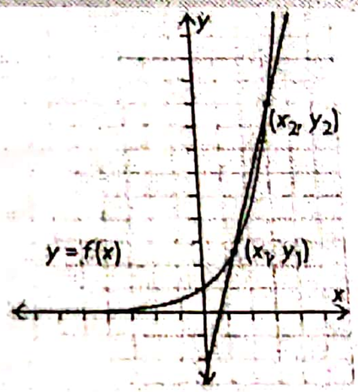
Determining Average Rate of Change

- The average rate of change is the change in the quantity represented by the dependent variable (Δy) divided by the corresponding change in the quantity represented by the independent variable (Δx) over an interval. Algebraically, the average rate of change for any function $y = f(x)$ over the interval $x_1 \le x \le x_2$ can be determined by

$$\text{Average rate of change} = \frac{change\ \ in\ \ y}{change\ \ in\ \ x} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

- Graphically, the average rate of change for any function $y = f(x)$ over the interval $x_1 \le x \le x_2$ is equivalent to the slope of the secant line passing through two points (x_1, y_1) and (x_2, y_2).



Average rate of change

$$= m_{secant} = \frac{change\ \ in\ \ y}{change\ \ in\ \ x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

- Average rate of change is expressed using the units of the two quantities that are related to each other.
- A ***positive average rate of change*** indicates that the quantity represented by the ***dependent variable is increasing on the specified interval***, compared with the quantity represented by the independent variable. Graphically, this is indicated by a ***secant line that has a positive slope*** (the secant line rises from left to right).
- A ***negative average rate of change*** indicates that the quantity represented by the ***dependent variable is decreasing on the specified interval,*** compared with the quantity represented by the independent variable. Graphically, this is indicated by ***a secant line that has a negative slope*** (the secant line falls from left to right).
- ***All linear relationships have a constant rate of change.*** Average rate of change calculations over different intervals of the independent variable give the same result.
- ***Nonlinear relationships do not have a constant rate of change.*** Average rate of change calculations over different intervals of the independent variable give different results.

Example 1

Which of the following are examples of average rates of change?

a) The average height of the players on the basketball team is 2.1 m.
b) A child grows 8 cm in 6 months.
c) A plane travelled 650 km in 3 h.
d) The snowboarder raced across the finish line at 60 km/h

Example 2

If a ball is dropped from the top of a 120-m cliff, its height, h, in metres, after t seconds can be modelled by $h(t) = 120 - 4.9t^2$.

a) Find the average rate of change of the height of the ball with respect to time over the intervals
   i. 1 s to 4 s
   ii. 4 s to 6 s
   iii. 6 s to 7 s
b) What does the average rate of change represent in this situation?

Example 3 Find the slope of the secant over the specified intervals

a) $f(x) = 2^x + 3;\ x \in [3,5]$

c) $f(x) = log_3 x;\ x \in \left[\frac{1}{9}, 27\right]$

b) $f(x) = cos(2x);\ x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

d) $f(x) = cot\, x;\ x \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

Radical Expressions: Rationalizing Denominators

- To rewrite a radical expression with a one-term radical in the denominator, multiply the numerator and denominator by the one-term denominator.

$$\frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a}}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{\sqrt{ab}}{b}$$

- When the denominator of a radical expression is a two-term expression, rationalize the denominator by multiplying the numerator and denominator by the conjugate, and then simplify.

$$\frac{1}{\sqrt{a}-\sqrt{b}} = \frac{1}{\sqrt{a}-\sqrt{b}} \times \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a}+\sqrt{b}}{a-b}$$

Note that $\sqrt{a}+\sqrt{b}$ is the conjugate of $\sqrt{a}-\sqrt{b}$, and vice versa.

Example 1 Write the conjugate of each radical expression.

a) $\sqrt{3}-\sqrt{7}$ b) $8\sqrt{2}+\sqrt{6}$ c) $-1-4\sqrt{5}$

Example 2 Rationalize each denominator.

a) $\dfrac{1}{\sqrt{10}}$ b) $\dfrac{9}{\sqrt{7}-\sqrt{5}}$ c) $\dfrac{\sqrt{10}}{5\sqrt{2}+4\sqrt{3}}$

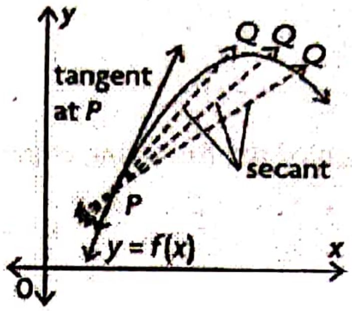
Example 3 Rationalize each numerator.

a) $\dfrac{3\sqrt{6}}{16}$ b) $\dfrac{2\sqrt{3}-4\sqrt{5}}{\sqrt{45}+\sqrt{75}}$ c) $\dfrac{\sqrt{x+81}-9}{x}$

The Slope of a Tangent

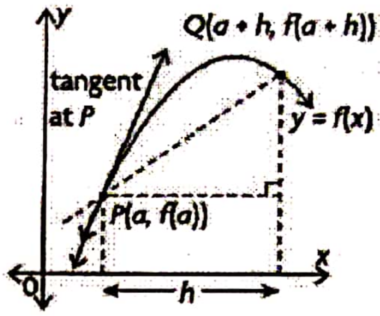
- The slope of the tangent to a curve at a point P is the limit of the slopes of the secants PQ as Q moves closer to P.

  $m_{tangent} = \lim_{Q \to P}$ (slope of secant PQ)



- The slope of the tangent to the graph of $y = f(x)$ at $P(a, f(a))$ is given by

  $$m_{tangent} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



- To find the slope of the tangent at a point $P(a, f(a))$,
  - find the value of $f(a)$
  - find the value of $f(a+h)$
  - evaluate $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Example 1 a) Copy and complete the table for $f(x) = 3x - x^2$ and a tangent at the point where $x = 4$.

Tangent Point $(a, f(a))$	**Increment** h	**Second Point** $(a+h, f(a+h))$	**Slope of Secant** $\frac{f(a+h)-f(a)}{h}$
	1		
	0.1		
	0.01		
	0.001		
	0.0001		

b) What do the values in the last column indicate about the slope of the tangent?

Example 2 Determine the slope of the tangent to each curve at the given value of x.

a) $f(x)=x^3$, $x=2$ b) $g(x)=\sqrt{10-x^2}$, $x=-1$ c) $k(x)=\dfrac{x+1}{x-4}$, $x=3$

Example 3 Determine the equation of the tangent to $f(x)=6x^2-2x$ where $x=1$.

Example 4 Find the slope of the demand curve $D(p)=\dfrac{20}{\sqrt{p-1}}$, $p>1$ at the point $(5,10)$.

Rates of Change

- The average velocity can be found in the same way that we found the slope of the secant.

$$average\ \ velocity = \frac{change\ \ in\ \ position}{change\ \ in\ \ time}$$

- To find the average velocity (average rate of change) form $t = a$ to $t = a + h$, we can use the difference quotient and the position function $s(t)$.

$$\frac{\Delta s}{\Delta t} = \frac{s(a+h) - s(a)}{h}$$

- The instantaneous velocity is the slope of the tangent to the graph of the position function and is found in the same way that we found the slope of the tangent.
- The instantaneous rate of change in the position function, $s(t)$, at $t = a$ is the instantaneous velocity at $t = a$. We can find this value by computing the limiting value of the average velocity as $h \to 0$.

$$v(a) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$

Example 1 If a ball is thrown into the air with a velocity of 30 m/s, its height in metres after t seconds is given by $s(t) = 30t - 4.9t^2$.

a) Find the average velocity for the time period beginning when $t = 2$ and lasting
   i) 1 s ii) 0.5 s iii) 0.1 s iv) 0.01 s

b) Find the instantaneous velocity when $t = 2$.

Example 2 The total cost, in dollars, of manufacturing x calculators is given by $C(x) = 10\sqrt{x} + 1000$.

a) What is the total cost of manufacturing 100 calculators?

b) What is the rate of change in the total cost with respect to the number of calculators, x, being produced when $x = 100$?

Example 3 Suppose that the temperature, T, in degrees Celsius, varies with the height h, in kilometres, above the earth's surface according to the equation $T(h) = \frac{60}{h+2}$. Find the rate of change of temperature with respect to height at a height of 3 km.

Limit of a Function

- The limit of a function $y = f(x)$ at $x = a$ is written as $\lim_{x \to a} f(x) = L$, which means that the function f approaches the value L as x approaches the value a from both the left and right side. In other words, if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$, then L is the limit of f as x approaches a, that is $\lim_{x \to a} f(x) = L$.
- $\lim_{x \to a} f(x)$ can be equal to $f(a)$. In this case, the graph of f passes through the point $(a, f(a))$.
- $\lim_{x \to a} f(x)$ may exist even if $f(a)$ is not defined.

Example 1 a) Construct a table of values to determine each limit in parts (i) and (ii), then use your results to determine the limit in part (iii).

i) $\lim_{x \to 4^-} \sqrt{4-x}$ ii) $\lim_{x \to 4^+} \sqrt{4-x}$ iii) $\lim_{x \to 4} \sqrt{4-x}$

b) Graph the function from part (a). How does the graph support your results in part (a)?

Example 2 Sketch the graph of the piecewise function $f(x)=\begin{cases} x^2, & x \le 1 \\ \dfrac{2}{x}, & 1 < x < 2 \\ x-1, & x \ge 2 \end{cases}$

State the value of each limit.

a) $\lim_{x\to 1^-} f(x)$ b) $\lim_{x\to 1^+} f(x)$ c) $\lim_{x\to 1} f(x)$

d) $\lim_{x\to 2^-} f(x)$ e) $\lim_{x\to 2^+} f(x)$ f) $\lim_{x\to 2} f(x)$

Example 3 Evaluate each limit. If the limit does not exist, explain why.

a) $\lim_{x\to -2} x^2-5x-2$ b) $\lim_{x\to 4} \dfrac{1}{x-4}$ c) $\lim_{x\to -3} \dfrac{x^2-9}{x+3}$

Example 4 Sketch the graph of a function $y = f(x)$ that satisfies each set of conditions.

a) $\lim_{x \to 5^-} f(x) = 0$, $\lim_{x \to 5^+} f(x) = -2$, and $f(5) = -1$

b) $\lim_{x \to 2^-} f(x) = -1$, $\lim_{x \to 2^+} f(x) = -1$, and $f(2) = 4$

c) $\lim_{x \to -3^-} f(x) = 2$, $\lim_{x \to -3^+} f(x)$ does not exist, and $f(-3) = 6$

Properties of Limits

1. $\lim\limits_{x\to a} c = c$

2. $\lim\limits_{x\to a} x = a$

3. $\lim\limits_{x\to a}[c f(x)] = c\lim\limits_{x\to a} f(x)$

4. $\lim\limits_{x\to a}[f(x) \pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$

5. $\lim\limits_{x\to a}[f(x)g(x)] = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x)$

6. $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}, \quad \lim\limits_{x\to a} g(x) \neq 0$

7. $\lim\limits_{x\to a}[f(x)]^n = [\lim\limits_{x\to a} f(x)]^n$, where $n \in \mathrm{N}$

8. $\lim\limits_{x\to a} x^n = a^n$

9. $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)}$, $\lim\limits_{x\to a} f(x) > 0$ if n is even.

Example 1 Evaluate each limit.

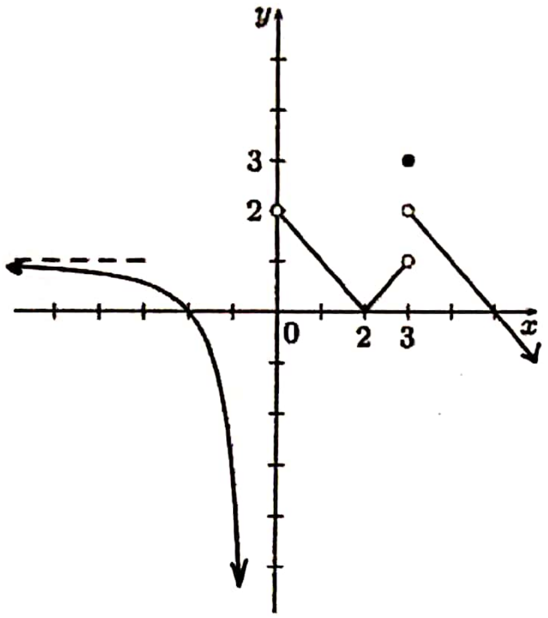
a) $\lim\limits_{x\to 0} x^3 - 6x^2 + 2x - 10$

b) $\lim\limits_{x\to 3} \dfrac{x^2 + x - 12}{x^2 + 3x}$

c) $\lim\limits_{x\to -1} \sqrt{\dfrac{15 - x}{2x^2 + 7}}$

WORKSHEET: LIMITS

1. Use the graph of the function $f(x)$ to answer each question.
   Use ∞, $-\infty$ or DNE where appropriate.



(a) $f(0) =$

(b) $f(2) =$

(c) $f(3) =$

(d) $\lim_{x\to 0^-} f(x) =$

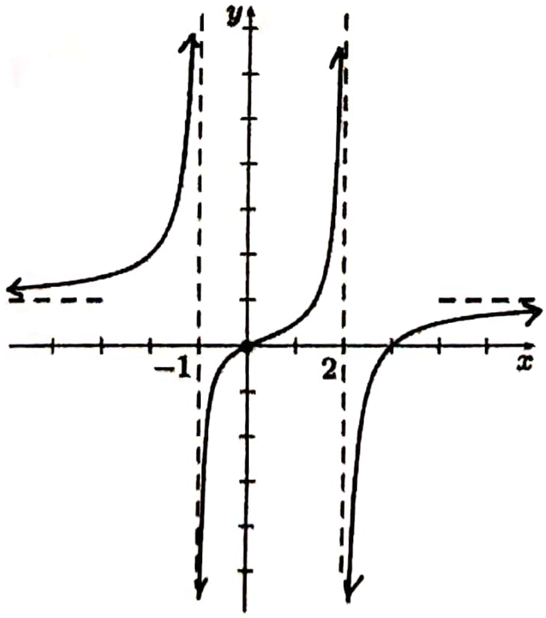
(e) $\lim_{x\to 0} f(x) =$

(f) $\lim_{x\to 3^+} f(x) =$

(g) $\lim_{x\to 3} f(x) =$

(h) $\lim_{x\to -\infty} f(x) =$

2. Use the graph of the function $f(x)$ to answer each question.
   Use ∞, $-\infty$ or DNE where appropriate.



(a) $f(0) =$

(b) $f(2) =$

(c) $f(3) =$

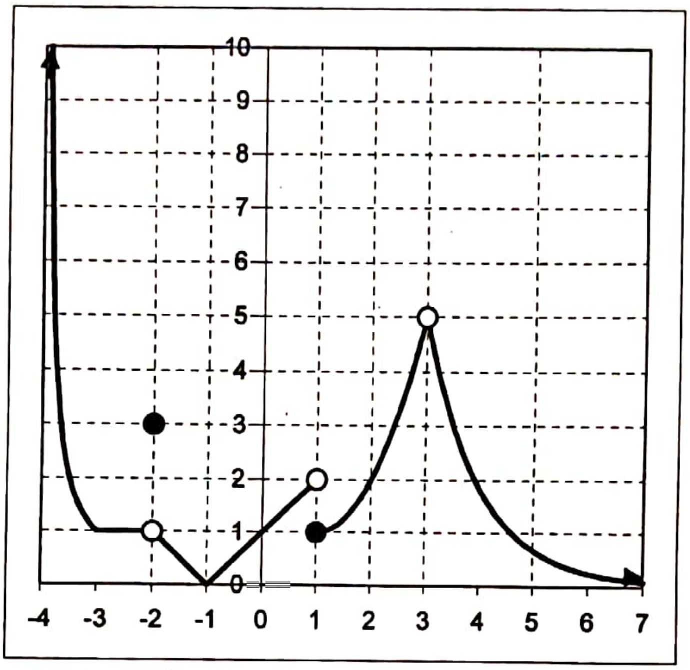
(d) $\lim_{x\to -1} f(x) =$

(e) $\lim_{x\to 0} f(x) =$

(f) $\lim_{x\to 2^+} f(x) =$

(g) $\lim_{x\to \infty} f(x) =$

3. **Limits**



Using the above graph, find each of the following (You should assume that y=0 is a horizontal asymptote and x = -4 is a vertical asymptote):

1) f(-2) = ____ 2) $\lim_{x \to -2^+} f(x) =$ ____ 3) $\lim_{x \to -2} f(x) =$ ____

4) $\lim_{x \to -1^+} f(x) =$ ____ 5) $\lim_{x \to -1^-} f(x) =$ ____ 6) $\lim_{x \to -1} f(x) =$ ____

7) $\lim_{x \to 1^+} f(x) =$ ____ 8) $\lim_{x \to 1^-} f(x) =$ ____ 9) $\lim_{x \to 1} f(x) =$ ____

10) f(3) = ____ 11) $\lim_{x \to 3^+} f(x) =$ ____ 12) $\lim_{x \to 3^-} f(x) =$ ____

13) $\lim_{x \to 3} f(x) =$ ____ 14) $\lim_{x \to -4^+} f(x) =$ ____ 15) $\lim_{x \to \infty} f(x) =$ ____

16) f(1) = ____ 17) $\lim_{x \to -3} f(x) =$ ____ 18) f(-4) = ____