$\because \frac{dV}{dr} = V'(r) \quad \therefore \frac{\Delta V}{\Delta r} \approx V'(r) \Rightarrow \Delta V \approx V'(r)\Delta r$

AP Calculus Homework Six – Applications of Differential Calculus

3.4 Curve Sketching; 3.5 Optimization Problems; 3.6 Local Linear Approximations

1. Find the best approximation, in cubic inches, to the increase in volume of a sphere when the radius is increased from 3 to 3.1 inches.

$V = \frac{4}{3}\pi r^3$, where V is the volume, r is the radius,

$V'(r) = \frac{4}{3}\pi(3)r^2 = 4\pi r^2$, $\because \Delta V \approx V'(r)\Delta r = V'(3)(3.1-3) = 4\pi(3)^2(0.1) = 3.6\pi$ (At $r = 3$ inches) (in^3)

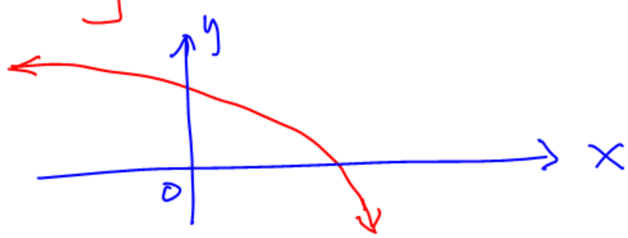
2. If the side e of a square is increased by 1%, find the increase of the area in terms of e.

$A = x^2$, where A is the area, x is the side length.

$\because \frac{\Delta A}{\Delta x} \approx A'(x)$, $\therefore \Delta A \approx A'(x)\Delta x = 2x\Delta x$; (at $x = e$,) $\therefore \Delta A \approx 2(e)(1.01e - e) = 2e(0.01e) = 0.02e^2$ ($unit^2$)

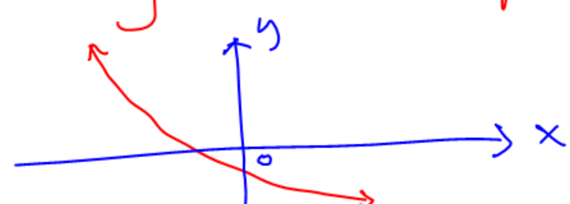
3. Sketch a curve for which both $f'(x)$ and $f''(x)$ are negative.

so $f(x)$ is decreasing and concave downward:

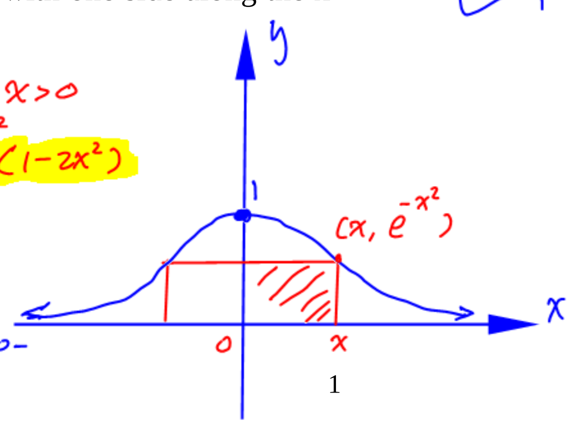


4. Sketch a curve for which $f'(x)$ is negative but $f''(x)$ is positive.

So $f(x)$ is decreasing and concave upward:



5. What is the area of the largest rectangle that can be drawn with one side along the x-axis and two vertices on the curve $y = e^{-x^2}$?

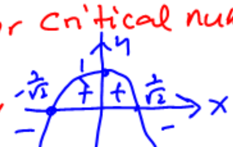


The area $f(x) = 2x \cdot e^{-x^2}$ to be maximized. where $x > 0$

$f'(x) = 2(x'e^{-x^2} + x(e^{-x^2})') = 2(e^{-x^2} - 2x^2e^{-x^2}) = 2e^{-x^2}(1-2x^2)$

let $f'(x) = 0$, solving for critical numbers: $1 - 2x^2 = 0$

$\Rightarrow x = \pm\frac{1}{\sqrt{2}} \approx \pm\frac{1}{1.4}$,

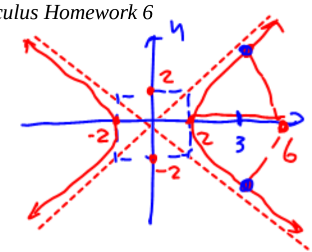


, $f'(x)$ changes from + to − at $x = \frac{1}{\sqrt{2}}$.

$\therefore f(\frac{1}{\sqrt{2}}) = 2(\frac{1}{\sqrt{2}})e^{-(\frac{1}{\sqrt{2}})^2} = \sqrt{2}\,e^{-\frac{1}{2}} = \frac{\sqrt{2}}{\sqrt{e}} = \frac{\sqrt{2e}}{e}$ is the maximum area.

$\frac{x^2}{2^2} - \frac{y^2}{2^2} = 1$

$y^2 = x^2 - 4$



6. Find the point(s) on the curve $x^2 - y^2 = 4$ closest to the point (6, 0).

Let $A(x, y)$ be the points closed to $B(6, 0)$. To minimize AB or AB^2, so $f(x) = (x-6)^2 + (y-0)^2 = x^2 - 12x + 36 + x^2 - 4$

so $f(x) = 2x^2 - 12x + 32$; $f'(x) = 4x - 12$, solve $f'(x) = 0$: $4x - 12 = 0 \Rightarrow x = 3$.

$f''(x) = (4x-12)' = 4 > 0$, $\therefore f(3)$ is the minimum. $\therefore y = \pm\sqrt{x^2-4} = \pm\sqrt{3^2-4} = \pm\sqrt{5}$

$\therefore (3, -\sqrt{5})$ and $(3, \sqrt{5})$ are the points required.

7. The sum of the squares of two positive numbers is 200; what is their ~~minimum~~ Maximum product?

Let x and y be the two positive numbers, then $x^2 + y^2 = 200$ ---- ①

let $f(x) = xy$, by ①, $y = \pm\sqrt{200-x^2}$. but $y > 0$, $\therefore y = \sqrt{200-x^2}$;

so $f(x) = x\sqrt{200-x^2}$; $f'(x) = x'\sqrt{200-x^2} + x(\sqrt{200-x^2})' = \sqrt{200-x^2} + \frac{-\cancel{2}x^2}{\cancel{2}\sqrt{200-x^2}}$; let $f'(x) = 0$

$\Rightarrow \sqrt{200-x^2} = \frac{x^2}{\sqrt{200-x^2}} \Rightarrow 200 - x^2 = x^2 \Rightarrow 100 = x^2 \Rightarrow x = 10$ and $y = \sqrt{200-10^2} = 10$.

$\because f'(x) = \frac{200-x^2-x^2}{\sqrt{200-x^2}} = \frac{200-2x^2}{\sqrt{200-x^2}}$; -10 $+$ 10 $-$ x. $\therefore f'(x)$ changes from $+$ to $-$ at $x = 10$

$\therefore f(10) = 10 \times 10 = 100$ is the maximum.

8. What is the best linear approximation for $f(x) = \tan x$ near $x = \pi/4$?

$f'(x) = (\tan x)' = \sec^2 x$; $f'(\frac{\pi}{4}) = \sec^2(\frac{\pi}{4}) = (\frac{1}{\frac{1}{\sqrt{2}}})^2 = 2$; $f(\frac{\pi}{4}) = \tan\frac{\pi}{4} = 1$;

$f(x) \approx f(\frac{\pi}{4}) + f'(\frac{\pi}{4})(x - \frac{\pi}{4}) = 1 + 2(x - \frac{\pi}{4})$

a linear function

9. If $f(6) = 30$ and $f'(x) = x^2/(x+3)$, what is the estimate of $f(6.02)$, using the local linearization?

$f'(6) = \frac{6^2}{6+3} = \frac{36}{9} = 4$;

$\therefore f(6.02) \approx f(6) + f'(6)(6.02 - 6) = 30 + 4(0.02) = 30 + 0.08 = 30.08$

10. Find the tangent line approximation for $f(x) = \sqrt{x^2 + 16}$ near $x = -3$.

$f'(x) = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+16}}$; $f'(-3) = \frac{-3}{\sqrt{(-3)^2+16}} = -\frac{3}{5}$; $f(-3) = \sqrt{(-3)^2+16} = 5$

$\therefore f(x) \approx f(-3) + f'(-3)(x-(-3)) = 5 - \frac{3}{5}(x+3)$ where x is near -3.

a line