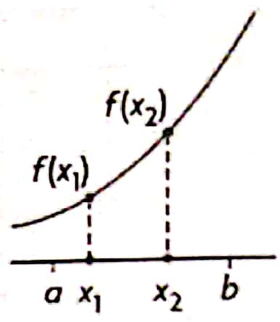
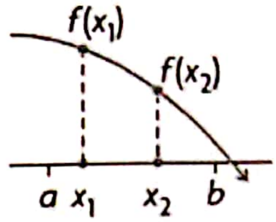
Increasing and Decreasing Functions

- A function increases on an interval if the graph rises from left to right. That is, f is increasing on an interval if, for any value of $x_1 < x_2$ in the interval, $f(x_1) < f(x_2)$.
- For a function f that is continuous and differentiable on an interval I, f is increasing on I if $f'(x) > 0$ for all values of x in I. That is, the slope of the tangent at a point on a section of a curve that is increasing is always positive.



- A function decreases on an interval if the graph falls from left to right. That is, f is decreasing on an interval if, for any value of $x_1 < x_2$ in the interval, $f(x_1) > f(x_2)$.
- For a function f that is continuous and differentiable on an interval I, f is decreasing on I if $f'(x) < 0$ for all values of x in I. That is, the slope of the tangent at a point on a section of a curve that is decreasing is always negative.



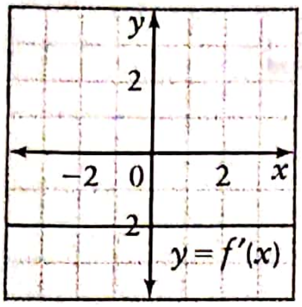
Example 1 Sketch a continuous function for each set of conditions.

a) $f'(x) > 0$ when $x < 0$, $f'(x) < 0$ when $x > 0$, and $f(0) = 4$.

b) $f'(x) > 0$ when $x < -1$ and when $x > 2$, $f'(x) < 0$ when $-1 < x < 2$, and $f(0) = 0$.

Example 2 For each function, use the graph of $y = f'(x)$ to sketch a possible function $y = f(x)$.

a)



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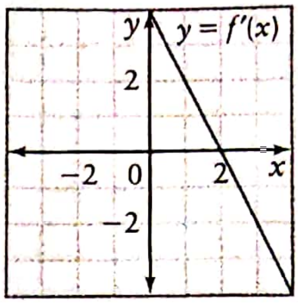

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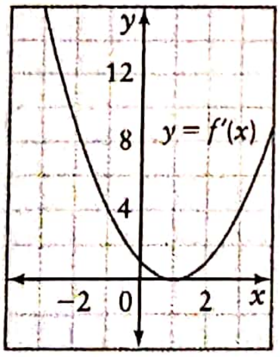

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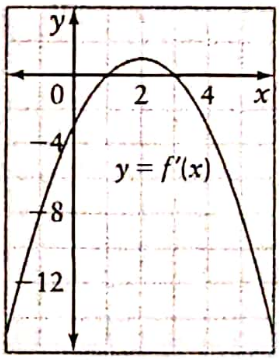


Example 3 Find the intervals of increase and decrease for each of the following functions.

a) $f(x) = 2x^3 + 3x^2 - 36x + 5$

b) $g(x) = \frac{1 - x^2}{x}$

Example 4 Show that the function $f(x) = x^3 + 12x - 3$ is increasing for all values of x.

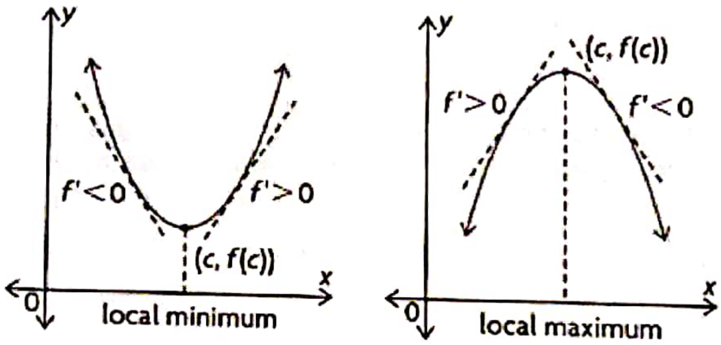
Example 5 A cell culture experiencing changing environmental conditions has a rate of growth modelled by the function $r(t) = 32t^2 - t^4$, $0 < t < 6$, with $r(t)$ measured in cells per hour.

a) When is the rate of growth of the cell culture increasing?

b) Does the rate of growth ever decrease? Explain.

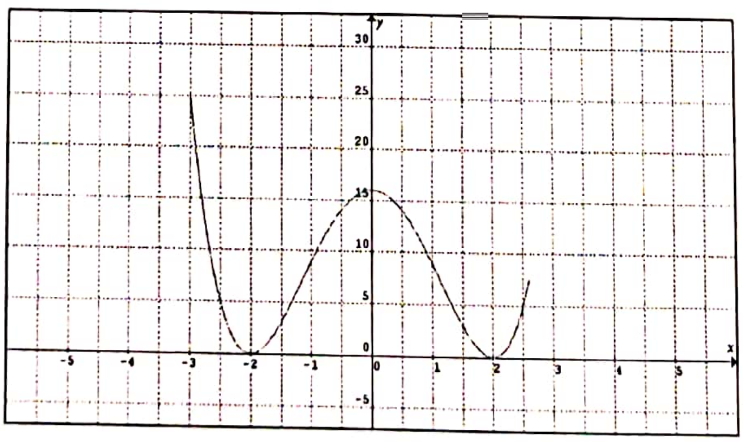
Critical Points, Local Maxima, and Local Minima

- For a function f, a critical number is a number, c, in the domain of f such that $f'(x)=0$ or $f'(x)$ is undefined. As a result, $(c, f(c))$ is called a critical point and usually corresponds to local or absolute extrema.
- First Derivative Test
  Let c be a critical number of a function f.
  When moving through x-values from left to right:
  1. if $f'(x)$ changes from negative to positive at c, then $(c, f(c))$ is a local minimum of f
  2. if $f'(x)$ changes from positive to negative at c, then $(c, f(c))$ is a local maximum of f
  3. if $f'(x)$ does not change its sign at c, then $(c, f(c))$ is neither a local minimum nor a local maximum

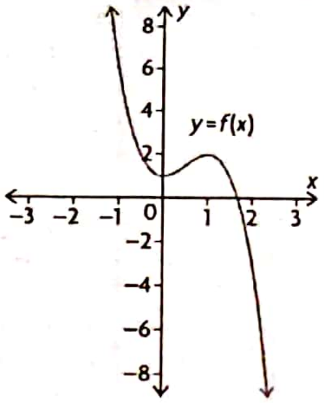


- Algorithm for Finding Local Maximum and Minimum Values of a Function f
  1. Find critical numbers of the function (that is, determine where $f'(x)=0$ and where $f'(x)$ is undefined) for all x-values in the domain of f.
  2. Use the first derivative to analyze whether f is increasing or decreasing on either side of each critical number.
  3. Based on your finding in step 2, conclude whether each critical number locates a local maximum value of the function f, a local minimum value, or neither.

Example 1 Determine the local and absolute extrema of the following function.



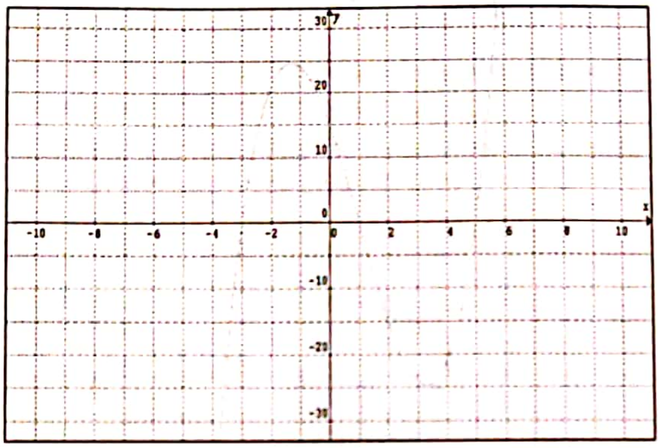
Example 2 Given the graph of a polynomial function $y = f(x)$, graph $y = f'(x)$.



Example 3 Determine the critical points for each of the following functions, and determine whether the function has a local maximum value, a local minimum value, or neither at the critical points. Sketch the graph of each function (graphing calculator).

a) $f(x) = 8x^3 - 6x^4$ b) $g(x) = (x+2)^{\frac{2}{3}}$ c) $h(x) = \dfrac{7x}{x^2 + 25}$

Example 4 Given the graph of the derivative function, $y = f'(x)$, for what values of x does the graph of $y = f(x)$ have a local maximum? a local minimum?

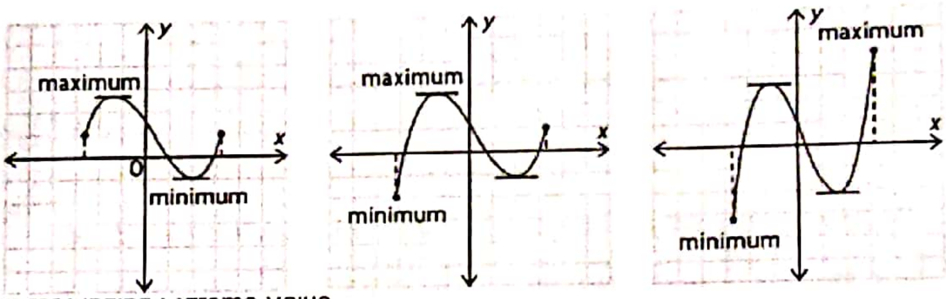


Example 5 The shape of a section of a roller coaster can be modelled by the function $f(x) = -0.5x^3 - 3x^2 + 2x + 12$, where x represents the horizontal distance, in metres, and $-7 \le x \le 3$. Find all the local extrema. Explain what portions of the roller coaster the extrema represent.

Example 6 For the quartic function $f(x) = ax^4 + bx^2 + cx + d$, find the values of a, b, c, and d such that there is a local maximum at $(0, -6)$ and a local minimum at $(1, -8)$.

Minimum and Maximum on an Interval (Extreme Values)

- The maximum and minimum values of a function on an interval are also called extreme values, or absolute extrema.
- The maximum value of a function that has a derivative at all points in an interval occurs at a "peak" $(f'(c)=0)$ or at an endpoint of the interval, $[a, b]$.
- The minimum value of a function that has a derivative at all points in an interval occurs at a "valley" $(f'(c)=0)$ or at an endpoint of the interval, $[a, b]$.



- Algorithm for Finding Extreme Value
  For a function f that is continuous over the interval $[a, b]$, the maximum or minimum values can be found by using the following procedure:
  1. Determine $f'(x)$. Find all points in the interval $a \leq x \leq b$ where $f'(x)=0$ or where $f'(x)$ does not exist.
  2. Evaluate $f(x)$ at the endpoints a and b, and at the points where $f'(x)=0$ and where $f'(x)$ does not exist.
  3. Compare all the values found in step 2.
     - The largest of these values is the maximum value of f on the interval $a \leq x \leq b$.
     - The smallest of these values is the minimum value of f on the interval $a \leq x \leq b$.

Example 1 State, with reasons, why the algorithm for finding extreme values can or cannot be used to determine the maximum and minimum values of the following functions.

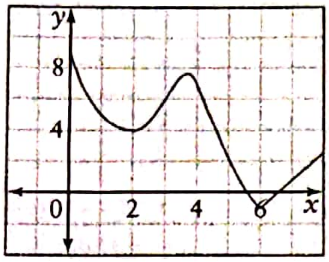
a) $y = x^5 + 10x^3 - 2x + 8$, $-6 \leq x \leq 0$

b) $y = \dfrac{x+4}{x^2-9}$, $-1 \leq x < 5$

c) $y = \dfrac{x+4}{x^2-9}$, $-2 \leq x \leq 2$

Example 2 State the absolute maximum value and the absolute minimum value of each function, if the function is defined on the interval shown.

a)



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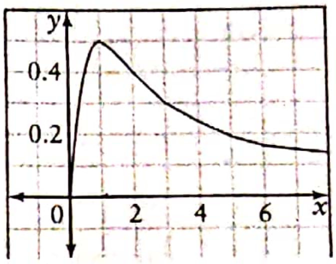

Example 3 Determine the absolute extrema of each function on the given interval.

a) $f(x) = x - \sqrt{x}$ for $x \in [0, 6]$ b) $g(x) = \dfrac{10}{x^2 - 3x + 4}$ for $x \in [-5, 1]$

Example 4 The position function $s = \dfrac{2 + 3t^2}{t + 4}$, where t is in seconds and s is in metres, describes the motion of a particle. Determine the maximum and minimum velocities of the particle over the interval $0 \leq t \leq 3$.