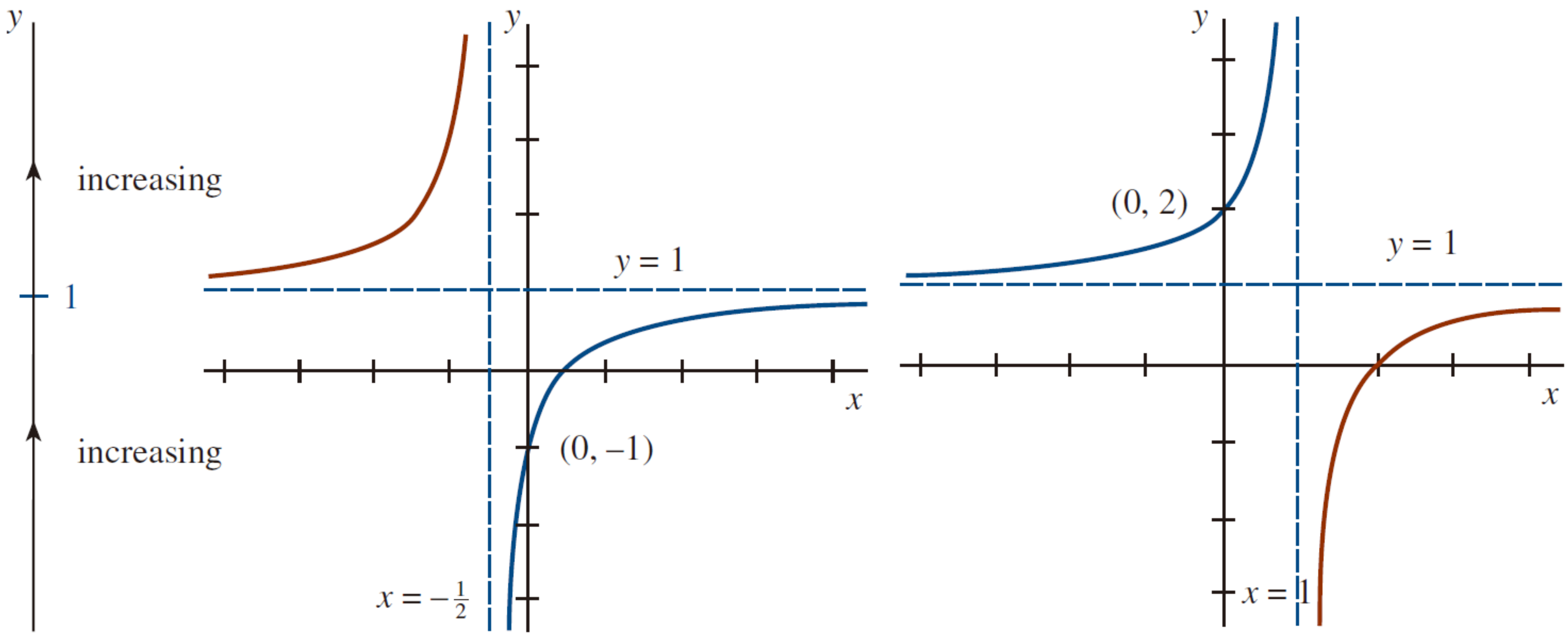
Example 5 Solution Curves of an Autonomous DE

The DE: $dy/dx = (y - 1)^2$ possesses the single critical point 1. From Fig 2.1.7(a), we conclude a solution $y(x)$ is increasing in $-\infty < y < 1$ and $1 < y < \infty$, where $-\infty < x < \infty$. See Fig 2.1.7.

The solutions of the following IVPs are shown in Fig.s 2.1.7(b) and 2.1.7(c), respectively.

$$dy/dx = (y - 1)^2,\ y(0) = -1\ (< 1)$$

$$dy/dx = (y - 1)^2,\ y(0) = 2\ (> 1)$$



(a) Phase line

(b) xy-plane
$y(0) < 1$

(c) xy-plane
$y(0) > 1$

FIGURE 2.1.7 Behavior of solutions near $y = 1$ in Example 5

Attractors and Repellers

- See Fig 2.1.8(a). When y_0 lies on either side of c, it will approach c. This kind of critical point is said to be **asymptotically stable**, also called an **attractor**.

- See Fig 2.1.8(b). When y_0 lies on either side of c, it will move away from c. This kind of critical point is said to be **unstable**, also called a **repeller**.

- See Fig 2.1.8(c) and (d). When y_0 lies on one side of c, it will be attracted to c and repelled from the other side. This kind of critical point is said to be **semi-stable**.

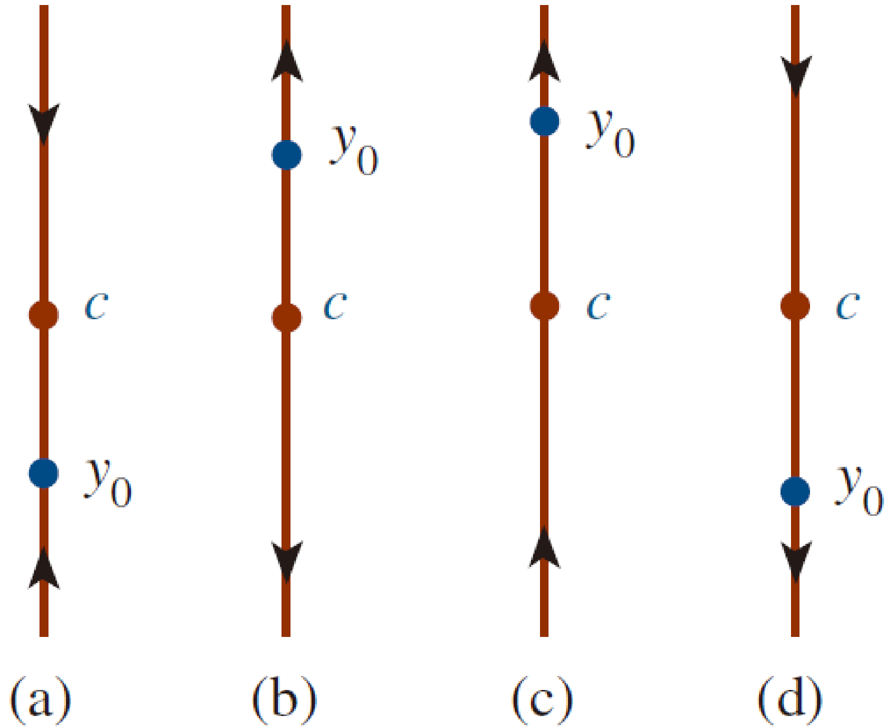


FIGURE 2.1.8 Critical point c is an attractor in (a), a repeller in (b), and semi-stable in (c) and (d)

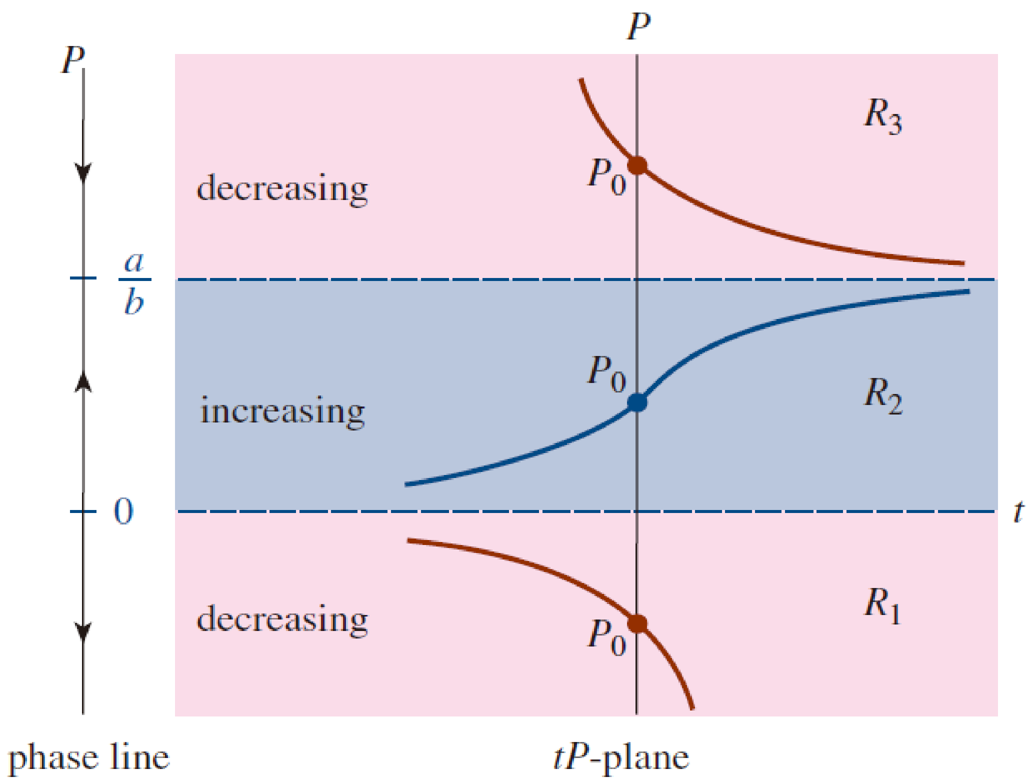


FIGURE 2.1.6 Phase portrait and solution curves in each of the three subregions in Example 4

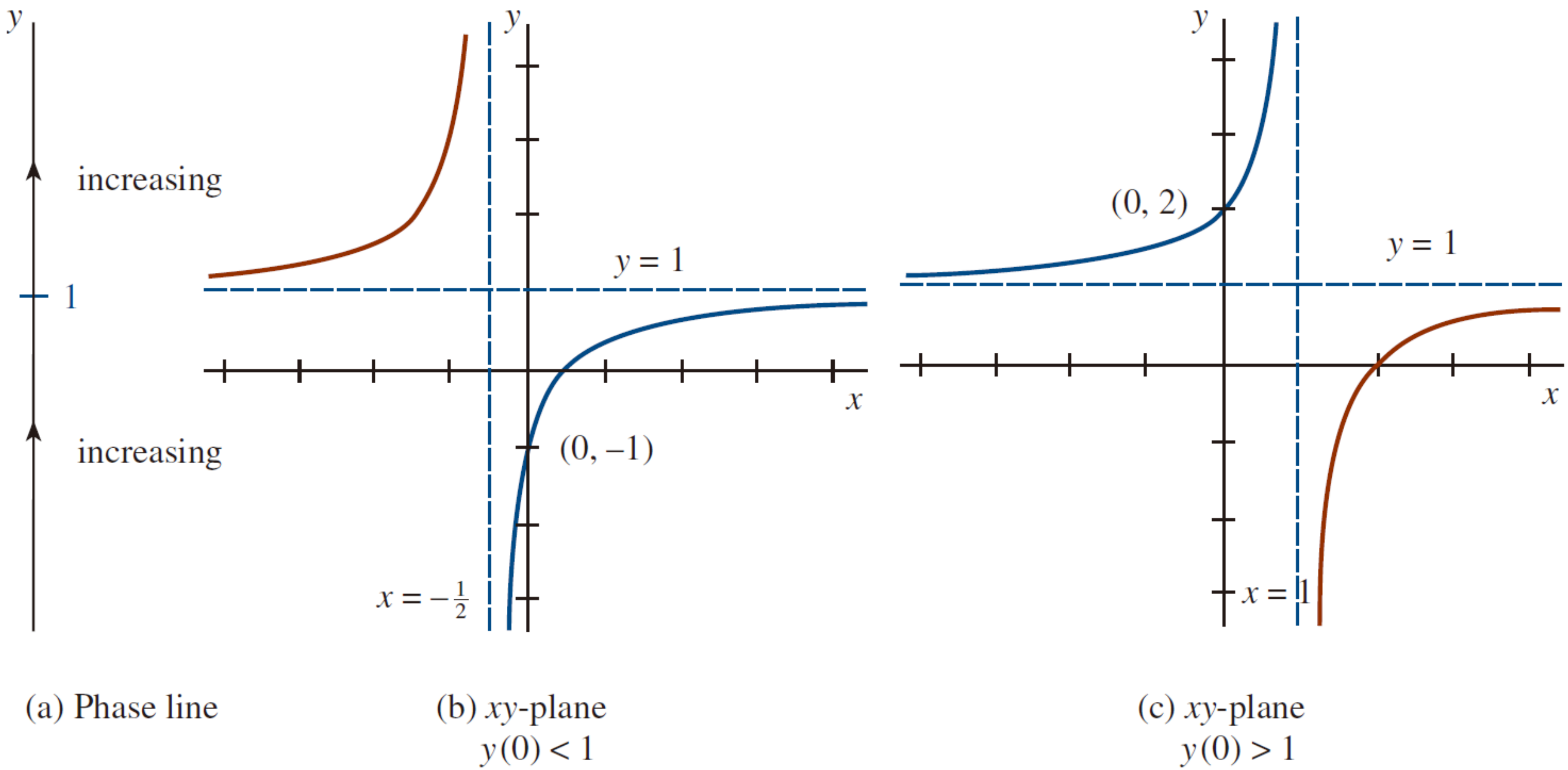


FIGURE 2.1.7 Behavior of solutions near $y = 1$ in Example 5

- **Autonomous DEs and Direction Fields**

  Fig 2.1.9 shows the direction field of $dy/dx = 2(y - 1)$.

  It can be seen that lineal elements passing through points on any **horizontal** line must have the same slope. Since the DE has the form $dy/dx = f(y)$, the slope depends only on y.

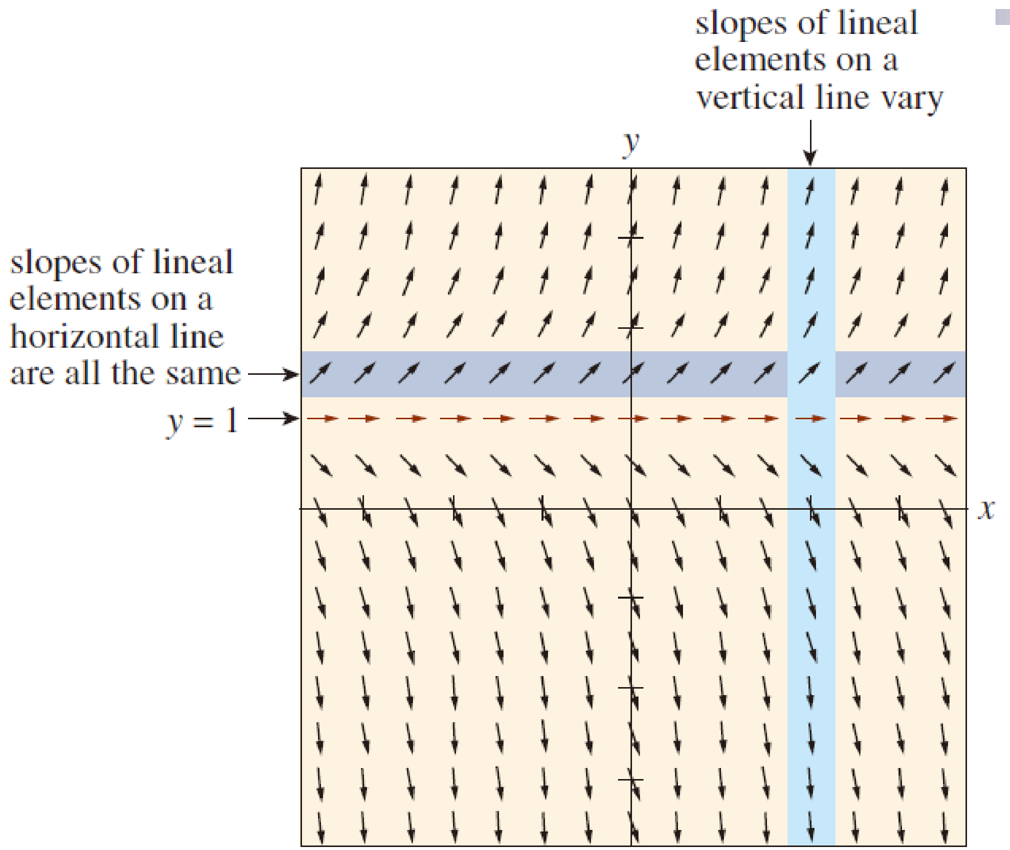


FIGURE 2.1.9 Direction field for an autonomous DE