Chapter-1

Integration

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Indefinite Integrals

Definite Integrals

1.1Indefinite Integrals

1.1.1 Definition of Indefinite Integrals

Indefinite integration may be regarded as the inverse operation to differentiation. This means that the derivative of an indefinite integral of a function is the function itself.

Given a function $f(x)$ and $F(x)$ is an anti-derivative of $f(x)$ such that $\boldsymbol{F'(x) = f(x)}$.
Then $F(x)$ is said to **be indefinite integral** or **anti derivative** of $f(x)$,which can be written as

$$\int f(x)\,dx = F(x) + C$$

Here, $\int$ is called integral symbol, $f(x)$ is called integrand, x is called integration variable and C is called constant of integration.

Example 1.1.1

$$\int \sin x\,dx = -\cos x + C, \text{ since } \frac{d}{dx}(-\cos x + C) = \sin x$$

$$\int \sinh x\,dx = \cosh x + C\,, \text{ since } \frac{d}{dx}(\cosh x + C) = \sinh x$$

$$\int \sec^2 x\,dx = tanx + C, \text{ since } \frac{d}{dx}(tanx + C) = \sec^2 x$$

$$\int x^3\,dx = \frac{x^4}{4} + C, \text{ since } \frac{d}{dx}\left(\frac{x^4}{4} + C\right) = x^3$$

Integral properties and table of indefinite integrals

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

Useful Technique

If $\frac{d}{dx}F(x) = f(x)$, then using chain rule of differentiation

$$\frac{d}{dx}F(ax+b) = af(ax+b)$$

Thus

1. $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$
2. $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C,$
3. $\int \cosh(ax)\,dx = \frac{1}{a}\sinh(ax) + C$,
4. $\int \sinh(ax)\,dx = \frac{1}{a}\cosh(ax) + C,$
5. $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)a} + C\ (n \neq -1),$
6. $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b) + C,$
7. $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
8. $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Example 1.1.2

1. $\int e^{3x}\, dx = \frac{1}{3} e^{3x} + C$;

2. $\int \cos 2x\, dx = \frac{1}{2} \sin 2x + C$;

3. $\int \frac{1}{\sqrt{9-x^2}}\, dx = \sin^{-1}\frac{x}{3} + C$;

4. $\int \frac{1}{16+x^2}\, dx = \frac{1}{4}\tan^{-1}\frac{x}{4} + C$;

5. $\int (2x+3)^3 dx = \frac{(2x+3)^4}{4\times 2} + C = \frac{(2x+3)^4}{8} + C$.

Example set-1.1.1

1. $$\int\left(3x^4-\frac{1}{\sqrt{x}}+1\right)dx=3\int x^4dx-\int x^{-1/2}dx+\int dx \quad =3\left(\frac{x^5}{5}\right)-\left(\frac{x^{1/2}}{(1/2)}\right)+x+C$$

$$=\frac{3}{5}x^5-2\sqrt{x}+x+C$$

2. $$\int(3\cos 4x-5e^{3x})dx=3\int\cos 4x\,dx-5\int e^{3x}dx$$

$$=\frac{3}{4}\sin 4x-\frac{5}{3}e^{3x}+C$$

3. $$\int(10x^4-2\sec^2 x)\,dx \quad =10\int x^4\,dx-2\int\sec^2 x\,dx$$

$$=10\frac{x^5}{5}-2\tan x+C$$

$$=2x^5-2\tan x+C$$

4. $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right) d\theta = \int \csc\theta \cot\theta \, d\theta = -\csc\theta + C$

5. $\int \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx = 2\int x^3 dx - 6\int x dx + 3\int \frac{1}{x^2+1} dx$

$= 2.\frac{x^4}{4} - 6.\frac{x^2}{2} + 3.\tan^{-1} x + c$

6. $\int \left(\frac{2t^2+t^2\sqrt{t}+2}{t^2}\right) dt = 2\int dt + \int t^{\frac{1}{2}} dt + 2\int t^{-2}\, dt$

$= 2.t + \frac{2}{3} t^{\frac{3}{2}}. - \frac{1}{3} t^{-3} + C$

7. $\int (x^3 - 6x) dx = ?$

Exercise set-1.1.1

(a) $\int dx$, (b) $\int x^5\,dx$, (c) $\int x^{3/2}\,dx$, (d) $\int \sin(-3x)\,dx$,

(e) $\int \cos(2x)\,dx$, (f) $\int e^{5x}dx$, (g) $\int e^{2x/3}dx$, (h) $\int \exp(-3x)\,dx$,

(i) $\int x^{-1}dx$, (j) $\int \frac{1}{x^3}\,dx$, (k) $\int \frac{1}{\sqrt{1-x^2}}dx$, (l) $\int \sqrt[3]{y^2}dy$,

(m) $\int \frac{1}{r}dr$, (n) $\int \sinh(2x)\,dx$, (o) $\int \cosh(-3x)\,dx$, (p) $\int (2x+3)^{3/2}dx$

(q) $\int (1-2x)^5dx$, (r) $\int \left(\frac{x^3+3\,x^2+3}{x}\right)dx$ (s) $\int (x+2)^{-3/2}dx$, (t) $\int \frac{1}{3x-1}dx$,

(u) $\int \cos\ (3x-2)\,dx$ (v) $\int \sin(1-2x)\,dx$ (w) $\int \exp(-3x+1)\,dx$,

(x) $\int \frac{3}{9+(x-2)^2}dx$, (y) $\int \frac{1}{\sqrt{4-(x+1)^2}}dx$, (z) $\int \frac{1}{\sqrt{4-9x^2}}dx$.

Exercise set-1.1.2

$(i) \int \left(\frac{1}{x} - 3\right) dx,$

(ii) $\int (x^3 + x^{-3}) dx,$

$(iii) \int \left(x^{-1} + \frac{1}{e^x}\right) dx,$

(iv) $\int (\sin(-3x) + \cos(2x))\, dx\,,$

$(v) \int (e^{-x} + \cos(3x - 2)) dx\,,$

(vi) $\int \left(3e^{3x+1} + \frac{1}{2x+3}\right) dx,$

(vii) $\int \left(\frac{x^2+1}{x}\right) dx,$

(viii) $\int \left(\frac{\sqrt{x}+1}{\sqrt{x}}\right) dx.$

$(ix) \int \left(4x^3 + \frac{2}{x^2} - 1\right) dx,$

(x) $\int \left(1 + \frac{3}{x} - 7 \sin 2x\right) dx\,,$

(xi) $\int (3e^{2x} + 3e^{-4x} + \sqrt[3]{x})\, dx\,,$

(xii) $\int (2 \cos 2x - \sin 3x) dx,$

(xiii) $\int 5^{y+1}\, dy\,,$

(xiv) $\int \left(\frac{4x^3 - 2x^2 + 15x^5}{x^2}\right) dx.$

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Sample MCQ

1. Evaluate $\int \left(x^{-1} + \frac{1}{e^x}\right) dx$

   (a) …..   (b) $\ln x - e^{-x} + C$ (c) ……..

2. Evaluate $\int \left(\frac{x^2+1}{x}\right) dx$

   (a) $\frac{x^2}{2} + lnx + c$ (b) ….   (c)…

3. Evaluate $\int \frac{1}{25+x^2} dx$

   (a) …   (b) ….   (c) $\frac{1}{5}\tan^{-1}\frac{x}{5} + C$

4. Evaluate . $\int \sin 3x \, dx$

   (a) $-\frac{1}{3} cos3x + C$ (b)…   (c)…

1.1.2 **Integration by Substitution**

If $u = g(x)$ is a differentiable function whose range is an interval I and $f(x)$ is continuous on I then

$\int f(g(x))g'(x)dx = \int f(u)du$ Where If $u = g(x)$ then d$u = g'(x)dx$

An integral of the form $\int \frac{f\prime(x)}{f(x)}\, dx$

$$\text{Let, } u = f(x) \Rightarrow \frac{du}{dx} = f'(x) \Rightarrow du = f'(x)\,dx$$

$$\text{Then, } \int \frac{f'(x)}{f(x)}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|f(x)| + C$$

$$\text{Thus, } \int \frac{f\prime(x)}{f(x)}\, dx = \ln|f(x)| + C$$

Example set 1.1.2

1. $$\int \left(1+x^{3/2}\right)^3 x^{1/2}dx \qquad \text{Set, } u = 1+x^{3/2}$$

$$\text{Then, } du = \tfrac{3}{2}x^{1/2}dx \qquad \therefore x^{1/2}dx = \tfrac{2}{3}du$$

$$\text{Thus, } \int \left(1+x^{3/2}\right)^3 x^{1/2}dx = \frac{2}{3}\int u^3 du = \frac{1}{6}u^4 + C = \frac{1}{6}\left(1+x^{3/2}\right)^4 + C$$

2. $$\int \frac{x}{4+x^4}dx \qquad \text{Set, } u = x^2$$

$$\text{Then, } du = 2xdx \qquad \therefore xdx = \tfrac{du}{2}du$$

$$\text{Thus, } \int \frac{x}{4+x^4}dx = \frac{1}{2}\int \frac{1}{2^2+u^2}du = \frac{1}{4}\tan^{-1}\left(\tfrac{u}{2}\right) + C = \frac{1}{4}\tan^{-1}\left(\tfrac{x^2}{2}\right) + C$$

3.

$$\int x^3\cos(x^4+2)dx \quad \text{Set, } u=x^4$$

$$\text{Then, } du=4x^3dx \quad \therefore x^3dx=\tfrac{du}{4}$$

$$\text{Thus, } \int x^3\cos(x^4+2)dx=\int \cos u.\frac{du}{4}=\frac{1}{4}\int \cos u du=\frac{1}{4}\sin u+C=\frac{1}{4}\sin(x^4+2)+C$$

4.

$$\int \sqrt{2x+1}dx \qquad \text{Set, } u=2x+1$$

$$\text{Then, } du=2dx \qquad \therefore dx=\tfrac{1}{2}du$$

$$\text{Thus, } \int \sqrt{2x+1}dx=\int \sqrt{u}.\frac{1}{2}du=\frac{1}{2}.\frac{u^{\frac{3}{2}}}{\frac{3}{2}}+C=\frac{1}{3}u^{\frac{3}{2}}+C=\frac{1}{3}(2x+1)^{\frac{3}{2}}+C$$

5.

$$\int \frac{x}{\sqrt{1-4x^2}}dx \quad \text{Set, } u=1-4x^2$$

$$\text{Then, } du=-8xdx \qquad \therefore xdx=-\tfrac{1}{8}du$$

$$\text{Thus, } \int \frac{x}{\sqrt{1-4x^2}}dx=\int \frac{1}{\sqrt{u}}.-\frac{1}{8}du=-\frac{1}{8}\int u^{-\frac{1}{2}}du=-\frac{1}{8}.\frac{u^{\frac{1}{2}}}{\frac{1}{2}}+C=-\frac{1}{4}u^{\frac{1}{2}}+C=-\frac{1}{4}(1-4x^2)^{\frac{1}{2}}+C$$

6. $$\int \frac{2x+3}{x^2+3x+5}\,dx = \ln\left| x^2+3x+5 \right| + C \qquad \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$f(x) = x^2+3x+5,\ f'(x) = 2x+3$$

7. $$\int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx = \ln\left| \sin x + \cos x \right| + C$$

8. $$\int \frac{\sin 3x}{1+\cos 3x}\,dx = -\frac{1}{3}\int \frac{-3\sin 3x}{1+\cos 3x}\,dx = -\frac{1}{3}\ln\left|(1+\cos 3x)\right| + C$$

9. $$\int \frac{\sec^2 2x}{5+\tan 2x}\,dx = \frac{1}{2}\ln\left| 5+\tan 2x \right| + C$$

10. $$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\ln\left|\cos x\right| + C$$

Exercise set-1.1.3

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(a) $\int \frac{1}{x+2}\,dx,$ (b) $\int \frac{3x^2+2}{x^3+2x}\,dx,$ (c) $\int \frac{2x-1}{x^2-x+3}\,dx,$

(d) $\int \frac{2x+\sin x}{x^2-\cos x}\,dx,$ (e) $\int \frac{1+e^{-t}}{t-e^{-t}}\,dt,$ (f) $\int \frac{1}{2x+3}\,dx,$

(g) $\int \frac{x^2+2x}{x^3+3x^2+1}\,dx,$ (h) $\int \frac{\cos 3x}{3+\sin 3x}\,dx,$ (i) $\int \frac{\sec^2 3x}{2+\tan 3x}\,dx,$

(j) $\int \frac{e^{3x}}{3-2e^{3x}}\,dx,$ (k) $\int \cot 3z\,dz$. (l)$\int \frac{1}{y(1+\ln y)}\,dy$

Exercise set-1.1.4

(a) $\int 2x(x^2+1)^{21}dx; u = x^2+1$

(b) $\int \frac{\sec^2(\ln x)}{x} dx$; $u = \ln x$

(c) $\int \frac{e^{3x}}{e^{3x}+5} dx$; $u = e^{3x}+5$

(d) $\int \cos^3 x \sin x \, dx$; $u = \cos x$

(e) $\int \frac{x^3}{(x^4+1)^5} dx$,

(f) $\int \frac{(1+\ln x)^3}{x} dx$

(g) $\int \frac{\cos x}{(1+\sin x)^5} dx$,

(h) $\int \sin 3x \sqrt{2+\cos 3x} \, dx$,

(i) $\int \frac{\cos(2/x)}{x^2} dx$,

(j) $\int \left(\frac{e^{\sqrt{x}}}{\sqrt{x}}\right) dx$,

(k) $\int \frac{1}{x(1+\ln x)^3} dx$,

(l) $\int \frac{e^{-3x}}{\sqrt{3+e^{-3x}}} dx$,

(m) $\int \frac{e^{m(\arctan x)}}{1+x^2} dx$,

(n) $\int \frac{e^x}{e^x+1} dx$,

$(o) \int 4\tan^3 x \sec^2 x \, dx; u = \tan x$

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Sample MCQ

1. Evaluate $\int \frac{e^x}{e^x+1}\,dx$

   (a) ..... (b) $\ln(1+e^x)+C$ (c) ....

2. Evaluate $\int \frac{x^2+2x}{x^3+3x^2+1}\,dx$,

   (a) $3\ln(x^3+3x^2+1)+c$ (b) .... (c)...

3. Evaluate $\int \frac{1}{x(1+\ln x)^3}\,dx$,

   (a) ... (b) .... (c) $-\frac{1}{2}(1+lnx)^{-2}+C$

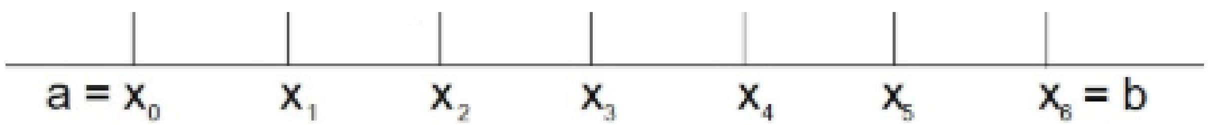
1.2 Definite Integrals

1.2.1 Riemann and Trapezoidal Sum

Consider a function $f(x)$ which is defined (i.e. bounded) over the closed interval$[a, b]$.

Consider a partition *P* of $[a, b]$ into n subintervals by the points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$



This partition corresponds to the subintervals

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \cdots, [x_{n-1}, x_n]$$

In each $[x_{r-1},\ x_r]$ choose any point c_r such that $x_{r-1} \leq c_r \leq x_r$. Then the sum

$$S_n = \sum_{r=1}^{n} f(c_r)(x_r - x_{r-1})$$

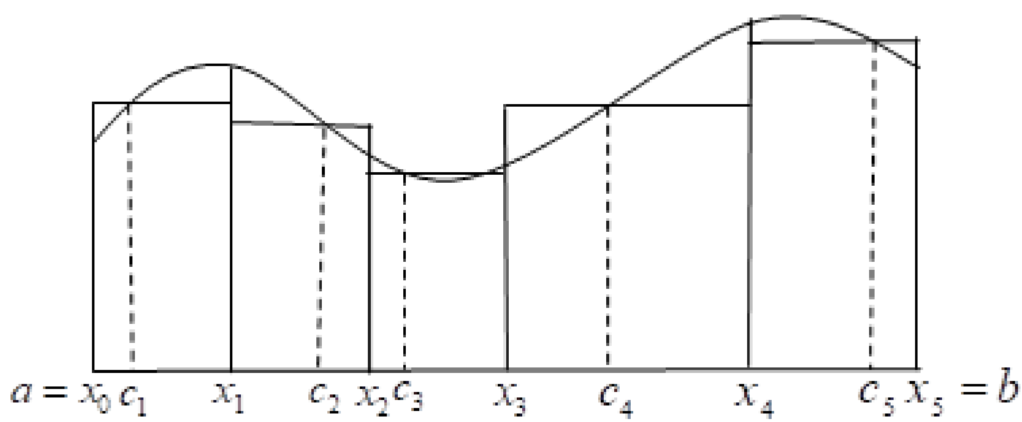
$$= \sum_{r=1}^{n} f(c_r)\Delta x_r\,, \qquad\qquad \Delta x_r = x_r - x_{r-1}$$

is called a **Riemann sum** for the function $f(x)$ on $[a, b]$.

Suppose $f(x) \geq 0$, on $[a, b]$. Then the Riemann sum

$$S_n = \sum_{r=1}^{n} f(c_r)\Delta x_r$$

is the sum of the areas of the *n* rectangles shown below, and thus represents an approximation to the area under the graph on $[a, b]$. Figure below illustrates the case where *n* = 5.



Different choice of the nodal points c_r give different values of the Riemann sums.

Commonly used Riemann sums are :

left Riemann sum ($c_r = x_{r-1}$),

right Riemann sum($c_r = x_r$) and

middle Riemann sum($c_r = \frac{x_{r-1}+x_r}{2}$).

If we use

$$f(c_r) = \frac{f(x_{r-1}) + f(x_r)}{2},$$

average of the heights at end points of the subinterval, it is called the **Trapezoidal Riemann sum**.

Summary of Riemann Sum:

Let a function $f(x)$ is defined in the closed interval $[a, b]$.

In evaluation of Riemann sums we commonly use equal subintervals. Dividing $[a, b]$ into n equal sub-intervals of the length

$$\Delta x = \frac{b-a}{n}$$

Riemann sum of $f(x)$ over the interval $[a, b]$ is

$$S_n = \sum_{r=1}^{n} f(c_r)\,\Delta x = \Delta x \sum_{r=1}^{n} f(c_r)$$

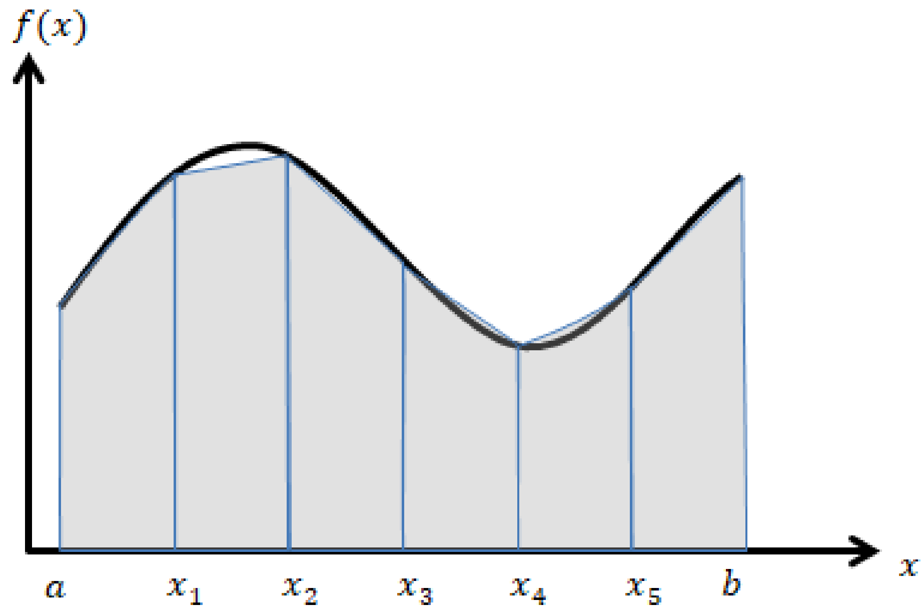
For all r if	The sum S_n will be called
$c_r = x_{r-1}$	left Riemann sum
$c_r = x_r$	right Riemann sum
$c_r = (x_r + x_{r-1})/2$	middle Riemann sum
$f(c_r) = \dfrac{f(x_{r-1}) + f(x_r)}{2}$	Trapezoidal Riemann sum

1.2.2 **Numerical Integration (The Trapezoidal Rule)**

First we subdivide the interval $[a, b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$.

Then on each interval we will approximate the function by a straight line joining the function values at either endpoint on the interval.

The following figure illustrates the case for $n = 6$.



Each of these shaded objects is a trapezoid (hence the rule's name) and as we can see some of them do a very good approximation to the actual area under the corresponding segment of the curve.

The area of the trapezoid in the interval $[x_r, x_{r+1}]$ is given by,

$$A_r = \left(f(x_r) + f(x_{r+1})\right) \times \frac{\Delta x}{2}.$$

Then sum of the area of the n trapeziums (e.g. 6 in the above figure) will approximate the area under the curve and is given by,

$$\int_a^b f(x)dx$$
$$\approx \left(f(x_0) + f(x_1)\right) \times \frac{\Delta x}{2} + \left(f(x_1) + f(x_2)\right) \times \frac{\Delta x}{2} + \cdots + \left(f(x_{n-1}) + f(x_n)\right) \times \frac{\Delta x}{2}$$
$$\approx \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$$

Which is known as the composite Trapezoidal rule.

Example set 1.2.1

Example 1:

Find the area under the curve $f(x) = x^4 - 3x^2 + 3$ by using different Riemann sum over the interval [0,1.6] using 8 subintervals.

Solution:

The following table shows the estimated area, using different Riemann sum, under the curve $f(x) = x^4 - 3x^2 + 3$ over the interval $[0,1.6]$using 8 equal subintervals .

Here, $a = 0$ and $b = 1.6$ and $n = 8$

Then the length of each subintervals is,

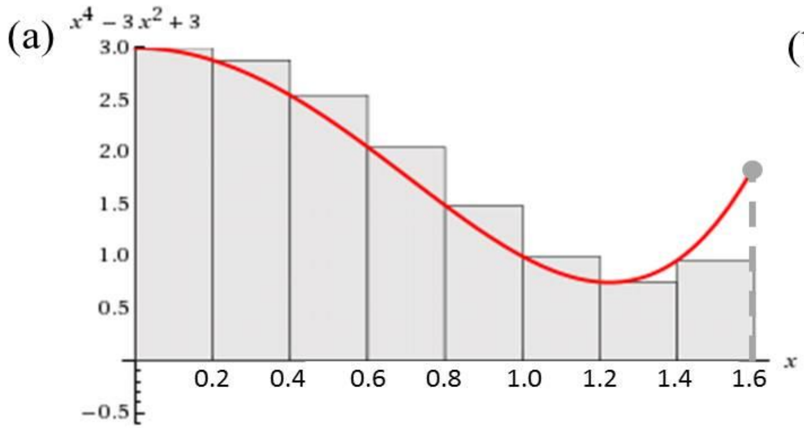
$$\Delta x = \frac{b-a}{n} = \frac{1.6}{8} = 0.2$$

Riemann sum $[x_{r-1}, x_r]$	left Riemann sum		right Riemann sum		middle Riemann sum	
	c_r	$f(c_r)$	c_r	$f(c_r)$	c_r	$f(c_r)$
$[0.0, 0.2]$	0.0	3	0.2	2.8816	0.1	2.9701
$[0.2, 0.4]$	0.2	2.8816	0.4	2.5456	0.3	2.7381
$[0.4, 0.6]$	0.4	2.5456	0.6	2.0496	0.5	2.3125
$[0.6, 0.8]$	0.6	2.0496	0.8	1.4896	0.7	1.7701
$[0.8, 1.0]$	0.8	1.4896	1.0	1.0000	0.9	1.2261
$[1.0, 1.2]$	1.0	1.0000	1.2	0.7536	1.1	0.8341
$[1.2, 1.4]$	1.2	0.7536	1.4	0.9616	1.3	0.7861
$[1.4, 1.6]$	1.4	0.9616	1.6	1.8736	1.5	1.3125
$\sum f(c_r)$	14.6816		13.5552		13.9496	
$\Delta x * \sum f(c_r)$	2.9363		2.7110		2.7899	

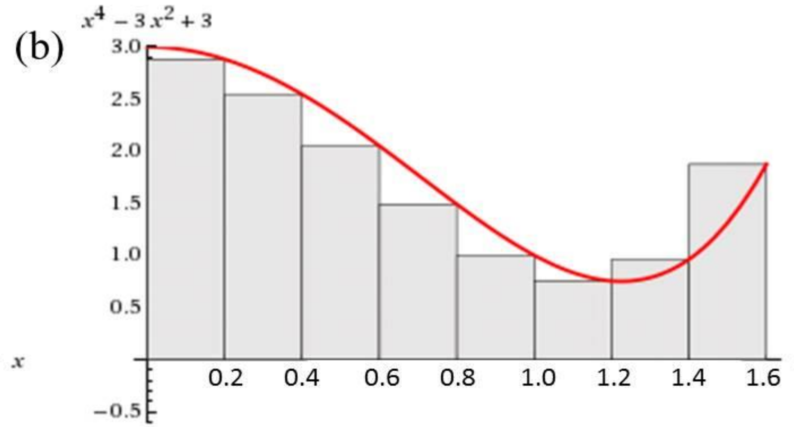
The Trapezoidal Riemann sum is,

$$S_n = [3 + 2 \times (2.8816 + 2.5456 + 2.0496 + 1.4896 + 1.0000 + 0.7536 + 0.9616) + 1.8736] \times \frac{0.2}{2}$$
$$= 2.8237.$$

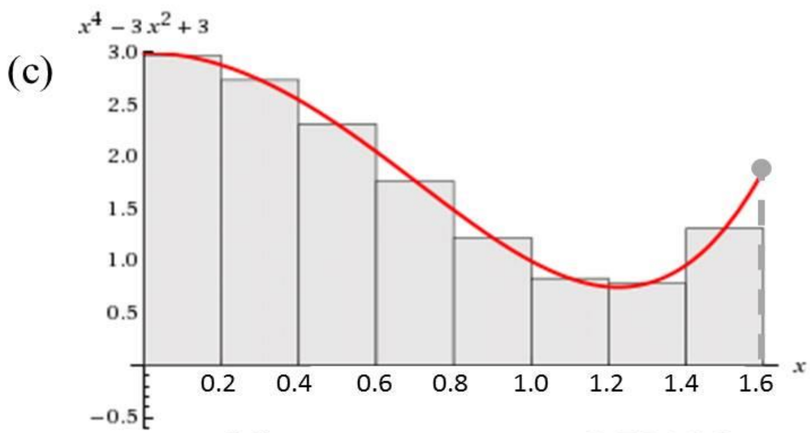
The following figures show the geometrical interpretation of the above Riemann sums,



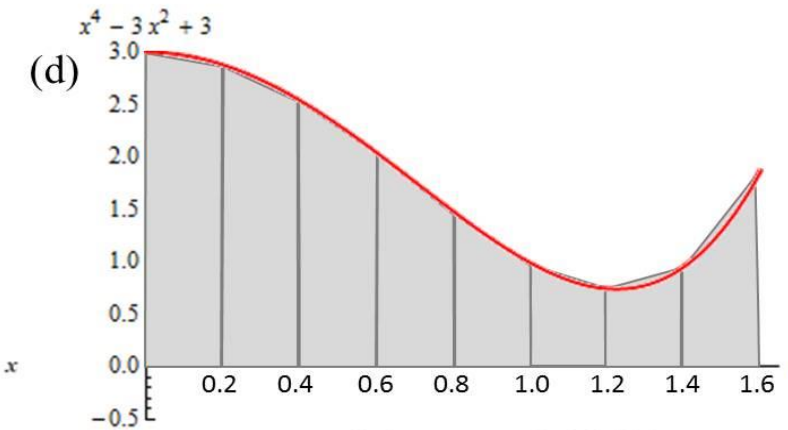
Left Riemann sum $= 2.93632$



Right Riemann sum $= 2.71104$



Middle Riemann sum$= 2.78992$



Trapezoidal sum $= 2.8237$

Note that the exact value of the area is 2.80115 which is calculated using the integration will be considered later.

Example 2:

Use the Trapezoidal rule with $n = 5$ to approximate the integral $\int_{0.5}^{1} \sqrt{1 + e^{x^2}}\ dx$ to 3 decimal places.

Solution:

Here $a = 0.5$, $b = 1$ and $n = 5$. So $\Delta x = \frac{1-0.5}{5} = 0.1$ and $f(x) = \sqrt{1 + e^{x^2}}$.

Hence,

x	0.5	0.6	0.7	0.8	0.9	1
f(x)	1.5113	1.5599	1.6224	1.7019	1.8022	1.9283

Using the Trapezoidal rule, we have

$$\int_{0.5}^{1} \sqrt{1 + e^{x^2}}\ dx \approx \frac{0.1}{2}[1.5113 + 2(1.5599) + 2(1.6224)$$

$$+2(1.7019) + 2(1.8022) + 1.9283]$$

$$=0.8406 \approx 0.841$$

Example 3:

Evaluate $\int_0^{\pi} \sqrt{(3+\cos x)}\ dx$. to three decimal places using Trapezoidal rule with four subintervals.[Note that in calculating the values of cos *x* use radian mode]

Solution:

Here $a = 0$, $b = \pi$ and $n = 4$. So,$\Delta x = \frac{\pi}{4}$ and $f(x) = \sqrt{3+\cos x}$.

Hence,

x	0	π/4	π/2	3π/4	π
f(x)	2.0000	1.9254	1.7321	1.5142	1.4142

Using Trapezoidal rule we have

$$\int_0^{\pi} \sqrt{(3+\cos x)}\ dx$$

$$\approx \frac{1}{2}.\frac{\pi}{4}[2 + 2 * (1.9254 + 1.7321 + 1.5142) + 1.4142]$$

$$\approx \frac{\pi}{8} \times 13.7576 \ \approx 5.403$$

Exercise set 1.2.1

1. Estimate the value the following integrals to 3 decimal places using 'n' subintervals of equal length using **(i) left Riemann sum, (ii) right Riemann sum, (iii) middle Riemann sum and (iv) Trapezoidal rule.**

(a) $\int_0^2 e^{-3x}dx$ (n = 4), (b) $\int_1^7 \frac{1}{\sqrt{x^3+1}}\,dx$ (n = 6), (c) $\int_3^5 \frac{1}{1-\ln x}\,dx$ (n = 4),

(d) $\int_0^1 \sin(x)\cos(x^2)\,dx$ (n = 4), (e) $\int_0^1 \sin(x^2)\,dx$ (n = 5).

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P- 388 Ex # 1, 3, 7, 9
P- 524 Ex # 7, 8, 9 (n= 4, 6) LRS, RRS, MRS, TR

Sample MCQ

1. If n is the number of subintervals then how to calculate the length of subintervals of the interval $[a, b]$

   (a) .....   (b)$\Delta x = \frac{b-a}{n}$ (c) ....

2. When Riemann sum of $f(x)$ over the interval $[a, b]$

$$S_n = \Delta x \sum_{r=1}^{n} f(c_r)$$

is called

(a) Left Riemann sum if $c_r = x_{r-1}$ (b) ....   (c)...

3. Evaluate $\int_0^{\pi} \sqrt{(3 + \cos x)}\, dx$. to three decimal places using Trapezoidal rule with n=4.

   (a) ...   (b) ....   (c)5.403

1.2.3 **The Fundamental Theorem of Calculus**

Theorem: If $f(x)$ is a **continuous** function on $[a, b]$ and $f(x)$ is an indefinite integral of $f(x)$ then

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a) \qquad \text{where } \int f(x)dx = F(x).$$

Example:

1. $$\int_1^2 (3x^2 + 4x + 5)dx = \left[x^3 + 2x^2 + 5x\right]_1^2 = 8 + 8 + 10 - (1 + 2 + 5) = 18$$

2. $$\int_1^e \frac{1}{x} dx = [\ln x]_1^e = \ln e - \ln 1 = 1$$

3. $$\int_0^{\pi/2} \sin 2x \, dx = [-\tfrac{1}{2} \cos 2x]_0^{\pi/2} = -\tfrac{1}{2}(-1 - 1) = 1$$

1.2.4 **Substitution Rule For Definite Integrals**

6 The Substitution Rule for Definite Integrals If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Example:

$\int_0^4 \sqrt{2x+1}dx$ Set, $u = 2x+1$ Then, $du = 2dx$ $\therefore dx = \frac{1}{2}du$

Changing Limit

x	u
0	1
4	9

$$\text{Thus, } \int_0^4 \sqrt{2x+1}dx = \int_1^9 \frac{1}{2}\sqrt{u}dx = \frac{1}{2}.\left.\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right|_1^9 = \frac{1}{3}\left(9^{\frac{3}{2}} - 1\right) = \frac{26}{3}$$

Exercise set 1.2.2

1.

(a) $\int_1^3 (x^2\sqrt{x} + 2e^x + 1)\ dx,$

(b) $\int_0^{\frac{\pi}{2}} (\sin 3x + \cos 3x)\, dx$,

(c) $\int_1^2 \frac{(1+\ln x)^5}{x} dx$

(d) $\int_0^1 \frac{e^x}{1+e^{2x}} dx$,

(e) $\int_0^5 \frac{dx}{25+x^2}$,

(f) $\int_{-1}^2 \sqrt{3-x}\, dx$

(g) $\int_0^1 \frac{4(\arctan x)^3}{1+x^2} dx,$

(h) $\int_0^1 \frac{1}{\sqrt{64-x^2}} dx$,

(i) $\int_1^9 \frac{dx}{\sqrt{x}(1+\sqrt{x})^2}$

2.

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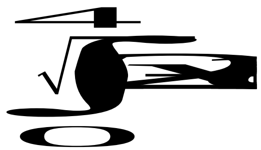
P-400 Ex # 19, 23, 24, 25, 27, 29, 35, 43
P-409 Ex # 21 – 30
P- 419 Ex # 53, 54, 55, 59, 60, 70, 71

Sample MCQ

1. Evaluate $\int_0^3 (3x^2 + 6x + 2)dx$

(a) ..... (b)60 (c) ....

2. Evaluate 

(a) $\frac{26}{3}$ (b) .... (c)...

3. Evaluate $\int_0^3 (3x^2 + 6x + 2)dy$

(a)3$(3x^2 + 6x + 2)$ (b) (c)

1.2.5 **Even and Odd Function**

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x)$$

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

Example: State whether the following functions are odd, even or neither.

(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

Solution: (a)

$$f(-x) = (-x)^5 + (-x) = (-1)^5 x^5 + (-x)$$
$$= -x^5 - x = -(x^5 + x)$$
$$= -f(x)$$

So, f(x) is an odd function

(b)

$$g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$

So, g(x) is an even function

(c)

$$h(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

So, h(x) is neither even nor odd function

1.2.6 **Some properties of definite integral**

1. If $f(x)$ is any integrable function then

$$\int_a^a f(x)dx = 0 .$$

2. If $f(x)$ is integrable on any interval containing three points a, b, c and $a<c<b$ then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx .$$

3. If $f(x)$ is an even function, that is, if $f(-x) = f(x)$ for all x then

$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx .$$

4. If $f(x)$ is an odd function, that is, if $f(-x) = -f(x)$ for all x then

$$\int_{-a}^a f(x)dx = 0 .$$

For example:

(a) $\int_{-2}^{2}(2x^2+3)dx = 2\int_{0}^{2}(2x^2+3)dx$ since $(2x^2+3)$ is an even function.

(b) $\int_{-\pi/4}^{\pi/4} \cos x\, dx = 2\int_{0}^{\pi/4} \cos x\, dx$ since $\cos x$ is an even function ($\cos(-x) = \cos x$).

(c) $\int_{-\pi/4}^{\pi/4} \sin x\, dx = 0$ since $\sin x$ is an odd function ($\sin(-x) = -\sin x$).

Exercise set 1.2.3

1. State whether the following functions are odd, even or neither.

(a) $f(x) = x + x^3$, (b) $f(x) = (x^2 + 25)^{\,2}$, (c) $f(x) = x^6 + \mathrm{e}^{4x}$,

(d) $f(x) = \sin^3 x\cos^6 x$, (e) $f(x) = \sin^4 x \cos^5 x$, (f) $f(x) = tanx + cotx$

2. Evaluate the following integrals by using integral properties

(a) $\int_{-1}^{1}(x^3 + 5x^4)dx$, (b) $\int_{-2}^{2} x(1 + x + x^2)\, dx$ (c) $\int_{-4}^{4}(2 + 3x^2)\, dx$,

(d) $\int_{-5}^{5} x^5 e^{x^4} dx$, (e) $\int_{-\pi}^{\pi} x^8 \sin x\, dx$, (f) $\int_{-\pi}^{\pi} x \cos x\, dx$,

(g) $\int_{-\pi}^{\pi} \sin^3 x \cos^5 x\ \, dx$, (h) $\int_{-\pi/2}^{\pi/2} x^4 \sin^3 x \cos^3 x\, dx$, (i) $\int_{-\pi}^{\pi} \frac{x^3}{\sqrt{1+x^2}}\, dx$,

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P- 419 Ex # 61, 66, P-422 Ex # 22, 23

Sample MCQ

1. Which one is even function

(a) .....　　(b) $f(x) = x^4 + x^2$　　(c) ....

2. Which one is odd function

(a) $f(x) = sin2x$　　(b) ....　　(c)...

3. Evaluate $\int_{-3}^{3}(sin2y)dy$

(a) 0　　(b)　　(c)