4.1 Anti Derivatives.

Definition:

$F(x)$ is the antiderivative of $f(x)$

if $F'(x) = f(x)$

Properties of Antiderivatives.

1) If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$, then

$G(x) = F(x) + C$, where C is a real number constant.

Notations of antiderivatives:

If $F(x)$ is the antiderivative of $f(x)$,

then $F(x) = \int f(x)dx$

2) If $f(x) = x^n$, a power function.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \longleftarrow$$

$$\because \left(\frac{x^{n+1}}{n+1} + C\right)' = \frac{1}{n+1}(n+1)x^n = x^n.$$

3) $$\int cf(x)\,dx = c\int f(x)\,dx.$$

4) $$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

5) $$\int [f(x)-g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx.$$

6) $$\int a\,dx = ax + C$$

$$\therefore \int a\,dx = a\int x^0\,dx = a\left(\frac{x^{0+1}}{0+1}\right) + C$$
$$= ax + C.$$

Now, we could do antiderivative of a polynomial function. For example.

$$\int [2x^5 - x^4 + 8x^3 + 6x^2 - x + 6]\,dx$$
$$= 2\int x^5dx - \int x^4dx + 8\int x^3dx + 6\int x^2dx - \int x\,dx + 6\int dx$$
$$= 2\left(\frac{x^6}{6}\right) - \frac{x^5}{5} + 8\left(\frac{x^4}{4}\right) + 6\left(\frac{x^3}{3}\right) - \frac{x^2}{2} + 6x + C$$
$$= \frac{1}{3}x^6 - \frac{1}{5}x^5 + 2x^4 + 2x^3 - \frac{1}{2}x^2 + 6x + C$$

7) $$\int \cos x\,dx = \sin x + C.$$

8) $$\int \sin x\,dx = -\cos x + C.$$

9) $$\int \sec^2 x\,dx = \tan x + C.$$

10) $\int \csc^2 x \, dx = -\cot x + C$;

11) $\int \sec x \tan x \, dx = \sec x + C$;

12) $\int \csc x \cot x \, dx = -\csc x + C$;

13) $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

$\because \quad [\ln(\sec x + \tan x) + c]'$

$= \dfrac{1}{\sec x + \tan x} \cdot (\sec x + \tan x)'$

$= \dfrac{1}{\sec x + \tan x} (\sec x \tan x + \sec^2 x)$

$= \dfrac{\sec x\,(\cancel{\tan x + \sec x})}{\sec x + \cancel{\tan x}} = \sec x.$

14) $\int \csc x \, dx = \ln|\csc x - \tan x| + C$.

15) $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$;

16) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$;

17) $\int e^x\,dx = e^x + C$;

18) $\int b^x\,dx = \frac{b^x}{\ln b} + C$;

19) $\int \frac{1}{x}\,dx = \ln|x| + C$;

$\because \quad \ln|x| = \begin{cases} \ln(-x) & \text{if } x<0. \\ \ln x & \text{if } x>0. \end{cases}$

and $[\ln(-x)]' = \frac{1}{-x}(-x)' = \frac{1}{-x}(-1) = \frac{1}{x}$;

$\therefore \quad (\ln|x|)' = \frac{1}{x}$.

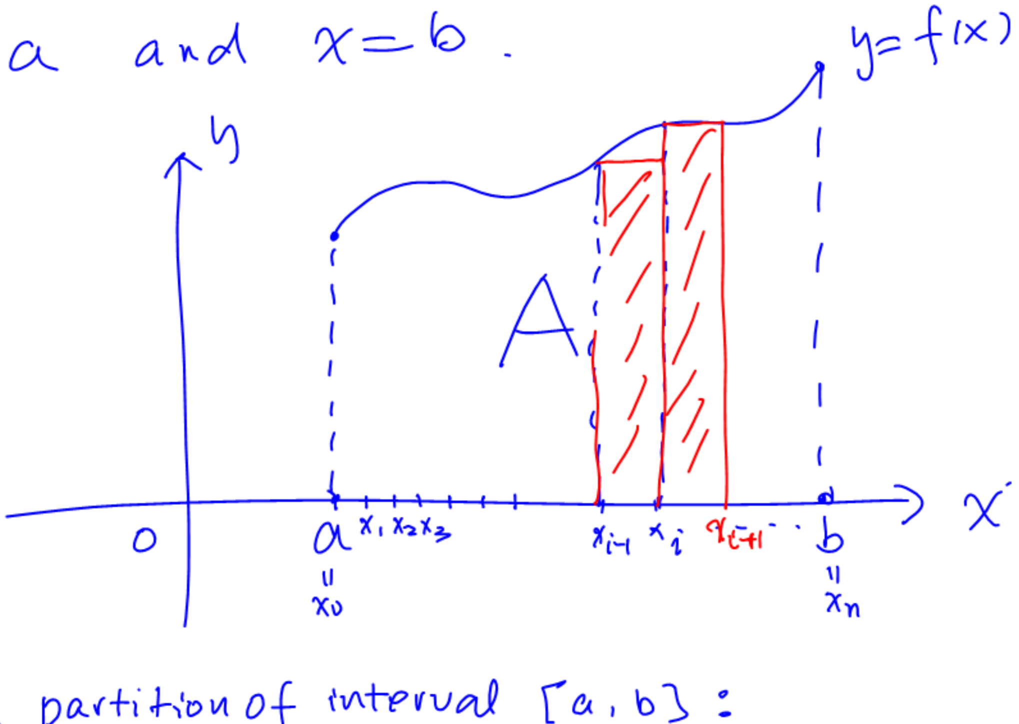
Two common skills for antiderivatives:

① Integration by parts.

② Variable substitutions.

4.2. Area — Definite Integral.

We want to find the area of a region bounded by a curve $y=f(x)$, the x-axis, $x=a$ and $x=b$.



Make a partition of interval $[a,b]$:

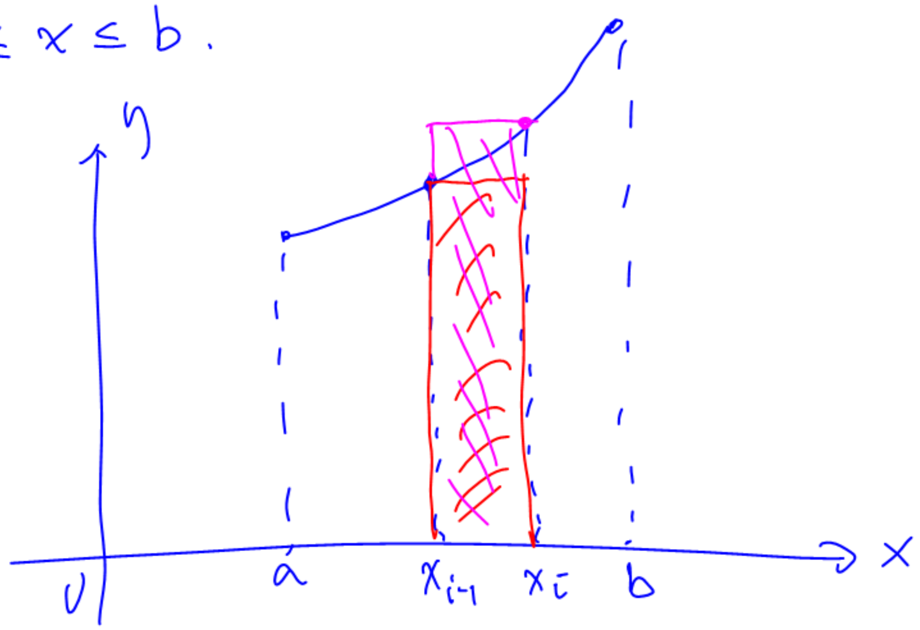
$$x_0=a < x_1 < x_2 < \cdots < x_{i-1} < x_i < \cdots < x_n = b.$$

$$A \approx f(w_1)(x_1-x_0)+f(w_2)(x_2-x_1)+\cdots+f(w_i)(x_i-x_{i-1})+\cdots+f(w_n)(x_n-x_{n-1});$$

$$A \approx f(w_1)\Delta x_1+f(w_2)\Delta x_2+\cdots+f(w_i)\Delta x_i+\cdots+f(w_n)\Delta x_n.$$

where $\Delta x_i = x_i - x_{i-1}$, $w_i \in [x_{i-1}, x_i]$;

If $y=f(x)$ is strictly increasing over $a \le x \le b$.



If $w_i = x_{i-1}$; $\Delta A_i = f(x_{i-1}) \cdot \Delta x_i$; inscribed rectangle.

or if $w_i = x_i$; $\Delta A_i = f(x_i) \Delta x_i$; circumscribed rectangle.

If let $\|P\| = \max\{\Delta x_1, \Delta x_2, \cdots, \Delta x_n\}$;

and $\|P\| \to 0$;

Then $A = \lim\limits_{\|P\| \to 0} \sum\limits_{i=1}^{n} \Delta A_i = \lim\limits_{\|P\| \to 0} \sum\limits_{i=1}^{n} f(w_i) \Delta x_i$

$= \lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} f(w_i) \left(\frac{b-a}{n}\right)$

Since $\Delta x_i = \frac{b-a}{n}$

For example. find the area of the region bounded by $f(x) = x^3$, the x-axis. $x=0$ and $x=b$.

Sol. let the partition of $[0, b]$ as follows:

$$x_0 = 0,\ x_1 = 0+\frac{b}{n},\ x_2 = 0+\frac{2b}{n},\ \cdots\ \underline{x_i = 0+\frac{ib}{n}};$$

$$x_n = 0+\frac{nb}{n} = b; \quad \text{where } \Delta x_i = \frac{b}{n}, \quad i=1,2,3,\cdots,n.$$

$$\Delta A_i = f(w_i)\Delta x_i = (w_i)^3\frac{b}{n} = (x_i)^3\frac{b}{n}.$$

$$= \left(\frac{ib}{n}\right)^3\frac{b}{n}. \qquad \text{where } w_i = x_i$$

$$A = \lim_{n\to\infty}\sum_{i=1}^{n}\Delta A_i = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{ib}{n}\right)^3\frac{b}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{b^4}{n^4}i^3$$

$$= (b^4)\lim_{n\to\infty}\frac{1}{n^4}\underbrace{\sum_{i=1}^{n} i^3} = (b^4)\lim_{n\to\infty}\underbrace{\left(\frac{n^2(1+n)^2}{4}\right)}\frac{1}{n^4}$$

$$= \frac{b^4}{4}\lim_{n\to\infty}\frac{n^2(1+n)^2}{n^4} = \frac{b^4}{4}(1) = \frac{b^4}{4}.$$