Lesson 7

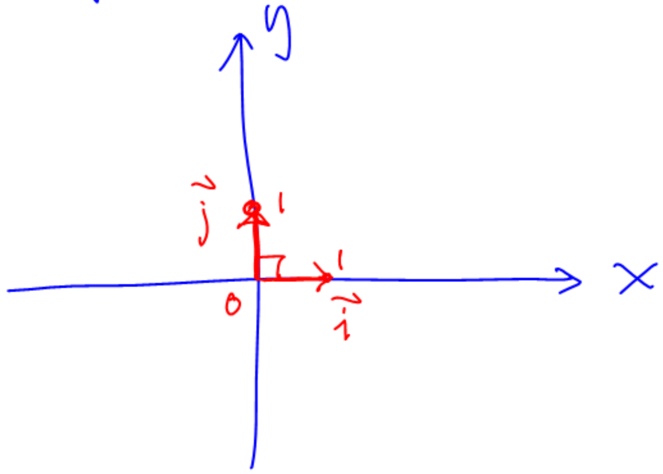
1) Motion along a Curve

$\vec{R}(t) = (x(t), y(t))$, a 2-dimension vector.

$= x(t)\vec{i} + y(t)\vec{j}$

where $\vec{i} = (1, 0)$; $\vec{j} = (0, 1)$ are unit vectors

$|\vec{i}| = \sqrt{1^2 + 0^2} = 1$; $|\vec{j}| = \sqrt{0^2 + 1^2} = 1$



$\vec{R}(t)$ is the position vector.

$\vec{v}(t) = \vec{R}'(t) = (x'(t), y'(t)) = (V_x(t), V_y(t))$

$= x'(t)\vec{i} + y'(t)\vec{j}$

$\vec{v}(t)$ is the velocity vector.

$|\vec{v}(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$ is the speed function.

≥ 0

$\vec{a}(t) = \vec{v}'(t) = \vec{R}''(t) = (x''(t), y''(t)) = (V_x'(t), V_y'(t))$

$\vec{a}(t)$ is the acceleration vector.

$$|\vec{a}(t)| = \sqrt{[x''(t)]^2+[y''(t)]^2}.$$

is the magnitude of acceleration.

For example,

$$\vec{R}(t) = \underline{[3\cos t, \ 2\sin t]}$$

a) $\because \begin{cases} x(t) = 3\cos t \\ y(t) = 2\sin t \end{cases}$ — a parametric function depending on t.

$\therefore \ x^2(t) = 9\cos^2 t \quad \Rightarrow \quad \frac{x^2}{9} = \cos^2 t \quad \cdots ①$

$\quad\ \ y^2(t) = 4\sin^2 t \qquad\quad \frac{y^2}{4} = \sin^2 t \quad \cdots ②$

① + ② : $\frac{x^2}{9} + \frac{y^2}{4} = \cos^2 t + \sin^2 t = 1$

or $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$ is an ellipse.

b) $\vec{V}(t) = \vec{R}'(t) = [(3\cos t)', \ (2\sin t)']$

$= [-3\sin t, \ 2\cos t]$;

$\vec{a}(t) = \vec{V}'(t) = [-3(\sin t)', \ 2(\cos t)']$

$= [-3\cos t, \ -2\sin t] = -\vec{R}(t)$

(c) If $t = \frac{\pi}{6}$,

$\vec{R}(\frac{\pi}{6}) = [3\cos\frac{\pi}{6}, \ 2\sin\frac{\pi}{6}] = [3(\frac{\sqrt{3}}{2}), 2(\frac{1}{2})] = (\frac{3\sqrt{3}}{2}, 1)$

$\vec{V}(\frac{\pi}{6}) = [-3\sin\frac{\pi}{6}, \ 2\cos\frac{\pi}{6}] = [-3(\frac{1}{2}), \ 2(\frac{\sqrt{3}}{2})] = [-\frac{3}{2}, \sqrt{3}]$;

$\vec{a}(\frac{\pi}{6}) = -\vec{R}(\frac{\pi}{6}) = [-\frac{3\sqrt{3}}{2}, -1]$;

If $t=\pi$

$\vec{R}(\pi) = [3\cos\pi, 2\sin\pi] = [-3, 0]$;

$\vec{v}(\pi) = [-3\sin\pi, 2\cos\pi] = [0, -2]$;

$\vec{a}(\pi) = -\vec{R}(\pi) = [+3, 0]$;

(d) speed $= |\vec{v}(t)| = \sqrt{(-3\sin t)^2 + (2\cos t)^2}$

$= \sqrt{9\sin^2 t + 4\cos^2 t}$

$|\vec{v}(\frac{\pi}{6})| = \sqrt{9\sin^2(\frac{\pi}{6}) + 4\cos^2(\frac{\pi}{6})} = \sqrt{9(\frac{1}{2})^2 + 4(\frac{\sqrt{3}}{2})^2} = \sqrt{\frac{9}{4}+3} = \frac{\sqrt{21}}{2}$

$|\vec{v}(\pi)| = \sqrt{9\sin^2(\pi) + 4\cos^2(\pi)} = \sqrt{0+4} = 2$.

$|\vec{a}(t)| = |\vec{R}(t)| = \sqrt{(3\cos t)^2 + (2\sin t)^2} = \sqrt{9\cos^2 t + 4\sin^2 t}$

$|\vec{a}(\frac{\pi}{6})| = \sqrt{9\cos^2(\frac{\pi}{6}) + 4\sin^2(\frac{\pi}{6})} = \sqrt{9(\frac{\sqrt{3}}{2})^2 + 4(\frac{1}{2})^2} = \sqrt{\frac{27}{4}+1}$

$= \frac{\sqrt{31}}{2}$

$|\vec{a}(\pi)| = \sqrt{9\cos^2(\pi) + 4\sin^2(\pi)} = \sqrt{9+0} = 3$;

2) Velocity and Acceleration.

If $y=S(t)$ is a position or displacement function.

Then $v(t) = S'(t)$ is a velocity function.

$|v(t)|$ is the speed;

$a(t) = v'(t) = s''(t)$ is the acceleration function;

If $v(t_0) = 0$, and $a(t) \neq 0$ at t_0 or near t_0, then the motion changes the direction at t_0.

Usually, a horizontal motion has positive velocity if moving to east, and negative velocity if moving to west.

a vertical motion has positive velocity if moving upward, and negative velocity if moving downward;

The speed increases when $v(t) \cdot a(t) > 0$;

and decreases when $v(t) \cdot a(t) < 0$;

3) Related Rates Problems.

In a related rates problem, there are serval variables, say x, y and z.
(could be two variables or more than three variables). Each of the variables is a function of a common independent variable, say t, "the time"; so
$x = f(t)$, $y = g(t)$ and $z = h(t)$.

and x, y and z are related.
Hence their rates $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$.
(or $x'(t)$, $y'(t)$, and $z'(t)$) are also related.

The first step is to define x, y, and z,
The second step is to find out all the relations that relating x, y and z.
The third step is to ~~take~~ derivatives of x, y and z with respect to t,

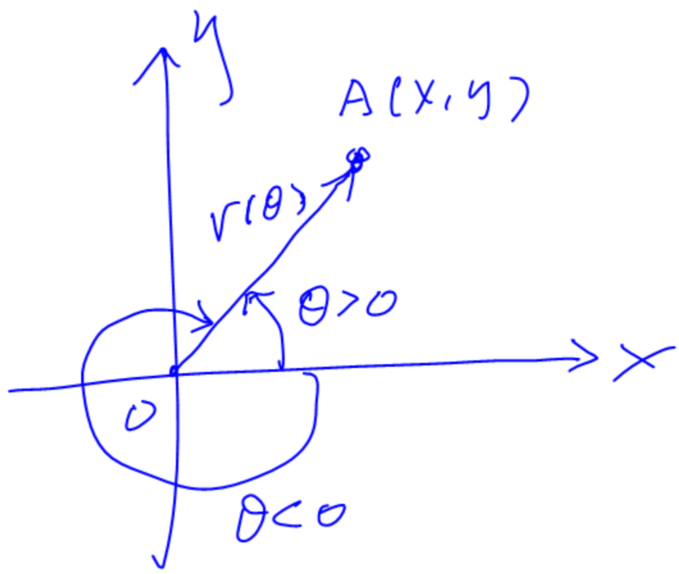
respectively.

$x'(t)$, $y'(t)$ and $z'(t)$ in the relations. So that the rates (or derivatives) are still related to each other.

Finally, We want to find unknown rate(s), based on given rates and other given information or conditions.

4) Slope of a Polar Curve.

$$\begin{cases} x = r(\theta)\cos\theta \\ y = r(\theta)\sin\theta \end{cases}$$



$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{\frac{d}{d\theta}(r(\theta)\sin\theta)}{\frac{d}{d\theta}(r(\theta)\cos\theta)} = \frac{r'(\theta)\sin\theta + r(\theta)\cdot\cos\theta}{r'(\theta)\cos\theta - r(\theta)\cdot\sin\theta}$$

For example, $r(\theta)=3\sin\theta$, $0\le\theta\le 2\pi$;

is a polar curve.