

AP Calculus Homework Seven – Applications of Differential Calculus

3.7 Motion along a Curve: Velocity and Acceleration; 3.8 Related Rates; 3.9 Slope of a Polar Curve

1. If the position of a particle moving along a straight line is given by

$$s = t^3 - 6t^2 + 12t + 8.$$

- Find the values of t for which s is increasing.
- What is the minimum value of the speed?
- Find the values of t for which the acceleration is positive.
- Find the values of t for which the speed is decreasing.

$$a) \quad v(t) = s'(t) = 3t^2 - 12t + 12 = 3(t^2 - 4t + 4) = 3(t-2)^2 > 0 \text{ for all } t \text{ except } t=2.$$

So $s(t)$ is increasing for all $t > 0$ except $t=2$.

$$b) \quad \text{The speed} = |v(t)| = |3(t-2)^2| = 3(t-2)^2; \text{ when } t=2, \text{ the minimum speed is } |v(2)| = 3(2-2)^2 = 0.$$

$$c) \quad a(t) = v'(t) = [3(t-2)^2]' = 6(t-2) > 0 \text{ when } t > 2.$$

d) When $0 < t < 2$, $v(t) > 0$ and $a(t) < 0$, and the speed is decreasing.

2. The displacement from the origin of a particle
- moving on a line
- is given by

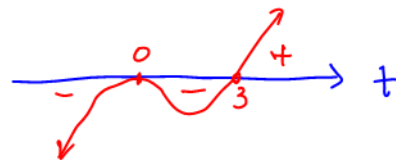
$$s = t^4 - 4t^3. \text{ Find the maximum displacement during the time interval } -2 \leq t \leq 4.$$

$$v(t) = s'(t) = 4t^3 - 12t^2 = 4t^2(t-3);$$

$$s(3) = 3^4 - 4(3)^3 = 3^3(3-4) = -27, \text{ a local min.}$$

$$\text{Two end points: } s(-2) = (-2)^4 - 4(-2)^3 = (-2)^3(-2-4) = (-8)(-6) = 48$$

$$s(4) = 4^4 - 4(4)^3 = 0$$



Therefore, the maximum displacement is 48.

$$d) \frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\frac{2\pi}{3} \cos \frac{\pi}{3} t}{-\pi \sin \frac{\pi}{3} t} = -\frac{2}{3} \cot \frac{\pi}{3} t;$$

$$\frac{dy}{dx} \Big|_{t=\frac{1}{2}} = -\frac{2}{3} \cot \frac{\pi}{6} = -\frac{2}{3} \sqrt{3}$$

3. If $\vec{R} = 3 \cos \frac{\pi}{3} t \vec{i} + 2 \sin \frac{\pi}{3} t \vec{j}$ is the (position) vector from the origin to a moving point $P(x, y)$ at time t .

- What is the equation in x and y for the path of the point?
- Find the speed of the point at $t = 3$.
- Find the magnitude of the acceleration at $t = 3$.
- At the point where $t = 0.5$, what is the slope of the curve along which the point moves?

$$a) \quad x(t) = 3 \cos \left(\frac{\pi}{3} t \right); \quad y(t) = 2 \sin \left(\frac{\pi}{3} t \right) \Rightarrow \frac{x}{3} = \cos \left(\frac{\pi}{3} t \right), \quad \frac{y}{2} = \sin \left(\frac{\pi}{3} t \right)$$

$$\Rightarrow \left(\frac{x}{3} \right)^2 + \left(\frac{y}{2} \right)^2 = \cos^2 \left(\frac{\pi}{3} t \right) + \sin^2 \left(\frac{\pi}{3} t \right) = 1 \Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1, \text{ ellipse.}$$

$$b) \quad \vec{v}(t) = \vec{R}'(t) = \left[(3 \cos \frac{\pi}{3} t)', (2 \sin \frac{\pi}{3} t)' \right] = \left[-3 \sin \frac{\pi}{3} t \left(\frac{\pi}{3} \right)', 2 \cos \frac{\pi}{3} t \left(\frac{\pi}{3} \right)' \right]$$

$$= \left[-\pi \sin \frac{\pi}{3} t, \frac{2\pi}{3} \cos \frac{\pi}{3} t \right]; \text{ The speed at } t=3 \text{ is:}$$

$$|\vec{v}(3)| = \sqrt{(-\pi \sin \frac{\pi}{3}(3))^2 + (\frac{2\pi}{3} \cos \frac{\pi}{3}(3))^2} = \frac{2\pi}{3}$$

$$c) \quad \vec{a}(t) = \vec{v}'(t) = \left[-\frac{\pi^2}{3} \cos \frac{\pi}{3} t, -\frac{2\pi^2}{3} \sin \frac{\pi}{3} t \right]; \quad |\vec{a}(3)| = \sqrt{\left(-\frac{\pi^2}{3} \cos \frac{\pi}{3}(3) \right)^2 + \left(-\frac{2\pi^2}{3} \sin \frac{\pi}{3}(3) \right)^2}$$

4. A balloon is being filled with helium at the rate of $4 \text{ ft}^3/\text{min}$. Find the rate, in square feet per minute, at which the surface area is increasing when the volume is $32\pi/3 \text{ ft}^3$.

$$= \frac{\pi^2}{3}$$

Let V be the volume, S be the surface area, r be the radius of the balloon.

Then, $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$; Given $\frac{dV}{dt} = 4 \text{ ft}^3/\text{min}$. Find $\frac{dS}{dt}$ when $V = \frac{32\pi}{3} \text{ ft}^3$.

$\therefore r = \sqrt{\frac{S}{4\pi}}$, $\therefore V = \frac{4}{3}\pi \left(\frac{S}{4\pi} \right)^{3/2} = \frac{1}{3\sqrt{4\pi}} S^{3/2}$, $\frac{dV}{dt} = \frac{1}{3\sqrt{4\pi}} \left(\frac{3}{2} \right) S^{1/2} \frac{dS}{dt}$, sub. given values

$4 = \frac{1}{4\sqrt{\pi}} (16\pi)^{1/2} \frac{dS}{dt} \Rightarrow \frac{dS}{dt} = 4 \text{ ft}^2/\text{min.}$

$\frac{32\pi}{3} = \frac{1}{6\pi} S^{3/2}$
 $64\pi^{3/2} = S^{3/2}$
 $\Rightarrow S = 16\pi$

5. The height of a rectangular box is 10 cm. Its length increases at the rate of 2 cm/sec; its width decreases at the rate of 4 cm/sec. When the length is 8 cm and the width is 6 cm, what is the rate, in cubic cm per second, at which the volume of the box is changing?

Let V be the volume, L be the length, w be the width of the box.

Given $\frac{dL}{dt} = 2 \text{ cm/s}$; $\frac{dw}{dt} = -4 \text{ cm/s}$; Find $\frac{dV}{dt}$ at $L = 8 \text{ cm}$, $w = 6 \text{ cm}$;

$$\therefore V = 10Lw \quad \therefore \frac{dV}{dt} = 10 \left(w \cdot \frac{dL}{dt} + L \cdot \frac{dw}{dt} \right) = 10(6 \cdot (2) + (8)(-4))$$

$$= 10(12 - 32) = 10(-20) = -200 \text{ cm}^3/\text{s}$$

So the volume is decreasing at a rate of $200 \text{ cm}^3/\text{s}$.

6. The table shows the velocity at time t of an object moving along a line. Estimate the acceleration (in cm/sec^2) at $t = 6$ sec.

t (sec)	0	4	8	10
velocity	18	16	10	0

$$a(6) \approx \frac{V(8) - V(4)}{8 - 4} = \frac{10 - 16}{+4} = -\frac{6}{4} = -\frac{3}{2} \text{ (cm/s}^2\text{)}$$

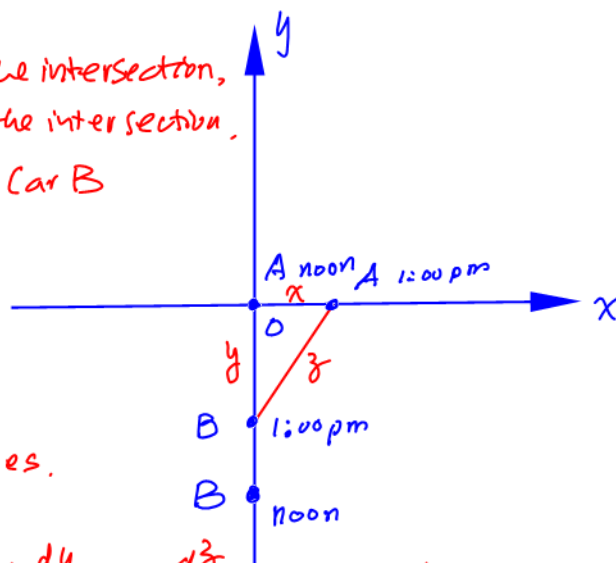
7. Two cars are traveling along perpendicular roads, car A at 40 mph, car B at 60 mph. At noon, when car A reaches the intersection, car B is 90 miles away, and moving toward it. What is the rate, in miles per hour, at which the distance between the cars is changing at 1:00 P.M.?

Let x be the distance between Car A and the intersection,
 y be the distance between Car B and the intersection,
 z be the distance between Car A and Car B

$$\text{Then } x^2 + y^2 = z^2$$

$$\text{Given } \frac{dx}{dt} = 40 \text{ mph, } \frac{dy}{dt} = -60 \text{ mph.}$$

$$\text{Find } \frac{dz}{dt} \text{ at } x = 40 \text{ miles, } y = 30 \text{ miles.}$$



$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(z^2) \Rightarrow 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}, \text{ sub. all the given values,}$$

$$40(40) + (30)(-60) = 50 \frac{dz}{dt}$$

$$\text{where } z = \sqrt{x^2 + y^2} = \sqrt{40^2 + 30^2} = 50$$

$$\frac{dz}{dt} = \frac{160 - 180}{5} = -\frac{20}{5} = -4 \text{ (mph)}$$

So the distance between the cars is decreasing at the rate of 4 mph. at 1:00 pm.

8. Find the points at which the curve given by $r = 1 + \cos\theta$ has a vertical or horizontal tangent line for $0 < \theta \leq 2\pi$.

$$\therefore \begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad \therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r'(\theta) \sin \theta + r(\theta) \cos \theta}{r'(\theta) \cos \theta - r(\theta) \sin \theta}$$

where $r(\theta) = 1 + \cos \theta$, $r'(\theta) = -\sin \theta$

For a vertical tangent line, $r'(\theta) \cos \theta - r(\theta) \sin \theta = 0$

$$\Rightarrow -\sin \theta \cos \theta - (1 + \cos \theta) \sin \theta = 0$$

$$\Rightarrow -\sin \theta (\cos \theta + 1 + \cos \theta) = 0 \Rightarrow \sin \theta = 0 \text{ or } \cos \theta = -\frac{1}{2}$$

$$\therefore \theta = \pi \text{ or } 2\pi; \quad \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \text{ or } \pi + \frac{\pi}{3} = \frac{4\pi}{3};$$

$$r(\theta) = 0 \text{ or } 2;$$

$$\frac{1}{2} \text{ or } \frac{1}{2}$$

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \text{ or } \begin{cases} x = -2 \\ y = 0 \end{cases} \text{ or } \begin{cases} x = \frac{1}{2} \\ y = \frac{\sqrt{3}}{4} \end{cases} \text{ or } \begin{cases} x = \frac{1}{2} \\ y = -\frac{\sqrt{3}}{4} \end{cases}$$

For the horizontal tangent line, $r'(\theta) \sin \theta + r(\theta) \cos \theta = 0$

$$\Rightarrow -\sin^2 \theta + (1 + \cos \theta) \cos \theta = 0$$

$$\Rightarrow -(1 - \cos^2 \theta) + \cos \theta + \cos^2 \theta = 0 \Rightarrow 2\cos^2 \theta + \cos \theta - 1 = 0$$

$$\Rightarrow \cos \theta = \frac{1}{2} \text{ or } \cos \theta = -1$$

$$\begin{matrix} 2\cos \theta & -1 \\ \cos \theta & +1 \end{matrix} \times$$

$$\Rightarrow \theta = \frac{\pi}{3} \text{ or } 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}; \text{ or } \theta = \pi.$$

$$r = \frac{3}{2} \text{ or } 0$$

$$(r, \theta) = \left(\frac{3}{2}, \frac{\pi}{3}\right), \left(\frac{3}{2}, \frac{5\pi}{3}\right), (0, \pi).$$

Answers

Q1: (a) all t except $t = 2$, (b) 0, (c) $t > 2$, (d) $t < 2$; Q2: 48;

Q3: (a) $4x^2 + 6y^2 = 36$, Ellipse; (b) $2\pi/3$, (c) $\pi^2/3$, (d) $-2\sqrt{3}/3$; Q4: 4 ft²/min;

Q5: $-200 \text{ cm}^3/\text{sec}$; Q6: -1.5 ; Q7: -4 mph ;

Q8: $(r, \theta) = (2, 0), (0, \pi), (2, 2\pi), (1/2, 4\pi/3), (1/2, 2\pi/3), (3/2, \pi/3), (3/2, 5\pi/3).$