Lesson 1 — Chapter 1

Permutations and Organized Counting

1.1 Organized Counting

Two Counting rules:

(1) Rule of Product

or Multiplicative Counting Principle.

or Fundamental Counting Principle.

If k stages are needed to go through to complete a task. and there are,

n_1 ways to do Stage 1,

n_2 ways to do Stage 2,

⋮ ⋮

n_k ways to do Stage k,

Then, in total, there are $n_1 \times n_2 \times \cdots \times n_k$ ways to complete the task.

For example, how many possible outcomes if a coin is flipped three times?

Sol. 3 flipps are 3 stages; $k=3$; $n_1=n_2=n_3=2$;

so in total, $n_1 \times n_2 \times n_3 = 2 \times 2 \times 2 = 8$ (outcomes).

(2) Rule of Sum

or Additive Counting Principle.

If only one of K actions is needed to complete a task, and the K actions are mutually exclusive, and there are,

n_1 ways to do Action 1,

n_2 ways to do Action 2,

$\vdots$

n_K ways to do Action 3,

Then, in total, there are $n_1 + n_2 + \cdots + n_K$ ways to complete the task.

For example. Q4 on Page 229.

Action 1 - draw a 10.

Action 2 - draw a Queen.

$n_1 = 4$, $n_2 = 4$.

$\therefore n_1 + n_2 = 4 + 4 = 8$ ways.

1.2 Factorials and Permutations.

(1) Factorials

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$

the product of the first n natural numbers

$n!$ is called "n factorial".

$n = 0, 1, 2, 3, \cdots$

$0! = 1$

$1! = 1$

$2! = 2 \times 1 = 2$

$3! = 3 \times 2 \times 1 = 6$, or $3! = 3 \times 2!$

$4! = 4 \times (3!) = 4 \times 3 \times (2!) = 24.$

$\vdots$

$$\begin{aligned} n! &= n \times ((n-1)!) \\ &= n \times (n-1) \times ((n-2)!) \\ &= n \times (n-1) \times (n-2) \times ((n-3)!). \end{aligned}$$

$\cdots$

On a calculator, there is a key $n!$ or $x!$

$69! \approx 1.711224524 \times 10^{98}$

$70!$ shows "Math Error"

For example,

$$\frac{500!}{498!} = \frac{500 \times 499 \times \cancel{498!}}{\cancel{498!}}$$

$$= 500 \times 499 = 249500$$

Another example.

Solve for n, if $n! = 56(n-2)!$

Sol. $\because n(n-1)\cancel{(n-2)!} = 56\cancel{(n-2)!}$

$n^2 - n - 56 = 0$

$(n-8)(n+7) = 0$

$\therefore n = 8$ or $n = -7$.

Accept $n=8$. reject $n=-7$.

Definition of Permutation:

Permutation is the number of arrangements of n items taken r items at a time. the order of the items is important.

For example, in how many ways can we choose three letters from a set of five, say $S = \{a, b, c, d, e\}$ and arrange them in different order along a line?

Sol.

abc, abd, abe, acd, ace, ade, bcd, bce, bde, cde,

acb	adb		ced
bac	bad	- - - - - - - - -	dce
bca	bda		dec
cab	dab		ecd
cba	dba		edc

$3! = 6$ ... - - - - - - $3! = 6$

In total, $10 \times 6 = 60$ ways.

or using Rule of Product:

$$\underline{5} \times \underline{4} \times \underline{3} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = \frac{5!}{2!} = \frac{5!}{(5-3)!}$$

is the permutation of 5 items taken 3 at a time.

$$5 \times 4 \times 3 = \frac{5!}{(5-3)!} = {}_5P_3$$

Generally, permutation of n items, taken r at a time is ${}_nP_r$ or $P(n,r)$

$${}_nP_r = P(n,r) = \frac{n!}{(n-r)!}$$

${}_nP_r$ consists of two stages:

1st stage: choose r items from n items.

2nd stage: arrange the r items in different order along a line.

Actually, the 1st stage is combination of n items, taken r at a time.

${}_nC_r$ or $C(n,r)$ or $\binom{n}{r}$.

the 2nd stage, using Rule of Product, is $r! = r \times (r-1) \times (r-2) \times \cdots \times 3 \times 2 \times 1$.

Therefore, ${}_nP_r = {}_nC_r \cdot r!$

$${}_nP_r = \frac{n!}{(n-r)!} \quad \text{where } r = 0, 1, 2, \cdots, n.$$

$${}_nP_0 = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$$ is the zero permutation.

$${}_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$ is the total permutation.