CHAPTER 5

Combinations and the Binomial Theorem

Specific Expectations	Section
Use Venn diagrams as a tool for organizing information in counting problems.	5.1
Solve introductory counting problems involving the additive and multiplicative counting principles.	5.1, 5.2, 5.3
Express answers to permutation and combination problems, using standard combinatorial symbols.	5.1, 5.2, 5.3
Evaluate expressions involving factorial notation, using appropriate methods.	5.2, 5.3
Solve problems, using techniques for counting combinations.	5.2, 5.3
Identify patterns in Pascal's triangle and relate the terms of Pascal's triangle to values of $\binom{n}{r}$, to the expansion of a binomial, and to the solution of related problems.	5.4
Communicate clearly, coherently, and precisely the solutions to counting problems.	5.1, 5.2, 5.3, 5.4

Chapter Problem

Radio Programming

Jeffrey works as a DJ at a local radio station. He does the drive shift from 16 00 to 20 00, Monday to Friday. Before going on the air, he must choose the music he will play during these four hours.

The station has a few rules that Jeffrey must follow, but he is allowed quite a bit of leeway. Jeffrey must choose all his music from the top 100 songs for the week and he must play at least 12 songs an hour. In his first hour, all his choices must be from the top-20 list.

1. In how many ways can Jeffrey choose the music for his first hour?
2. In how many ways can he program the second hour if he chooses at least 10 songs that are in positions 15 to 40 on the charts?
3. Over his 4-h shift, he will play at least 48 songs from the top 100. In how many ways can he choose these songs?

In these questions, Jeffrey can play the songs in any order. Such questions can be answered with the help of combinatorics, the branch of mathematics introduced in Chapter 4. However, the permutations in Chapter 4 dealt with situations where the order of items was important. Now, you will learn techniques you can apply in situations where order is not important.

Review of Prerequisite Skills

If you need help with any of the skills listed in purple below, refer to Appendix A.

1. Factorials (section 4.2) Evaluate.

a) $8!$ b) $\frac{8!}{5!}$
c) $\frac{24!}{22!}$ d) $3! \times 4!$

2. Permutations (section 4.2) Evaluate mentally.

a) ${}_5P_5$ b) ${}_{10}P_2$
c) ${}_{12}P_1$ d) ${}_7P_3$

3. Permutations (section 4.2) Evaluate manually.

a) ${}_{10}P_5$ b) $P(16, 2)$
c) ${}_{10}P_{10}$ d) $P(8, 5)$

4. Permutations (section 4.2) Evaluate using software or a calculator.

a) ${}_{50}P_{25}$ b) $P(37, 16)$
c) ${}_{29}P_{29}$ d) ${}_{46}P_{23}$

5. Organized counting (section 4.1) Every Canadian aircraft has five letters in its registration. The first letter must be C, the second letter must be F or G, and the last three letters have no restrictions. If repeated letters are allowed, how many aircraft can be registered with this system?

6. Applying permutations (Chapter 4)

a) How many arrangements are there of three different letters from the word *kings*?
b) How many arrangements are there of all the letters of the word *management*?
c) How many ways could first, second, and third prizes be awarded to 12 entrants in a mathematics contest?

7. Exponent laws Use the exponent laws to simplify each of the following.

a) $(-3y)^0$ b) $(-4x)^3$
c) $15(7x)^4(4y)^2$ d) $21(x^3)^2\left(\frac{1}{x^2}\right)^5$
e) $(4x^0y)^2(3x^2y)^3$ f) $\left(\frac{1}{2}\right)^4(3x^2)(2y)^3$
g) $(-3xy)(-5x^2y)^2$ h) $\left(\frac{1}{3}\right)^0(-2xy)^3$

8. Simplifying expressions Expand and simplify.

a) $(x-5)^2$ b) $(5x-y)^2$
c) $(x^2+5)^2$ d) $(x+3)(x-5)^2$
e) $(x^2-y)^2$ f) $(2x+3)^2$
g) $(x-4)^2(x-2)$ h) $(2x^2+3y)^2$
i) $(2x+1)^2(x-2)$ j) $(x+y)(x-2y)^2$

9. Sigma notation Rewrite the following using sigma notation.

a) $1 + 2 + 4 + 8 + 16$
b) $x + 2x^2 + 3x^3 + 4x^4 + 5x^5$
c) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$

10. Sigma notation Expand.

a) $\sum_{n=2}^{5} 2n$
b) $\sum_{n=1}^{4} \frac{x^n}{n!}$
c) $\sum_{n=1}^{5} (2^n + n^2)$

5.1 Organized Counting With Venn Diagrams

In Chapter 4, you used tree diagrams as a tool for counting items when the order of the items was important. This section introduces a type of diagram that helps you organize data about groups of items when the order of the items is not important.

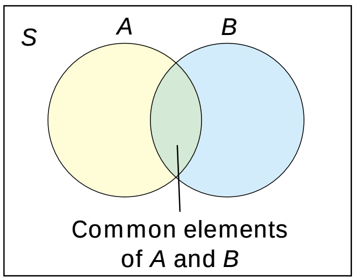
INVESTIGATE & INQUIRE: Visualizing Relationships Between Groups

A group of students meet regularly to plan the dances at Vennville High School. Amar, Belinda, Charles, and Danica are on the dance committee, and Belinda, Charles, Edith, Franco, and Geoff are on the students' council. Hans and Irena are not members of either group, but they attend meetings as reporters for the school newspaper.

1. Draw two circles to represent the dance committee and the students' council. Where on the diagram would you put initials representing the students who are
   a) on the dance committee?
   b) on the students' council?
   c) on the dance committee and the students' council?
   d) not on either the dance committee or the students' council?
2. Redraw your diagram marking on it the number of initials in each region. What relationships can you see among these numbers?

Your sketch representing the dance committee and the students' council is a simple example of a **Venn diagram**. The English logician John Venn (1834–1923) introduced such diagrams as a tool for analysing situations where there is some overlap among groups of items, or **sets**. Circles represent different sets and a rectangular box around the circles represents the **universal set**, ***S***, from which all the items are drawn. This box is usually labelled with an *S* in the top left corner.

The items in a set are often called the **elements** or **members** of the set. The size of a circle in a Venn diagram does not have to be proportional to the number of elements in the set the circle represents. When some items in a set are also elements of another set, these items are **common elements** and the sets are shown as overlapping circles. If *all* elements of a set C are also elements of set A, then C is a **subset** of A. A Venn diagram would show this set C as a region contained within the circle for set A.



The common elements are a subset of both A and B.

WEB CONNECTION

www.mcgrawhill.ca/links/MDM12

To learn more about Venn diagrams, visit the above web site and follow the links. Describe an example of how Venn diagrams can be used to organize information.

You can use Venn diagrams to organize information for situations in which the number of items in a group are important but the order of the items is not.

Example 1 Common Elements

There are 10 students on the volleyball team and 15 on the basketball team. When planning a field trip with both teams, the coach has to arrange transportation for a total of only 19 students.

a) Use a Venn diagram to illustrate this situation.

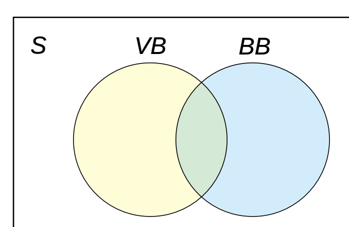
b) Explain why you cannot use the additive counting principle to find the total number of students on the teams.

c) Determine how many students are on both teams.

d) Determine the number of students in the remaining regions of your diagram and explain what these regions represent.

Solution

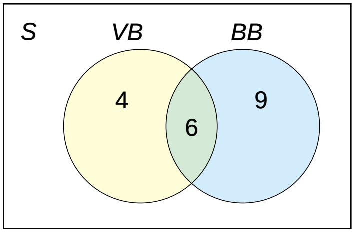
a) Some students must be on both the volleyball and the basketball team. Draw a box with an S in the top left-hand corner. Draw and label two overlapping circles to represent the volleyball and basketball teams.



b) The additive counting principle (or rule of sum) applies only to mutually exclusive events or items. However, it is possible for students to be on both teams. If you simply add the 10 students on the volleyball team to 15 students on the basketball team, you get a total of 25 students because the students who play on both teams have been counted twice.

c) The difference between the total in part b) and the total number of students actually on the two teams is equal to the number of students who are members of both teams. Thus, $25 - 19 = 6$ students play on both teams. In the Venn diagram, these 6 students are represented by the area where the two circles overlap.

d) There are $10 - 6 = 4$ students in the section of the *VB* circle that does not overlap with the *BB* circle. These are the students who play only on the volleyball team. Similarly, the non-overlapping portion of the *BB* circle represents the $15 - 6 = 9$ students who play only on the basketball team.



Example 1 illustrates the **principle of inclusion and exclusion**. If you are counting the total number of elements in two groups or sets that have common elements, you must subtract the common elements so that they are not included twice.

Principle of Inclusion and Exclusion for Two Sets

For sets A and B, the total number of elements in either A or B is the number in A plus the number in B minus the number in both A and B.

$n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)$,
where $n(X)$ represents the numbers of elements in a set X.

The set of all elements in either set A or set B is the **union** of A and B, which is often written as $A \cup B$. Similarly, the set of all elements in both A and B is the **intersection** of A and B, written as $A \cap B$. Thus the principle of inclusion and exclusion for two sets can also be stated as

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Note that the additive counting principle (or rule of sum) could be considered a special case of the principle of inclusion and exclusion that applies only when sets A and B have no elements in common, so that $n(A \text{ and } B) = 0$. The principle of inclusion and exclusion can also be applied to three or more sets.

Example 2 Applying the Principle of Inclusion and Exclusion

A drama club is putting on two one-act plays. There are 11 actors in the Feydeau farce and 7 in the Molière piece.

a) If 3 actors are in both plays, how many actors are there in all?

b) Use a Venn diagram to calculate how many students are in only one of the two plays.

Solution

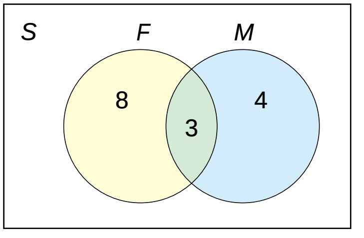
a) Calculate the number of students in both plays using the principle of inclusion and exclusion.

$$\begin{aligned} n(\text{total}) &= n(\text{Feydeau}) + n(\text{Molière}) - n(\text{Feydeau and Molière}) \\ &= 11 + 7 - 3 \\ &= 15 \end{aligned}$$

There are 15 students involved in the two one-act plays.

b) There are 3 students in the overlap between the two circles. So, there must be $11 - 3 = 8$ students in the region for Feydeau only and $7 - 3 = 4$ students in the region for Molière only.

Thus, a total of $8 + 4 = 12$ students are in only one of the two plays.



As in the first example, using a Venn diagram can clarify the relationships between several sets and subsets.

Example 3 Working With Three Sets

Of the 140 grade 12 students at Vennville High School, 52 have signed up for biology, 71 for chemistry, and 40 for physics. The science students include 15 who are taking both biology and chemistry, 8 who are taking chemistry and physics, 11 who are taking biology and physics, and 2 who are taking all three science courses.

a) How many students are not taking any of these three science courses?

b) Illustrate the enrolments with a Venn diagram.

Solution

a) Extend the principle of inclusion and exclusion to three sets. Total the numbers of students in each course, subtract the numbers of students taking two courses, then add the number taking all three. This procedure subtracts out the students who have been counted twice because they are in two courses, and then adds back those who were subtracted twice because they were in all three courses.

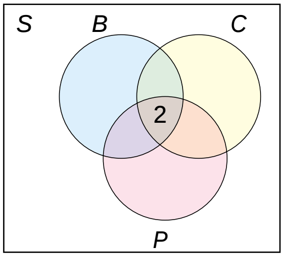
For simplicity, let B stand for biology, C stand for chemistry, and P stand for physics. Then, the total number of students taking at least one of these three courses is

$$\begin{aligned} n(\text{total}) &= n(B) + n(C) + n(P) - n(B \text{ and } C) - n(C \text{ and } P) - n(B \text{ and } P) + n(B \text{ and } C \text{ and } P) \\ &= 52 + 71 + 40 - 15 - 8 - 11 + 2 \\ &= 131 \end{aligned}$$

There are 131 students taking one or more of the three science courses. To find the number of grade 12 students who are not taking any of these science courses, subtract 131 from the total number of grade 12 students.

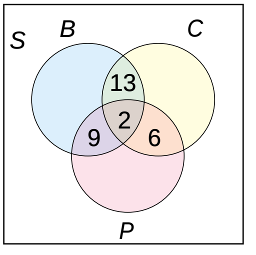
Thus, $140 - 131 = 9$ students are not taking any of these three science courses in grade 12.

b) For this example, it is easiest to start with the overlap among the three courses and then work outward. Since there are 2 students taking all three courses, mark 2 in the centre of the diagram where the three circles overlap.



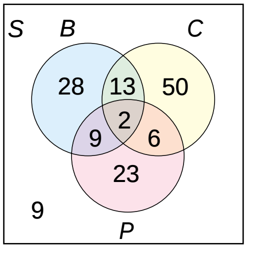
Next, consider the adjacent regions representing the students who are taking exactly two of the three courses.

Biology and chemistry: Of the 15 students taking these two courses, 2 are also taking physics, so 13 students are taking only biology and chemistry.
Chemistry and physics: 8 students less the 2 in the centre region leaves 6.
Biology and physics: $11 - 2 = 9$.



Now, consider the regions representing students taking only one of the science courses.

Biology: Of the 52 students taking this course, $13 + 2 + 9 = 24$ are in the regions overlapping with the other two courses, leaving 28 students who are taking biology only.
Chemistry: 71 students less the $13 + 2 + 6$ leaves 50.
Physics: $40 - (9 + 2 + 6) = 23$.



Adding all the numbers within the circles gives a total of 131. Thus, there must be $140 - 131 = 9$ grade 12 students who are not taking any of the three science courses, which agrees with the answer found in part a).

Key Concepts

- Venn diagrams can help you visualize the relationships between sets of items, especially when the sets have some items in common.
- The principle of inclusion and exclusion gives a formula for finding the number of items in the union of two or more sets. For two sets, the formula is $n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)$.

Communicate Your Understanding

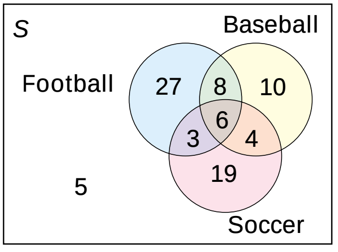
1. Describe the principal use of Venn diagrams.
2. Is the universal set the same for all Venn diagrams? Explain why or why not.
3. Explain why the additive counting principle can be used in place of the principle of inclusion and exclusion for mutually exclusive sets.

Practise

A

1. Let set A consist of an apple, an orange, and a pear and set B consist of the apple and a banana.
   a) List the elements of
      i) A and B
      ii) A or B
      iii) S
      iv) $S \cap B$
      v) $A \cup B \cup S$
   b) List the value of
      i) $n(A) + n(B)$
      ii) $n(A \text{ or } B)$
      iii) $n(S)$
      iv) $n(A \cup B)$
      v) $n(S \cap A)$
   c) List all subsets containing exactly two elements for
      i) A
      ii) B
      iii) $A \cup B$

2. A recent survey of a group of students found that many participate in baseball, football, and soccer. The Venn diagram below shows the results of the survey.



a) How many students participated in the survey?

b) How many of these students play both soccer and baseball?

c) How many play only one sport?

d) How many play football and soccer?

e) How many play all three sports?

f) How many do not play soccer?

Apply, Solve, Communicate

B

3. Of the 220 graduating students in a school, 110 attended the semi-formal dance and 150 attended the formal dance. If 58 students attended both events, how many graduating students did not attend either of the two dances? Illustrate your answer with a Venn diagram.

4. **Application** A survey of 1000 television viewers conducted by a local television station produced the following data:

- 40% watch the news at 12 00
- 60% watch the news at 18 00
- 50% watch the news at 23 00
- 25% watch the news at 12 00 and at 18 00
- 20% watch the news at 12 00 and at 23 00
- 20% watch the news at 18 00 and at 23 00
- 10% watch all three news broadcasts

a) What percent of those surveyed watch at least one of these programs?

b) What percent watch none of these news broadcasts?

c) What percent view the news at 12 00 and at 18 00, but not at 23 00?

d) What percent view only one of these shows?

e) What percent view exactly two of these shows?

5. Suppose the Canadian Embassy in the Netherlands has 32 employees, all of whom speak both French and English. In addition, 22 of the employees speak German and 15 speak Dutch. If there are 10 who speak both German and Dutch, how many of the employees speak neither German nor Dutch? Illustrate your answer with a Venn diagram.

6. **Application** There are 900 employees at CantoCrafts Inc. Of these, 615 are female, 345 are under 35 years old, 482 are single, 295 are single females, 187 are singles under 35 years old, 190 are females under 35 years old, and 120 are single females under 35 years old. Use a Venn diagram to determine how many employees are married males who are at least 35 years old.

7. **Communication** A survey of 100 people who volunteered information about their reading habits showed that

- 75 read newspapers daily
- 35 read books at least once a week
- 45 read magazines regularly
- 25 read both newspapers and books
- 15 read both books and magazines
- 10 read newspapers, books, and magazines

a) Construct a Venn diagram to determine the maximum number of people in the survey who read both newspapers and magazines.

b) Explain why you cannot determine exactly how many of the people surveyed read both newspapers and magazines.

8. Jeffrey works as a DJ at a local radio station. On occasion, he chooses some of the songs he will play based on the phone-in requests received by the switchboard the previous day. Jeffrey's list of 200 possible selections includes

- all the songs in the top 100
- 134 hard-rock songs
- 50 phone-in requests
- 45 hard-rock songs in the top 100
- 20 phone-in requests in the top 100
- 24 phone-in requests for hard-rock songs

Use a Venn diagram to determine

a) how many phone-in requests were for hard-rock songs in the top 100

b) how many of the songs in the top 100 were neither phone-in requests nor hard-rock selections

C

9. Inquiry/Problem Solving The Vennville junior hockey team has 12 members who can play forward, 8 who can play defence, and 2 who can be goalies. What is the smallest possible size of the team if

a) no one plays more than one position?

b) no one plays both defence and forward?

c) three of the players are able to play defence or forward?

d) both the goalies can play forward but not defence?

10. Inquiry/Problem Solving Use the principle of inclusion and exclusion to develop a formula for the number of elements in

a) three sets **b)** four sets **c)** n sets

Career Connection

Forensic Scientist

The field of forensic science could be attractive to those with a mathematics and science background. The job of a forensic scientist is to identify, analyse, and match items collected from crime scenes.

Forensic scientists most often work in a forensic laboratory. Such laboratories examine and analyse physical evidence, including controlled substances, biological materials, firearms and ammunition components, and DNA samples.

Forensic scientists may have specialities such as fingerprints, bullistics, clothing and fibres, footprints, tire tracks, DNA profiling, or crime scene analysis.

Modern forensic science combines mathematics and computers. A forensic scientist should have a background in combinatorics, biology, and the physical sciences. Forensic scientists work for a wide variety of organizations including police forces, government offices, and the military.

WEB CONNECTION

www.mcgrawhill.ca/links/MDM12

For more information about forensic science and other careers related to mathematics, visit the above web site and follow the links. Write a brief description of how combinatorics could be used by forensic scientists.

5.2 Combinations

In Chapter 4, you learned about permutations—arrangements in which the order of the items is specified. However, in many situations, order does not matter. For example, in many card games, what is in your hand is important, but the order in which it was dealt is not.

INVESTIGATE & INQUIRE: Students' Council

Suppose the students at a secondary school elect a council of eight members, two from each grade. This council then chooses two of its members as co-chairpersons. How could you calculate the number of different pairs of members who could be chosen as the co-chairs?

Choose someone in the class to record your answers to the following questions on a blackboard or an overhead projector.

a) Start with the simplest case. Choose two students to stand at the front of the class. In how many ways can you choose two co-chairs from this pair of students?

b) Choose three students to be at the front of the class. In how many ways can you choose two co-chairs from this trio?

c) In how many ways can you choose two co-chairs from a group of four students?

d) In how many ways can you choose two co-chairs from a group of five students? Do you see a pattern developing? If so, what is it? If not, try choosing from a group of six students and then from a group of seven students while continuing to look for a pattern.

e) When you see a pattern, predict the number of ways two co-chairs can be chosen from a group of eight students.

f) Can you suggest how you could find the answers for this investigation from the numbers of permutations you found in the investigation in section 4.2?

In the investigation on the previous page, you were dealing with a situation in which you were selecting two people from a group, but the order in which you chose the two did not matter. In a permutation, there is a difference between selecting, say, Bob as president and Margot as vice-president as opposed to selecting Margot as president and Bob as vice-president. If you select Bob and Margot as co-chairs, the order in which you select them does not matter since they will both have the same job.

A selection from a group of items without regard to order is called a **combination**.

Example 1 Comparing Permutations and Combinations

a) In how many ways could Alana, Barbara, Carl, Domenic, and Edward fill the positions of president, vice-president, and secretary?

b) In how many ways could these same five people form a committee with three members? List the ways.

c) How are the numbers of ways in parts a) and b) related?

Solution

a) Since the positions are different, order is important. Use a permutation, ${}_nP_r$. There are five people to choose from, so $n = 5$. There are three people being chosen, so $r = 3$. The number of permutations is ${}_5P_3 = 60$.

There are 60 ways Alana, Barbara, Carl, Domenic, and Edward could fill the positions of president, vice-president, and secretary.

b) The easiest way to find all committee combinations is to write them in an ordered fashion. Let A represent Alana, B represent Barbara, C represent Carl, D represent Domenic, and E represent Edward.

The possible combinations are:

A B C	A B D	A B E	A C D	A C E
A D E	B C D	B C E	B D E	C D E

All other possible arrangements include the same three people as one of the combinations listed above. For example, ABC is the same as ACB, BAC, BCA, CAB, and CBA since order is not important.

So, there are only ten ways Alana, Barbara, Carl, Domenic and Edward can form a three-person committee.

c) In part a), there were 60 possible permutations, while in part b), there were 10 possible combinations. The difference is a factor of 6. This factor is ${}_3P_3 = 3!$, the number of possible arrangements of the three people in each combination. Thus,

$$\begin{aligned}\text{number of combinations} &= \frac{\text{number of permutations}}{\text{number of permutations of the objects selected}}\\ &= \frac{{}_5P_3}{3!}\\ &= \frac{60}{6}\\ &= 10\end{aligned}$$

Combinations of *n* distinct objects taken *r* at a time

The number of combinations of r objects chosen from a set of n distinct objects is

$$\begin{aligned}{}_nC_r &= \frac{{}_nP_r}{r!}\\ &= \frac{\frac{n!}{(n-r)!}}{r!}\\ &= \frac{n!}{(n-r)!r!}\end{aligned}$$

The notations ${}_nC_r$, $C(n, r)$, and $\binom{n}{r}$ are all equivalent. Many people prefer the form $\binom{n}{r}$ when a number of combinations are multiplied together. The symbol ${}_nC_r$ is used most often in this text since it is what appears on most scientific and graphing calculators.

Example 2 Applying the Combinations Formula

How many different sampler dishes with 3 different flavours could you get at an ice-cream shop with 31 different flavours?

Solution

There are 31 flavours, so $n = 31$. The sampler dish has 3 flavours, so $r = 3$.

$$\begin{aligned}{}_{31}C_3 &= \frac{31!}{(31-3)!\,3}\\ &= \frac{31!}{28!3!}\\ &= \frac{31 \times 30 \times 29}{3 \times 2}\\ &= 4495\end{aligned}$$

There are 4495 possible sampler combinations.

Note that the number of combinations in Example 2 was easy to calculate because the number of items chosen, r, was quite small.

Example 3 Calculating Numbers of Combinations Manually

A ballet choreographer wants 18 dancers for a scene.

a) In how many ways can the choreographer choose the dancers if the company has 20 dancers? 24 dancers?

b) How would you advise the choreographer to choose the dancers?

Solution

a) When n and r are close in value, ${}_nC_r$ can be calculated mentally.
With $n = 20$ and $r = 18$,

$$\begin{aligned} {}_{20}C_{18} &= \frac{20!}{(20-18)!\,18!} \\ &= \frac{20 \times 19}{2!} && 20 \div 2 = 10 \\ &= 190 && \text{Then, } 10 \times 19 = 190 \end{aligned}$$

The choreographer could choose from 190 different combinations of the 20 dancers.

With $n = 24$ and $r = 18$, ${}_nC_r$ can be calculated manually or with a basic calculator once you have divided out the common terms in the factorials.

$$\begin{aligned} {}_{24}C_{18} &= \frac{24!}{(24-18)!\,18!} \\ &= \frac{24 \times 23 \times 22 \times 21 \times 20 \times 19}{6!} \\ &= \frac{24 \times 23 \times 22 \times 21 \times 20 \times 19}{6 \times 5 \times 4 \times 3 \times 2 \times 1} \\ &= 23 \times 11 \times 7 \times 4 \times 19 \\ &= 134\ 596 \end{aligned}$$

With the 4 additional dancers, the choreographer now has a choice of 134 596 combinations.

b) From part a), you can see that it would be impractical for the choreographer to try every possible combination. Instead the choreographer could use an indirect method and try to decide which dancers are least likely to be suitable for the scene.

Even though there are fewer permutations of n objects than there are combinations, the numbers of combinations are often still too large to calculate manually.

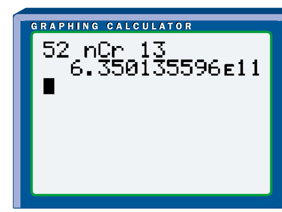
Example 4 Using Technology to Calculate Numbers of Combinations

Each player in a bridge game receives a hand of 13 cards dealt from a standard deck. How many different bridge hands are possible?

For details of calculator and software functions, refer to Appendix B.

Solution 1 Using a Graphing Calculator

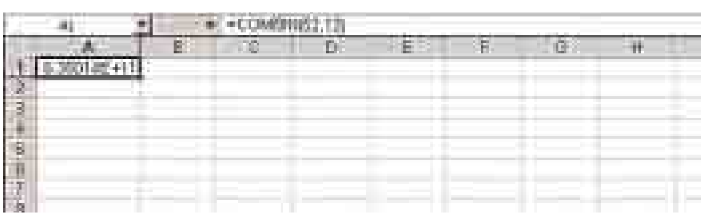
Here, the order in which the player receives the cards does not matter. What you want to determine is the number of different combinations of cards a player could have once the dealing is complete. So, the answer is simply ${}_{52}C_{13}$. You can evaluate ${}_{52}C_{13}$ by using the **nCr function** on the MATH PRB menu of a graphing calculator. This function is similar to the **nPr function** used for permutations.



There are about 635 billion possible bridge hands.

Solution 2 Using a Spreadsheet

Most spreadsheet programs have a **combinations function** for calculating numbers of combinations. In Microsoft® Excel, this function is the COMBIN(n,r) function. In Corel® Quattro® Pro, this function is the COMB(r,n) function.



You now have a variety of methods for finding numbers of combinations—paper-and-pencil calculations, factorials, scientific or graphing calculators, and software. When appropriate, you can also apply both of the counting principles described in Chapter 4.

Project Prep

Techniques for calculating numbers of combinations could be helpful for designing the game in your probability project, especially if your game uses cards.

Example 5 Using the Counting Principles With Combinations

Ursula runs a small landscaping business. She has on hand 12 kinds of rose bushes, 16 kinds of small shrubs, 11 kinds of evergreen seedlings, and 18 kinds of flowering lilies. In how many ways can Ursula fill an order if a customer wants

a) 15 different varieties consisting of 4 roses, 3 shrubs, 2 evergreens, and 6 lilies?

b) either 4 different roses or 6 different lilies?

Solution

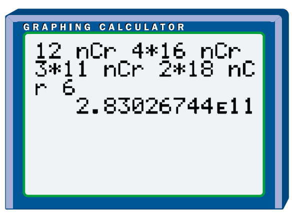
a) The order in which Ursula chooses the plants does not matter.

The number of ways of choosing the roses is ${}_{12}C_4$.
The number of ways of choosing the shrubs is ${}_{16}C_3$.
The number of ways of choosing the evergreens is ${}_{11}C_2$.
The number of ways of choosing the lilies is ${}_{18}C_6$.

Since varying the rose selection for each different selection of the shrubs, evergreens, and lilies produces a different choice of plants, you can apply the fundamental (multiplicative) counting principle. Multiply the series of combinations to find the total number of possibilities.

$$\begin{aligned} {}_{12}C_4 \times {}_{16}C_3 \times {}_{11}C_2 \times {}_{18}C_6 &= 495 \times 560 \times 55 \times 18\,564 \\ &= 2.830\,267\,44 \times 10^{11} \end{aligned}$$

Ursula has over 283 billion ways of choosing the plants for her customer.



b) Ursula can choose the 4 rose bushes in ${}_{12}C_4$ ways.

She can choose the 6 lilies in ${}_{18}C_6$ ways.

Since the customer wants *either* the rose bushes *or* the lilies, you can apply the additive counting principle to find the total number of possibilities.

$$\begin{aligned} {}_{12}C_4 + {}_{18}C_6 &= 495 + 18\,564 \\ &= 19\,059 \end{aligned}$$

Ursula can fill the order for either roses or lilies in 19 059 ways.

As you can see, even relatively simple situations can produce very large numbers of combinations.

Key Concepts

- A combination is a selection of objects in which order is not important.
- The number of combinations of n distinct objects taken r at a time is denoted as ${}_nC_r$, $C(n, r)$, or $\binom{n}{r}$ and is equal to $\dfrac{n!}{(n-r)!\,r!}$.
- The multiplicative and additive counting principles can be applied to problems involving combinations.

Communicate Your Understanding

1. Explain why n objects have more possible permutations than combinations. Use a simple example to illustrate your explanation.
2. Explain whether you would use combinations, permutations, or another method to calculate the number of ways of choosing
   a) three items from a menu of ten items
   b) an appetizer, an entrée, and a dessert from a menu with three appetizers, four entrées, and five desserts
3. Give an example of a combination expression you could calculate
   a) by hand
   b) algebraically
   c) only with a calculator or computer

Practise

A

1. Evaluate using a variety of methods. Explain why you chose each method.
   a) ${}_{21}C_{19}$ b) ${}_{30}C_{28}$
   c) ${}_{18}C_{5}$ d) ${}_{16}C_{3}$
   e) ${}_{19}C_{4}$ f) ${}_{25}C_{20}$
2. Evaluate the following pairs of combinations and compare their values.
   a) ${}_{11}C_{1}$, ${}_{11}C_{10}$
   b) ${}_{11}C_{2}$, ${}_{11}C_{9}$
   c) ${}_{11}C_{3}$, ${}_{11}C_{8}$

Apply, Solve, Communicate

B

3. **Communication** In how many ways could you choose 2 red jellybeans from a package of 15 red jellybeans? Explain your reasoning.
4. How many ways can 4 cards be chosen from a deck of 52, if the order in which they are chosen does not matter?
5. How many groups of 3 toys can a child choose to take on a vacation from a toy box containing 11 toys?
6. How many sets of 6 questions for a test can be chosen from a list of 22 questions?
7. In how many ways can a teacher select 5 students from a class of 23 to make a bulletin-board display? Explain your reasoning.
8. As a promotion, a video store decides to give away posters for recently released movies.
   a) If posters are available for 27 recent releases, in how many ways could the video-store owner choose 8 different posters for the promotion?
   b) Are you able to calculate the number of ways mentally? Why or why not?

9. **Communication** A club has 11 members.
   a) How many different 2-member committees could the club form?
   b) How many different 3-member committees could the club form?
   c) In how many ways can a club president, treasurer, and secretary be chosen?
   d) By what factor do the answers in parts b) and c) differ? How do you account for this difference?

10. Fritz has a deck of 52 cards, and he may choose any number of these cards, from none to all. Use a spreadsheet or Fathom™ to calculate and graph the number of combinations for each of Fritz's choices.

11. **Application** A track club, a swim club, and a cycling club are forming a joint committee to organize a triathlon. The committee will have two members from each club. In how many ways can the committee be formed if ten runners, eight swimmers, and seven cyclists volunteer to serve on it?

12. In how many ways can a jury of 6 men and 6 women be chosen from a group of 10 men and 15 women?

13. **Inquiry/Problem Solving** There are 15 technicians and 11 chemists working in a research laboratory. In how many ways could they form a 5-member safety committee if the committee
   a) may be chosen in any way?
   b) must have exactly one technician?
   c) must have exactly one chemist?
   d) must have exactly two chemists?
   e) may be all technicians or all chemists?

Chapter Problem

14. Jeffrey, a DJ at a local radio station, is choosing the music he will play on his shift. He must choose all his music from the top 100 songs for the week and he must play at least 12 songs an hour. In his first hour, all his choices must be from the top-20 list.
   a) In how many ways can Jeffrey choose the songs for his first hour if he wants to choose exactly 12 songs?
   b) In how many ways can Jeffrey choose the 12 songs if he wants to pick 8 of the top 10 and 4 from the songs listed from 11 to 20 on the chart?
   c) In how many ways can Jeffrey choose either 12 or 13 songs to play in the first hour of his shift?
   d) In how many ways can Jeffrey choose the songs if he wants to play up to 15 songs in the first hour?

15. The game of euchre uses only 24 of the cards from a standard deck. How many different five-card euchre hands are possible?

16. **Application** A taxi is shuttling 11 students to a concert. The taxi can hold only 4 students. In how many ways can 4 students be chosen for
   a) the taxi's first trip?
   b) the taxi's second trip?

17. Diane is making a quilt. She needs three pieces with a yellow undertone, two pieces with a blue undertone, and four pieces with a white undertone. If she has six squares with a yellow undertone, five with a blue undertone, and eight with a white undertone to choose from, in how many ways can she choose the squares for the quilt?

18. **Inquiry/Problem Solving** At a family reunion, everyone greets each other with a handshake. If there are 20 people at the reunion, how many handshakes take place?

ACHIEVEMENT CHECK

Knowledge/Understanding	Thinking/Inquiry/Problem Solving	Communication	Application

19. A basketball team consists of five players—one centre, two forwards, and two guards. The senior squad at Vennville Central High School has two centres, six forwards, and four guards.

a) How many ways can the coach pick the two starting guards for a game?

b) How many different starting lineups are possible if all team members play their specified positions?

c) How many of these starting lineups include Dana, the team's 185-cm centre?

d) Some coaches designate the forwards as power forward and small forward. If all six forwards are adept in either position, how would this designation affect the number of possible starting lineups?

e) As the league final approaches, the centre Dana, forward Ashlee, and guard Hollie are all down with a nasty flu. Fortunately, the five healthy forwards can also play the guard position. If the coach can assign these players as either forwards or guards, will the number of possible starting lineups be close to the number in part b)? Support your answer mathematically.

f) Is the same result achieved if the forwards play their regular positions but the guards can play as either forwards or guards? Explain your answer.

20. In the game of bridge, each player is dealt a hand of 13 cards from a standard deck of 52 cards.

a) By what factor does the number of possible bridge hands differ from the number of ways a bridge hand could be dealt to a player? Explain your reasoning.

b) Use combinations to write an expression for the number of bridge hands that have exactly five clubs, two spades, three diamonds, and three hearts.

c) Use combinations to write an expression for the number of bridge hands that have exactly five hearts.

d) Use software or a calculator to evaluate the expressions in parts b) and c).

C

21. There are 18 students involved in the class production of *Arsenic and Old Lace*.

a) In how many ways can the teacher cast the play if there are five male roles and seven female roles and the class has nine male and nine female students?

b) In how many ways can the teacher cast the play if Jean will play the young female part only if Jovane plays the male lead?

c) In how many ways can the teacher cast the play if all the roles could be played by either a male or a female student?

22. A large sack contains six basketballs and five volleyballs. Find the number of combinations of four balls that can be chosen from the sack if

a) they may be any type of ball

b) two must be volleyballs and two must be basketballs

c) all four must be volleyballs

d) none may be volleyballs

5.3 Problem Solving With Combinations

In the last section, you considered the number of ways of choosing r items from a set of n distinct items. This section will examine situations where you want to know the total number of possible combinations of any size that you could choose from a given number of items, some of which may be identical.

INVESTIGATE & INQUIRE: Combinations of Coins

1. a) How many different sums of money can you create with a penny and a nickel? List these sums.
   b) How many different sums can you create with a penny, a nickel, and a dime? List them.
   c) Predict how many different sums you can create with a penny, a nickel, a dime, and a quarter. Test your conjecture by listing the possible sums.
2. a) How many different sums of money can you create with two pennies and a dime? List them.
   b) How many different sums can you create with three pennies and a dime?
   c) Predict how many sums you can create with four pennies and a dime. Test your conjecture. Can you see a pattern developing? If so, what is it?

Example 1 All Possible Combinations of Distinct Items

An artist has an apple, an orange, and a pear in his refrigerator. In how many ways can the artist choose one or more pieces of fruit for a still-life painting?

Solution

The artist has two choices for each piece of fruit: either include it in the painting or leave it out. Thus, the artist has a total of $2 \times 2 \times 2 = 8$ choices. Note that one of these choices is to leave out the apple, the orange, *and* the pear. However, the artist wants at least one piece of fruit in his painting. Thus, he has $2^3 - 1 = 7$ combinations to choose from.

You can apply the same logic to any group of distinct items.

The total number of combinations containing at least one item chosen from a group of n distinct items is $2^n - 1$.

Remember that combinations are subsets of the group of n objects. A **null set** is a set that has no elements. Thus,

A set with n distinct elements has 2^n subsets including the null set.

Example 2 Applying the Formula for Numbers of Subsets

In how many ways can a committee with at least one member be appointed from a board with six members?

Solution

The board could choose 1, 2, 3, 4, 5, or 6 people for the committee, so $n = 6$. Since the committee must have at least one member, use the formula that excludes the null set.

$$\begin{aligned} 2^6 - 1 &= 64 - 1 \\ &= 63 \end{aligned}$$

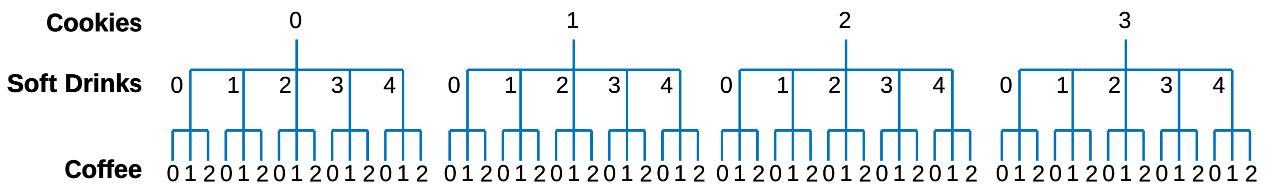
There are 63 ways to choose a committee of at least one person from a six-member board.

Example 3 All Possible Combinations With Some Identical Items

Kate is responsible for stocking the coffee room at her office. She can purchase up to three cases of cookies, four cases of soft drinks, and two cases of coffee packets without having to send the order through the accounting department. How many different direct purchases can Kate make?

Solution

Kate can order more than one of each kind of item, so this situation involves combinations in which some items are alike.
- Kate may choose to buy three or two or one or no cases of cookies, so she has four ways to choose cookies.
- Kate may choose to buy four or three or two or one or no cases of soft drinks, so she has five ways to choose soft drinks.
- Kate may choose to buy two or one or no cases of coffee packets, so she has three ways to choose coffee.



As shown on the first branch of the diagram above, one of these choices is purchasing *no* cookies, *no* soft drinks, and *no* coffee. Since this choice is not a purchase at all, subtract it from the total number of choices.

Thus, Kate can make $4 \times 5 \times 3 - 1 = 59$ different direct purchases.

In a situation where you can choose all, some, or none of the p items available, you have $p + 1$ choices. You can then apply the fundamental (multiplicative) counting principle if you have successive choices of different kinds of items. Always consider whether the choice of not picking any items makes sense. If it does not, subtract 1 from the total.

Combinations of Items in Which Some are Alike

If at least one item is chosen, the total number of selections that can be made from p items of one kind, q items of another kind, r items of another kind, and so on, is $(p + 1)(q + 1)(r + 1) \ldots - 1$

Having identical elements in a set reduces the number of possible combinations when you choose r items from that set. You cannot calculate this number by simply dividing by a factorial as you did with permutations in section 4.3. Often, you have to consider a large number of cases individually. However, some situations have restrictive conditions that make it much easier to count the number of possible combinations.

Example 4 Combinations With Some Identical Items

The director of a short documentary has found five rock songs, two blues tunes, and three jazz pieces that suit the theme of the film. In how many ways can the director choose three pieces for the soundtrack if she wants the film to include some jazz?

Solution 1 Counting Cases

The director can select exactly one, two, or three jazz pieces.

Case 1: One jazz piece

The director can choose the one jazz piece in ${}_3C_1$ ways and two of the seven non-jazz pieces in ${}_7C_2$ ways. Thus, there are ${}_3C_1 \times {}_7C_2 = 63$ combinations of music with one jazz piece.

Case 2: Two jazz pieces

The director can choose the two jazz pieces in ${}_3C_2$ ways and one of the seven non-jazz pieces in ${}_7C_1$ ways. There are ${}_3C_2 \times {}_7C_1 = 21$ combinations with two jazz pieces.

Case 3: Three jazz pieces

The director can choose the three jazz pieces and none of the seven non-jazz pieces in only one way: ${}_3C_3 \times {}_7C_0 = 1$.

The total number of combinations with at least one jazz piece is $63 + 21 + 1 = 85$.

Solution 2 Indirect Method

You can find the total number of possible combinations of three pieces of music and subtract those that do not have any jazz.

The total number of ways of choosing any three pieces from the ten available is ${}_{10}C_3 = 120$. The number of ways of not picking any jazz, that is, choosing only from the non-jazz pieces is ${}_7C_3 = 35$.

Thus, the number of ways of choosing at least one jazz piece is $120 - 35 = 85$.

Here is a summary of ways to approach questions involving choosing or selecting objects.

Is order important?

Yes: Use permutations. Can the same objects be selected more than once (like digits for a telephone number)?

- **Yes:** Use the fundamental counting principle.
- **No:** Are some of the objects identical?
  - **Yes:** Use the formula $\dfrac{n!}{a!b!c!\ldots}$.
  - **No:** Use ${}_nP_r = \dfrac{n!}{(n-r)!}$.

No: Use combinations. Are you choosing exactly r objects?

- **Yes:** Could some of the objects be identical?
  - **Yes:** Count the individual cases.
  - **No:** Use ${}_nC_r = \dfrac{n!}{(n-r)!r!}$
- **No:** Are some of the objects identical?
  - **Yes:** Use $(p + 1)(q + 1)(r + 1) - 1$ to find the total number of combinations with at least one object.
  - **No:** Use 2^n to find the total number of combinations; subtract 1 if you do not want to include the null set.

Key Concepts

- Use the formula $(p + 1)(q + 1)(r + 1) \ldots - 1$ to find the total number of selections of at least one item that can be made from p items of one kind, q of a second kind, r of a third kind, and so on.
- A set with n distinct elements has 2^n subsets including the null set.
- For combinations with some identical elements, you often have to consider all possible cases individually.
- In a situation where you must choose *at least* one particular item, either consider the total number of choices available minus the number without the desired item or add all the cases in which it is possible to have the desired item.

Communicate Your Understanding

1. Give an example of a situation where you would use the formula $(p + 1)(q + 1)(r + 1) \ldots - 1$. Explain why this formula applies.
2. Give an example of a situation in which you would use the expression $2^n - 1$. Explain your reasoning.
3. Using examples, describe two different ways to solve a problem where *at least* one particular item must be chosen. Explain why both methods give the same answer.

Practise

A

1. How many different sums of money can you make with a penny, a dime, a one-dollar coin, and a two-dollar coin?
2. How many different sums of money can be made with one \$5 bill, two \$10 bills, and one \$50 bill?
3. How many subsets are there for a set with
   a) two distinct elements?
   b) four distinct elements?
   c) seven distinct elements?
4. In how many ways can a committee with eight members form a subcommittee with at least one person on it?

B

5. Determine whether the following questions involve permutations or combinations and list any formulas that would apply.
   a) How many committees of 3 students can be formed from 12 students?
   b) In how many ways can 12 runners finish first, second, and third in a race?
   c) How many outfits can you assemble from three pairs of pants, four shirts, and two pairs of shoes?
   d) How many two-letter arrangements can be formed from the word *star*?

Apply, Solve, Communicate

6. Seven managers and eight sales representatives volunteer to attend a trade show. Their company can afford to send five people. In how many ways could they be selected
   a) without any restriction?
   b) if there must be at least one manager and one sales representative chosen?

7. **Application** A cookie jar contains three chocolate-chip, two peanut-butter, one lemon, one almond, and five raisin cookies.
   a) In how many ways can you reach into the jar and select some cookies?
   b) In how many ways can you select some cookies, if you must include at least one chocolate-chip cookie?

8. A project team of 6 students is to be selected from a class of 30.
   a) How many different teams can be selected?
   b) Pierre, Gregory, and Miguel are students in this class. How many of the teams would include these 3 students?
   c) How many teams would not include Pierre, Gregory, and Miguel?

9. The game of euchre uses only the 9s, 10s, jacks, queens, kings, and aces from a standard deck of cards. How many five-card hands have
   a) all red cards?
   b) at least two red cards?
   c) at most two red cards?

10. If you are dealing from a standard deck of 52 cards,
    a) how many different 4-card hands could have at least one card from each suit?
    b) how many different 5-card hands could have at least one spade?
    c) how many different 5-card hands could have at least two face cards (jacks, queens, or kings?

11. The number 5880 can be factored into prime divisors as $2 \times 2 \times 2 \times 3 \times 5 \times 7 \times 7$.
    a) Determine the total number of divisors of 5880.
    b) How many of the divisors are even?
    c) How many of the divisors are odd?

12. **Application** A theme park has a variety of rides. There are seven roller coasters, four water rides, and nine story rides. If Stephanie wants to try one of each type of ride, how many different combinations of rides could she choose?

13. Shuwei finds 11 shirts in his size at a clearance sale. How many different purchases could Shuwei make?

14. **Communication** Using the summary on page 285, draw a flow chart for solving counting problems.

15. a) How many different teams of 4 students could be chosen from the 15 students in the grade-12 Mathematics League?
    b) How many of the possible teams would include the youngest student in the league?
    c) How many of the possible teams would exclude the youngest student?

16. **Inquiry/Problem Solving**
    a) Use combinations to determine how many diagonals there are in
       i) a pentagon ii) a hexagon
    b) Draw sketches to verify your answers in part a).

17. A school is trying to decide on new school colours. The students can choose three colours from gold, black, green, blue, red, and white, but they know that another school has already chosen black, gold, and red. How many different combinations of three colours can the students choose?

18. **Application** The social convenor has 12 volunteers to work at a school dance. Each dance requires 2 volunteers at the door, 4 volunteers on the floor, and 6 floaters. Joe and Jim have not volunteered before, so the social convenor does not want to assign them to work together. In how many ways can the volunteers be assigned?

19. Jeffrey is a DJ at a local radio station. For the second hour of his shift, he must choose all his music from the top 100 songs for the week. Jeffery will play exactly 12 songs during this hour.

Chapter Problem

a) How many different stacks of discs could Jeffrey pull from the station's collection if he chooses at least 10 songs that are in positions 15 to 40 on the charts?

b) Jeffrey wants to start his second hour with a hard-rock song and finish with a pop classic. How many different play lists can Jeffrey prepare if he has chosen 4 hard rock songs, 5 soul pieces, and 3 pop classics?

c) Jeffrey has 8 favourite songs currently on the top 100 list. How many different subsets of these songs could he choose to play during his shift?

ACHIEVEMENT CHECK

Knowledge / Understanding	Thinking / Inquiry / Problem Solving	Communication	Application

20. There are 52 white keys on a piano. The lowest key is A. The keys are designated A, B, C, D, E, F, and G in succession, and then the sequence of letters repeats, ending with a C for the highest key.

a) If five notes are played simultaneously, in how many ways could the notes all be

i) As? ii) Gs?

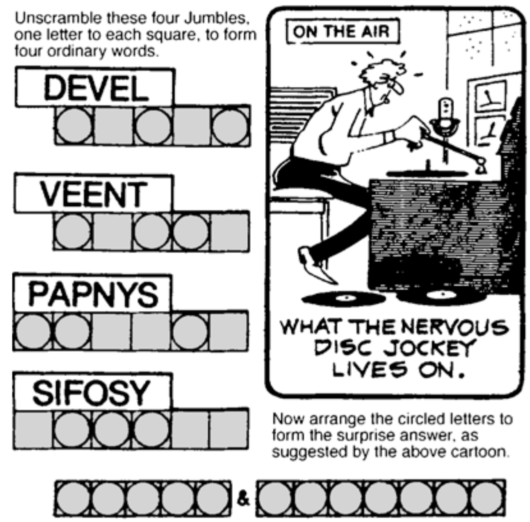
iii) the same letter? iv) different letters?

b) If the five keys are played in order, how would your answers in part a) change?

21. **Communication**

a) How many possible combinations are there for the letters in the three circles for each of the clue words in this puzzle?

b) Explain why you cannot answer part a) with a single $_nC_r$ calculation for each word.



22. Determine the number of ways of selecting four letters, without regard for order, from the word *parallelogram*.

C

23. **Inquiry / Problem Solving** Suppose the artist in Example 1 of this section had two apples, two oranges, and two pears in his refrigerator. How many combinations does he have to choose from if he wants to paint a still-life with

a) two pieces of fruit?

b) three pieces of fruit?

c) four pieces of fruit?

24. How many different sums of money can be formed from one \$2 bill, three \$5 bills, two \$10 bills, and one \$20 bill?

5.4 The Binomial Theorem

Blaise Pascal

Recall that a binomial is a polynomial with just two terms, so it has the form $a + b$. Expanding $(a + b)^n$ becomes very laborious as n increases. This section introduces a method for expanding powers of binomials. This method is useful both as an algebraic tool and for probability calculations, as you will see in later chapters.

INVESTIGATE & INQUIRE: Patterns in the Binomial Expansion

1. Expand each of the following and simplify fully.

a) $(a + b)^1$ **b)** $(a + b)^2$ **c)** $(a + b)^3$

d) $(a + b)^4$ **e)** $(a + b)^5$

2. Study the terms in each of these expansions. Describe how the degree of each term relates to the power of the binomial.

3. Compare the terms in Pascal's triangle to the expansions in question 1. Describe any pattern you find.

4. Predict the terms in the expansion of $(a + b)^6$.

In section 4.4, you found a number of patterns in Pascal's triangle. Now that you are familiar with combinations, there is another important pattern that you can recognize. Each term in Pascal's triangle corresponds to a value of ${}_nC_r$.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

$$
\begin{array}{ccccccccccc}
 & & & & & {}_0C_0 & & & & & \\
 & & & & {}_1C_0 & & {}_1C_1 & & & & \\
 & & & {}_2C_0 & & {}_2C_1 & & {}_2C_2 & & & \\
 & & {}_3C_0 & & {}_3C_1 & & {}_3C_2 & & {}_3C_3 & & \\
 & {}_4C_0 & & {}_4C_1 & & {}_4C_2 & & {}_4C_3 & & {}_4C_4 & \\
{}_5C_0 & & {}_5C_1 & & {}_5C_2 & & {}_5C_3 & & {}_5C_4 & & {}_5C_5
\end{array}
$$

Comparing the two triangles shown on page 289, you can see that $t_{n,r} = {}_nC_r$. Recall that Pascal's method for creating his triangle uses the relationship

$t_{n,r} = t_{n-1,\, r-1} + t_{n-1,\, r}$.

So, this relationship must apply to combinations as well.

Pascal's Formula

$${}_nC_r = {}_{n-1}C_{r-1} + {}_{n-1}C_r$$

Proof:

$$\begin{aligned}
{}_{n-1}C_{r-1} + {}_{n-1}C_r &= \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-r-1)!} \\
&= \frac{r(n-1)!}{r(r-1)!(n-r)!} + \frac{(n-1)!(n-r)}{r!(n-r)(n-r-1)!} \\
&= \frac{r(n-1)!}{r!(n-r)!} + \frac{(n-1)!(n-r)}{r!(n-r)!} \\
&= \frac{(n-1)!}{r!(n-r)!}[r + (n-r)] \\
&= \frac{(n-1)! \times n}{r!(n-r)!} \\
&= \frac{n!}{r!(n-r)!} \\
&= {}_nC_r
\end{aligned}$$

This proof shows that the values of ${}_nC_r$ do indeed follow the pattern that creates Pascal's triangle. It follows that ${}_nC_r = t_{n,r}$ for all the terms in Pascal's triangle.

Example 1 Applying Pascal's Formula to Combinations

Rewrite each of the following using Pascal's formula.

a) ${}_{12}C_8$

b) ${}_{19}C_5 + {}_{19}C_6$

Solution

a) ${}_{12}C_8 = {}_{11}C_7 + {}_{11}C_8$

b) ${}_{19}C_5 + {}_{19}C_6 = {}_{20}C_6$

As you might expect from the investigation at the beginning of this section, the coefficients of each term in the expansion of $(a + b)^n$ correspond to the terms in row n of Pascal's triangle. Thus, you can write these coefficients in combinatorial form.

The Binomial Theorem

$$(a+b)^n = {}_nC_0a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + \ldots + {}_nC_ra^{n-r}b^r + \ldots + {}_nC_nb^n$$
$$\text{or } (a+b)^n = \sum_{r=0}^{n} {}_nC_ra^{n-r}b^r$$

Example 2 Applying the Binomial Theorem

Use combinations to expand $(a+b)^6$.

Solution

$$\begin{aligned}(a+b)^6 &= \sum_{r=0}^{6} {}_6C_ra^{6-r}b^r\\ &= {}_6C_0a^6 + {}_6C_1a^5b + {}_6C_2a^4b^2 + {}_6C_3a^3b^3 + {}_6C_4a^2b^4 + {}_6C_5ab^5 + {}_6C_6b^6\\ &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6\end{aligned}$$

Example 3 Binomial Expansions Using Pascal's Triangle

Use Pascal's triangle to expand

a) $(2x-1)^4$

b) $(3x-2y)^5$

Solution

a) Substitute $2x$ for a and -1 for b. Since the exponent is 4, use the terms in row 4 of Pascal's triangle as the coefficients: 1, 4, 6, 4, and 1. Thus,

$$\begin{aligned}(2x-1)^4 &= 1(2x)^4 + 4(2x)^3(-1) + 6(2x)^2(-1)^2 + 4(2x)(-1)^3 + 1(-1)^4\\ &= 16x^4 + 4(8x^3)(-1) + 6(4x^2)(1) + 4(2x)(-1) + 1\\ &= 16x^4 - 32x^3 + 24x^2 - 8x + 1\end{aligned}$$

b) Substitute $3x$ for a and $-2y$ for b, and use the terms from row 5 as coefficients.

$$\begin{aligned}(3x-2y)^5 &= 1(3x)^5 + 5(3x)^4(-2y) + 10(3x)^3(-2y)^2 + 10(3x)^2(-2y)^3 + 5(3x)(-2y)^4 + 1(-2y)^5\\ &= 243x^5 - 810x^4y + 1080x^3y^2 - 720x^2y^3 + 240xy^4 - 32y^5\end{aligned}$$

Example 4 Expanding Binomials Containing Negative Exponents

Use the binomial theorem to expand and simplify $\left(x + \frac{2}{x^2}\right)^4$.

Solution

Substitute x for a and $\frac{2}{x^2}$ for b.

$$\begin{aligned}\left(x+\frac{2}{x^2}\right)^4 &= \sum_{r=0}^{4} {}_4C_r x^{4-r}\left(\frac{2}{x^2}\right)^r \\ &= {}_4C_0x^4 + {}_4C_1x^3\left(\frac{2}{x^2}\right) + {}_4C_2x^2\left(\frac{2}{x^2}\right)^2 + {}_4C_3x\left(\frac{2}{x^2}\right)^3 + {}_4C_4\left(\frac{2}{x^2}\right)^4 \\ &= 1x^4 + 4x^3\left(\frac{2}{x^2}\right) + 6x^2\left(\frac{4}{x^4}\right) + 4x\left(\frac{8}{x^6}\right) + 1\left(\frac{16}{x^8}\right) \\ &= x^4 + 8x + 24x^{-2} + 32x^{-5} + 16x^{-8}\end{aligned}$$

Example 5 Patterns With Combinations

Using the patterns in Pascal's triangle from the investigation and Example 4 in section 4.4, write each of the following in combinatorial form.

a) the sum of the terms in row 5 and row 6

b) the sum of the terms in row n

c) the first 5 triangular numbers

d) the nth triangular number

Solution

a) *Row 5:*

$$\begin{aligned}&1 + 5 + 10 + 10 + 5 + 1 \\ &= {}_5C_0 + {}_5C_1 + {}_5C_2 + {}_5C_3 + {}_5C_4 + {}_5C_5 \\ &= 32 \\ &= 2^5\end{aligned}$$

Row 6:

$$\begin{aligned}&1 + 6 + 15 + 20 + 15 + 6 + 1 \\ &= {}_6C_0 + {}_6C_1 + {}_6C_2 + {}_6C_3 + {}_6C_4 + {}_6C_5 + {}_6C_6 \\ &= 64 \\ &= 2^6\end{aligned}$$

b) From part a) it appears that ${}_nC_0 + {}_nC_1 + \dots + {}_nC_n = 2^n$.

Using the binomial theorem,

$$\begin{aligned}2^n &= (1+1)^n \\ &= {}_nC_0 \times 1^n + {}_nC_1 \times 1^{n-1} \times 1 + \dots + {}_nC_n \times 1^n \\ &= {}_nC_0 + {}_nC_1 + \dots + {}_nC_n\end{aligned}$$

c)

n	Triangular Numbers	Combinatorial Form
1	1	${}_2C_2$
2	3	${}_3C_2$
3	6	${}_4C_2$
4	10	${}_5C_2$
5	15	${}_6C_2$

d) The nth triangular number is ${}_{n+1}C_2$.

Example 6 Factoring Using the Binomial Theorem

Rewrite $1 + 10x^2 + 40x^4 + 80x^6 + 80x^8 + 32x^{10}$ in the form $(a + b)^n$.

Solution

There are six terms, so the exponent must be 5.
The first term of a binomial expansion is a^n, so a must be 1.
The final term is $32x^{10} = (2x^2)^5$, so $b = 2x^2$.
Therefore, $1 + 10x^2 + 40x^4 + 80x^6 + 80x^8 + 32x^{10} = (1 + 2x^2)^5$

Key Concepts

- The coefficients of the terms in the expansion of $(a + b)^n$ correspond to the terms in row n of Pascal's triangle.
- The binomial $(a + b)^n$ can also be expanded using combinatorial symbols:
  $(a+b)^n = {}_nC_0a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + \ldots + {}_nC_nb^n$ or $\sum_{r=0}^{n} {}_nC_ra^{n-r}b^r$
- The degree of each term in the binomial expansion of $(a + b)^n$ is n.
- Patterns in Pascal's triangle can be summarized using combinatorial symbols.

Communicate Your Understanding

1. Describe how Pascal's triangle and the binomial theorem are related.
2. **a)** Describe how you would use Pascal's triangle to expand $(2x + 5y)^9$.
   b) Describe how you would use the binomial theorem to expand $(2x + 5y)^9$.
3. Relate the sum of the terms in the nth row of Pascal's triangle to the total number of subsets of a set of n elements. Explain the relationship.

Practise

A

1. Rewrite each of the following using Pascal's formula.
   a) ${}_{17}C_{11}$ **b)** ${}_{43}C_{36}$
   c) ${}_{n+1}C_{r+1}$ **d)** ${}_{32}C_4 + {}_{32}C_5$
   e) ${}_{15}C_{10} + {}_{15}C_9$ **f)** ${}_nC_r + {}_nC_{r+1}$
   g) ${}_{18}C_9 - {}_{17}C_9$ **h)** ${}_{24}C_8 - {}_{23}C_7$
   i) ${}_nC_r - {}_{n-1}C_{r-1}$

2. Determine the value of k in each of these terms from the binomial expansion of $(a + b)^{10}$.
   a) $210a^6b^k$ **b)** $45a^kb^8$ **c)** $252a^kb^k$

3. How many terms would be in the expansion of the following binomials?
   a) $(x + y)^{12}$ **b)** $(2x - 3y)^5$ **c)** $(5x - 2)^{20}$

4. For the following terms from the expansion of $(a + b)^{11}$, state the coefficient in both ${}_nC_r$ and numeric form.
   a) a^2b^9 **b)** a^{11} **c)** a^6b^5

Apply, Solve, Communicate

B

5. Using the binomial theorem and patterns in Pascal's triangle, simplify each of the following.

a) ${}_9C_0 + {}_9C_1 + \ldots + {}_9C_9$

b) ${}_{12}C_0 - {}_{12}C_1 + {}_{12}C_2 - \ldots - {}_{12}C_{11} + {}_{12}C_{12}$

c) $\sum_{r=0}^{15} {}_{15}C_r$ **d)** $\sum_{r=0}^{n} {}_{n}C_r$

6. If $\sum_{r=0}^{n} {}_{n}C_r = 16\ 384$, determine the value of n.

7. a) Write formulas in combinatorial form for the following. (Refer to section 4.4, if necessary.)

i) the sum of the squares of the terms in the nth row of Pascal's triangle

ii) the result of alternately adding and subtracting the squares of the terms in the nth row of Pascal's triangle

iii) the number of diagonals in an n-sided polygon

b) Use your formulas from part a) to determine

i) the sum of the squares of the terms in row 15 of Pascal's triangle

ii) the result of alternately adding and subtracting the squares of the terms in row 12 of Pascal's triangle

iii) the number of diagonals in a 14-sided polygon

8. How many terms would be in the expansion of $(x^2 + x)^8$?

9. Use the binomial theorem to expand and simplify the following.

a) $(x + y)^7$ **b)** $(2x + 3y)^6$

c) $(2x - 5y)^5$ **d)** $(x^2 + 5)^4$

e) $(3a^2 + 4c)^7$ **f)** $5(2p - 6c^2)^5$

10. Communication

a) Find and simplify the first five terms of the expansion of $(3x + y)^{10}$.

b) Find and simplify the first five terms of the expansion of $(3x - y)^{10}$.

c) Describe any similarities and differences between the terms in parts a) and b).

11. Use the binomial theorem to expand and simplify the following.

a) $\left(x^2 - \frac{1}{x}\right)^5$ **b)** $\left(2y + \frac{3}{y^2}\right)^4$

c) $(\sqrt{x} + 2x^2)^6$ **d)** $\left(k + \frac{k}{m^2}\right)^5$

e) $\left(\sqrt{y} - \frac{2}{\sqrt{y}}\right)^7$ **f)** $2\left(3m^2 - \frac{2}{\sqrt{m}}\right)^4$

12. Application Rewrite the following expansions in the form $(a + b)^n$, where n is a positive integer.

a) $x^6 + 6x^5y + 15x^4y^2 + 20\, x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

b) $y^{12} + 8y^9 + 24y^6 + 32y^3 + 16$

c) $243a^5 - 405a^4b + 270a^3b^2 - 90a^2b^3 + 15ab^4 - b^5$

13. Communication Use the binomial theorem to simplify each of the following. Explain your results.

a) $\left(\frac{1}{2}\right)^5 + 5\left(\frac{1}{2}\right)^5 + 10\left(\frac{1}{2}\right)^5 + 10\left(\frac{1}{2}\right)^5 + 5\left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^5$

b) $(0.7)^7 + 7(0.7)^6(0.3) + 21(0.7)^5(0.3)^2 + \ldots + (0.3)^7$

c) $7^9 - 9 \times 7^8 + 36 \times 7^7 - \ldots - 7^0$

14. a) Expand $\left(x + \frac{2}{x}\right)^4$ and compare it with the expansion of $\frac{1}{x^4}(x^2 + 2)^4$.

b) Explain your results.

15. Use your knowledge of algebra and the binomial theorem to expand and simplify each of the following.

a) $(25x^2 + 30xy + 9y^2)^3$

b) $(3x - 2y)^5(3x + 2y)^5$

16. Application

a) Calculate an approximation for $(1.2)^9$ by expanding $(1 + 0.2)^9$.

b) How many terms do you have to evaluate to get an approximation accurate to two decimal places?

17. In a trivia contest, Adam has drawn a topic he knows nothing about, so he makes random guesses for the ten true/false questions. Use the binomial theorem to help find

a) the number of ways that Adam can answer the test using exactly four *true*s

b) the number of ways that Adam can answer the test using at least one *true*

ACHIEVEMENT CHECK

Knowledge / Understanding	Thinking / Inquiry / Problem Solving	Communication	Application

18. a) Expand $(h + t)^5$.

b) Explain how this expansion can be used to determine the number of ways of getting exactly h heads when five coins are tossed.

c) How would your answer in part b) change if six coins are being tossed? How would it change for n coins? Explain.

C

19. Find the first three terms, ranked by degree of the terms, in each expansion.

a) $(x + 3)(2x + 5)^4$

b) $(2x + 1)^2(4x - 3)^5$

c) $(x^2 - 5)^9(x^3 + 2)^6$

20. Inquiry / Problem Solving

a) Use the binomial theorem to expand $(x + y + z)^2$ by first rewriting it as $[x + (y + z)]^2$.

b) Repeat part a) with $(x + y + z)^3$.

c) Using parts a) and b), predict the expansion of $(x + y + z)^4$. Verify your prediction by using the binomial theorem to expand $(x + y + z)^4$.

d) Write a formula for $(x + y + z)^n$.

e) Use your formula to expand and simplify $(x + y + z)^5$.

21. a) In the expansion of $(x + y)^5$, replace x and y with B and G, respectively. Expand and simplify.

b) Assume that a couple has an equal chance of having a boy or a girl. How would the expansion in part a) help find the number of ways of having k girls in a family with five children?

c) In how many ways could a family with five children have exactly three girls?

d) In how many ways could they have exactly four boys?

22. A simple code consists of a string of five symbols that represent different letters of the alphabet. Each symbol is either a dot (•) or a dash (–).

a) How many different letters are possible using this code?

b) How many coded letters will contain exactly two dots?

c) How many different coded letters will contain at least one dash?

Review of Key Concepts

5.1 Organized Counting With Venn Diagrams

Refer to the Key Concepts on page 270.

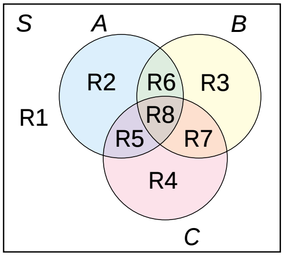
1. Which regions in the diagram below correspond to

a) the union of sets A and B?

b) the intersection of sets B and C?

c) $A \cap C$?

d) either B or S?



2. a) Write the equation for the number of elements contained in either of two sets.

b) Explain why the principle of inclusion and exclusion subtracts the last term in this equation.

c) Give a simple example to illustrate your explanation.

3. A survey of households in a major city found that

- 96% had colour televisions
- 65% had computers
- 51% had dishwashers
- 63% had colour televisions and computers
- 49% had colour televisions and dishwashers
- 31% had computers and dishwashers
- 30% had all three

a) List the categories of households not included in these survey results.

b) Use a Venn diagram to find the proportion of households in each of these categories.

5.2 Combinations

Refer to the Key Concepts on page 278.

4. Evaluate the following and indicate any calculations that could be done manually.

a) ${}_{41}C_8$	**b)** ${}_{33}C_{15}$
c) ${}_{25}C_{17}$	**d)** ${}_{50}C_{10}$
e) ${}_{10}C_8$	**f)** ${}_{15}C_{13}$
g) ${}_5C_4$	**h)** ${}_{25}C_{24}$
i) ${}_{15}C_{11}$	**j)** ${}_{25}C_{20}$
k) ${}_{16}C_8$	**l)** ${}_{30}C_{26}$

5. A track and field club has 12 members who are runners and 10 members who specialize in field events. The club has been invited to send a team of 3 runners and 2 field athletes to an out-of-town meet. How many different teams could the club send?

6. A bridge hand consists of 13 cards. How many bridge hands include 5 cards of one suit, 6 cards of a second, and 2 cards of a third?

7. Explain why combination locks should really be called permutation locks.

5.3 Problem Solving With Combinations

Refer to the Key Concepts on page 286.

8. At Subs Galore, you have a choice of lettuce, onions, tomatoes, green peppers, mushrooms, cheese, olives, cucumbers, and hot peppers on your submarine sandwich. How many ways can you "dress" your sandwich?

9. Ballots for municipal elections usually list candidates for several different positions. If a resident can vote for a mayor, two councillors, a school trustee, and a hydro commissioner, how many combinations of positions could the resident choose to mark on the ballot?

10. There are 12 questions on an examination, and each student must answer 8 questions including at least 4 of the first 5 questions. How many different combinations of questions could a student choose to answer?

11. Naomi invites eight friends to a party on short notice, so they may not all be able to come. How many combinations of guests could attend the party?

12. In how many ways could 15 different books be divided equally among 3 people?

13. The camera club has five members, and the mathematics club has eight. There is only one member common to both clubs. In how many ways could a committee of four people be formed with at least one member from each club?

5.4 The Binomial Theorem

Refer to the Key Concepts on page 293.

14. Without expanding $(x + y)^5$, determine

a) the number of terms in the expansion

b) the value of k in the term $10x^ky^2$

15. Use Pascal's triangle to expand

a) $(x + y)^8$

b) $(4x - y)^6$

c) $(2x + 5y)^4$

d) $(7x - 3)^5$

16. Use the binomial theorem to expand

a) $(x + y)^6$

b) $(6x - 5y)^4$

c) $(5x + 2y)^5$

d) $(3x - 2)^6$

17. Write the first three terms of the expansion of

a) $(2x + 5y)^7$

b) $(4x - y)^6$

18. Describe the steps in the binomial expansion of $(2x - 3y)^6$.

19. Find the last term in the binomial expansion of $\left(\frac{1}{x^2} + 2x\right)^5$.

20. Find the middle term in the binomial expansion of $\left(\sqrt{x} + \frac{5}{\sqrt{x}}\right)^8$.

21. In the expansion of $(a + x)^6$, the first three terms are $1 + 3 + 3.75$. Find the values of a and x.

22. Use the binomial theorem to expand and simplify $(y^2 - 2)^6(y^2 + 2)^6$.

23. Write $1024x^{10} - 3840x^8 + 5760x^6 - 4320x^4 + 1620x^2 - 243$ in the form $(a + b)^n$. Explain your steps.

Chapter Test

ACHIEVEMENT CHART				
Category	Knowledge / Understanding	Thinking / Inquiry / Problem Solving	Communication	Application
Questions	**All**	**12**	**6, 12**	**5, 6, 7, 8, 9**

1. Evaluate each of the following. List any calculations that require a calculator.

a) ${}_{25}C_{25}$

b) ${}_{52}C_{1}$

c) ${}_{12}C_{3}$

d) ${}_{40}C_{15}$

2. Rewrite each of the following as a single combination.

a) ${}_{10}C_{7} + {}_{10}C_{8}$

b) ${}_{23}C_{15} - {}_{22}C_{14}$

3. Use Pascal's triangle to expand

a) $(3x - 4)^4$

b) $(2x + 3y)^7$

4. Use the binomial theorem to expand

a) $(8x - 3)^5$

b) $(2x - 5y)^6$

5. A student fundraising committee has 14 members, including 7 from grade 12. In how many ways can a 4-member subcommittee for commencement awards be formed if

a) there are no restrictions?

b) the subcommittee must be all grade-12 students?

c) the subcommittee must have 2 students from grade 12 and 2 from other grades?

d) the subcommittee must have no more than 3 grade-12 students?

6. A track club has 20 members.

a) In how many ways can the club choose 3 members to help officiate at a meet?

b) In how many ways can the club choose a starter, a marshal, and a timer?

c) Should your answers to parts a) and b) be the same? Explain why or why not.

7. Statistics on the grade-12 courses taken by students graduating from a secondary school showed that

- 85 of the graduates had taken a science course
- 75 of the graduates had taken a second language
- 41 of the graduates had taken mathematics
- 43 studied both science and a second language
- 32 studied both science and mathematics
- 27 had studied both a second language and mathematics
- 19 had studied all three subjects

a) Use a Venn diagram to determine the minimum number of students who could be in this graduating class.

b) How many students studied mathematics, but neither science nor a second language?

8. A field-hockey team played seven games and won four of them. There were no ties.
   a) How many arrangements of the four wins and three losses are possible?
   b) In how many of these arrangements would the team have at least two wins in a row?

9. A restaurant offers an all-you-can-eat Chinese buffet with the following items:
   - egg roll, wonton soup
   - chicken wings, chicken balls, beef, pork
   - steamed rice, fried rice, chow mein
   - chop suey, mixed vegetables, salad
   - fruit salad, custard tart, almond cookie

   a) How many different combinations of items could you have?
   b) The restaurant also has a lunch special with your choice of one item from each group. How many choices do you have with this special?

10. In the expansion of $(1 + x)^n$, the first three terms are $1 - 0.9 + 0.36$. Find the values of x and n.

11. Use the binomial theorem to expand and simplify $(4x^2 - 12x + 9)^3$.

12. A small transit bus has 8 window seats and 12 aisle seats. Ten passengers board the bus and select seats at random. How many seating arrangements have all the window seats occupied if which passenger is in a seat
   a) does not matter? b) matters?

ACHIEVEMENT CHECK

Knowledge / Understanding	Thinking / Inquiry / Problem Solving	Communication	Application

13. The students' council is having pizza at their next meeting. There are 20 council members, 6 of whom are vegetarian. A committee of 3 will order six pizzas from a pizza shop that has a special price for large pizzas with up to three toppings. The shop offers ten different toppings.
   a) How many different pizza committees can the council choose if there must be at least one vegetarian and one non-vegetarian on the committee?
   b) In how many ways could the committee choose *exactly* three toppings for a pizza?
   c) In how many ways could the committee choose *up to* three toppings for a pizza?
   d) The committee wants as much variety as possible in the toppings. They decide to order each topping exactly once and to have at least one topping on each pizza. Describe the different cases possible when distributing the toppings in this way.
   e) For one of these cases, determine the number of ways of choosing and distributing the ten toppings.