let $x^2-3=1$ $\quad x^2=4 \quad x=\pm 2$

$-\frac{(x-\mu)^2}{2\sigma^2}$

$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e$

$f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

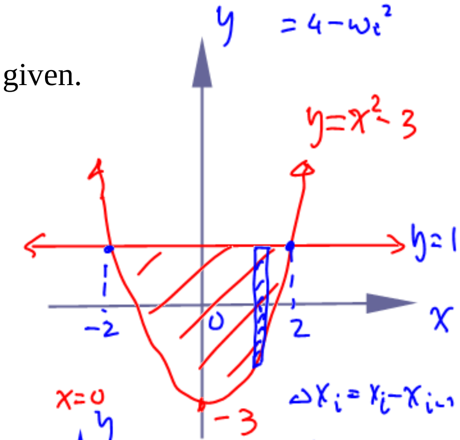
$h_i = f(w_i) - g(w_i) = 1-(x^2-3) = 1-x^2+3 = 4-x^2 = 4-w_i^2$

AP Calculus Homework Ten – Applications of Definite Integral and Polar Coordinates

5.1 Area and Solids of Revolution

In Questions 1 - 6, evaluate the area of the region whose boundaries are given.

1. The parabola of $y = x^2 - 3$ and the line $y = 1$.

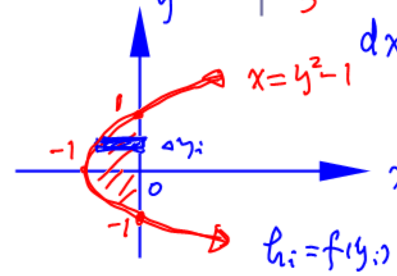


$\Delta x_i = x_i - x_{i-1}$ $\quad dx$

$$A=\int_a^b [f(x)-g(x)]\,dx=\int_{-2}^{2}[1-(x^2-3)]\,dx=\int_{-2}^{2}(4-x^2)\,dx$$

$$=2\int_0^2(4-x^2)\,dx=2\left[4x-\frac{x^3}{3}\right]_0^2=2\left(\left[4(2)-\frac{2^3}{3}\right]-0\right)=\frac{32}{3}$$

2. The curve of $x = y^2 - 1$ and the y – axis.



$$A=\int_{-1}^{1}[f(y)-g(y)]\,dy=\int_{-1}^{1}[0-(y^2-1)]\,dy=\int_{-1}^{1}(1-y^2)\,dy$$

$$=2\int_0^1(1-y^2)\,dy=2\left[y-\frac{y^3}{3}\right]_0^1=2\left\{\left[1-\frac{1^3}{3}\right]-\left(0-\frac{0^3}{3}\right)\right\}=\frac{4}{3}$$

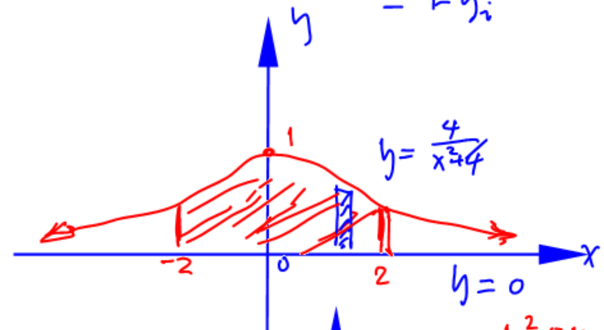
$h_i = f(y_i) - g(y_i) = 0-(y_i^2-1) = 1-y_i^2$

3. The curve of y = $4/(x^2 + 4)$, the x-axis, and the vertical lines $x = -2$, and $x = 2$.

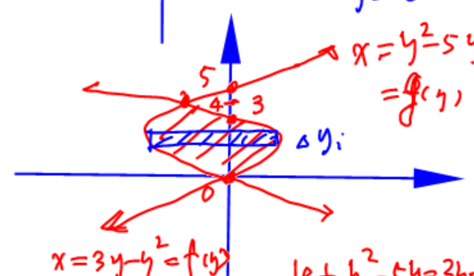
$$A=\int_{-2}^{2}\left[\frac{4}{x^2+4}-0\right]dx=2\times4\int_0^2\frac{1}{x^2+4}\,dx=\frac{2\times4}{4}\int_0^2\frac{1}{(\frac{x}{2})^2+1}\,dx$$

$$=2\int_0^2\frac{1\times2}{(\frac{x}{2})^2+1}\,d\left(\frac{x}{2}\right)=4\int_0^2\frac{d(\frac{x}{2})}{(\frac{x}{2})^2+1}=4\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$=4\left\{\tan^{-1}\left(\frac{2}{2}\right)-\tan^{-1}\left(\frac{0}{2}\right)\right\}=4\left(\frac{\pi}{4}-0\right)=\pi$$



4. The parabolas $x = y^2 - 5y$ and $x = 3y - y^2$.



$$A=\int_0^4[f(y)-g(y)]\,dy=\int_0^4[3y-y^2-(y^2-5y)]\,dy=\int_0^4[8y-2y^2]\,dy$$

$$=\left[8\left(\frac{y^2}{2}\right)-2\left(\frac{y^3}{3}\right)\right]_0^4=\left[4(4)^2-\frac{2}{3}(4^3)\right]=\frac{1}{3}(4)^3=\frac{64}{3}$$

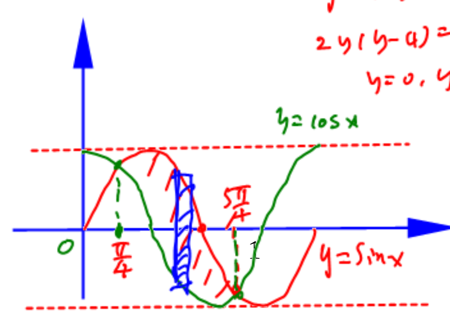
let $y^2-5y=3y-y^2$
$2y^2-8y=0$
$2y(y-4)=0$
$y=0,\ y=4$

5. Bounded above by the curve $y = \sin x$ and below by $y = \cos x$ from $x = \pi/4$ to $x = 5\pi/4$.

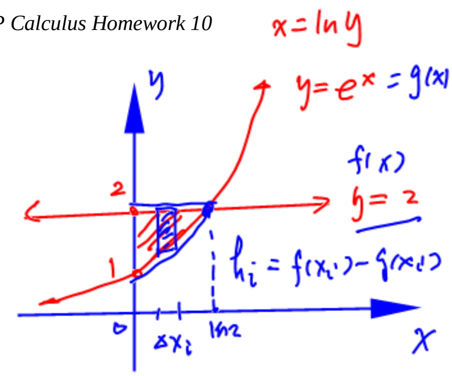
$$A=\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}[\sin x-\cos x]\,dx=\left[-\cos x-\sin x\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$=-\left[\left(\cos\frac{5\pi}{4}+\sin\frac{5\pi}{4}\right)-\left(\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\right)\right]$$

$$=-\left\{-\frac{\sqrt2}{2}-\frac{\sqrt2}{2}-\frac{\sqrt2}{2}-\frac{\sqrt2}{2}\right\}=2\sqrt2$$



6. Find the area bounded by $y = e^x$, $y = 2$, and the y-axis.

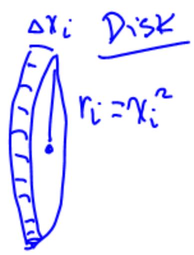


$$A=\int_0^{\ln 2}[2-e^x]\,dx = [2x-e^x]_0^{\ln 2}$$
$$= [2\ln 2 - e^{\ln 2}] - [2(0) - e^0]$$
$$= 2\ln 2 - \cancel{e}^{\ln 2} + e^0 = \ln 4 - 2 + 1 = \ln 4 - 1$$

$e^x = 2$

$x = \ln 2$

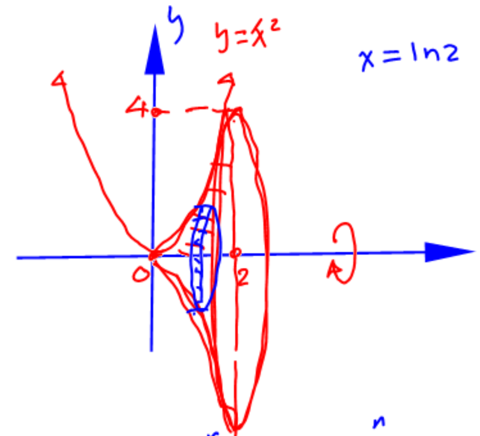
Disk



In Questions 7 - 10, the region whose boundaries are given is rotated about the line indicated. Calculate the volume of the solid generated.

7. $y = x^2$, $x = 2$, and $y = 0$; about the x-axis.

$\Delta V_i = \pi r_i^2$



$$V=\int_0^2 \pi r^2\,dx = \int_0^2 \pi (x^2)^2\,dx = \pi\int_0^2 x^4\,dx$$
$$= \pi\left[\frac{x^5}{5}\right]_0^2 = \frac{\pi}{5}(2)^5 = \frac{32\pi}{5}$$

$$V = \lim_{n\to\infty}\sum_{i=1}^{n} \Delta V_i = \lim_{n\to\infty}\sum_{i=1}^{n} \pi r_i^2$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n} \pi [f(x_i)]^2$$

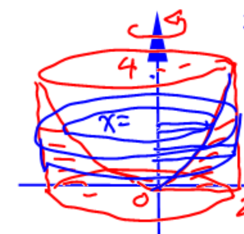
$x = \sqrt{y}$

Washer

R_i

8. $y = x^2$, $x = 2$, and $y = 0$; about the y-axis.

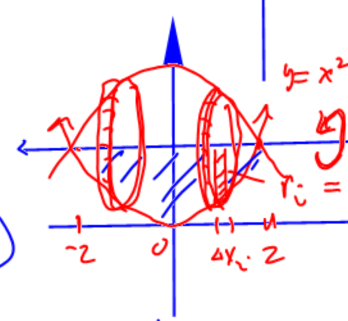
$$V=\int_0^4 \pi[R^2-r^2]\,dy = \pi\int_0^4 [2^2-(\sqrt{y})^2]\,dy$$
$$= \pi\int_0^4 (4-y)\,dy = \pi\left[4y-\frac{y^2}{2}\right]_0^4$$
$$= \pi\left[4(4)-\frac{4^2}{2}\right] = \pi(16-8) = 8\pi$$



$x = f(y) = 2 = R_i$

$x = g(y_i) = \sqrt{y_i} = r_i$

9. $y = x^2$ and $y = 4$; about the line $y = 4$.



$y = x^2 = g(x)$

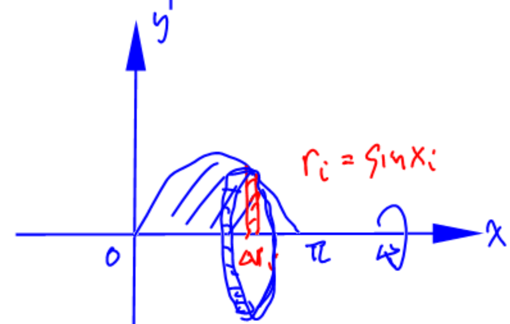
$y = 4 = f(x)$

$r_i = f(x_i) - g(x_i) = 4 - x^2$

$$V=\int_{-2}^{2} \pi r^2\,dx = \pi\int_{-2}^{2}(4-x^2)^2\,dx = 2\pi\int_0^2 (16+x^4-8x^2)\,dx$$
$$= 2\pi\left[16x+\frac{x^5}{5}-\frac{8}{3}x^3\right]_0^2 = 2\pi\left[16(2)+\frac{2^5}{5}-\frac{8}{3}(2)^3\right]$$
$$= 2\pi\left(32-\frac{2\times 32}{15}\right) = 64\left(\frac{13}{15}\right)\pi$$

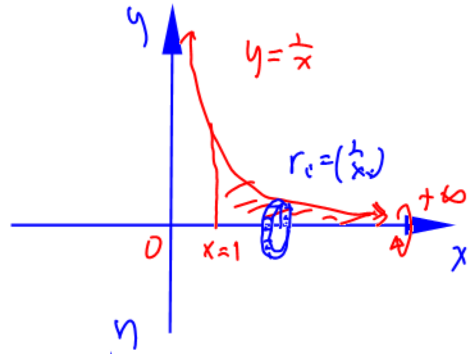
10. An arch of $y = \sin x$ and the x-axis; about the x-axis.

$$V=\int_0^{\pi} \pi(\sin x)^2\,dx = \pi\int_0^{\pi} \frac{1}{2}(1-\cos 2x)\,dx$$
$$= \frac{\pi}{2}\left[x-\frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi}{2}(\pi) = \frac{\pi^2}{2}$$



In Questions 11 and 12, calculate the volume, if it exists, of the solid generated.

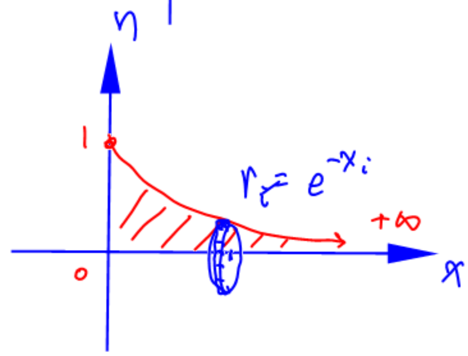
11. $y = 1/x$, at the left by $x = 1$, and below by y = 0; about the x –axis.



$$V=\int_1^{\infty}\pi\left(\frac{1}{x}\right)^2 dx = \pi\left[\frac{x^{-2+1}}{-2+1}\right]_1^{\infty} = \pi\left(x^{-1}\right)\Big|_{\infty}^{1}$$

$$=\pi\left(1^{-1}-\infty^{-1}\right) = \pi(1-0) = \pi$$

12. The first-quadrant region under $y = e^{-x}$; about the x-axis.



$$V=\int_0^{\infty}\pi\left(e^{-x}\right)^2 dx = \frac{\pi}{-2}\int_0^{\infty} e^{-2x}\, d(-2x)$$

$$= -\frac{\pi}{2}\left[e^{-2x}\right]_0^{\infty} = \frac{\pi}{2}\left[e^{-2x}\right]_{\infty}^{0} = \frac{\pi}{2}\left[e^{0}-e^{-\infty}\right] = \frac{\pi}{2}(1-0)$$

$$= \frac{\pi}{2}$$

In Questions 13 - 15, the region whose boundaries are given is rotated about the line indicated. Derive a definite integral that gives the volume of the solid generated.

13. $y = x^2$ and $y = 4$; about the line $y = -1$.

14. $y = \ln x$, $y = 0$, $x = e$; about the line $x = e$.

15. The curve with parametric equation $x = \tan\theta$, $y = \cos^2\theta$, and the lines $x = 0$, $x = 1$, and $y = 0$; about the x-axis.