Olympiads School Calculus Class 8

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Test 1 /51

Show your work!

1. Compute each limit. (12marks)

(a) $\lim_{x\to 1}\frac{1-x^3}{x^4-1} = \lim_{x\to 1}\frac{(1-x)(1+x+x^2)}{(x^2-1)(x^2+1)}$

$= \lim_{x\to 1}\frac{-\cancel{(x-1)}(1+x+x^2)}{\cancel{(x-1)}(x+1)(x^2+1)}$

$= \frac{1+1+1}{(1+1)(1+1)} = \frac{3}{4}$

(d) $\lim_{x\to -1}\frac{x-1}{x+1} = DNE$

$x\to -1^- \quad \frac{x-1}{x+1} = +\infty$

$x\to -1^+ \quad \frac{x-1}{x+1} = -\infty$

(b) $\lim_{x\to -\infty}\frac{-2x^3-7}{3x^2+1}$

$\lim_{x\to -\infty}\frac{\cancel{x^3}(-2-\frac{7}{x^3})}{\cancel{x^3}(\frac{3}{x}+\frac{1}{x^3})} = \frac{-2}{0}$

$= -\infty$

(e) $\lim_{x\to 2}\frac{x-2}{x+2} = \frac{2-2}{2+2} = 0$

(c) $\lim_{x\to 4}\frac{\sqrt{5-x}-1}{x-4} = \frac{(\sqrt{5-x}-1)(\sqrt{5-x}+1)}{x-4(\sqrt{5-x}+1)}$

$\lim_{x\to 4}\frac{-\cancel{(x-4)}}{\cancel{x-4}(\sqrt{5-x}+1)} = -\frac{1}{2}$

(f) $\lim_{x\to -\infty}(-2x^3+17x^2+x+3)$

$= \lim_{x\to -\infty} x^3(-2+\frac{17}{x}+\frac{1}{x^2}+\frac{3}{x^3})$

$= \lim_{x\to -\infty} -\infty(-2+0+0+0)$

$= \infty$

2. For each case find $f'(x)$: (12marks)

(a) f(x) = $\frac{x-1}{x+1}$

$$f'(x) = \frac{\cancel{x}+1-(\cancel{x}-1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

(b) f(x) = $2^x + x^2$

$$f'(x) = 2^x \ln 2 + 2x$$

(c) f(x) = $\sin^2(x^2+1)$

$$f'(x) = 2\sin(x^2+1)\cdot(\cos(x^2+1)(2x))$$
$$= 4x\sin(x^2+1)\cos(x^2+1)$$

(d) f(x) = $e^x(x^3+x^2-1)$

$$f'(x) = e^x(x^3+x^2-1) + e^x(3x^2+2x)$$
$$= e^x(x^3+4x^2+2x-1)$$

(e) f(x) = $\ln(e^x)$

$$f'(x) = \frac{e^x}{e^x} = 1$$

(f) f(x)= $\ln(\cos x)$

$$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$$

$(x-1)(x+3)$

3. Use the first principles to find the derivatives of f(x) = x^2 + 2x – 3. (3marks)

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2+2(x+h)-3-(x^2+2x-3)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2+\cancel{2x}+2h\cancel{-3}-\cancel{x^2}-\cancel{2x}\cancel{+3}}{h} = \frac{2x\cancel{h}+h^{\cancel{2}}+2\cancel{h}}{\cancel{h}}$$

$$= 2x+2$$

(6marks)

4. For each case, use the first derivative sign to find the intervals of increase or decrease, LM, Lm.

(a) f(x) = x^2 – 2x

$f'(x) = 2x-2$

$0 = 2x-2$

$x = 1$

$- \quad 1 \quad +$

Min at $(1, 1^2-2(1))$

$(1,-1)$

Decreasing: $x \in (-\infty, 1)$

increasing: $x \in (1, \infty)$

(b) f(x) = $x^3(x-1)^4$

$f'(x) = 3x(x-1)^4 + x^3 4(x-1)^3$

$= (x-1)^3(3x^2-3x+4x^3)$

$= x(x-1)^3(4x^2+3x-3)$

$x = 0, 1, -\frac{3 \pm 2\sqrt{284}}{8}$

$-4.5 \quad 0 \quad 1 \quad 3.8$

5. Find the intervals of concavity and the points of inflection for f(x) = x^2 - 4x + 3. (3marks)

$f'(x) = 2x-4$

$f''(x) = 2$

There are no points of inflection

Concavity is 2 (upwards) for $x \in (-\infty, \infty)$

(3marks)

6. Find a function *f* such that *f′(x)* = $6x^2$-*12x+6* and *(1, 3)* is a point of inflection of the graph of *f*.

$f(x) = 2x^3 - 6x^2 + 6x + C$

$3 = 2(1)^3 - 6(1)^2 + 6 + C$

$C = 1$

$f(x) = 2x^3 - 6x^2 + 6x + 1$

(3marks)

7. Find the equation of the tangent line to the curve defined by $x^2 + xy + y^2 = 7$ at the point (1, -3).

$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$x\frac{dy}{dx} + 2y\frac{dy}{dx} = -2x - y$

$\frac{dy}{dx}(x + 2y) = -2x - y$

$\frac{dy}{dx} = -\frac{2x+y}{x+2y}$

$\frac{dy}{dx} = -\frac{2-3}{1-6} = -\frac{1}{5}$

$y+3 = -\frac{1}{5}(x-1)$

$y = -\frac{1}{5}x - \frac{14}{5}$

8. What is the maximum slope of a tangent to the curve y = $-x^3 + 3x^2 + 9x - 27$? (3marks)

$y' = -3x^2 + 6x + 9$ ← slope

$y'' = -6x + 6$

$0 = -6x + 6$

$x = 1$

y' at $1 = -3 + 6 + 9 = 12$

The maximum slope is 12

(3marks)

9. Find the points on the curve y = $x^3 - 3x^2$ at which the tangent is parallel to the line y = 9x + 7.

$y' = 3x^2 - 6x$

$m = 9$

$9 = 3x^2 - 6x$

$3x^2 - 6x - 9 = 0$

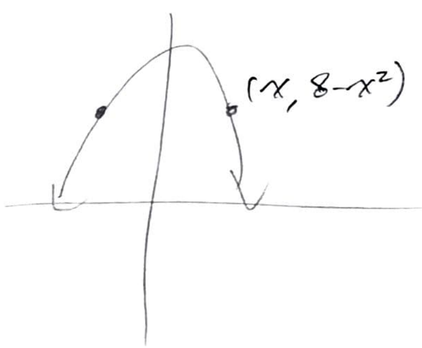
$x^2 - 2x - 3 = 0$

$(x+1)(x-3) = 0$

$x = -1, 3$

Points: $(-1,-4)$ and $(3,0)$

10. Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above x-axis and lying on the parabola y = $8 - x^2$. (3marks)



$A(x) = x(8-x^2) \cdot 2$

$= 8x - x^3 \cdot 2$

$= -2x^3 + 16x$

$A'(x) = -6x^2 + 16$

$0 = -6x^2 + 16$

$6x^2 = 16$

$x^2 = \frac{8}{3}$

$x = \pm \frac{2\sqrt{6}}{3}$

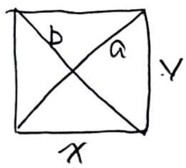
$y = \frac{16}{3}$

Dimensions: $\frac{4\sqrt{6}}{3} \times \frac{16}{3}$

Bonus question

11. The diagonals of a quadrilateral are perpendicular. The sum of the diagonals is 8 cm.

What is the maximum area of such quadrilateral? (3marks)



$a+b=8$

$x^2+y^2=a^2$