AP Calculus Homework Eight – Antiderivatives and the Definite Integral

4.1 Antiderivatives; 4.2 Area

1. Find the most general antiderivatives of the functions.

(a) $f(x) = 6/\sqrt[3]{x} - \sqrt[3]{x}/6 + 7$

$\int f(x)dx = \int (6x^{-\frac{1}{3}} - \frac{1}{6}x^{\frac{1}{3}} + 7)\,dx$

$= 6\int x^{-\frac{1}{3}}dx - \frac{1}{6}\int x^{\frac{1}{3}}dx + 7\int dx$

$= 6\left(\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}\right) - \frac{1}{6}\left(\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right) + 7x + C$

$= 9x^{2/3} - \frac{1}{8}x^{4/3} + 7x + C$

(b) $f(x) = (4 + 3x^2 \cos 4x)/x^2$ $= 4x^{-2} + 3\cos 4x$

$\int f(x)dx = \int (4x^{-2} + 3\cos 4x)dx$

$= 4\int x^{-2}dx + 3\int \cos 4x\, dx$

$= 4\left(\frac{x^{-2+1}}{-2+1}\right) + 3\left(\frac{1}{4}\right)\int \cos 4x\, d4x$

$= -\frac{4}{x} + \frac{3}{4}\sin(4x) + C$

(c) $f(x) = 2\cos 3x - 3\sin 2x$

$\int f(x)\,dx = 2\int \cos 3x\, dx - 3\int \sin 2x\, dx$

$= 2\left(\frac{1}{3}\right)\int \cos 3x\, d3x - 3\left(\frac{1}{2}\right)\int \sin 2x\, d2x$

$= \frac{2}{3}\sin(3x) + \frac{3}{2}\cos(2x) + C$

(d) $f(x) = \sin(4x)/\cos(2x)$ $= \frac{2\sin(2x)\cancel{\cos(2x)}}{\cancel{\cos(2x)}}$

$\int f(x)\,dx = \int 2\sin(2x)dx = \int \sin(2x)\,d(2x)$

$= -\cos(2x) + C$

2. Solve the differential equations subject to the given boundary conditions. initial conditions

(a) $f'''(x) = 6x, \quad f''(0) = 2, \quad f'(0) = -1, \quad f(0) = 4$

$f''(x) = \int f'''(x)dx = \int 6x\,dx = 6\left(\frac{x^2}{2}\right) + C = 3x^2 + C$. $\because f''(0) = 2$, $\therefore 2 = 3(0)^2 + C \Rightarrow C = 2$;

$\therefore f''(x) = 3x^2 + 2$; $f'(x) = \int f''(x)dx = \int (3x^2 + 2)dx = 3\left(\frac{x^{2+1}}{2+1}\right) + 2x + C = x^3 + 2x + C$;

$\because f'(0) = -1$, $\therefore -1 = 0^3 + 2(0) + C$, $\therefore C = -1$; $f'(x) = x^3 + 2x - 1$; $f(x) = \int f'(x)dx = \int (x^3 + 2x + 1)dx$

$= \frac{x^4}{4} + x^2 + x + C$

$\because f(0) = 4$, $\therefore 4 = 0 + 0 + 0 + C$, $\therefore C = 4$; $\therefore f(x) = \frac{1}{4}x^4 + x^2 + x + 4$

(b) $f''(x) = 4\sin 2x + 16\cos 4x, \quad f'(0) = 1, \quad f(0) = 6$

$f'(x) = \int f''(x)dx = \int 4\sin 2x\,dx + \int 16\cos 4x\,dx = 2\int \sin 2x\,d2x + 4\int \cos 4x\,d4x = -2\cos 2x + 4\sin 4x + C$

$\because f'(0) = 1$, $\therefore 1 = -2\cos(0) + 4\sin(0) + C$, $\therefore C = 1 + 2 - 0 = 3$; $\therefore f'(x) = -2\cos 2x + 4\sin 4x + 3$

$f(x) = \int (-2\cos 2x + 4\sin 4x + 3)dx = -\sin 2x - \cos 4x + 3x + C$; $\because f(0) = 6$, $\therefore 6 = -\sin(0) - \cos(0) + 0 + C$

$\therefore C = 7$, $f(x) = -\sin 2x - \cos 4x + 3x + 7$

3. Evaluate the integrals without using your calculator.

(a) $\int (2 - 3x)^5\, dx$

$\int x^5 dx$

$\therefore u = 2 - 3x$

let $u = 2 - 3x$

$du = (2 - 3x)'dx$

$du = -3dx$

$dx = -\frac{1}{3}du$

$= \int u^5\left(-\frac{1}{3}\right)du$

$= -\frac{1}{3}\int u^5 du = -\frac{1}{3}\frac{u^6}{6} + C$

$= -\frac{1}{18}u^6 + C$

$= -\frac{1}{18}(2 - 3x)^6 + C$

(b) $\int \frac{1 - 3y}{\sqrt{2y - 3y^2}}\, dy$

let $u = 2y - 3y^2 \Rightarrow$

$du = (2y - 3y^2)'dy$

$du = (2 - 6y)dy$

$dy = \frac{1}{2 - 6y}du$

$= \int \frac{(1 - 3y)}{\sqrt{u}} \cdot \frac{1}{(2 - 6y)}du$

$= \frac{1}{2}\int u^{-\frac{1}{2}}du = \frac{1}{2}\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$

$= u^{\frac{1}{2}} + C = \sqrt{2y - 3y^2} + C$

$2 - 6y = 2(1 - 3y)$

4. Evaluate the integrals without using your calculator.

(a) $\int \frac{xdx}{1+4x^2}$

let $u = 1+4x^2 \Rightarrow x = \pm\frac{\sqrt{u-1}}{2}$

$du = (1+4x^2)'dx$

$du = \underline{8x}\, dx$

$dx = \frac{1}{8x} du$

$= \int \frac{\cancel{x}}{u} \cdot \frac{1}{8\cancel{x}} du$

$= \frac{1}{8}\int \frac{1}{u} du = \frac{1}{8}\ln|u| + C$

$= \frac{1}{8}\ln(1+4x^2) + C$

(b) $\int \frac{dx}{1+4x^2} = \int \frac{1}{1+(2x)^2} dx$

let $u = 2x$

$du = 2dx$

$dx = \frac{1}{2} du$

$= \int \frac{\frac{1}{2}du}{1+u^2} = \frac{1}{2}\int \frac{du}{1+u^2}$

$= \frac{1}{2}\tan^{-1}(u) + C = \frac{1}{2}\tan^{-1}(2x) + C$

(c) $\int \frac{xdx}{(1+4x^2)^2}$

let $u = 1+4x^2$

$xdx = \frac{1}{8}du$

$= \int \frac{\frac{1}{8}du}{u^2} = \frac{1}{8}\int u^{-2} du$

$= \frac{1}{8} \frac{u^{-2+1}}{-2+1} + C = -\frac{1}{8}\cdot\frac{1}{u} + C$

$= -\frac{1}{8(1+4x^2)} + C$

(d) $\int \frac{xdx}{\sqrt{1+4x^2}}$

let $u = 1+4x^2$

$xdx = \frac{1}{8}du$

$= \int \frac{\frac{1}{8}du}{\sqrt{u}}$

$= \frac{1}{8}\int u^{-\frac{1}{2}} du = \frac{1}{8}\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = \frac{1}{4}\sqrt{u} + C$

$= \frac{1}{4}\sqrt{1+4x^2} + C$

5. Evaluate the integrals without using your calculator.

$\cos 2\theta = 1-2\sin^2\theta = 2\cos^2\theta - 1 \Rightarrow \cos^2\theta = \frac{1+\cos 2\theta}{2}$

(a) $\int \sin\theta\cos\theta\, d\theta = \frac{1}{2}\int \sin 2\theta\, d\theta$;

let $u = \sin\theta$

$du = \cos\theta\, d\theta$

$= \int u\, du$

$= \frac{1}{2}u^2 du + C$

$= \frac{1}{2}\sin^2\theta + C$

let $u = 2\theta$

$du = 2d\theta$

$d\theta = \frac{1}{2}du$

$= \frac{1}{2}\int \sin u \cdot \frac{1}{2}du$

$= \frac{1}{4}\int \sin u\, du = -\frac{1}{4}\cos u + C$

$= -\frac{1}{4}\cos(2\theta) + C = -\frac{1}{4}(1-2\sin^2\theta) + C$

$= \frac{1}{2}\sin^2\theta + C - \frac{1}{4}$

(b) $\int \frac{\sin\sqrt{x}\, dx}{\sqrt{x}}$

let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}dx$

$dx = 2\sqrt{x}\, du$

$= \int \frac{\sin u}{\cancel{\sqrt{x}}} \cdot (2\cancel{\sqrt{x}}) du$

$= 2\int \sin u\, du$

$= -2\cos u + C$

$= -2\cos\sqrt{x} + C$

(c) $\int \cos^2 2x\, dx$

$= \int \frac{1+\cos 4x}{2} dx$

$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 4x\, dx = \frac{1}{2}x + \frac{1}{2}\cdot(\frac{1}{4})\int \cos 4x\, d(4x)$

$= \frac{1}{2}x + \frac{1}{8}\sin(4x) + C$

(d) $\int \sin 2\theta\, d\theta$

$= \frac{1}{2}\int \sin 2\theta\, d2\theta$

$= -\frac{1}{2}\cos(2\theta) + C$

6. Evaluate the integrals without using your calculator.

(a) $\int \frac{\sin 2xdx}{\sqrt{1+\cos^2 x}}$

let $u = 1+\cos^2 x$

$du = -2\cos x \cdot \sin x\, dx$

$= -\sin 2x\, dx$

$dx = \frac{1}{-\sin 2x} du$

$= \int \frac{\cancel{\sin 2x}}{\sqrt{u}} \cdot \frac{1}{-\cancel{\sin 2x}} du$

$= -\int u^{-\frac{1}{2}} du$

$= -\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = -2\sqrt{u} + C$

$= -2\sqrt{1+\cos^2 x} + C$

(b) $\int \sec^{3/2} x \tan x dx$

let $u = \sec x$

$du = \sec x \cdot \tan x\, dx$

$dx = \frac{1}{\sec x \tan x} du$

$= \int \sec^{1/2} x \cdot \sec x \cdot \tan x\, dx$

$= \int u^{\frac{1}{2}} \cdot \cancel{\sec x \tan x} \cdot \frac{1}{\cancel{\sec x \tan x}} du$

$= \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{2}{3}u^{3/2} + C = \frac{2}{3}\sec^{3/2} x + C$

Let $u = \cos\theta$

$du = -\sin\theta\, d\theta \Rightarrow \sin\theta\, d\theta = -du$

(c) $\int \tan\theta\, d\theta = \int \frac{\sin\theta}{\cos\theta} d\theta$

$= \int \frac{-du}{u} = -\ln|u| + C$

$= -\ln|\cos\theta| + C$

$= \ln|\sec\theta| + C$

let $u = 2x$

$du = 2dx$

$dx = \frac{1}{2}du$

(d) $\int \frac{dx}{\sin^2 2x} = \int \csc^2(2x)\, dx$

$= \int \csc^2 u \cdot \frac{1}{2} \cdot du = -\frac{1}{2}\cot u + C$

$= -\frac{1}{2}\cot(2x) + C$

7. Evaluate the integrals without using your calculator.

(a) $\int \frac{\tan^{-1} y}{1+y^2} dy$

let $u = \tan^{-1} y$

$du = \frac{1}{1+y^2} dy$

$= \int u \cdot du$

$= \frac{1}{2}u^2 + C = \frac{1}{2}[\tan^{-1} y]^2 + C$

(b) $\int \sin 2\theta \cos\theta\, d\theta = \int 2\sin\theta\cos\theta\cos\theta\, d\theta$;

let $u = \cos\theta$

$du = -\sin\theta\, d\theta$

$= -2\int u^2\, du$

$= -2\left(\frac{u^3}{3}\right) + C = -\frac{2}{3}u^3 + C$

$= -\frac{2}{3}\cos^3\theta + C$

let $t = \sin 2u$

$dt = 2\cos 2u\, du$

$\cos 2u\, du = \frac{1}{2}dt$

(c) $\int \cot 2u\, du = \int \frac{\cos 2u}{\sin 2u} du$

$= \int \frac{\frac{1}{2}dt}{t} = \frac{1}{2}\ln|t| + C$

$= \frac{1}{2}\ln|\sin 2u| + C$

(d) $\int e^{2\theta} \sin e^{2\theta}\, d\theta$

let $u = e^{2\theta}$

$du = 2e^{2\theta} d\theta$

$e^{2\theta} d\theta = \frac{1}{2}du$

$= \int \frac{1}{2}\sin u\, du$

$= -\frac{1}{2}\cos u + C$

$= -\frac{1}{2}\cos e^{2\theta} + C$

8. Evaluate the integrals without using your calculator.

$\because \frac{d}{dx}(e^x) = e^x$

$\therefore de^x = e^x dx$

$\because \frac{dx^2}{dx} = 2x$

$\therefore dx^2 = 2x\, dx$

(a) $\int x^2 e^x dx = \int x^2\, de^x$

$= x^2 e^x - \int e^x dx^2 = x^2 e^x - \int e^x (2x) dx$

$= x^2 e^x - 2\int x\, de^x = x^2 e^x - 2[xe^x - \int e^x dx]$

$= x^2 e^x - 2[xe^x - e^x] + C$

$= e^x(x^2 - 2x + 2) + C$

let $u = e^x - e^{-x}$,

$du = (e^x + e^{-x})\, dx$

(b) $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \int \frac{du}{u} = \ln|u| + C$

$= \ln|e^x - e^{-x}| + C$

(c) $\int \frac{\ln v\, dv}{v}$

let $u = \ln v$

$du = \frac{1}{v} dv$

$= \int u\, du$

$= \frac{u^2}{2} + C$

$= \frac{1}{2}[\ln v]^2 + C$

(d) $\int x^3 \ln x\, dx = \int \frac{1}{4}\ln x\, dx^4$

$\because \frac{d}{dx}x^4 = 4x^3$

$\therefore dx^4 = 4x^3 dx$

$= \frac{1}{4}[x^4 \ln x - \int x^4\, d\ln x]$

$\because \frac{d}{dx}\ln x = \frac{1}{x}$

$\therefore d\ln x = \frac{1}{x}dx$

$= \frac{1}{4}[x^4 \ln x - \int x^4 (\frac{1}{x}) dx]$

$= \frac{1}{4}[x^4 \ln x - \int x^3 dx]$

$= \frac{1}{4}(x^4 \ln x - \frac{1}{4}x^4) + C$

$= \frac{x^4}{4}(\ln x - \frac{1}{4}) + C$

$\because \ln x^3 = 3\ln x$ $\quad\because \ln(x^2) = 2\ln x$ $\quad\because \frac{d}{dy}\left(\frac{1}{y}\right) = -\frac{1}{y^2}$ $\therefore \frac{1}{y^2}dy = -d\left(\frac{1}{y}\right)$

9. Evaluate the integrals without using your calculator.

(a) $\int \ln x^3\, dx = 3\int \ln x\, dx$

$= 3\left[x\ln x - \int x\, d\ln x\right]$

$= 3\left[x\ln x - \int x\cdot\frac{1}{x}dx\right]$

$= 3\left[x\ln x - x\right] + C = 3x(\ln x - 1) + C$

(b) $\int \frac{\ln y}{y^2}\, dy = \int -\ln y\, d\left(\frac{1}{y}\right) = -\left[\frac{1}{y}\ln y - \int \frac{1}{y}\, d\ln y\right]$

$= -\left[\frac{1}{y}\ln y - \int \frac{1}{y}\cdot\frac{1}{y}dy\right]$

$= -\left[\frac{1}{y}\ln y - \int y^{-2}dy\right] = -\left[\frac{1}{y}\ln y - \frac{y^{-2+1}}{-2+1}\right] + C$

$= -\left[\frac{1}{y}\ln y + \frac{1}{y}\right] + C = -\frac{1}{y}\left[\ln y + 1\right] + C$

(c) $\int \frac{dx}{x^2 + 2x + 2} = \int \frac{dx}{(1+x)^2 + 1}$

let $u = 1+x$
$du = dx$

$= \int \frac{du}{1+u^2} = \tan^{-1}u + C$

$= \tan^{-1}(1+x) + C$

(d) $\int u\sec^2 u\, du$

$= \int u\, d\tan u = u\tan u - \int \tan u\, du$

$= u\tan u - \int \frac{\sin u}{\cos u}du = u\tan u + \int \frac{d\cos u}{\cos u}$

$= u\tan u + \ln|\cos u| + C;$

$\because -\sin u\, du = d\cos u$

10. Evaluate the integrals without using your calculator.

(a) $\int \frac{2x-1}{\sqrt{4x-4x^2}}\, dx$

let $u = 4x - 4x^2$
$du = (4-8x)dx$
$= -4(2x-1)dx$
$(2x-1)dx = -\frac{1}{4}du$

$= \int \frac{-\frac{1}{4}du}{\sqrt{u}}$

$= -\frac{1}{4}\left(\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right) + C$

$= \frac{1}{2}\sqrt{u} + C = \frac{1}{2}\sqrt{4x-4x^2} + C = \sqrt{x(1-x)} + C$

(b) $\int \frac{dx}{1-e^x}$

let $u = 1 - e^x \Rightarrow e^x = 1-u$
$du = -e^x dx$
$dx = -\frac{1}{e^x}du$

$= \int \frac{-\frac{1}{e^x}du}{u}$

$= -\int \frac{1}{u(1-u)}du = \int \frac{1}{u(u-1)}du$

$\frac{1}{u} - \frac{1}{u-1} = \frac{u-1-u}{(u-1)u} = -\frac{1}{(u-1)u}$

$= \int\left(\frac{1}{u-1} - \frac{1}{u}\right)du$

$= \ln|u-1| - \ln|u| + C$

$= \ln\left|\frac{u-1}{u}\right| + C$

$= \ln\left|\frac{-e^x}{1-e^x}\right| + C$

$= \ln\left|\frac{e^x}{e^x-1}\right| + C$

(c) $\int e^{2\ln u}\, du$

$= \int \cancel{e}^{\cancel{\ln}\, u^2} du = \int u^2 du$

$= \frac{1}{3}u^3 + C$

(d) $\int (\tan\theta - 1)^2\, d\theta$

$= \int(\tan^2\theta + 1 - 2\tan\theta)d\theta$

$= \int(\sec^2\theta - 2\tan\theta)d\theta$

$= \tan\theta - 2\ln|\sec\theta| + C$

11. A projectile is fired vertically upward from ground level with a velocity of 500 m/s. If air resistance is neglected, find its distance s(t) above ground at time t. What is its maximum height?

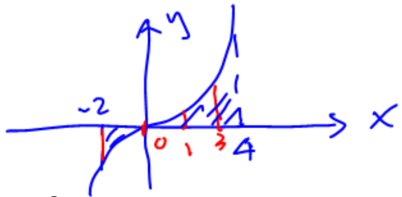
$a(t) = -9.8\ m/s^2.\quad v(t) = \int a(t)dt = \int -9.8\,dt = -9.8t + C.$

$\because v(0) = 500 \quad \therefore 500 = -9.8(0) + C \Rightarrow C = 500;\quad v(t) = -9.8t + 500;$

$h(t) = \int v(t)dt = \int(-9.8t + 500)dt = -4.9t^2 + 500t + C.\quad \because h(0) = 0,\quad \therefore C = 0.$

$h(t) = -4.9t^2 + 500t;\quad$ let $v(t) = 0 \Rightarrow -9.8t + 500 = 0.\quad t = \frac{500}{9.8} \doteq 51\ (s)$

$h(51) = -4.9(51)^2 + 500(51) \doteq 12744.90\ (m)$



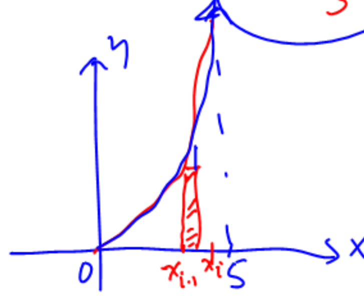
12. Suppose $f(x) = x^3$ and P is the partition of [-2, 4] into the four subintervals determined by $x_0 = -2$, $x_1 = 0$, $x_2 = 1$, $x_3 = 3$ and $x_4 = 4$. Find the Riemann sun R_p of $f(x)$ if w_i is the right-hand endpoint of the interval $[x_{i-1}, x_i]$.

$$R_p = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + f(x_3)\Delta x_3 + f(x_4)\Delta x_4$$
$$= (0)^3(0+2) + (1)^3(1-0) + 3^3(3-1) + 4^3(4-3)$$
$$= 0 + 1 + 27\times 2 + 64\times 1 = 119$$

$$\int_0^5 x^2\,dx = \frac{x^3}{3}\Big|_0^5 = \frac{125}{3}$$

13. Find the area under the graph of $f(x)$ from a to b using inscribed rectangles. In each case sketch the graph and typical rectangles, labeling the drawing.

(a) $f(x) = x^2; \quad a = 0, b = 5$



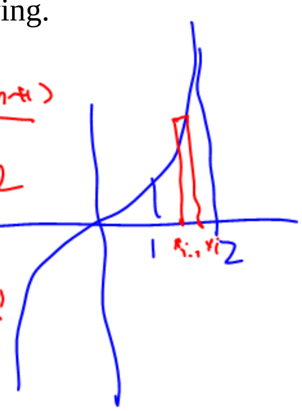
$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_{i-1})\Delta x_i = \lim_{n\to\infty}\sum_{i=1}^{\infty}(x_{i-1})^2\left(\frac{5}{n}\right)$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{5}{n}(i-1)\right)^2\frac{5}{n} = 125\lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}(i-1)^2$$
$$= 125\lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}(i^2-2i+1)$$
$$= 125\lim_{n\to\infty}\frac{1}{n^3}\left(\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + n\right)$$
$$= 125\lim_{n\to\infty}\frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} + n\right]$$
$$= 125\left(\frac{1}{3}+0+0\right) = \frac{125}{3}$$

$$\Delta x_i = x_i - x_{i-1} = \frac{5-0}{n} = \frac{5}{n}$$
$$x_i = 0 + \frac{i(5)}{n}$$
$$x_0 = 0 + \frac{0(5)}{n} = 0$$
$$x_n = 0 + \frac{n(5)}{n} = 5$$

14. Find the area under the graph of $f(x)$ from a to b using circumscribed rectangles. In each case sketch the graph and typical rectangles, labeling the drawing.

(a) $f(x) = x^3 + 1; \quad a = 1, \ b = 2$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$



$$A = \lim_{n\to\infty}\sum_{i=1}^{\infty} f(x_i)\Delta x_i$$
$$= \lim_{n\to\infty}\sum_{i=1}^{\infty}\left(\left(1+\frac{i}{n}\right)^3+1\right)\left(\frac{1}{n}\right)$$
$$= \ldots$$

$$x_i = 1 + \frac{i}{n}$$
$$\Delta x_i = \frac{2-1}{n} = \frac{1}{n}$$

$$\frac{19}{4}$$

$$\int_1^2 (x^3+1)\,dx = \left(\frac{x^4}{4}+x\right)\Big|_1^2 = \frac{2^4}{4}+2-\frac{1}{4}-1 =$$