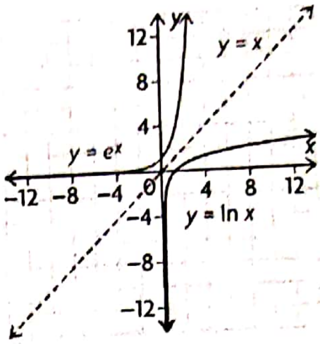
Derivatives of Exponential Functions, $y=e^x$

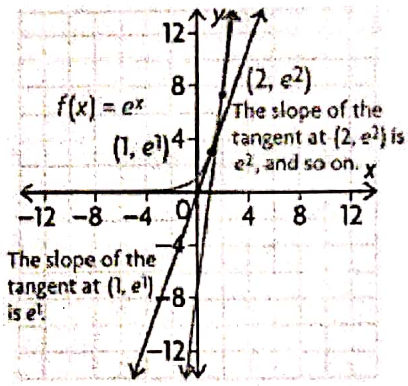
- e is called Euler's number or the natural number, where $e \approx 2.718$.
- The exponential function, $y=e^x$, has an inverse, $y=\log_e x$.
  Their graphs are reflections in the line $y=x$.
  The function $y=\log_e x$ can be written as $y=\ln x$ and is called the natural logarithm function.



- All the properties of exponential functions and logarithmic functions that you are familiar with also apply to $y=e^x$ and $y=\ln x$.

$y=e^x$	$y=\ln x$
The domain is $\{x \in \Re\}$.	The domain is $\{x \in \Re \mid x>0\}$.
The range is $\{y \in \Re \mid y>0\}$.	The range is $\{y \in \Re\}$.
The function passes through $(0,1)$.	The function passes through $(1,0)$.
$e^{\ln x}=x,\ x>0$.	$\ln e^x = x,\ x \in \Re$.
The line $y=0$ is the horizontal asymptote.	The line $x=0$ is the vertical asymptote.

- For $f(x)=e^x$, $f'(x)=e^x$.
  In Leibniz notation, $\frac{d}{dx}\left(e^x\right)=e^x$.
  That is, the slope of the tangent at a point on the graph of $f(x)=e^x$ equals the value of the function at this point.



- The rules for differentiating functions, such as the product, quotient, and chain rules, also apply to combinations involving exponential functions.
  In particular, for $f(x)=e^{g(x)}$, $f'(x)=e^{g(x)} \times g'(x)$.
  In Leibniz notation, $\frac{d}{dx}\left(e^{g(x)}\right)=\frac{d\left(e^{g(x)}\right)}{d(g(x))}\times\frac{d(g(x))}{dx}$.

Example 1 Differentiate each of the following functions. Simplify your answers.

a) $f(x) = e^{\sqrt{x}}$

b) $y = \left(1 + 5e^{3x}\right)^2$

c) $s(t) = t^6 e^t$

d) $g(x) = \dfrac{e^{-x}}{1 - e^{-2x}}$

Example 2 Determine the equation of the normal to the curve defined by $y = 3x - e^{-x}$ that is parallel to the line represented by $x + 5y - 2 = 0$.

Example 3 Determine the coordinates of all points at which the tangent to the curve defined by $y = 4x^3e^{-2x}$ is horizontal.

Example 4

a) Find the first, second, third, and fourth derivatives of $y = e^{3x}$.

b) State a formula for the *n*th derivative, $y^{(n)}$.

The Derivative of the General Exponential Function $y = b^x$

$$\lim_{h \to 0} \frac{b^h - 1}{h} = \ln b$$

- If $f(x) = b^x$, then $f'(x) = b^x \ln b$. In Leibniz notation, $\frac{d}{dx}(b^x) = b^x \ln b$.
- When you are differentiating a function that involves an exponential function, use the rule given above, along with the sum, difference, product, quotient, and chain rules as required.
  In particular, if $f(x) = b^{g(x)}$, then $f'(x) = b^{g(x)} \ln b \times g'(x)$.

  In Leibniz notation, $\frac{d}{dx}\left(b^{g(x)}\right) = \frac{d\left(b^{g(x)}\right)}{d(g(x))} \times \frac{d(g(x))}{dx}$.

Example 1 Differentiate each of the following functions. Simplify your answers.

a) $f(x) = x^4 - 4^x$

b) $m(x) = 3^{6x^2 - 5x + 10}$

c) $y = 5^x e^{-x}$

d) $h(t) = \frac{8^t}{t^2}$

Example 2 Determine the equation of the tangent to the function $f(x)=2(6^x)$ where the curve meets the y-axis.

Example 3 If $f(x)=3^{7x+2}\times e^{9x^2}$, determine the value of x such that $f'(x)=0$.

Example 4

a) Determine the first, second, and third derivatives for each function.

i) $y=2^x$

ii) $y=3^x$

iii) $y=4^x$

b) What is the *n*th derivative of $y=2^x$?

c) What is the *n*th derivative of $y=k^x$, where $k>0$, $k\neq 1$?

The Derivative of the Logarithm Function

The fundamental limit

$$\lim_{t \to 0} (1 + t)^{\frac{1}{t}} = e$$

enables us to find the derivative of the logarithm function

$$f(x) = \log_a x, \quad x > 0$$

By the definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\log_a (x + h) - \log_a x}{h}$$

$$= \lim_{h \to 0} \frac{\log_a \left(\frac{x + h}{x}\right)}{h} \quad \text{(quotient property for logarithms)}$$

$$= \lim_{h \to 0} \left[\frac{1}{h} \log_a \left(1 + \frac{h}{x}\right)\right]$$

In order to apply the fundamental limit, let

$$t = \frac{h}{x}$$

Since x is a fixed positive number, it follows that $t \to 0$ as $h \to 0$. Thus, on eliminating h in the expression for $f'(x)$, we obtain

$$f'(x) = \lim_{t \to 0} \left[\left(\frac{1}{x}\right)\left(\frac{1}{t} \log_a (1 + t)\right)\right]$$

The second expression in parentheses can be rewritten as

$$\log_a (1 + t)^{\frac{1}{t}}$$

by using the power property of logarithms, and the factor $\frac{1}{x}$ can be taken outside the limit as a constant. Therefore,

$$f'(x) = \frac{1}{x} \lim_{t \to 0} \left[\log_a (1 + t)^{\frac{1}{t}}\right]$$

$$= \frac{1}{x} \log_a \left[\lim_{t \to 0} (1 + t)^{\frac{1}{t}}\right]$$

$$= \frac{1}{x} \log_a e$$

by the definition of e. Interchanging the limit and logarithm operations is justified by the fact that the logarithm function is continuous (as shown by its graph). This point is discussed in Problem 8. In summary, for $x > 0$ and for any $a > 0$, $a \neq 1$, we have shown that

$$\frac{d}{dx} (\log_a x) = \frac{1}{x} \log_a e$$

The value of the constant $\log_a e$ depends on the **base a of the logarithm function**. This derivative formula would be particularly simple if $\log_a e$ were equal to one. Since $\log_a a = 1$, we have

$$\log_e e = 1$$

and thus if we choose $a = e$ in the derivative formula, we obtain

$$\frac{d}{dx} (\log_e x) = \frac{1}{x}$$

When one is working with logarithm functions in Calculus, it is customary to choose base e, since then the derivative formula is simple. For this reason the logarithm function with base e has been given a special name, the **natural logarithm** of x, and a special notation, $\ln x$, pronounced "lon x".

> **The Natural Logarithm**
>
> For $x > 0$, the natural logarithm of x is defined by
>
> $$\ln x = \log_e x$$

We now give the derivative formula for the natural logarithm function in this notation.

> **Derivative of ln x**
>
> $$\frac{d}{dx} (\ln x) = \frac{1}{x}, \text{ for } x > 0$$

The exponential function that corresponds to the natural logarithm function is denoted e^x. The inverse identities that relate $\ln x$ and e^x are obtained by setting $a = e$ in the inverse identities in section 6.1.

> **The Inverse Identities**
>
> $e^{\ln x} = x$, for all $x > 0$
>
> $\ln (e^y) = y$, for all real y
>
> $\ln e = 1$

1. Find the derivative of each function.

a. $f(x) = \ln (x - 2)$
b. $f(x) = 3 \ln (4 - 3x)$
c. $f(x) = \ln (x^2 + 5)$
d. $f(x) = \ln x^2 + \ln 5$
e. $g(t) = \ln \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)$
f. $h(t) = t^2 \ln t$
g. $f(x) = (\ln x)^4$
h. $f(x) = \ln (x^4)$
i. $f(t) = (t \ln t)^4$
j. $f(v) = (\ln v + v)^3$
k. $g(u) = (\sin u)(\ln u)$
l. $f(y) = \ln (\sin y)$
m. $f(x) = \frac{\ln x}{3 + \ln x}$
n. $g(t) = \ln (\sin t + \cos t)$
o. $y = \cos^2 (\ln x)$
p. $y = \sin (\ln u)$
q. $y = \ln \sqrt{9 - x^2}$
r. $y = \ln \left[\frac{1 + x}{1 - x}\right]$

1. a. $\frac{1}{x - 2}$ b. $\frac{-9}{4 - 3x}$ c. $\frac{2x}{x^2 + 5}$ d. $\frac{2}{x}$
e. $\frac{t - 1}{2t(t + 1)}$ f. $t(2 \ln t + 1)$ g. $\frac{4}{x}(\ln x)^3$
h. $\frac{4}{x}$ i. $4(t \ln t)^3(\ln t + 1)$
j. $3(\ln v + v)^2\left(\frac{1}{v} + 1\right)$ k. $\frac{\sin u}{u} + (\cos u) \ln u$
l. $\cot y$ m. $\frac{3}{x(3 + \ln x)^2}$ n. $\frac{\cos t - \sin t}{\sin t + \cos t}$
o. $\frac{-2 \cos (\ln x) \sin (\ln x)}{x}$ p. $\frac{\cos (\ln u)}{u}$ q. $\frac{-x}{9 - x^2}$
r. $\frac{2}{(1 + x)(1 - x)}$

Derivatives of Logarithmic Functions

1. $f(x) = x^2 \ln x$

2. $f(x) = \dfrac{\ln x}{x}$

3. $f(x) = e^x \ln x$

4. $f(x) = \ln\left(x^3 + x^2\right)$

5. $f(x) = \ln\left(\dfrac{x-1}{x+1}\right)$

6. $f(x) = \ln\left(e^x + e^{-x}\right)$

Derivatives of Logarithmic Functions

1. $f(x) = \log x$

2. $f(x) = x^2 \log_3 x$

3. $f(x) = \dfrac{\log x}{10^x}$

4. $f(x) = \log\left(x^2 + 1\right)$

5. $f(x) = \log_2\left(x^2 2^x\right)$

6. $f(x) = \log(\ln x)$

Derivatives of the Sine and Cosine Functions

The following trigonometric derivative formulas are valid only for radian measure.

1. $\frac{d}{dx}\sin x = \cos x$

2. $\frac{d}{dx}\cos x = -\sin x$

Proofs

1. If $f(x) = \sin x$, then, by the definition of the derivative,

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{(\sin x\cos h + \cos x \sin h) - \sin x}{h}$$ using the addition formula for sine

$$f'(x) = \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$f'(x) = \lim_{h\to 0} \left(\sin x \frac{\cos h - 1}{h} + +\cos x \frac{\sin h}{h} \right)$$

$$f'(x) = \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}$$

$$f'(x) = \sin x(0) + \cos x(1)$$

$$f'(x) = \cos x$$

2. The derivative of the cosine function can be obtained using the Pythagorean trigonometric identity and differentiating implicitly. (Implicit differentiation will be discussed in detail in the next unit.)

$$\sin^2 x + \cos^2 x = 1$$

$$2\sin x \cos x + 2\cos x \frac{d}{dx}\cos x = 0$$

$$\frac{d}{dx}\cos x = -\frac{2\sin x \cos x}{2\cos x}$$

$$\frac{d}{dx}\cos x = -\sin x$$

Example Differentiate

a) $f(x)=x\cos x$

b) $y=\dfrac{x}{\sin x}$

c) $g(x)=\sin^3 x+\cos^3 x$

d) $y=\left(1+\cos^2 x\right)^6$

e) $h(x)=\sin(\ln x)$

f) $F(x)=e^{2x}\sin 3x$

Example Find an equation of the tangent line to the given curve at the given point.

a) $y = \dfrac{1}{\cos x} - 2\cos x$, at $\left(\dfrac{\pi}{3}, 1\right)$

b) $f(x) = \sin x + \cos 2x$, at $\left(\dfrac{\pi}{6}, 1\right)$

Derivatives of $y = \sin x$ and $y = \cos x$

- The derivatives of sinusoidal functions are found as follows:

$$\frac{d}{dx}(\sin x) = \cos x \qquad \text{and} \qquad \frac{d}{dx}(\cos x) = -\sin x$$

- When you are differentiating a function that involves sinusoidal functions, use the rules given above, along with the sum, difference, product, quotient, and chain rules as required.

  In particular, if $y = \sin f(x)$, then $\frac{dy}{dx} = \cos f(x) \times f'(x)$.

  Furthermore, if $y = \cos f(x)$, then $\frac{dy}{dx} = -\sin f(x) \times f'(x)$.

Example 1 Differentiate each of the following functions. Simplify your answers.

a) $f(x) = -2\sin 3x$

b) $y = \sqrt{2 + \sin^2 5x}$

c) $h(t) = -t^2 \sin(3t - \pi)$

d) $f(\theta) = \dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$

Example 2 Determine the equation of the line tangent to $y = 4\sin x \cos(2x)$ when $x = \frac{5\pi}{6}$.

Derivatives $y = \tan x$

- The derivative of the tangent function is found as follows:

  $$\frac{d}{dx}(\tan x) = \sec^2 x$$

- When you are differentiating a function that involves a tangent function, use the rule given above, along with the sum, difference, product, quotient, and chain rules as required.

  In particular, if $y = \tan f(x)$, then $\frac{dy}{dx} = \sec^2 f(x) \times f'(x)$.

- Trigonometric identities can be used to write one expression as an equivalent expression and then differentiate. In some cases, the new function will be easier to work with.

Example 1 Differentiate each of the following functions. Simplify your answers.

a) $y = e^{\tan x}$

b) $f(\theta) = \tan^4 3\theta$

c) $s(t) = \frac{\tan t}{t}$

d) $g(x) = x\cos(\tan x)$

Example 2 Determine an equation for the normal to the graph of $f(x) = \frac{1}{\tan^2 x}$ at $x = \frac{\pi}{4}$.

Derivatives of Reciprocal Trigonometric Functions

The following trigonometric derivative formulas are valid only for radian measure.

1. $\frac{d}{dx}\tan x = \sec^2 x$

2. $\frac{d}{dx}\csc x = -\csc x \cot x$

3. $\frac{d}{dx}\sec x = \sec x \tan x$

4. $\frac{d}{dx}\cot x = -\csc^2 x$

Proofs

1. To find the derivative of the function $y = \tan x$,

   we rewrite the function as $y = \frac{\sin x}{\cos x}$ and use the quotient rule.

   $$\frac{dy}{dx} = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

   Therefore, $\frac{d}{dx}\tan x = \sec^2 x$.

2. To find the derivative of the function $y = \csc x$,

   we rewrite the function as $y = \frac{1}{\sin x}$ and use the quotient rule.

   $$\frac{dy}{dx} = \frac{(0)(\sin x) - (1)(\cos x)}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = \left(\frac{-1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right) = -\csc x \cot x$$

   Therefore, $\frac{d}{dx}\csc x = -\csc x \cot x$.

3. To find the derivative of the function $y = \sec x$

   we rewrite the function as $y = \frac{1}{\cos x}$ and use the quotient rule.

   $$\frac{dy}{dx} = \frac{(0)(\cos x) - (1)(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) = \sec x \tan x$$

   Therefore, $\frac{d}{dx}\sec x = \sec x \tan x$.

4. To find the derivative of the function $y = \cot x$

we rewrite the function as $y = \dfrac{\cos x}{\sin x}$ and use the quotient rule.

$$\frac{dy}{dx} = \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= -\frac{1}{\sin^2 x} = -\csc^2 x$$

Therefore, $\dfrac{d}{dx}\cot x = -\csc^2 x$.

Example Differentiate the following functions.

a) $y = -\frac{1}{4}\csc(8x)$

b) $y = 2x(\sqrt{x} - \cot x)$

c) $y = \dfrac{x^2 \tan x}{\sec x}$

d) $y = \tan^2(\cos x)$

Example Determine the derivative of the following functions.

a) $y = \ln(\sec x + \tan x)$

b) $y = xe^{\cot 4x}$

Example For the function $y = \sec x + \csc x$, find the equation of the tangent line at the point where $x = \dfrac{3\pi}{4}$.

Logarithmic Differentiation

1. Take the natural logarithm of both sides of an equation $y = f(x)$.
2. Use the properties of logarithms to simplify the equation.
3. Differentiate implicitly with respect to x.
4. Solve the resulting equation for $\frac{dy}{dx}$.

Logarithmic differentiation can be used to simplify the calculation of derivatives of complicated functions involving products, quotients and powers. This method can also be used to differentiate functions where both the base and the exponent are variable.

Example Use logarithmic differentiation to find the derivative of each function.

a) $y = \dfrac{(x+1)^3}{(x+2)^5(x+3)^7}$

b) $y = x^5 e^x \sqrt{x^2 - x + 1}$

c) $y = x^{\cos x}$, for x > 0

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Example Find the equation of the tangent line to the curve $y = x^x$ at the point $(2, 4)$.