$V=\int_a^b 2\pi R h\,dx = \int_a^b 2\pi x f(x)\,dx$

AP Calculus Homework Eleven – Applications of Definite Integral and Polar Coordinates

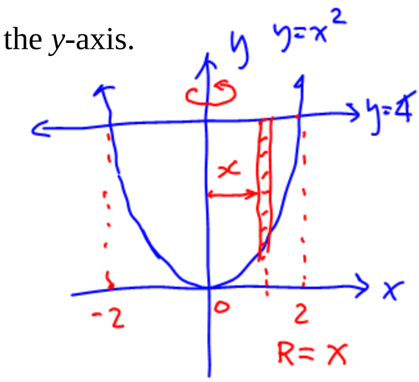
5.2 Volumes Using Cylindrical Shells and Volumes by Slicing; 5.3 Work and Arc Length

In Questions 1 - 2, the region whose boundaries are given is rotated about the line indicated. Calculate the volume of the solid generated using cylindrical shells if possible.

1. The first quadrant region bounded by $y = x^2$, the y-axis, and $y = 4$; about the y-axis.

$V=\int_0^2 2\pi R h\,dx = 2\pi\int_0^2 x(4-x^2)\,dx = 2\pi\int_0^2 (4x-x^3)\,dx$

$=2\pi\left[2x^2-\frac{1}{4}x^4\right]_0^2 = 2\pi\left[2(2)^2-\frac{1}{4}(2)^4\right] = 2\pi[8-4] = 8\pi$

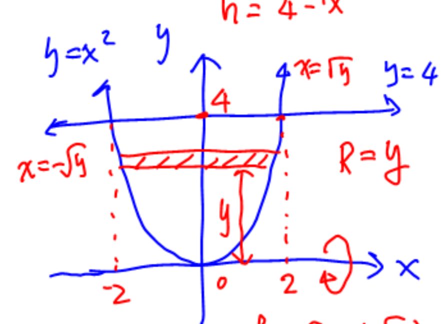


$h=4-x^2$

2. $y = x^2$ and $y = 4$; about the x-axis.

$V=\int_0^4 2\pi R h\,dy = 2\pi\int_0^4 y(2\sqrt{y})\,dy = 4\pi\int_0^4 y^{3/2}\,dy$

$=4\pi\left[\frac{y^{5/2}}{5/2}\right]_0^4 = \frac{8\pi}{5}\left[4^{5/2}\right] = \frac{8\pi}{5}(32) = \frac{256\pi}{8}$



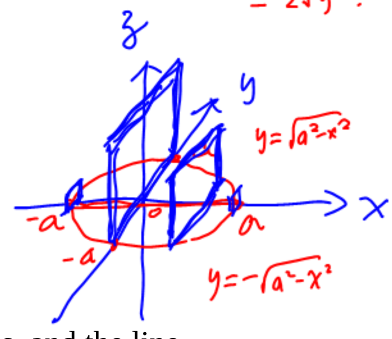
$h=\sqrt{y}-(-\sqrt{y})$
$=2\sqrt{y}$

3. The base of a solid is a circle of radius a, and every plane section perpendicular to a diameter is a square. Find the volume of the solid.

$A(x)=\left(\sqrt{a^2-x^2}-(-\sqrt{a^2-x^2})\right)^2 = \left(2\sqrt{a^2-x^2}\right)^2 = 4(a^2-x^2)$

$V=2\int_0^a A(x)\,dx = 2\int_0^a 4(a^2-x^2)\,dx = 8\left[a^2x-\frac{1}{3}x^3\right]_0^a$

$=8\left[a^3-\frac{1}{3}a^3\right] = 8a^3\left(\frac{2}{3}\right) = \frac{16a^3}{3}$

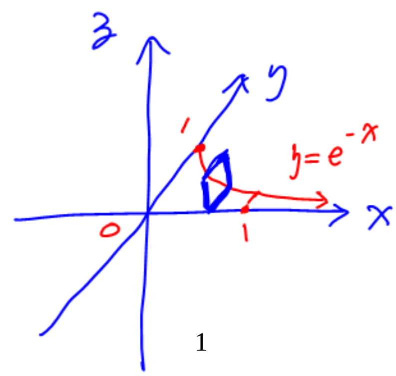


4. The base of a solid is the region bounded by $y = e^{-x}$, the x-axis, the y-axis, and the line $x = 1$. Each cross section perpendicular to the x-axis is a square. Find the volume of the solid.

$A(x)=(e^{-x})^2 = e^{-2x}$

$\therefore V=\int_0^1 A(x)\,dx = \int_0^1 e^{-2x}\,dx = \frac{1}{-2}\int_0^1 e^{-2x}\,d(-2x)$

$=\frac{1}{-2}\left[e^{-2x}\right]_0^1 = \frac{1}{-2}\left[e^{-2}-e^0\right] = \frac{1}{2}\left[1-e^{-2}\right]$



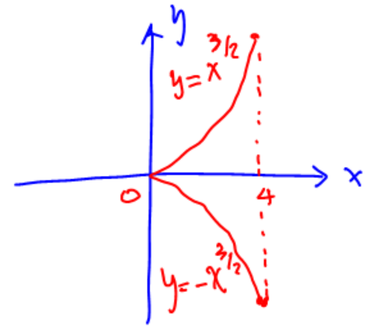
$L=\int_a^b \sqrt{1+[y']^2}\,dx$ or $L=\int_{t_1}^{t_2}\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt$

5. Find the length of the arc of the curve $y^2 = x^3$ cut off by the line $x = 4$.

$y'=(x^{3/2})'=\frac{3}{2}x^{\frac{1}{2}}$ for $0\le x\le 4$ $\therefore$

$\therefore L = 2\int_0^4\sqrt{1+(y')^2}\,dx = 2\int_0^4\sqrt{1+(\tfrac{3}{2}x^{\frac{1}{2}})^2}\,dx = 2\int_0^4\sqrt{1+\tfrac{9}{4}x}\,dx$

$= 2(\tfrac{4}{9})\int_0^4 (1+\tfrac{9}{4}x)^{\frac{1}{2}}\,d(1+\tfrac{9}{4}x) = \tfrac{8}{9}\left[\frac{(1+\frac{9}{4}x)^{3/2}}{3/2}\right]_0^4 = \frac{16}{27}[10\sqrt{10}-1]$



6. What is the length of the arc of $y = \ln\cos x$ from $x = \pi/4$ to $x = \pi/3$.

In Questions 7 - 8, write a definite integral to represent the arc length.

7. The length of one arc of the cycloid $x = t - \sin t$ and $y = 1 - \cos t$.

8. Find the arc length of $x = e^t\cos t$, $y = e^t\sin t$ from $t = 2$ to $t = 3$.

9. Suppose the current world population is 6 billion and the population t years from now is estimated to be $P(t) = 6e^{0.024t}$. On the basis of this supposition, estimate the average population of the world, in billions, over the next 25 years.

10. A beach opens at 8 A.M. and people arrive at a rate of $R(t) = 10 + 40t$ people per hour, where t represents the number of hours the beach has been open. Assuming no one leaves before noon, at what time will there be 100 people there?

11. Suppose the amount of a drug in a patient's bloodstream t hours after intravenous administration is $30/(t + 1)^2$ mg. What is the average amount in the bloodstream during the first 4 hours?

12. A rumor spreads through a town at the rate of $(t^2 + 10t)$ new people per day. Approximately how many people hear the rumor during the second week after it was first heard?

13. The population density of Winnipeg, which is located in the middle of the Canadian prairie, drops dramatically as distance from the centre of town increase. This is shown in the following table:

x distance from the centre (miles)	0	2	4	6	8	10
$f(x)$ density (100 people/mile^2)	50	45	40	30	15	5

Using a Riemann sum, calculate the population living within a 10-mile radius of the centre approximately.