AP Calculus Homework Three – Differentiation

2.1 Definition of Derivative; 2.2 Differentiation Rules

1. Find $\frac{dy}{dx}$

(a) $y = x^5 \tan x$

$y' = (x^5)' \tan x + x^5 (\tan x)' = 5x^4 \tan x + x^5 \sec^2 x$
$= x^4 (5\tan x + x\sec^2 x)$

(b) $y = \sqrt{3-2x}$

$y' = \frac{1}{2}(3-2x)^{-\frac{1}{2}}(3-2x)' = \frac{1}{2}(3-2x)^{-\frac{1}{2}}(-2) = -\frac{\sqrt{3-2x}}{3-2x}$

(c) $y = \frac{2}{(5x+1)^3}$, $y' = \frac{-2[((5x+1)^3)']}{(5x+1)^6} = \frac{-2[3\cancel{(5x+1)^2}(5x+1)']}{(5x+1)^{\cancel{6}\,4}} = \frac{-30}{(5x+1)^4}$

(d) $y = 3x^{2/3} - 4x^{1/2} - 2$, $y' = 3(x^{2/3})' - 4(x^{\frac{1}{2}})' - (2)' = 3(\frac{2}{3})x^{-\frac{1}{3}} - 4(\frac{1}{2})x^{-\frac{1}{2}}$
$= 2x^{-\frac{1}{3}} - 2x^{-\frac{1}{2}}$
$= 2(x^{-\frac{1}{3}} - x^{-\frac{1}{2}})$

(e) $y = \frac{x^2}{\cos x}$, $y' = \frac{(x^2)'\cos x - x^2(\cos x)'}{\cos^2 x} = \frac{2x\cos x + x^2 \sin x}{\cos^2 x}$
$= \frac{x(2\cos x + x\sin x)}{\cos^2 x}$

$(\tan^{-1}x)' = \frac{1}{1+x^2}$

(f) $y = \ln\frac{e^x}{e^x - 1}$, $y' = \frac{1}{\frac{e^x}{e^x-1}}\left(\frac{e^x}{e^x-1}\right)' = \frac{e^x-1}{e^x}\left(\frac{(e^x)'(e^x-1) - e^x(e^x-1)'}{(e^x-1)^2}\right)$
$= \frac{\cancel{e^x-1}}{\cancel{e^x}}\left(\frac{\cancel{e^x}(e^x-1) - \cancel{e^x}\cdot e^x}{(e^x-1)^{\cancel{2}}}\right) = \frac{1}{1-e^x}$

(g) $y = \tan^{-1}\frac{x}{2}$, $y' = \frac{1}{1+(\frac{x}{2})^2}\left(\frac{x}{2}\right)' = \frac{\frac{1}{2}}{1+\frac{x^2}{4}} = \frac{2}{4+x^2}$

(h) $y = \ln(\sec x + \tan x)$, $y' = \frac{1}{\sec x + \tan x}(\sec x + \tan x)' = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x$

(i) $y = \sin\left(\frac{1}{x}\right)$, $y' = \cos(\frac{1}{x})\cdot(\frac{1}{x})' = -\frac{1}{x^2}\cos(\frac{1}{x})$

(j) $y = e^{-x}\cos 2x$, $y' = (e^{-x})'\cos 2x + e^{-x}(\cos 2x)' = -e^{-x}\cos 2x + e^{-x}(-2\sin 2x)$
$= -e^{-x}(\cos 2x + 2\sin 2x)$

(k) $y = \sec^2(x)$, $y' = 2\sec x\cdot(\sec x)' = 2\sec x\cdot\sec x\cdot\tan x = 2\sec^2 x\cdot\tan x$

$(\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}}$

(l) $y = \sin^{-1}x - \sqrt{1-x^2}$, $y' = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{1-x^2}}(1-x^2)' = \frac{1+x}{\sqrt{1-x^2}}$
$= \frac{(1+x)\sqrt{1-x^2}}{1-x^2} = \frac{\sqrt{1-x^2}}{1-x}$

2. Find limits. where $f(x)=\sqrt[3]{x}$, $\therefore f'(x)=\frac{1}{3}x^{-2/3}$

(a) $\lim_{h\to 0}\frac{\sqrt[3]{8+h}-2}{h} = f'(8)=\frac{1}{12}$; $f'(8)=\frac{1}{3}(8)^{-\frac{2}{3}}=\frac{1}{3}(2)^{-2}=\frac{1}{12}$

(b) $\lim_{h\to 0}\frac{\ln(e+h)-1}{h} = f'(e)=\frac{1}{e}$; where $f(x)=\ln x$, $f'(x)=\frac{1}{x}$
$f'(e)=\frac{1}{e}$;

$\because \lim_{x\to 0}\frac{\sin ax}{ax}=1$

(c) $\lim_{x\to 0}\frac{\cos x-1}{x} = \lim_{x\to 0}\frac{[\cos x-1][\cos x+1]}{x[\cos x+1]} = \lim_{x\to 0}\frac{-\sin^2 x}{x(\cos x+1)} = -\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{\sin x}{\cos x+1}$
$=(-1)(1)\left(\frac{0}{1+1}\right)=0$

(d) $\lim_{x\to 0}\frac{\sin 3x}{\sin 4x} = \lim_{x\to 0}\frac{\frac{3\sin 3x}{3x}}{\frac{4\sin 4x}{4x}} = \frac{3}{4}\frac{\lim_{x\to 0}\frac{\sin 3x}{3x}}{\lim_{x\to 0}\frac{\sin 4x}{4x}} = \frac{3}{4}\left(\frac{1}{1}\right)=\frac{3}{4}$

$\lim_{x\to 0}\frac{\sin x}{x}=1$

(e) $\lim_{x\to 0}\frac{\tan \pi x}{x} = \lim_{x\to 0}\frac{\sin \pi x}{x}\cdot\frac{1}{\cos \pi x} = \pi\lim_{x\to 0}\frac{\sin \pi x}{\pi x}\lim_{x\to 0}\frac{1}{\cos \pi x} = \pi(1)\left(\frac{1}{1}\right)=\pi$

$\Leftrightarrow \lim_{x\to \infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}=1$

(f) $\lim_{x\to\infty} x^2\sin\frac{1}{x} = \lim_{x\to\infty} x\frac{\sin\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} x\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}} = \infty\cdot(1)=\infty$

3. At how many points on the interval [–5, 5] is a tangent to the curve of $y=x+\cos x = f(x)$ parallel to the secant line that passes the two endpoints of the curve?

$m_{secant}=\frac{f(5)-f(-5)}{5-(-5)}=\frac{5+\cancel{\cos(5)}-[(-5)+\cancel{\cos(-5)}]}{10}=\frac{10}{10}=1$; $y'=(x+\cos x)'=1-\sin x \overset{let}{=} 1$

$\Rightarrow \sin x=0 \Rightarrow x=-\pi, 0, \pi$ $\therefore$ three points.

$\pi\simeq 3.14$
$2\pi\approx 6.28$

4. If f is differentiable and $\boxed{\text{difference quotients}}$ overestimate the slope of f at $x = a$ for all $h > 0$, which must be true?
(A) $f'(a)>0$ (B) $f'(a)<0$ (C) $f''(a)>0$ (E) $f''(a)<0$ (D) none of these

$m_s=\frac{f(x)-f(a)}{x-a}$

5. If $f(u)=\sin u$ and $u=g(x)=x^2-9$, find $(f\circ g)'(3)$.

$\because f\circ g(x)=\sin(x^2-9)$, $\therefore (f\circ g)'(x)=\cos(x^2-9)\cdot(x^2-9)'=2x\cos(x^2-9)$
$\Rightarrow \therefore (f\circ g)'(3)=2(3)\cos(3^2-9)=6$

6. If $f(x)=\frac{x}{(x-1)^2}$, find the set of x's for which $f'(x)$ exists.

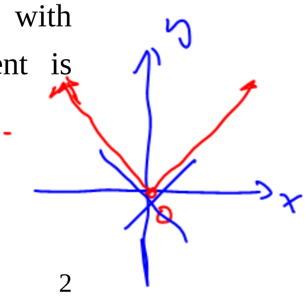
$f'(x)=\frac{x'(x-1)^2-x((x-1)^2)'}{(x-1)^4}=\frac{(x-1)^{\cancel{2}}-x(2)\cancel{(x-1)}}{(x-1)^{\cancel{4}\,3}}=\frac{-1-x}{(x-1)^3}$, $\therefore x\in R$ and $x\neq 1$.

7. If $y=\sqrt{x^2+1}$, find the derivative of y^2 with respect to x^2.

$\because y^2=x^2+1$, $\therefore (y^2)'_{x^2}=1$

8. Find the value of $f'(0)$ obtained using the symmetric difference quotient with $f(x)=|x|$ and h = 0.001. (the formula of symmetric difference quotient is $\frac{1}{2}\left[\frac{f(a+h)-f(a)}{h}+\frac{f(a)-f(a-h)}{h}\right]$)

$f'(0)$ D.N.E.



$f'(0)\approx\frac{1}{2}\left[\frac{f(0+h)-\cancel{f(0)}}{h}+\frac{\cancel{f(0)}-f(0-h)}{h}\right]=\frac{1}{2}\left[\frac{f(0.001)-f(-0.001)}{0.001}\right]$

$=\frac{1}{2}\left[\frac{|0.001|-|-0.001|}{0.001}\right]=\frac{1}{2}\left(\frac{0}{0.001}\right)=0$;

$\therefore f(x)=|x|$ is not differentiable at $x=0$.