1.2 Initial-Value Problems

- **Introduction**

  A solution $y(x)$ of a DE satisfies an initial condition.

- **Initial-Value Problem**

  On some interval I containing x_0,

$$Solve: \frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}) \qquad (1)$$

$$subjuct\ to: \ y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$$

- This is called an **Initial-Value Problem (IVP).**

$$y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$$

  are called **initial conditions**.

- **First and Second IVPs**

$$\begin{aligned} &solve: \quad \frac{dy}{dx} = f(x, y) \\ &subject\ \ to: \ y(x_0) = y_0 \end{aligned} \qquad (2)$$

and

$$\begin{aligned} &solve: \quad \frac{d^2y}{dx^2} = f(x, y, y') \\ &subject\ \ to: \ y(x_0) = y_0, y'(x_0) = y_1 \end{aligned} \qquad (3)$$

are first and second order initial-value problems, respectively. See Fig. 1.2.1 and 1.2.2.

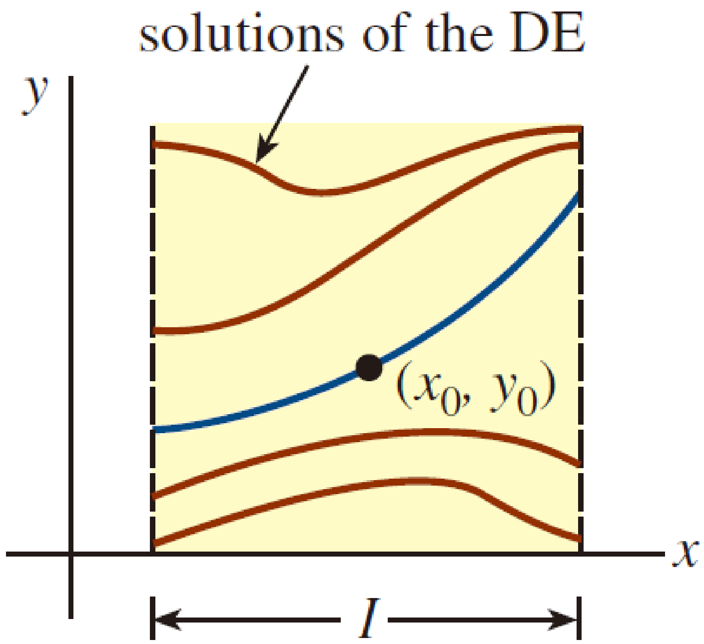


FIGURE 1.2.1 First-order IVP

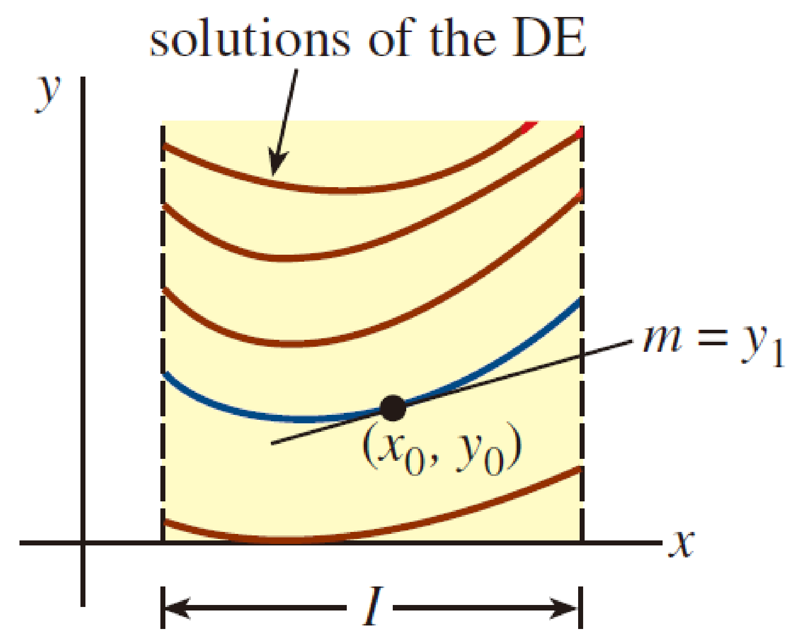


FIGURE 1.2.2 Second-order IVP

Example 1 First-Order IVPs

We know $y = ce^x$ is the solutions of $y' = y$ on $(-\infty, \infty)$. If $y(0) = 3$, then $3 = ce^0 = c$. Thus $y = 3e^x$ is a solution of this initial-value problem

$$y' = y,\ \ y(0) = 3.$$

If we want a solution pass through (1, -2), that is

$$y(1) = -2,\ -2 = ce, \text{ or } \ c = -2e^{-1}.$$

The function $y = -2e^{x-1}$ is a solution of the initial-value problem $y' = y,\ \ y(1) = -2.$

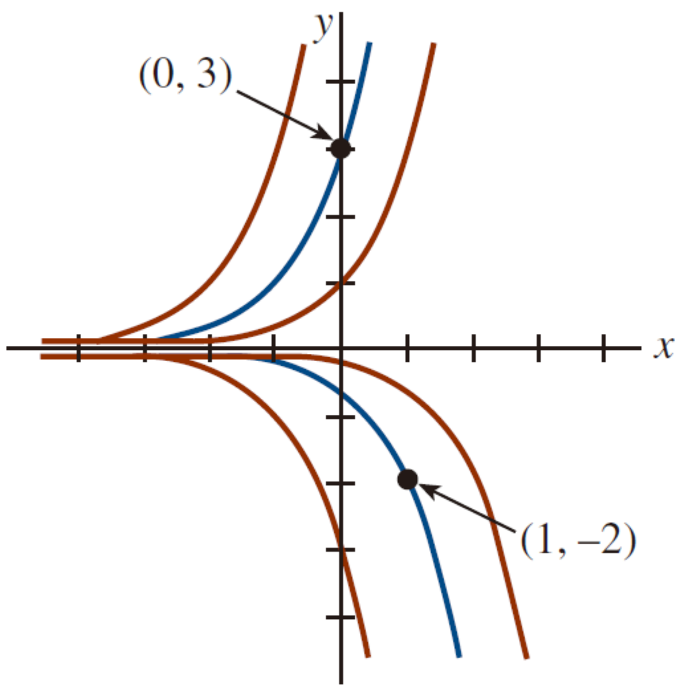


FIGURE 1.2.3 Solutions of IVPs in Example 1

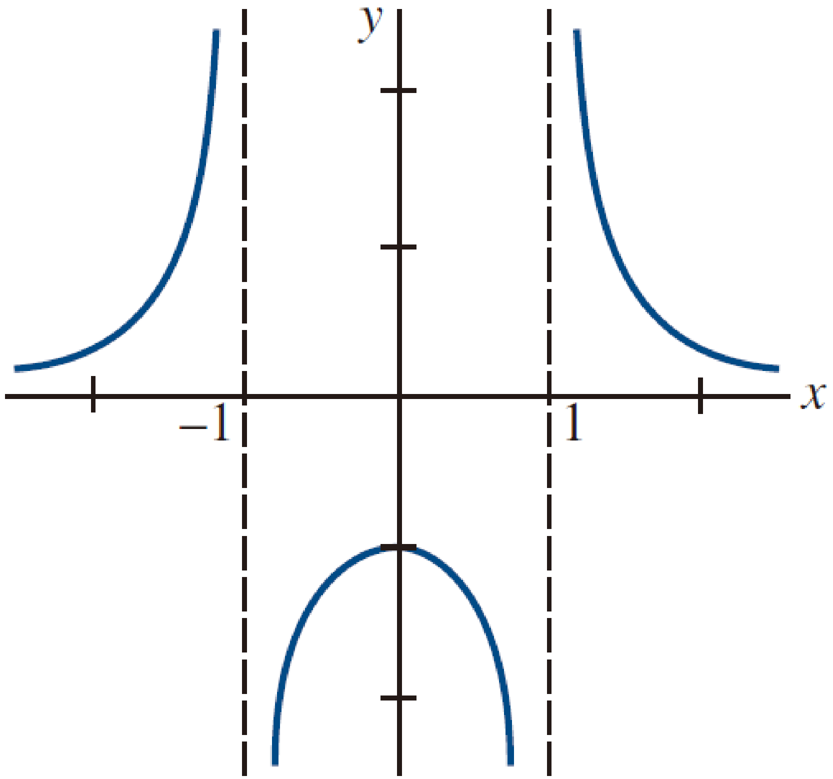
Example 2 Interval *I* of Definition of a Solution

In Exercise # 6 of Sec. 2.2, we have the solution of $y' + 2xy^2 = 0$, which is $y = 1/(x^2 + c)$. If we impose $y(0) = -1$, it gives $c = -1$.

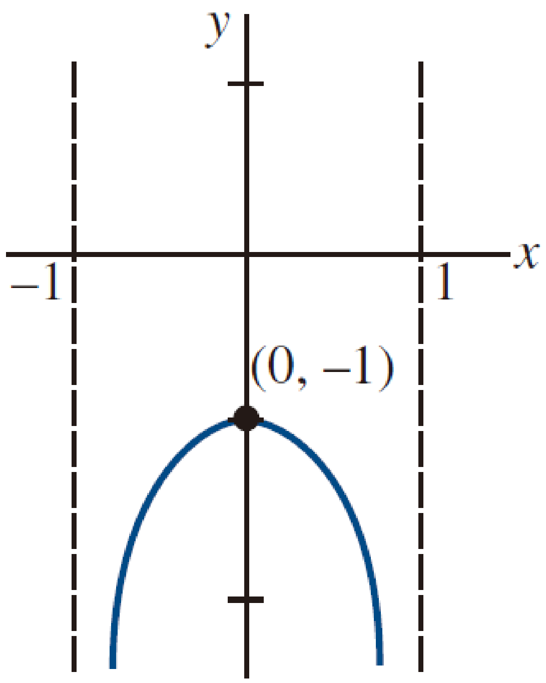
Consider the following distinctions.

1) As a *function*, the domain of $y = 1/(x^2 - 1)$ is the set of all real numbers except $x = -1$ and 1. See Fig. 1.2.4(a).
2) As a *solution*, the intervals of definition are $(-\infty, 1)$, $(-1, 1)$, $(1, \infty)$
3) As a *initial-value problem*, $y(0) = -1$, the interval of definition is $(-1, 1)$. See Fig. 1.2.4(b).

Fig 1.2.4 Graph of the function and solution of IVP in Ex 2



(a) Function defined for all x except $x = \pm 1$



(b) Solution defined on interval containing $x = 0$

Example 3 Second-Order IVP

In Example 7 of Sec. 1.1, $x = c_1\cos 4t + c_2\sin 4t$ is a solution of $x'' + 16x = 0$

Find a solution of the following IVP:

$$x'' + 16x = 0,\ x(\pi/2) = -2,\ x'(\pi/2) = 1 \qquad (4)$$

Solution:

Substitute $x(\pi/2) = -2$ into $x = c_1\cos 4t + c_2\sin 4t$, we find $c_1 = -2$. In the same manner, from $x'(\pi/2) = 1$ we have $c_2 = ¼$. Thus, the solution of the IVP is

$$x = -2\cos 4t + (¼)\sin 4t$$

- **Existence and Uniqueness**

Does a solution of the IVP exist?
If a solution exists, is it unique?

Example 4 An IVP Can Have Several Solutions

Since $y = x^4/16$ and $y = 0$ satisfy the DE $dy/dx = xy^{1/2}$, and also initial-value $y(0) = 0$, this DE has at least two solutions, See Fig 1.2.5.

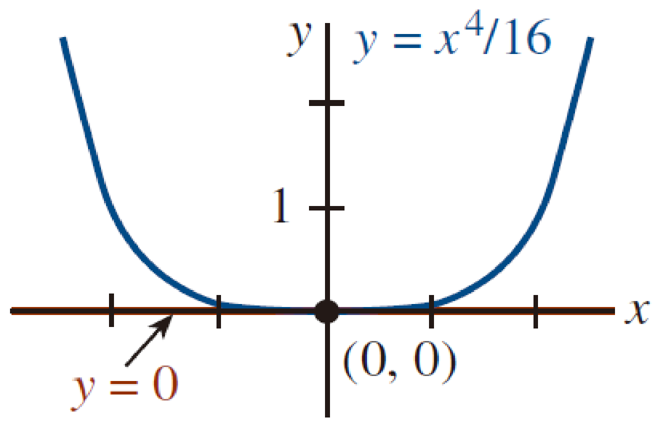


FIGURE 1.2.5 Two solutions of the same IVP in Example 4

Theorem 1.2.1 Existence of a Unique Solution

Let R be the region defined by $a \le x \le b$, $c \le y \le d$ that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\partial f/\partial y$ are continuous in R, then there exists some interval I_0: $(x_0 - h, x_0 + h)$, $h > 0$, contained in $[a, b]$ and a unique function $y(x)$ defined on I_0 that is a solution of the IVP (2).

- The geometry of Theorem 1.2.1 shows in Fig 1.2.6.

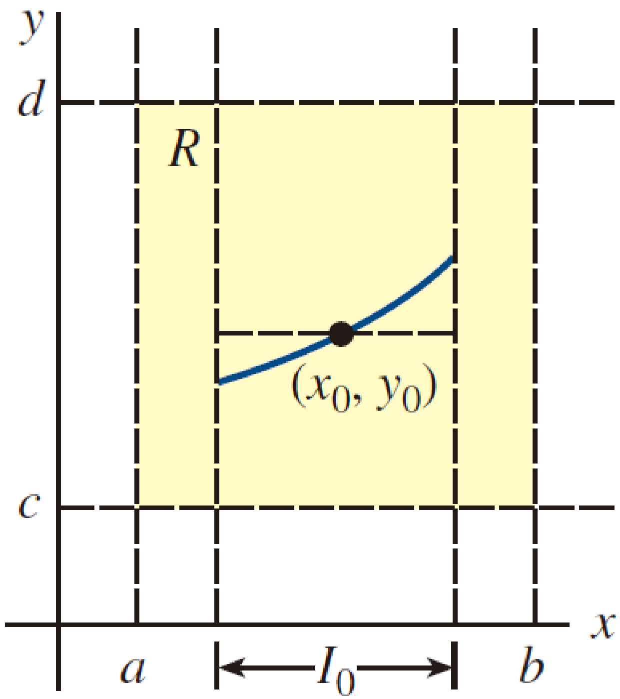


FIGURE 1.2.6 Rectangular region R

Example 5 Example 4 Revisited

For the DE: $dy/dx = xy^{1/2}$, inspection of the functions $f(x, y) = xy^{1/2}$ and $\dfrac{\partial f}{\partial y} = \dfrac{x}{2y^{1/2}}$

we find they are continuous in $y > 0$. From Theorem 1.2.1, we conclude that through any point (x_0, y_0), $y_0 > 0$, there is some interval centered at x_0 on which this DE has a unique solution.

- **Interval of Existence / Uniqueness**

  Suppose $y(x)$ is a solution of IVP (2), the following sets may not be the same:

  the **domain** of $y(x)$,

  the **interval of definition** of $y(x)$ as a solution,

  the **interval I_0 of existence and uniqueness**.