AP Calculus Homework Seven – Applications of Differential Calculus

3.7 Motion along a Curve: Velocity and Acceleration; 3.8 Related Rates; 3.9 Slope of a Polar Curve

1. If the position of a particle moving along a straight line is given by $s = t^3 - 6t^2 + 12t + 8$.
   a) Find the values of t for which s is increasing.
   b) What is the minimum value of the speed?
   c) Find the values of t for which the acceleration is positive.
   d) Find the values of t for which the speed is decreasing.

a) $v(t) = s'(t) = 3t^2 - 12t + 12 = 3(t^2 - 4t + 4) = 3(t-2)^2 > 0$ for all t except $t=2$.
So $s(t)$ is increasing for all $t > 0$ except $t = 2$.

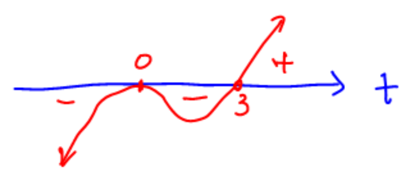
b) The speed $= |v(t)| = |3(t-2)^2| = 3(t-2)^2$; when $t = 2$, the minimum speed is $|v(2)| = 3(2-2)^2 = 0$.

c) $a(t) = v'(t) = [3(t-2)^2]' = 6(t-2) > 0$ when $t > 2$.

d) when $0 < t < 2$, $v(t) > 0$ and $a(t) < 0$, and the speed is decreasing.

2. The displacement from the origin of a particle moving on a line is given by $s = t^4 - 4t^3$. Find the maximum displacement during the time interval $-2 \le t \le 4$.

$v(t) = s'(t) = 4t^3 - 12t^2 = 4t^2(t-3)$;



$s(3) = 3^4 - 4(3)^3 = 3^3(3-4) = -27$, a local min.

Two end points: $s(-2) = (-2)^4 - 4(-2)^3 = (-2)^3(-2-4) = (-8)(-6) = 48$

$s(4) = 4^4 - 4(4)^3 = 0$

Therefore, the maximum displacement is 48.

d) $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\frac{2\pi}{3}\cos\frac{\pi}{3}t}{-\pi\sin\frac{\pi}{3}t} = -\frac{2}{3}\cot\frac{\pi}{3}t$;

$\left.\frac{dy}{dx}\right|_{t=\frac{1}{2}} = -\frac{2}{3}\cot\frac{\pi}{6} = -\frac{2}{3}\sqrt{3}$

3. If $\vec{R} = 3\cos\frac{\pi}{3}t\,\vec{\imath} + 2\sin\frac{\pi}{3}t\,\vec{\jmath}$ is the (position) vector from the origin to a moving point P(x, y) at time t.
   a) What is the equation in x and y for the path of the point?
   b) Find the speed of the point at t = 3.
   c) Find the magnitude of the acceleration at t = 3.
   d) At the point where t = 0.5, what is the slope of the curve along which the point moves?

a) $x(t) = 3\cos(\frac{\pi}{3}t)$; $y(t) = 2\sin(\frac{\pi}{3}t) \Rightarrow \frac{x}{3} = \cos(\frac{\pi}{3}t)$, $\frac{y}{2} = \sin(\frac{\pi}{3}t)$

$\Rightarrow (\frac{x}{3})^2 + (\frac{y}{2})^2 = \cos^2(\frac{\pi}{3}t) + \sin^2(\frac{\pi}{3}t) = 1 \Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1$, ellipse

b) $\vec{v}(t) = \vec{R}'(t) = [(3\cos\frac{\pi}{3}t)', (2\sin\frac{\pi}{3}t)'] = [-3\sin\frac{\pi}{3}t(\frac{\pi}{3}t)', 2\cos\frac{\pi}{3}t(\frac{\pi}{3}t)']$

$= [-\pi\sin\frac{\pi}{3}t, \frac{2\pi}{3}\cos\frac{\pi}{3}t]$; The speed at $t=3$ is :

$|\vec{v}(3)| = \sqrt{(-\pi\sin\frac{\pi}{3}(3))^2 + (\frac{2\pi}{3}\cos\frac{\pi}{3}(3))^2} = \frac{2\pi}{3}$

c) $\vec{a}(t) = \vec{v}'(t) = [-\frac{\pi^2}{3}\cos\frac{\pi}{3}t, -\frac{2\pi^2}{9}\sin\frac{\pi}{3}t]$; $|\vec{a}(3)| = \sqrt{(-\frac{\pi^2}{3}\cos\frac{\pi}{3}(3))^2 + (-\frac{2\pi^2}{9}\sin\frac{\pi}{3}(3))^2} = \frac{\pi^2}{3}$

4. A balloon is being filled with helium at the rate of 4 ft^3/min. Find the rate, in square feet per minute, at which the surface area is increasing when the volume is $32\pi/3$ ft^3.

Let V be the volume, S be the surface area, r be the radius of the balloon

Then, $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$; Given $\frac{dV}{dt} = 4$ ft^3/min. Find $\frac{dS}{dt}$ when $V = \frac{32\pi}{3}$ ft^3

$\because r = \sqrt{\frac{S}{4\pi}}$, $\therefore V = \frac{4}{3}\pi(\frac{S}{4\pi})^{3/2} = \frac{1}{3\sqrt{4\pi}}S^{3/2}$, $\frac{dV}{dt} = \frac{1}{3\sqrt{4\pi}}(\frac{3}{2})S^{\frac{1}{2}}\frac{dS}{dt}$, sub. given values

$\frac{32\pi}{3} = \frac{1}{6\sqrt{\pi}}S^{3/2}$

$64\pi^{3/2} = S^{3/2}$

$\Rightarrow S = 16\pi$

$4 = \frac{1}{4\sqrt{\pi}}(16\pi)^{\frac{1}{2}}\frac{dS}{dt} \Rightarrow \frac{dS}{dt} = 4$ ft^2/min.

5. The height of a rectangular box is 10 cm. Its length increases at the rate of 2 cm/sec; its width decreases at the rate of 4 cm/sec. When the length is 8 cm and the width is 6 cm, what is the rate, in cubic cm per second, at which the volume of the box is changing?

Let V be the volume, L be the length, W be the width of the box.

Given $\frac{dL}{dt} = 2$ cm/s ; $\frac{dW}{dt} = -4$ cm/s ; Find $\frac{dV}{dt}$ at $L = 8$cm, $W = 6$cm;

$\because V = 10LW$ $\therefore \frac{dV}{dt} = 10(W\cdot\frac{dL}{dt} + L\cdot\frac{dW}{dt}) = 10(6\cdot(2) + (8)(-4))$

$= 10(12-32) = 10(-20) = -200$ (cm^3/s)

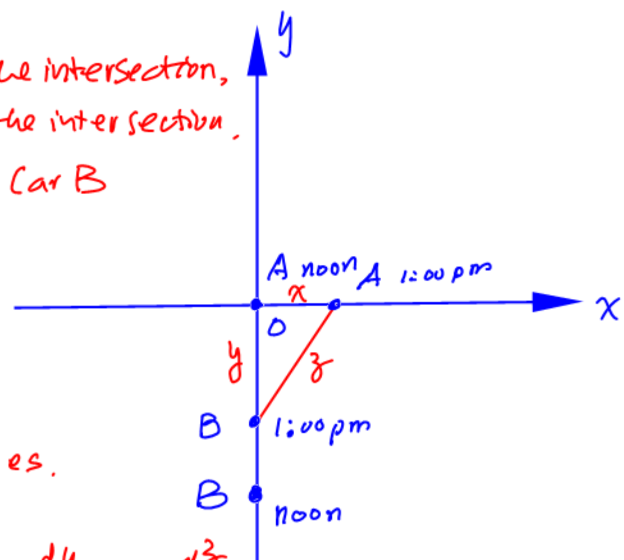
So the volume is decreasing at a rate of 200 cm^3/s.

6. The table shows the velocity at time *t* of an object moving along a line. Estimate the acceleration (in cm/sec^2) at $t = 6$ sec.

t (sec)	0	4	8	10
velocity	18	16	10	0

$$a(6) \approx \frac{v(8)-v(4)}{8-4} = \frac{10-16}{+4} = -\frac{6}{4} = -\frac{3}{2} \ (\text{cm}/\text{s}^2)$$

7. Two cars are traveling along perpendicular roads, car A at 40 mph, car B at 60 mph. At noon, when car A reaches the intersection, car B is 90 miles away, and moving toward it. What is the rate, in miles per hour, at which the distance between the cars is changing at 1:00 P.M.?



Let x be the distance between Car A and the intersection,
y be the distance between Car B and the intersection,
z be the distance between Car A and Car B

Then $x^2+y^2=z^2$

Given $\frac{dx}{dt} = 40$ mph, $\frac{dy}{dt} = -60$ mph.

Find $\frac{dz}{dt}$ at $x = 40$ miles, $y = 30$ miles.

$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(z^2) \Rightarrow \cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} = \cancel{2}z\frac{dz}{dt}$, sub. all the given values:

$40(40) + (30)(-60) = 50\frac{dz}{dt}$ where $z = \sqrt{x^2+y^2} = \sqrt{40^2+30^2} = 50$

$$\frac{dz}{dt} = \frac{160-180}{5} = \frac{-20}{5} = -4 \ (\text{mph})$$

So the distance between the cars is decreasing at the rate of 4 mph. at 1:00 pm.

8. Find the points at which the curve given by $r = 1 + \cos\theta$ has a vertical or horizontal tangent line for $0 < \theta \leq 2\pi$.

$\because \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\quad \therefore \dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{r'(\theta)\sin\theta + r(\theta)\cos\theta}{r'(\theta)\cos\theta - r(\theta)\sin\theta}$

where $r(\theta) = 1+\cos\theta$, $r'(\theta) = -\sin\theta$

For a vertical tangent line, $r'(\theta)\cos\theta - r(\theta)\sin\theta = 0$

$\Rightarrow -\sin\theta\cos\theta - (1+\cos\theta)\sin\theta = 0$

$\Rightarrow -\sin\theta(\cos\theta + 1 + \cos\theta) = 0 \Rightarrow \sin\theta = 0$ or $\cos\theta = -\frac{1}{2}$

$\therefore \theta = \pi$ or 2π; $\quad \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ or $\pi + \frac{\pi}{3} = \frac{4\pi}{3}$;

$r(\theta) = 0$ or 2; $\quad \frac{1}{2}$ or $\frac{1}{2}$

$\begin{cases} x = 0 \\ y = 0 \end{cases}$ or $\begin{cases} x = -2 \\ y = 0 \end{cases}$ or $\begin{cases} x = \frac{1}{2} \\ y = \frac{\sqrt{3}}{4} \end{cases}$ or $\begin{cases} x = \frac{1}{2} \\ y = -\frac{\sqrt{3}}{4} \end{cases}$

For the horizontal tangent line, $r'(\theta)\sin\theta + r(\theta)\cos\theta = 0$

$\Rightarrow -\sin^2\theta + (1+\cos\theta)\cos\theta = 0$

$\Rightarrow -(1-\cos^2\theta) + \cos\theta + \cos^2\theta = 0 \Rightarrow 2\cos^2\theta + \cos\theta - 1 = 0$

$\Rightarrow \cos\theta = \frac{1}{2}$ or $\cos\theta = -1$

$\begin{matrix} 2\cos\theta & & -1 \\ & \times & \\ \cos\theta & & +1 \end{matrix}$

$\Rightarrow \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ or $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$; or $\theta = \pi$.

$r = \frac{3}{2}$ or $\frac{3}{2}$ or 0

$(r, \theta) = (\frac{3}{2}, \frac{2\pi}{3}), (\frac{3}{2}, \frac{5\pi}{3}), (0, \pi)$.

Answers

Q1: (a) all t except $t = 2$, (b) 0, (c) $t > 2$, (d) $t < 2$; Q2: 48;
Q3: (a) $4x^2 + 6y^2 = 36$, Ellipse; (b) $2\pi/3$, (c) $\pi^2/3$, (d) $-2\sqrt{3}/3$; Q4: 4 ft^2/min;
Q5: –200 cm^3/sec; Q6: –1.5; Q7: –4 mph;
Q8: $(r, \theta) = (2, 0), (0, \pi), (2, 2\pi), (1/2, 4\pi/3), (1/2, 2\pi/3), (3/2, \pi/3), (3/2, 5\pi/3)$.