Lecture-02

Complex Differentiation and The Cauchy-Riemann Equation

Analytic Functions:

If a single valued function $f(z)$ is differentiable i.e. $f'(z)$ exists at every point of a domain D except possibly at a finite number of exceptional points then the function is said to be **analytic** in the domain D. These exceptional point at which $f'(z)$ does not exist are called **singular points** or **singularities of the function**.

Necessary conditions for $f(z)$ to be analytic:

RECTANGULAR FORM:

If $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$ satisfies the **Cauchy-Riemann equations(C-R)** i.e.,

$$u_x = v_y \text{ and } u_y = -v_x$$

$$\text{i.e., } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

then $f(z)$ is said to be **analytic.**

Hence, at points where $f'(z)$ exists may be obtained from either of

$$f'(z) = u_x + iv_x \text{ or } f'(z) = v_y - iu_y.$$

POLAR FORM:

If $z = re^{i\theta}$ and $f(z) = u(r, \theta) + iv(r, \theta)$ satisfies the **Cauchy-Riemann equations(C-R)** i.e.

$$u_r = \frac{1}{r} v_\theta \text{ and } v_r = -\frac{1}{r} u_\theta$$

$$\text{i.e., } \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

then $f(z)$ is said to be **analytic.**

Hence, at points where $f'(z)$ exists may be obtained from either of

$$f'(z) = e^{-i\theta}(u_r + iv_r).$$

Important Formulae:

$$\sin at = \frac{e^{iat} - e^{-iat}}{2i}$$

$$\cos at = \frac{e^{iat} + e^{-iat}}{2}$$

$$\cos iay = \cosh ay$$

$$\sinh at = \frac{\mathrm{e}^{\mathrm{at}} - \mathrm{e}^{-\mathrm{at}}}{2}$$

$$\cosh at = \frac{\mathrm{e}^{\mathrm{at}} + \mathrm{e}^{-\mathrm{iat}}}{2}$$

$$\sin iay = i\sinh ay$$

Example: 1

Verify **C-R** equations for the function $f(z) = e^{z^2}$ and hence find $\dfrac{df}{dz}$ or $f'(z)$.

Solution: Given,

$f(z) = e^{z^2} = e^{(x+iy)^2}$

or, $u + iv = e^{x^2-y^2+i\,2xy}$

or, $u + iv = e^{x^2-y^2}\, e^{i\,2xy}$

or, $u + iv = e^{x^2-y^2}(\cos 2xy + i \sin 2xy\,)$

or, $u + iv = e^{x^2-y^2} \cos 2xy + i\, e^{x^2-y^2} \sin 2xy$

Here, $u(x, y) = e^{x^2-y^2} \cos 2xy$ and $v(x, y) = e^{x^2-y^2} \sin 2xy$.

Now, partially differentiating u and v with respect to x and y, we get

$u_x = 2x\, e^{x^2-y^2} \cos 2xy - 2y\, e^{x^2-y^2} \sin 2xy$

$u_y = -2y\, e^{x^2-y^2} \cos 2xy - 2x\, e^{x^2-y^2} \sin 2xy$

$v_x = 2y\, e^{x^2-y^2} \cos 2xy + 2x\, e^{x^2-y^2} \sin 2xy$

$v_y = -2y\, e^{x^2-y^2} \sin 2xy + 2x\, e^{x^2-y^2} \cos 2xy$

From the above result, we can write

$$u_x = v_y \quad \text{and} \quad v_x = -u_y$$

Since $f(z)$ satisfies Cauchy-Riemann equations, so $f(z)$ is analytic.

$f'(z) = u_x + i\, v_x$

$= 2x\, e^{x^2-y^2} \cos 2xy - 2y\, e^{x^2-y^2} \sin 2xy \;+$

$i\,(2y\, e^{x^2-y^2} \cos 2xy + 2x\, e^{x^2-y^2} \sin 2xy)$

$= 2x\, e^{x^2-y^2}(\cos 2xy + i\, \sin 2xy) + i\, 2y\, e^{x^2-y^2}(\cos 2xy - \frac{1}{i} \sin 2xy\,)$

$= 2x\, e^{x^2-y^2}(\cos 2xy + i\, \sin 2xy) + i\, 2y\, e^{x^2-y^2}(\cos 2xy + \frac{i^2}{i} \sin 2xy\,)$

$= 2x\, e^{x^2-y^2}(\cos 2xy + i\, \sin 2xy) + i\, 2y\, e^{x^2-y^2}(\cos 2xy + i \sin 2xy\,)$

$= 2x\, e^{x^2-y^2} e^{i\,2xy} + i\, 2y\, e^{x^2-y^2} e^{i\,2xy}$

$= 2(x + i\, y)\, e^{x^2-y^2+i\,2xy} = 2(x + i\, y) e^{(x+iy)^2} = 2z\, e^{z^2}$.

Example: 2

Verify **C-R** equations for the function $f(z) = z^5$ and hence find $\dfrac{df}{dz}$ or $f'(z)$.

Solution:

Given

$f(z) = z^5$

or, $u + iv = (r\, e^{i\theta})^5$

or, $u + iv = r^5 e^{i\,5\theta}$

or, $u + iv = r^5(\cos 5\theta + i \sin 5\theta)$

or, $u + iv = r^5 \cos 5\theta + i\; r^5 \sin 5\theta$

Here , $u = r^5 \cos 5\theta$ and $v = r^5 \sin 5\theta$

Partially differentiating u and v with respect to r and θ, we get
$u_r = 5r^4 \cos 5\theta$
$v_r = 5r^4 \sin 5\theta$
$u_\theta = -5r^5 \sin 5\theta$
$v_\theta = 5r^5 \cos 5\theta$
From the above result, we can write

$$u_r = \frac{1}{r} v_\theta \quad \text{and} \quad v_r = -\frac{1}{r} u_\theta.$$

Since $f(z)$ satisfies Cauchy-Riemann equations, so $f(z)$ is an analytic function.

$$\begin{aligned} f'(z) &= e^{-i\theta}(u_r + i\, v_r) \\ &= e^{-i\theta}(5r^4 \cos 5\theta + i5r^4 \sin 5\theta) \\ &= 5r^4 e^{-i\theta} \, (\cos 5\theta + i \sin 5\theta) \\ &= 5r^4 e^{-i\theta} \, e^{i\,5\theta} \\ &= 5r^4 e^{i\,4\theta} \\ &= 5(r\, e^{i\,\theta})^4 \\ &= 5\, z^4. \end{aligned}$$

Exercise set: 2.1

1. Write Cauchy-Riemann (**C-R**) equations in rectangular and polar forms.
2. For the following functions:
   a) $f(z) = iz\overline{z}$
   b) $f(z) = z^2$
   c) $f(z) = e^x(\cos y - i \sin y)$
   d) $f(z) = e^{2x}(\cos 2y + i \sin 2y)$
   e) $f(z) = Re(z^2) - i\, Im(z^2)$

   (i) separate real and imaginary parts,
   (ii) verify **C-R** equations,
   (iii) find $\frac{df}{dz}$ or $f'(z)$.
3. Are the following functions analytic? If analytic, then find $f'(z) = u_x + iv_x$
   $f(z) = \overline{z}$, $2z^2 + 3e^z$, $2ze^z$ and $3z^3$.
4. Are the following functions analytic? If analytic, then find $f'(z) = e^{-i\theta}\left(u_r + iv_r\right)$
   $f(z) = z^2$, $\frac{1}{z^9}$, $z^{-\frac{2}{3}}$ and $z^{\frac{3}{5}}$.

Reference Book: Advanced Engineering Mathematics (10th edition) by Erwin Kreyszig, Herbert Kreyszig, Edward J. Norminton, published by John Wiley & Sons, Inc