3.4 Binomial Theorem.

or Binomial Expansion.

$(a+b)^0 = 1$

$(a+b)^1 = a+b$

$(a+b)^2 = a^2+2ab+b^2$

$(a+b)^3 = 1a^3+3a^2b+3ab^2+1b^3$

$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
{}_4C_0 & & {}_4C_1 & & {}_4C_2 & & {}_4C_3 & & {}_4C_4
\end{array}
$$

$\vdots$

$$(a+b)^n = {}_nC_0 a^n + {}_nC_1 a^{n-1}b + {}_nC_2 a^{n-2}b^2 + \cdots + {}_nC_r a^{n-r}b^r + \cdots + {}_nC_n b^n$$

$$= \sum_{r=0}^{n} {}_nC_r a^{n-r} b^r = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

$$(1+1)^n = {}_nC_0 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_r + \cdots + {}_nC_n = 2^n$$

$$(1-1)^n = (1+(-1))^n = {}_nC_0 - {}_nC_1 + {}_nC_2 - {}_nC_3 + \cdots + (-1)^r {}_nC_r + \cdots + (-1)^n {}_nC_n$$

$$= 0$$

$${}_4C_0 - {}_4C_1 + {}_4C_2 - {}_4C_3 + {}_4C_4$$

$$= 1 - 4 + 6 - 4 + 1 = 0$$

$${}_5C_0 - {}_5C_1 + {}_5C_2 - {}_5C_3 + {}_5C_4 - {}_5C_5$$

$$= 1 - 5 + 10 - 10 + 5 - 1 = 0$$

For example, P293. Q4.

In the expansion of $(a+b)^{11}$.

Find coefficients of in nCr and in numeric value

a) $a^2b^{9=r}$ $\longrightarrow$ ${}_{11}C_9 = {}_{11}C_2 = \frac{11\times10}{2} = 55$

$2+9=n$

b) a^{11} $\longrightarrow$ ${}_{11}C_0 = 1$

c) a^6b^5 $\longrightarrow$ ${}_{11}C_5 = \frac{11\times\cancel{10}\times\cancel{9}^{3}\times\cancel{8}^{2}\times7}{\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}\times1} = 11\times42 = 462$

$a^{n-r}b^r \longrightarrow nC_r$

P295. Q19 b)

Find the first three terms of $(2x+1)^2(4x-3)^5$

Method 1.

$\because (2x+1)^2 = (2x)^2+2(2x)(1)+1$
$= \underline{4x^2}+4x+1$

$(4x-3)^5 = (4x)^5+5(4x)^4(-3)+\underline{10}(\underline{4x})^3(\underline{-3})^2+10(4x)^2(-3)^3+5(4x)(-3)^4+(-3)^5$
$= \underline{1024x^5}-3840x^4+\underline{5760}x^3-1080x^2+1620x-243.$

The first three terms are: $\underline{4096x^7}$, $(-3840\times4+1024\times4)x^6$
$= \underline{-11264x^6}$,
$(4\times5760-4\times3840+1024)x^5 = \underline{8704\ x^5}$

Method 2. using general term.

$(2x+1)^2(4x-3)^5$ has general term:

$$2C_r(2x)^{2-r}(1)^r\, 5C_k(4x)^{5-k}(-3)^k$$

$$= 2C_r(2)^{2-r}\, 5C_k(4)^{5-k}(-3)^k\, x^{2-r+5-k}$$

$$= 2C_r(2)^{2-r}\, 5C_k(4)^{5-k}(-3)^k\, x^{7-r-k}$$

The first term: let $r=0, k=0$,

$$2C_0(2)^2\, 5C_0(4)^5(-3)^0 = 4\times4^5 = 4^6 = 4096.$$

$$\therefore \underline{4096x^7}$$

The second term: let $\begin{cases} r=0 \\ k=1 \end{cases}$ or $\begin{cases} r=1 \\ k=0 \end{cases}$.

$$2C_0(2)^2\, 5C_1(4)^4(-3) + 2C_1(2)^1\, 5C_0(4)^5(-3)^0$$

$$= 4\times5\times4^4(-3) + 2\times2\times4^5 = -15360+4096$$

$$= -11264$$

$$\therefore \underline{-11264x^6}$$

The third term: let $\begin{cases} r=0 \\ k=2 \end{cases}$ or $\begin{cases} r=1 \\ k=1 \end{cases}$ or $\begin{cases} r=2 \\ k=0 \end{cases}$

$$2C_0(2)^2\, 5C_2(4)^3(-3)^2 + 2C_1(2)\, 5C_1(4)^4(-3) + 2C_2(4)^5$$

$$= 4\times10\times4^3\times9 + 2\times2\times5\times4^4(-3) + 4^5$$

$$= 8704, \quad \therefore \underline{8704x^5}$$