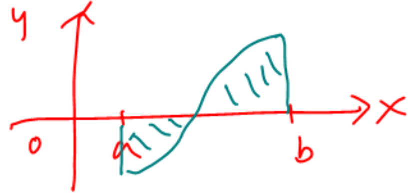

AP Calculus Homework Nine – Antiderivatives and the Definite Integral

4.4 Definition of Definite Integral and Properties of Definite Integral

4.4 The Mean Value Theorem for Definite Integral; 4.5 The Fundamental Theorem of Calculus

1. Evaluate the definite integrals.

(a) $\int_1^4 (x^2 - 4x - 3)dx$

$= \int_1^4 x^2 dx - 4\int_1^4 x\,dx - 3\int_1^4 dx$

$= \frac{x^3}{3}\Big|_1^4 - 4\left(\frac{x^2}{2}\right)\Big|_1^4 - 3(x)\Big|_1^4$

$= \frac{1}{3}[4^3 - 1^3] - 2(4^2 - 1^2) - 3(4-1) = -18$

(b) $\int_1^2 [5/(8x^6)]dx = \frac{5}{8}\int_1^2 x^{-6}dx$

$= \frac{5}{8}\,\frac{x^{-6+1}}{-6+1}\Big|_1^2 = -\frac{1}{8}(x^{-5})\Big|_1^2$

$= -\frac{1}{8}(2^{-5} - 1^{-5}) = \frac{31}{256}$

(c) $\int_{-3}^6 |x-4|dx$

$= \int_{-3}^4 (4-x)dx + \int_4^6 (x-4)dx$

$= \left(4x - \frac{x^2}{2}\right)\Big|_{-3}^4 + \left(\frac{x^2}{2} - 4x\right)\Big|_4^6$

$= \left(4(4) - \frac{4^2}{2}\right) - \left(4(-3) - \frac{(-3)^2}{2}\right) + \left(\frac{6^2}{2} - 4(6)\right) - \left(\frac{4^2}{2} - 4(4)\right)$

$= \frac{49}{2} + 2 = \frac{53}{2}$

(d) $\int_{-\pi/6}^{\pi/6} (x + \sin 5x)dx$

$= \frac{x^2}{2}\Big|_{-\pi/6}^{\pi/6} - \frac{1}{5}(\cos 5x)\Big|_{-\pi/6}^{\pi/6}$

$= 0 - \frac{1}{5}(0) = 0$

(e) $\int_0^1 \frac{1}{(3-2v)^2}dv$

$= -\frac{1}{2}\int_0^1 (3-2v)^{-2}\,d(3-2v)$

$= -\frac{1}{2}\,\frac{(3-2v)^{-2+1}}{-2+1}\Big|_0^1 = \frac{1}{2}(3-2v)^{-1}\Big|_0^1$

$= \frac{1}{2}\left[(3-2(1))^{-1} - (3-2(0))^{-1}\right]$

$= \frac{1}{2}\left(1 - \frac{1}{3}\right) = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$

(f) $\int_1^4 \frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)^3}dx$

let $u = \sqrt{x} + 1$
if $x=1$. $u = \sqrt{1}+1 = 2$
if $x=4$. $u = \sqrt{4}+1 = 3$
$du = \frac{1}{2\sqrt{x}}dx$
$\frac{1}{\sqrt{x}}dx = 2du$

$= \int_2^3 \frac{2du}{u^3}$

$= 2\left(\frac{u^{-3+1}}{-3+1}\right)\Big|_2^3 = -(u^{-2})\Big|_2^3 = u^{-2}\Big|_3^2$

$= 2^{-2} - 3^{-2} = \frac{5}{36}$

2. Let f be continuous on [-a, a]. if f is an even (odd) function, show that $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ and interpret this result geometrically. Verify the result for the special case $f(x) = \cos x$.

$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$.

∵ $f(x)$ is even,
∴ $f(-x) = f(x)$.

let $u = -x$
then if $x=-a$, $u=a$
if $x=0$, $u=0$
$du = -dx$
$x = -u$

$\int_{-a}^{0} f(x)dx = \int_a^0 f(-u)(-du) = \int_0^a f(u)du = \int_0^a f(x)dx$

∴ $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.

If $f(x)$ is odd,
$\int_{-a}^{a} f(x)dx = 0$

$\int_{-a}^{a} \cos x\,dx = \sin x\Big|_{-a}^{a} = \sin a - \sin(-a)$
$= \sin a + \sin a = 2\sin a$.

3. Evaluate the definite integrals.

(a) $\int_{-1}^{4}(x^2 - x - 1)dx$

$= \left(\frac{x^3}{3} - \frac{x^2}{2} - x\right)\Big|_{-1}^{4}$

$= \left(\frac{4^3}{3} - \frac{4^2}{2} - 4\right) - \left(\frac{(-1)^3}{3} - \frac{(-1)^2}{2} - (-1)\right)$

$= \frac{28}{3} - \frac{1}{6} = \frac{55}{6}$

(b) $\int_0^3 \frac{dt}{\sqrt{4-t}} = -\int_0^3 (4-t)^{-\frac{1}{2}} d(4-t)$

$= \frac{(4-t)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\Big|_3^0 = 2(4-t)^{\frac{1}{2}}\Big|_3^0$

$= 2\left[(4-0)^{\frac{1}{2}} - (4-3)^{\frac{1}{2}}\right] = 2[2-1] = 2$

(c) $\int_0^1 (2t-1)^3 dt = \frac{1}{2}\int_0^1 (2t-1)^3 d(2t-1)$

$= \frac{1}{2}\left(\frac{(2t-1)^4}{4}\right)\Big|_0^1 = \frac{1}{8}\left[(2(1)-1)^4 - (2(0)-1)^4\right]$

$= \frac{1}{8}(1-16) = -\frac{15}{8}$

(d) $\int_4^9 \frac{2+x}{2\sqrt{x}}dx = \int_4^9 \frac{1}{\sqrt{x}}dx + \frac{1}{2}\int_4^9 \sqrt{x}\,dx$

$= \left(\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right)\Big|_4^9 + \frac{1}{2}\left(\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right)\Big|_4^9$

$= 2\left(x^{\frac{1}{2}}\right)\Big|_4^9 + \frac{1}{3}\left(x^{3/2}\right)\Big|_4^9$

$= 2\left[9^{\frac{1}{2}} - 4^{\frac{1}{2}}\right] + \frac{1}{3}\left[9^{3/2} - 4^{3/2}\right] = 2 + \frac{19}{3} = \frac{25}{3}$

4. Evaluate the definite integrals.

(a) $\int_0^1 e^{-x}dx = -\int_0^1 e^{-x} d(-x)$

$= e^{-x}\Big|_1^0 = e^0 - e^1$

$= 1 - e$

(b) $\int_0^1 xe^{x^2}dx$

let $u = x^2$, $du = 2x\,dx$, $x\,dx = \frac{1}{2}du$; $x=0$, $u=0$; $x=1$, $u=1$

$= \frac{1}{2}\int_0^1 e^u du$

$= \frac{1}{2}\left(e^u\right)\Big|_0^1 = \frac{1}{2}(e^1 - e^0)$

$= \frac{1}{2}(e-1).$

(c) $\int_0^{\pi/6} \frac{\cos\theta}{1+2\sin\theta}d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{6}} \frac{1}{1+2\sin\theta}\, d(1+2\sin\theta)$

$= \frac{1}{2}\ln|1+2\sin\theta|\Big|_0^{\frac{\pi}{6}} = \frac{1}{2}\left(\ln|1+2\sin\tfrac{\pi}{6}| - \ln|1+2\sin 0|\right)$

$= \frac{1}{2}\left(\ln(1+2\times\tfrac{1}{2}) - \ln(1)\right) = \frac{1}{2}\ln(2) = \ln\sqrt{2}$;

(d) $\int_{\sqrt{2}}^{2} \frac{u}{u^2-1}du$

let $t = u^2 - 1$, $dt = 2u\,du$, $u\,du = \frac{1}{2}dt$; $u = \sqrt{2}, t = 1$; $u = 2, t = 3$

$= \int_1^3 \frac{\frac{1}{2}dt}{t}$

$= \frac{1}{2}\ln t\Big|_1^3$

$= \frac{1}{2}(\ln 3 - \ln 1)$

$= \ln\sqrt{3}$.

5. Evaluate the definite integrals.

(a) $\int_{\sqrt{2}}^{2} \frac{u}{(u^2-1)^2}du$ $\because$ Q4 (d)

$= \frac{1}{2}\int_1^3 \frac{dt}{t^2} = \frac{1}{2}\left(\frac{t^{-2+1}}{-2+1}\right)\Big|_1^3$

$= -\frac{1}{2}(t^{-1})\Big|_1^3 = \frac{1}{2}(t^{-1})\Big|_3^1$

$= \frac{1}{2}(1^{-1} - 3^{-1}) = \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3}$.

(b) $\int_0^{\pi/4} \cos^2\theta\, d\theta$

$\because \cos 2\theta = 2\cos^2\theta - 1$

$\therefore \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (1+\cos 2\theta)d\theta$

$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/4}$

$= \frac{1}{2}\left[\left(\frac{\pi}{4} + \frac{1}{2}\sin 2(\tfrac{\pi}{4})\right) - \left(0 + \frac{1}{2}\sin(2(0))\right)\right]$

$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right) = \frac{1}{8}(\pi + 2)$

let $u=\sin 2x$
$du = 2\cos 2x\,dx$
$\cos 2x\,dx = \frac{1}{2}du$

if $x=\frac{\pi}{12}$, $u=\sin\frac{\pi}{6}=\frac{1}{2}$
if $x=\frac{\pi}{4}$, $u=\sin\frac{\pi}{2}=1$

(c) $\int_{\pi/12}^{\pi/4}\frac{\cos 2x dx}{\sin^2 2x}$

$= \int_{\frac{1}{2}}^{1}\frac{\frac{1}{2}u}{u^2} = \frac{1}{2}\left(\frac{u^{-2+1}}{-2+1}\right)\Big|_{\frac{1}{2}}^{1}$

$= -\frac{1}{2}(u^{-1})\Big|_{\frac{1}{2}}^{1} = \frac{1}{2}(u^{-1})\Big|_{1}^{\frac{1}{2}} = \frac{1}{2}\left(\left(\frac{1}{2}\right)^{-1}-1\right)$

$= \frac{1}{2}$

(d) $\int_0^{\pi/2}\sin^2\frac{x}{2}dx$

$\because \cos x = 1-2\sin^2\frac{x}{2}$
$\therefore \sin^2\frac{x}{2} = \frac{1}{2}(1-\cos x)$

$= \int_0^{\frac{\pi}{2}}\frac{1}{2}(1-\cos x)\,dx$

$= \frac{1}{2}(x-\sin x)\Big|_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\left(\frac{\pi}{2}-\sin\frac{\pi}{2}\right)-(0-\sin 0)\right] = \frac{1}{2}\left[\frac{\pi}{2}-1\right]$

6. Evaluate the definite integrals with substitutions.

(a) $\int_1^2\frac{\sqrt{4-x^2}}{x}dx$, let $x=2\sin\theta$

$dx = 2\cos\theta d\theta$
$\sqrt{4-x^2} = 2\cos\theta$
if $x=1$, $1=2\sin\theta$, $\theta=\pi/6$
if $x=2$, $2=2\sin\theta$, $\theta=\pi/2$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{2\cos\theta\cdot 2\cos\theta d\theta}{2\sin\theta}$

$= 2\int_{\frac{\pi}{6}}^{\pi/2}(\csc\theta-\sin\theta)d\theta$

$= 2\left[\ln|\csc\theta-\cot\theta|+\cos\theta\right]\Big|_{\pi/6}^{\pi/2}$

$= -2\left(\ln(2-\sqrt{3})-\frac{\sqrt{3}}{2}\right)$

(b) $\int_1^{\sqrt{3}}\sqrt{1+x^2}dx$, let $x=\tan\theta$, $dx=\sec^2\theta d\theta$

$\sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \sec\theta$
if $x=1$, $1=\tan\theta$, $\theta=\pi/4$
if $x=\sqrt{3}$, $\sqrt{3}=\tan\theta$, $\theta=\pi/3$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\sec\theta\cdot\sec^2\theta d\theta$

$= \frac{1}{2}\left[\int_{\frac{\pi}{4}}^{\pi/3}\sec\theta d\theta + (\tan\theta\sec\theta)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}\right]$

$= \frac{1}{2}\left(\left[\ln|\sec\theta+\tan\theta|+\tan\theta\sec\theta\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$= \frac{1}{2}\left(\left[\ln\left|\sec\frac{\pi}{3}+\tan\frac{\pi}{3}\right|+\tan\frac{\pi}{3}\sec\frac{\pi}{3}\right]-\left[\ln\left|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right|+\tan\frac{\pi}{4}\sec\frac{\pi}{4}\right]\right)$

$= \frac{1}{2}\left(\left[\ln(2+\sqrt{3})+2\sqrt{3}\right]-\left[\ln(\sqrt{2}+1)+\sqrt{2}\right]\right)$

$= \frac{1}{2}\ln\left(\frac{2+\sqrt{3}}{1+\sqrt{2}}\right)+\frac{1}{2}(2\sqrt{3}-\sqrt{2})$

(c) $\pi\int_0^{2a}y^2dx$, let $y=2a\cos^2\theta$ and $x=2a\tan\theta$, where $0\le\theta\le\pi$

$dx = 2a\sec^2\theta d\theta$.
if $x=0$, $0=2a\tan\theta$, $\theta=0$
if $x=2a$, $2a=2a\tan\theta$, $\theta=\pi/4$

$= \pi\int_0^{\frac{\pi}{4}}4a^2\cos^4\theta(2a\sec^2\theta)d\theta$.

$= 8a^3\pi\int_0^{\pi/4}\cos^2\theta d\theta$

$= 8a^3\pi\int_0^{\frac{\pi}{4}}\frac{1}{2}(1+\cos 2\theta)d\theta$

$= 4a^3\pi\left(\theta+\frac{1}{2}\sin 2\theta\right)\Big|_0^{\frac{\pi}{4}} = 4a^3\pi\left[\left(\frac{\pi}{4}+\frac{1}{2}\right)-(0+0)\right] = a^3\pi(\pi+2)$

7. In each sub-question, find numbers that satisfy the conclusion of the Mean Value Theorem for Definite Integral.

(a) $\int_0^3 3x^2dx = 27$

$\Rightarrow 3c^2(3-0) = 27$

$\Rightarrow c^2 = 3$

$c = \sqrt{3}$

$0 < \sqrt{3} < 3$.

(b) $\int_0^a\sqrt{a^2-x^2}dx = (\pi a^2)/4,\quad a>0$

$\Rightarrow \sqrt{a^2-c^2}(\cancel{a}-0) = \frac{\pi a^{\cancel{2}}}{4}$

$a^2-c^2 = \frac{\pi^2}{16} \Rightarrow c^2 = a^2-\frac{\pi^2}{16}$

$c = \frac{1}{4}\sqrt{16a^2-\pi^2}$