

AP Calculus Homework Seven – Applications of Differential Calculus
 3.7 Motion along a Curve: Velocity and Acceleration; 3.8 Related Rates; 3.9 Slope of a Polar Curve

1. If the position of a particle moving along a straight line is given by $s = t^3 - 6t^2 + 12t + 8$.

- a) Find the values of t for which s is increasing.
- b) What is the minimum value of the speed?
- c) Find the values of t for which the acceleration is positive.
- d) Find the values of t for which the speed is decreasing.

a) $v(t) = s'(t) = 3t^2 - 12t + 12 = 3(t^2 - 4t + 4) = 3(t-2)^2 > 0$ for all t except $t=2$.

so $s(t)$ is increasing for all $t > 0$ except $t=2$.

b) The speed $= |v(t)| = |3(t-2)^2| = 3(t-2)^2$; when $t=2$, the minimum speed is $|v(2)| = 3(2-2)^2 = 0$.

c) $a(t) = v'(t) = [3(t-2)^2]' = 6(t-2) > 0$ when $t > 2$.

d) When $0 < t < 2$, $v(t) > 0$ and $a(t) < 0$, and the speed is decreasing.

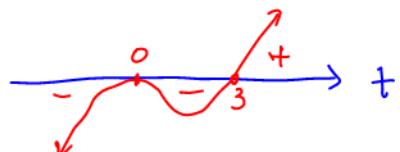
2. The displacement from the origin of a particle moving on a line is given by $s = t^4 - 4t^3$. Find the maximum displacement during the time interval $-2 \leq t \leq 4$.

$$v(t) = s'(t) = 4t^3 - 12t^2 = 4t^2(t-3);$$

$$s(3) = 3^4 - 4(3)^3 = 3^3(3-4) = -27, \text{ a local min.}$$

Two end points: $s(-2) = (-2)^4 - 4(-2)^3 = (-2)^3(-2-4) = (-8)(-6) = 48$

$$s(4) = 4^4 - 4(4)^3 = 0$$



Therefore, the maximum displacement is 48.

$$d) \frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\frac{2\pi}{3} \cos \frac{\pi}{3} t}{-\pi \sin \frac{\pi}{3} t} = -\frac{2}{3} \cot \frac{\pi}{3} t;$$

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$$\frac{dy}{dx} \Big|_{t=\frac{1}{2}} = -\frac{2}{3} \cot \frac{\pi}{6} = -\frac{2}{3}\sqrt{3}$$

3. If $\vec{R} = 3 \cos \frac{\pi}{3} t \vec{i} + 2 \sin \frac{\pi}{3} t \vec{j}$ is the (position) vector from the origin to a moving point $P(x, y)$ at time t .

- What is the equation in x and y for the path of the point?
- Find the speed of the point at $t = 3$.
- Find the magnitude of the acceleration at $t = 3$.
- At the point where $t = 0.5$, what is the slope of the curve along which the point moves?

a) $x(t) = 3 \cos(\frac{\pi}{3}t); y(t) = 2 \sin(\frac{\pi}{3}t) \Rightarrow x = \cos(\frac{\pi}{3}t), y = \sin(\frac{\pi}{3}t)$

$$\Rightarrow (\frac{x}{3})^2 + (\frac{y}{2})^2 = \cos^2(\frac{\pi}{3}t) + \sin^2(\frac{\pi}{3}t) = 1 \Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1, \text{ ellipse.}$$

b) $\vec{v}(t) = \vec{R}'(t) = [(3 \cos \frac{\pi}{3}t)', (2 \sin \frac{\pi}{3}t)'] = [-3 \sin \frac{\pi}{3}t (\frac{\pi}{3}t)', 2 \cos \frac{\pi}{3}t (\frac{\pi}{3}t)']$

$$= [-\pi \sin \frac{\pi}{3}t, \frac{2\pi}{3} \cos \frac{\pi}{3}t]; \text{ the speed at } t=3 \text{ is:}$$

$$|\vec{v}(3)| = \sqrt{(-\pi \sin \frac{\pi}{3}(3))^2 + (\frac{2\pi}{3} \cos \frac{\pi}{3}(3))^2} = \frac{2\pi}{3}$$

c) $\vec{a}(t) = \vec{v}'(t) = [-\frac{\pi^2}{3} \cos \frac{\pi}{3}t, -\frac{2\pi^2}{9} \sin \frac{\pi}{3}t]; |\vec{a}(3)| = \sqrt{(-\frac{\pi^2}{3} \cos \frac{\pi}{3}(3))^2 + (-\frac{2\pi^2}{9} \sin \frac{\pi}{3}(3))^2}$

4. A balloon is being filled with helium at the rate of $4 \text{ ft}^3/\text{min}$. Find the rate, in square feet per minute, at which the surface area is increasing when the volume is $32\pi/3 \text{ ft}^3$.

$$= \frac{\pi^2}{3}$$

Let V be the volume, S be the surface area, r be the radius of the balloon

Then, $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$; Given $\frac{dV}{dt} = 4 \text{ ft}^3/\text{min}$. Find $\frac{dS}{dt}$ when $V = \frac{32\pi}{3} \text{ ft}^3$

$\therefore r = \sqrt{\frac{V}{\frac{4}{3}\pi}}$, $\therefore V = \frac{4}{3}\pi \left(\frac{S}{4\pi}\right)^{3/2} = \frac{1}{3\sqrt{4\pi}} S^{3/2}$, $\frac{dV}{dt} = \frac{1}{3\sqrt{4\pi}} (\frac{3}{2}) S^{1/2} \frac{dS}{dt}$, sub. given values

$$\therefore \frac{1}{4\sqrt{\pi}} (\frac{3}{2}) S^{1/2} \frac{dS}{dt} \Rightarrow \frac{dS}{dt} = 4 \text{ ft}^2/\text{min.}$$

$$\frac{32\pi}{3} = \frac{1}{6\sqrt{\pi}} S^{3/2}$$

$$64\pi^{3/2} = S^{3/2}$$

$$\Rightarrow S = 16\pi$$

5. The height of a rectangular box is 10 cm. Its length increases at the rate of 2 cm/sec; its width decreases at the rate of 4 cm/sec. When the length is 8 cm and the width is 6 cm, what is the rate, in cubic cm per second, at which the volume of the box is changing?

Let V be the volume, L be the length, w be the width of the box.

Given $\frac{dL}{dt} = 2 \text{ cm/s}$; $\frac{dw}{dt} = -4 \text{ cm/s}$; Find $\frac{dV}{dt}$ at $L = 8 \text{ cm}$, $w = 6 \text{ cm}$;

$$\therefore V = 10Lw \quad \therefore \frac{dV}{dt} = 10 (w \cdot \frac{dL}{dt} + L \cdot \frac{dw}{dt}) = 10 (6 \cdot 2 + 8 \cdot (-4)) \\ = 10 (12 - 32) = 10(-20) = -200 \text{ cm}^3/\text{s}$$

So the volume is decreasing at a rate of $200 \text{ cm}^3/\text{s}$.

6. The table shows the velocity at time t of an object moving along a line. Estimate the acceleration (in cm/sec²) at $t = 6$ sec.

t (sec)	0	4	8	10
velocity	18	16	10	0

$$a(6) \approx \frac{v(8) - v(4)}{8 - 4} = \frac{10 - 16}{+4} = -\frac{6}{4} = -\frac{3}{2} \text{ (cm/s}^2\text{)}$$

7. Two cars are traveling along perpendicular roads, car A at 40 mph, car B at 60 mph. At noon, when car A reaches the intersection, car B is 90 miles away, and moving toward it. What is the rate, in miles per hour, at which the distance between the cars is changing at 1:00 P.M.?

Let x be the distance between Car A and the intersection,

y be the distance between Car B and the intersection,

z be the distance between Car A and Car B

$$\text{Then } x^2 + y^2 = z^2$$

$$\text{Given } \frac{dx}{dt} = 40 \text{ mph}, \quad \frac{dy}{dt} = -60 \text{ mph.}$$

$$\text{Find } \frac{dz}{dt} \text{ at } x = 40 \text{ miles, } y = 30 \text{ miles.}$$

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(z^2) \Rightarrow 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}, \text{ sub. all the given values,}$$

$$40(40) + (30)(-60) = 50 \frac{dz}{dt}$$

$$\text{where } z = \sqrt{x^2 + y^2} = \sqrt{40^2 + 30^2} = 50$$

$$\frac{dz}{dt} = \frac{160 - 180}{5} = -\frac{20}{5} = -4 \text{ (mph)}$$

So the distance between the cars is decreasing at the rate of 4 mph. at 1:00 pm.

8. Find the points at which the curve given by $r=1+\cos\theta$ has a vertical or horizontal tangent line for $0 < \theta \leq 2\pi$.

$$\therefore \begin{cases} x = r \cos\theta \\ y = r \sin\theta \end{cases} \quad \therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r(\cos\theta) + r'(\theta)\sin\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}$$

$$\text{where } r(\theta) = 1 + \cos\theta, \quad r'(\theta) = -\sin\theta$$

For a vertical tangent line, $r'(\theta)\cos\theta - r(\theta)\sin\theta = 0$

$$\Rightarrow -\sin\theta\cos\theta - (1 + \cos\theta)\sin\theta = 0$$

$$\Rightarrow -\sin\theta(\cos\theta + 1 + \cos\theta) = 0 \Rightarrow \sin\theta = 0 \text{ or } \cos\theta = -\frac{1}{2}$$

$$\therefore \theta = \pi \text{ or } 2\pi; \quad \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \quad \text{or} \quad \pi + \frac{\pi}{3} = \frac{4\pi}{3};$$

$$r(\theta) = 0 \quad \text{or} \quad \frac{1}{2}; \quad \frac{1}{2} \quad \text{or} \quad \frac{1}{2}$$

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \quad \text{or} \quad \begin{cases} x = -2 \\ y = 0 \end{cases} \quad \text{or} \quad \begin{cases} x = \frac{1}{2} \\ y = \frac{\sqrt{3}}{4} \end{cases} \quad \text{or} \quad \begin{cases} x = \frac{1}{2} \\ y = -\frac{\sqrt{3}}{4} \end{cases}$$

For the horizontal tangent line, $r'(\theta)\sin\theta + r(\theta)\cos\theta = 0$

$$\Rightarrow -\sin^2\theta + (1 + \cos\theta)\cos\theta = 0$$

$$\Rightarrow -(1 - \cos^2\theta) + \cos\theta + \cos^2\theta = 0 \Rightarrow 2\cos^2\theta + \cos\theta - 1 = 0$$

$$\Rightarrow \cos\theta = \frac{1}{2} \quad \text{or} \quad \cos\theta = -1$$

$$\Rightarrow \theta = \frac{\pi}{3} \quad \text{or} \quad 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}; \quad \text{or} \quad \theta = \pi. \quad \begin{matrix} 2\cos\theta & -1 \\ \cos\theta & +1 \end{matrix}$$

$$r = \frac{3}{2} \quad \text{or} \quad \frac{3}{2} \quad \text{or} \quad 0$$

Answers

$$(r, \theta) = \left(\frac{3}{2}, \frac{\pi}{3}\right), \left(\frac{3}{2}, \frac{5\pi}{3}\right), (0, \pi)$$

Q1: (a) all t except $t = 2$, (b) 0, (c) $t > 2$, (d) $t < 2$; Q2: 48;

Q3: (a) $4x^2 + 6y^2 = 36$, Ellipse; (b) $2\pi/3$, (c) $\pi^2/3$, (d) $-2\sqrt{3}/3$; Q4: 4 ft²/min;

Q5: $-200 \text{ cm}^3/\text{sec}$; Q6: -1.5 ; Q7: -4 mph ;

Q8: $(r, \theta) = (2, 0), (0, \pi), (2, 2\pi), (1/2, 4\pi/3), (1/2, 2\pi/3), (3/2, \pi/3), (3/2, 5\pi/3)$.