Mappings

Geometrical Representation:

To draw curve of complex variable (x, y)we take two axes i.e., one real axis and the other imaginary axis. A number of points (x, y) are plotted on z-plane, by taking different value of z (different value of x and y). The curve C is drawn by joining the plotted points. The diagram obtained is called **Argand diagram.**

Transformation:

For every point (x, y) in the z-plane, the relation $w = f(z)$ defines a corresponding point (u, v) in the w-plane. We call this "transformation or mapping of z-plane into w-plane". If a point z_0 maps into the point w_0, $w_0$is known as the image of z_0.

If the point $P(x, y)$ moves along a curve C in z-plane, the point $P^{'}(u, v)$ will move along a corresponding curve C_1 in the w-plane. We, then, say that a curve C in the z-plane is mapped into the corresponding curve C_1 in the w- plane by the relation $w = f(z)$.

Translation, Rotation and reflection are the standard transformations. Terms such as **translation, rotation** and **reflection** are used to convey dominant geometric characteristics of certain mappings.

Translation

$w = z + C$,

where,

$$C = a + ib$$
$$z = x + iy$$
$$w = u + iv$$

Hence, $$u + iv = x + iy + a + ib$$

So, $$u = x + a \text{ and } v = y + b$$
$$x = u - a \text{ and } y = v - b$$

On substituting the values of x and y in the equation of the curve to be transformed we get the equation of the image in the w-pane.

As an example the mapping $w = z + 1$ where $z = x + iy$, can be thought of as a translation of each point of z one unit to the right.

Example:

Let the rectangular region R in z-plane which is bounded by the lines

$$x = 0, y = 0, x = 2, y = 1.$$

Determine the region R' of the w-plane into which R is mapped under the transformation $w = z + 1$.

Solution:

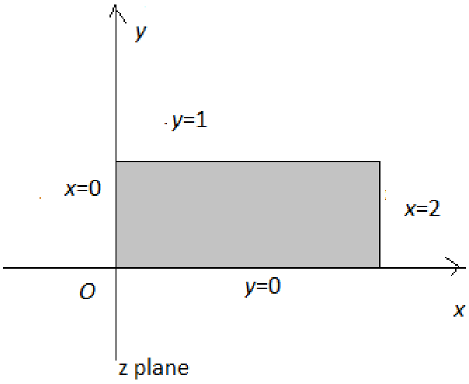
Given $w = z + 1$

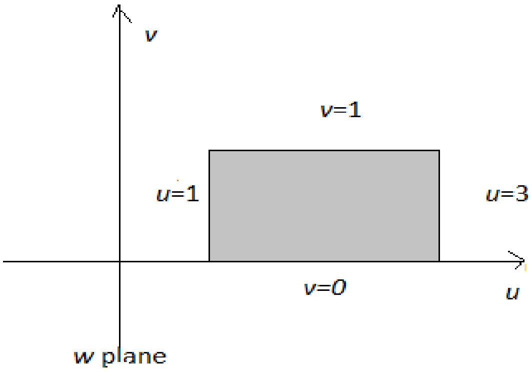
when $x = 0,\ \ u = 1,$

$y = 0,\ \ v = 0$

or, $u + iv = (x + 1) + iy$.
Hence $u = x + 1$ and $v = y$.

$x = 2,\ \ u = 3,$
$y = 1,\ \ v = 1$





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Rotation:

The mapping $w = iz$ where $z = re^{i\theta}$ can be thought of as a rotation of the radius vector for each non-zero point z through a right angle about the origin in the counter clock wise direction.

Example:

Let the rectangular region R in z-plane which is bounded by the lines

$$x = 0, y = 0, x = 2, y = 1.$$

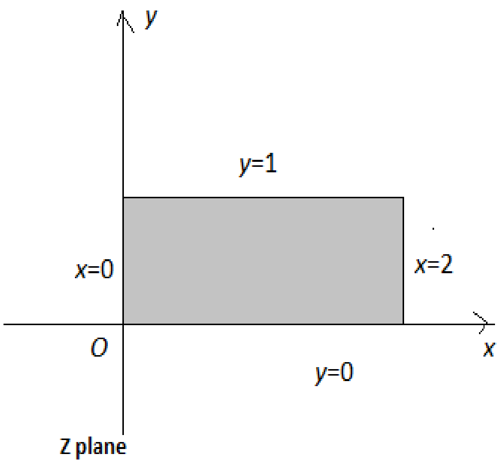
Determine the region R' of the w-plane into which R is mapped under the transformation

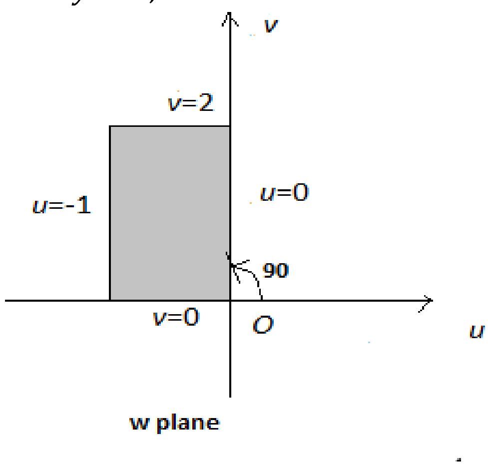
$$w = iz.$$

Solution:

Given $w = iz$
or, $u + iv = -y + ix$.
Hence $u = -y$ and $v = x$.

when $x = 0,\ \ v = 0,$
$y = 0,\ \ u = 0$
$x = 2,\ \ v = 2,$
$y = 1,\ \ u = -1$





Reflection:

The mapping $w = \bar{z}$ transforms each point of $z = x + iy$ into its reflection in the real axis.

Example:

Let the rectangular region R in z-plane which is bounded by the lines

$$x = 0, y = 0, x = 2, y = 1.$$

Determine the region R' of the w-plane into which R is mapped under the transformation

$$w = \bar{z}\ .$$

Solution:

Given $w = \bar{z}$

or, $u + iv = x - iy$.

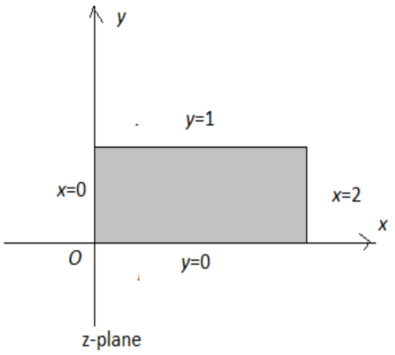
Hence $u = x$ and $v = -y$.

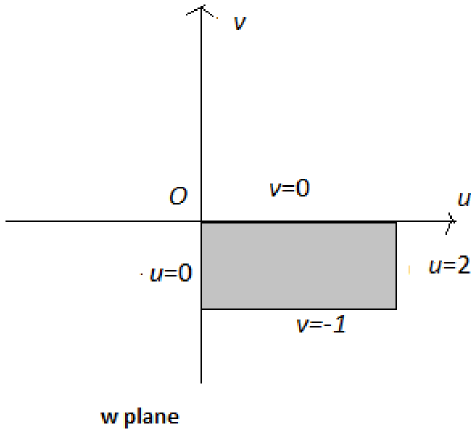
when $x = 0,\ \ u = 0,$

$y = 0,\ \ v = 0$

$x = 2,\ \ u = 2,$

$y = 1,\ \ v = -1$





Example:

Given triangle T in the z-plane with vertices at $-1 + 2i, 1 - 2i$ and $1 + 2i$. Determine the triangle T' of the w-plane into which T is mapped under the transformation $w = \sqrt{2}e^{\frac{\pi i}{4}}z$.

Solution:

Given $w = \sqrt{2}e^{\frac{\pi i}{4}}z = (1 + i)(x + iy)$

or, $u + iv = (x - y) + i(x + y)$.

Hence $u = x - y$ and $v = x + y$.

When

$x = 1,$

$y = 2$

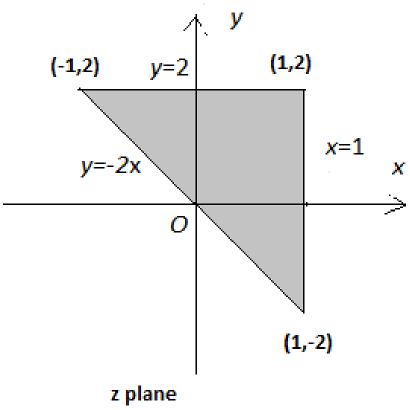
$y = -2x$

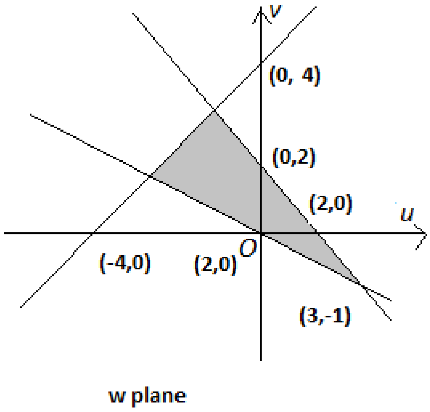
The vertices of the triangle are $-1 + 2i, 1 - 2i, 1 + 2i$. Hence the sides are $x = 1,\ \ y = 2$ and $y = -2x$.

$u = 1 - y, v = 1 + y \quad => \ u + v = 2$

$u = x - 2, v = x + 2 \quad => \ u - v = -4$

$u = 3x, v = -x \quad => \ u + 3v = 0$





Exercise Set

1. Let the rectangular region R in z-plane which is bounded by the lines

   $x = 2, y = 0, x = 5$ and $y = 4$. Determine the region R' of the w-plane into which R is mapped under the following transformations:

   (i) $w = 3z - (2 + 3i)$, (ii) $w = \frac{1}{2}e^{\frac{\pi i}{2}}z + 2i$,

   (iii) $w = \sqrt{2}e^{\frac{\pi i}{4}}z - (1 - i)$, (iv) $w = e^{i\pi}z + 3 + i$,

   (v) $w = \frac{1}{\sqrt{2}}e^{\frac{\pi i}{4}}z + 1 - 3i$.

2. Given triangle T in the z -plane with vertices at $1, 1 - 3i$ and $3 - i$. Determine the triangle T' of the w-plane into which T is mapped under the following transformations:

   (i) $w = 3z + 1 - 3i$, (ii) $w = iz + 3 + 2i$,

   (iii) $w = (1 + 2i)z - i$, (iv) $w = \frac{1}{2}e^{\frac{\pi i}{2}}z - 4$.