Calculus

Lesson #4

The Derivatives of Composite Functions

Rule	Function Notation	Leibniz Notation
Chain Rule	If $h(x)=f(g(x))$, then $h'(x)=f'(g(x))g'(x)$.	$\frac{dy}{dx}=\frac{dy}{du}*\frac{du}{dx}$, where u is a function of x

- When the outer function is a power function of the form $y=[g(x)]^n$, we have the special case of the chain rule called the power of a function rule.

Example 1 If $f(x)=x^2+2x$ and $g(x)=10-3x$, find the following.

a) $f(g(-4))$

b) $(g\circ f)(6)$

c) $(f\circ f)(1)$

d) $g(g(0))$

Example 2 Let $f(x)=\frac{1}{x-1}$ and $g(x)=\sqrt{x+3}$. Determine $f\circ g$, and find its domain.

Example 3 If $f(x) = 3x + 5$ and $g(x) = x^2 + 2x - 3$, determine x such that $f(g(x)) = g(f(x))$.

Example 4 Differentiate each function. Express your answer in simplified factored form.

a) $f(x) = \sqrt{x^2 - 5x}$

b) $g(x) = \dfrac{1}{(x^3 - 4x^2 + 10)^5}$

c) $y = (3x - 2)^3 (2x^2 + 5)^4$

d) $h(x) = \left(\dfrac{2x - 1}{x^2 + 1}\right)^3$

Example 5 If $y = u^2 + \frac{u}{5}$ and $u = 9\sqrt{x} - 23$, find $\frac{dy}{dx}$ when $x = 4$.

Example 6 If $F(x) = f(g(x))$, where $g(8) = 5$, $g'(8) = -\frac{1}{2}$, and $f'(5) = 84$, find $F'(8)$.

Example 7 Determine the equation of the normal to the curve $y = \frac{1}{\sqrt{x^2 - 9}}$ at $x = -5$.

Example 8 Determine the coordinates of the point(s) on the curve $y = -2x^4(x^2 - 1)^2$ at which the tangent is horizontal.

Example 9 A manufacturer determines that the demand for one of its newest products can be modelled by the function $p(x) = 100 - \sqrt{x^2 + 20}$, where p is the price per unit, in dollars, and x is the number of units sold. Determine $p'(4)$ and interpret the meaning of this value

Implicit Differentiation

Example 1. Find the equation of the tangent line to the curve

$$y^5 + x^2y - 2x^2 = -1$$

at the point $(\sqrt{2}, 1)$.

Example 2. Find the slope of the tangent line to the curve

$$y = \sqrt[3]{6 + x^2}$$

at the point $(\sqrt{2}, 2)$.

1. Let $y = f(x)$ be a function that satisfies the given equation. Find $\frac{dy}{dx}$ in terms of x and y, using implicit differentiation.

a. $4x^2 + y^2 = 8$
b. $3x - 4y^2 = 2$
c. $x^2 + y^2 + 5y = 10$
d. $xy^2 = 4$
e. $x^2 + 2xy - y^2 = 13$
f. $y^3 + y = 4x$
g. $y(x^2 + 3) = y^4 + 1$
h. $(x - 1)^2 + (y - 1)^2 = 4$
i. $xy^3 + x^3y = 2$
j. $\sqrt{x} + \sqrt{y} = \sqrt{b}$ where b is a positive constant

2. Find the equation of the tangent line to the hyperbola $x^2 - 4y^2 = 5$ at the point $(3, -1)$.

a. by using implicit differentiation,
b. by solving explicitly for y.

3. For each curve, find the equations of the tangent line and the normal line at the given point.

a. $\dfrac{x^2}{100} + \dfrac{y^2}{25} = 1, \quad (-8, -3)$
b. $4x^2 - 9y^2 = 36, \quad \left(-5, \frac{8}{3}\right)$
c. $xy = 64, \quad (16, 4)$
d. $x^3 + y^3 - 3xy = 17, \quad (2, 3)$
e. $y^2 = \dfrac{x^3}{2 - x}, \quad (1, -1)$
f. $y = \dfrac{75}{x^2 + y^2}, \quad (4, 3)$
g. $8\dfrac{y}{x^2} - 8\dfrac{x}{y^2} = 7, \quad (2, 4)$

4. Write each function in an implicit form without radicals, and hence find $\frac{dy}{dx}$ in terms of x and y.

a. $y = -2\sqrt{x}$
b. $y = \sqrt{3 - x}$
c. $y = \sqrt[3]{x}$
d. $y = -\sqrt{4 - x^2}$
e. $y = \dfrac{3}{\sqrt{x}}$
f. $y = \sqrt{x} + 5$

5. Show that, if $15x = 15y + 5y^3 + 3y^5$, then

$$\frac{dy}{dx} = (1 + y^2 + y^4)^{-1}$$

6. Let $y = -(65 - x^6)^{\frac{1}{6}}$. Find the equation of the tangent line at $(1, -2)$ by expressing the curve in a simple implicit form, and then using implicit differentiation.

7. **At what points on the curve $y^3 - 3x = 5$ is the slope of the tangent line equal to 1?**

8. **At what points on the ellipse**

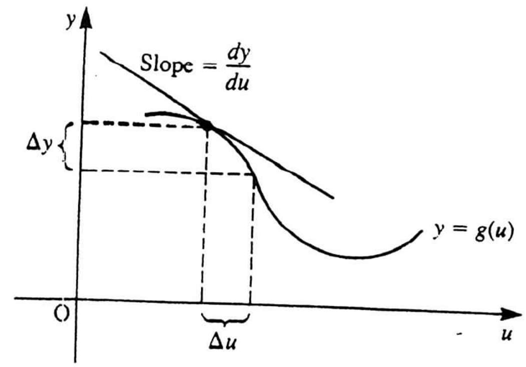
$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

is the slope of the tangent line equal to 1? Illustrate your answer with a sketch.

1. a. $\dfrac{-4x}{y}$ **b.** $\dfrac{3}{8y}$ **c.** $-\dfrac{2x}{2y + 5}$ **d.** $-\dfrac{y}{2x}$
e. $\dfrac{x + y}{y - x}$ **f.** $\dfrac{4}{1 + 3y^2}$ **g.** $\dfrac{2xy}{4y^3 - x^2 - 3}$ **h.** $\dfrac{x - 1}{1 - y}$
i. $\dfrac{-y^3 - 3x^2y}{3xy^2 + x^3}$ **j.** $-\sqrt{\dfrac{y}{x}}$ **2. a.** $\dfrac{x}{4y}$;
$3x + 4y - 5 = 0$ **b.** $-\dfrac{x}{2\sqrt{x^2 - 5}}$;
$3x + 4y - 5 = 0$ **3. a.** $2x + 3y + 25 = 0$, tangent; $3x - 2y + 18 = 0$, normal
b. $5x + 6y + 9 = 0$, tangent; $18x - 15y + 130 = 0$, normal
c. $x + 4y - 32 = 0$, tangent; $4x - y - 60 = 0$, normal
d. $x + 7y - 23 = 0$, tangent; $7x - y - 11 = 0$, normal **e.** $2x + y - 1 = 0$, tangent; $x - 2y - 3 = 0$, normal
f. $24x + 43y - 225 = 0$, tangent; $43x - 24y - 100 = 0$, normal
g. $17x - 5y - 14 = 0$, tangent; $5x + 17y - 78 = 0$, normal **4. a.** $y^2 = 4x$, $\dfrac{dy}{dx} = \dfrac{2}{y}$ **b.** $y^2 = 3 - x$, $\dfrac{dy}{dx} = -\dfrac{1}{2y}$ **c.** $y^3 = x$, $\dfrac{dy}{dx} = \dfrac{1}{3y^2}$ **d.** $y^2 = 4 - x^2$, $\dfrac{dy}{dx} = -\dfrac{x}{y}$
e. $xy^2 = 9$, $\dfrac{dy}{dx} = -\dfrac{y}{2x}$ **f.** $y^2 - 10y + 25 = x$, $\dfrac{dy}{dx} = \dfrac{1}{2y - 10}$ **6.** $y^6 = 65 - x^6$,
$x + 32y - 65 = 0$ **7.** $\left(-\frac{4}{3}, 1\right)$ and $(-2, -1)$ **8.** $\left(-\frac{9}{5}, \frac{16}{5}\right)$ and $\left(\frac{9}{5}, -\frac{16}{5}\right)$

Proof of the Chain Rule

Let y be a function of u, such that $y = g(u)$ where u is a function of x, such that $u = h(x)$. A change Δx in x will produce a change Δu in u, which in turn will produce a change Δy in y.



By the definition of the derivative,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}\right)$$

(multiply and divide by Δu)

$$= \left(\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u}\right)\left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\right)$$

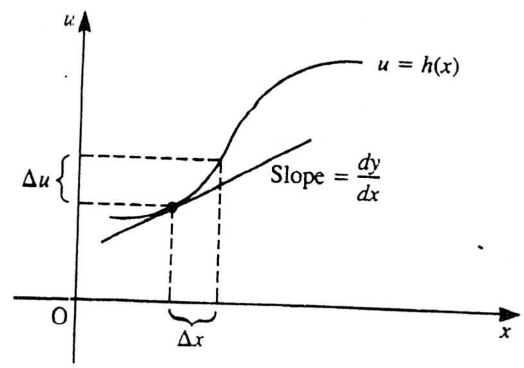
(product property of limits)

$$= \left(\lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u}\right)\left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\right)$$

(since $\Delta u \to 0$ as $\Delta x \to 0$)

$$= \frac{dy}{du}\frac{du}{dx}$$

(by the definition of the derivative)



This proof is not valid in all circumstances, since in dividing by Δu we assumed that $\Delta u \neq 0$ whenever $\Delta x \neq 0$, as is the case in the situation illustrated in the diagram. A more advanced proof is needed to avoid this difficulty.