AP Calculus In-Class Eleven – Applications of Definite Integral and Polar Coordinates

5.2 Volumes Using Cylindrical Shells and Volumes by Slicing; 5.3 Work and Arc Length

In Questions 1 - 2, the region whose boundaries are given is rotated about the line indicated. Derive a definite integral that gives the volume of the solid generated, using cylindrical shell if possible.

1. $y = 3x - x^2$ and $y = 0$; about the x-axis.

2. $y = 3x - x^2$ and $y = x$; about the x-axis.

By disks

$R = x = \sqrt{r^2-y^2}$

$V_s = \int_a^b \pi R^2\, dy = \int_h^r \pi (r^2-y^2)\, dy$

$= \pi \left[r^2 \cdot y - \frac{1}{3}y^3 \right]_h^r = \pi\left[\left(r^2\cdot r - \frac{1}{3}r^3\right) - r^2 h + \frac{1}{3}h^3\right]$

$= \pi\left(\frac{2}{3}r^3 - r^2 h + \frac{1}{3}h^3\right) = \frac{2\pi}{3}(r^3 - h^3)$

h^2 ?

$V = \int_a^b 2\pi x f(x)\, dx$

3. A sphere of radius r is divided into two parts by a plane at distance h ($0 < h < r$) from the centre. What is the volume of the smaller part?

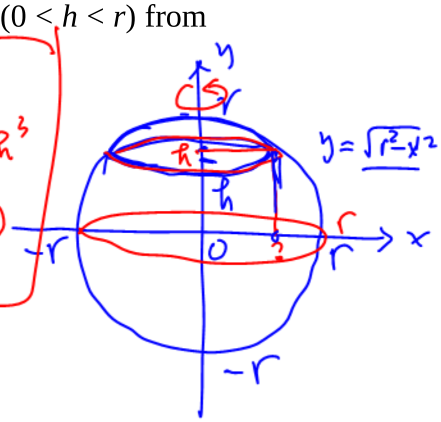
let $h = \sqrt{r^2-x^2} \Rightarrow x = \sqrt{r^2-h^2}$

$V_B = \int_{\sqrt{r^2-h^2}}^{r} 2\pi x \sqrt{r^2-x^2}\, dx$, where $R = x$, $h = \sqrt{r^2-x^2}$

$= -\pi \int_{\sqrt{r^2-h^2}}^{r} (r^2-x^2)^{\frac{1}{2}}\, d(r^2-x^2)$

$= \pi \left(\frac{(r^2-x^2)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right)_r^{\sqrt{r^2-h^2}} = \frac{2\pi}{3}\left[(r^2-x^2)^{\frac{3}{2}}\right]_r^{\sqrt{r^2-h^2}} = \frac{2\pi}{3}(\cancel{r^2}-\cancel{r^2}+h^2)^{3/2} = \frac{2\pi}{3}h^3$

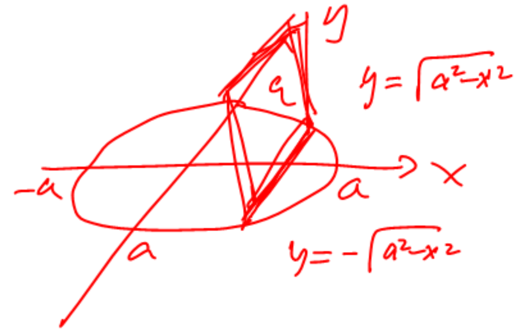
$$\boxed{\begin{aligned} V_s &= \frac{1}{2}\cdot\frac{4}{3}\pi r^3 - V_B \\ &= \frac{2}{3}\pi r^3 - \frac{2}{3}\pi h^3 \\ &= \frac{2\pi}{3}(r^3 - h^3) \end{aligned}}$$



4. The base of a solid is a circle of radius a, and every plane section perpendicular to a diameter is a square. Find the volume of the solid.

$A(x) = \left(2\sqrt{a^2-x^2}\right)^2 = 4(a^2-x^2)$

$V = \int_{-a}^{a} A(x)\,dx = 2\int_0^a 4(a^2-x^2)\,dx$

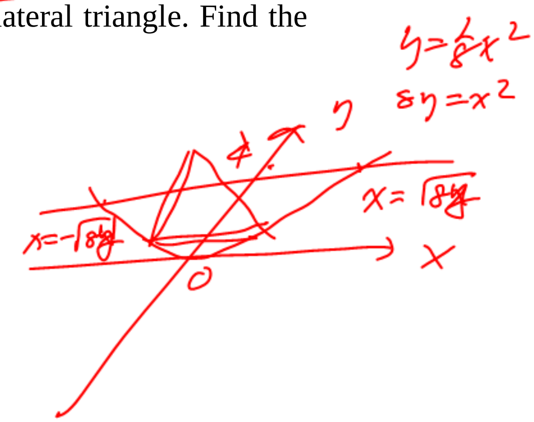


5. The base of a solid is the region bounded by the parabola $x^2 = 8y$ and the line $y = 4$, and each plane section perpendicular to the y-axis is an equilateral triangle. Find the volume of the solid.

$A(y) = \frac{\sqrt{3}}{4}\left(2\sqrt{8y}\right)^2 = \frac{\sqrt{3}}{\cancel{4}}\cancel{(4)}\,8y$

$V = \int_0^4 A(y)\,dy = \int_0^4 8\sqrt{3}\,y\,dy = 8\sqrt{3}\left(\frac{y^2}{2}\right)\Big|_0^4$

$= \cancel{8}4\sqrt{3}\,(4)^2 = 64\sqrt{3}.$



6. A particle moves along a line in such a way that its position at time t is given by $s = t^3 - 6t^2 + 9t + 3$. When does the direction of the motion change?

7. A particle moves along a line with velocity $v = 3t^2 - 6t$. Find the total distance traveled from $t = 0$ to $t = 3$.

8. What is the net change in the position of the particle in Q7?

9. The acceleration of a particle moving on a straight line is given by $a = \cos t$, and when $t = 0$ the particle is at rest. What is the distance the particle covers from $t = 0$ to $t = 2$?

10. During the worst 4-hr period of a hurricane the wind velocity, in miles per hour, is given by $v = 5t - t^2 + 100$, $0 \le t \le 4$. What is the average wind velocity during this period (in mph)?

11. A car accelerates from 0 to 60 mph in 10 seconds, with constant acceleration. (Note that 60 mph = 88 ft/sec.) Determine the acceleration (in ft/sec^2).

For Questions 12 – 14 use the following information: The velocity $\vec{v}$ of a particle moving on a curve is given, at time t, by $\vec{v} = t\vec{\imath} - (1 - t)\vec{\jmath}$. When t = 0, the particle is at point (0, 1).

12. Determine the position vector $\vec{R}$ at time t.

13. What is the acceleration vector at time $t = 2$?

14. Find time t when the speed of the particle is at a minimum.

15. A particle moves along a curve in such a way that its position vector and velocity vector are perpendicular at all times. If the particle passes through the point (4, 3), what is the equation of the curve?

16. The acceleration of an object in motion is given by the vector $\vec{a}(t) = (2t,\ e^{t})$. If the object's initial velocity was $\vec{v}(0) = (2,\ 0)$, find the velocity vector at any time *t*.

17. The velocity of an object is given by $\vec{v}(t) = (3\sqrt{t},\ 4)$. If this object is at the origin when $t = 1$, where was it at $t = 0$?

18. If a quantity $Q(t)$ is growing at the rate of 5% per year and Q now equals Q_0, then find Q in t years.

19. A stone is thrown upward from the ground with an initial velocity of 96 ft/sec. Determine its average velocity (given that $a(t) = -32$ ft/sec^2) during the first two seconds.

20. Oil is leaking from a tanker at the rate of $1000e^{-0.3t}$ gal/hr, where t is given in hours. Write a general Riemann sum for the amount of oil that leaks out in the next 8 hours, where the interval [0, 8] has been partitioned into n subintervals.

21. In Question 20, what is the total number of gallons of oil that will leak out during the next 8 hours approximately?

22. Assume that the density of vehicles (number per mile) during morning rush hour, for the 20-mile stretch along the New York State Thruway southbound from the Tappan Zee Bridge, is given by $f(x)$, where x is the distance, in miles, south of the bridge. Write a definite integral in terms of $f(x)$to represent the number of vehicles (on this 20-mile stretch) from the bridge to a point x miles south of the bridge.

23. The centre of a city that we will assume is circular in on a straight highway. The radius of the city is 3 miles. The density of the population, in thousands of people per square mile, is given approximately by $f(r) = 12 - 2r$ at a distance r miles from the highway. Write a definite integral to represent the population of the city.

24. If a factory continuously dumps pollutants into a river at the rate of $\sqrt{t}/180$ tones per day, then what is approximately the amount dumped after 7 weeks.

25. A roast at 160ºF is put into a refrigerator whose temperature is 45ºF. the temperature of the roast is cooling at time t at the rate of $(-9e^{-0.08t})$ºF per minute. Find the temperature, to the nearest degree F, of the roast 20 minutes after it is put in the refrigerator.

26. How long will it take to release 9 tones of pollutant if the rate at which pollutant is being released is $te^{-0.3t}$ tone per week?

27. If you deposit \$1000 today at 8% interest compounded continuously, it will grow at the rate of $80e^{0.08t}$ dollars per year. How much will it be worth in 6 years (in dollars)?

28. Water is leaking from a tank at the rate of $(-0.1t^2 - 0.3t + 2)$ gal/hr. Find the total amount, in gallons, that will leak out in the next 3 hours approximately.

29. A bacteria culture is growing at the rate of $1000e^{0.03t}$ bacteria in t hours. Determine the total increase in the bacteria population during the second hour approximately.

30. An 18-wheeler traveling at speed v mph gets about $(4 + 0.01v)$ mpg (miles per gallon) of diesel fuel. If its speed is $80(t + 1)/(t + 2)$ mph at time t, then what is the approximate amount, in gallons, of diesel fuel used during the first 2 hours?

$2\int_0^2 \sqrt{1+\frac{1}{x}}\,dx$

$y = 2\sqrt{x}$

$y' = 2\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}}$

$y'^2 = \frac{1}{x}$

31. The length of arc of the parabola $4x = y^2$ cut off by the line $x = 2$.

$4(2) = y^2$

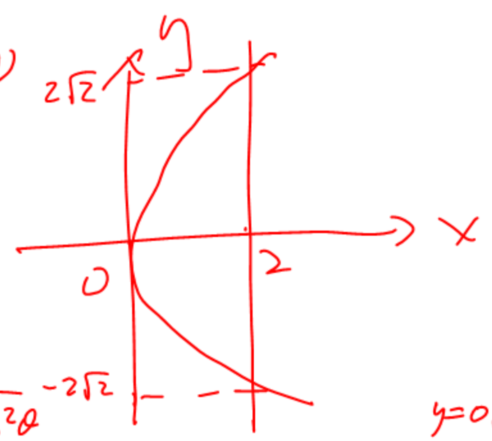
$L = 2\int_0^{2\sqrt{2}} \sqrt{1+(f'(y))^2}\,dy$,

$x = \frac{1}{4}y^2 = f(y)$

$f'(y) = \frac{1}{2}y$

$= 2\int_0^{2\sqrt{2}} \sqrt{1+(\frac{1}{2}y)^2}\,dy$

let $\frac{1}{2}y = \tan\theta$



$= 2\int_0^{\tan^{-1}(\sqrt{2})} \sec\theta \cdot 2\sec^2\theta\,d\theta$

$\sqrt{1+(\frac{1}{2}y)^2} = \sqrt{1+\tan^2\theta}$

$= \sec\theta$

$y=0,\ \theta=0$

$y=2\sqrt{2},$

$\theta=\tan^{-1}(\sqrt{2})$

$= 4\int_0^{\tan^{-1}(\sqrt{2})} \sec^3\theta\,d\theta = \dots$

$dy = 2\sec^2\theta\,d\theta$