3.6 Probabilities using Matrices.

If a sequence of random events $S_0, S_1, S_2, \cdots, S_n, \cdots$ are dependent so that $S_n = S_{n-1} P$. where P is the probability transition matrix. Then $\{S_n\}$ is a Markov Chain.

S_0 is called "the initial probability vector".

S_n is called "the n^{th}-step probability vector".

and if $S = SP$. then

S is called "long-term" or "steady-state" probability vector.

For example. P345 Ex. 1.

a) $$S_0 = \begin{array}{cc} VV & MM \\ \end{array}$$
$$S_0 = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}_{1\times 2}$$

b) $$P = \begin{array}{c} \begin{array}{cc} VV & MM \end{array} \\ \begin{bmatrix} 0.6 & 0.4 \\ 0.7 & 0.3 \end{bmatrix}_{2\times 2} \end{array} \begin{array}{c} VV \\ MM \end{array}$$, where the sum of each row is 1.

c) $$\underset{1\times 2}{S_1} = S_0 P = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}_{1\times 2} \begin{bmatrix} 0.6 & 0.4 \\ 0.7 & 0.3 \end{bmatrix}_{2\times 2}$$

$$S_1 = S_0 P = [0.5 \quad 0.5]\begin{bmatrix} 0.6 & 0.4 \\ 0.7 & 0.3 \end{bmatrix}$$

$$= [0.5\times0.6 + 0.5\times0.7 \quad 0.5\times0.4+0.5\times0.3]$$

$$= [0.65 \quad 0.35]$$

d) $$S_2 = S_1 P = [0.65 \quad 0.35]\begin{bmatrix} 0.6 & 0.4 \\ 0.7 & 0.3 \end{bmatrix}$$

$$= [0.65\times0.6 + 0.35\times0.7 \quad 0.65\times0.4+0.35\times0.3]$$

$$= [\underline{\underline{0.635}} \quad 0.365]$$

e) Assume that the transition matrix P remains unchanged,

By the way, let's find the steady-state vector S:

Let $S = [p \quad q]$, where $p+q=1$ ---- ①

and $S = SP$.

$$\therefore [p \quad q] = [p \quad q]\begin{bmatrix} 0.6 & 0.4 \\ 0.7 & 0.3 \end{bmatrix}$$

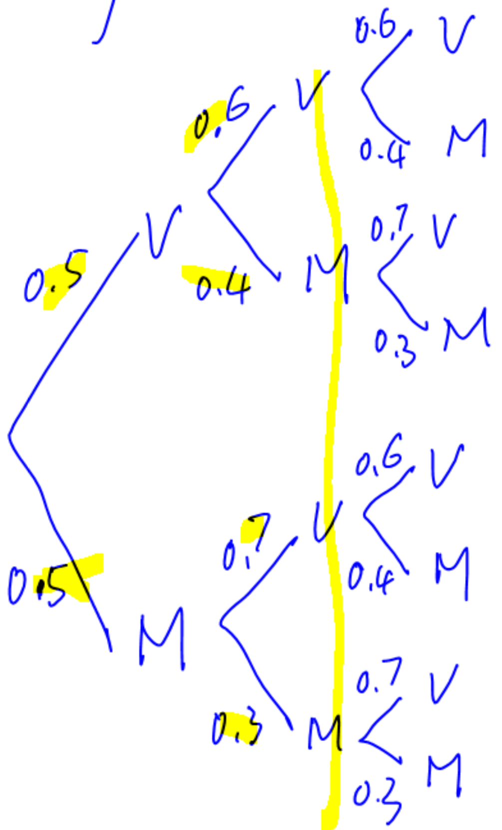
$$[p \quad q] = [0.6p+0.7q \quad 0.4p+0.3q]$$

$$\Rightarrow \begin{cases} p = 0.6p+0.7q \\ q = 0.4p+0.3q \end{cases} \Rightarrow \begin{cases} 0.4p = 0.7q \\ 0.7q = 0.4p \end{cases} \Rightarrow 4p = 7q \quad \text{---} ②$$

By ②, $p = \frac{7}{4}q$. sub into ①: $\frac{7}{4}q + q = 1 \Rightarrow \frac{11}{4}q = 1 \Rightarrow q = \frac{4}{11}$. $\therefore p = \frac{7}{11}$.

$$S = [\tfrac{7}{11}, \tfrac{4}{11}]$$

We could solve this kind of questions using tree diagrams, instead of matrices.



— $0.5 \times 0.6 \times 0.6 = 0.18$
— $0.5 \times 0.6 \times 0.4 = 0.12$
— $0.5 \times 0.4 \times 0.7 = 0.14$
— $0.5 \times 0.4 \times 0.3 = 0.06$
— $0.5 \times 0.7 \times 0.6 = 0.21$
— $0.5 \times 0.7 \times 0.4 = 0.14$
— $0.5 \times 0.3 \times 0.7 = 0.105$
— $0.5 \times 0.3 \times 0.3 = 0.045$

0.32

$0.32 + 0.315 = \underline{\underline{0.635}}$

0.315

Chapter 4 Probability Distributions

4.1 Uniform Distributions

In this chapter, we are going to discuss four discrete probability distributions. (p.d.s)

A p.d. consists of a random variable x, all the values of X and all the probabilities of the x-values, that could be organized into a table as follows:

X	x_1	x_2	$\cdots$	x_n	$\cdots$
$p(X)$	$p(x_1)$	$p(x_2)$	$\cdots$	$p(x_n)$	$\cdots$

where $p(x_1)+p(x_2)+\cdots+p(x_n)+\cdots = 1$

and the expectation of X or expected value of X is defined as $\mu = E(X) = x_1 p(x_1) + x_2 p(x_2) + \cdots + x_n p(x_n) + \cdots$,

that is a kind of "weighted average of X".

Also the variance of X is denoted as $Var(X)$,

$$\sigma^2 = Var(X) = E(X-E(X))^2 = E(X^2) - (E(X))^2$$

where $E(X^2) = x_1^2 p(x_1) + x_2^2 p(x_2) + \cdots + x_n^2 p(x_n) + \cdots$.

The standard deviation of X is denoted as

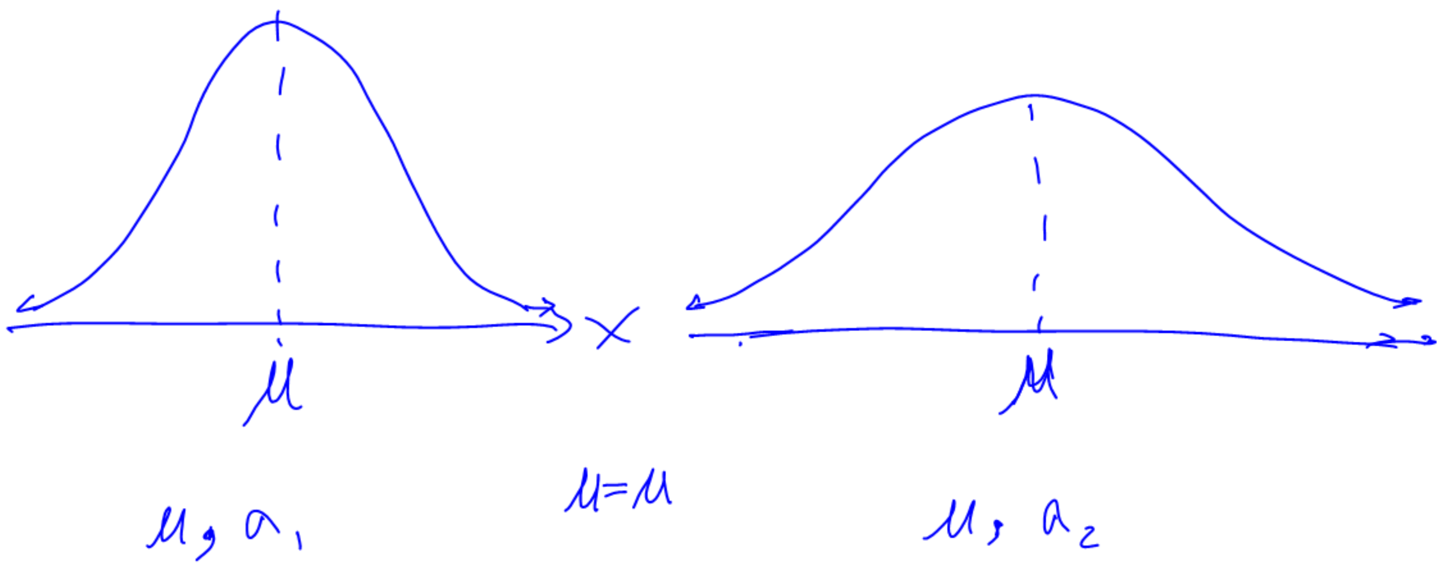
$$\sqrt{Var(X)} = \sigma$$

$\mu = E(X)$; $\quad \sigma^2 = Var(X)$; $\quad \sigma = \sqrt{Var(X)}$;

$\mu = E(X)$ is a measure of the centre of $\{x_i\}$;

σ^2 and σ is measures of the spread of $\{x_i\}$ from its centre μ.

If X follows a normal distribution.



$\mu=\mu$

$\sigma_2 > \sigma_1$

If a random variable X takes discrete values, such as $X = 1, 2, 3, 4, \cdots$, or $X = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots$ and so on, then X is a discrete random variable, the p.d. of X is a discrete p.d..

If X takes any value from an interval, such as $x \in [0, 10]$, or $x \in (-2.5, 6)$, or $x \in (-\infty, \infty)$, ... and so on. then X is a continuous random variable, and the p.d. of X is a continuous p.d.

① Uniform Distribution (discrete)

X	x_1	x_2	...	x_n
$p(x)$	$\frac{1}{n}$	$\frac{1}{n}$	...	$\frac{1}{n}$

$$E(x) = x_1\left(\tfrac{1}{n}\right) + x_2\left(\tfrac{1}{n}\right) + \cdots + x_n\left(\tfrac{1}{n}\right)$$

$$= \frac{x_1 + x_2 + \cdots + x_n}{n}$$

$$Var(x) = E(x - E(x))^2 = E(x^2) - (E(x))^2$$

$$= \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} - \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^2$$

For example, rolling a 6-side die once, let x be the point rolled. then the p.d. of X is given as:

3.5

x	1	2	3	4	5	6
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(x) = \frac{1+2+3+4+5+6}{6} = \frac{7}{2} = 3.5$$

Another example. P372 Example 3.