AP Calculus Homework Two – Limit and Continuity

1.4 Other Basic Limits; 1.5 Asymptotes

1. Use the Sandwich Theorem and the fact that $\lim_{x\to 0}(|x|+1)=1$ to prove that $\lim_{x\to 0}(x^2+1)=1$.

$f(x)=x^2+1$, $g(x)=1$, $h(x)=|x|+1$

$g(x)\le f(x)\le h(x)$ for $-\frac{1}{2}\le x\le \frac{1}{2}$; $\lim_{x\to 0} g(x)=\lim_{x\to 0} h(x)=1$;

Hence, $\lim_{x\to 0}(x^2+1)=1$.

2. Find limits.

(a) $\lim_{x\to -\infty}\dfrac{5x^3+27}{20x^2+10x+9}$ $= \lim_{x\to -\infty}\dfrac{5x^3}{20x^2} = \frac{1}{4}\lim_{x\to -\infty} x = -\infty$

(b) $\lim_{x\to \infty}\dfrac{2^{-x}}{2^x}$ $= \lim_{x\to \infty}\dfrac{1}{2^x 2^x} = \lim_{x\to\infty}\dfrac{1}{2^{2x}} = \lim_{x\to\infty}\dfrac{1}{4^x} = \dfrac{1}{4^\infty} = \dfrac{1}{\infty} = 0$

(c) $\lim_{x\to 0}\dfrac{4x^2+3x\sin x}{x^2}$ $= \lim_{x\to 0}\dfrac{4x^2}{x^2} + \lim_{x\to 0}\dfrac{3x\sin x}{x^2} = \lim_{x\to 0} 4 + 3\lim_{x\to 0}\dfrac{\sin x}{x}$

$= 4+3(1) = 7$

(d) $\lim_{t\to 0}\dfrac{1-\cos t}{t^{2/3}}$ $= \lim_{t\to 0}\dfrac{(1-\cos t)(1+\cos t)}{t^{2/3}(1+\cos t)} = \lim_{t\to 0}\dfrac{\sin^2 t}{t^{2/3}(1+\cos t)}$

$= \lim_{t\to 0}\dfrac{t^{1/3}\sin^2 t}{t(1+\cos t)} = \lim_{t\to 0}\dfrac{\sin t}{t}\lim_{t\to 0}\dfrac{t^{1/3}\sin t}{1+\cos t} = (1)\dfrac{0}{1+1} = 0$

1^∞

(e) $\lim_{x\to +\infty}\left(1+\dfrac{2}{x}\right)^x$

$\because \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$; let $\frac{1}{t}=\frac{2}{x} \Rightarrow x=2t$ and $x\to+\infty \Leftrightarrow t\to+\infty$.

$\therefore \lim_{x\to+\infty}\left(1+\frac{2}{x}\right)^x = \lim_{t\to+\infty}\left(1+\frac{1}{t}\right)^{2t}$

$= \left[\lim_{t\to+\infty}\left(1+\frac{1}{t}\right)^t\right]^2 = e^2$

indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty-\infty$, 1^{∞},

$\lim_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x}} = 1$

(f) $\lim_{x\to+\infty}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x+1}} = \lim_{x\to+\infty}\frac{\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x}}}{\frac{\sqrt{x+1}}{\sqrt{x}}} = \lim_{x\to+\infty}\frac{\sqrt{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}}}{\sqrt{1+\frac{1}{x}}}$

$= \frac{\sqrt{1+\sqrt{0+\sqrt{0}}}}{\sqrt{1+0}} = 1$

(g) $\lim_{x\to+\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right) = \lim_{x\to+\infty}\frac{(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x})(\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x})}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$

$= \lim_{x\to+\infty}\frac{\cancel{x}+\sqrt{x+\sqrt{x}}\cancel{-x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}} = \lim_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x}+\sqrt{x}} = \frac{1}{2}$

3. Find a value of k such that $g(x)$ is continuous at $x = 0$.

$$g(x) = \begin{cases} \ln(x+k), & \text{if } 0 < x < 3 \\ \cos(kx), & \text{if } x \le 0 \end{cases}$$

$\lim_{x\to 0} g(x) = g(0)$

$\because g(0) = \cos(k(0)) = \cos(0) = 1$.

and $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-}\cos(kx) = \cos(k(0)) = 1$; $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+}\ln(x+k) = \ln(0+k)$
$= \ln(k) \overset{\text{let}}{=} 1$

$\therefore k = e$.

4. Find all asymptotes for the graph of $f(x) = \frac{2x^2+4}{2+7x-4x^2}$.

$\because \lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\frac{2x^2+4}{2+7x-4x^2} = \lim_{x\to\pm\infty}\frac{2x^2}{-4x^2} = -\frac{1}{2}$, $\therefore y = -\frac{1}{2}$ is a H.A.

$\begin{matrix} 4x & & 1 \\ -x & & +2 \end{matrix}$

$\because 2+7x-4x^2 = (4x+1)(-x+2)$, $\therefore x = -\frac{1}{4}$ and $x = 2$ are two V.A.s

5. Find all vertical and horizontal asymptotes for the graph of $h(x) = \frac{e^{-x}}{x}$. $= \frac{1}{xe^x}$

$\because \lim_{x\to+\infty}\frac{1}{xe^x} = 0$, $\therefore y = 0$ is a H.A.

$\because \lim_{x\to 0}\frac{1}{xe^x} = \pm\infty$, $\therefore x = 0$ is a V.A.

6. For what vales of k will $\lim_{x\to 3}\frac{x-3}{x^2-6x+k}$ exist?

$\because \lim_{x\to 3}\frac{x-3}{x^2-6x+k} = \frac{3-3}{3^2-18+k}$

If $k \neq 9$, $\lim_{x\to 3}\frac{x-3}{x^2-6x+k} = 0$.

If $k = 9$, $\lim_{x\to 3}\frac{x-3}{x^2-6x+9} = \lim_{x\to 3}\frac{x-3}{(x-3)^2} = \lim_{x\to 3}\frac{1}{x-3} = \pm\infty$;

$\therefore k \in \mathbb{R}$ but $k \neq 9$

7. Show that $f(x) = \frac{x^2-5}{x+1}$ has a root between $x = 2$ and $x = 3$.

By the intermediate value theorem, $f(x) = \frac{x^2-5}{x+1}$ is continuous for $2 \le x \le 3$

$f(2) = \frac{2^2-5}{2+1} = -\frac{1}{3}$, $f(3) = \frac{3^2-5}{3+1} = \frac{4}{4} = 1$. let $M = 0$. $f(2) < M < f(3)$

then there exist at least one c, such that $f(c) = M = 0$.

c is the root. between 2 and 3. Actually, $c = \sqrt{5}$