AP Calculus Homework Four – Differentiation

2.3 Implicit Differentiation; 2.4 Estimating a Derivative; 2.5 Derivative of the Inverse of a Function

1. Find $\dfrac{dy}{dx}$

a) $(x)' + [\cos(x+y)]' = (0)'$

$\Rightarrow 1 - \sin(x+y)(x'+y') = 0 \Rightarrow 1 + y' = \dfrac{1}{\sin(x+y)}$

$\boxed{\therefore y' = \csc(x+y) - 1}$

(a) $x + \cos(x+y) = 0$

b) $(\sin x)' - ((\cos y))' - (2)' = 0 \Rightarrow \cos x + \sin y \cdot y' = 0$

$\therefore y' = -\dfrac{\cos x}{\sin y}$

(b) $\sin x - \cos y - 2 = 0$

d) $(\sin(xy))' = x'$

$\cos(xy)(xy)' = 1$

$\cos(xy)(y + xy') = 1$

$y + xy' = \sec(xy)$

$\boxed{\therefore y' = \dfrac{\sec(xy) - y}{x}}$

(c) $3x^2 - 2xy + 5y^2 = 1$

c) $3(x^2)' - 2(xy)' + 5(y^2)' = (1)'$

$\Rightarrow 3(2x) - 2(x'y + xy') + 5(2y \cdot y') = 0$

$\Rightarrow 6x - 2y - 2xy' + 10yy' = 0 \Rightarrow y' = \dfrac{2y - 6x}{10y - 2x} = \dfrac{y - 3x}{5y - x}$

(d) $\sin(xy) = x$

(e) $\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases}$

e) $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{(1-\cos t)'}{(t - \sin t)'} = \dfrac{\sin t}{1 - \cos t}$

(f) $\begin{cases} x = 1 - e^{-t} \\ y = t + e^{-t} \end{cases}$

f) $\dfrac{dy}{dx} = \dfrac{(t + e^{-t})'}{(1 - e^{-t})'} = \dfrac{1 - e^{-t}}{e^{-t}} = e^t - 1$

(g) $\begin{cases} x = \dfrac{1}{1-t} \\ y = 1 - \ln(1-t) \end{cases}$

g) $\dfrac{dy}{dx} = \dfrac{[1 - \ln(1-t)]'}{\left(\frac{1}{1-t}\right)'} = \dfrac{\frac{1}{1-t}}{\frac{1}{(1-t)^2}} = 1 - t$

2. Find $\dfrac{d^2y}{dx^2}$

a) $(x^2)' + (y^2)' = (25)' \Rightarrow 2x + 2y \cdot y' = 0 \Rightarrow y' = -\dfrac{x}{y}$

$y'' = -\dfrac{x'y - xy'}{y^2} = \dfrac{x\left(-\frac{x}{y}\right) - y}{y^2} = \dfrac{-x^2 - y^2}{y^3} = \dfrac{-25}{y^3}$

(a) $x^2 + y^2 = 25$

(b) $\begin{cases} x = t^2 - 1 \\ y = t^4 - 2t^3 \end{cases}$

b) $\dfrac{dy}{dx} = \dfrac{(t^4 - 2t^3)'}{(t^2 - 1)'} = \dfrac{4t^3 - 6t^2}{2t} = 2t^2 - 3t$; $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \dfrac{(2t^2 - 3t)'}{(t^2 - 1)'}$

$= \dfrac{4t - 3}{2t} = 2 - \dfrac{3}{2t}$

(c) $\begin{cases} x = \cos t \\ y = \cos 2t \end{cases}$

c) $\dfrac{dy}{dx} = \dfrac{(\cos 2t)'}{(\cos t)'} = \dfrac{-\sin 2t\,(2)}{-\sin t} = \dfrac{4\cancel{\sin t}\cos t}{\cancel{\sin t}} = 4\cos t.$

$\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \dfrac{(4\cos t)'}{(\cos t)'} = \dfrac{-4\sin t}{-\sin t} = 4$

3. In the following eight sub questions, differentiable functions *f* and *g* have the values shown in the table.

x	f	f'	g	g'
0	2	1	5	–4
1	3	2	3	–3
2	5	3	1	–2
3	10	4	0	–1

(a) If $A = f + 2g$, find $A'(3)$

$A'(x) = f'(x) + 2g'(x)$

$\therefore A'(3) = f'(3) + 2g'(3) = 4 + 2(-1) = 2$

(b) If $B = f \times g$, find $B'(2)$

$B'(2) = f'(2)g(2) + f(2)g'(2)$

$= (3)(1) + (5)(-2) = -7$

(c) If $D = \dfrac{1}{g}$, find $D'(1)$

$D'(1) = -\dfrac{1}{(g(1))^2} \cdot g'(1) = -\dfrac{-3}{3^2} = \dfrac{1}{3}$

$\because D' = (g^{-1})' = (-1)g^{-2} \cdot g'$

(d) If $H(x) = \sqrt{f(x)}$, find $H'(3)$

$H'(x) = \frac{1}{2}(f(x))^{-\frac{1}{2}} f'(x)$

$\therefore H'(3) = \frac{1}{2}[f(3)]^{-\frac{1}{2}} f'(3) = \frac{1}{2}(10)^{-\frac{1}{2}}(4) = \frac{2}{\sqrt{10}} = \frac{\sqrt{10}}{5}$

(e) If $K(x) = \left(\dfrac{f}{g}\right)(x)$, find $K'(0)$

$K'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

$K'(0) = \dfrac{f'(0)g(0) - f(0)g'(0)}{[g(0)]^2} = \dfrac{(1)(5) - (2)(-4)}{5^2} = \dfrac{13}{25}$

(f) If $M(x) = f(g(x))$, find $M'(1)$

$M'(x) = f'(g(x))g'(x)$ $\therefore M'(1) = f'(g(1))g'(1)$

$= f'(3) \cdot (-3)$

$= 4(-3) = -12$

(g) If $P(x) = f(x^3)$, find $P'(1)$

$\because P'(x) = f'(x^3)(x^3)' = f'(x^3)(3x^2)$

$\therefore P'(1) = f'(1^3)(3(1)^2) = f'(1) \cdot 3 = 2(3) = 6$

(h) If $S(x) = f^{-1}(x)$, find $S'(3)$

$\because S'(x) = \dfrac{1}{f'(y)}$ where $x = f(y)$ $\therefore S'(3) = \dfrac{1}{f'(1)}$

$= \dfrac{1}{2}$

4. From the values of *f* shown in the table below, estimate $f'(2)$.

x	1.92	1.94	1.96	1.98	2.00
$f(x)$	6.00	5.00	4.40	4.10	4.00

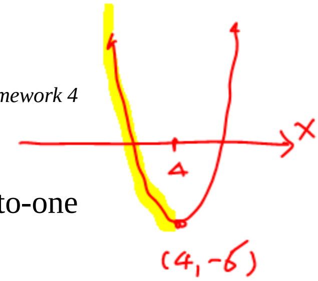
$f'(2) = \lim_{x \to 2} \dfrac{f(x) - f(2)}{x - 2} \approx \dfrac{f(1.98) - f(2)}{1.98 - 2} = \dfrac{4.10 - 4.00}{1.98 - 2} = \dfrac{0.1}{-0.02} = \dfrac{10}{-2} = -5$

5. Using the values shown in the table in Q7, estimate $\left(f^{-1}\right)'(4)$.

$(f^{-1})'(4) = \dfrac{1}{f'(2)} \approx \dfrac{1}{-5}$

$\because$ If $b = f(a)$, then $(f^{-1})'(b) = \dfrac{1}{f'(a)}$

$y = x^2 - 8x + 10 = x^2 - 2(4)x + 4^2 - 4^2 + 10 = (x-4)^2 - 6$



6. The "left half" of the parabola defined by $y = x^2 - 8x + 10$ for $x \le 4$ is a one-to-one function; therefore its inverse is also a function. Call that inverse g. Find $g'(3)$.

let $x^2 - 8x + 10 = 3 \Rightarrow x^2 - 8x + 7 = 0 \Rightarrow (x-1)(x-7) = 0$

$\Rightarrow x = 1$ or $x = 7$. $\because x \le 4 \quad \therefore x = 1$.

and $f'(x) = (x^2 - 8x + 10)' = 2x - 8$; $\quad f'(1) = 2(1) - 8 = -6$.

$\therefore g'(3) = \dfrac{1}{f'(1)} = \dfrac{1}{-6}$

7. At how many points on the interval [a, e] does the function graphed satisfy the Mean Value Theorem?