Lesson 10.

Definite Integral.

For a function $y=f(x)$ defined over interval $x \in [a, b]$. if

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(w_i)\Delta x_i \text{ exists.}$$

where $x_0 = a < x_1 < x_2 < \cdots < x_n = b$.

$\Delta x_i = x_i - x_{i-1}$, $w_i \in [x_{i-1}, x_i]$, for $i = 1, 2, 3, \cdots, n$.

Then $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(w_i)\Delta x_i$$

is called "definite integral of $f(x)$" over $x \in [a, b]$.

where $f(x)$ is the integrand.

x is the integration variable.

a and b are the lower and upper Limits of the integration.

Fundamental theorem of Calculus.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F'(x) = f(x)$.

Properties of Definite Integral.

1) $$\int_a^a f(x)\,dx = 0$$

2) $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

3) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

4) $$\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$$

5) $$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

6) $$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

7) If $f(x) \geq 0$, for $a \leq x \leq b$,

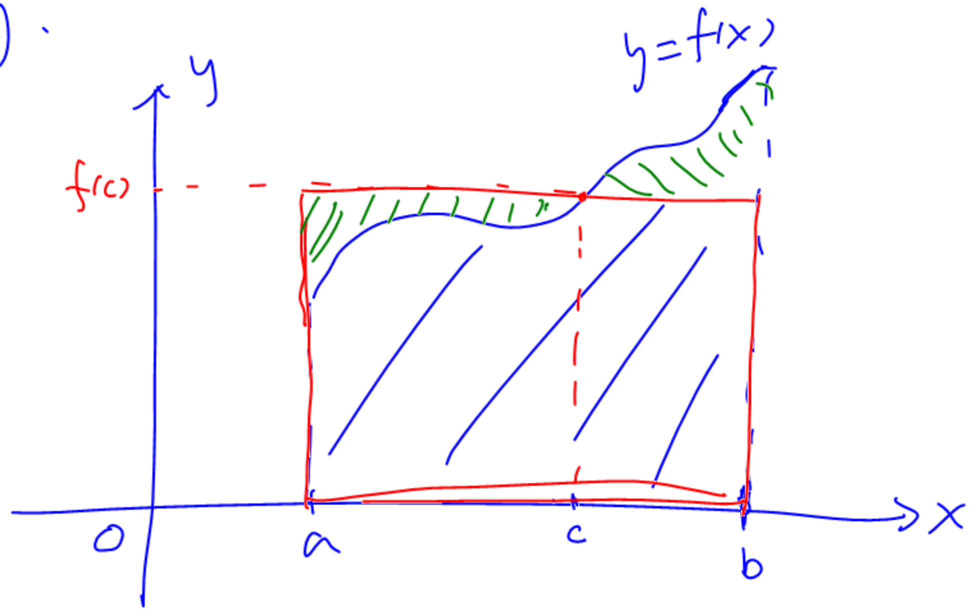
then $\int_a^b f(x)\,dx \geq 0$.

Mean Value Theorem for Definite Integral

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

where $a \leq c \leq b$.

Graphically.



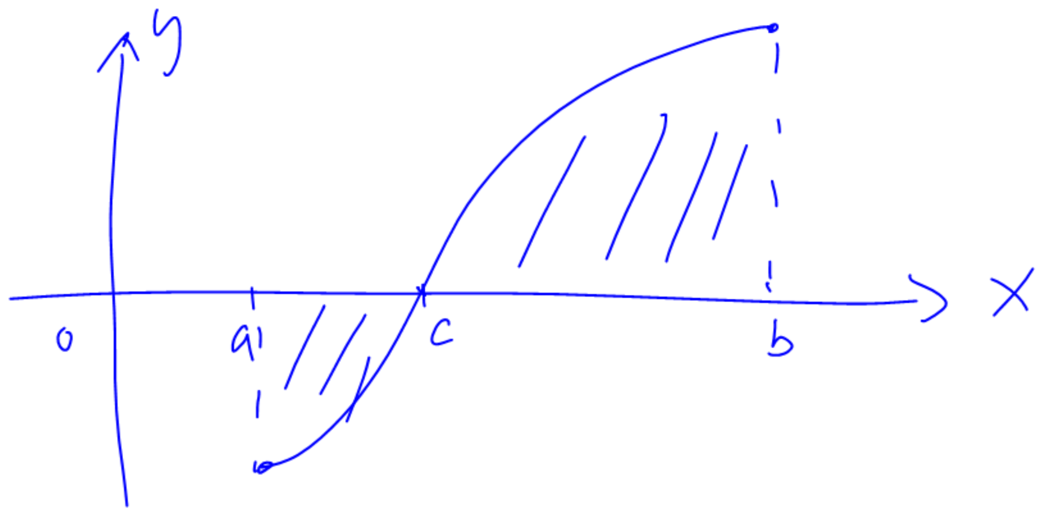
Evaluate $\int_a^b f(x)\,dx$

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

or $$\left[F(x) \right]_a^b = F(b) - F(a).$$

Area of the region bounded by $y = f(x)$,

$x = a$ and $x = b$.



$$\text{Area} = \int_a^b |f(x)|\,dx$$

$$= \int_a^c -f(x)\,dx + \int_c^b f(x)\,dx.$$

$$\int \sec^3\theta \, d\theta = \int \sec\theta \,(1+\tan^2\theta)\, d\theta$$

$$= \int \sec\theta \, d\theta + \int \sec\theta \tan^2\theta \, d\theta$$

$$= \int \sec\theta \, d\theta + \int \tan\theta \, d\sec\theta$$

$$= \int \sec\theta \, d\theta + \left(\tan\theta \sec\theta - \int \sec\theta \, d\tan\theta\right)$$

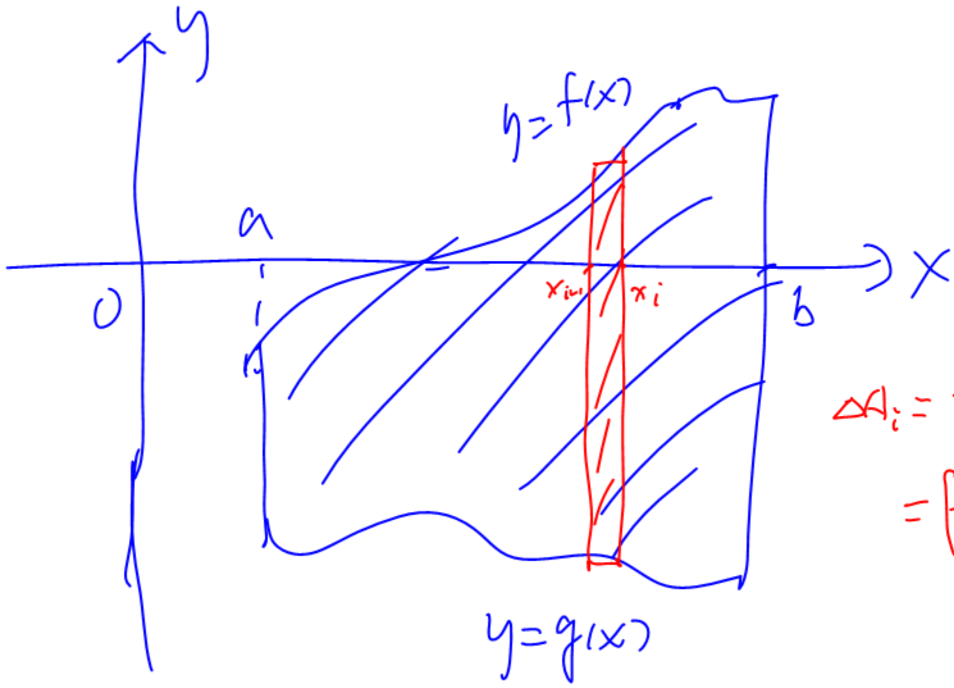
$$= \int \sec\theta \, d\theta + \tan\theta \sec\theta - \int \sec\theta \sec^2\theta \, d\theta$$

$$\therefore 2\int \sec^3\theta \, d\theta = \int \sec\theta \, d\theta + \tan\theta \sec\theta$$

$$\int \sec^3\theta \, d\theta = \frac{1}{2}\left[\int \sec\theta \, d\theta + \tan\theta \sec\theta\right]$$

Area of region bounded by

$y = f(x)$, $y = g(x)$, $x = a$, $x = b$.

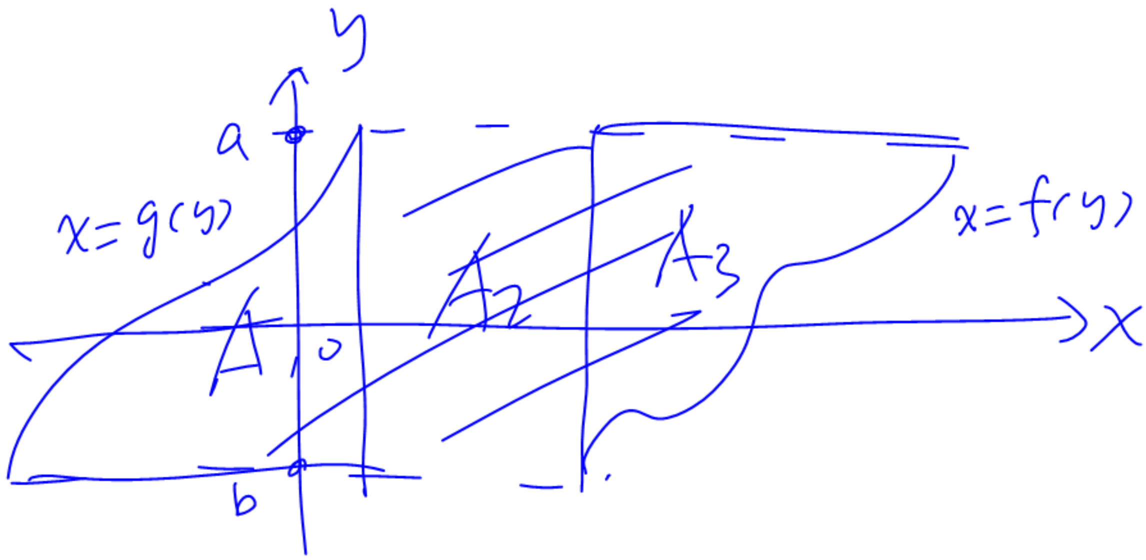


$\Delta A_i = h_i \Delta x_i$

$= [f(x_i) - g(x_i)] \Delta x_i$

> 0

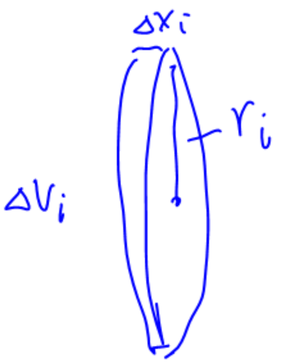
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta A_i = \int_a^b [f(x) - g(x)]\,dx$$

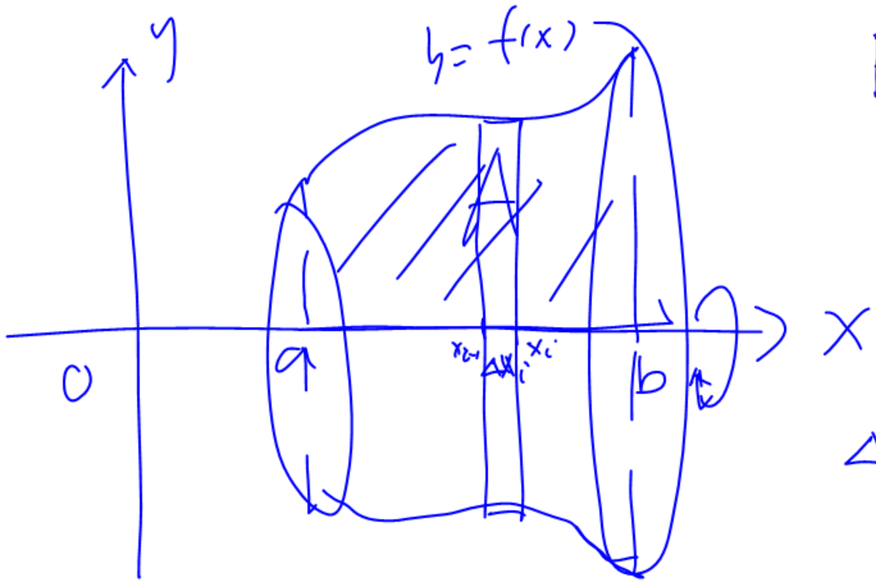


$$A = \int_a^b [f(y) - g(y)]\,dy$$

Volume of Rotating Solid.

By disks





$\Delta V_i = \pi r_i^2 \Delta x_i$

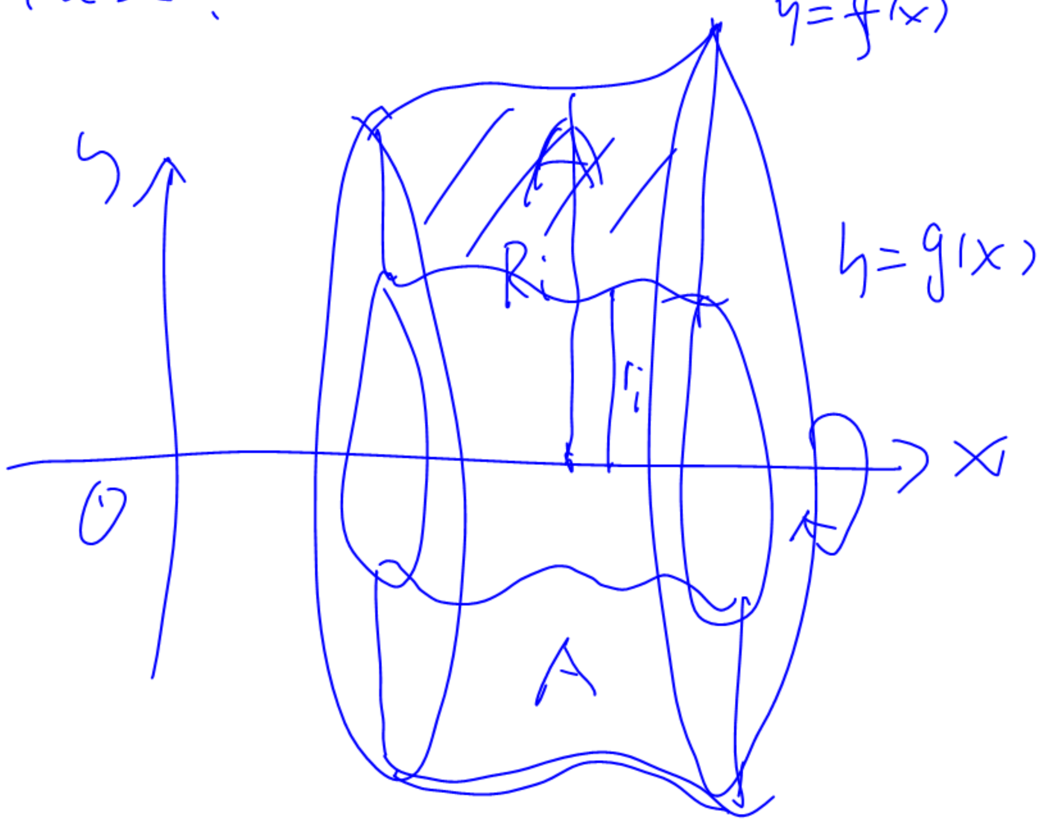
$r_i = f(x_i)$

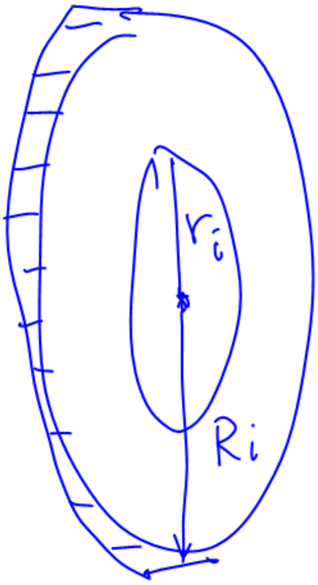
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta V_i = \lim_{n\to\infty} \sum_{i=1}^{n} \pi r_i^2 \Delta x_i$$

$$V = \int_a^b \pi [f(x)]^2 dx$$

General case.

Washers





$$\Delta V_i = \pi\left(R_i^2 - r_i^2\right)\Delta x_i$$

$$V = \int_a^b \pi\left[[f(x)]^2 - [g(x)]^2\right]dx$$

rotating with respect to y axis

$$V = \int_a^b \pi(f(y))^2\,dy$$

or $V = \int_a^b \pi\left[(f(y))^2 - (g(x))^2\right]dy$