Planar Graphs & Graph Coloring

Course Code: 00090 Course Title: Discrete Mathematics

Dept. of Computer Science
Faculty of Science and Technology

Lecturer No:	**19**	**Week No:**	**11**	**Semester:**	**Summer 21-22**
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Lecture Outline

8.7 Planar Graphs

8.8 Graph Coloring

Objectives and Outcomes

- Objectives: To understand the terms planar graph, graph coloring, chromatic number, Euler formula; to determine whether a graph is planar; to determine the chromatic number of a graph, to understand applications of graph coloring.
- Outcomes: The students are expected to be able to explain the terms planar graph, graph coloring, chromatic number, Euler formula; be able to determine whether a graph is planar; be able to determine the chromatic number of a graph, be able to solve the problem of Scheduling Final Exams at a university using graph coloring model.

Planar Graphs

- A graph is called *planar* if it can be drawn in the plane without any edges crossing.
  - Such a drawing is called a *planar representation* of the graph

- ***Note***: A graph may be planar even if it is usually drawn with crossings, since it may be possible to draw it in another way without crossings.

Example 1

Example 1: **Is K_4** (shown in Figure 2 with two edges crossing) **planar**?

Solution: **K_4 is planar** because it can be drawn without crossings, as shown in Figure 3

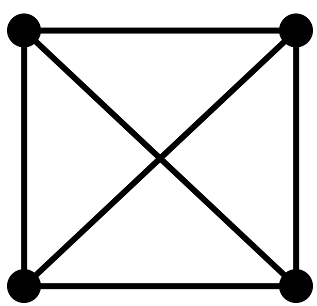

Figure 2

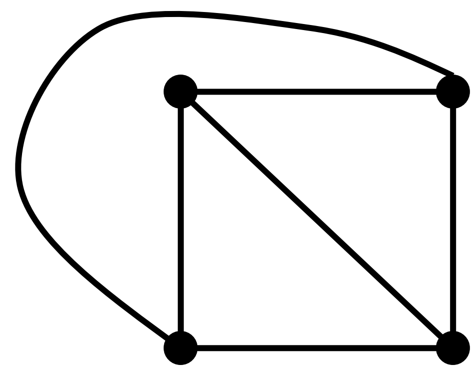

Figure 3

Example 2

- <u>Example 2:</u> **Is Q_3** (shown in Figure 4) **planar**?

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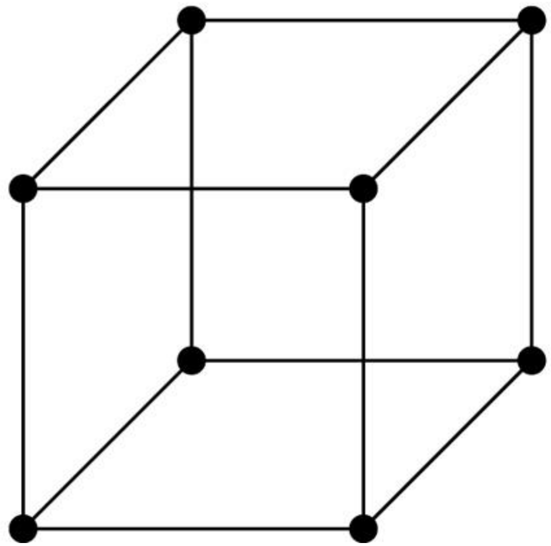

Figure 4

Solution of Example 2

Solution: Q_3 **is planar**, because it can be drawn without any edges crossing, as shown in Figure 5.

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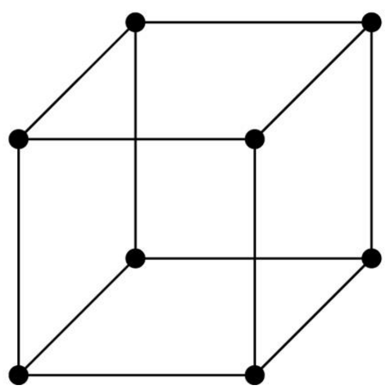

Figure 4

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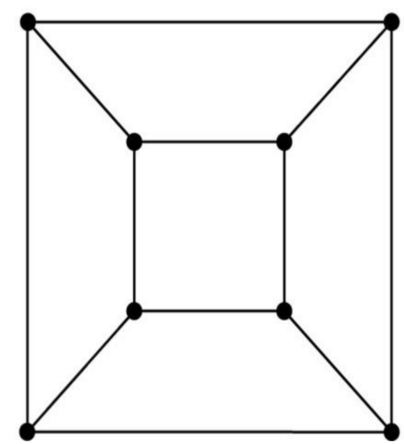


Figure 5

Example 3

- <u>Example 3:</u> **Is $K_{3,3}$**, shown in Figure 6, **planar**?

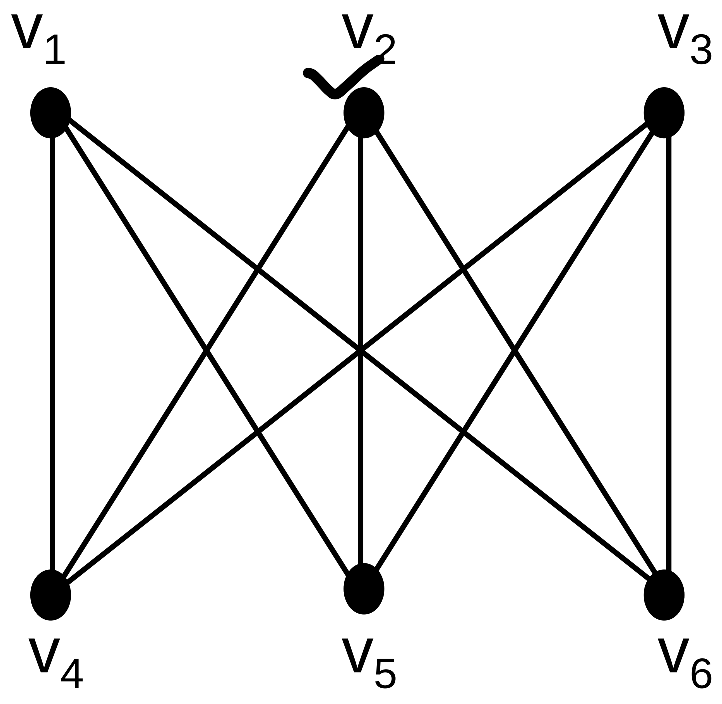
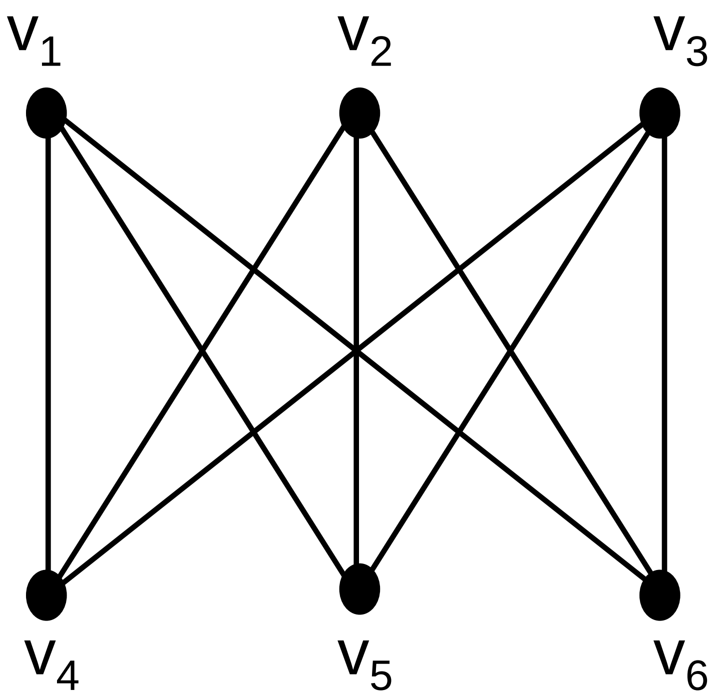


Figure 6

Solution of Example 3

- In any planar representation of $K_{3,3}$, vertex v_1 must be connected to both v_4 and v_5, and v_2 also must be connected to both v_4 and v_5.



Solution of Example 3 (cont.)

- The four edges $\{v_1, v_4\}$, $\{v_4, v_2\}$, $\{v_2, v_5\}$, $\{v_5, v_1\}$ form a closed curve that splits the plane into two regions, $\mathbf{R_1}$ and $\mathbf{R_2}$, as shown in Figure 7(a).

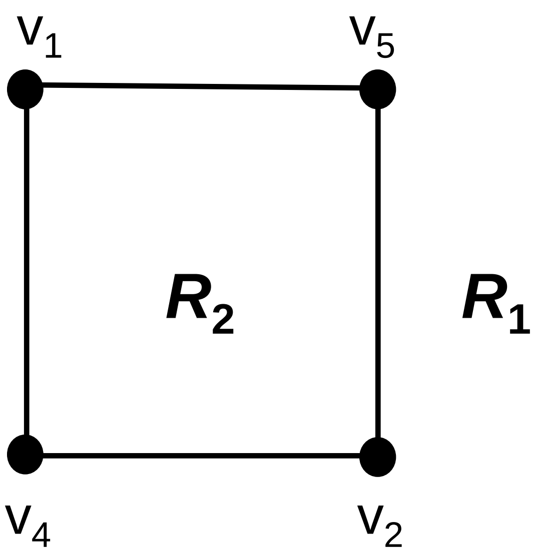
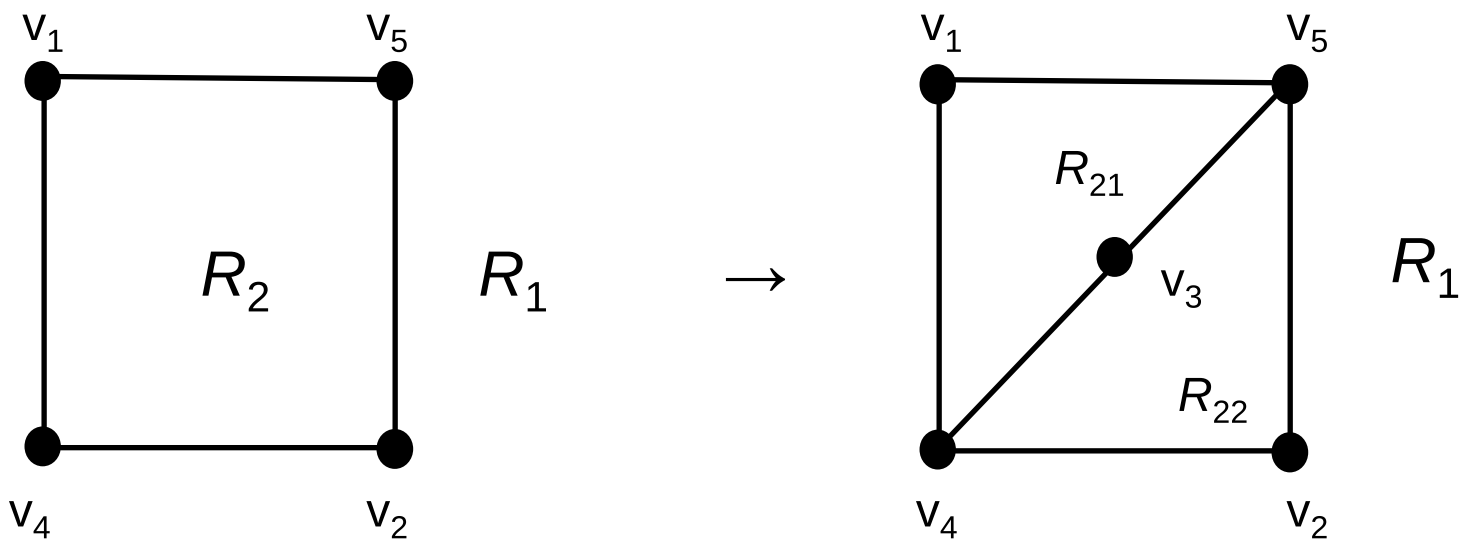


Figure 7 (a)

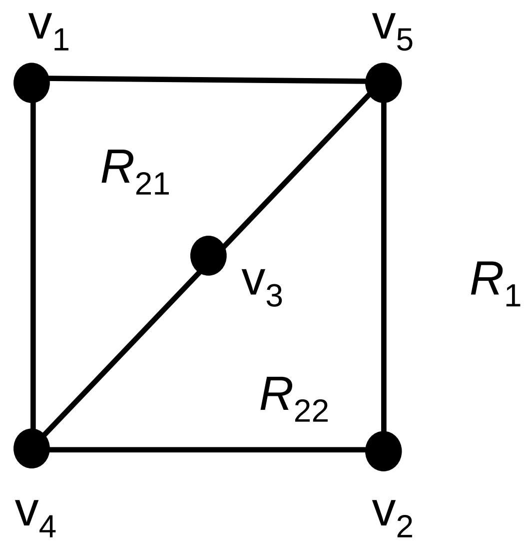
Solution of Example 3 (cont.)

- Next, we note that **v_3 must be in either R_1 or R_2.**
- **When v_3 is in R_2**, then the edges $\{v_3, v_4\}$ and $\{v_3, v_5\}$ separate R_2 into two sub-regions, R_{21} and R_{22}.



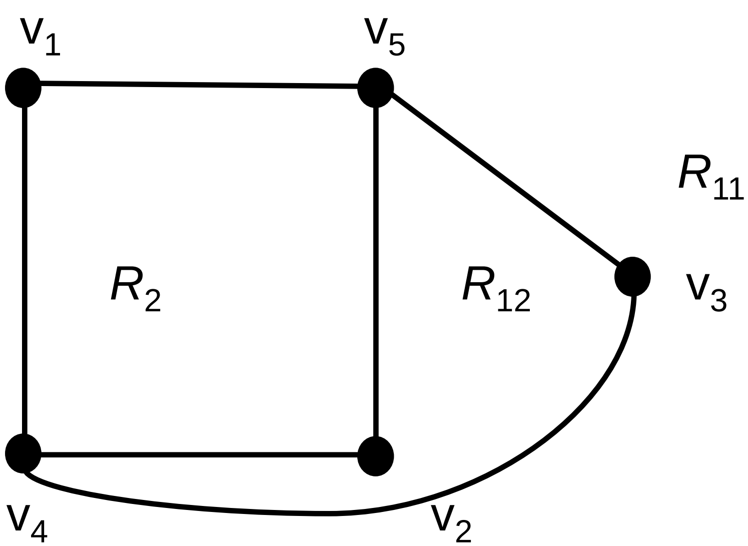
Solution of Example 3 (cont.)

- Now there is no way to place vertex v_6 without forcing a crossing:
  - If v_6 is in R_1, then $\{v_6, v_3\}$ must cross an edge
  - If v_6 is in R_{21}, then $\{v_6, v_2\}$ must cross an edge
  - If v_6 is in R_{22}, then $\{v_6, v_1\}$ must cross an edge



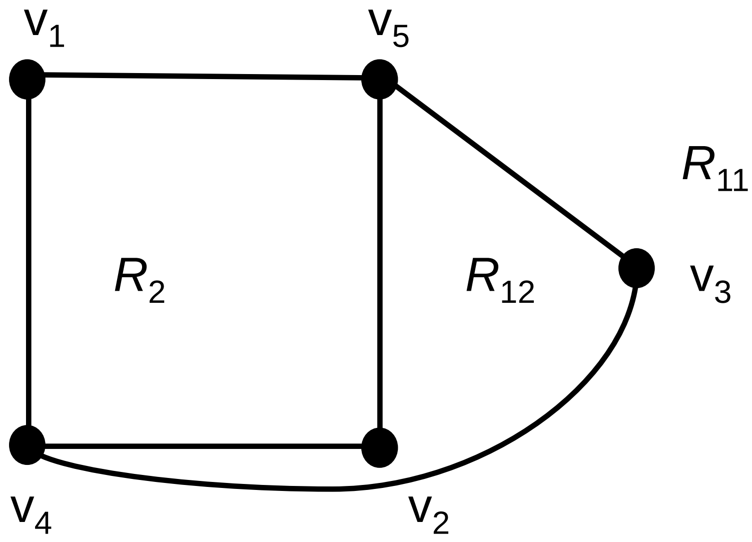
Solution of Example 3 (cont.)

- **When v_3 is in R_1**, then the edges $\{v_3, v_4\}$ and $\{v_4, v_5\}$ separate R_1 into two sub-regions, R_{11} and R_{12}.



Solution of Example 3 (cont.)

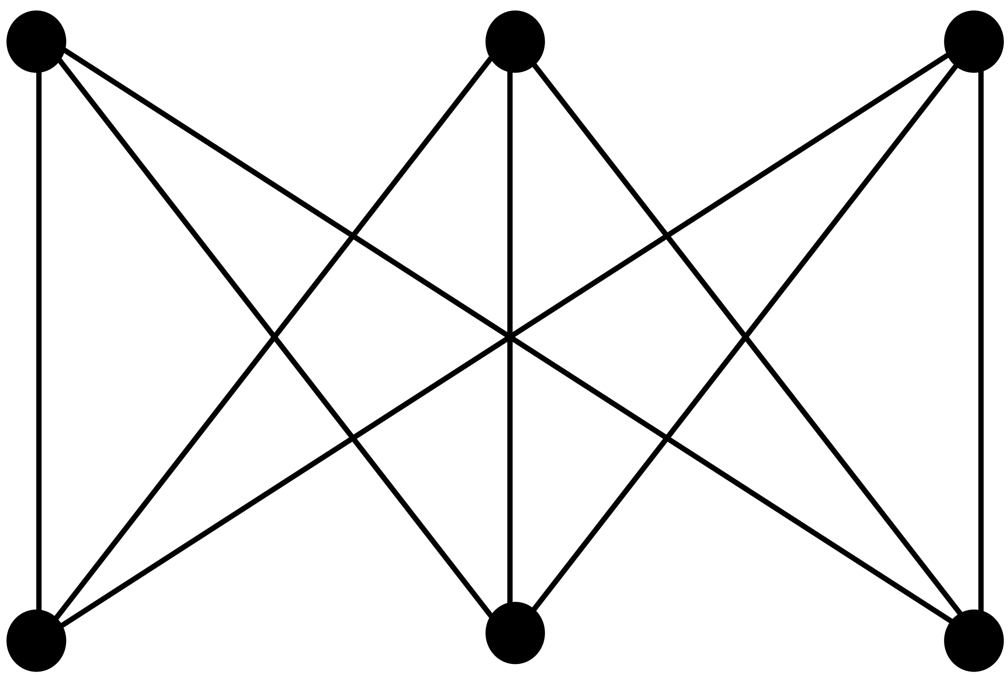
- Now there is no way to place vertex v_6 without forcing a crossing:
  - If v_6 is in R_2, then $\{v_6, v_3\}$ must cross an edge
  - If v_6 is in R_{11}, then $\{v_6, v_2\}$ must cross an edge
  - If v_6 is in R_{12}, then $\{v_6, v_1\}$ must cross an edge



Solution of Example 3 (cont.)

- **Consequently, the graph $K_{3,3}$ must be nonplanar.**

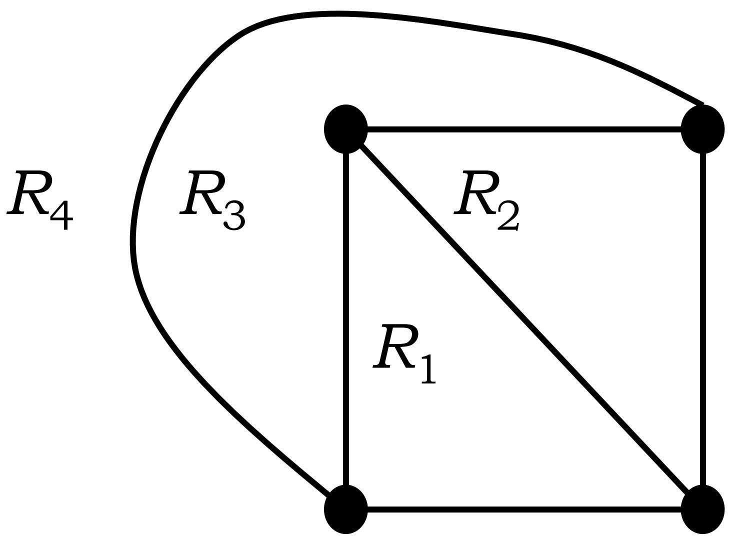
Note: See an easier solution by Corollary 3 *later*…



$\mathbf{K_{3,3}}$

Regions

- Euler showed that all planar representations of a graph split the plane into the **same** number of *regions*, including an unbounded region.



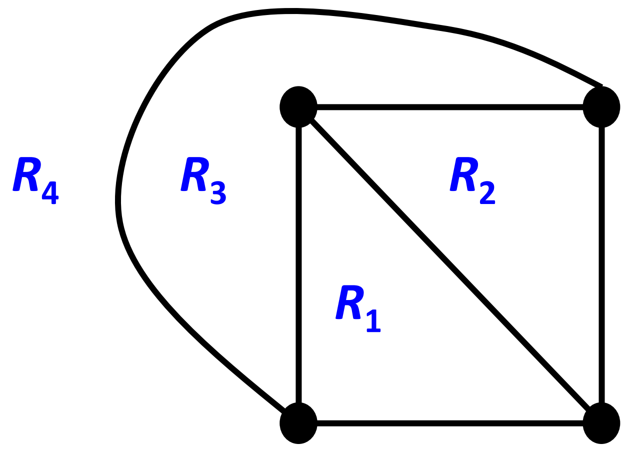
Here, R_4 is the unbounded region

Regions

- **Euler** devised a formula for expressing the **relationship between the number of vertices, edges, and regions** of a planar graph.
- These *may* help us determine if a graph can be planar or not.

Euler's Formula

- Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $\boldsymbol{r = e - v + 2}$



# of edges, $e = 6$
# of vertices, $v = 4$
# of regions, $r = e - v + 2 = 4$

Example 4

- Suppose that a planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

- **Solution:**

$2e = 20.3 = 60$ [Since sum of the degrees of the vertices is equal to

$e = 30$ twice the number of edges]

From Euler's formula, the number of regions is

$\boldsymbol{r = e - v + 2} = 30 - 20 + 2 = \mathbf{12}$

Class Work

- Suppose that a connected planner graph has 30 edges. If a planner representation of this graph divides the plane into 20 regions, how many vertices does this graph have?

- Solution:

  From **Euler's Formula, $r = e - v + 2$**

  $20 = 30 - v + 2$

  $v = 12$

  So, the graph has 12 vertices.

Euler's Formula (Cont.)

- Corollary 1: If G is a connected planar simple graph with e edges and v vertices where $v \geq 3$, then $e \leq 3v - 6$

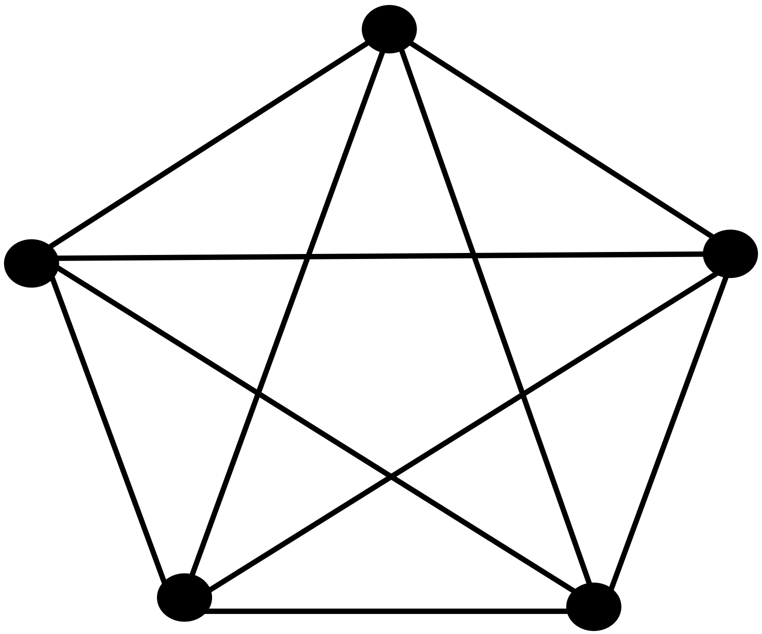
- **Warning!** Do not interpret the corollary as meaning: If $e \leq 3v - 6$, then a connected graph is planar, because there are many nonplanar graphs which also satisfy this equation!

  For example, $K_{3,3}$ has 6 vertices and 9 edges. So when you substitute into the equation, you get: $9 \leq 3.6 - 6$, which holds. However, $K_{3,3}$ is not planar.

Example 5

- Show that K_5 is nonplanar ***using Corollary 1***.

K_5



Solution of Example 5

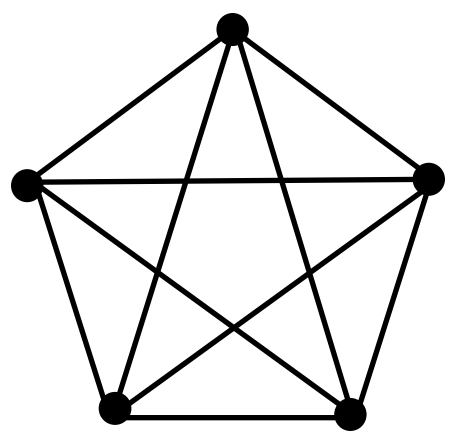
- The graph K_5 **has 5 vertices and 10 edges.**

  However, the inequality $e \leq 3v - 6$ is not satisfied for this graph, because $e = 10$ and $3v - 6 = 3*5 - 6 =$

  $15 - 6 = 9$

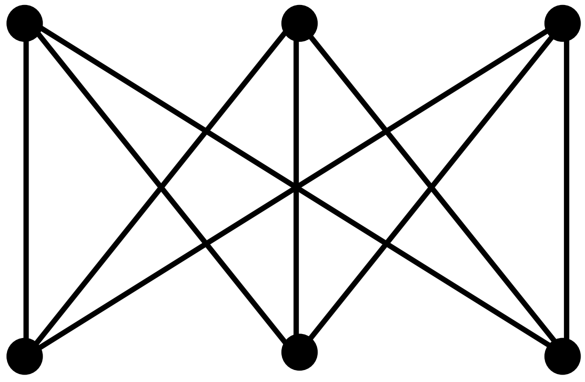
  Therefore, K_5 is not planar.

K_5



Euler’s Formula (Cont.)

- **Corollary 3:** If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length 3, then $\boldsymbol{e \leq 2v - 4}$
- Example 6: Use ***Corollary 3*** to show that $K_{3,3}$ is nonplanar.



Solution of Example 6

***$K_{3,3}$* has 6 vertices and 9 edges**. [So, ***v* = 6, *e* = 9**]

In graph $K_{3,3}$, v $\geq$ 3 and there is no circuit of length 3.

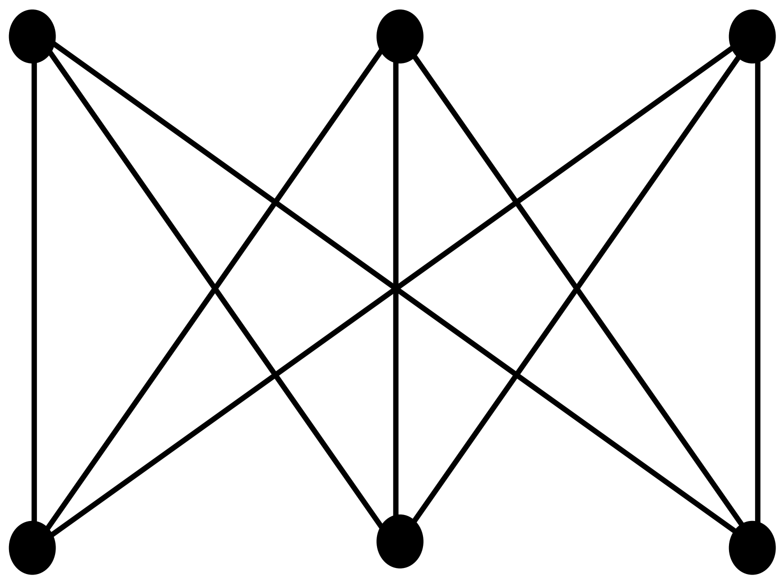
If $K_{3,3}$ were planar, then $\boldsymbol{e \leq 2v - 4}$ would have to be true.

$2v - 4 = 2*6 - 4 = 8$

So e must be ≤ 8.

But $e = 9$.

Therefore, $K_{3,3}$ is nonplanar.



$K_{3,3}$

Graph Coloring

- A ***coloring*** of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

- The ***chromatic number*** of a graph is the least number of colors needed for a coloring of this graph.
  - The chromatic number of a graph G is denoted by $\boldsymbol{\chi(G)}$

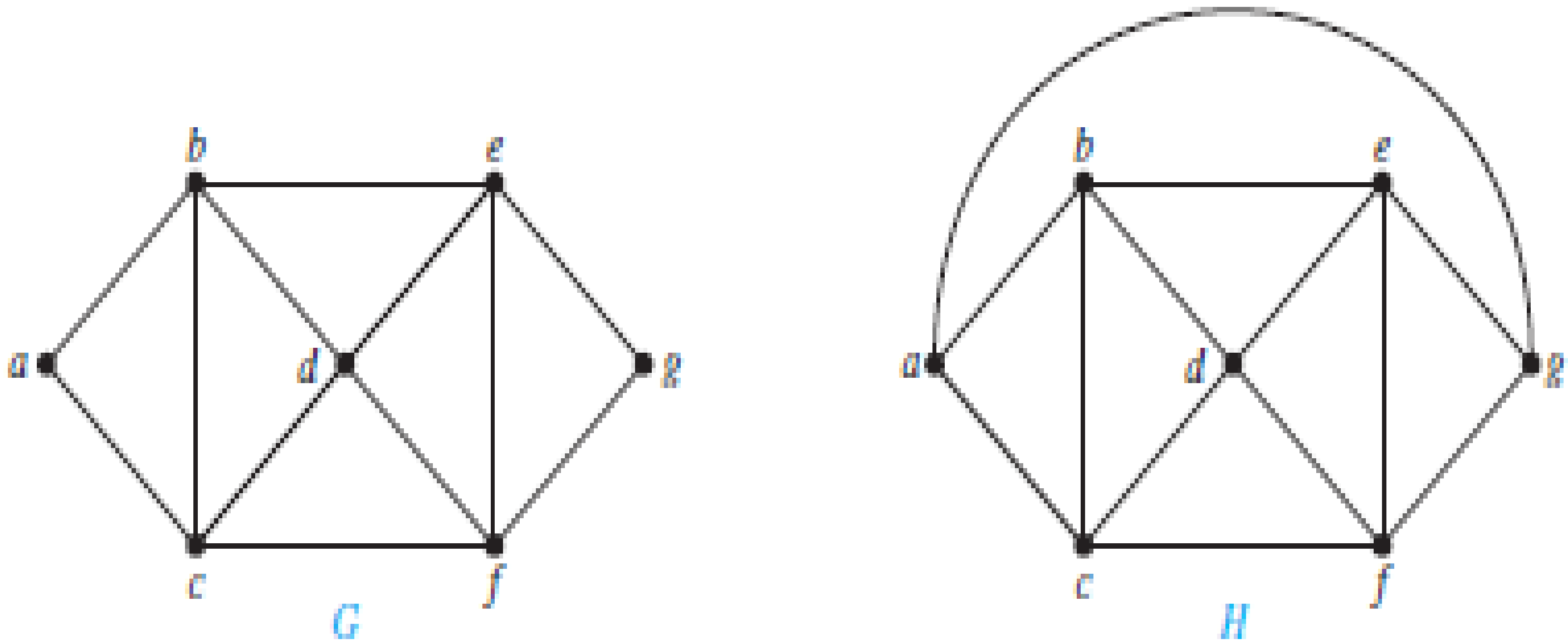
The Four Color Theorem

- *The **chromatic number** of a planar graph is no greater than four.*

Example 1

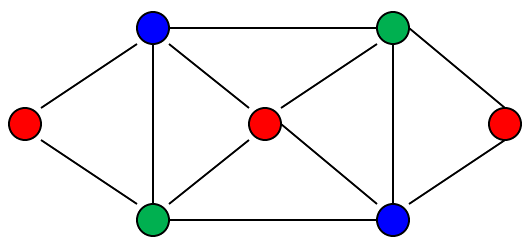
Example 1: What are the chromatic numbers of the graphs G and H?

[We have done for the first graph in the last slide]

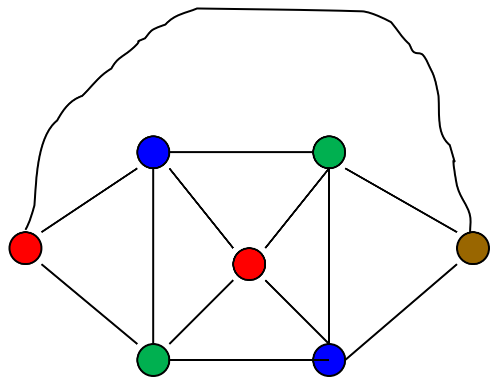


Solution of Example 1

- **Solution:** $\chi(G) = 3$; $\chi(H) = 4$



G



H

An Application of Graph Coloring: Scheduling Final Exams at a university

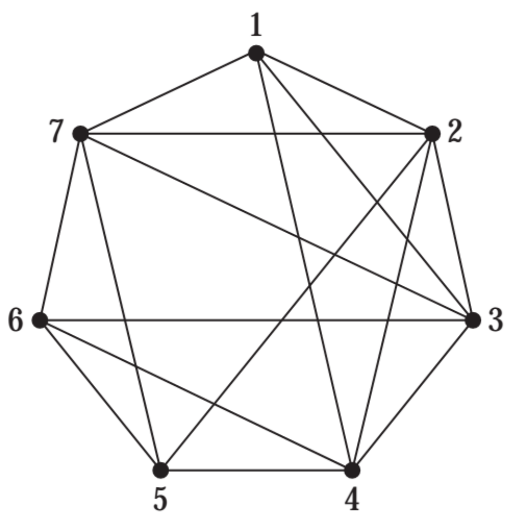
- How can the final exams at a university be scheduled so that no student has two exams at the same time?
- **Solution:** This scheduling problem can be solved using a graph model, with **vertices** representing **courses** and with an **edge** between two vertices if there is a **common student** in the courses they represent. **Each time slot** for a final exam is represented by a **different color.**
- A scheduling of the exams corresponds to a coloring of the associated graph.

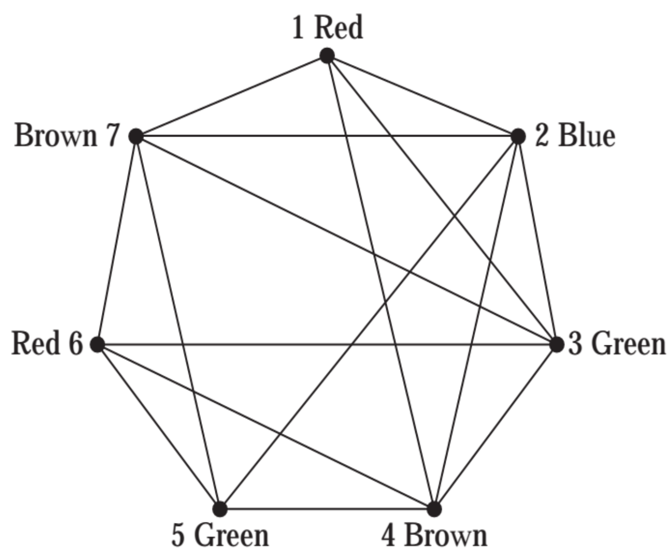
Example 5: Scheduling Final Exam

Suppose there are seven finals to be scheduled. Suppose that the following pairs of courses have common students: 1 and 2, 1 and 3, 1 and 4, 1 and 7, 2 and 3, 2 and 4, 2 and 5, 2 and 7, 3 and 4, 3 and 6, 3 and 7, 4 and 5, 4 and 6, 5 and 6, 5 and 7, and 6 and 7.

How the final exams can be scheduled so that no student has two exams at the same time?

Solution





Because the chromatic number of this graph is 4, four time slots are needed.

Time Period	Courses
I	1, 6
II	2
III	3, 5
IV	4, 7

Books

- Rosen, K. H., & Krithivasan, K. (2012). Discrete mathematics and its applications: with combinatorics and graph theory. Tata McGraw-Hill Education. (7th Edition)

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http://discrete.openmathbooks.org/dmoi3/sec_planar.html