First Name:____________________ Last Name: __________________ Student ID: ________

Rational Functions (1)

A **rational function** is a function of the form $\boxed{f(x)=\frac{g(x)}{h(x)}}$ where $g(x)$ and $h(x)$ are polynomials and $h(x) \neq 0$.

In the case where $h(x) = k$, $k \in$ R, $k \neq 0$ (i.e., a constant polynomial of degree 0), the rational function reduces to the polynomial function $f(x) = \frac{1}{k}g(x)$.

Examples of rational functions:

$y = \frac{1}{x^2-3x+2}$, $x \neq -1,-2$ $\quad$ $f(x)= \frac{x^2}{x-1}$, $x \neq 1$ $\quad$ $y = \frac{x^7+3x^2+10}{x^2-2x}$, $x \neq 0, 2$

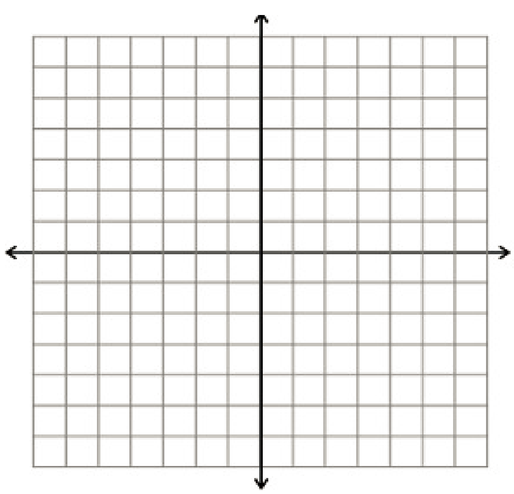
Reciprocal Functions

Knowing the graph of y = f(x) and understanding the general relationship between a function and its reciprocal function y= $\frac{1}{f(x)}$ can help when sketching the graph of the reciprocal function.

Reciprocal Linear Functions

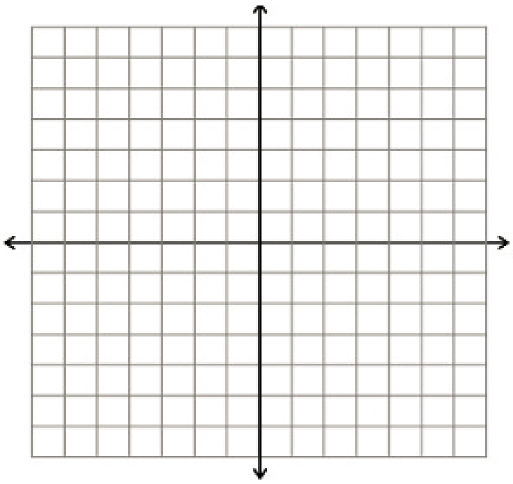
Example 1: Sketch each of the following functions. State all asymptotes and the end behaviour.

a)



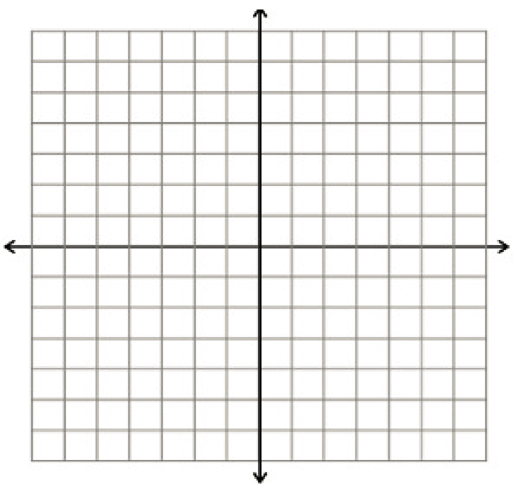
Equation	$y = \frac{1}{x}$
Domain	
Range	
End Behaviour	As $x \to \infty$ $y \to$ As $x \to -\infty$ $y \to$ As $x \to$ $y \to$ As $x \to$ $y \to$
H. Asymptote	V. Asymptote

b)



Equation	y = $\frac{1}{x+3}$
Domain	
Range	
End Behaviour	As x→ ∞ y→ As x→ −∞ y→ As x→ y→ As x→ y→
H. Asymptote	V. Asymptote

c)

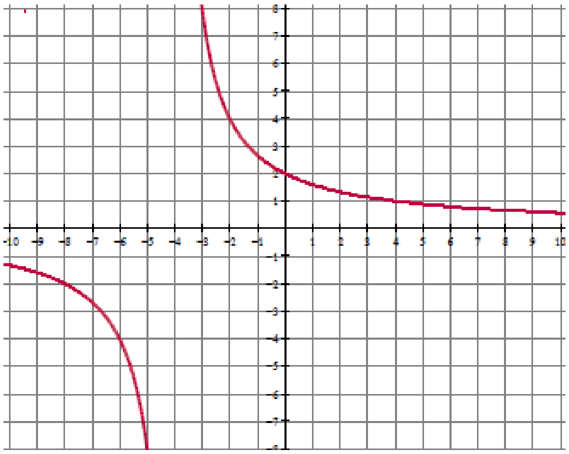


Equation	y = $\frac{1}{-2x-2}$
Domain	
Range	
End Behaviour	As x→ ∞ y→ As x→ −∞ y→ As x→ y→ As x→ y→
H. Asymptote	V. Asymptote

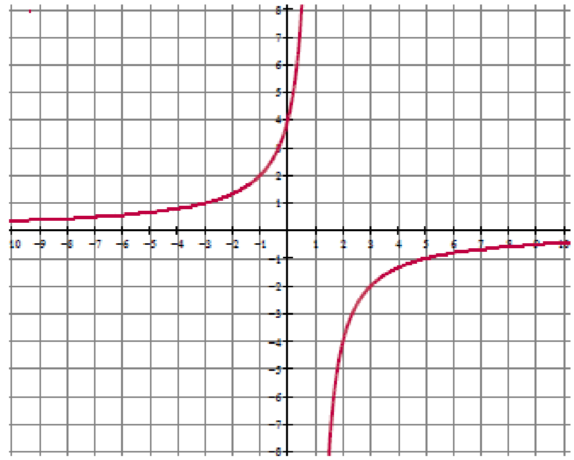
How can you tell if the leading coefficient is positive or negative?

Example 2: Determine the equation of each of the following functions

a)



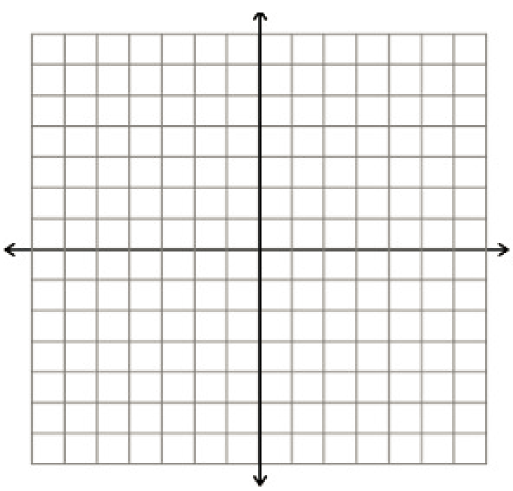
b)



Reciprocal Quadratic Functions

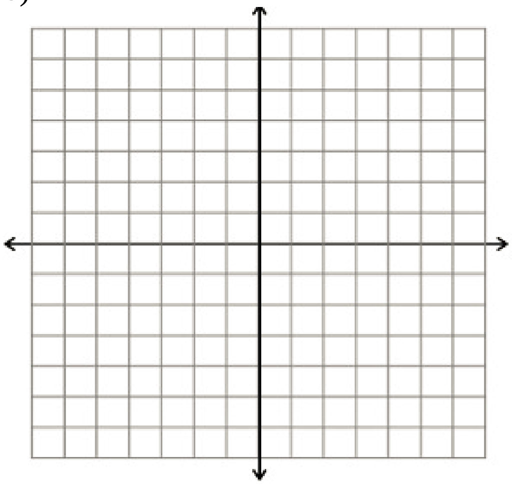
Example 1: Determine the following information about each of the functions below.

a)



Equation	$y = \frac{1}{x^2}$
Domain	
Range	
End Behaviour	As $x \to \infty$ $y \to$ As $x \to -\infty$ $y \to$ As $x \to$ $y \to$ As $x \to$ $y \to$ As $x \to$ $y \to$ As $x \to$ $y \to$
H. Asymptote	V. Asymptote

b)

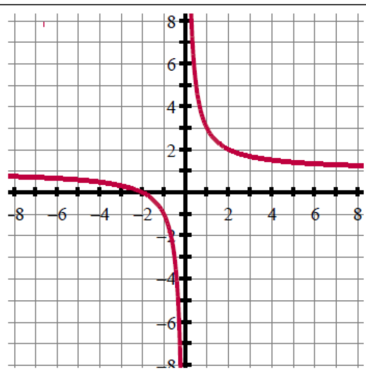


Equation	$y = \frac{1}{x^2-1}$
Domain	
Range	
End Behaviour	As $x\to\infty$ $y\to$ As $x\to-\infty$ $y\to$ As $x\to$ $y\to$ As $x\to$ $y\to$ As $x\to$ $y\to$ As $x\to$ $y\to$
H. Asymptote	V. Asymptote

Graphing Rational Functions

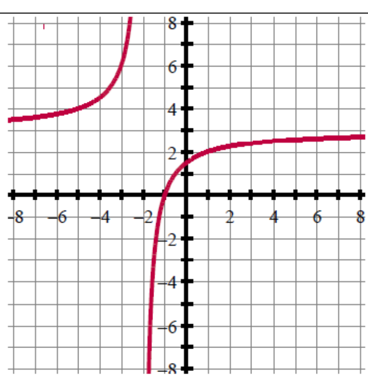
Example 1: Using the graphs of each of the following functions, draw in the horizontal and vertical asymptotes and state the x-intercept(s), y-intercept(s), vertical asymptote(s) and horizontal asymptote(s).

a.



Equation	$y = \frac{x+2}{x}$
X-Intercept(s)	
Y-Intercept(s)	
Vertical Asymptote(s)	
Horizontal Asymptote(s)	

b.

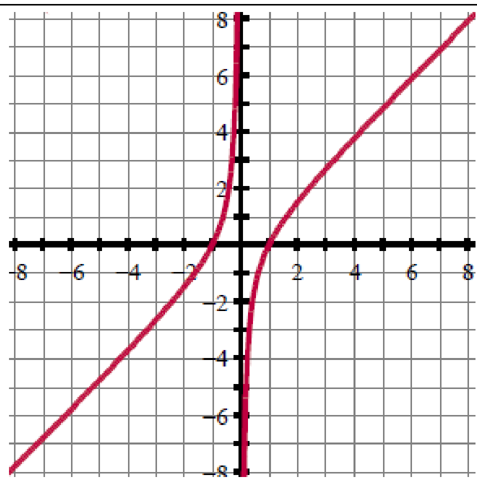


Equation	$y = \frac{3x+3}{x+2}$
X-Intercept(s)	
Y-Intercept(s)	
Vertical Asymptote(s)	
Horizontal Asymptote(s)	

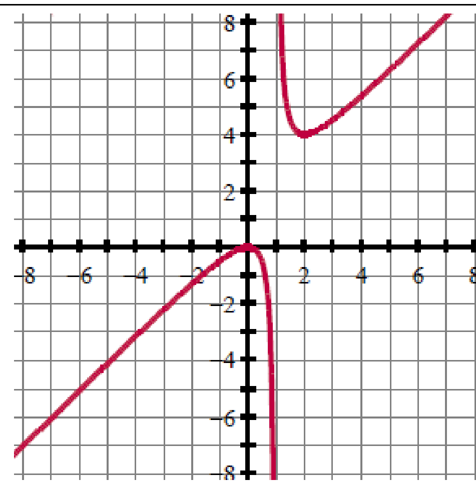
Think! How can you determine the location of the horizontal asymptote from the equation of the function?

Example 2: Draw in the asymptotes on each of the graphs of each of the following functions.

a) $y = \dfrac{x^2 - 1}{x}$



b) $y = \dfrac{x^2}{x - 1}$



What is different about the asymptotes for these functions compared to the ones on the previous page?

Asymptotes

Vertical Asymptote

- A vertical line that a graph approaches, but does not cross.
- Occurs at values of x that make the denominator equal zero.

Horizontal Asymptote

- A horizontal line that the ends of a graph approach but do not cross.
- Can be found using long division (quotient).

Oblique Asymptote

- A diagonal line that the ends of a graph approach but do not cross.
- Can be found using long division (quotient).

Example 3: Sketch the graph of the function y = $\frac{x^2}{x}$.

For a rational function $y = \frac{g(x)}{h(x)}$, $h(x) \neq 0$:

- The function will be discontinuous at $x = a$ if $h(a) = 0$.
- The function has a vertical asymptote at $x = a$ if $h(a) = 0$ and $g(a) \neq 0$, when the function is in simplest form.
- **If $h(a) = 0$ and $g(a) =$** 0 for some value of $a \in$ R, then $x-a$ is a factor of the numerator and denominator of the function, and a point of discontinuity (a hole) may occur at $x = a$. To verify this, express the function in simplified form and then determine if it generates a **single point of discontinuity (Hole),** or a vertical asymptote.

Example 4: Identify the domain, intercepts, points of discontinuity, holes, vertical asymptotes, and horizontal asymptote of each function. Then sketch the graph.

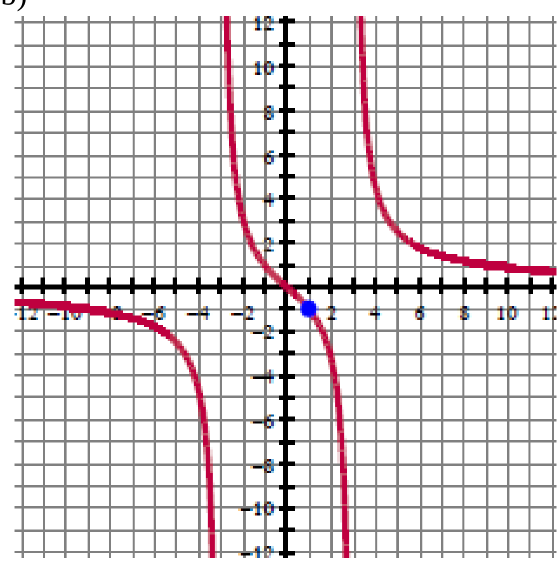
a) f(x) = - $\frac{4}{x^2-3x}$

b) g(x) = $\frac{x^3-9x}{3x^2-6x-9}$

Example 5: Determine the equation of each of the following.

a)
x-intercept: x = -6
y-intercept: y = -4
vertical asymptote: x = 3
horizontal asymptote: y = 2

b)



Extra Practice – Graphing Rational Functions

Identify the holes, vertical asymptotes, and horizontal asymptote of each. Then sketch the graph.

1) $f(x)=\dfrac{x^3+x^2-6x}{4x^2+4x-8}$

2) $f(x)=\dfrac{x^2+3x}{x^2-x}$

3) $f(x)=\dfrac{-3x^2-12x-9}{x^2+5x+4}$

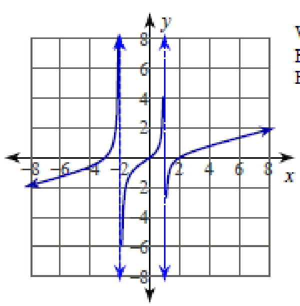
4) $f(x)=\dfrac{x^2-5x+6}{-4x-4}$

5) $f(x)=\dfrac{x^3-16x}{-3x^2+3x+18}$

6) $f(x)=\dfrac{2x-6}{x^2-3x}$

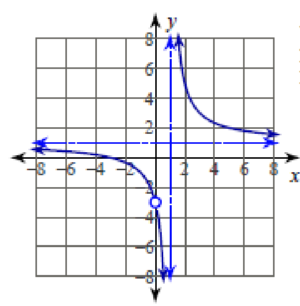
Answer

1)



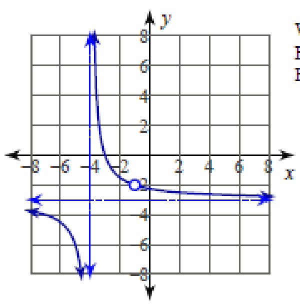
Vertical Asym.: $x=1$, $x=-2$
Holes: None
Horz. Asym.: None

2)



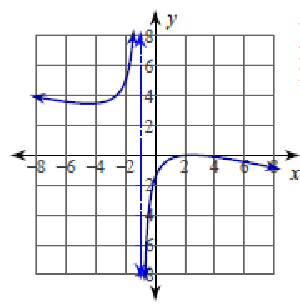
Vertical Asym.: $x=1$
Holes: $x=0$
Horz. Asym.: $y=1$

3)



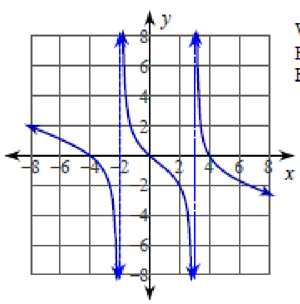
Vertical Asym.: $x=-4$
Holes: $x=-1$
Horz. Asym.: $y=-3$

4)



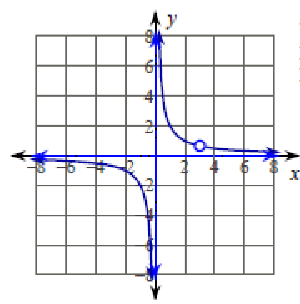
Vertical Asym.: $x=-1$
Holes: None
Horz. Asym.: None

5)



Vertical Asym.: $x=3$, $x=-2$
Holes: None
Horz. Asym.: None

6)



Vertical Asym.: $x=0$
Holes: $x=3$
Horz. Asym.: $y=0$