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Title: RF Gap Transformation in PARMILA

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Technical Note

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1. Introduction

The PARMILA (Phase And Radial Motion in Ion Linear Accelerators) code is used to design ion linacs and simulate the beam dynamics therein. The original code was developed in the 1960's and later the Los Alamos Accelerator Code Group developed a PC version that has been used worldwide. Recently, in an effort to modernize the code and use it as the basis for a high-performance beam dynamics simulator with the LANSCE linac, the details of the code have come under additional scrutiny. PARMILA is a "z code" that uses various transformations to approximately represent the effect of the beam line element on the beam. In the case of the RF gap, it treats the particle dynamics inside the gap as a combination of drift + kick at the mid-plane + drift [1][2]. Recently, we uncovered an undocumented keyword "offexact"[3] in PARMILA that enables the code to use the full modified-Bessel functions instead of low-order approximations in the RF gap transformation. This change improved the accuracy of the transformation for off-axis particles. However, a discrepancy between the analytical formula and the code was found for the transformation of the transverse coordinates. The purpose of this tech note is to document the derivation of the gap transformation and the related DTL cell length calculation used in the PARMILA code.

2. RF gap transformation

The TM field inside a gap takes the form of

$$E_{r,z}(r,z,t) = E_{r,z}(r,z)\cos(\omega t + \varphi)\,, \tag{1}$$

where ω is the angular frequency of the RF, $\varphi = kz + \varphi_0$ and φ_0 is the phase of the RF field when the synchronous particles cross the center of the gap at $z = 0$. And $E_r(r,z)$ and $E_z(r,z)$ have the following symmetries

$$E_r(r,-z) = -E_r(r,z),\;\; E_z(r,-z) = E_z(r,z)\,. \tag{2}$$

The transverse and longitudinal transit time factors of the gap are defined as

$$\begin{aligned} T_l(k,r) &= \frac{\int_{-L/2}^{L/2} E_z(z,r)\cos(kz)\,dz}{V_0} \\ T_r(k,r) &= \frac{\int_{-L/2}^{L/2} E_r(z,r)\sin(kz)\,dz}{V_0} \\ S_l(k,r) &= \frac{2\int_0^{L/2} E_z(z,r)\sin(kz)\,dz}{V_0} \\ S_r(k,r) &= \frac{2\int_0^{L/2} E_r(z,r)\cos(kz)\,dz}{V_0} \end{aligned} \tag{3}$$

with $V_0 = \int_{-L/2}^{L/2} E_z(z,r)dz$, where the integral cover one cell length from *–L/2* to *L/2*. Using these definitions for the transit time factors and the wave equations for the electric fields in cylindrical coordinates [4], the transit time factors for off-axis particles can be derived and are given by

$$T_l(k,r) = T_0(k) I_0(k_r r)$$
$$T_r(k,r) = T_0(k) k I_1(k_r r)\,/k_r\ , \qquad (4)$$

where $T_0(k)$ is the transit time factor for on-axis particles and ${k_r}^2 = k^2 - (\frac{2\pi}{\lambda})^2$, $k = \frac{2\pi}{\beta\lambda}$, $k_r = \frac{2\pi}{\beta\gamma\lambda}$. The calculations in the following subsections are done first for nonrelativistic cases. The corrections for relativistic cases are added in using Table 2 of [2].

2.1 Energy change of a particle crossing an RF gap

The energy gain of a particle across the gap is

$$\Delta W = \int_{-L/2}^{L/2} eE_z(r,z)\cos\varphi\, dz\ . \qquad (5)$$

A Taylor expansion for $E_z(r,z)$ around a particle's displacement r_0 gives

$$E_z(r,z) = E_z(r_0,z) + \frac{\partial E_z(r,z)}{\partial r}|_{r=r_0}(r-r_0) + O(r-r_0)\ . \qquad (6)$$

Here $r - r_0 = \frac{dr}{dz}|_{r=r_0} z = {r_0}'z$. By using the symmetry in Eq. (2) and $\cos\varphi = \cos(kz)\cos\varphi_0 - \sin(kz)\sin\varphi_0$, ΔW can be written as

$$\Delta W = \int_{-L/2}^{L/2} eE_z(r_0,z)\cos(kz)\cos\varphi_0\, dz - \int_{-L/2}^{L/2} e\frac{\partial E_z(r,z)}{\partial r}|_{r=r_0}\, {r_0}'z\sin(kz)\sin\varphi_0\, dz + O({r_0}')$$
$$\approx eV_0 T_l(k,r_0)\cos\varphi_0 + eV_0\frac{d}{dk}\frac{\partial T_l(k,r_0)}{\partial r_0}{r_0}'\sin\varphi_0\ . \qquad (7)$$

From Eq. (4), one can get

$$\frac{\partial T_l(k,r_0)}{\partial r_0} = T_0(k) k_r I_1(k_r r)\ . \qquad (8)$$

If the relativistic case is considered, then an extra term should be added to the energy gain according to that shown in Table 2 of [2], then

$$\Delta W \approx eV_0 T_l(k,r_0)\cos\varphi_0 + eV_0\frac{d}{dk}(T_0(k)k_r I_1){r_0}'\sin\varphi_0 - eV_0 T_0(k) I_1 \gamma {r_0}'\sin\varphi_0$$
$$= eV_0 T_0(k) I_0(k_r r_0)\cos\varphi_0 + eV_0\left(\frac{dT_0}{dk}k_r I_1 + \frac{T_0(k)I_1}{\gamma} + \frac{T_0(k)k_r {I_1}' r_0}{\gamma} - T_0(k) I_1\gamma\right){r_0}'\sin\varphi_0\ , \qquad (9)$$

where $I_1 = I_1(k_r r)$. PARMILA uses the transit time factor definitions from the SUPERFISH code [5], i.e.

$$T_{SF} = T_0(k_0),\ S_{SF} = S_0(k_0),\ {T_{SF}}' = -\frac{1}{\beta\lambda}\frac{dT_0}{dk},\ {S_{SF}}' = \frac{1}{\beta\lambda}\frac{dS_0}{dk}\ . \qquad (10)$$

So $T_0(k)$ in PARMILA is

$$T_0(k) = T_0(k_0) + \frac{dT_0}{dk}\Delta k = T_{SF} - \frac{2\pi\Delta\beta}{\beta} T_{SF}{}' .$$

Note that Eq. (9) gives the energy change across the gap for a synchronous particle. For an asynchronous particle, the synchronous phase should be replaced with the particle's phase at the mid-plane of the gap (see Eq. (19)), which will be discussed in the next section.

2.2 Phase change of a particle crossing an RF gap

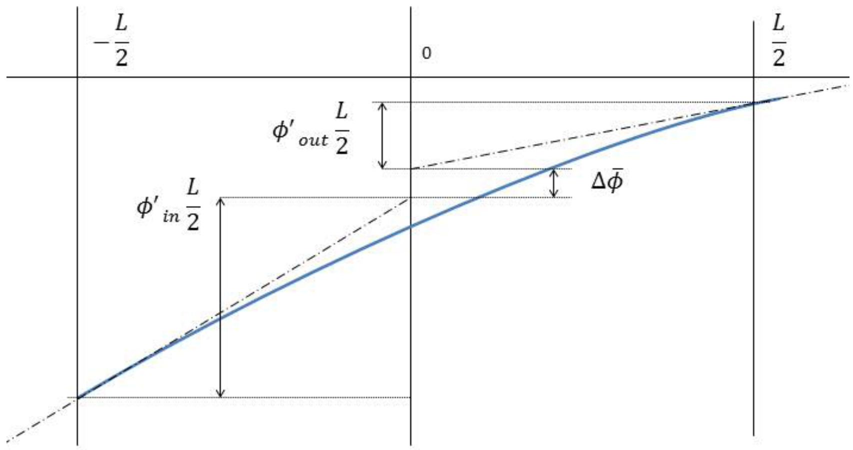

Figure 1: Phase change of a particle crossing an RF gap.

A particle's phase inside a cell is:

$$\varphi(z) = \varphi_{in} + \int_{-L/2}^{z} \varphi'(\zeta) d\zeta = \varphi_{in} + \int_{-L/2}^{z} d\zeta \left(\varphi'_{in} + \int_{-L/2}^{\zeta} \varphi''(\eta) d\eta\right)$$
$$= \varphi_{in} + (z + \tfrac{L}{2})\varphi'_{in} + \int_{-L/2}^{z} d\zeta \int_{-L/2}^{\zeta} \varphi''(\eta) d\eta .$$

With an interchange of the integration, the last term becomes

$$\int_{-L/2}^{z} d\zeta \int_{-L/2}^{\zeta} \varphi''(\eta) d\eta = \int_{-L/2}^{z} \varphi''(\eta) d\eta \int_{\eta}^{z} d\zeta = z\, \varphi'(z) - z\varphi'\left(-\tfrac{L}{2}\right) - \int_{-L/2}^{z} \eta\varphi''(\eta) d\eta .$$

Therefore at the exit of the cell where $z = \frac{L}{2}$

$$\varphi_{out} - \varphi_{in} = \varphi'_{in} L + \frac{L}{2}\left[\varphi'\left(\frac{L}{2}\right) - \varphi'\left(-\frac{L}{2}\right)\right] - \int_{-L/2}^{L/2} \eta\varphi''(\eta) d\eta = \frac{L}{2}\varphi'_{in} + \frac{L}{2}\varphi'_{out} + \Delta\bar{\varphi} \quad (11)$$

where

$$\Delta\bar{\varphi} = -\int_{-L/2}^{L/2} \eta\varphi''(\eta) d\eta . \quad (12)$$

Since $\frac{d\varphi}{dz} = \frac{\omega}{dz/dt} = \omega(\frac{m}{2W})^{1/2}$, one can get

$$\frac{d^2\varphi}{dz^2} = -\omega(\frac{m}{8W^3})^{1/2}\ \frac{dW}{dz}\,. \tag{13}$$

With $W = \frac{m\omega^2}{2k^2}$, φ'' becomes

$$\varphi'' = -\frac{k}{2W}\frac{dW}{dz}\,.$$

Eq. (12) can now be rewritten as

$$\Delta\bar{\varphi} = \frac{k}{2W}\int_{-L/2}^{L/2} z dW\ .$$

Using Eqs. (2), (5), (6) and $\cos\phi = \cos(kz)\cos\phi_0 - \sin(kz)\sin\phi_0$, the equation above can be rewritten as

$$\Delta\bar{\varphi} = -\frac{ek}{2W}\int_{-L/2}^{L/2} E_z(r_0,z) z\sin(kz)\sin\varphi_0\, dz + \frac{ek}{2W} r_0{}' \int_{-L/2}^{L/2} \frac{\partial E_z(r,z)}{\partial r}|_{r=r_0} z^2\cos(kz)\cos\varphi_0\, dz + O(r_0{}')$$

$$\approx \frac{eV_0k}{2W}\frac{d}{dk}T_l(k,r_0)\sin\varphi_0 - \frac{eV_0k}{2W}\frac{d^2}{dk^2}\frac{\partial T_l(k,r_0)}{\partial r_0} r_0{}'\cos\varphi_0\ .$$

Including the relativistic term from Table 2 of [2], the phase change at the center of the gap is

$$\Delta\bar{\varphi} = \frac{eV_0k}{2W\gamma^3}\frac{d}{dk}T_l(k,r_0)\sin\varphi_0 - \frac{eV_0k}{2W\gamma^3}\frac{d^2}{dk^2}\frac{\partial T_l(k,r_0)}{\partial r_0} r_0{}'\cos\varphi_0 - \frac{eV_0k}{2W\gamma}\beta^2 T_0\frac{I_1}{k_r} r_0{}'\cos\varphi_0\ .$$

In PARMILA, the $\frac{d^2}{dk^2}$ term is neglected. With $W = \frac{1}{2}m\beta^2c^2$, the equation above gives

$$\Delta\bar{\varphi} = \frac{eV_0k}{m\beta^2c^2\gamma^3}\left[T_0{}' I_0(k_r r) + T_0 I_1(k_r r)\frac{r}{\gamma}\right]\sin\varphi_0 - \frac{eV_0}{mc^2}T_0 I_1 r_0{}'\cos\varphi_0$$

Substituting $T_0{}'$ with $-\beta\lambda T_{SF}{}'$ and k with $\frac{2\pi}{\beta\lambda}$, the phase correction at the center of the gap can be expressed as

$$\Delta\bar{\varphi} = \frac{2\pi eV_0}{m\beta^2c^2\gamma^3}\left[-T_{SF}{}' I_0(k_r r) + \frac{T_0 I_1(k_r r) r}{\beta\lambda\gamma}\right]\sin\varphi_0 - \frac{eV_0}{mc^2}T_0 I_1 r_0{}'\cos\varphi_0\ \ . \tag{14}$$

The phase change across the gap, $\Delta\bar{\varphi}$, is the sum of $\Delta\bar{\varphi}_{left}$ and $\Delta\bar{\varphi}_{right}$ and

$$\Delta\bar{\varphi}_{left} = \frac{k}{2W}\int_{-L/2}^{0} z dW\ .$$

Using Eq. (6) and ignoring the $r_0{}'$ term, the phase correction up to the left mid plane can be expressed as

$$\begin{aligned}\Delta\bar{\varphi}_{left} &= -\frac{ek}{2W}\int_{-\frac{L}{2}}^{0} E_z(r_0,z)\cos\phi\, dz \\ &= -\frac{ek}{2W}\int_{-\frac{L}{2}}^{0} E_z(r_0,z)z\sin(kz)\sin\varphi_0\, dz + \frac{ek}{2W}\int_{-\frac{L}{2}}^{0} E_z(r_0,z)z\cos(kz)\cos\phi_0\, dz \\ &= -\frac{ekV_0\beta\lambda}{4W}(\sin\varphi_0 T_{SF}{}' + \cos\varphi_0 S_{SF}{}').\end{aligned}$$

Here the definitions of $T_{SF}{}'$ and $S_{SF}{}'$ in Eq. (10) are used. They give,

$$\frac{\beta\lambda V_0}{2}T_{SF}{}' = \int_{-L/2}^{0} E_z(r_0,z)z\sin(kz)\, dz$$

$$\frac{\beta\lambda V_0}{2}S_{SF}{}' = \int_{0}^{L/2} E_z(r_0,z)z\cos(kz)\, dz = -\int_{-L/2}^{0} E_z(r_0,z)z\cos(kz)\, dz$$

Considering the relativistic case and ignoring the $r_0{}'$ term in Table 2 of [2], $\Delta\bar{\varphi}_{left}$ becomes

$$\Delta\bar{\varphi}_{left} = -\frac{e\pi V_0}{mc^2\beta^2\gamma^3}(\sin\varphi_0 T_{SF}{}' + \cos\varphi_0 S_{SF}{}') \tag{15}$$

Up to now, all the calculations are done for the synchronous particles. In PARMILA, since the phase advance of a synchronous particle is known from the design process, the phase change of an asynchronous particle is calculated relative to the synchronous particle. To calculate the difference in phase advance between an asynchronous particle and a synchronous one, one needs to look at the differences generated in the left half-cell, at the mid-plane and in the right half-cell.

Let $\Delta\varphi_{0,left}$ and $\Delta\varphi_{0,right}$ denotes the phase change of a synchronous particle in the left and right half gap. An asynchronous particle's phase change is

$$\Delta\varphi_{i,left} = \Delta\varphi_{0,left} - \frac{\beta_{i,in}-\beta_{0,in}}{\beta_{i,in}}\Delta\varphi_{0,left} \tag{16}$$

$$\Delta\varphi_{i,right} = \Delta\varphi_{0,right} - \frac{\beta_{i,out}-\beta_{0,out}}{\beta_{i,out}}\Delta\varphi_{0,right} \tag{17}$$

where $\beta_{i,in}$ and $\beta_{i,out}$ is the beta of the asynchronous particle at the entrance and exit of the cell. So its phase up to this point is

$$\varphi_{i,left-mid} = \varphi_{i,entrance} + \Delta\varphi_{i,left}$$

Using Eq. (15), the phase correction for an asynchronous particle at the left mid-plane of a gap is

$$\Delta\bar{\varphi}_{i,left} = \Delta\bar{\varphi}_{left} - \frac{e\pi V_0}{mc^2\beta^2\gamma^3}\left((\sin\varphi_{i,left-mid} - \sin\varphi_0)T_{SF}{}' + (\cos\varphi_{i,left-mid} - \cos\varphi_0)S_{SF}{}'\right). \tag{18}$$

At the center of the mid plane, the phase of an asynchronous particle is

$$\varphi_{i,mid} = \varphi_{i,entrance} + \Delta\varphi_{i,left} + \Delta\bar{\varphi}_{i,left} \tag{19}$$

This value is used in the calculations of all the correction terms at the mid-plane of the gap. Similar to Eq. (18), one can get the phase correction for an asynchronous particle across the gap using Eq. (14)

$$\Delta\bar{\varphi}_i = \Delta\bar{\varphi} + \frac{2\pi eV_0}{m\beta^2c^2\gamma^3}\left[-T_{SF}{}'I_0(k_r r) + \frac{T_0 I_1(k_r r)r}{\beta\lambda\gamma}\right](\sin\varphi_{i,mid} - \sin\varphi_0)$$
$$-\frac{eV_0}{mc^2}T_0 I_1 r_0{}'\,(\cos\varphi_{i,mid} - \cos\varphi_0)\ . \tag{20}$$

Finally, by combining Eq. (16), (17) and (20) the phase advance of an asynchronous particle across the gap can be obtained

$$\varphi_{i,out} - \varphi_{i,in} = \Delta\varphi_{i,left} + \Delta\bar{\varphi}_i + \Delta\varphi_{i,right}$$

2.3 Change in a particle's r and r' coordinates when crossing an RF gap

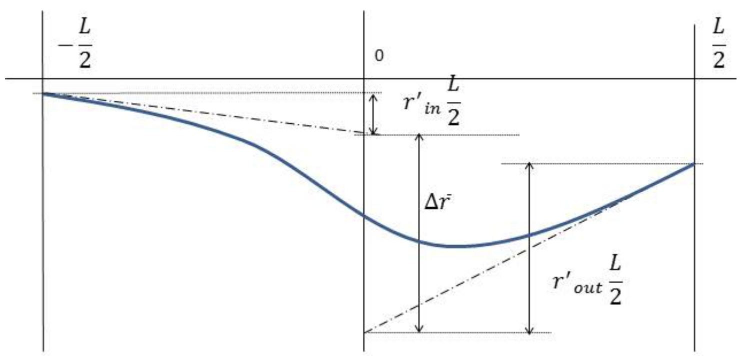

Figure 2: Change in transverse coordinate of a particle crossing an RF gap.

The change of the transverse coordinates in the left and right half-cells can be treated as transporting through two drifts. We only need to focus on calculating the displacement and angle corrections at the mid-plane of the gap.

Similar to Eq. (12), one can write the displacement correction at the mid-plane of the gap as

$$\Delta\bar{r} = -\int_{-L/2}^{L/2} z r''(z)dz\ . \tag{21}$$

Because $\frac{dr}{dz} = \frac{1}{\dot{z}}\dot{r}$, one has

$$r'' = \frac{d^2r}{d^2z} = \frac{1}{\dot{z}^2}\ddot{r} + \dot{r}\frac{d^2t}{d^2z}\,.$$

With $\dot{r} = r'\dot{z}$, $\frac{d^2t}{d^2z} = \frac{1}{\omega}\frac{d^2\varphi}{d^2z}$ and Eq. (13), the equation above becomes

$$r'' = \frac{1}{\dot{z}^2}\ddot{r} - \frac{r'}{2W}\frac{dW}{dz} = \frac{eE_r(r_0,z)\cos(kz+\varphi_0)}{2W} - \frac{r'}{2W}\frac{dW}{dz}$$

Therefore,

$$\Delta\bar{r} = -\frac{e}{2W}\int_{-L/2}^{L/2} zE_r(r_0,z)\cos(kz+\varphi_0)\,dz + \frac{r'}{2W}\int_{-L/2}^{L/2} zdW\ . \tag{22}$$

Expanding $E_r(r,z)$ around $r = r_0$, one can get

$$E_r(r,z) = E_r(r_0,z) + \frac{\partial E_r(r,z)}{\partial r}|_{r=r_0}(r-r_0) + O(r-r_0)\ .$$

With the symmetry in Eq. (2) and the definition of T_r in Eq. (3), the first term of Eq. (22) reduces to

$$-\frac{e}{2W}\int_{-L/2}^{L/2} zE_r(r_0,z)\cos(kz)\cos\varphi_0\,dz + \frac{er_0{}'}{2W}\int_{-L/2}^{L/2} z^2\frac{\partial E_r(r,z)}{\partial r}|_{r=r_0}\sin(kz)\sin\varphi_0\,dz$$

$$= -\frac{eV_0}{2W}\frac{dT_r}{dk}\cos\varphi_0 - \frac{eV_0r_0{}'}{2W}\frac{d^2}{dk^2}\frac{\partial T_r}{\partial r}|_{r=r_0}\sin\varphi_0\ . \tag{23}$$

According to Eq. (6), the energy change can be written as

$$\mathrm{d}W = eE_z(r_0,z)\cos(kz+\varphi_0)dz + e\frac{\partial E_z(r,z)}{\partial r}|_{r=r_0}r_0{}'z\cos(kz+\varphi_0)\,dz$$

By using the symmetry of $E_z(r_0,z)$ as shown in Eq. (2), the second term of Eq. (22) becomes

$$\frac{r'}{2W}\int_{-\frac{L}{2}}^{\frac{L}{2}} zdW = -\frac{er'}{2W}\int_{-\frac{L}{2}}^{\frac{L}{2}} zE_z(r_0,z)\sin(kz)\sin\varphi_0 dz + O(r') \approx \frac{er'V_0}{2W}\frac{dT_l}{dk}\sin\varphi_0\ . \tag{24}$$

Plugging Eq. (22) and (23) into Eq. (21), one can get the displacement correction as

$$\Delta\bar{r} = -\frac{eV_0}{2W}\frac{dT_r}{dk}\cos\varphi_0 - \frac{eV_0}{2W}\left(\frac{d^2}{dk^2}\frac{\partial T_r}{\partial r}|_{r=r_0} - \frac{dT_l}{dk}\right)r_0{}'\sin\varphi_0\ .$$

The $r_0{}'$ term is omitted in PARMILA. After adding the correction for the relativistic case [2],

$$\begin{aligned}\Delta\bar{r} &= -\frac{eV_0}{2W}\left[\frac{1}{\gamma^3}\frac{d}{dk}\left(\frac{T_0(k)kI_1(k_rr)}{k_r}\right) + \left(1-\frac{1}{\gamma^2}\right)\frac{T_0(k)I_1(k_rr)}{k}\right]\cos\varphi_0\\ &= -\frac{eV_0}{2W}\left[\frac{T_0(k)'I_1(k_rr)}{\gamma^2} + \frac{I_1(k_rr)'T_0(k)r}{\gamma^3} + \left(1-\frac{1}{\gamma^2}\right)\frac{T_0(k)I_1(k_rr)}{k}\right]\cos\varphi_0\ .\end{aligned} \tag{25}$$

To calculate $\Delta\bar{r}'$, one can use

$$\begin{aligned}\Delta\bar{r}' &= \int_{-L/2}^{L/2} r''(z)dz = \frac{e}{2W}\int_{-L/2}^{L/2} E_r(r_0,z)\cos(kz+\varphi_0)\,dz - \frac{r'}{2W}\int_{-L/2}^{L/2} dW\\ &= -\frac{e}{2W}\int_{-L/2}^{L/2} E_r(r_0,z)\sin(kz)\sin\varphi_0\,dz + \frac{er_0{}'}{2W}\int_{-L/2}^{L/2} z\frac{\partial E_r(r,z)}{\partial r}|_{r=r_0}\cos(kz)\cos\varphi_0\,dz\\ &\quad - \frac{er'}{2W}\int_{-L/2}^{L/2} E_z(r_0,z)\cos(kz)\cos\varphi_0 dz\\ &= -\frac{eV_0}{2W}T_r\sin\varphi_0 + \frac{eV_0}{2W}\left(\frac{d}{dk}\frac{\partial T_r}{\partial r}|_{r=r_0} - T_l\right)r'\cos\varphi_0\end{aligned}$$

Considering the relativistic case [2] and ignoring the r' terms,

$$\Delta\bar{r}' = -\frac{eV_0}{2W}\frac{T_0(k)I_1(k_r r)}{\gamma^2}\sin\varphi_0 \ . \tag{26}$$

The transverse corrections can be obtained by using the following equations

$$\Delta x' = \frac{x}{r}\Delta\bar{r}',\ \Delta y' = \frac{y}{r}\Delta\bar{r}',\ \Delta\bar{x} = \frac{x}{r}\Delta\bar{r},\ \Delta\bar{y} = \frac{y}{r}\Delta\bar{r}$$

So in the second half of the cell, the horizontal transformation is

$$x_1{}' = -\frac{eV_0}{2W}\frac{T_0(k)I_1(k_r r)}{\gamma^2 r}\sin\varphi_0\, x_0$$

$$x_1 = -\frac{eV_0}{2Wr}\left[\frac{T_0(k)'I_1(k_r r)}{\gamma^2} + \frac{I_1(k_r r)'T_0(k)r}{\gamma^3} + \left(1-\frac{1}{\gamma^2}\right)\frac{T_0(k)I_1(k_r r)}{k}\right]\cos\varphi_0\, x_0 + x_1{}'d_2$$

where d_2 is the length of the second half gap. An extra term should be added to $x_1{}'$ in order to preserve the horizontal unnormalized emittance so that the Jacobian of the horizontal coordinates equals to the ratio of the $\beta\gamma$. With the definitions of

$$A = \frac{eV_0}{2W}\frac{T_0(k)I_1(k_r r)}{\gamma^2 r}\sin\varphi_0$$
$$B = -\frac{eV_0}{2W}\left[\frac{T_0(k)'I_1(k_r r)}{\gamma^2 r} + \frac{I_1(k_r r)'T_0(k)}{\gamma^3} + \left(1-\frac{1}{\gamma^2}\right)\frac{T_0(k)}{k}\frac{I_1(k_r r)}{r}\right]\cos\varphi_0 \tag{27}$$

the transformation can be simplified to

$$x_1{}' = -Ax_0 + \frac{(\beta\gamma)_0}{(\beta\gamma)_1}\frac{1}{B}x_0{}' \ . \tag{28}$$
$$x_1 = Bx_0 + x_1{}'d_2$$

One can verify that the Jacobian of the coordinates does give the ratio between the initial and final $\beta\gamma$.

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial x_0} & \frac{\partial x_1}{\partial x_0{}'} \\ \frac{\partial x_1{}'}{\partial x_0} & \frac{\partial x_1{}'}{\partial x_0{}'} \end{vmatrix} = \begin{vmatrix} B - d_2 A & d_2\frac{(\beta\gamma)_0}{(\beta\gamma)_1}\frac{1}{B} \\ -A & \frac{(\beta\gamma)_0}{(\beta\gamma)_1}\frac{1}{B} \end{vmatrix} = \frac{(\beta\gamma)_0}{(\beta\gamma)_1}$$

However, the current version (2.34) of the PARMILA code seems to mistakenly drop the $\frac{1}{r}$ from the last term in the bracket of coefficient B as shown in Eq. (27). Fig.3 implies that with the $\frac{1}{r}$ in coefficient B, PARMILA results agree better with the "t code" PARMELA in predicting emittance growth for a mismatched input beam in the first tank of an SNS-like DTL. Similarly, Fig. 4 shows the PARMILA simulation results of the horizontal emittance growth in the SNS Linac with and without the $\frac{1}{r}$ in coefficient B.

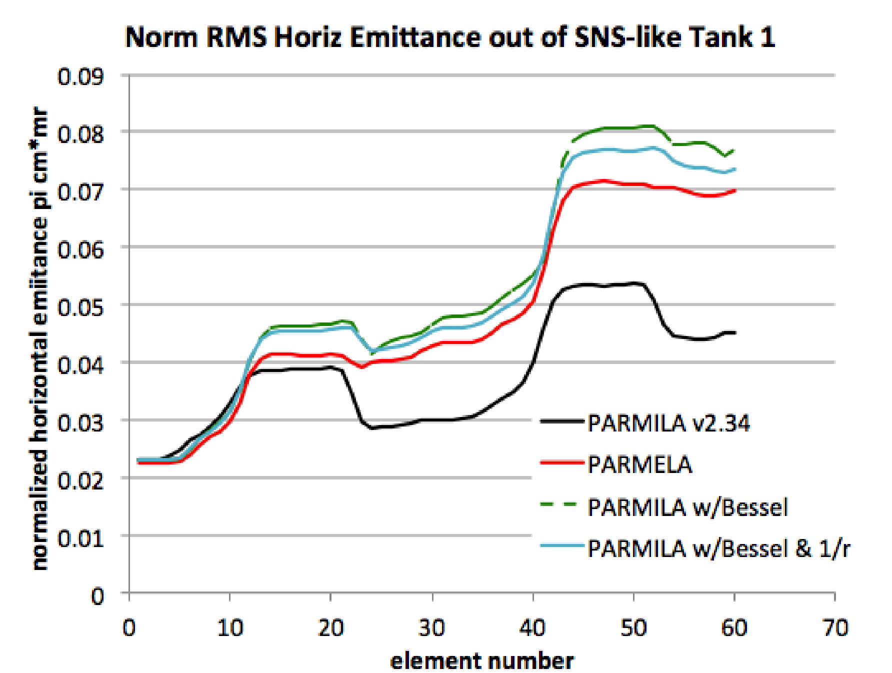

Figure 3 Simulated emittance growth in one first tank of an SNS-like DTL for a horizontally mismatched input beam using PARMELA and PARMILA with different corrections.

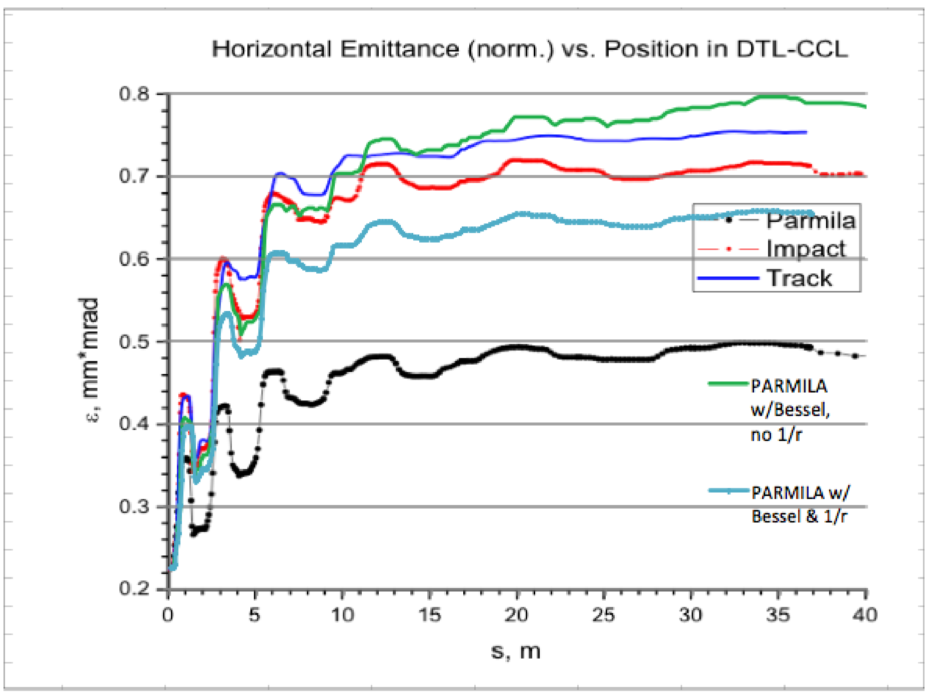

Figure 4: Simulated horizontal beam emittance (rms, n) in SNS linac for an input beam that was horizontally mismatched. The black, green and turquoise lines represent results from PARMILA for the cases where the Bessel function approximation is used, is not used, and is not used and the 1/r correction is included.

3. Approximations used for buncher elements

In buncher cavities, changes are applied only to x', y' and energy in PARMILA code. Using just the first term of Eq. (7), the energy gain across the buncher is

$$\Delta W = \ eV_0 T_0(k) I_0(k_r r_0) \cos\phi_0$$

Since $I_0 = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+1)} (\frac{x}{2})^{2m}$, one can write

$$I_0(k_r r_0) = 1 + \frac{(k_r r_0)^2}{4} + \frac{(k_r r_0)^4}{64} + \frac{(k_r r_0)^6}{36\times 64} + O(k_r r_0) \ .$$

In PARMILA, the buncher voltage is defined as $V_{buncher} = V_0 T_0(k)$, so the energy gain equals

$$\Delta W = \ e\, V_{buncher} \cos\phi_0 \, [1 + \frac{(k_r r_0)^2}{4} + \frac{(k_r r_0)^4}{64} + \frac{(k_r r_0)^6}{36\times 64}]$$

To calculate the transformation for x', y', one can apply some approximations to Eq. (28). Since

$$I_1 = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+2)} (\frac{x}{2})^{2m+1} = \frac{x}{2} + \frac{x^3}{16} + \frac{x^5}{24\times 16} + O(x)$$

the coefficient A in Eq. (28) approximately equals to

$$A = \ \frac{eV_{buncher}}{mc^2\beta^2} \frac{2\pi}{\beta\lambda\gamma} (1-\beta^2) \sin\varphi_0 \, [\frac{1}{2} + \frac{(k_r r_0)^2}{16} + \frac{(k_r r_0)^4}{24\times 16}]$$

and the transformation of $x'$is

$$x_1{}' = -A x_0 + \frac{(\beta\gamma)_0}{(\beta\gamma)_1} x_0{}'$$

Notice here $x_1 = x_0$, and the $\frac{1}{B}$ in the second term of Eq. (28) is dropped to preserve the emittance.

4. Determination of DTL cell length

In PARMILA, the DTL cell length was determined in a way that the phase of the synchronous particle at the entrance and exit of the cell is $-\pi + \frac{1}{2}(\varphi_{0,n-1}+\varphi_{0,n})$ and $\pi + \frac{1}{2}(\varphi_{0,n}+\varphi_{0,n+1})$, where $\varphi_{0,n}$denotes the synchronous phase at the electric center of the n^{th} cell. So the phase advance of a synchronous particle through the left and right half-cells is

$$\Delta\varphi_{0,left} = \pi + \frac{1}{2}(\varphi_{0,n}\text{-}\varphi_{0,n-1}) = \frac{2\pi}{\beta_{in}\lambda} Z_l + \Delta\bar{\varphi}_{left}$$

and

$$\Delta\varphi_{0,right} = \pi + \frac{1}{2}(\varphi_{0,n+1}\text{-}\varphi_{0,n}) = \ \frac{2\pi}{\beta_{out}\lambda} Z_r + \Delta\bar{\varphi}_{right}$$

where $\Delta\bar{\varphi} = \Delta\bar{\varphi}_{left} + \Delta\bar{\varphi}_{right}$. For a particle with constant phase, $\varphi_{0,n}\text{-}\varphi_{0,n-1} = 0$ and $\varphi_{0,n+1}\text{-}\varphi_{0,n} = 0$. Therefore,

$$Z_l = \ \frac{\beta_{in}\lambda}{2\pi} (\pi - \Delta\bar{\varphi}_{left})$$

$$Z_r = \frac{\beta_{out}\lambda}{2\pi}(\pi - \Delta\bar{\varphi}_{right}) \,.$$

Assuming $k_r r \ll 1$, then Eq. (14) becomes

$$\Delta\bar{\varphi} = -\frac{2\pi e V_0}{m\beta^2 c^2 \gamma^3} T_{SF}{}' \sin\varphi_0$$

Together with Eq. (15), one can get $\Delta\bar{\varphi}_{right}$

$$\Delta\bar{\varphi}_{right} = -\frac{e\pi V_0}{mc^2\beta^2\gamma^3}(\sin\varphi_0 T_{SF}{}' - \cos\varphi_0 S_{SF}{}') \,. \quad (29)$$

Using Eq. (15) and Eq. (29), the left and right half cell length can finally be written as

$$Z_l = \frac{\beta_{in}\lambda}{2}[1 + \frac{eV_0}{mc^2{\beta_{in}}^2{\gamma_g}^3}(\sin\varphi_0 T_{SF}{}' + \cos\varphi_0 S_{SF}{}')]$$

$$Z_r = \frac{\beta_{out}\lambda}{2}[1 + \frac{eV_0}{mc^2{\beta_{out}}^2{\gamma_g}^3}(\sin\varphi_0 T_{SF}{}' - \cos\varphi_0 S_{SF}{}')]$$

5. Conclusion

RF transformations are derived in this document and a discrepancy between the analytical formula and the current PARMILA code (v2.34) is found in the transformation of the transverse coordinates. This can lead to overestimation of the emittance growth in an RF gap. The correction of this error is recommended.

6. References

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