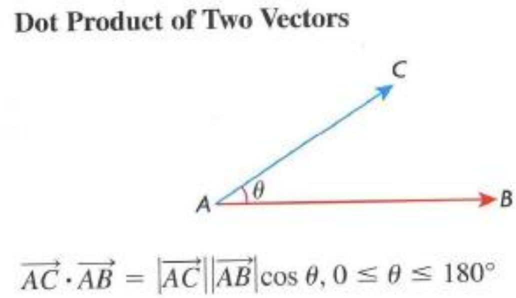
The Dot Product of Two Geometric Vectors

- The dot product between two geometric vectors $\vec{a}$ and $\vec{b}$ is a scalar quantity defined as $\vec{a} \bullet \vec{b} = \left|\vec{a}\right|\left|\vec{b}\right| \cos\theta$, where θ is the angle between the two vectors.



- If $0^\circ \le \theta \le 90^\circ$, then $\vec{a} \bullet \vec{b} > 0$.
- If $\theta = 90^\circ$, then $\vec{a} \bullet \vec{b} = 0$.
- If $90^\circ \le \theta \le 180^\circ$, then $\vec{a} \bullet \vec{b} < 0$

- $\vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a}$
- $\vec{a} \bullet \left(\vec{b} + \vec{c}\right) = \vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{c}$
- $\vec{a} \bullet \vec{a} = \left|\vec{a}\right|^2$
- $\vec{i} \bullet \vec{i} = 1$, $\vec{j} \bullet \vec{j} = 1$, and $\vec{k} \bullet \vec{k} = 1$.
- $\left(k\vec{a}\right) \bullet \vec{b} = \vec{a} \bullet \left(k\vec{b}\right) = k\left(\vec{a} \bullet \vec{b}\right)$

Example 1 State whether each expression has any meaning. Justify your answers.

a) $\vec{u} \bullet \left(\vec{v} \bullet \vec{w}\right)$ b) $\left|\vec{u} \bullet \vec{v}\right|$ c) $\vec{u}\left(\vec{v} \bullet \vec{w}\right)$ d) $\left(\vec{v}\right)^2$

Example 2 Calculate the dot product $\vec{a} \bullet \vec{b}$, given the magnitudes of the two vectors and the angle θ between them.

a) $\left|\vec{a}\right| = 1.3$, $\left|\vec{b}\right| = 8.5$, and $\theta = 90^{\circ}$

b) $\left|\vec{a}\right| = 4.7$, $\left|\vec{b}\right| = 13.9$, and $\theta = 126^{\circ}$

Example 3 Calculate, to the nearest degree, the angle between the given vectors.

a) $\left|\vec{p}\right| = 9.1$, $\left|\vec{q}\right| = 7.2$, and $\vec{p} \bullet \vec{q} = 53$

b) $\left|\vec{f}\right| = 5.8$, $\left|\vec{g}\right| = 13.4$, and $\vec{f} \bullet \vec{g} = -77.72$

Example 4 Determine $\left(3\vec{a} - 4\vec{b}\right) \bullet \left(2\vec{a} + \vec{b}\right)$ given that $\vec{a}$ and $\vec{b}$ are unit vectors and the angle between them is 30°.

The Dot Product of Algebraic Vectors

- The dot product is defined as follows for algebraic vectors in R^2 and R^3, respectively:
  - if $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$, then $\vec{a} \bullet \vec{b} = a_1 b_1 + a_2 b_2$
  - if $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, then $\vec{a} \bullet \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

- The properties of the dot product hold for both geometric and algebraic vectors.
  For example, two nonzero vectors $\vec{a}$ and $\vec{b}$ are perpendicular if $\vec{a} \bullet \vec{b} = 0$.

- For two nonzero vectors $\vec{a}$ and $\vec{b}$, $\cos\theta = \dfrac{\vec{a} \bullet \vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|}$, where θ is the angle between the vectors.

Example 1 Determine the angle θ between the following pair of vectors.

$$\vec{u} = 3\vec{i} + \vec{j} - 6\vec{k} \quad \text{and} \quad \vec{v} = 9\vec{i} - \vec{k}$$

Example 2 Determine the value of c so that $\vec{p} = (c, 3, 8)$ and $\vec{q} = (c, 2c, -5)$ are perpendicular.

Example 3 A triangle has vertices $L(0, 3, 4)$, $M(4, -1, 1)$, and $N(6, 1, 1)$.

a) Verify that ΔLMN is a right triangle.

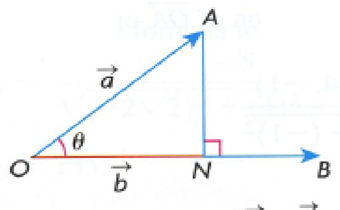
b) Determine a fourth vertex P such that $LMNP$ is a rectangle.

Example 4 Determine the coordinates of a vector $\vec{r}$ such that $\vec{r} \bullet (-3, 2) = 1$ and $(1, 4) \bullet \vec{r} = 5$.

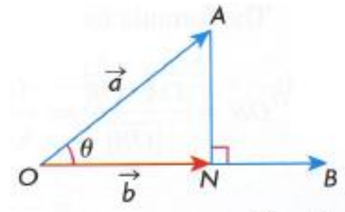
Example 5 Given $\vec{a} = 3\vec{i} + k\vec{j}$ and $\vec{b} = -4\vec{i} + 2\vec{j}$, determine the value of k such that the angle between $\vec{a}$ and $\vec{b}$, when placed tail to tail, is 60°.

Scalar and Vector Projections

- A projection of one vector onto another can be either a scalar or a vector. The difference is the vector projection has a direction.



Scalar projection of $\vec{a}$ on $\vec{b}$



Vector projection of $\vec{a}$ on $\vec{b}$

The scalar projection of $\vec{a}$ on $\vec{b}$ is

$$\frac{\vec{a} \bullet \vec{b}}{\left|\vec{b}\right|} = \left|\vec{a}\right| \cos\theta,$$

The vector projection of $\vec{a}$ on $\vec{b}$ is

$$\frac{\vec{a} \bullet \vec{b}}{\left|\vec{b}\right|^2}\vec{b} = \left(\frac{\vec{a} \bullet \vec{b}}{\vec{b} \bullet \vec{b}}\right)\vec{b}.$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

- The direction cosines for $\overrightarrow{OP} = (a, b, c)$ are $\cos\alpha = \frac{a}{\sqrt{a^2+b^2+c^2}}$, $\cos\beta = \frac{b}{\sqrt{a^2+b^2+c^2}}$, and $\cos\gamma = \frac{c}{\sqrt{a^2+b^2+c^2}}$, where α, β, and γ are the direction angles between the position vector $\overrightarrow{OP}$ and the positive x-axis, y-axis, and z-axis, respectively. Note that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Example 1 Calculate the scalar projection of $\vec{a} = \left(-3, 4, 5\sqrt{3}\right)$ on $\vec{b} = (-2, 2, -1)$.

Example 2 Determine the vector projection of $\vec{d} = (2, -3)$ on $\vec{c} = (1, 4)$.

Example 3 The scalar projection of $(1, m, 0)$ on $(2, 2, 1)$ is -4. Determine the value of m.

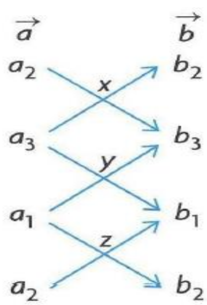
Example 4 Find the direction cosines and the direction angles of the vector $\vec{p} = (0, 5, -3)$.

Example 5 The vector $\vec{q}$ makes angles of 60° and 105°, respectively, with the x- and y-axes. What is the angle between $\vec{q}$ and the z-axis?

The Cross Product of Two Vectors

- The cross product $\vec{a} \times \vec{b}$ between two vectors $\vec{a}$ and $\vec{b}$ results in a third vector that is perpendicular to the plane in which the given vectors lie.
- Method for Calculating $\vec{a} \times \vec{b}$, where $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$

   1. List the components of vector $\vec{a}$ in column form on the left side, starting with a_2 and then writing a_3, a_1, and a_2 below each other as shown.
   2. Write the components of vector $\vec{b}$ in a column to the right of $\vec{a}$, starting with b_2 and then writing b_3, b_1, and b_2 in exactly the same way as the components of $\vec{a}$.
   3. The required formula is now a matter of following the arrows and doing the calculation. To find the x component, for example, we take the down product a_2b_3 and subtract the up product a_3b_2 from it to get $a_2b_3 - a_3b_2$. The other components are calculated in exactly the same way, and the formula for each component is listed below.



$$x = a_2b_3 - a_3b_2$$

$$y = a_3b_1 - a_1b_3$$

$$z = a_1b_2 - a_2b_1$$

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$$

- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- $\left(k\vec{a}\right) \times \vec{b} = \vec{a} \times \left(k\vec{b}\right) = k\left(\vec{a} \times \vec{b}\right)$

Example 1 State if each expression results in a vector, a scalar, or neither. Justify your answers.

a) $\vec{a} \bullet \left(\vec{b} \times \vec{c}\right)$

b) $\vec{a} \times \left(\vec{b} \bullet \vec{c}\right)$

c) $\left(\vec{a} + \vec{b}\right) \times \vec{c}$

d) $\left(\vec{a} \bullet \vec{b}\right) - \vec{c} \times \vec{d}$

Example 2 Determine two vectors that are orthogonal to both $\vec{p} = (5, 3, -1)$ and $\vec{q} = (4, -2, 7)$. Verify your answers using the dot product.

Example 3 Use $\vec{a} = (2, 1, -1)$, $\vec{b} = (4, -5, 3)$, and $\vec{c} = (-1, -2, 5)$ to verify the following properties of the cross product.

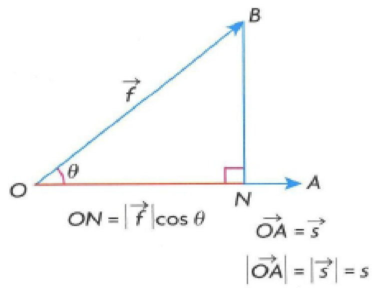
a) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

b) $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

Example 4 Determine the value of m and n for $\vec{u} = (m, -12, 9)$ and $\vec{v} = (5, n, -3)$ such that $\vec{u} \times \vec{v} = \vec{0}$.

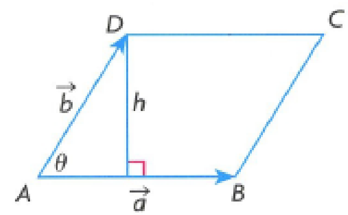
Applications of the Dot Product and Cross Product

- $W = \vec{F} \bullet \vec{s}$, where $\vec{F}$ is the force applied to an object, measured in newtons (N); $\vec{s}$ is the displacement of the object, measured in metres (m); and W is work, measured in joules (J).



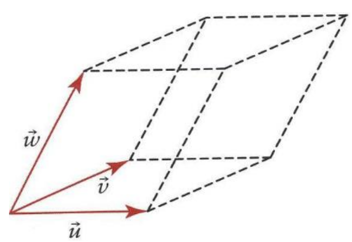
- Area of a triangle, with sides $\vec{a}$ and $\vec{b}$, equals

  $$\frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta$$

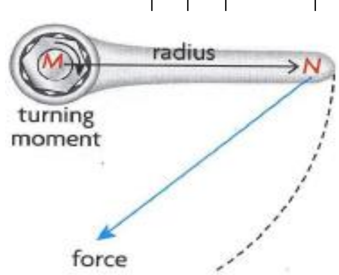


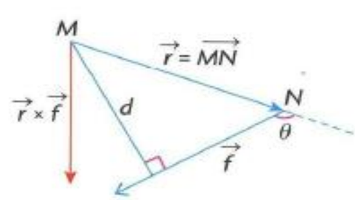
- Area of a parallelogram, with sides $\vec{a}$ and $\vec{b}$, equals
  $\left|\vec{a} \times \vec{b}\right| = \left|\vec{a}\right|\left|\vec{b}\right|\sin\theta$

- The volume of the parallelepiped defined by $\vec{u}$, $\vec{v}$, and $\vec{w}$, as shown, is $V = \left|\vec{w} \bullet \vec{u} \times \vec{v}\right|$.



- Torque, $\vec{\tau}$, equals $\vec{r} \times \vec{f}$, where $\vec{r}$ is the vector determined by the lever arm acting from the axis of rotation, $\vec{f}$ is the applied force, and θ is the angle between the force and the lever arm. The magnitude of the torque equals $\left|\vec{\tau}\right| = \left|\vec{r} \times \vec{f}\right| = \left|\vec{r}\right|\left|\vec{f}\right|\sin\theta$. It measures the overall twisting effect of applied force.





Example 1 Angela has entered the wheelchair division of a marathon race. While training, she races her wheelchair up a 300-m hill with a constant force of 500 N applied at an angle of 30° to the surface of the hill. Find the work done by Angela, to the nearest 100 J.

Example 2 Determine the area of ΔPQR with vertices $P(5, -4, 6)$, $Q(-2, 8, 3)$, and $R(1, 7, 1)$.

Example 3 A parallelogram has area 85 cm^2. The side lengths are 10 cm and 9 cm. What are the measures of the interior angles?

Example 4 Find the volume of the parallelepiped defined by the vectors $\vec{u} = (8, -3, 4)$, $\vec{v} = (-2, 9, 1)$, and $\vec{w} = (0, -4, -7)$.

Example 5 A wrench is used to tighten a bolt.
A force of 60 N is applied in a clockwise direction at 80° to the handle, 20 cm from the centre of the bolt.

a) Calculate the magnitude of the torque.
b) In what direction does the torque vector point?