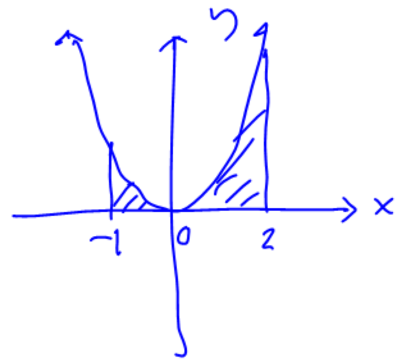
AP Calculus In-Class Ten – Applications of Definite Integral and Polar Coordinates

5.1 Area and Solids of Revolution

In Questions 1 - 11, evaluate the area of the region whose boundaries are given.

1. The curve of $y = x^2$, $y = 0$, $x = -1$, and $x = 2$.

$A=\int_{-1}^{2} x^2\,dx = \left(\frac{x^3}{3}\right)\Big|_{-1}^{2} = \frac{1}{3}\left(2^3-(-1)^3\right) = \frac{9}{3} = 3$



2. The parabola of $y^2 = x$ and the line $x + y = 2$.

3. The curve of $y = 2/x$ and $x + y = 3$.

let $\frac{2}{x} = 3-x$

$2 = 3x - x^2$

$x^2 - 3x + 2 = 0$

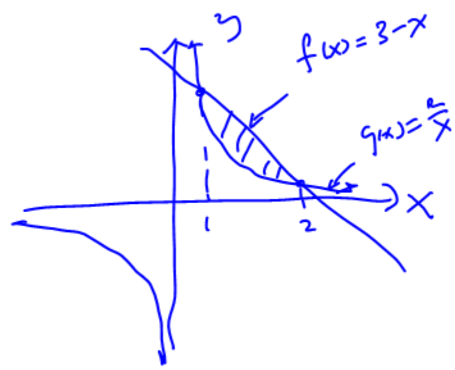
$(x-2)(x-1) = 0$

$x=1,\ x=2$

$A=\int_{1}^{2} [f(x)-g(x)]\,dx$

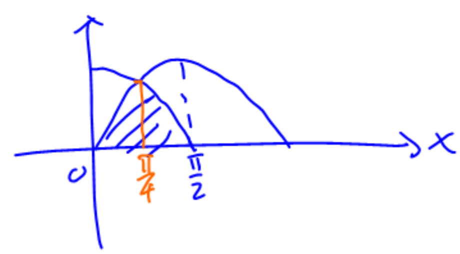
$=\int_{1}^{2} \left(3-x-\frac{2}{x}\right) dx = \left(3x-\frac{x^2}{2}-2\ln x\right)\Big|_{1}^{2}$

$=\left(3(2)-\frac{2^2}{2}-2\ln 2\right)-\left(3-\frac{1}{2}-0\right) = \frac{3}{2}-\ln 4$



4. In the first quadrant, bounded below by the x-axis and above by the curves of $y = \sin x$ and $y = \cos x$.

$A=\int_{0}^{\frac{\pi}{4}} \sin x\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\,dx$



5. The curve of $y = \cot x$, the line $x = \pi/4$, and the x-axis.

6. The curve of $y = x^3 - 2x^2 - 3x$ and the x-axis.

7. Find the total area bounded by the cubic $x = y^3 - y$ and the line $x = 3y$.

8. What is the area enclosed by the ellipse with parametric equations
$x = 2\cos\theta$ and $y = 3\sin\theta$?

9. Find the area enclosed by one arch of the cycloid with parametric equations
$x = \theta - \sin\theta$ and $y = 1 - \cos\theta$.

10. Write a definite integral to represent the area enclosed by the curve $y^2 = x(1 - x)$.

11. Find the area bounded by the curve $y = x^3$, $y = c^3$, and the y-axis.

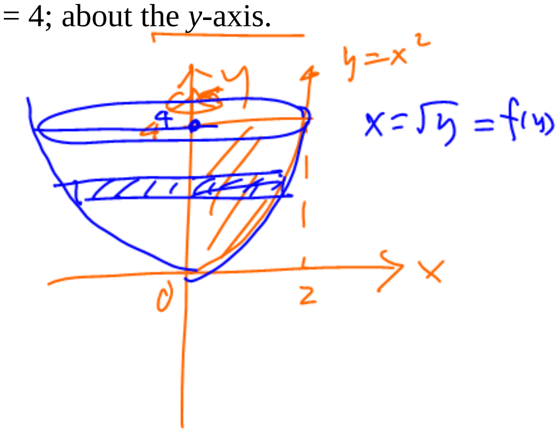
In Questions 12 - 15, the region whose boundaries are given is rotated about the line indicated. Calculate the volume of the solid generated.

12. The first quadrant region bounded by $y = x^2$, the y-axis, and $y = 4$; about the y-axis.

$$V = \int_0^4 \pi [f(y)]^2 dy = \int_0^4 \pi (\sqrt{y})^2 dy$$

$$= \pi \int_0^4 y\,dy = \pi \left(\frac{y^2}{2}\right)\Big|_0^4$$

$$= \frac{\pi}{2}(4^2 - 0^2) = 8\pi$$



13. $y = x^2$ and $y = 4$; about the x-axis.

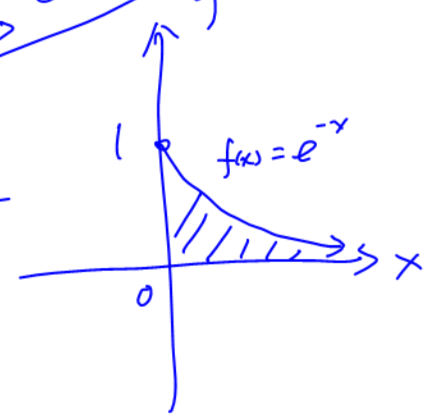
$$\because \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$$

$$\Rightarrow \int_0^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{\frac{\pi}{2}}$$

14. The first-quadrant region under $y = e^{-x}$; about the x-axis.

$$V = \int_0^{\infty} \pi [f(x)]^2 dx = \int_0^{\infty} \pi (e^{-x})^2 dx = \pi \int_0^{\infty} e^{-x^2} dx$$

$$= \sqrt{2}\pi \int_0^{\infty} e^{-\frac{x^2}{2}} d\left(\frac{x}{\sqrt{2}}\right) = \sqrt{2}\pi \cdot \sqrt{\frac{\pi}{2}} = \pi\sqrt{\pi}$$



15. A trapezoid with vertices at (2, 0), (2, 2), (4, 0), and (4, 4); about the x-axis.

In Questions 15 - 18, evaluate the area, if it exists, of the region described.

16. In the first quadrant under the curve of $y = xe^{-x^2}$.

17. In the first quadrant above $y = 1$, between by the y-axis and the curve $xy = 1$.

18. Between the curve $y = 4/(1 + x^2)$ and the x-axis.

19. Above the x-axis, between the curve $y = \dfrac{4}{\sqrt{1-x^2}}$ and its asymptotes.

In Questions 20 - 21, write a definite integral that gives the area of the region whose boundaries are given.

20. The area bounded by the parabola $y = 2 - x^2$ and the line $y = x - 4$.

21. The area enclosed by the hypocycloid with parametric equations $x = \cos^3 t$ and $y = \sin^3 t$.

22. Suppose the following is a table of ordinates for $y = f(x)$, given that f is continuous on [1, 5]:

x	1	2	3	4	5
y	1.62	4.15	7.5	9.0	12.13

If a trapezoid sum in used, when $n = 4$, then find the area under the curve, from $x = 1$ to $x = 5$, rounding up (or down) the answer to two decimal places.

23. What is the exact total area bounded by the curve $y = x^3 - 4x^2 + 3x$ and the x-axis?

24. If the curve of $f(x)$ and $g(x)$ intersect for $x = a$ and $x = b$ and if $f(x) > g(x) > 0$ for all x on (a, b), then what is the volume obtained when the region bounded by the curves is rotated about the x-axis?

$$V = \int_a^b \pi \left[(f(x))^2 - (g(x))^2 \right] dx$$