

# Permutations and Organized Counting

Specific Expectations	Section
Represent complex tasks or issues, using diagrams.	4.1
Solve introductory counting problems involving the additive and multiplicative counting principles.	4.1, 4.2, 4.3
Express the answers to permutation and combination problems, using standard combinatorial symbols.	4.2, 4.3
Evaluate expressions involving factorial notation, using appropriate methods.	4.2, 4.3
Solve problems, using techniques for counting permutations where some objects may be alike.	4.3
Identify patterns in Pascal's triangle and relate the terms of Pascal's triangle to values of $\binom{n}{r}$ , to the expansion of a binomial, and to the solution of related problems.	4.4, 4.5
Communicate clearly, coherently, and precisely the solutions to counting problems.	4.1, 4.2, 4.3, 4.4, 4.5



## Chapter Problem

### Students' Council Elections

Most high schools in Ontario have a students' council comprised of students from each grade. These students are elected representatives, and a part of their function is to act as a liaison between the staff and the students. Often, these students are instrumental in fundraising and in coordinating events, such as school dances and sports.

A students' council executive could consist of a president, vice-president, secretary, treasurer, social convener, fundraising chair, and four grade representatives. Suppose ten students have been nominated to fill these positions. Five of the nominees are from grade 12, three are from grade 11, and the other two are a grade 9 and a grade 10 student.

1. In how many ways could the positions of president and vice-president be filled by these ten students if all ten are eligible for these positions? How many ways are there if only the grades 11 and 12 students are eligible?
2. The grade representatives must represent their current grade level. In how many ways could the grade representative positions be filled?

You could answer both of these questions by systematically listing all the possibilities and then counting them. In this chapter, you will learn easier and more powerful techniques that can also be applied to much more complex situations.

# Review of Prerequisite Skills

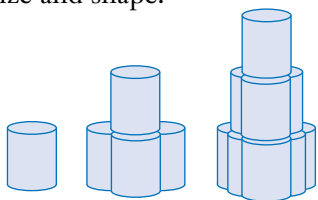
If you need help with any of the skills listed in **purple** below, refer to Appendix A.

1. **Tree diagrams** Draw a tree diagram to illustrate the number of ways a quarter, a dime, and a nickel can come up heads or tails if you toss one after the other.

2. **Tree diagrams**

- a) Draw a tree diagram to illustrate the possible outcomes of tossing a coin and rolling a six-sided die.  
b) How many possible outcomes are there?

3. **Number patterns** The manager of a grocery store asks a stock clerk to arrange a display of canned vegetables in a triangular pyramid like the one shown. Assume all cans are the same size and shape.



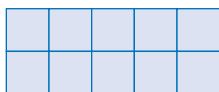
- a) How many cans is the tallest complete pyramid that the clerk can make with 100 cans of vegetables?  
b) How many cans make up the base level of the pyramid in part a)?  
c) How many cans are in the full pyramid in part a)?  
d) What is the sequence of the numbers of cans in the levels of the pyramid?

4. **Number patterns** What is the greatest possible number of rectangles that can be drawn on a

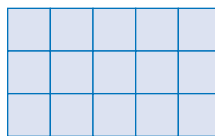
a) 1 by 5 grid?



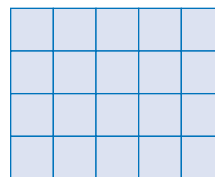
b) 2 by 5 grid?



c) 3 by 5 grid?



d) 4 by 5 grid?



5. **Evaluating expressions** Evaluate each expression given  $x = 5$ ,  $y = 4$ , and  $z = 3$ .

a)  $\frac{8y(x+2)(y+2)(z+2)}{(x-3)(y+3)(z+2)}$

b)  $\frac{(x-2)^3(y+2)^2(z+1)^2}{y(x+1)(y-1)^2}$

c)  $\frac{(x+4)(y-2)(z+3)}{(y-1)(x-3)z} + \frac{(x-1)^2(z+1)y}{(x-3)^4(y+4)}$

6. **Order of operations** Evaluate.

a)  $5(4) + (-1)^3(3)^2$

b)  $\frac{(10-2)^2(10-3)^2}{(10-2)^2 - (10-3)^2}$

c)  $\frac{6(6-1)(6-2)(6-3)(6-4)(6-5)}{3(3-1)(3-2)}$

d)  $\frac{50(50-1)(50-2)\dots(50-49)}{48(48-1)(48-2)\dots(48-47)}$

e)  $\frac{12 \times 11 \times 10 \times 9}{6^2} + \frac{10 \times 9 \times 8 \times 7}{2^4} - \frac{8 \times 7 \times 6 \times 5}{42}$

7. **Simplifying expressions** Simplify.

a)  $\frac{x^2 - xy + 2x}{2x}$

b)  $\frac{(4x+8)^2}{16}$

c)  $\frac{14(3x^2+6)}{7 \times 6}$

d)  $\frac{x(x-1)(x-2)(x-3)}{x^2-2x}$

e)  $\frac{2y+1}{x} + \frac{16y+4}{4x}$

## Organized Counting

The techniques and mathematical logic for counting possible arrangements or outcomes are useful for a wide variety of applications. A computer programmer writing software for a game or industrial process would use such techniques, as would a coach planning a starting line-up, a conference manager arranging a schedule of seminars, or a school board trying to make the most efficient use of its buses.

**Combinatorics** is the branch of mathematics dealing with ideas and methods for counting, especially in complex situations. These techniques are also valuable for probability calculations, as you will learn in Chapter 6.

### INVESTIGATE & INQUIRE: Licence Plates

Until 1997, most licence plates for passenger cars in Ontario had three numbers followed by three letters. Suppose the provincial government had wanted all the vehicles registered in Ontario to have plates with the letters O, N, and T.

1. Draw a diagram to illustrate all the possibilities for arranging these three letters assuming that the letters can be repeated. How many possibilities are there?
2. How could you calculate the number of possible three-letter groups without listing them all?
3. Predict how many three-letter groups the letters O, N, T, and G can form.
4. How many three-letter groups do you think there would be if you had a choice of five letters?
5. Suggest a general strategy for counting all the different possibilities in situations like those above.



When you have to make a series of choices, you can usually determine the total number of possibilities without actually counting each one individually.

**Example 1 Travel Itineraries**

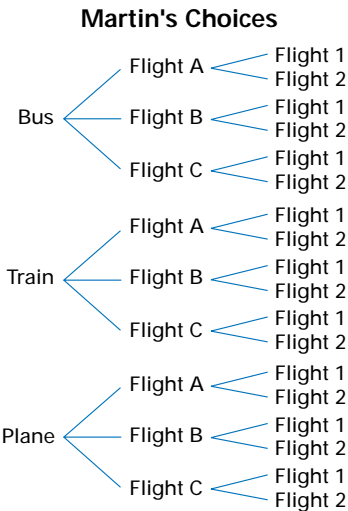
Martin lives in Kingston and is planning a trip to Vienna, Austria. He checks a web site offering inexpensive airfares and finds that if he travels through London, England, the fare is much lower. There are three flights available from Toronto to London and two flights from London to Vienna. If Martin can take a bus, plane, or train from Kingston to Toronto, how many ways can he travel from Kingston to Vienna?

**Solution**

You can use a tree diagram to illustrate and count Martin’s choices. This diagram suggests another way to determine the number of options Martin has for his trip.

Choices for the first portion of trip: 3  
Choices for the second portion of trip: 3  
Choices for the third portion of trip: 2  
Total number of choices:  $3 \times 3 \times 2 = 18$

In all, Martin has 18 ways to travel from Kingston to Vienna.



**Example 2 Stereo Systems**

Javon is looking at stereos in an electronics store. The store has five types of receivers, four types of CD players, and five types of speakers. How many different choices of stereo systems does this store offer?

**Solution**

For each choice of receiver, Javon could choose any one of the CD players. Thus, there are  $5 \times 4 = 20$  possible combinations of receivers and CD players. For each of these combinations, Javon could then choose one of the five kinds of speakers.

The store offers a total of  $5 \times 4 \times 5 = 100$  different stereo systems.

These types of counting problems illustrate the **fundamental** or **multiplicative counting principle**:

If a task or process is made up of stages with separate choices, the total number of choices is  $m \times n \times p \times \dots$ , where  $m$  is the number of choices for the first stage,  $n$  is the number of choices for the second stage,  $p$  is the number of choices for the third stage, and so on.

### • **Example 3 Applying the Fundamental Counting Principle**

A school band often performs at benefits and other functions outside the school, so its members are looking into buying band uniforms. The band committee is considering four different white shirts, dress pants in grey, navy, or black, and black or grey vests with the school crest. How many different designs for the band uniform is the committee considering?

#### **Solution**

First stage: choices for the white shirts,  $m = 4$

Second stage: choices for the dress pants,  $n = 3$

Third stage: choices for the vests,  $p = 2$

The total number of possibilities is

$$\begin{aligned} m \times n \times p &= 4 \times 3 \times 2 \\ &= 24 \end{aligned}$$

The band committee is considering 24 different possible uniforms.

#### **Project Prep**

You can use the fundamental or multiplicative counting principle to help design the game for your probability project.

In some situations, an **indirect method** makes a calculation easier.

### • **Example 4 Indirect Method**

Leora, a triathlete, has four pairs of running shoes loose in her gym bag.

In how many ways can she pull out two unmatched shoes one after the other?

#### **Solution**

You can find the number of ways of picking unmatched shoes by subtracting the number of ways of picking matching ones from the total number of ways of picking any two shoes.

There are eight possibilities when Leora pulls out the first shoe, but only seven when she pulls out the second shoe. By the fundamental counting principle, the number of ways Leora can pick any two shoes out of the bag is  $8 \times 7 = 56$ . She could pick each of the matched pairs in two ways: left shoe then right shoe or right shoe then left shoe. Thus, there are  $4 \times 2 = 8$  ways of picking a matched pair.

Leora can pull out two unmatched shoes in  $56 - 8 = 48$  ways.

Sometimes you will have to count several subsets of possibilities separately.

### Example 5 Signal Flags

Sailing ships used to send messages with signal flags flown from their masts. How many different signals are possible with a set of four distinct flags if a minimum of two flags is used for each signal?

#### Solution

A ship could fly two, three, or four signal flags.

Signals with two flags:  $4 \times 3 = 12$

Signals with three flags:  $4 \times 3 \times 2 = 24$

Signals with four flags:  $4 \times 3 \times 2 \times 1 = 24$

Total number of signals:  $12 + 24 + 24 = 60$

Thus, the total number of signals possible with these flags is 60.

In Example 5, you were counting actions that could not occur at the same time. When counting such **mutually exclusive** actions, you can apply the **additive counting principle** or **rule of sum**:

If one mutually exclusive action can occur in  $m$  ways, a second in  $n$  ways, a third in  $p$  ways, and so on, then there are  $m + n + p \dots$  ways in which one of these actions can occur.

### Key Concepts

- Tree diagrams are a useful tool for organized counting.
- If you can choose from  $m$  items of one type and  $n$  items of another, there are  $m \times n$  ways to choose one item of each type (fundamental or multiplicative counting principle).
- If you can choose from either  $m$  items of one type or  $n$  items of another type, then the total number of ways you can choose an item is  $m + n$  (additive counting principle).
- Both the multiplicative and the additive counting principles also apply to choices of three or more types of items.
- Sometimes an indirect method provides an easier way to solve a problem.

## Communicate Your Understanding

1. Explain the fundamental counting principle in your own words and give an example of how you could apply it.
2. Are there situations where the fundamental counting principle does not apply? If so, give one example.
3. Can you always use a tree diagram for organized counting? Explain your reasoning.

## Practise

### A

1. Construct a tree diagram to illustrate the possible contents of a sandwich made from white or brown bread, ham, chicken, or beef, and mustard or mayonnaise. How many different sandwiches are possible?
2. In how many ways can you roll either a sum of 4 or a sum of 11 with a pair of dice?
3. In how many ways can you draw a 6 or a face card from a deck of 52 playing cards?
4. How many ways are there to draw a 10 or a queen from the 24 cards in a euchre deck, which has four 10s and four queens?
5. Use tree diagrams to answer the following:
  - a) How many different soccer uniforms are possible if there is a choice of two types of shirts, three types of shorts, and two types of socks?
  - b) How many different three-scoop cones can be made from vanilla, chocolate, and strawberry ice cream?
  - c) Suppose that a college program has six elective courses, three on English literature and three on the other arts. If the college requires students to take one of the English courses and one of the other arts courses, how many pairs of courses will satisfy these requirements?

## Apply, Solve, Communicate

### B

6. Ten different books and four different pens are sitting on a table. One of each is selected. Should you use the rule of sum or the product rule to count the number of possible selections? Explain your reasoning.
7. **Application** A grade 9 student may build a timetable by selecting one course for each period, with no duplication of courses. Period 1 must be science, geography, or physical education. Period 2 must be art, music, French, or business. Periods 3 and 4 must each be mathematics or English.
  - a) Construct a tree diagram to illustrate the choices for a student's timetable.
  - b) How many different timetables could a student choose?
8. A standard die is rolled five times. How many different outcomes are possible?
9. A car manufacturer offers three kinds of upholstery material in five different colours for this year's model. How many upholstery options would a buyer have? Explain your reasoning.
10. **Communication** In how many ways can a student answer a true-false test that has six questions. Explain your reasoning.



11. The final score of a soccer game is 6 to 3. How many different scores were possible at half-time?
12. A large room has a bank of five windows. Each window is either open or closed. How many different arrangements of open and closed windows are there?
13. **Application** A Canadian postal code uses six characters. The first, third, and fifth are letters, while the second, fourth, and sixth are digits. A U.S.A. zip code contains five characters, all digits.
- How many codes are possible for each country?
  - How many more possible codes does the one country have than the other?
14. When three-digit area codes were introduced in 1947, the first digit had to be a number from 2 to 9 and the middle digit had to be either 1 or 0. How many area codes were possible under this system?
15. Asha builds new homes and offers her customers a choice of brick, aluminium siding, or wood for the exterior, cedar or asphalt shingles for the roof, and radiators or forced-air for the heating system. How many different configurations is Asha offering?
16.
  - In how many ways could you choose two fives, one after the other, from a deck of cards?
  - In how many ways could you choose a red five and a spade, one after the other?
  - In how many ways could you choose a red five or a spade?
  - In how many ways could you choose a red five or a heart?
  - Explain which counting principles you could apply in parts a) to d).
17. Ten students have been nominated for a students' council executive. Five of the nominees are from grade 12, three are from grade 11, and the other two are from grades 9 and 10.
- In how many ways could the nominees fill the positions of president and vice-president if all ten are eligible for these senior positions?
  - How many ways are there to fill these positions if only grade 11 and grade 12 students are eligible?
18. **Communication**
- How many different licence plates could be made using three numbers followed by three letters?
  - In 1997, Ontario began issuing licence plates with four letters followed by three numbers. How many different plates are possible with this new system?
  - Research the licence plate formats used in the other provinces. Compare and contrast these formats briefly and suggest reasons for any differences between the formats.
19. In how many ways can you arrange the letters of the word *think* so that the *t* and the *h* are separated by at least one other letter?
20. **Application** Before the invention of the telephone, Samuel Morse (1791–1872) developed an efficient system for sending messages as a series of dots and dashes (short or long pulses). International code, a modified version of Morse code, is still widely used.
- How many different characters can the international code represent with one to four pulses?
  - How many pulses would be necessary to represent the 72 letters of the Cambodian alphabet using a system like Morse code?



## ACHIEVEMENT CHECK

Knowledge/  
Understanding

Thinking/Inquiry/  
Problem Solving

Communication

Application

- 21.** Ten finalists are competing in a race at the Canada Games.
- a)** In how many different orders can the competitors finish the race?
  - b)** How many ways could the gold, silver, and bronze medals be awarded?
  - c)** One of the finalists is a friend from your home town. How many of the possible finishes would include your friend winning a medal?
  - d)** How many possible finishes would leave your friend out of the medal standings?
  - e)** Suppose one of the competitors is injured and cannot finish the race. How does that affect your previous answers?
  - f)** How would the competitor's injury affect your friend's chances of winning a medal? Explain your reasoning. What assumptions have you made?



- 22.** A locksmith has ten types of blanks for keys. Each blank has five different cutting positions and three different cutting depths at each position, except the first position, which only has two depths. How many different keys are possible with these blanks?
- 23. Communication** How many 5-digit numbers are there that include the digit 5 and exclude the digit 8? Explain your solution.

- 24. Inquiry/Problem Solving** Your school is purchasing a new type of combination lock for the student lockers. These locks have 40 positions on their dials and use a three-number combination.
- a)** How many combinations are possible if consecutive numbers cannot be the same?
  - b)** Are there any assumptions that you have made? Explain.
  - c)** Assuming that the first number must be dialled clockwise from 0, how many different combinations are possible?
  - d)** Suppose the first number can also be dialled counterclockwise from 0. Explain the effect this change has on the number of possible combinations.
  - e)** If you need four numbers to open the lock, how many different combinations are possible?
- 25. Inquiry/Problem Solving** In chess, a knight can move either two squares horizontally plus one vertically or two squares vertically plus one horizontally.
- a)** If a knight starts from one corner of a standard  $8 \times 8$  chessboard, how many different squares could it reach after
    - i)** one move?
    - ii)** two moves?
    - iii)** three moves?
  - b)** Could you use the fundamental counting principle to calculate the answers for part a)? Why or why not?

## Factorials and Permutations

In many situations, you need to determine the number of different orders in which you can choose or place a set of items.

### INVESTIGATE & INQUIRE: Numbers of Arrangements

Consider how many different ways a president and a vice-president could be chosen from eight members of a students' council.

1. a) Have one person in your class make two signs, writing *President* on one and *Vice-President* on the other. Now, choose two people to stand at the front of the class. Using the signs to indicate which person holds each position, decide in how many ways you can choose a president and a vice-president from the two people at the front of the class.
- b) Choose three students to be at the front of the class. Again using the signs to indicate who holds each position, determine how many ways you can choose a president and a vice-president from the three people at the front of the class.
- c) Repeat the process with four students. Do you see a pattern in the number of ways a president and a vice-president can be chosen from the different sizes of groups? If so, what is the pattern? If not, continue the process with five students and then with six students.
- d) When you see a pattern, predict how many ways a president and a vice-president can be chosen from the eight members of the students' council.
- e) Suggest other ways of simulating the selection of a president and a vice-president for the students' council.



2. Suppose that each of the eight members of the students' council has to give a brief speech at an assembly. Consider how you could determine the number of different orders in which they could speak.
- a) Choose two students from your class and list all the possible orders in which they could speak.
  - b) Choose three students and list all the possible orders in which they could speak.
  - c) Repeat this process with four students.
  - d) Is there an easy method to organize the list so that you could include all the possibilities?
  - e) Is this method related to your results in question 1? Explain.
  - f) Can you use your method to predict the number of different orders in which eight students could give speeches?

Many counting and probability calculations involve the product of a series of consecutive integers. You can use **factorial** notation to write such expressions more easily. For any natural number  $n$ ,

$$n! = n \times (n - 1) \times (n - 2) \times (n - 3) \times \dots \times 3 \times 2 \times 1$$

This expression is read as  $n$  *factorial*.

### Example 1 Evaluating Factorials

Calculate each factorial.

- a)  $2!$                       b)  $4!$                       c)  $8!$

#### Solution

a)  $2! = 2 \times 1$   
 $= 2$

b)  $4! = 4 \times 3 \times 2 \times 1$   
 $= 24$

c)  $8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$   
 $= 40\,320$

As you can see from Example 1,  $n!$  increases dramatically as  $n$  becomes larger. However, calculators and computer software provide an easy means of calculating the larger factorials. Most scientific and graphing calculators have a factorial key or function.

## Example 2 Using Technology to Evaluate Factorials

Calculate.

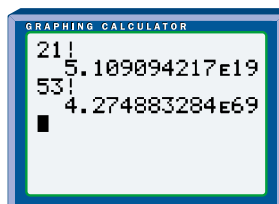
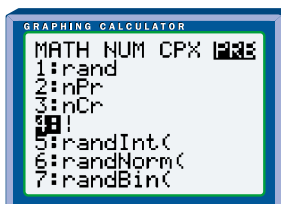
- a)  $21!$       b)  $53!$       c)  $70!$

### Solution 1 Using a Graphing Calculator

Enter the number on the home screen and then use the **!** function on the MATH PRB menu to calculate the factorial.

a)  $21! = 21 \times 20 \times 19 \times 18 \times \dots \times 2 \times 1$   
 $= 5.1091 \times 10^{19}$

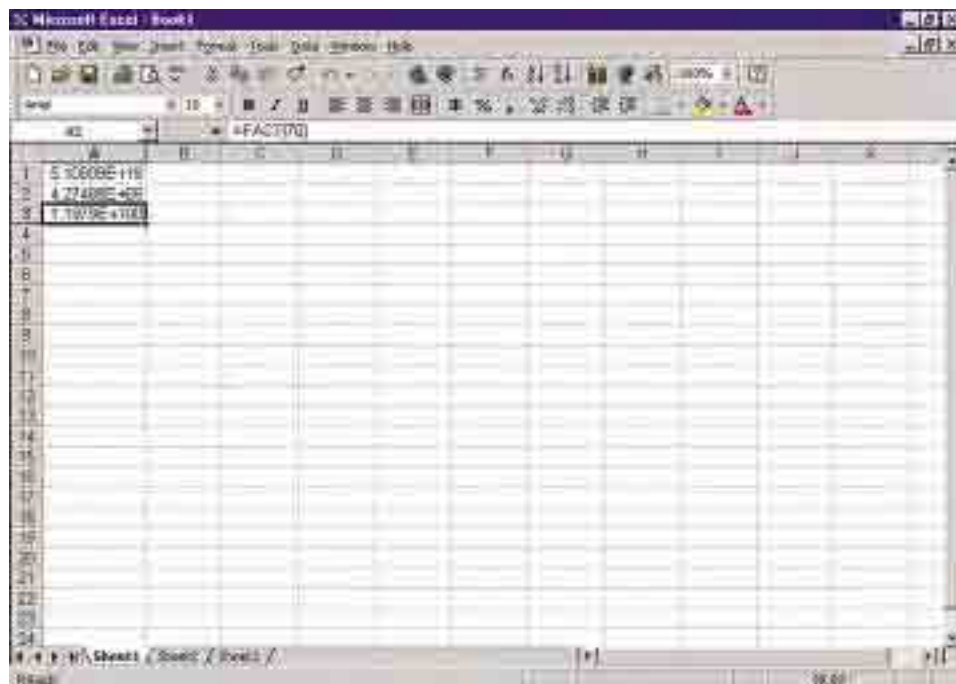
b)  $53! = 53 \times 52 \times 51 \times \dots \times 3 \times 2 \times 1$   
 $= 4.2749 \times 10^{69}$



- c) Entering  $70!$  on a graphing calculator gives an ERR:OVERFLOW message since  $70! > 10^{100}$  which is the largest number the calculator can handle. In fact,  $69!$  is the largest factorial you can calculate directly on TI-83 series calculators.

### Solution 2 Using a Spreadsheet

Both Corel® Quattro® Pro and Microsoft® Excel have a built-in **factorial function** with the syntax **FACT( $n$ )**.



### Example 3 Evaluating Factorial Expressions

Evaluate.

a)  $\frac{10!}{5!}$

b)  $\frac{83!}{79!}$

#### Solution

In both these expressions, you can divide out the common terms in the numerator and denominator.

$$\begin{aligned}\text{a) } \frac{10!}{5!} &= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} \\ &= 10 \times 9 \times 8 \times 7 \times 6 \\ &= 30\,240\end{aligned}$$

$$\begin{aligned}\text{b) } \frac{83!}{79!} &= \frac{83 \times 82 \times 81 \times 80 \times 79 \times 78 \times \dots \times 2 \times 1}{79 \times 78 \times \dots \times 2 \times 1} \\ &= 83 \times 82 \times 81 \times 80 \\ &= 44\,102\,880\end{aligned}$$

Note that by dividing out the common terms, you can use a calculator to evaluate this expression even though the factorials are too large for the calculator.

### Example 4 Counting Possibilities

The senior choir has rehearsed five songs for an upcoming assembly. In how many different orders can the choir perform the songs?

#### Solution

There are five ways to choose the first song, four ways to choose the second, three ways to choose the third, two ways to choose the fourth, and only one way to choose the final song. Using the fundamental counting principle, the total number of different ways is

$$\begin{aligned}5 \times 4 \times 3 \times 2 \times 1 &= 5! \\ &= 120\end{aligned}$$

The choir can sing the five songs in 120 different orders.

### Example 5 Indirect Method

In how many ways could ten questions on a test be arranged, if the easiest question and the most difficult question

a) are side-by-side?

b) are not side-by-side?

### Solution

- a) Treat the easiest question and the most difficult question as a unit making nine items that are to be arranged. The two questions can be arranged in  $2!$  ways within their unit.

$$9! \times 2! = 725\,760$$

The questions can be arranged in 725 760 ways if the easiest question and the most difficult question are side-by-side.

- b) Use the indirect method. The number of arrangements with the easiest and most difficult questions separated is equal to the total number of possible arrangements less the number with the two questions side-by-side:

$$\begin{aligned} 10! - 9! \times 2! &= 3\,628\,800 - 725\,760 \\ &= 2\,903\,040 \end{aligned}$$

The questions can be arranged in 2 903 040 ways if the easiest question and the most difficult question are not side-by-side.

A **permutation** of  $n$  distinct items is an arrangement of all the items in a definite order. The total number of such permutations is denoted by  ${}_nP_n$  or  $P(n, n)$ .

There are  $n$  possible ways of choosing the first item,  $n - 1$  ways of choosing the second,  $n - 2$  ways of choosing the third, and so on. Applying the fundamental counting principle as in Example 5 gives

$$\begin{aligned} {}_nP_n &= n \times (n - 1) \times (n - 2) \times (n - 3) \times \dots \times 3 \times 2 \times 1 \\ &= n! \end{aligned}$$

### Example 6 Applying the Permutation Formula

In how many different orders can eight nominees for the students' council give their speeches at an assembly?

#### Solution

$$\begin{aligned} {}_8P_8 &= 8! \\ &= 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 40\,320 \end{aligned}$$

There are 40 320 different orders in which the eight nominees can give their speeches.

### Example 7 Student Government

In how many ways could a president and a vice-president be chosen from a group of eight nominees?

#### Solution

Using the fundamental counting principle, there are  $8 \times 7$ , or 56, ways to choose a president and a vice-president.

A permutation of  $n$  distinct items taken  $r$  at a time is an arrangement of  $r$  of the  $n$  items in a definite order. Such permutations are sometimes called  $r$ -arrangements of  $n$  items. The total number of possible arrangements of  $r$  items out of a set of  $n$  is denoted by  ${}_nP_r$  or  $P(n, r)$ .

There are  $n$  ways of choosing the first item,  $n - 1$  ways of choosing the second item, and so on down to  $n - r + 1$  ways of choosing the  $r$ th item. Using the fundamental counting principle,

$${}_nP_r = n(n - 1)(n - 2) \dots (n - r + 1)$$

It is often more convenient to rewrite this expression in terms of factorials.

$${}_nP_r = \frac{n!}{(n - r)!}$$

The denominator divides out completely, as in Example 3, so these two ways of writing  ${}_nP_r$  are equivalent.

### Project Prep

The permutations formula could be a useful tool for your probability project.

### Example 8 Applying the Permutation Formula

In a card game, each player is dealt a face down “reserve” of 13 cards that can be turned up and used one by one during the game. How many different sequences of reserve cards could a player have?

#### Solution 1 Using Pencil and Paper

Here, you are taking 13 cards from a deck of 52.

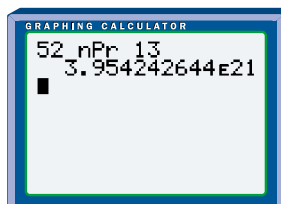
$$\begin{aligned} {}_{52}P_{13} &= \frac{52!}{(52 - 13)!} \\ &= \frac{52!}{39!} \\ &= 52 \times 51 \times 50 \times \dots \times 41 \times 40 \\ &= 3.9542 \times 10^{21} \end{aligned}$$

There are approximately  $3.95 \times 10^{21}$  different sequences of reserve cards a player could have.

#### Solution 2 Using a Graphing Calculator

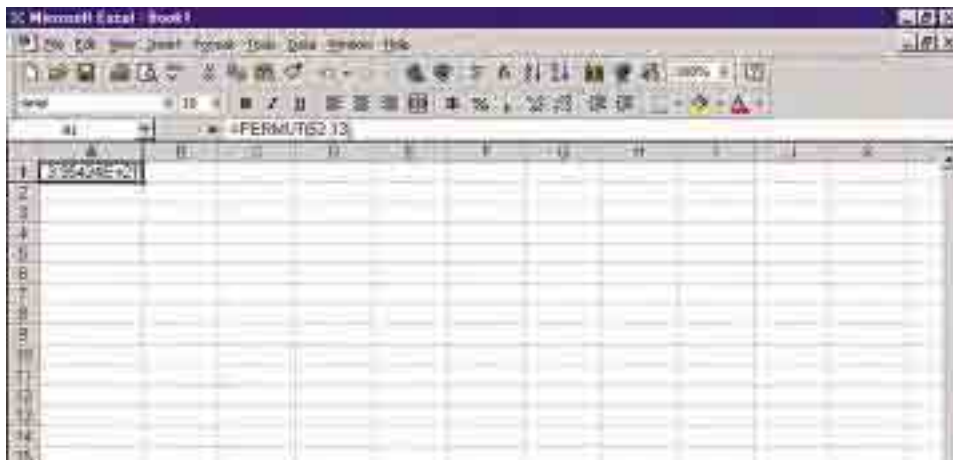
Use the **nPr** function on the MATH PRB menu.

There are approximately  $3.95 \times 10^{21}$  different sequences of reserve cards a player could turn up during one game.



### Solution 3 Using a Spreadsheet

Both Corel® Quattro® Pro and Microsoft® Excel have a **permutations function** with the syntax PERMUT( $n, r$ ).



There are approximately  $3.95 \times 10^{21}$  different sequences of reserve cards a player could turn up during one game.

### Key Concepts

- A factorial indicates the multiplication of consecutive natural numbers.  
$$n! = n(n-1)(n-2) \times \dots \times 1.$$
- The number of permutations of  $n$  distinct items chosen  $n$  at a time in a definite order is  ${}_nP_n = n!$
- The number of permutations of  $r$  items taken from  $n$  distinct items is

$${}_nP_r = \frac{n!}{(n-r)!}.$$

### Communicate Your Understanding

1. Explain why it is convenient to write the expression for the number of possible permutations in terms of factorials.
2. a) Is  $(-3)!$  possible? Explain your answer.  
b) In how many ways can you order an empty list, or zero items? What does this tell you about the value of  $0!$ ? Check your answer using a calculator.

## Practise

### A

- Express in factorial notation.
  - $6 \times 5 \times 4 \times 3 \times 2 \times 1$
  - $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$
  - $3 \times 2 \times 1$
  - $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$
- Evaluate.
  - $\frac{7!}{4!}$
  - $\frac{11!}{9!}$
  - $\frac{8!}{5! 2!}$
  - $\frac{15!}{3! 8!}$
  - $\frac{85!}{82!}$
  - $\frac{14!}{4! 5!}$
- Express in the form  ${}_nP_r$ .
  - $6 \times 5 \times 4$
  - $9 \times 8 \times 7 \times 6$
  - $20 \times 19 \times 18 \times 17$
  - $101 \times 100 \times 99 \times 98 \times 97$
  - $76 \times 75 \times 74 \times 73 \times 72 \times 71 \times 70$
- Evaluate without using technology.
  - $P(10, 4)$
  - $P(16, 4)$
  - ${}_5P_2$
  - ${}_9P_4$
  - $7!$
- Use either a spreadsheet or a graphing or scientific calculator to verify your answers to question 4.

## Apply, Solve, Communicate

- How many ways can you arrange the letters in the word *factor*?
  - How many ways can Ismail arrange four different textbooks on the shelf in his locker?
  - How many ways can Laura colour 4 adjacent regions on a map if she has a set of 12 coloured pencils?
- Simplify each of the following in factorial form. Do not evaluate.
    - $12 \times 11 \times 10 \times 9!$
    - $72 \times 7!$
    - $(n+4)(n+5)(n+3)!$
  - Communication** Explain how a factorial is an iterative process.
  - Seven children are to line up for a photograph.
    - How many different arrangements are possible?
    - How many arrangements are possible if Brenda is in the middle?
    - How many arrangements are possible if Ahmed is on the far left and Yen is on the far right?
    - How many arrangements are possible if Hanh and Brian must be together?
  - A 12-volume encyclopedia is to be placed on a shelf. How many incorrect arrangements are there?
  - In how many ways can the 12 members of a volleyball team line up, if the captain and assistant captain must remain together?
  - Ten people are to be seated at a rectangular table for dinner. Tanya will sit at the head of the table. Henry must not sit beside either Wilson or Nancy. In how many ways can the people be seated for dinner?
  - Application** Joanne prefers classical and pop music. If her friend Charlene has five classical CDs, four country and western CDs, and seven pop CDs, in how many orders can Joanne and Charlene play the CDs Joanne likes?
  - In how many ways can the valedictorian, class poet, and presenter of the class gift be chosen from a class of 20 students?

### B

15. **Application** If you have a standard deck of 52 cards, in how many different ways can you deal out
- 5 cards?
  - 10 cards?
  - 5 red cards?
  - 4 queens?
16. **Inquiry/Problem Solving** Suppose you are designing a coding system for data relayed by a satellite. To make transmissions errors easier to detect, each code must have no repeated digits.
- If you need 60 000 different codes, how many digits long should each code be?
  - How many ten-digit codes can you create if the first three digits must be 1, 3, or 6?
17. Arnold Schoenberg (1874–1951) pioneered serialism, a technique for composing music based on a tone row, a sequence in which each of the 12 tones in an octave is played only once. How many tone rows are possible?



18. Consider the students' council described on page 223 at the beginning of this chapter.
- In how many ways can the secretary, treasurer, social convenor, and fundraising chair be elected if all ten nominees are eligible for any of these positions?
  - In how many ways can the council be chosen if the president and vice-president must be grade 12 students and the grade representatives must represent their current grade level?
19. **Inquiry/Problem Solving** A student has volunteered to photograph the school's championship basketball team for the yearbook. In order to get the perfect picture, the student plans to photograph the ten players and their coach lined up in every possible order. Determine whether this plan is practical.



#### ACHIEVEMENT CHECK

Knowledge/ Understanding	Thinking/Inquiry/ Problem Solving	Communication	Application
<p>20. Wayne has a briefcase with a three-digit combination lock. He can set the combination himself, and his favourite digits are 3, 4, 5, 6, and 7. Each digit can be used at most once.</p> <ol style="list-style-type: none"> <li>How many permutations of three of these five digits are there?</li> <li>If you think of each permutation as a three-digit number, how many of these numbers would be odd numbers?</li> <li>How many of the three-digit numbers are even numbers and begin with a 4?</li> <li>How many of the three-digit numbers are even numbers and do <i>not</i> begin with a 4?</li> <li>Is there a connection among the four answers above? If so, state what it is and why it occurs.</li> </ol>			



21. TI-83 series calculators use the definition  $\left(-\frac{1}{2}\right)! = \sqrt{\pi}$ . Research the origin of this definition and explain why it is useful for mathematical calculations.
22. **Communication** How many different ways can six people be seated at a round table? Explain your reasoning.
23. What is the highest power of 2 that divides evenly into 100! ?
24. A committee of three teachers are to select the winner from among ten students nominated for special award. The teachers each make a list of their top three choices in order. The lists have only one name in common, and that name has a different rank on each list. In how many ways could the teachers have made their lists?

## Permutations With Some Identical Items

Often, you will deal with permutations in which some items are identical.

### INVESTIGATE & INQUIRE: What Is in a Name?

1. In their mathematics class, John and Jenn calculate the number of permutations of all the letters of their first names.
  - a) How many permutations do you think John finds?
  - b) List all the permutations of John's name.
  - c) How many permutations do you think Jenn finds?
  - d) List all the permutations of Jenn's name.
  - e) Why do you think there are different numbers of permutations for the two names?
2.
  - a) List all the permutations of the letters in your first name. Is the number of permutations different from what you would calculate using the  ${}_nP_n = n!$  formula? If so, explain why.
  - b) List and count all the permutations of a word that has two identical pairs of letters. Compare your results with those your classmates found with other words. What effect do the identical letters have on the number of different permutations?
  - c) Predict how many permutations you could make with the letters in the word *googol*. Work with several classmates to verify your prediction by writing out and counting all of the possible permutations.
3. Suggest a general formula for the number of permutations of a word that has two or more identical letters.



As the investigation above suggests, you can develop a general formula for permutations in which some items are identical.

### Example 1 Permutations With Some Identical Elements

Compare the different permutations for the words *DOLE*, *DOLL*, and *LOLL*.

#### Solution

The following are all the permutations of *DOLE*:

DOLE	DOEL	DLOE	DLEO	DEOL	DELO
ODLE	ODEL	OLDE	OLED	OEDL	OELD
LODE	LOED	LDOE	LDEO	LEOD	LEDO
EOLD	EODL	ELOD	ELDO	EDOL	EDLO

There are 24 permutations of the four letters in *DOLE*. This number matches what you would calculate using  ${}_4P_4 = 4!$

To keep track of the permutations of the letters in the word *DOLL*, use a subscript to distinguish the one *L* from the other.

DOLL <sub>1</sub>	<b>DOL<sub>1</sub>L</b>	DLO L <sub>1</sub>	DL L <sub>1</sub> O	<b>DL<sub>1</sub>OL</b>	<b>DL<sub>1</sub>LO</b>
ODLL <sub>1</sub>	<b>ODL<sub>1</sub>L</b>	OLDL <sub>1</sub>	OLL <sub>1</sub> D	<b>OL<sub>1</sub>DL</b>	<b>OL<sub>1</sub>LD</b>
LODL <sub>1</sub>	<b>LOL<sub>1</sub>D</b>	LDOL <sub>1</sub>	LDL <sub>1</sub> O	<b>LL<sub>1</sub>OD</b>	<b>LL<sub>1</sub>DO</b>
L <sub>1</sub> OLD	<b>L<sub>1</sub>ODL</b>	L <sub>1</sub> LOD	L <sub>1</sub> LDO	<b>L<sub>1</sub>DOL</b>	<b>L<sub>1</sub>DLO</b>

Of the 24 arrangements listed here, only 12 are actually different from each other. Since the two *L*s are in fact identical, each of the permutations shown in black is duplicated by one of the permutations shown in red. If the two *L*s in a permutation trade places, the resulting permutation is the same as the original one. The two *L*s can trade places in  ${}_2P_2 = 2!$  ways.

Thus, the number of different arrangements is

$$\frac{4!}{2!} = \frac{24}{2} = 12$$

In other words, to find the number of permutations, you divide the total number of arrangements by the number of ways in which you can arrange the identical letters. For the letters in *DOLL*, there are four ways to choose the first letter, three ways to choose the second, two ways to choose the third, and one way to choose the fourth. You then divide by the  $2!$  or 2 ways that you can arrange the two *L*s.

Similarly, you can use subscripts to distinguish the three *L*s in *LOLL*, and then highlight the duplicate arrangements.

L <sub>2</sub> OLL <sub>1</sub>	<b>L<sub>2</sub>OL<sub>1</sub>L</b>	L <sub>2</sub> LOL <sub>1</sub>	L <sub>2</sub> LL <sub>1</sub> O	<b>L<sub>2</sub>L<sub>1</sub>OL</b>	<b>L<sub>2</sub>L<sub>1</sub>LO</b>
OL <sub>2</sub> LL <sub>1</sub>	<b>OL<sub>2</sub>L<sub>1</sub>L</b>	OLL <sub>2</sub> L <sub>1</sub>	OLL <sub>1</sub> L <sub>2</sub>	<b>OL<sub>1</sub>L<sub>2</sub>L</b>	<b>OL<sub>1</sub>LL<sub>2</sub></b>
<b>LOL<sub>2</sub>L<sub>1</sub></b>	<b>LOL<sub>1</sub>L<sub>2</sub></b>	<b>LL<sub>2</sub>OL<sub>1</sub></b>	<b>LL<sub>2</sub>L<sub>1</sub>O</b>	<b>LL<sub>1</sub>OL<sub>2</sub></b>	<b>LL<sub>1</sub>L<sub>2</sub>O</b>
<b>L<sub>1</sub>OLL<sub>2</sub></b>	<b>L<sub>1</sub>OL<sub>2</sub>L</b>	<b>L<sub>1</sub>LOL<sub>2</sub></b>	<b>L<sub>1</sub>LL<sub>2</sub>O</b>	<b>L<sub>1</sub>L<sub>2</sub>OL</b>	<b>L<sub>1</sub>L<sub>2</sub>LO</b>

The arrangements shown in black are the only different ones. As with the other two words, there are 24 possible arrangements if you distinguish between the identical *L*s. Here, the three identical *L*s can trade places in  ${}_3P_3 = 3!$  ways.

Thus, the number of permutations is  $\frac{4!}{3!} = 4$ .

You can generalize the argument in Example 1 to show that the number of permutations of a set of  $n$  items of which  $a$  are identical is  $\frac{n!}{a!}$ .

### Example 2 Tile Patterns

Tanisha is laying out tiles for the edge of a mosaic. How many patterns can she make if she uses four yellow tiles and one each of blue, green, red, and grey tiles?

#### Solution

Here,  $n = 8$  and  $a = 4$ .

$$\begin{aligned}\frac{8!}{4!} &= 8 \times 7 \times 6 \times 5 \\ &= 1680\end{aligned}$$

Tanisha can make 1680 different patterns with the eight tiles.

### Example 3 Permutation With Several Sets of Identical Elements

The word *bookkeeper* is unusual in that it has three consecutive double letters. How many permutations are there of the letters in *bookkeeper*?

#### Solution

If each letter were different, there would be  $10!$  permutations, but there are two *os*, two *ks*, and three *es*. You must divide by  $2!$  twice to allow for the duplication of the *os* and *ks*, and then divide by  $3!$  to allow for the three *es*:

$$\begin{aligned}\frac{10!}{2!2!3!} &= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4}{2 \times 2} \\ &= 151\,200\end{aligned}$$

There are 151 200 permutations of the letters in *bookkeeper*.

The number of permutations of a set of  $n$  objects containing  $a$  identical objects of one kind,  $b$  identical objects of a second kind,  $c$  identical objects of a third kind, and so on is  $\frac{n!}{a!b!c!\dots}$ .

#### Example 4 Applying the Formula for Several Sets of Identical Elements

Barbara is hanging a display of clothing imprinted with the school's crest on a line on a wall in the cafeteria. She has five sweatshirts, three T-shirts, and four pairs of sweatpants. In how many ways can Barbara arrange the display?

##### Solution

Here,  $a = 5$ ,  $b = 3$ ,  $c = 4$ , and the total number of items is 12.

So,

$$\begin{aligned}\frac{n!}{a!b!c!} &= \frac{12!}{5!3!4!} \\ &= 27\,720\end{aligned}$$

Barbara can arrange the display in 27 720 different ways.

##### Project Prep

The game you design for your probability project could involve permutations of identical objects.

#### Key Concepts

- When dealing with permutations of  $n$  items that include  $a$  identical items of one type,  $b$  identical items of another type, and so on, you can use the formula  $\frac{n!}{a!b!c!\dots}$ .

#### Communicate Your Understanding

- Explain why there are fewer permutations of a given number of items if some of the items are identical.
- Explain why the formula for the numbers of permutations when some items are identical has the denominator  $a!b!c!\dots$  instead of  $a \times b \times c \dots$ .
  - Will there ever be cases where this denominator is larger than the numerator? Explain.
  - Will there ever be a case where the formula does not give a whole number answer? What can you conclude about the denominator and the numerator? Explain your reasoning.

## Practise

### A

- Identify the indistinguishable items in each situation.
  - The letters of the word *mathematics* are arranged.
  - Dina has six notebooks, two green and four white.
  - The cafeteria prepares 50 chicken sandwiches, 100 hamburgers, and 70 plates of French fries.
  - Thomas and Richard, identical twins, are sitting with Marianna and Megan.
- How many permutations are there of all the letters in each name?
  - Inverary    b) Beamsville
  - Mattawa    d) Penetanguishene
- How many different five-digit numbers can be formed using three 2s and two 5s?
- How many different six-digit numbers are possible using the following numbers?
  - 1, 2, 3, 4, 5, 6    b) 1, 1, 1, 2, 3, 4
  - 1, 3, 3, 4, 4, 5    d) 6, 6, 6, 6, 7, 8

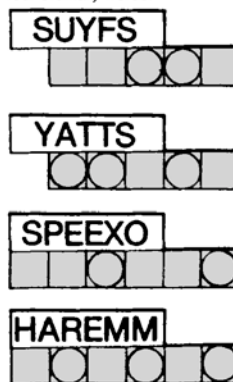
## Apply, Solve, Communicate

### B

- Communication** A coin is tossed eight times. In how many different orders could five heads and three tails occur? Explain your reasoning.
- Inquiry/Problem Solving** How many 7-digit even numbers less than 3 000 000 can be formed using all the digits 1, 2, 2, 3, 5, 5, 6?
- Kathryn's soccer team played a good season, finishing with 16 wins, 3 losses, and 1 tie. In how many orders could these results have happened? Explain your reasoning.

- Calculate the number of permutations for each of the jumbled words in this puzzle.
  - Estimate how long it would take to solve this puzzle by systematically writing out the permutations.

Unscramble these four Jumbles, one letter to each square, to form four ordinary words.



Now arrange the circled letters to form the surprise answer, as suggested by the above cartoon.



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## WEB CONNECTION

[www.mcgrawhill.ca/links/MDM12](http://www.mcgrawhill.ca/links/MDM12)

For more word jumbles and other puzzles, visit the above web site and follow the links. Find or generate two puzzles for a classmate to solve.

- Application** Roberta is a pilot for a small airline. If she flies to Sudbury three times, Timmins twice, and Thunder Bay five times before returning home, how many different itineraries could she follow? Explain your reasoning.
- After their training run, six members of a track team split a bag of assorted doughnuts. How many ways can the team share the doughnuts if the bag contains
  - six different doughnuts?
  - three each of two varieties?
  - two each of three varieties?

11. As a project for the photography class, Haseeb wants to create a linear collage of photos of his friends. He creates a template with 20 spaces in a row. If Haseeb has 5 identical photos of each of 4 friends, in how many ways can he make his collage?
12. **Communication** A used car lot has four green flags, three red flags, and two blue flags in a bin. In how many ways can the owner arrange these flags on a wire stretched across the lot? Explain your reasoning.
13. **Application** Malik wants to skateboard over to visit his friend Gord who lives six blocks away. Gord's house is two blocks west and four blocks north of Malik's house. Each time Malik goes over, he likes to take a different route. How many different routes are there for Malik if he only travels west or north?



#### ACHIEVEMENT CHECK

Knowledge/ Understanding	Thinking/Inquiry/ Problem Solving	Communication	Application
<p>14. Fran is working on a word puzzle and is looking for four-letter "scrambles" from the clue word <i>calculate</i>.</p> <p>a) How many of the possible four-letter scrambles contain four different letters?</p> <p>b) How many contain two <i>as</i> and one other pair of identical letters?</p> <p>c) How many scrambles consist of any two pairs of identical letters?</p> <p>d) What possibilities have you not yet taken into account? Find the number of scrambles for each of these cases.</p> <p>e) What is the total number of four-letter scrambles taking all cases into account?</p>			



15. Ten students have been nominated for the positions of secretary, treasurer, social convenor, and fundraising chair. In how many ways can these positions be filled if the Norman twins are running and plan to switch positions on occasion for fun since no one can tell them apart?
16. **Inquiry/Problem Solving** In how many ways can all the letters of the word *CANADA* be arranged if the consonants must always be in the order in which they occur in the word itself?
17. Glen works part time stocking shelves in a grocery store. The manager asks him to make a pyramid display using 72 cans of corn, 36 cans of peas, and 57 cans of carrots. Assume all the cans are the same size and shape. On his break, Glen tries to work out how many different ways he could arrange the cans into a pyramid shape with a triangular base.
- a) Write a formula for the number of different ways Glen could stack the cans in the pyramid.
- b) Estimate how long it will take Glen to calculate this number of permutations by hand.
- c) Use computer software or a calculator to complete the calculation.
18. How many different ways are there of arranging seven green and eight brown bottles in a row, so that exactly one pair of green bottles is side-by-side?
19. In how many ways could a class of 18 students divide into groups of 3 students each?



# Pascal's Triangle

The array of numbers shown below is called Pascal's triangle in honour of French mathematician, Blaise Pascal (1623–1662). Although it is believed that the 14th century Chinese mathematician Chu Shi-kie knew of this array and some of its applications, Pascal discovered it independently at age 13. Pascal found many mathematical uses for the array, especially in probability theory.

Pascal's method for building his triangle is a simple iterative process similar to those described in, section 1.1. In Pascal's triangle, each term is equal to the sum of the two terms immediately above it. The first and last terms in each row are both equal to 1 since the only term immediately above them is also always a 1.

If  $t_{n,r}$  represents the term in row  $n$ , position  $r$ , then  $t_{n,r} = t_{n-1,r-1} + t_{n-1,r}$ .

For example,  $t_{6,2} = t_{5,1} + t_{5,2}$ . Note that both the row and position labelling begin with 0.

			1					Row 0
		1		1				Row 1
	1		2		1			Row 2
	1	3		3		1		Row 3
	1	4	6		4		1	Row 4
1	5	10		10	5		1	Row 5
1	6	15	20	15	6		1	Row 6



Chu Shi-kie's triangle

								$t_{0,0}$
							$t_{1,0}$	$t_{1,1}$
						$t_{2,0}$	$t_{2,1}$	$t_{2,2}$
					$t_{3,0}$	$t_{3,1}$	$t_{3,2}$	$t_{3,3}$
				$t_{4,0}$	$t_{4,1}$	$t_{4,2}$	$t_{4,3}$	$t_{4,4}$
			$t_{5,0}$	$t_{5,1}$	$t_{5,2}$	$t_{5,3}$	$t_{5,4}$	$t_{5,5}$
		$t_{6,0}$	$t_{6,1}$	$t_{6,2}$	$t_{6,3}$	$t_{6,4}$	$t_{6,5}$	$t_{6,6}$

## WEB CONNECTION

[www.mcgrawhill.ca/links/MDM12](http://www.mcgrawhill.ca/links/MDM12)

Visit the above web site and follow the links to learn more about Pascal's triangle. Write a brief report about an application or an aspect of Pascal's triangle that interests you.



## INVESTIGATE & INQUIRE: Row Sums

- Find the sums of the numbers in each of the first six rows of Pascal's triangle and list these sums in a table.
- Predict the sum of the entries in
  - row 7
  - row 8
  - row 9
- Verify your predictions by calculating the sums of the numbers in rows 7, 8, and 9.
- Predict the sum of the entries in row  $n$  of Pascal's triangle.
- List any other patterns you find in Pascal's triangle. Compare your list with those of your classmates. Do their lists suggest further patterns you could look for?

In his book *Mathematical Carnival*, Martin Gardner describes Pascal's triangle as "so simple that a 10-year old can write it down, yet it contains such inexhaustible riches and links with so many seemingly unrelated aspects of mathematics, that it is surely one of the most elegant of number arrays."

### Example 1 Pascal's Method

- The first six terms in row 25 of Pascal's triangle are 1, 25, 300, 2300, 12 650, and 53 130. Determine the first six terms in row 26.
- Use Pascal's method to write a formula for each of the following terms:
  - $t_{12,5}$
  - $t_{40,32}$
  - $t_{n+1,r+1}$

### Solution

- $$t_{26,0} = 1$$

$$t_{26,1} = 1 + 25 = 26$$

$$t_{26,2} = 25 + 300 = 325$$

$$t_{26,3} = 300 + 2300 = 2600$$

$$t_{26,4} = 2300 + 12\ 650 = 14\ 950$$

$$t_{26,5} = 12\ 650 + 53\ 130 = 65\ 780$$
- $t_{12,5} = t_{11,4} + t_{11,5}$
  - $t_{40,32} = t_{39,31} + t_{39,32}$
  - $t_{n+1,r+1} = t_{n,r} + t_{n,r+1}$

### Example 2 Row Sums

Which row in Pascal's triangle has the sum of its terms equal to 32 768?

#### Solution

From the investigation on page 248, you know that the sum of the terms in any row  $n$  is  $2^n$ . Dividing 32 768 by 2 repeatedly, you find that  $32\,768 = 2^{15}$ . Thus, it is row 15 of Pascal's triangle that has terms totalling 32 768.

### Example 3 Divisibility

Determine whether  $t_{n,2}$  is divisible by  $t_{n,1}$  in each row of Pascal's triangle.

#### Solution

Row	$\frac{t_{n,2}}{t_{n,1}}$	Divisible?
0 and 1	n/a	n/a
2	0.5	no
3	1	yes
4	1.5	no
5	2	yes
6	2.5	no
7	3	yes

It appears that  $t_{n,2}$  is divisible by  $t_{n,1}$  only in odd-numbered rows. However,  $2t_{n,2}$  is divisible by  $t_{n,1}$  in all rows that have three or more terms.

### Example 4 Triangular Numbers

Coins can be arranged in the shape of an equilateral triangle as shown.

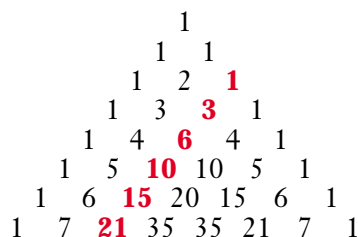


- Continue the pattern to determine the numbers of coins in triangles with four, five, and six rows.
- Locate these numbers in Pascal's triangle.
- Relate Pascal's triangle to the number of coins in a triangle with  $n$  rows.
- How many coins are in a triangle with 12 rows?

### Solution

- a) The numbers of coins in the triangles follow the pattern  $1 + 2 + 3 + \dots$  as shown in the table below.
- b) The numbers of coins in the triangles match the entries on the third diagonal of Pascal's triangle.

Number of Rows	Number of Coins	Term in Pascal's Triangle
1	1	$t_{2,2}$
2	3	$t_{3,2}$
3	6	$t_{4,2}$
4	10	$t_{5,2}$
5	15	$t_{6,2}$
6	21	$t_{7,2}$



- c) Compare the entries in the first and third columns of the table. The row number of the term from Pascal's triangle is always one greater than the number of rows in the equilateral triangle. The position of the term in the row,  $r$ , is always 2. Thus, the number of coins in a triangle with  $n$  rows is equal to the term  $t_{n+1,2}$  in Pascal's triangle.
- d)  $t_{12+1,2} = t_{13,2}$   
 $= 78$

A triangle with 12 rows contains 78 coins.

Numbers that correspond to the number of items stacked in a triangular array are known as **triangular numbers**. Notice that the  $n$ th triangular number is also the sum of the first  $n$  positive integers.

### Example 5 Perfect Squares

Can you find a relationship between perfect squares and the sums of pairs of entries in Pascal's triangle?

### Solution

Again, look at the third diagonal in Pascal's triangle.

$n$	$n^2$	Entries in Pascal's Triangle	Terms in Pascal's Triangle
1	1	1	$t_{2,2}$
2	4	$1 + 3$	$t_{2,2} + t_{3,2}$
3	9	$3 + 6$	$t_{3,2} + t_{4,2}$
4	16	$6 + 10$	$t_{4,2} + t_{5,2}$

Each perfect square greater than 1 is equal to the sum of a pair of adjacent terms on the third diagonal of Pascal's triangle:  $n^2 = t_{n,2} + t_{n+1,2}$  for  $n > 1$ .

## Key Concepts

- Each term in Pascal's triangle is equal to the sum of the two adjacent terms in the row immediately above:  $t_{n,r} = t_{n-1,r-1} + t_{n-1,r}$  where  $t_{n,r}$  represents the  $r$ th term in row  $n$ .
- The sum of the terms in row  $n$  of Pascal's triangle is  $2^n$ .
- The terms in the third diagonal of Pascal's triangle are triangular numbers. Many other number patterns occur in Pascal's triangle.

## Communicate Your Understanding

- Describe the symmetry in Pascal's triangle.
- Explain why the triangular numbers in Example 4 occur in Pascal's triangle.

## Practise

### A

- For future use, make a diagram of the first 12 rows of Pascal's triangle.
- Express as a single term from Pascal's triangle.
  - $t_{7,2} + t_{7,3}$
  - $t_{51,40} + t_{51,41}$
  - $t_{18,12} - t_{17,12}$
  - $t_{n,r} - t_{n-1,r}$
- Determine the sum of the terms in each of these rows in Pascal's triangle.
  - row 12
  - row 20
  - row 25
  - row  $(n - 1)$
- Determine the row number for each of the following row sums from Pascal's triangle.
  - 256
  - 2048
  - 16 384
  - 65 536

## Apply, Solve, Communicate

### B

### 5. Inquiry/Problem Solving

- Alternately add and subtract the terms in each of the first seven rows of Pascal's triangle and list the results in a table similar to the one below.

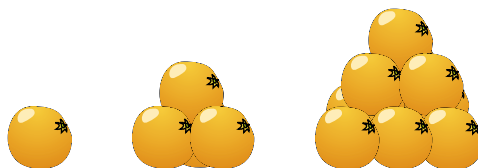
Row	Sum/Difference	Result
0	1	1
1	$1 - 1$	0
2	$1 - 2 + 1$	0
3	$1 - 3 + 3 - 1$	0
$\vdots$		

- Predict the result of alternately adding and subtracting the entries in the eighth row. Verify your prediction.
  - Predict the result for the  $n$ th row.
- Predict the sum of the squares of the terms in the  $n$ th row of Pascal's triangle.
    - Predict the result of alternately adding and subtracting the squares of the terms in the  $n$ th row of Pascal's triangle.

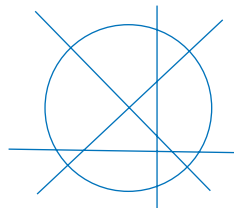
## 7. Communication

- a) Compare the first four powers of 11 with entries in Pascal's triangle. Describe any pattern you notice.
  - b) Explain how you could express row 5 as a power of 11 by regrouping the entries.
  - c) Demonstrate how to express rows 6 and 7 as powers of 11 using the regrouping method from part b). Describe your method clearly.
8. a) How many diagonals are there in
- i) a quadrilateral?
  - ii) a pentagon?
  - iii) a hexagon?
- b) Find a relationship between entries in Pascal's triangle and the maximum number of diagonals in an  $n$ -sided polygon.
- c) Use part b) to predict how many diagonals are in a heptagon and an octagon. Verify your prediction by drawing these polygons and counting the number of possible diagonals in each.
9. Make a conjecture about the divisibility of the terms in prime-numbered rows of Pascal's triangle. Confirm that your conjecture is valid up to row 11.
10. a) Which rows of Pascal's triangle contain only odd numbers? Is there a pattern to these rows?
- b) Are there any rows that have only even numbers?
- c) Are there more even or odd entries in Pascal's triangle? Explain how you arrived at your answer.

11. **Application** Oranges can be piled in a tetrahedral shape as shown. The first pile contains one orange, the second contains four oranges, the third contains ten oranges, and so on. The numbers of items in such stacks are known as **tetrahedral numbers**.



- a) Relate the number of oranges in the  $n$ th pile to entries in Pascal's triangle.
- b) What is the 12th tetrahedral number?
12. a) Relate the sum of the squares of the first  $n$  positive integers to entries in Pascal's triangle.
- b) Use part a) to predict the sum of the squares of the first ten positive integers. Verify your prediction by adding the numbers.
13. **Inquiry/Problem Solving** A straight line drawn through a circle divides it into two regions.
- a) Determine the maximum number of regions formed by  $n$  straight lines drawn through a circle. Use Pascal's triangle to help develop a formula.



- b) What is the maximum number of regions inside a circle cut by 15 lines?
14. Describe how you would set up a spreadsheet to calculate the entries in Pascal's triangle.



15. The Fibonacci sequence is 1, 1, 2, 3, 5, 8, 13, 21, ... . Each term is the sum of the previous two terms. Find a relationship between the Fibonacci sequence and the following version of Pascal's triangle.

```

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
...

```

16. **Application** Toothpicks are laid out to form triangles as shown below. The first triangle contains 3 toothpicks, the second contains 9 toothpicks, the third contains 18 toothpicks, and so on.



- a) Relate the number of toothpicks in the  $n$ th triangle to entries in Pascal's triangle.
- b) How many toothpicks would the 10th triangle contain?
17. Design a 3-dimensional version of Pascal's triangle. Use your own criteria for the layers. The base may be any regular geometric shape, but each successive layer must have larger dimensions than the one above it.

18. a) Write the first 20 rows of Pascal's triangle on a sheet of graph paper, placing each entry in a separate square.
- b) Shade in all the squares containing numbers divisible by 2.
- c) Describe, in detail, the patterns produced.
- d) Repeat this process for entries divisible by other whole numbers. Observe the resulting patterns and make a conjecture about the divisibility of the terms in Pascal's triangle by various whole numbers.

### 19. Communication

- a) Describe the iterative process used to generate the terms in the triangle below.

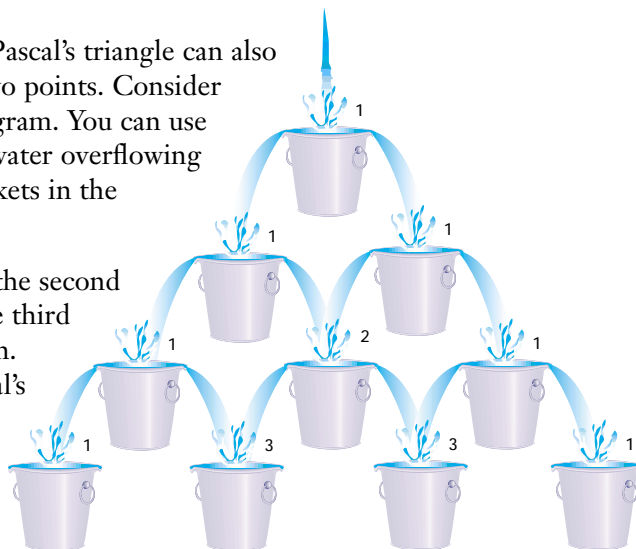
$$\begin{array}{ccccccc}
 & & & & & & \frac{1}{1} \\
 & & & & & & 1 \\
 & & & & & \frac{1}{2} & \frac{1}{2} \\
 & & & \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \\
 & & \frac{1}{4} & \frac{1}{12} & \frac{1}{12} & \frac{1}{4} \\
 & \frac{1}{5} & \frac{1}{20} & \frac{1}{30} & \frac{1}{20} & \frac{1}{5} \\
 \frac{1}{6} & \frac{1}{30} & \frac{1}{60} & \frac{1}{60} & \frac{1}{30} & \frac{1}{6}
 \end{array}$$

- b) Write the entries for the next two rows.
- c) Describe three patterns in this triangle.
- d) Research why this triangle is called the harmonic triangle. Briefly explain the origin of the name, listing your source(s).

## Applying Pascal's Method

The iterative process that generates the terms in Pascal's triangle can also be applied to counting paths or routes between two points. Consider water being poured into the top bucket in the diagram. You can use Pascal's method to count the different paths that water overflowing from the top bucket could take to each of the buckets in the bottom row.

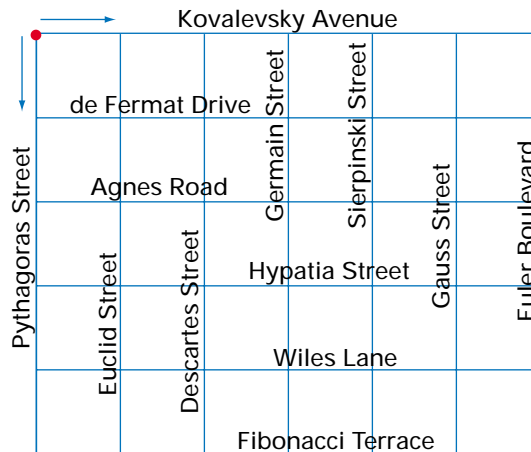
The water has one path to each of the buckets in the second row. There is one path to each outer bucket of the third row, but two paths to the middle bucket, and so on. The numbers in the diagram match those in Pascal's triangle because they were derived using the same method—Pascal's method.



### INVESTIGATE & INQUIRE: Counting Routes

Suppose you are standing at the corner of Pythagoras Street and Kovalevsky Avenue, and want to reach the corner of Fibonacci Terrace and Euler Boulevard. To avoid going out of your way, you would travel only east and south. Notice that you could start out by going to the corner of either Euclid Street and Kovalevsky Avenue or Pythagoras Street and de Fermat Drive.

1. How many routes are possible to the corner of Euclid Street and de Fermat Drive from your starting point? Sketch the street grid and mark the number of routes onto it.
2. a) Continue to travel only east or south. How many routes are possible from the start to the corner of
  - i) Descartes Street and Kovalevsky Avenue?
  - ii) Pythagoras Street and Agnes Road?
  - iii) Euclid Street and Agnes Road?
  - iv) Descartes Street and de Fermat Drive?
  - v) Descartes Street and Agnes Road?
- b) List the routes you counted in part a).



3. Consider your method and the resulting numbers. How do they relate to Pascal's triangle?
4. Continue to mark the number of routes possible on your sketch until you have reached the corner of Fibonacci Terrace and Euler Boulevard. How many different routes are possible?
5. Describe the process you used to find the number of routes from Pythagoras Street and Kovalevsky Avenue to Fibonacci Terrace and Euler Boulevard.

### Example 1 Counting Paths in an Array

Determine how many different paths will spell *PASCAL* if you start at the top and proceed to the next row by moving diagonally left or right.

```

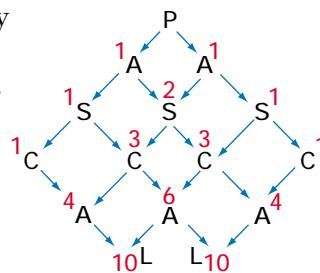
      P
     A A
    S S S
   C C C C
  A A A
 L L

```

#### Solution

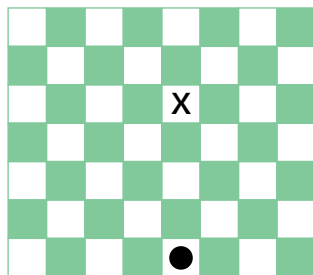
Starting at the top, record the number of possible paths moving diagonally to the left and right as you proceed to each different letter. For instance, there is one path from *P* to the left *A* and one path from *P* to the right *A*. There is one path from an *A* to the left *S*, two paths from an *A* to the middle *S*, and one path from an *A* to the right *S*.

Continuing with this counting reveals that there are 10 different paths leading to each *L*. Therefore, a total of 20 paths spell *PASCAL*.



### Example 2 Counting Paths on a Checkerboard

On the checkerboard shown, the checker can travel only diagonally upward. It cannot move through a square containing an X. Determine the number of paths from the checker's current position to the top of the board.



## Solution

Use Pascal's method to find the number of paths to each successive position. There is one path possible into each of the squares diagonally adjacent to the checker's starting position. From the second row there are four paths to the third row: one path to the third square from the left, two to the fifth square, and one to the seventh square. Continue this process for the remaining four rows. The square containing an X gets a zero or no number since there are no paths through this blocked square.

5		9		8		8	
	5		4		4		4
1		4		X		4	
	1		3		3		1
		1		2		1	
			1		1		
				●			

From left to right, there are 5, 9, 8, and 8 paths to the white squares at the top of the board, making a total of 30 paths.

## Key Concepts

- Pascal's method involves adding two neighbouring terms in order to find the term below.
- Pascal's method can be applied to counting paths in a variety of arrays and grids.

## Communicate Your Understanding

1. Suggest a context in which you could apply Pascal's method, other than those in the examples above.
2. Which of the numbers along the perimeter of a map tallying possible routes are always 1? Explain.

## Practise



1. Fill in the missing numbers using Pascal's method.

		495	
			825
3003	2112		

2. In the following arrangements of letters, start from the top and proceed to the next row by moving diagonally left or right. How many different paths will spell each word?

a)

```

      P
     A A
    T T T
   T T T T
  E E E E E
 R R R R R R
N N N N N N N
S S S S S S S

```

b)

```

      M
    A A
  T T T
H H H H
E E E E E
M M M M M M
A A A A A A A
  T T T T T T
    I I I I I
      C C C C
        S S S

```

c)

```

      T
    R R
  I I I
A A A A
  N N N
    G G
  L L L
E E E E

```

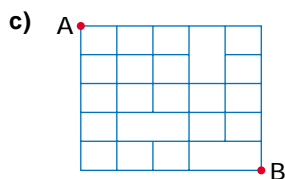
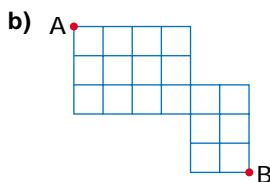
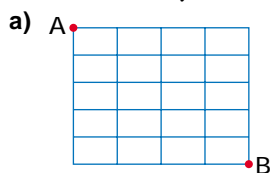
3. The first nine terms of a row of Pascal's triangle are shown below. Determine the first nine terms of the previous and next rows.

1 16 120 560 1820 4368 8008 11 440 12 870

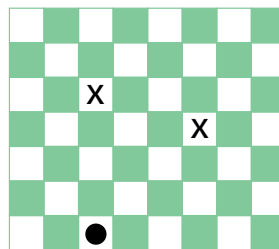
## Apply, Solve, Communicate



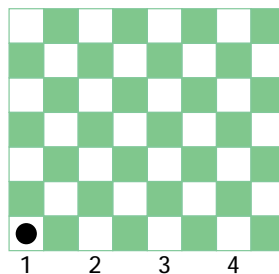
4. Determine the number of possible routes from A to B if you travel only south or east.



5. Sung is three blocks east and five blocks south of her friend's home. How many different routes are possible if she walks only west or north?
6. Ryan lives four blocks north and five blocks west of his school. Is it possible for him to take a different route to school each day, walking only south and east? Assume that there are 194 school days in a year.
7. A checker is placed on a checkerboard as shown. The checker may move diagonally upward. Although it cannot move into a square with an X, the checker may jump over the X into the diagonally opposite square.

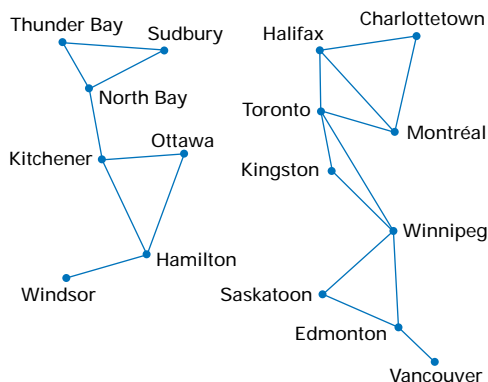


- a) How many paths are there to the top of the board?
- b) How many paths would there be if the checker could move both diagonally and straight upward?
8. **Inquiry/Problem Solving**
- a) If a checker is placed as shown below, how many possible paths are there for that checker to reach the top of the game board? Recall that checkers can travel only diagonally on the white squares, one square at a time, moving upward.

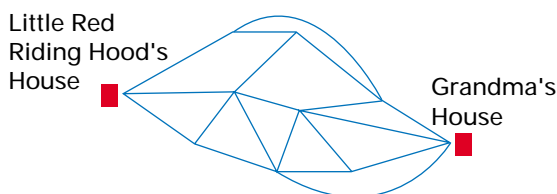


- b) When a checker reaches the opposite side, it becomes a “king.” If the starting squares are labelled 1 to 4, from left to right, from which starting square does a checker have the most routes to become a king? Verify your statement.

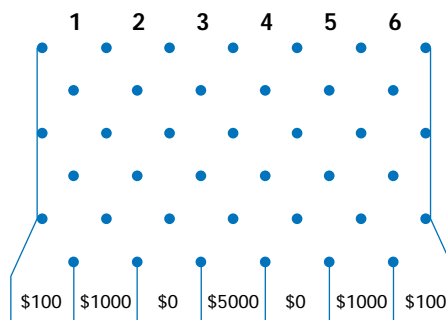
9. **Application** The following diagrams represent communication networks between a company’s computer centres in various cities.



- How many routes are there from Windsor to Thunder Bay?
  - How many routes are there from Ottawa to Sudbury?
  - How many routes are there from Montréal to Saskatoon?
  - How many routes are there from Vancouver to Charlottetown?
  - If the direction were reversed, would the number of routes be the same for parts a) to d)? Explain.
10. To outfox the Big Bad Wolf, Little Red Riding Hood mapped all the paths through the woods to Grandma’s house. How many different routes could she take, assuming she always travels from left to right?



11. **Communication** A popular game show uses a more elaborate version of the Plinko board shown below. Contestants drop a peg into one of the slots at the top of the upright board. The peg is equally likely to go left or right at each post it encounters.



- Into which slot should contestants drop their pegs to maximize their chances of winning the \$5000 prize? Which slot gives contestants the least chance of winning this prize? Justify your answers.
  - Suppose you dropped 100 pegs into the slots randomly, one at a time. Sketch a graph of the number of pegs likely to wind up in each compartment at the bottom of the board. How is this graph related to those described in earlier chapters?
12. **Inquiry/Problem Solving**
- Build a new version of Pascal’s triangle, using the formula for  $t_{n,r}$  on page 247, but start with  $t_{0,0} = 2$ .
  - Investigate this triangle and state a conjecture about its terms.
  - State a conjecture about the sum of the terms in each row.
13. **Inquiry/Problem Solving** Develop a formula relating  $t_{n,r}$  of Pascal’s triangle to the terms in row  $n - 3$ .



## ACHIEVEMENT CHECK

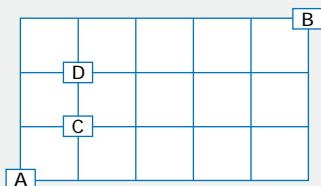
Knowledge/  
Understanding

Thinking/Inquiry/  
Problem Solving

Communication

Application

14. The grid below shows the streets in Anya's neighbourhood.

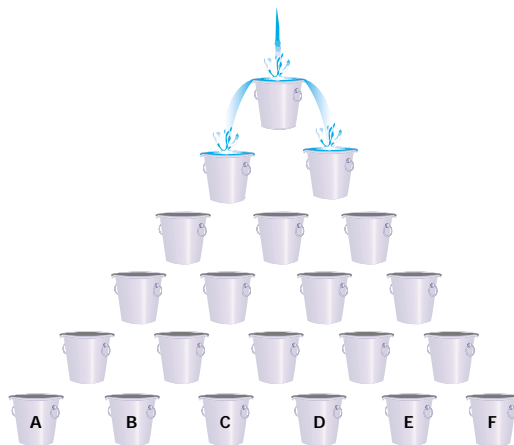


- If she only travels east and north, how many different routes can Anya take from her house at intersection A to her friend's house at intersection B?
- How many of the routes in part a) have only one change of direction?
- Suppose another friend lives at intersection C. How many ways can Anya travel from A to B, meeting her friend at C along the way?
- How many ways can she travel to B without passing through C? Explain your reasoning.
- If Anya takes any route from A to B, is she more likely to pass through intersection C or D? Explain your reasoning.



15. Develop a general formula to determine the number of possible routes to travel  $n$  blocks north and  $m$  blocks west.
16. **Inquiry/Problem Solving** In chess, a knight moves in L-shaped jumps consisting of two squares along a row or column plus one square at a right angle. On a standard  $8 \times 8$  chessboard, the starting position for a knight is the second square of the bottom row. If the knight travels upward on every move, how many routes can it take to the top of the board?

17. **Inquiry/Problem Solving** Water is poured into the top bucket of a triangular stack of 2-L buckets. When each bucket is full, the water overflows equally on both sides into the buckets immediately below. How much water will have been poured into the top bucket when at least one of the buckets in the bottom row is full?



18. **Application** Is it possible to arrange a pyramid of buckets such that the bottom layer will fill evenly when water overflows from the bucket at the top of the pyramid?
19. **Application** Enya is standing in the centre square of a 9 by 9 grid. She travels outward one square at a time, moving diagonally or along a row or column. How many different paths can Enya follow to the perimeter?
20. **Communication** Describe how a chessboard path activity involving Pascal's method is related to network diagrams like those in section 1.5. Would network diagrams for such activities be planar? Explain.

# Review of Key Concepts

## 4.1 Organized Counting

Refer to the Key Concepts on page 228.

1. A restaurant has a daily special with soup or salad for an appetizer; fish, chicken, or a vegetarian dish for the entrée; and cake, ice cream, or fruit salad for dessert. Use a tree diagram to illustrate all the different meals possible with this special.
2. A theatre company has a half-price offer for students who buy tickets for at least three of the eight plays presented this season. How many choices of three plays would a student have?
3. In how many different orders can a photographer pose a row of six people without having the tallest person beside the shortest one?
4. A transporter truck has three compact cars, a station wagon, and a minivan on its trailer. In how many ways can the driver load the shipment so that one of the heavier vehicles is directly over the rear axle of the trailer?

## 4.2 Factorials and Permutations

Refer to the Key Concepts on page 238.

5. For what values of  $n$  is  $n!$  less than  $2^n$ ? Justify your answer.
6. A band has recorded five hit singles. In how many different orders could the band play three of these five songs at a concert?
7. In how many ways could a chairperson, treasurer, and secretary be chosen from a 12-member board of directors?

## 4.3 Permutations With Some Identical Items

Refer to the Key Concepts on page 244.

8. How many different ten-digit telephone numbers contain four 2s, three 3s, and three 7s?
9.
  - a) How many permutations are there of the letters in the word *baseball*?
  - b) How many begin with the letter *a*?
  - c) How many end with the letter *e*?
10. Find the number of  $4 \times 4$  patterns you can make using eight white, four grey, and four blue floor tiles.

## 4.4 Pascal's Triangle

Refer to the Key Concepts on page 251.

11. Write out the first five rows of Pascal's triangle.
12. What is the sum of the entries in the seventh row of Pascal's triangle?
13. Describe three patterns in Pascal's triangle.

## 4.5 Applying Pascal's Method

Refer to the Key Concepts on page 256.

14. Explain why Pascal's method can be considered an iterative process.
15. How many paths through the array shown will spell *SIERPINSKI*?

S  
I I  
E E E  
R R  
P P P  
I I I I  
N N N  
S S S S  
K K K  
I I

# Chapter Test

ACHIEVEMENT CHART				
Category	Knowledge/ Understanding	Thinking/Inquiry/ Problem Solving	Communication	Application
Questions	All	4, 7, 8	1, 3, 8	3, 4, 5, 6, 8

- Natasha tosses four coins one after the other.
  - In how many different orders could heads or tails occur.
  - Draw a tree diagram to illustrate all the possible results.
  - Explain how your tree diagram corresponds to your calculation in part a).
- Evaluate the following by first expressing each in terms of factorials.
  - ${}_{15}P_6$
  - $P(6, 2)$
  - ${}_7P_3$
  - ${}_9P_9$
  - $P(7, 0)$
- Suppose you are designing a remote control that uses short, medium, or long pulses of infrared light to send control signals to a device.
  - How many different control codes can you define using
    - three pulses?
    - one, two, or three pulses?
  - Explain how the multiplicative and additive counting principles apply in your calculations for part a).
- How many four-digit numbers can you form with the digits 1, 2, 3, 4, 5, 6, and 7 if no digit is repeated?
  - How many of these four-digit numbers are odd numbers?
  - How many of them are even numbers?
- How many ways are there to roll either a 6 or a 12 with two dice?
- How many permutations are there of the letters of each of the following words?
  - data
  - management
  - microwave
- A number of long, thin sticks are lying in a pile at odd angles such that the sticks cross each other.
  - Relate the maximum number of intersection points of  $n$  sticks to entries in Pascal's triangle.
  - What is the maximum number of intersection points with six overlapping sticks?



## ACHIEVEMENT CHECK

Knowledge/Understanding	Thinking/Inquiry/Problem Solving	Communication	Application
<ol style="list-style-type: none"> <li>At a banquet, four couples are sitting along one side of a table with men and women alternating.               <ol style="list-style-type: none"> <li>How many seating arrangements are possible for these eight people?</li> <li>How many arrangements are possible if each couple sits together? Explain your reasoning.</li> <li>How many arrangements are possible if no one is sitting beside his or her partner?</li> <li>Explain why the answers from parts b) and c) do not add up to the answer from part a).</li> </ol> </li> </ol>			