Application of Residue Theorem

Evaluation of Real Definite Integrals by Contour Integrals:

A large number of real definite integrals, whose evaluation by usual methods become sometimes very tedious, can be easily evaluated by using **Cauchy's Residue theorem**. For finding the integrals we take a suitable complex function $f(z)$ and closed curve C, then find the poles or singularity of the function $f(z)$ and calculate residues at those poles only which lie within the curve C. Then using Cauchy's residue theorem we have

$$\oint_C f(z)\,dz = 2\pi i\left[\text{sum of the residues of } f(z) \text{ at the poles within } C\right]$$

We call the curve, a contour and the process of integration along a contour is called contour integration.

(Improper Integral) Infinite real integrals of the form $\int_{-\infty}^{+\infty} \frac{f_1(x)}{f_2(x)}dx$ **or,** $\int_0^{+\infty} \frac{f_1(x)}{f_2(x)}dx$ where $f_1(x)$ and $f_2(x)$ are polynomials in x. Such integrals can be reduced to contour integrals, if

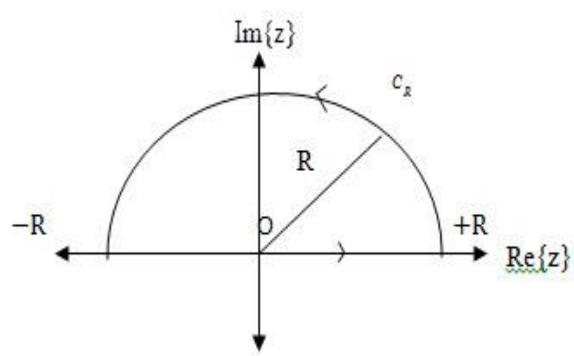
(i) $f_2(x)$ has no real roots.

(ii) The degree of $f_2(x)$ is greater than that of $f_1(x)$ by at least two.

Procedure to solve:

To evaluate such integrals we consider the contour integrals

$\oint_C \frac{f_1(z)}{f_2(z)}dz$ where C is the closed contour, consisting the real axis from $-R$ to R and the upper half C_R of the circle $|z| = R$ i.e.,



$$\oint_C \frac{f_1(z)}{f_2(z)}dz = \int_{-R}^{R} \frac{f_1(x)}{f_2(x)}dx + \int_{C_R} \frac{f_1(z)}{f_2(z)}dz \text{ - - - - } (1)$$

Now using CRT we get,

$$\oint_C \frac{f_1(z)}{f_2(z)} dz = 2\pi i \times (\text{sum of the residue at the poles within } C)$$

Then (1) becomes,

$$\int_{-R}^{R} \frac{f_1(x)}{f_2(x)} dx + \int_{C_R} \frac{f_1(z)}{f_2(z)} dz = 2\pi i \times (\text{sum of the residue at the poles within } C)$$

$$\Rightarrow \int_{-R}^{R} \frac{f_1(x)}{f_2(x)} dx = -\int_{C_R} \frac{f_1(z)}{f_2(z)} dz + 2\pi i \times (\text{sum of the residue at the poles within } C) \text{ - -- -- --- --- (2)}$$

$$\therefore \lim_{R\to\infty} \int_{-R}^{R} \frac{f_1(x)}{f_2(x)} dx = -\lim_{R\to\infty} \int_{C_R} \frac{f_1(z)}{f_2(z)} dz + 2\pi i \times (\text{sum of the residue at the poles within } C)$$

Now, on the semi circular path C_R, $|z| = R \Rightarrow z = \mathrm{Re}^{i\theta}, (0 \le \theta \le \pi) \therefore dz = i\,\mathrm{Re}^{i\theta}\, d\theta$. Then applying Jordan's Lemma,

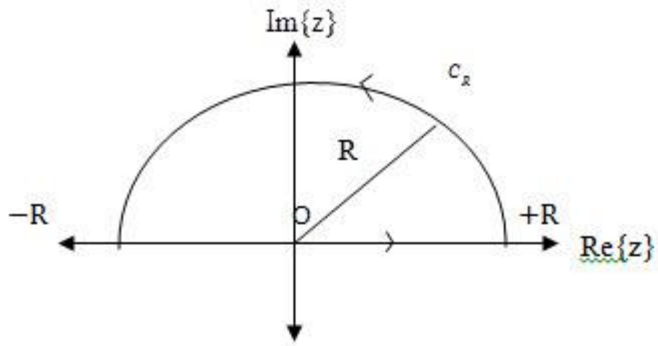
$$\lim_{R\to\infty} \int_{C_R} \frac{f_1(z)}{f_2(z)} dz = \lim_{R\to\infty} \int_0^{\pi} \frac{f_1(\mathrm{Re}^{i\theta})}{f_2(\mathrm{Re}^{i\theta})} Rie^{i\theta} d\theta = 0$$

Then (2) reduces to

$$\therefore \int_{-\infty}^{\infty} \frac{f_1(x)}{f_2(x)} dx = 2\pi i \times (\text{sum of the residues at the poles within } C)$$

Example 7.5: Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^2}$ by using contour integration.

Solution:



We consider $\oint_C \frac{dz}{\left(z^2+4\right)^2}$ where C is the closed contour consisting of the semi circle C_R of radius R together with the part of the real axis –R to +R. i.e.,

$$\oint_C \frac{dz}{\left(z^2+4\right)^2} = \int_{-R}^{R} \frac{dx}{\left(x^2+4\right)^2} + \int_{C_R} \frac{dz}{\left(z^2+4\right)^2} \quad \ldots\ldots(1)$$

Now the first integral has singularities or pole at $\left(z^2+4\right)^2 = 0$ i.e. $z = \pm 2i$ of order 2. But the only pole $z = +2i$ is inside the contour C. So,

$$\operatorname{Re} s(at\ z = +2i) = \lim_{z\to 2i} \frac{1}{1!}\frac{d}{dz}\left\{(z-2i)^2 . \frac{1}{(z-2i)^2 (z+2i)^2}\right\}$$

$$= \lim_{z\to 2i} \frac{d}{dz}\left\{\frac{1}{(z+2i)^2}\right\} = \lim_{z\to 2i}\left\{\frac{-2}{(z+2i)^3}\right\} = \frac{-2}{(4i)^3} = \frac{1}{32i}$$

So by CRT,

$$\oint_C \frac{dz}{\left(z^2+4\right)^2} = 2\pi i \times \frac{1}{32i} = \frac{\pi}{16}$$

So equation (1) becomes

$$\int_{-R}^{R} \frac{dx}{\left(x^2+4\right)^2} + \int_{C_R} \frac{dz}{\left(z^2+4\right)^2} = \frac{\pi}{16}$$

By Jordan Lemma letting $R \to \infty$ and noting that the second integral in left hand side would become zero.

$$\lim_{z\to R} z\, f(z) = \lim_{z\to R} z \frac{1}{\left(z^2+4\right)^2} = \lim_{z\to R} z \frac{1}{z^4\left(1+\frac{4}{z^2}\right)^2} = \lim_{z\to R} \frac{1}{z^3\left(1+\frac{4}{z^2}\right)^2} = 0$$

$$\therefore \lim_{R\to\infty} \int_{C_R} \frac{dz}{\left(z^2+4\right)^2} = 0.$$

Hence,

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{\left(x^2+4\right)^2} + \lim_{R\to\infty} \int_{C_R} \frac{dz}{\left(z^2+4\right)^2} = \frac{\pi}{16}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{\left(x^2+4\right)^2} + 0 = \frac{\pi}{16}$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{\left(x^2+4\right)^2} = \frac{\pi}{16}.$$

Example 7.6: Evaluate $\int_0^{\infty} \frac{dx}{\left(x^4+16\right)}$ by using contour integration.

Solution: We consider $\oint_C \frac{dz}{\left(z^4+16\right)}$ where C is the closed contour consisting of the semi circle C_R of radius R together with the part of the real axis $-R$ to $+R$. i.e.,

$$\oint_C \frac{dz}{\left(z^4+16\right)} = \int_{-R}^{R} \frac{dx}{\left(x^4+16\right)} + \int_{C_R} \frac{dz}{\left(z^4+16\right)} \quad \ldots\ldots(1)$$

The figure in the previous example should be considered here.

Now the first integral has singularities or poles at

$$z^4 + 16 = 0$$

$$\Rightarrow z^4 = -16 = 16e^{i\pi}$$

$$\therefore z_k = (16)^{\frac{1}{4}} e^{i\left(\frac{\pi+2k\pi}{4}\right)}, k = 0,1,2,3$$

When $k = 0,\ z_0 = 2e^{i\frac{\pi}{4}}$

$k = 1,\ z_1 = 2e^{i\frac{3\pi}{4}}$

$$k=2,\ z_2=2e^{i\frac{5\pi}{4}}$$

$$k=3,\ z_3=2e^{i\frac{7\pi}{4}}$$

i.e. there are four poles, but only two poles at z_0 and z_1 lie within the contour C. So,

$$\mathrm{B}_k=\mathrm{Res}\ (at\ z=z_k)=\lim_{z\to z_k}\left\{(z-z_k).\frac{1}{(z^4+16)}\right\}\quad,\ k=0,1$$

$$=\lim_{z\to z_k}\left\{\frac{1}{4z_k^3}\right\}$$

$$\mathrm{B}_0=\frac{1}{4z_0^3}=\frac{1}{4}z_0^{-3}=\frac{1}{4}e^{-i\frac{3\pi}{4}}$$

$$\mathrm{B}_1=\frac{1}{4z_1^3}=\frac{1}{4}z_1^{-3}=\frac{1}{4}e^{-i\frac{9\pi}{4}}$$

So by CRT,

$$\oint_C\frac{dz}{(z^4+16)}=2\pi i\times[\mathrm{B}_0+\mathrm{B}_1]=\frac{\sqrt{2}\pi}{16}$$

So equation (1) becomes

$$\int_{-R}^{R}\frac{dx}{(x^4+16)}+\int_{C_R}\frac{dz}{(z^4+16)}=\frac{\sqrt{2}\pi}{16}.$$

By Jordan Lemma letting $R\to\infty$ and noting that the second integral in left hand side would become zero. Hence,

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{(x^4+16)}+\lim_{R\to\infty}\int_{C_R}\frac{dz}{(z^4+16)}=\frac{\sqrt{2}\pi}{16}$$

$$\Rightarrow\int_{-\infty}^{\infty}\frac{dx}{(x^4+16)}+0=\frac{\sqrt{2}\pi}{16}$$

$$\Rightarrow\int_{-\infty}^{\infty}\frac{dx}{(x^4+16)}=\frac{\sqrt{2}\pi}{16}$$

$$\int_{0}^{\infty}\frac{dx}{(x^4+16)}=\frac{\sqrt{2}\pi}{32}.$$

Matlab command for improper integral:

1. Evaluate $\int_0^\infty \frac{dx}{x^2+1}$, >> fun=@(x) 1./(x.^2+1); >> q=integral(fun,0,inf) q = 1.5708	2. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2-2x+2)^2}$, >> f=@(x) 1./(x.^2-2.*x+2).^2; >> q=integral(f,-inf,inf) q = 1.5708

Sample Exercise Set on Improper Integral:

Integration of the form $\int_{-\infty}^{\infty} \frac{f_1(x)}{f_2(x)} dx$ (**improper integral**)

1. Evaluate the following improper integral using Cauchy's residue theorem (CRT):

(i) $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$, (ii) $\int_{0}^{\infty} \frac{dx}{x^2+1}$,

(iii) $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx$, (iv) $\int_{-\infty}^{\infty} \frac{dx}{\left(x^2-2x+2\right)^2}$,

(v) $\int_{0}^{\infty} \frac{x^2}{x^6+1} dx$.

Laurent series generalize Taylor series. Indeed, whereas a Taylor series has positive integer powers (and a constant term) and converges in a disc, a Laurent series is a series of positive and negative integer powers of $(z - z_0)$ and converges in an annulus (a circular ring) with center z_0 . Hence by a Laurent series we can represent a given function $f(z)$ that is analytic in an annulus and may have singularities outside the ring as well as in the "hole" of the annulus.

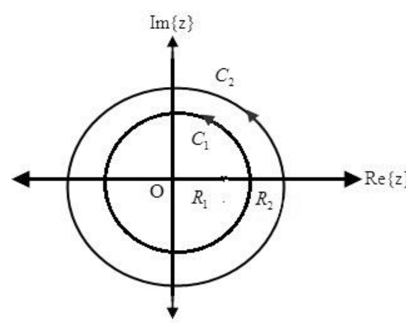
Laurent's Theorem:

Let $f(z)$ be analytic in a domain containing two concentric circles c_1 and c_2 with center z_0 , radii R_1 and R_2 , ($R_1 < R_2$) and the annulus between them. Then $f(z)$ can be represented by the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \frac{b_3}{(z-z_0)^3} + \cdots$$



The coefficients of Laurent series are given by the integrals

$$a_n = \frac{1}{2\pi i} \oint_c \frac{f(z^*)}{(z^*-z_0)^{n+1}}\, dz^* , \qquad b_n = \frac{1}{2\pi i} \oint_c (z^*-z_0)^{n-1}\, f(z^*)\, dz^*$$

The variable of integration is denoted by z^* , since z is used in Laurent series.

The existing negative power of $(z - z_0)$ is known as **principal part**. If there is finite number of terms in the principal part of $f(z)$ in the Laurent series expansion then the coefficient of $\left(\frac{1}{z-z_0}\right)$ is called the residue of $f(z)$ at pole $z = z_0$.

Laurent series expansion

Example: 7.7 Obtain Laurent series expansion of $f(z) = \dfrac{1}{(1+z^2)(z+2)}$ when (i) $1 < |z| < 2$, (ii) $|z| > 2$. .

Solution: (i) Since $1 < |z| < 2$

$$\Rightarrow \frac{1}{z} < 1 \text{ and } \frac{|z|}{2} < 1$$

$$\therefore \frac{1}{|z^2|} < 1 \text{ and } \frac{|z|}{2} < 1$$

Let $\dfrac{1}{(1+z^2)(z+2)} \equiv \dfrac{Az+B}{1+z^2} + \dfrac{C}{z+2}$

$\therefore 1 \equiv (\mathrm{A}z + \mathrm{B})(z+2) + \mathrm{C}(1+z^2)$

At, $z = -2$, $5C = 1$ $\therefore \mathrm{C} = \frac{1}{5}$

Equating coefficients of z^2 ; $A + C = 0$ $\therefore A = -C = -\frac{1}{5}$

Equating coefficients of z ; $2A + B = 0$ $\therefore \mathrm{B} = -2\mathrm{A} = \frac{2}{5}$

$$\therefore \frac{1}{(1+z^2)(z+2)} = \frac{-\frac{1}{5}z+\frac{2}{5}}{1+z^2} + \frac{\frac{1}{5}}{z+2}$$

$$= \frac{2}{5}\frac{1}{1+z^2} - \frac{1}{5}\frac{z}{1+z^2} + \frac{1}{5}\frac{1}{z+2}$$

$$= \frac{2}{5}\frac{1}{z^2(1+\frac{1}{z^2})} - \frac{1}{5}\frac{z}{z^2(1+\frac{1}{z^2})} + \frac{1}{5}\frac{1}{2(1+\frac{z}{2})}$$

$$= \frac{2}{5z^2}\left(1+\frac{1}{z^2}\right)^{-1} - \frac{1}{5z}\left(1+\frac{1}{z^2}\right)^{-1} + \frac{1}{10}\left(1+\frac{z}{2}\right)^{-1}$$

$$= \frac{2}{5z^2}\left(1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \cdots\right) - \frac{1}{5z}\left(1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \cdots\right)$$

$$+ \frac{1}{10}\left(1 - \frac{z}{2} + \frac{z^2}{4} - \cdots\right)$$

which is the required Laurent series.

(ii) For $|z| > 2$ we have $\frac{|z|}{2} > 1 \Rightarrow \frac{2}{|z|} < 1$

Also $\frac{1}{|z^2|} < 1$

$$\therefore \frac{1}{(1+z^2)(z+2)} = \frac{2}{5}\frac{1}{1+z^2} - \frac{1}{5}\frac{z}{1+z^2} + \frac{1}{5}\frac{1}{z+2}$$

$$= \frac{2}{5}\frac{1}{z^2(1+\frac{1}{z^2})} - \frac{1}{5}\frac{z}{z^2(1+\frac{1}{z^2})} + \frac{1}{5}\frac{1}{z(1+\frac{2}{z})}$$

$$= \frac{2}{5z^2}\left(1 + \frac{1}{z^2}\right)^{-1} - \frac{1}{5z}\left(1 + \frac{1}{z^2}\right)^{-1} + \frac{1}{5z}\left(1 + \frac{2}{z}\right)^{-1}$$

$$= \frac{2}{5z^2}\left(1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \cdots\right) - \frac{1}{5z}\left(1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \ \cdots\cdots\right)$$

$$+ \frac{1}{5z}\left(1 - \frac{2}{z} + \frac{4}{z^2} - \cdots\right).$$

which is the required Laurent series.

Sample Exercise Set on Laurent Series:

State Laurent series. Expand $f(z) = \dfrac{3z}{(z-1)(2-z)}$ in a Laurent series valid for

(a) $|z| < 1$, (b) $1 < |z| < 2$, (c) $|z| > 2$, (d) $|z-1| > 2$ and (e) $0 < |z-1| < 1$.

1. Expand $f(z) = \dfrac{1}{z(z-2)}$ in a Laurent series valid for

   (a) $0 < |z| < 2$ and (b) $|z| > 2$.

2. Expand $f(z) = \dfrac{5z}{(z^2+1)(z+2)}$ in a Laurent series valid for

   (a) $1 < |z| < 2$ and (b) $|z| > 2$

3. Find the function, $f(z)$ and the region of convergence for the following series:

   a. $1 + z + z^2 + z^3 + \cdots$.

   b. $1 - z + z^2 - z^3 + \cdots$.

c. $1+2z+3z^2+4z^3+\cdots.$

d. $1-2z+3z^2-4z^3+\cdots.$

4. Given functions (i) $f(z)=\dfrac{z}{(z-1)(3-z)}$ [Figure: (a) and (b)] and (ii) $f(z)=\dfrac{z}{(z-1)(2-z)}$.

[Figure: (a) and (c)] Determine the region of convergence and the series for the following figures:

(a)	(b)	(c)