An Algorithm for Curve Sketching

- Sketching the Graph of a Polynomial or Rational Function
    1. Use the function to
        - determine the domain and any discontinuities
        - determine the y-intercept
        - find any asymptotes, and determine function behaviour relative to these asymptotes
    2. Use the first derivative to
        - find the critical numbers
        - determine where the function is increasing and where it is decreasing
        - identify any local maxima or minima
    3. Use the second derivative to
        - determine where the graph is concave up and where it is concave down
        - find any points of inflection

        (Recall that the second derivative can also be used to identify local maxima and minima.)
    4. Use the information above to sketch the graph.

Example 1   Use the algorithm for curve sketching to sketch the graph of each function.

a) $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$
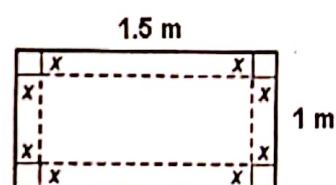
b) $g(x) = \dfrac{6}{x^2 - 9}$

c) $h(x) = \dfrac{x^2 - 2x + 10}{x - 1}$

Optimization Problems

- Algorithm for Solving Optimization Problems
    1. Understand the problem, and identify quantities that can vary. Whenever possible, draw a diagram, labelling the given and required quantities.
    2. Determine a function in one variable that represents the quantity to be optimized.
    3. Determine the domain of the function to be optimized, using the information given in the problem.
    4. Use the algorithm for extreme values to find the absolute maximum or minimum value in the domain. If the function to be optimized is not continuous over a closed interval $[a, b]$, then we must use the first derivative test (or the second derivative test) to find the maximum or minimum value of the function. [Note that these tests will be discussed again in greater detail in Chapter 4.]
    5. Use your result from step 4 to answer the original problem.

Example 1   A farmer wants to fence an area of 750 000 m² in a rectangular field and divide it in half with a fence parallel to one of the sides of the rectangle. How can this be done so as to minimize the amount of fence used?

Example 2   An open metal box for removing ashes from a fireplace is to be constructed from a rectangular piece of sheet metal that is 1.5 m by 1 m. Squares are to be cut from each corner of the sheet metal, the sides folded upward to form the box, and then the seams welded. Determine the dimensions that will give the box with the largest volume. What is the maximum capacity of a box that is constructed in this way?

Example 3   A car leaves a small town at 13:00 and travels due south at a speed of 80 km/h. Another car has been heading due west at 100 km/h and reaches the same town at 15:00. At what time were the two cars closest together?
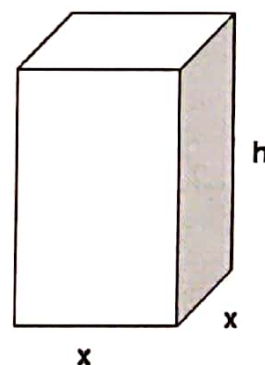
Example 4   A mirror in the shape of a rectangle capped by a semicircle is to have a perimeter of 30 feet. Choose the radius of the semicircular part so that the mirror has maximum area.

Optimization Problems in Economics and Science

- Profit, cost, and revenue are quantities whose rates of change are measured in terms of the number of units produced or sold.
- Economic situations usually involve maximizing profits or minimizing costs.
- To maximize revenue, we can use the revenue function.
  Revenue = total revenue from the sale of x units = (price per unit) $\times$ x.
- Practical constraints, as well as mathematical constraints, must always be considered when constructing a model.
- Once the constraints on the model have been determined – that is, the domain of the function – apply the extreme value algorithm to the function over the appropriately defined domain to determine the absolute extrema. If the function to be optimized is not continuous over a closed interval $[a, b]$, then use the first derivative test (or the second derivative test) to find the absolute extrema.

Example 1   In low season, 120 rooms in a 250-room hotel are occupied when the price for each room is $120 per night. When the price is reduced to $110 per night, 135 rooms are occupied. If the pattern persists, what room rate generates the greatest revenue?

Example 2   A cardboard box with a square base is to have a volume of 8 L. The cardboard for the box costs 0.1 ¢/cm², but the cardboard for the bottom is thicker, so it costs three times as much. Find the dimensions that will minimize the cost of the cardboard.

Example 3   A soup can of volume 500 cm³ is to be constructed. The material for the top costs 0.4 ¢/cm² while the material for the bottom and sides costs 0.2 ¢/cm². Find the dimensions that will minimize the cost of producing the can.