

## AP Calculus Lesson Six Notes

### Chapter Three – Applications of Differential Calculus

#### **3.4 Curve Sketching**

#### **3.5 Optimization Problems**

#### **3.6 Local Linear Approximations**

#### **3.4 Curve Sketching**

##### **Process for Graphing a Function**

1. Discuss the domain, symmetry and end-behaviours of the function.
2. Find the intercepts of a function  $y = f(x)$ , if there are any. Remember that the  $y$ -intercept is given by  $(0, f(0))$  and we find the  $x$ -intercepts by setting  $y = 0$  and solving.
3. Find the vertical asymptotes by setting the denominator, if it exists, equal to zero and solving.
4. Find the horizontal asymptote(s) and Oblique linear asymptote(s), if they exist.
5. The vertical asymptotes will divide the number line into regions. In each region graph at least one point in each region. This point will tell us whether the graph will be above or below the horizontal asymptote and if we need to we should get several points to determine the general shape of the graph.
6. Determine increasing and decreasing intervals, local maximum and minimum value(s), if there are any.
7. Discuss the concavity of the function and determine inflection point(s), if they exist.
8. Sketch the graph.

Here are the general definitions of the three asymptotes.

1. The line  $x = a$  is a **vertical asymptote** if the graph increases or decreases without bound on one or both sides of the line as  $x$  moves in closer and closer to  $x = a$ . This implies that  $\lim_{x \rightarrow a^+} f(x) = -\infty$ , or  $\lim_{x \rightarrow a^+} f(x) = +\infty$ , or  $\lim_{x \rightarrow a^-} f(x) = -\infty$ , or  $\lim_{x \rightarrow a^-} f(x) = +\infty$

2. The line  $y = b$  is a **horizontal asymptote** if the graph approaches  $y = b$  as  $x$  increases or decreases without bound. This implies that  $\lim_{x \rightarrow -\infty} f(x) = b$ , or  $\lim_{x \rightarrow +\infty} f(x) = b$ .
  3. The line  $y = ax + b$  is an **oblique linear asymptote** if the graph approaches  $y = ax + b$  as  $x$  increases or decreases without bound. This implies that
- $$\lim_{x \rightarrow -\infty} [f(x) - (ax + b)] = 0, \text{ or } \lim_{x \rightarrow +\infty} [f(x) - (ax + b)] = 0$$

Note that it doesn't have to approach an asymptote as  $x$  BOTH increases and decreases. It only needs to approach it on one side in order for it to be an asymptote.

We then have the following facts about asymptotes for a rational function,

$$f(x) = \frac{a_n x^n + \dots + a_0}{b_m x^m + \dots + b_0}$$

1. The graph will have a vertical asymptote at  $x = a$ , if the denominator is zero at  $x = a$  and the numerator isn't zero at  $x = a$ .
2. If  $n < m$  then the  $x$ -axis is the horizontal asymptote.
3. If  $n = m$  then the line  $y = \frac{a_n}{b_m}$  is the horizontal asymptote.
4. If  $n > m$  there will be no horizontal asymptotes.
5. If  $n = m + 1$  there will be an oblique linear asymptotes, which can be obtained by polynomial division.

**Example 3.4-1** For the following function identify the intervals where the function is increasing and decreasing and the intervals where the function is concave up and concave down. Use this information to sketch the graph.

$$h(x) = 3x^5 - 5x^3 + 3$$

### Solution

Okay, this is a polynomial function, the set of all real numbers is its domain. No symmetry. No asymptotes exist. The  $y$ -intercept is  $(0, 3)$ . It is a little bit difficult to find it  $x$ -intercept since the degree of the polynomial is 5. So we just leave it here now, but just go straight to get the first two derivatives first.

$$h'(x) = 15x^4 - 15x^2 = 15x^2(x+1)(x-1)$$

$$h''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$$

Let's start with the increasing/decreasing information since we should be fairly comfortable with that after the last section.

There are three critical points for this function:  $x = 0$ ,  $x = -1$  and  $x = 1$ . Below is the chart for the increasing/decreasing information.

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, +\infty)$
Test Value	-2	-0.5	0.5	2
$x + 1$	-	+	+	+
$x - 1$	-	-	-	+
$h'(x)$	+	-	-	+
$h(x)$	increasing	decreasing	decreasing	increasing

So, it looks like we've got the following intervals of increasing and decreasing.

Increasing intervals:  $x \in (-\infty, -1) \cup (1, +\infty)$

Decreasing intervals:  $x \in (-1, 0) \cup (0, 1)$

Note that from the first derivative test we can also say that at  $x = -1$ ,  $h(x)$  has a relative maximum value and at  $x = 1$ ,  $h(x)$  has a relative minimum value. Also at  $x = 0$ ,  $h(x)$  has neither a relative minimum or maximum value.

Now let's get the intervals where the function is concave up and concave down. If you think about it this process is almost identical to the process we use to identify the intervals of increasing and decreasing. This only difference is that we will be using the second derivative instead of the first derivative.

The first thing that we need to do is identify the possible inflection points. These will be where the second derivative is zero or doesn't exist. The second derivative in this case is a polynomial and so will exist everywhere. It will be zero at the following points.

$$x = 0, x = \pm \frac{1}{\sqrt{2}} \approx 0.7071$$

As with the increasing and decreasing part we can draw a number line up and use these points to divide the number line into regions. In these regions we know that the second derivative will always have the same value since these three points are the only places where the function *may* change sign. Therefore, all that we need to do is pick a point from each region and plug it into the second derivative. The second derivative will then have that sign in the whole region from which the point came from

Here is the chart for this second derivative.

Interval	$(-\infty, -0.7071)$	$(-0.7071, 0)$	$(0, 0.7071)$	$(0.7071, +\infty)$
Test Value	-2	-0.5	0.5	2
$x$	-	-	+	+
$x + 0.7071$	-	+	+	+
$x - 0.7071$	-	-	-	+
$h''(x)$	-	+	-	+
$h(x)$	concave down	concave up	concave down	concave up

So, it looks like we've got the following intervals of concavity.

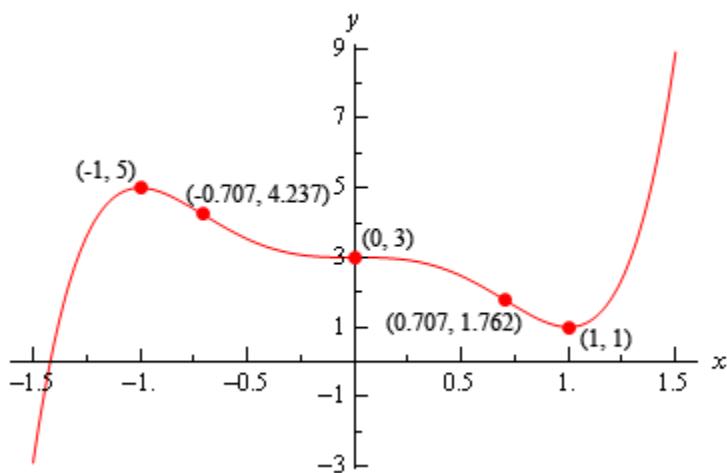
Concave up intervals:  $x \in (-0.7071, 0) \cup (0.7071, +\infty)$

Concave down intervals:  $x \in (-\infty, -0.7071) \cup (0, 0.7071)$

This also means that there are all inflection points at  $x = 0$ ,  $x = \pm \frac{1}{\sqrt{2}} \approx 0.7071$ .

All this information can be a little overwhelming when going to sketch the graph. The first thing that we should do is get some starting points. The critical points and inflection points are good starting points. So, first graph these points. Now, start to the left and start graphing the increasing/decreasing information as we did in the previous section when all we had was the increasing/decreasing information. As we graph this we will make sure that the concavity information matches up with what we're graphing.

Using all this information to sketch the graph gives the following graph.



Next let's look at an example of rational function.

**Example 3.4-2** Sketch the graph of the following function.

$$f(x) = \frac{x^2 - 4}{x^2 - 4x}$$

**Solution**

This time notice that if we were to plug in  $x = 0$  or  $x = 4$  into the denominator we would get division by zero. This means there will not be a  $y$ -intercept for this graph. We have however, managed to find two vertical asymptotes already. And the domain of the function is  $D = \{x \mid -\infty < x < 0, 0 < x < 4, 4 < x < +\infty\}$ . The function has no symmetry.

Now, let's see if we've got  $x$ -intercepts.

$$x^2 - 4 = 0 \quad \Rightarrow \quad x = \pm 2$$

So, we've got two of them.

Let's also go through the process to get the two vertical asymptotes.

$$x^2 - 4x = x(x - 4) = 0 \quad \Rightarrow \quad x = 0, x = 4$$

So, we've got three regions that are  $x < 0$ ,  $0 < x < 4$  and  $x > 4$ .

Next, the largest exponent in both the numerator and denominator is 2 so by the fact there will be a horizontal asymptote at the line,

$$y = \frac{1}{1} = 1$$

Now, one of the  $x$ -intercepts is in the far left region so we don't need any points there. The other  $x$ -intercept is in the middle region. So, we'll need a point in the far right region and as noted in the previous example we will want to get a couple more points in the middle region to completely determine its behavior.

$$\begin{array}{ll} f(1) = 1 & (1, 1) \\ f(3) = -\frac{5}{3} & \left(3, -\frac{5}{3}\right) \\ f(5) = \frac{21}{5} & \left(5, \frac{21}{5}\right) \end{array}$$

We are moving on to get the first two derivatives,

$$f'(x) = \frac{-4(x^2 - 2x + 4)}{(x^2 - 4x)^2}$$

$$f''(x) = \frac{8(x^3 - 3x^2 + 12x - 16)}{(x^2 - 4x)^3}$$

Let  $f'(x) = 0$ , but there is no solutions for critical numbers. As the matter of fact,  $f'(x) < 0$  for all the  $x$  values of the domain. This means that  $f(x)$  is always decreasing.

Let  $f''(x) = 0$ , with the help of calculator we get only one solution

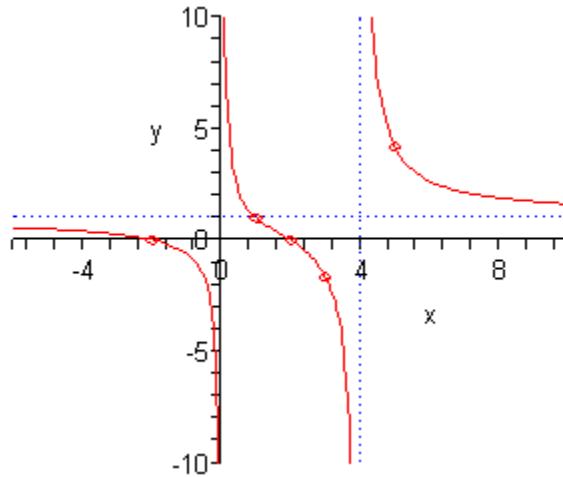
$$x = \sqrt[3]{9} - \frac{3}{\sqrt[3]{9}} + 1 \approx 1.64$$

Now, we construct a chart to test concavity and inflection point:

Interval	$(-\infty, 0)$	$(0, 1.64)$	$(1.64, 4)$	$(4, +\infty)$
Test value	-1	1	2	5
$x^3 - 3x^2 + 12x - 16$	-	-	+	+
$x$	-	+	+	+
$x - 4$	-	-	-	+
$f''(x)$	-	+	-	+
$f(x)$	concave down	concave up	concave down	concave up

Therefore, there is an inflection point at  $x = \sqrt[3]{9} - \frac{3}{\sqrt[3]{9}} + 1 \approx 1.64$ .

Here is the sketch for this function.



Notice that this time the middle region doesn't have the same behavior at the asymptotes as we saw in the previous example. This can and will happen fairly often. Sometimes the behavior at the two asymptotes will be the same as in the previous example and sometimes it will have the opposite behavior at each asymptote as we see in this example. Because of this we will always need to get a couple of points in these types of regions to determine just what the behavior will be.

### 3.5 Optimization Problems

In this section we are going to look at optimization problems. In optimization problems we will be looking for the largest or smallest value of a function subject to some kind of constraint. The constraint will be some condition (that can usually be described by some equation) that must absolutely, positively be true no matter what our solution is. On occasion, the constraint will not be easily described by an equation, but in these problems it will be easy to deal with as we'll see.

This section is generally one of the more difficult for students taking a Calculus course. One of the main reasons for this is that a subtle change of wording can completely change the problem. There is also the problem of identifying the quantity that we'll be optimizing and the quantity that is the constraint and writing down equations for each.

The first step in all of these problems should be to very carefully read the problem. Once you've done that the next step is to identify the quantity to be optimized and the constraint.

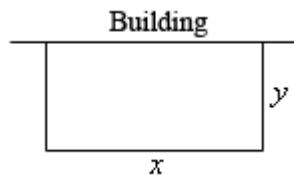
In identifying the constraint remember that the constraint is something that must true regardless of the solution. In almost every one of the problems we'll be looking at here one quantity will be clearly indicated as having a fixed value and so must be the constraint. Once you've got that identified the quantity to be optimized should be fairly simple to get. It is however easy to confuse the two if you just skim the problem so make sure you carefully read the problem first!

Let's start the section off with a simple problem to illustrate the kinds of issues will be dealing with here.

**Example 3.5-1** We need to enclose a field with a fence. We have 500 feet of fencing material and a building is on one side of the field and so won't need any fencing. Determine the dimensions of the field that will enclose the largest area.

**Solution**

In all of these problems we will have two functions. The first is the function that we are actually trying to optimize and the second will be the constraint. Sketching the situation will often help us to arrive at these equations so let's do that.



In this problem we want to maximize the area of a field and we know that will use 500 ft of fencing material. So, the area will be the function we are trying to optimize and the amount of fencing is the constraint. The two equations for these are,

$$\text{Maximize : } A = xy$$

$$\text{Constraint : } 500 = x + 2y$$

Okay, we know how to find the largest or smallest value of a function provided it's only got a single variable. The area function (as well as the constraint) has two variables in it and so what we know about finding absolute extrema won't work. However, if we solve the constraint for one of the two variables we can substitute this into the area and we will then have a function of a single variable.

So, let's solve the constraint for  $x$ . Note that we could have just as easily solved for  $y$  but that would have led to fractions and so, in this case, solving for  $x$  will probably be best.

$$x = 500 - 2y$$

Substituting this into the area function gives a function of  $y$ .

$$A(y) = (500 - 2y)y = 500y - 2y^2$$

Now we want to find the largest value this will have on the interval  $[0, 250]$ . Note that the interval corresponds to taking  $y = 0$  (*i.e.* no sides to the fence) and  $y = 250$  (*i.e.* only two sides and no width, also if there are two sides each must be 250 ft to use the whole 500ft...).

Note that the endpoints of the interval won't make any sense from a physical standpoint if we actually want to enclose some area because they would both give zero area. They do, however, give us a set of limits on  $y$  and so the Extreme Value Theorem tells us that we will have a maximum value of the area somewhere between the two endpoints. Having these limits will also mean that we can use the process we discussed in the

Finding Absolute Extrema section earlier in the chapter to find the maximum value of the area.

So, recall that the maximum value of a continuous function (which we've got here) on a closed interval (which we also have here) will occur at critical points and/or end points. As we've already pointed out the end points in this case will give zero area and so don't make any sense. That means our only option will be the critical points.

So let's get the derivative and find the critical points.

$$A'(y) = 500 - 4y$$

Setting this equal to zero and solving gives a lone critical point of  $y = 125$ . Plugging this into the area gives an area of  $31250 \text{ ft}^2$ . So according to the method from Absolute Extrema section this must be the largest possible area, since the area at either endpoint is zero.

Finally, let's not forget to get the value of  $x$  and then we'll have the dimensions since this is what the problem statement asked for. We can get the  $x$  by plugging in our  $y$  into the constraint.

$$x = 500 - 2(125) = 250$$

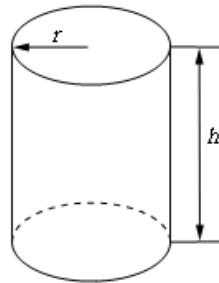
The dimensions of the field that will give the largest area, subject to the fact that we used exactly 500 ft of fencing material, are  $250 \times 125$ .

Don't forget to actually read the problem and give the answer that was asked for. These types of problems can take a fair amount of time/effort to solve and it's not hard to sometimes forget what the problem was actually asking for.

**Example 3.5-2** A manufacturer needs to make a cylindrical can that will hold 1.5 liters of liquid. Determine the dimensions of the can that will minimize the amount of material used in its construction.

### Solution

In this problem the constraint is the volume and we want to minimize the amount of material used. This means that what we want to minimize is the surface area of the can and we'll need to include both the walls of the can as well as the top and bottom "caps". Here is a quick sketch to get us started off.



We'll need the surface area of this can and that will be the surface area of the walls of the can (which is really just a cylinder) and the area of the top and bottom caps (which are just disks, and don't forget that there are two of them).

Note that if you think of a cylinder of height  $h$  and radius  $r$  as just a bunch of disks/circles of radius  $r$  stacked on top of each other the equations for the surface area and volume are pretty simple to remember. The volume is just the area of each of the disks times the height. Similarly, the surface area is just the circumference of the each circle times the height. The equations for the volume and surface area of a cylinder are then,

$$V = (\pi r^2)(h) = \pi r^2 h \quad A = (2\pi r)(h) = 2\pi r h$$

Next, we're also going to need the required volume in a better set of units than liters. Since we want length measurements for the radius and height we'll need to use the fact that 1 Liter = 1000 cm<sup>3</sup> to convert the 1.5 liters into 1500 cm<sup>3</sup>. This will in turn give a radius and height in terms of centimeters.

Here are the equations that we'll need for this problem and don't forget that there two caps and so we'll need the area from each.

$$\text{Minimize : } A = 2\pi r h + 2\pi r^2$$

$$\text{Constraint : } 1500 = \pi r^2 h$$

In this case it looks like our best option is to solve the constraint for  $h$  and plug this into the area function.

$$h = \frac{1500}{\pi r^2} \quad \Rightarrow \quad A(r) = 2\pi r \left( \frac{1500}{\pi r^2} \right) + 2\pi r^2 = 2\pi r^2 + \frac{3000}{r}$$

Notice that this formula will only make sense from a physical standpoint if  $r > 0$  which is a good thing as it is not defined at  $r = 0$

Next, let's get the first derivative.

$$A'(r) = 4\pi r - \frac{3000}{r^2} = \frac{4\pi r^3 - 3000}{r^2}$$

From this we can see that we have two critical points  $r = 0$  and  $r = \sqrt[3]{\frac{750}{\pi}} = 6.2035$ . The first critical point doesn't make sense from a physical standpoint and so we can ignore that one.

So we only have a single critical point to deal with here and notice that 6.2035 is the only value for which the derivative will be zero and hence the only place (with  $r > 0$  of course) that the derivative may change sign. It's not difficult to check that if  $r < 6.2035$ ,  $A'(x) < 0$  and likewise if  $r > 6.2035$  then  $A'(x) > 0$ . The variant of the First Derivative Test above then tells us that the absolute minimum value of the area (for  $r > 0$ ) must occur at  $r = 6.2035$ .

All we need to do this is determine height of the can and we'll be done.

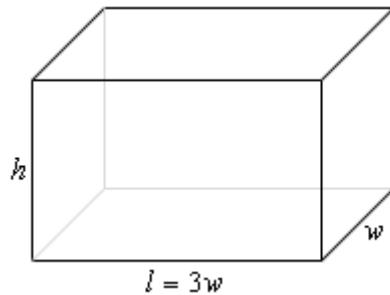
$$h = \frac{1500}{\pi(6.2035)^2} = 12.4070$$

Therefore if the manufacturer makes the can with a radius of 6.2035 cm and a height of 12.4070 cm the least amount of material will be used to make the can.

**Example 3.5-3** We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost \$10/ft<sup>2</sup> and the material used to build the sides cost \$6/ft<sup>2</sup>. If the box must have a volume of 50ft<sup>3</sup> determine the dimensions that will minimize the cost to build the box.

### Solution

First, a quick figure (probably not to scale...).



We want to minimize the cost of the materials subject to the constraint that the volume must be  $50\text{ft}^3$ . Note as well that the cost for each side is just the area of that side times the appropriate cost.

The two functions we'll be working with here this time are,

$$\text{Minimize : } C = 10(2hw) + 6(2wh + 2lh) = 60w^2 + 48wh$$

$$\text{Constraint : } 50 = lwh = 3w^2h$$

As with the first example, we will solve the constraint for one of the variables and plug this into the cost. It will definitely be easier to solve the constraint for  $h$  so let's do that.

$$h = \frac{50}{3w^2}$$

Plugging this into the cost gives,

$$C(w) = 60w^2 + 48w\left(\frac{50}{3w^2}\right) = 60w^2 + \frac{800}{w}$$

Now, let's get the first and second (we'll be needing this later...) derivatives,

$$C'(w) = 120w - 800w^{-2} = \frac{120w^3 - 800}{w^2} \quad C''(w) = 120 + 1600w^{-3}$$

So, it looks like we've got two critical points here. The first is obvious,  $w = 0$  and it's also just as obvious that this will not be the correct value. We are building a box here and  $w$  is the box's width and so since it makes no sense to talk about a box with zero width we will ignore this critical point. This does not mean however that you should just get into the habit of ignoring zero when it occurs. There are other types of problems where it will be a valid point that we will need to consider.

The next critical point will come from determining where the numerator is zero.

$$120w^3 - 800 = 0 \quad \Rightarrow \quad w = \sqrt[3]{\frac{800}{120}} = \sqrt[3]{\frac{20}{3}} = 1.8821$$

So, once we throw out  $w = 0$ , we've got a single critical point and we now have to verify that this is in fact the value that will give the absolute minimum cost.

In this case we can't use Method 1 from above. First, the function is not continuous at one of the endpoints,  $w = 0$ , of our interval of possible values. Secondly, there is no theoretical upper limit to the width that will give a box with volume of  $50\text{ ft}^3$ . If  $w$  is very large then we would just need to make  $h$  very small.

The second method listed above would work here, but that's going to involve some calculations, not difficult calculations, but more work nonetheless.

The third method however, will work quickly and simply here. First, we know that whatever the value of  $w$  that we get it will have to be positive and we can see second derivative above that provided  $w > 0$  we will have  $C''(w) > 0$  and so in the interval of possible optimal values the cost function will always be concave up and so  $w = 1.8821$  must give the absolute minimum cost.

All we need to do now is to find the remaining dimensions.

$$w = 1.8821$$

$$l = 3w = 3(1.8821) = 5.6463$$

$$h = \frac{50}{3w^2} = \frac{50}{3(1.8821)^2} = 4.7050$$

Also, even though it was not asked for, the minimum cost is

$$C(1.8821) = \$637.60$$

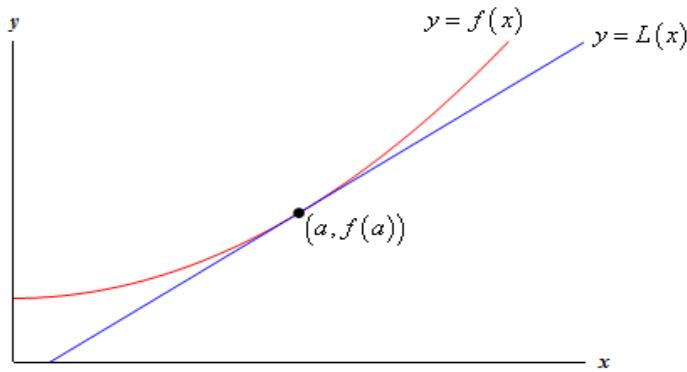
### 3.6 Local Linear Approximations

In this section we're going to take a look at an application not of derivatives but of the tangent line to a function. Of course, to get the tangent line we do need to take derivatives, so in some way this is an application of derivatives as well.

Given a function,  $f(x)$  we can find its tangent at  $x = a$ . The equation of the tangent line, which we'll call  $L(x)$  for this discussion, is,

$$L(x) = f(a) + f'(a)(x - a)$$

Take a look at the following graph of a function and its tangent line.



From this graph we can see that near  $x = a$  the tangent line and the function have nearly the same graph. On occasion we will use the tangent line,  $L(x)$  as an approximation to the function,  $f(x)$ , near  $x = a$ . In these cases we call the tangent line the **linear approximation** to the function at  $x = a$ .

So, why do we do this? Let's take a look at an example.

**Example 3.6-1** Determine the linear approximation for  $f(x) = \sqrt[3]{x}$  at  $x = 8$ . Use the linear approximation to approximate the value of  $\sqrt[3]{8.05}$  and  $\sqrt[3]{25}$ .

### Solution

Since this is just the tangent line there really isn't a whole lot to finding the linear approximation.

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}} \quad f(8) = 2 \quad f'(8) = \frac{1}{12}$$

The linear approximation is then,

$$L(x) = 2 + \frac{1}{12}(x - 8) = \frac{1}{12}x + \frac{4}{3}$$

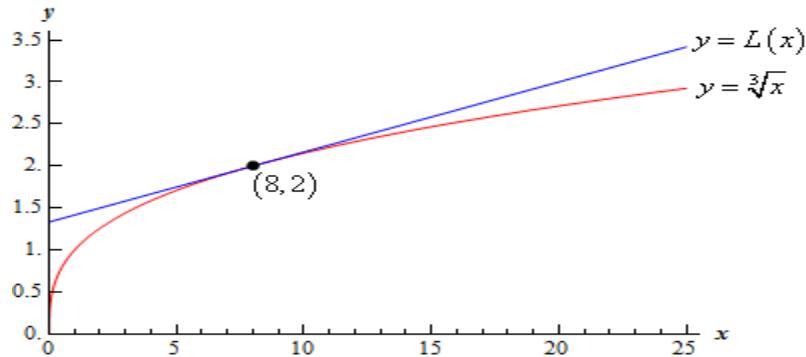
Now, the approximations are nothing more than plugging the given values of  $x$  into the linear approximation. For comparison purposes we'll also compute the exact values.

$$\begin{aligned} L(8.05) &= 2.00416667 & \sqrt[3]{8.05} &= 2.00415802 \\ L(25) &= 3.41666667 & \sqrt[3]{25} &= 2.92401774 \end{aligned}$$

So, at  $x = 8.05$  this linear approximation does a very good job of approximating the actual value. However, at  $x = 25$  it doesn't do such a good job.

This shouldn't be too surprising if you think about. Near  $x = 8$  both the function and the linear approximation have nearly the same slope and since they both pass through the point  $(8, 2)$  they should have nearly the same value as long as we stay close to  $x = 8$ . However, as we move away from  $x = 8$  the linear approximation is a line and so will always have the same slope while the functions slope will change as  $x$  changes and so the function will, in all likelihood, move away from the linear approximation.

Here's a quick sketch of the function and its linear approximation at  $x = 8$ .



As noted above, the farther from  $x = 8$  we get the more distance separates the function itself and its linear approximation.

Linear approximations do a very good job of approximating values of  $f(x)$  as long as we stay "near"  $x = a$ . However, the farther away from  $x = a$  we get the worse the approximation is liable to be. The main problem here is that how near we need to stay to  $x = a$  in order to get a good approximation will depend upon both the function we're using and the value of  $x = a$  that we're using. Also, there will often be no easy way of prediction how far away from  $x = a$  we can get and still have a "good" approximation.

Let's take a look at another example that is actually used fairly heavily in some places.

**Example 3.6-2** Determine the linear approximation for  $\sin\theta$  at  $\theta = 0$ .

### Solution

Again, there really isn't a whole lot to this example. All that we need to do is compute the tangent line to  $\sin\theta$  at  $\theta = 0$ .

$$\begin{aligned} f(\theta) &= \sin\theta & f'(\theta) &= \cos\theta \\ f(0) &= 0 & f'(0) &= 1 \end{aligned}$$

The linear approximation is,

$$\begin{aligned}L(\theta) &= f(0) + f'(0)(\theta - a) \\&= 0 + (1)(\theta - 0) \\&= \theta\end{aligned}$$

So, as long as  $\theta$  stays small we can say that  $\sin \theta \approx \theta$ . Here, we recall an important limit  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$  or  $\lim_{\theta \rightarrow 0} \sin \theta = \lim_{\theta \rightarrow 0} \theta$ . The linear approximation here may explain this limit from another point of view.