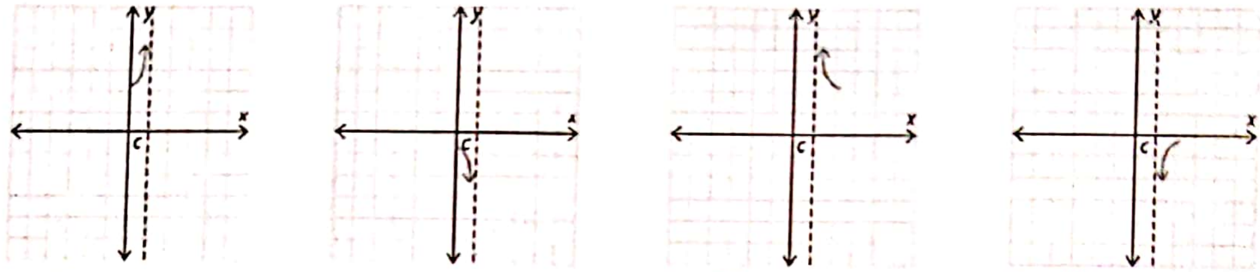
Vertical and Horizontal Asymptotes

- The graph of f has a vertical asymptote $x = c$ if any of the following is true

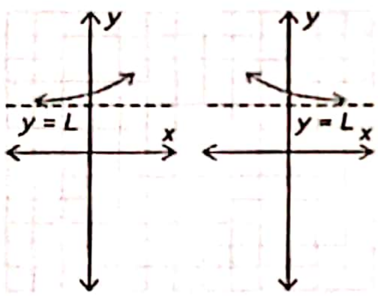
  $\lim_{x\to c^-} f(x) = +\infty$ $\quad$ $\lim_{x\to c^-} f(x) = -\infty$ $\quad$ $\lim_{x\to c^+} f(x) = +\infty$ $\quad$ $\lim_{x\to c^+} f(x) = -\infty$

  The following graphs correspond to each limit statement above:

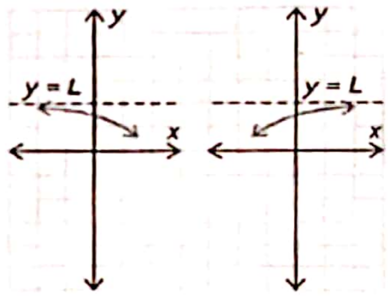


$\lim_{x\to c^-} f(x) = +\infty$ $\quad$ $\lim_{x\to c^-} f(x) = -\infty$ $\quad$ $\lim_{x\to c^+} f(x) = +\infty$ $\quad$ $\lim_{x\to c^+} f(x) = -\infty$

- A rational function of the form $f(x) = \dfrac{p(x)}{q(x)}$ has a vertical asymptote $x = c$ if $q(c) = 0$ and $p(c) \neq 0$.
- The line $y = L$ is a horizontal asymptote of the graph of f if $\lim_{x\to+\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.



$f(x) > L$, so the graph approaches from above.



$f(x) < L$, so the graph approaches from below.

- The reciprocal function and limits at infinity

  $\lim_{x\to\infty} \dfrac{1}{x} = 0$ $\quad$ and $\quad$ $\lim_{x\to-\infty} \dfrac{1}{x} = 0$

- In a rational function, an oblique asymptote occurs when the degree of the numerator is exactly one greater than the degree of the denominator.
- Algorithm for Curve Sketching (so far)
  1. Check for any discontinuities in the domain. Determine if there are vertical asymptotes at these discontinuities, and determine the direction from which the curve approaches these asymptotes.
  2. Test end behaviour by determining $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
  3. Find both intercepts.
  4. Find any critical points. Use the first derivative test to determine whether the critical points are local maxima, local minima, or neither.
  5. Sketch the curve.

Example 1 For each of the following, check for discontinuities and state the equation of any vertical asymptotes. Conduct a limit test to determine the behavior of the curve on either side of the asymptote.

a) $f(x)=\dfrac{x^2+3x+2}{x+1}$

b) $g(x)=-\dfrac{4}{x^2-12x+36}$

Example 2 For each of the following, determine the equations of any horizontal asymptotes. Then state whether the curve approaches the asymptote from above or below.

a) $f(x)=\dfrac{5x}{3x^2-x}$

b) $g(x)=\dfrac{8x^2+1}{7+4x-6x^2}$

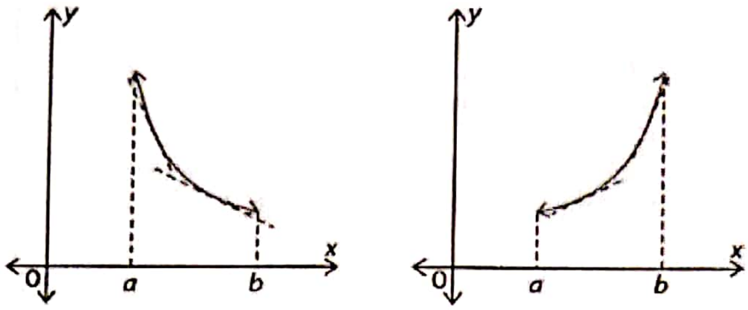
Example 3 Determine the equation of the oblique asymptote for each of the following. Then state whether the curve approaches the asymptote from above or below.

a) $f(x)=\dfrac{2x^2+4x+1}{x+1}$

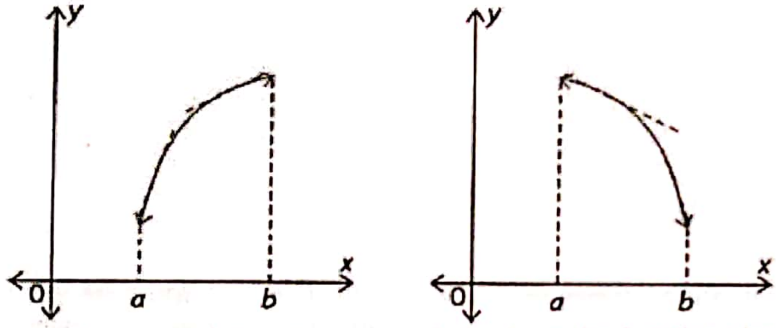
b) $g(x)=\dfrac{3-x-x^3}{x^2-x+2}$

Concavity and Points of Inflection

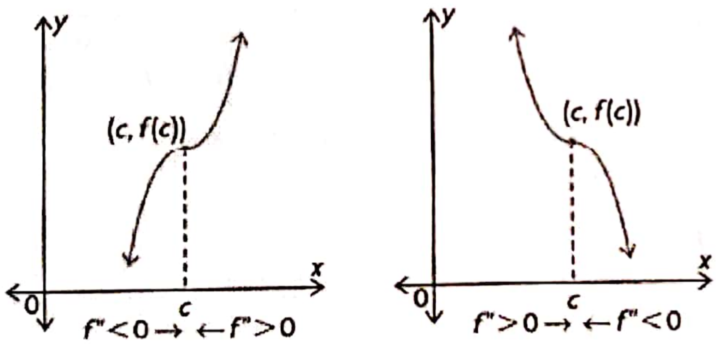
- The graph of a function f is concave up on an interval if f' is increasing on the interval. The graph of the function is above the tangent at every point on the interval.



- The graph of a function f is concave down on an interval if f' is decreasing on the interval. The graph of the function is below the tangent at every point on the interval.

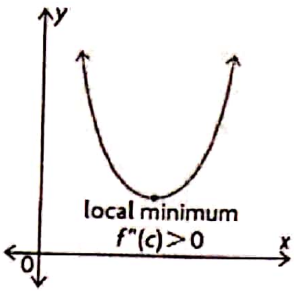


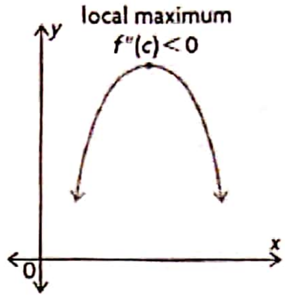
- A point of inflection is a point on the graph of f where the function changes from concave up to concave down, or vice versa. $f''(c)=0$ or is undefined if $(c,\ f(c))$ is a point of inflection on the graph of f.



- Test for concavity
  If f is a differentiable function whose second derivative exists on an open interval I, then
  - the graph of f is concave up on I if $f''(x)>0$ for all values of x in I
  - the graph of f is concave down on I if $f''(x)<0$ for all values of x in I

- The second derivative test
  Suppose that f is a function for which $f'(c)=0$ and the second derivative of f exists on an interval containing c.

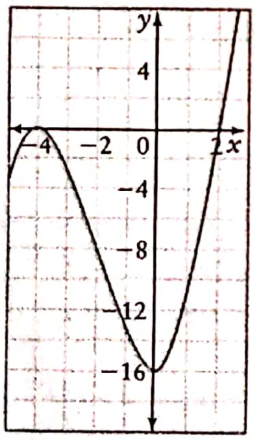




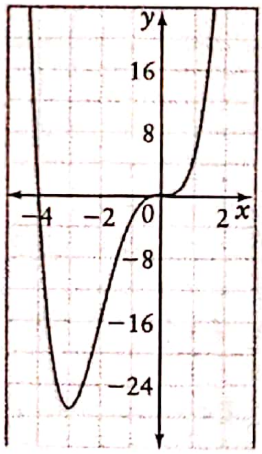
  - If $f''(c)>0$, then $f(c)$ is a local minimum value.
  - If $f''(c)<0$, then $f(c)$ is a local maximum value.
  - If $f''(c)=0$, then the test fails. Use the first derivative test.

Example 1 For each graph, identify the intervals over which the graph is concave up and the intervals over which it is concave down.

a)

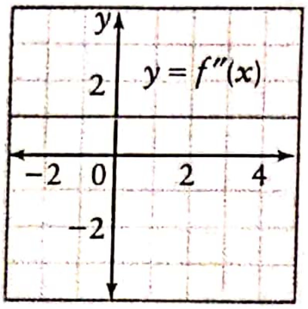


b)

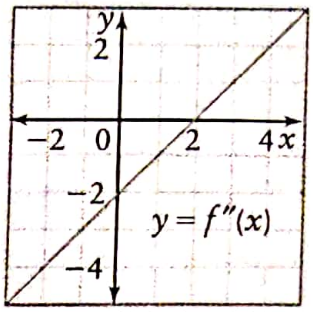


Example 2 Given each graph of $y = f''(x)$, state the intervals of concavity for the function $y = f(x)$. Also indicate where any points of inflection occur for $y = f(x)$.

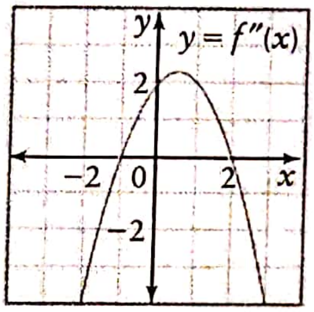
a)



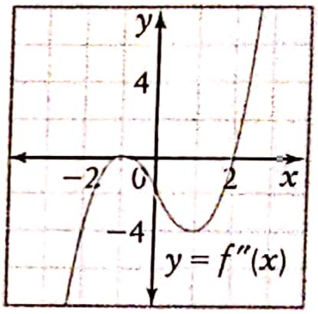
b)



c)



d)



Example 3 Sketch a graph of a function that satisfies each set of conditions

a) $f''(x) = 2$ for all x, $f'(2) = 0$, $f(2) = -3$

b) $f''(x) > 0$ when $x < -6$, $f''(x) < 0$ when $x > -6$, $f'(-6) = 3$, $f(-6) = 2$

c) $f''(x) < 0$ when $-2 < x < 1$, $f''(x) > 0$ when $x < -2$ and $x > 1$, $f(-2) = -3$, $f(0) = 0$

Example 4 Find the intervals on which each curve is concave upward and the intervals on which it is concave downward. State any points of inflection.

a) $f(x)=3x^4-4x^3-6x^2+2$

b) $g(x)=\dfrac{x-1}{3-x}$

Example 5 Use the second derivative test to find the local maximum and minimum points of each function, wherever possible.

a) $y=-x^4+4x^3+4$

b) $f(x)=\dfrac{x}{(x-1)^2}$

Example 6 A function is defined by $f(x) = ax^3 + bx^2 + cx + d$. Find the values of a, b, c, and d if $f(x)$ has a point of inflection at $(0, 2)$ and a local maximum at $(2, 6)$.

Optimization Problems

- Algorithm for Solving Optimization Problems
    1. Understand the problem, and identify quantities that can vary. Whenever possible, draw a diagram, labelling the given and required quantities.
    2. Determine a function in one variable that represents the quantity to be optimized.
    3. Determine the domain of the function to be optimized, using the information given in the problem.
    4. Use the algorithm for extreme values to find the absolute maximum or minimum value in the domain. If the function to be optimized is not continuous over a closed interval $[a, b]$, then we must use the first derivative test (or the second derivative test) to find the maximum or minimum value of the function. [Note that these tests will be discussed again in greater detail in Chapter 4.]
    5. Use your result from step 4 to answer the original problem.

Example 1 A farmer wants to fence an area of 750 000 m² in a rectangular field and divide it in half with a fence parallel to one of the sides of the rectangle. How can this be done so as to minimize the amount of fence used?

Example 2 An open metal box for removing ashes from a fireplace is to be constructed from a rectangular piece of sheet metal that is 1.5 m by 1 m. Squares are to be cut from each corner of the sheet metal, the sides folded upward to form the box, and then the seams welded. Determine the dimensions that will give the box with the largest volume. What is the maximum capacity of a box that is constructed in this way?