Lesson 4

1. Combinations

Combination is the number of selection of items without regard to order.

Then, permutation is the number of arrangement of items with regard to order.

We consider permutation consists of two stages:

Stage 1 is combination of n items, taken r items at a time.

Stage 2 is arrangement of the r items in different order in a line.

So $_nP_r = {_nC_r} \cdot r!$

or $_nC_r = \dfrac{_nP_r}{r!}$

where $_nC_r$ is combination of n items, taken r at a time.

Other notations: $_nP_r = P(n,r)$

$_nC_r = C(n,r) = \binom{n}{r}$.

P274. Ex. 1.

Permutation V.S. Combination

a) $_5P_3 = 5\times4\times3 = 60$ (ways)

b) $_5C_3 = \dfrac{_5P_3}{3!} = \dfrac{60}{6} = 10$ (ways)

$\{A, B, C, D, E\}$,

ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE,
ACB, ABD, CED,
BAC, BAD, DCE,
BCA, BDA, DEC,
CAB, DAB, ECD,
CBA, DBA, EDC,
- - - - -

c) $_5P_3 = {_5C_3} \times 3!$ or $_5C_3 = \dfrac{_5P_3}{3!}$

where $3! = 3\times2\times1 = 6$.

2. Solving Problems with Combination

Properties of Combination

① $_nC_r = {_nC_{n-r}}$, $\boxed{_nP_r \neq {_nP_{n-r}}}$

$\because {_nC_r} = \dfrac{n!}{r!(n-r)!} = {_nC_{n-r}}$

$_{100}C_{98} = {_{100}C_2} = \dfrac{_{100}P_2}{2!} = \dfrac{100\times99}{2}$

② $_nC_r = {}_{n-1}C_{r-1} + {}_{n-1}C_r$;

③ $_nC_0 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_n = 2^n$;

④ $_nC_0 - {}_nC_1 + {}_nC_2 - {}_nC_3 + \cdots + (-1)^n {}_nC_n = 0$;

For example, $_4C_0 + {}_4C_1 + {}_4C_2 + {}_4C_3 + {}_4C_4$

$= 1 + 4 + 6 + 4 + 1 = 16 = 2^4$

$_4C_0 - {}_4C_1 + {}_4C_2 - {}_4C_3 + {}_4C_4$

$= 1 - 4 + 6 - 4 + 1 = 8 - 8 = 0$

$_5C_0 + {}_5C_1 + {}_5C_2 + {}_5C_3 + {}_5C_4 + {}_5C_5$

$= 1 + 5 + 10 + 10 + 5 + 1 = 32 = 2^5$

$_5C_0 - {}_5C_1 + {}_5C_2 - {}_5C_3 + {}_5C_4 - {}_5C_5$

$= 1 - 5 + 10 - 10 + 5 - 1 = 16 - 16 = 0$

$_nC_0 = 1$, $\quad {}_nC_n = 1$

$_nP_0 = 1$, $\quad {}_nP_n = n!$

P282 Ex.1.

All Possible combinations of Distinct Items

$S = \{A, O, P\}$.

Method 1: $_3C_1 + {}_3C_2 + {}_3C_3$

$= 3 + 3 + 1 = 7$ (ways).

Method 2:

A	O	P
0	0	0
1	1	1

$2 \times 2 \times 2 = 8$

~~(0, 0, 0)~~
(1, 0, 0)
(0, 1, 0)
(0, 0, 1)
(1, 1, 0)
(1, 0, 1)
(0, 1, 1)
(1, 1, 1)

$\therefore 8 - 1 = 7$ (ways)

to choose at least one piece of fruit.

Generally, $S = \{a_1, a_2, \cdots, a_n\}$.

The number of ways to choose at least one element from S is:

$$_nC_1 + {}_nC_2 + \cdots + {}_nC_n = 2^n - 1$$

or

$$2 \times 2 \times 2 \times \cdots \times 2 - 1 = 2^n - 1$$

a_1	a_2	a_3	$\cdots$	a_n
0	0	0		0
1	1	1		1
2 ×	2 ×	2 ×	$\cdots$	× 2 $= 2^n$

$\therefore 2^n - 1$

P283. Ex.3.

All Possible Combination with Some Identical Items.

Cookies	Drinks	Coffee
0	0	0
1	1	1
2	2	2
3	3	
	4	

$(3+1) \times (4+1) \times (2+1)$

$= 4 \times 5 \times 3 = 60$

$\therefore 60 - 1 = 59$ (direct purchases)

Generally, if n items consist of k kinds,

and there are n_1 items of the 1^{st} kind,

n_2 items of the 2^{nd} kind,

$\vdots$

n_k items of the k^{th} kind,

$n_1 + n_2 + \cdots + n_k = n$. Then the number of ways

to choose at least one item from the n items

is $(n_1+1)(n_2+1)\cdots(n_k+1) - 1$

P284. Ex. 4. Combinations with Some Identical Items.

5 rocks; 2 blue tones. 3 Jazz

At least one Jazz, choose 3 out of 10;

Method 1. — Direct Method.

Case 1. 1-Jazz; $3C_1 \cdot 7C_2 = 3 \times 21 = 63$ (ways)

Case 2. 2-Jazz. $3C_2 \cdot 7C_1 = 3 \times 7 = 21$ (ways)

Case 3. 3-Jazz. $3C_3 \cdot 7C_0 = 1 \times 1 = 1$ (way)

Using Rule of Sum, $63 + 21 + 1 = 85$ (ways)

Method 2 — Indirect Method,

Total. $10C_3 = \frac{10 \times \cancel{9}^{3} \times \cancel{8}^{4}}{\cancel{3!}} = 120$ (ways)

0-Jazz, $3C_0 \cdot 7C_3 = 1 \times \frac{7 \times \cancel{6} \times 5}{\cancel{3!}} = 35$ (ways)

$120 - 35 = 85$ (ways)