

Difference Array & Prefix Sum Array

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Prefix Sum Array

- **Definition:** Given an array of numbers A , we replace each element with the sum of itself and all the elements preceding it.
- Example:
 - $A=[3, 5, 4, 1, 2]$
 - $P(A)=[3, 8, 12, 13, 15]$

Time complexity

- Linear time $O(N)$

```
// P must have enough space for n+1 ints
void prefix_sum_array(int c, int* A, int n, int* P)
{
    P[0] = c;
    for (int i = 0; i < n; i++)
        P[i+1] = P[i] + A[i];
}
```

Difference Array

- **Definition:** Given an array of numbers, we can construct a new array by replacing each element by the difference between itself and the previous element, except for the first element, which we simply ignore.
- Example:
- $A = [3, 5, 4, 1, 2]$
- $D(A) = [5-3, 4-5, 1-4, 2-1] = [2, -1, -3, 1]$
- $D(A) = [3, 2, -1, -3, 1]$

Time complexity

- Linear time $O(N)$

```
// D must have enough space for n-1 ints  
void difference_array(int* A, int n, int* D)  
{  
    for (int i = 0; i < n-1; i++)  
        D[i] = A[i+1] - A[i];  
}
```

Analysis

- Difference array and Prefix Sum array carry out **reverse process**
- Given an array A
 - $D(P(A)) = A$
 - $A=[9,2,6,3,1,5,0,7]$
 - $P(A)=[9, 11, 17, 20, 21, 26, 26, 33]$
 - $D(P(A)) = [9,2,6,3,1,5,0,7]$
 - $P(A0, D(A)) = A$

Application

- Prefix Sum Array
 - Used for frequent reference to range sums
 - Example:
<http://www.spoj.com/problems/SUBSEQ/>

Application(2)

- Difference array is used to keep track of an array when ranges of said array can be updated all at once.

Application(3)

- If we have array A and add an increment k to elements $A_i, A_{i+1}, \dots, A_{j-1}$, then notice that D_0, D_1, \dots, D_{i-2} are not affected; that $D_{i-1} = A_i - A_{i-1}$ is increased by k ; that $D_i, D_{i+1}, \dots, D_{j-2}$ are not affected; that $D_{j-1} = A_j - A_{j-1}$ is decreased by k ; and that D_j, D_{j+1}, \dots are unaffected.
- If we are required to update many ranges of an array in this manner, we should keep track of D rather than A itself, and then integrate at the end to reconstruct A .

Application(3)

- Difference Array Examples:
 - Battle Positions (<https://dmoj.ca/problem/seed3>)
 - Wireless (<https://dmoj.ca/problem/ccc09s5>)

How about 2D sum query

- PSA from 1D to 2D
 - What if instead 1D array, we have 2D one
- Query is to find a rectangle sum
 - E.g. $\text{Sum}((2, 4), (5, 7))$ where $(2, 4)$ is the top left corner of a sub-matrix?
- Can we still use prefix sum array?
 - Create new Array S.
 - For each row in A, create its cumulative sum in S
 - In S, in-place, create cumulative sum for each column
 - Now $S(i, j) = \text{Sum}((0, 0), (i, j))$

2D prefix sum array

Accumulate each row

Accumulate each column

1	2	2	4	1
3	4	1	5	2
2	3	3	2	4
4	1	5	4	6
6	3	2	1	3

1	3	5	9	10
3	7	8	13	15
2	5	8	10	14
4	5	10	14	20
6	9	11	12	15

1	3	5	9	10
4	10	13	22	25
6	15	21	32	39
10	20	31	46	59
16	29	42	58	74

$8 = \text{row } 3 + 4 + 1$

$32 = \text{col } 9 + 13 + 10$
 $32 = \text{sum}((0, 0), (2, 3))$

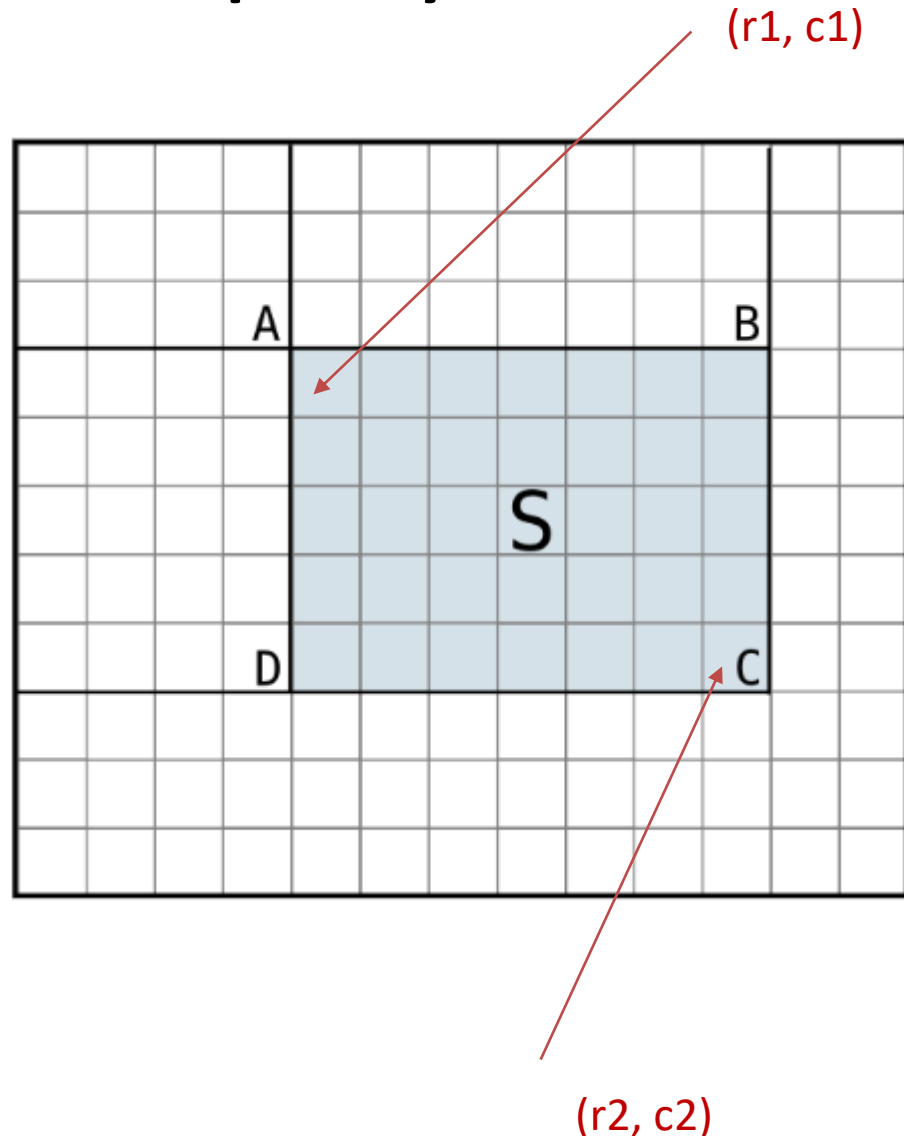
1	2	2	4
3	4	1	5
2	3	3	2

2D sum query

How to compute the sum in submatrix S?

- Sum of C is the sum from top-left corner to C
- Let's remove sum of B
- Let's remove sum of D
- Sum of A is removed twice
- Let's add A back
- $S = C - B - D + A$

- A: $\text{psa}[r1-1][c1-1]$
- B: $\text{psa}[r1-1][c2]$
- C: $\text{psa}[r2][c2]$
- D: $\text{psa}[r2][c1-1]$
- $S = \text{psa}[r2][c2] - \text{psa}[r1-1][c2] - \text{psa}[r2][c1-1] + \text{psa}[r1-1][c1-1]$



How about 3D?

- Same idea to calculate $\text{psa}[i][j][k]$
 - Accumulate sum over i
 - Accumulate sum over j
 - Accumulate sum over k

- How about 3D sum query?

Sum of (i_1, j_1, k_1) to (i_2, j_2, k_2) (inclusive)

$$\begin{aligned} &= \text{psa}[i_2][j_2][k_2] - \text{psa}[i_1-1][j_2][k_2] - \text{psa}[i_2][j_1-1][k_2] - \text{psa}[i_2][j_2][k_1-1] + \\ &\text{psa}[i_1-1][j_1-1][k_2] + \text{psa}[i_1-1][j_2][k_1-1] + \text{psa}[i_2][j_1-1][k_1-1] - \text{psa}[i_1-1][j_1-1][k_1-1] \end{aligned}$$

How about 2D update

- DA from 1D to 2D
 - What if instead 1D array update, we have 2D update
- Update is to add/subtract value in a submatrix
- Can we still use difference array?
 - Yes
 - Update 4 corners:
dif[r1][c1] += val dif[r2+1][c1] -= val
dif[r1][c2+1] -= val dif[r2+1][c2+1] += val