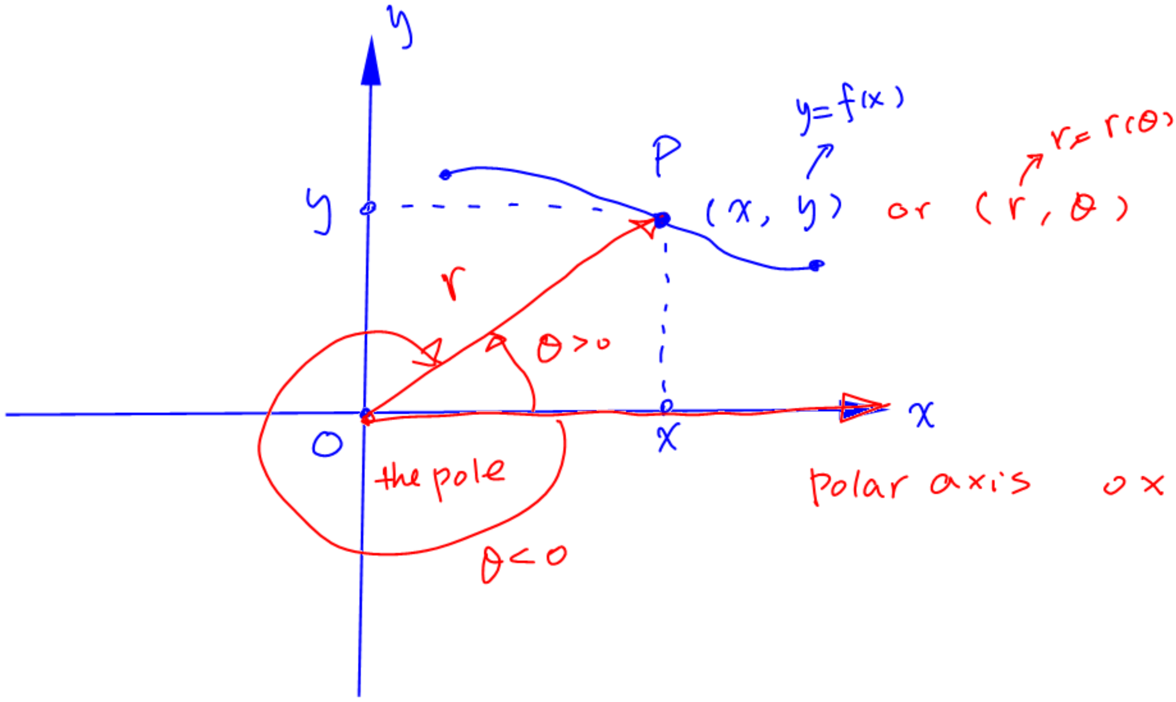
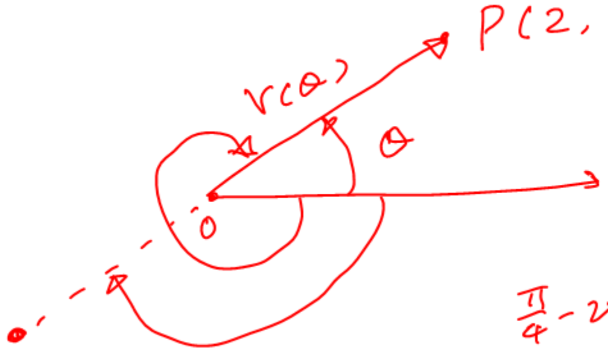
Polar Coordinates.



θ is positive if measured counterclockwise from OX to OP, or negative if measured clockwise.

θ could be in degree or radian.



$P(2, \frac{\pi}{4})$ or $(2, -\frac{7\pi}{4})$

or $(2, \frac{9\pi}{4})$

or $(-2, -\frac{3\pi}{4})$

⋮

$\frac{\pi}{4} - 2\pi = -\frac{7\pi}{4}$

$\frac{\pi}{4} + 2\pi = \frac{9\pi}{4}$

$-\pi + \frac{\pi}{4} = -\frac{3\pi}{4}.$

Relation between (x, y) and (r, θ).

$$\begin{cases} x = r(\theta)\cdot\cos\theta \\ y = r(\theta)\cdot\sin\theta \end{cases}$$

$$\begin{cases} r^2 = x^2 + y^2 . \\ \tan\theta = \dfrac{y}{x} \end{cases}$$

Some common polar curves:

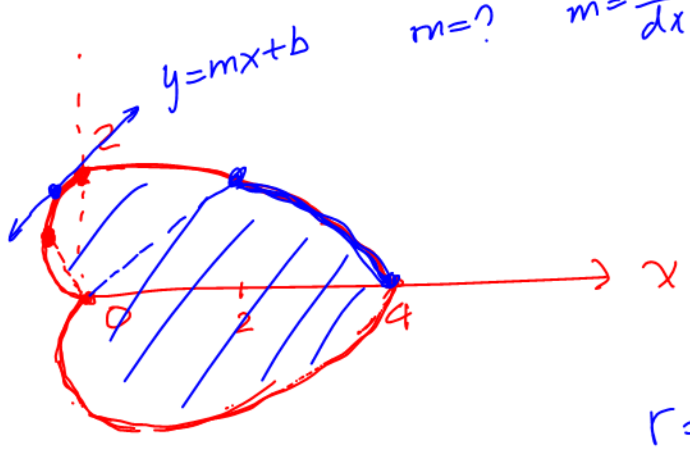
1) Cardioid

$r = a(1 \pm \cos\theta)$ or $r = a(1 \pm \sin\theta)$.

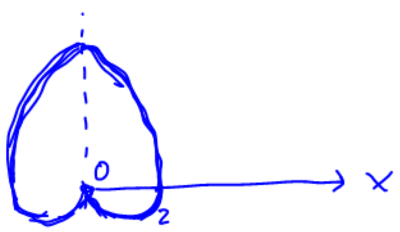
For example, $\underline{r = 2(1+\cos\theta)}$, $0 \le \theta \le 2\pi$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	4	$2+\sqrt{3}$	$2+\sqrt{2}$	3	2	1	$2-\sqrt{2}$	$2-\sqrt{3}$	0

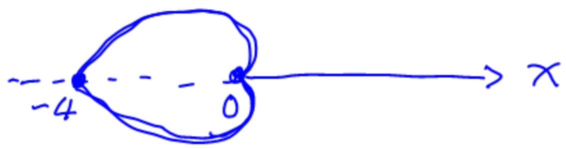
$m = ?$ $m = \frac{dy}{dx} = ?$



$r = 2(1+\sin\theta)$



$r = 2(1-\cos\theta)$



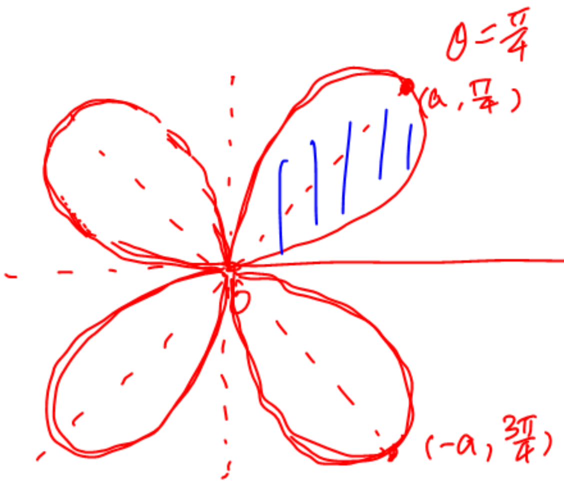
2) Rose

$r = a\sin(n\theta)$ or $r = a\cos(n\theta)$.

If n is even, there are $2n$ loops.

If n is odd, there are n loops.

For example. $r = a\sin(2\theta)$. $a > 0$, $0 \le \theta \le 2\pi$.



when $\frac{\pi}{2} \le \theta \le \pi$

$\pi \le 2\theta \le 2\pi$

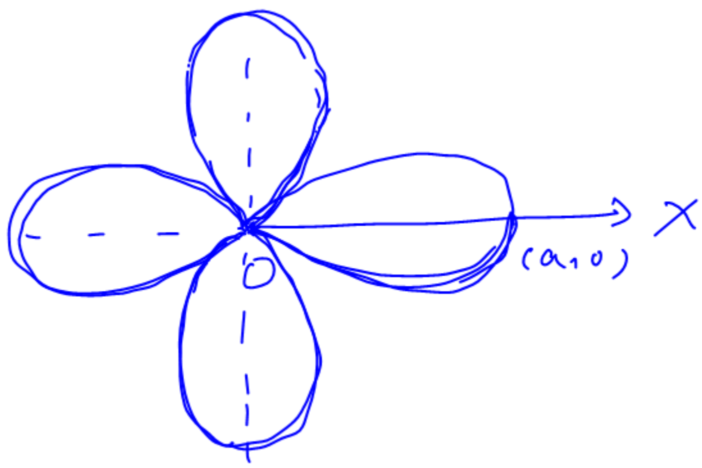
when $\pi \le \theta \le \frac{3\pi}{2}$

$2\pi \le 2\theta \le 3\pi$

when $\frac{3\pi}{2} \le \theta \le 2\pi$

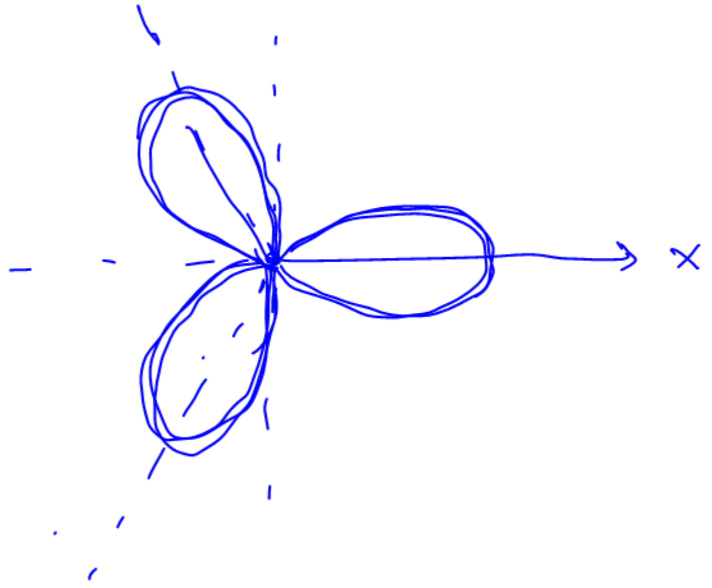
$3\pi \le 2\theta \le 4\pi$

$r = a\cos(2\theta)$



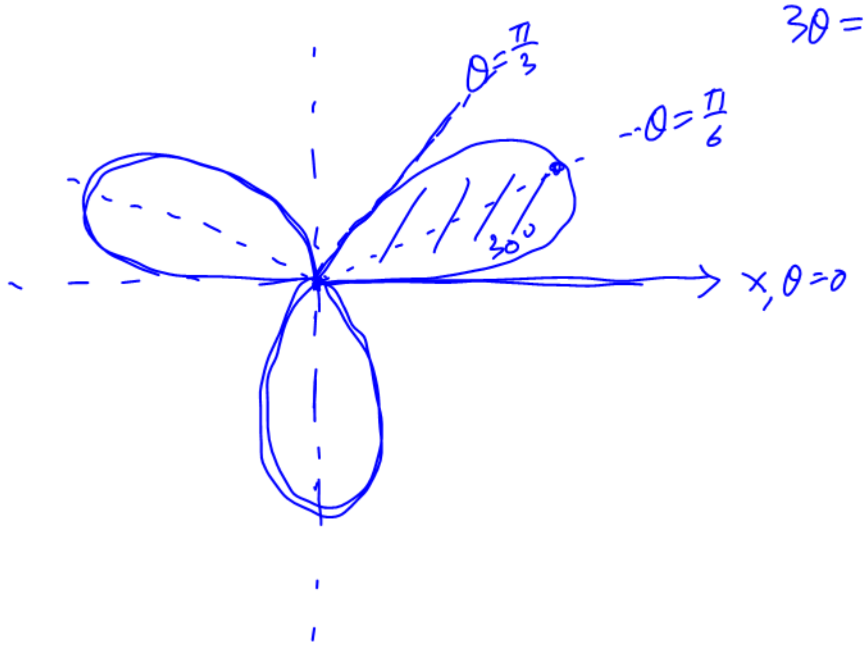
$r = \cos(3\theta)$

$\frac{360}{3} = 120$



$r = \sin(3\theta)$. when $\theta = \frac{\pi}{6}$.

$3\theta = \frac{\pi}{2}$.



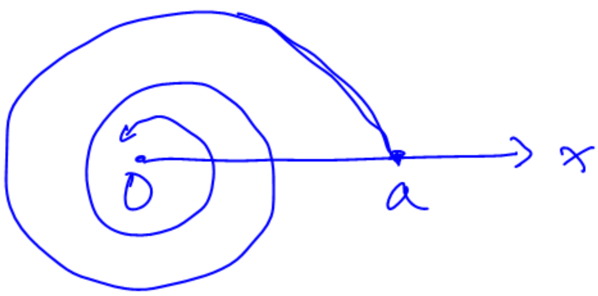
$30^\circ + 120^\circ = 150^\circ$

$150^\circ + 120^\circ = 270^\circ$

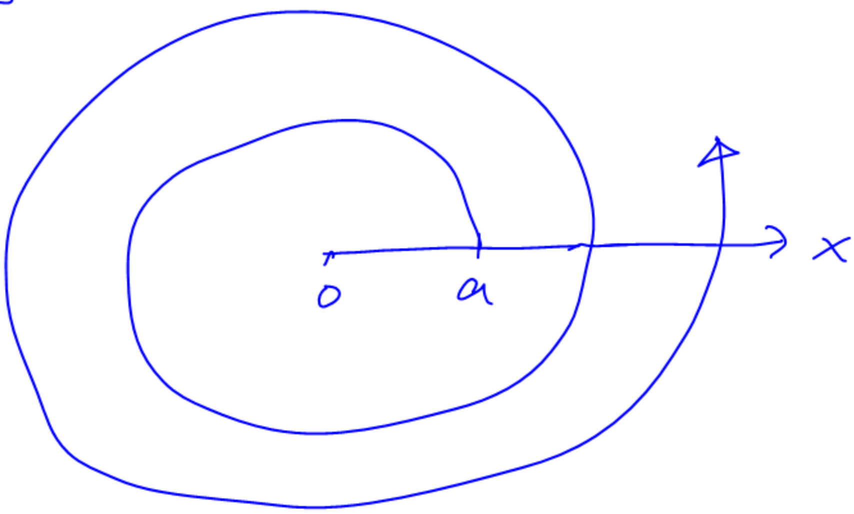
3) Spiral

$$r = a \cdot (b^\theta).$$

i) $0 < b < 1$,

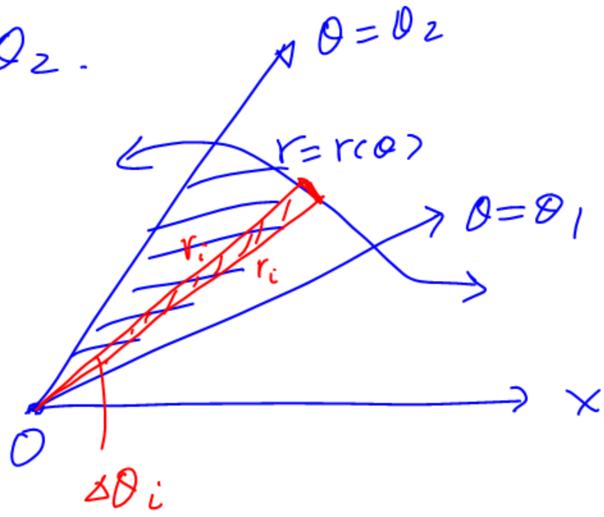


ii) $1<b<\infty$



※ Area of a region bounded by $r=r(\theta)$, $\theta=\theta_1$ and $\theta=\theta_2$.

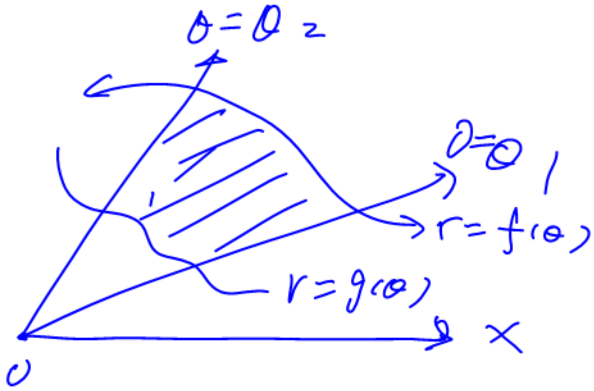
$$A=\int_{\theta_1}^{\theta_2}\frac{1}{2}[r(\theta)]^2\,d\theta$$



$$\Delta A_i=\frac{1}{2}\Delta\theta_i\,(r(\theta_i))^2$$

Generally, the area of a region bounded by $r=f(\theta)$, $r=g(\theta)$, $\theta=\theta_1$ and $\theta=\theta_2$

$$A=\int_{\theta_1}^{\theta_2}\frac{1}{2}\left[(f(\theta))^2-(g(\theta))^2\right]d\theta$$



** Arc length

$\because$ $x = r(\theta)\cos\theta$; $y = r(\theta)\sin\theta$

$\therefore$ $x'(\theta) = r'(\theta)\cos\theta - r(\theta)\sin\theta$

$y'(\theta) = r'(\theta)\sin\theta + r(\theta)\cos\theta$

$$(x'(\theta))^2 + (y'(\theta))^2 = [r'(\theta)\cos\theta - r(\theta)\sin\theta]^2 + [r'(\theta)\sin\theta + r(\theta)\cos\theta]^2$$

$$= (r'(\theta))^2\cos^2\theta + (r(\theta))^2\sin^2\theta - \cancel{2r'(\theta)\cos\theta\, r(\theta)\sin\theta}$$

$$+ (r'(\theta))^2\sin^2\theta + (r(\theta))^2\cos^2\theta + \cancel{2r'(\theta)\sin\theta\, r(\theta)\cos\theta}$$

$$= (r'(\theta))^2 + (r(\theta))^2.$$

$\therefore$ $L = \int_{\theta_1}^{\theta_2} \sqrt{(r'(\theta))^2 + (r(\theta))^2}\, d\theta$ is the arc length of $r(\theta)$ bounded by $\theta = \theta_1$ and $\theta = \theta_2$.

**. Slope of tangent to the polar curve at (r, θ) :

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}$$