AP Calculus Homework Five – Applications of Differential Calculus

3.1 Slope, Critical Points, Tangents and Normals; 3.2 Increasing and Decreasing Functions;
3.3 Maximum, Minimum, and Inflection Points

1. Find the slope of the curve $y^2 - xy - 3x = 1$ at the point (0, –1).

$(y^2)' - (xy)' - 3(x)' = (1)' \Rightarrow 2y\cdot y' - (x'y + xy') - 3(1) = 0,$

$\Rightarrow 2y\cdot y' - y - xy' - 3 = 0$, sub. $x=0$ and $y=-1$: $2(-1)y' - (-1) - (0)y' - 3 = 0$

$\Rightarrow -2y' = 3-1 \Rightarrow y' = -1$ is the slope required.

2. Find the equation of tangent to the curve $y = x\sin x$ at the point ($\pi/2$, $\pi/2$).

$y' = (x\sin x)' = x'\sin x + x(\sin x)' = (1)\sin x + x\cos x$; sub $\begin{cases} x = \pi/2 \\ y = \pi/2 \end{cases}$

$y'\big|_{x=\frac{\pi}{2},\, y=\frac{\pi}{2}} = \sin\frac{\pi}{2} + \frac{\pi}{2}\cos\frac{\pi}{2} = 1$; $\therefore y - \frac{\pi}{2} = (1)(x - \frac{\pi}{2})$

$\Rightarrow y = x$ is the equation of tangent required.

3. Find the value of x so that the tangent to the curve $y = xe^{-x}$ is horizontal.

$y' = (xe^{-x})' = x'e^{-x} + x(e^{-x})' = (1)e^{-x} + x(e^{-x})(-x)' = e^{-x} - xe^{-x}$

$= e^{-x}(1-x)$; let $y' = 0 \Rightarrow e^{-x}(1-x) = 0 \Rightarrow 1-x=0 \Rightarrow x = 1.$

4. What is the value of y for which the tangent to the curve $y^2 - xy + 9 = 0$ is vertical? (1)

$(y^2)' - (xy)' + (9)' = (0)' \Rightarrow 2y\cdot y' - (x'y + xy') + 0 = 0$

$\Rightarrow 2y\cdot y' - y - xy' = 0 \Rightarrow (2y-x)y' = y \Rightarrow y' = \frac{y}{2y-x} \Rightarrow 2y - x = 0.$

$\Rightarrow x = 2y$. sub. into (1) : $y^2 - 2y\cdot y + 9 = 0 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3.$

5. Find the local extrema and the inflection points of the function $y = x^4 - 4x^2$

$y' = (x^4 - 4x^2)' = 4x^3 - 8x$; $y'' = (4x^3 - 8x)' = 12x^2 - 8$;

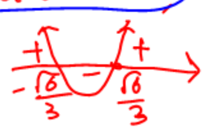
let $y' = 0 \Rightarrow 4x^3 - 8x = 0 \Rightarrow 4x(x^2-2) = 0 \Rightarrow x = 0$ or $x = \pm\sqrt{2}$;

$\therefore f(-\sqrt{2}) = f(\sqrt{2}) = (\pm\sqrt{2})^4 - 4(\pm\sqrt{2})^2 = 4 - 4(2) = -4$; local min.

$f(0) = 0^4 - 4(0)^2 = 0$ is local max.

let $f''(x) = 0 \Rightarrow 12x^2 - 8 = 0 \Rightarrow 4(3x^2 - 2) = 0$

$\Rightarrow x = \pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{6}}{3}$;



$f(\pm\frac{\sqrt{6}}{3}) = (\pm\frac{\sqrt{6}}{3})^4 - 4(\pm\frac{\sqrt{6}}{3})^2 = \frac{4}{9} - 4(\frac{2}{3}) = \frac{4-24}{9} = -\frac{20}{9}$

$\therefore (-\frac{\sqrt{6}}{3}, -\frac{20}{9})$ and $(\frac{\sqrt{6}}{3}, -\frac{20}{9})$ are P.O.I.s

6. What is the maximum value of the function $y = -4\sqrt{2-x}$? , where $2-x \geq 0 \Rightarrow x \leq 2$

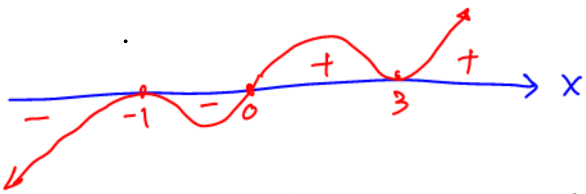
$f'(x) = -4(\sqrt{2-x})' = -4 \dfrac{-1}{2\sqrt{2-x}} = \dfrac{2}{\sqrt{2-x}}$, where $x < 2$;

so $f'(x) > 0$ for all $x < 2$, so $y = -4\sqrt{2-x}$ is increasing for all $x < 2$.

$\therefore f(2) = -4\sqrt{2-2} = 0$ is the global maximum value.

7. Find the total number of local maximum and minimum points of the function $f(x)$ whose derivative, for all x, is given by $f'(x) = x(x-3)^2(x+1)^4$.

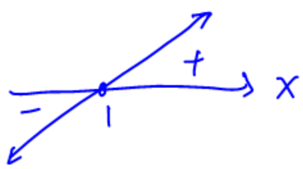
Sketch the graph of $f'(x)$:



Since $f'(x)$ changes its sign from − to + at $x = 0$ only, therefore, $f(x)$ has only one local minimum point at $x = 0$.

8. Find local minimum value(s) of the function $y = \dfrac{e^x}{x}$.

$y' = \left(\dfrac{e^x}{x}\right)' = \dfrac{(e^x)'x - e^x(x)'}{x^2} = \dfrac{e^x x - e^x}{x^2} = \dfrac{e^x(x-1)}{x^2}$; let $y' = 0$, $\Rightarrow x = 1$.



, $\therefore y'$ changes sign from − to + at $x = 1$,

$y(1) = \dfrac{e^1}{1} = e$ is a local minimum value.

9. If $f(x) = xe^{-x}$, then at $x = 0$

(A) f is increasing. (B) f is decreasing.
(C) f has a relative maximum. (D) f has a relative minimum.
(E) f' does not exist.

$f'(x) = (xe^{-x})' = x'e^{-x} + x(e^{-x})' = e^{-x} - xe^{-x} = e^{-x}(1-x)$; $f'(0) = e^0(1-0) = 1 > 0$

$\therefore$ (A)

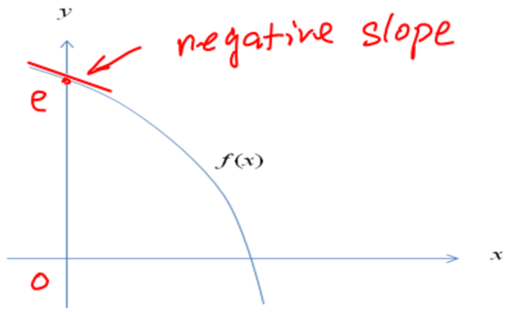
10. Find the equation of the tangent to the curve with parametric equations $x = 2t + 1$ and $y = 3 - t^3$ at the point where $t = 1$.

The tangency point is $\begin{cases} x(1) = 2(1) + 1 = 3 \\ y(1) = 3 - 1^3 = 2 \end{cases}$ or $(3, 2)$

$m = \left.\dfrac{dy}{dx}\right|_{t=1} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{(3-t^3)'}{(2t+1)'} = \left.\dfrac{-3t^2}{2}\right|_{t=1} = \dfrac{-3(1)^2}{2} = -\dfrac{3}{2}$.

$\therefore$ The equation of the tangent required is $y - 2 = -\frac{3}{2}(x-3)$.

11. If $f(x) = cx^2 + dx + e$ for the function shown in the graph, then



$\because f(0) = c(0)^2 + d(0) + e = e > 0$

$\because f'(x) = 2cx + d\ ;\ f'(0) = 2c(0) + d = d < 0$

and $f''(x) = 2c < 0$ for the curve given.
concave down

$\therefore$ (E)

(A) c, d, and e are all positive.
(B) $c > 0$, $d < 0$, $e < 0$.
(C) $c > 0$, $d < 0$, $e > 0$.
(D) $c < 0$, $d > 0$, $e > 0$.
(E) $c < 0$, $d < 0$, $e > 0$.

12. Find the point on the curve $y = \sqrt{2x+1}$ at which the normal is parallel to the line L: $y = -3x + 6$.

$\because m_N = m_L = -3\ ;\quad y' = (\sqrt{2x+1})' = \frac{(2x+1)'}{2\sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}} = m_T$

$\therefore m_T = \frac{1}{\sqrt{2x+1}} = \frac{1}{3} \Rightarrow \sqrt{2x+1} = 3 \Rightarrow 2x+1 = 9 \Rightarrow x = 4;\ y(4) = \sqrt{2(4)+1} = 3.$

$\therefore$ the point required is $(4, 3)$.

13. Find the value(s) of k such that the line $y = 3x + k$ is tangent to the curve $y = x^3$.

$y' = (x^3)' = 3x^2$, let $3x^2 = 3 \Rightarrow x = \pm 1$, sub. into $y = x^3$:

$\begin{cases} x = 1 \\ y = 1 \end{cases}$ or $\begin{cases} x = -1 \\ y = -1 \end{cases}$; Sub. into $y = 3x + k$, respectively:

$1 = 3(1) + k \Rightarrow k = -2$, or $-1 = 3(-1) + k \Rightarrow k = 2$;

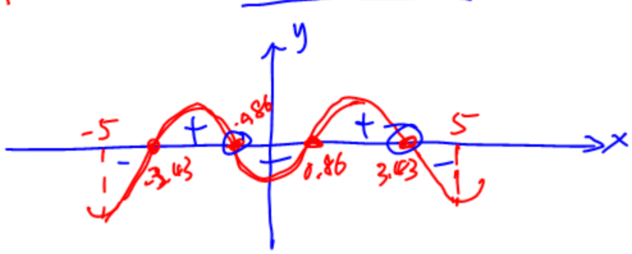
For Questions 14 and 15, $f'(x) = x \sin x - \cos x$ for $\boxed{0 < x < 4}$.

Q15. $f''(x) = (x \sin x - \cos x)'$
$= x' \sin x + x(\sin x)' - (\cos x)'$
$= \sin x + x \cos x + \sin x$
$= 2 \sin x + x \cos x$

14. Find the value of x for which f has a local maximum.

15. Find the value of x for which the graph of f has a point of inflection.

Q14. let $f'(x) = 0 \Rightarrow x \sin x - \cos x = 0 \Rightarrow x \approx \pm 0.86,\ \pm 3.43$;



Since $f'(x)$ changes from + to − ~~at $x = -0.86$ or~~ at $x = 3.43$.
so $f(x)$ has local maximum ~~at $x = -0.86$ or~~ at $x \approx 3.43$.

Q15.
let $f''(x) = 0$
$\Rightarrow 2 \sin x + x \cos x = 0$

$\Rightarrow x \approx 2.29$. and $f''(x)$ changes its sign from + to − at $x = 2.29$; $\therefore f(x)$ has a p.o.I at $x \approx 2.29$.