First Name:____________________ Last Name: _________________ Student ID: ________

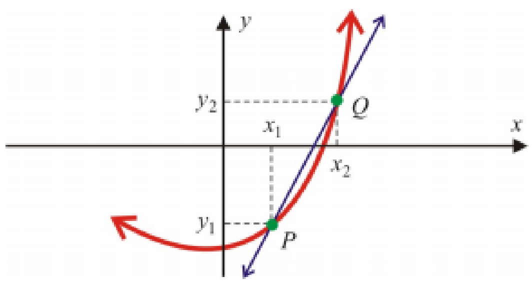
Rates of Change

Slopes of Secants and Average Rate of Change (ARC)

Secant Line

Let $y = f(x)$ be a function and $P(x_1, y_1)$ and $Q(x_2, y_2)$ two points on its graph.
The *slope* of the *secant line* that passes through the points P and Q is given by:

$$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$



Ex. 1

Consider f(x) = $\frac{2}{x+1}$. Find the equation of the secant line that passes through the points
A(0, 2) and B(-3, -1).

Average Rate of Change

$y = f(x), \quad y_1 = f(x_1), \quad y_2 = f(x_2)$

$\Delta x = x_2 - x_1$ (change in variable x)

$\Delta y = y_2 - y_1$ (change in variable y)

The *Average Rate of Change* (ARC) in y variable over the interval $[x_1, x_2]$ is given by:

$$ARC = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Note: The *Average Rate of Change* is the same as the *slope of the secant line* passing through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

If $x_1 = a$ and $x_2 = a + h$ then:

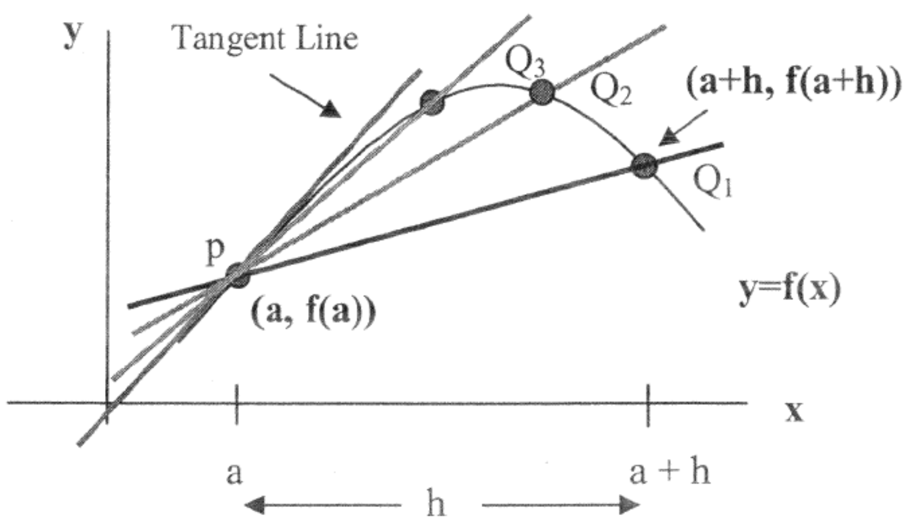
$$ARC = \frac{f(a+h) - f(a)}{h}$$

Ex. 2

Consider y = f (x) = $(x+1)^2$. Find the rate of change in the y variable over the interval [−1, 2].

Average Velocity	**Ex. 3**
Let $s = s(t)$ be the position function, where s is position in meters and t is the time in seconds. $s = s(t), \quad s_1 = s(t_1), \quad s_2 = s(t_2)$ $\Delta t = t_2 - t_1$ (time duration) $\Delta s = s_2 - s_1$ (displacement) The *Average Velocity* (AV) over the time interval $[t_1, t_2]$ is given by: $AV = \frac{rise}{run} = \frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$ Note: The *unit* of measurement for velocity is m/s.	A rock is launched vertically upward. The height of the rock is given by $s(t) = 100t - 10t^2$. Find the average velocity over the time interval [1, 2].

Moving from secant to tangent…



If $P(a, f(a))$ and $Q(a+h, f(a+h))$ then the *slope* of the *secant line* is given by:

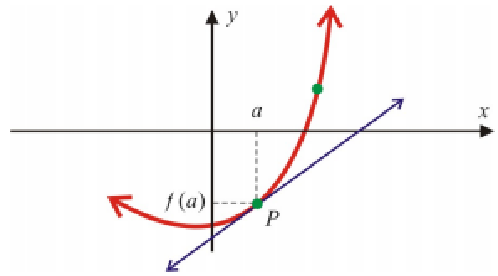
$$m = \frac{f(a+h) - f(a)}{h}$$

Slopes of Tangents and Instantaneous Rate of Change (IRC or RC)

Tangent Line

As the point Q approaches the point P , the secant line approaches the tangent line at P . See the diagram on the right side. The slope of the tangent line at P(a, f (a)) is:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



Algebraic Computation

1. Use the formula $m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.
2. Do not substitute h by 0 because you will get the indeterminate case $\frac{0}{0}$.
3. Compute algebraic the *difference quotient* $DQ = \frac{f(a+h) - f(a)}{h}$ until you succeed to cancel out the factor h .
4. *Substitute* in the remaining expression h by 0 .

Ex. 4

Find the slope of the tangent line to the graph of y = f (x) = $x^2 - 3x$ at the point P(1,−2) .

Ex. 5

a. Find the equation of the tangent line to the graph of y= f(x) = $\sqrt{x-2}$ at the point P(6, 2).

b. Graph the curve and the tangent line.

Instantaneous Rate of Change	**Ex. 6**
As $h \to 0$ the Average Rate of Change approaches to the *Instantaneous Rate of Change* (IRC): $IRC = RC = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$	Consider the following position function: $s(t) = t^2 - 4t$. a) Find the instantaneous velocity at t = 3s .
Note: The *Instantaneous Rate of Change* (IRC) is the same as the *slope of the tangent line* at the point $P(a, f(a))$.	b) Find the instantaneous velocity at the generic moment t = a
Similarly, the *Average Velocity* (AV) approaches *Instantaneous Velocity* (IV): $IV = v = \lim_{h\to 0} \frac{s(a+h)-s(a)}{h}$	c) Use the formula at part b) to compute the velocity at time t = 5s.
	d) Find the moment(s) of time at which the velocity is zero.

How can you estimate the Instantaneous Rate of Change?

Ex. 7
The population of a small town appears to be growing exponentially. Town planners think that the equation
$P(t) = 35\,000\,(1.05)^t$,
where P(t) is the number of people in the town and t is the number of years after 2000, models the size of the population. Estimate the instantaneous rate of change in the population in 2015.

Using a preceding interval in which $14 \leq t \leq 15$, $\frac{\Delta P}{\Delta t} =$	**Preceding Interval** an interval of the independent variable of the form $a - h \leq x \leq a$, where h is a small positive value; used to determine an average rate of change
Using a following interval in which $15 \leq t \leq 16$, $\frac{\Delta P}{\Delta t} =$	**Following Interval** an interval of the independent variable of the form $a \leq x \leq a + h$, where h is a small positive value; used to determine an average rate of change
Using a following interval in which $14 \leq t \leq 16$, $\frac{\Delta P}{\Delta t} =$	**Centred Interval** an interval of the independent variable of the form $a - h \leq x \leq a + h$, where h is a small positive value; used to determine an average rate of change

$\frac{\Delta P}{\Delta t} =$	**Difference Quotient** if $P(a, f(a))$ and $Q(a + h, f(a + h))$ are two points on the graph of $y = f(x)$, then the instantaneous rate of change of y with respect to x at P can be estimated using $\frac{\Delta y}{\Delta x} = \frac{f(a + h) - f(a)}{h}$, where h is a very small number. This expression is called the difference quotient.