Chapter-2

Applications of Definite Integrals

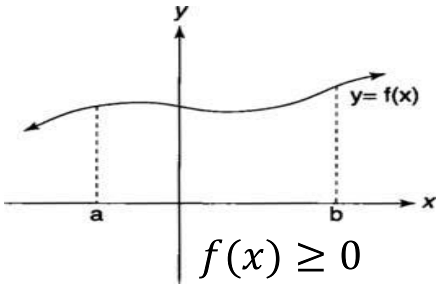
2.1 Area of Regions Between Two Graphs

Definite integrals could be used to determine the area of the region between the graph of a function and the *x*-axis or the *y*-axis.

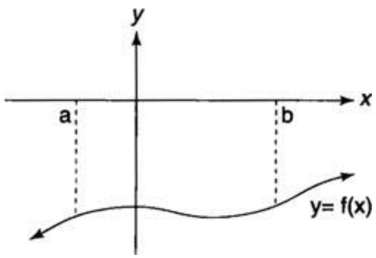
Recall that:

If $f(x) \geq 0$ or $f(x) \leq 0$ for $a \leq x \leq b$ then the area of the region bounded by the curve $y = f(x)$, the x −axis and the line $x = a$ and $x = b$ is

$$A = \int_a^b |f(x)|dx$$



$f(x) \geq 0$



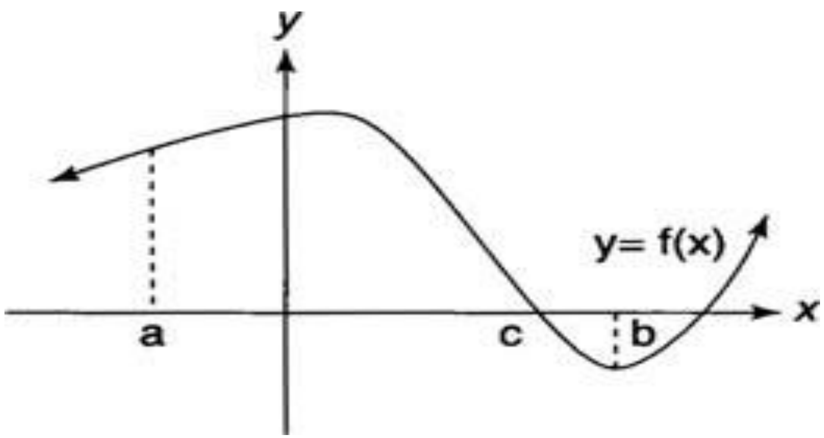
$f(x) \leq 0$

Similarly.
If $g(y) \geq 0$ or $g(y) \leq 0$ for c$\leq x \leq d$ then the area of the region bounded by the curve $x = f(y)$, the y −axis and the line $y = c$ and $y = d$ is

$$A = \int_c^d |g(y)|dy$$

Also,

If $f(x) \geq 0$ on $[a, c]$ and $f(x) \leq 0$ on $[c, b]$, then the area A of the region bounded by the graph of $f(x)$, the x −axis, and the lines $x = a$ and $x = b$ would be determined by the following definite integrals:

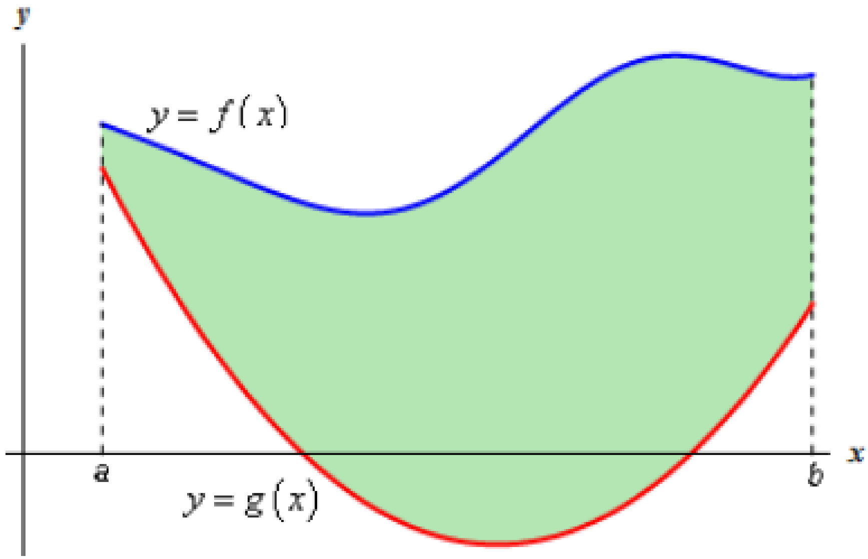


$$\mathrm{A} = \int_a^b |f(x)| dx \quad = \int_a^c f(x) dx - \int_c^b f(x) dx$$

Area Between Two Curves

First Case:

In the first case we want to determine the area between $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$. We are also going to assume that $f(x) \geq g(x)$. Take a look at the following sketch to get an idea of what we're initially going to look at.



So the Area is,

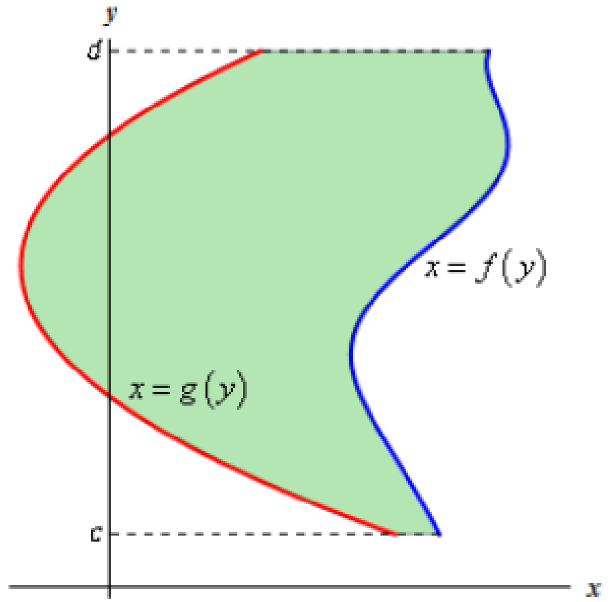
$$A = \int_a^b (f(x) - g(x))\,dx$$

In other words,

$$A = \int_a^b \begin{pmatrix} \text{upper} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{lower} \\ \text{function} \end{pmatrix} dx, \qquad a \leq x \leq b$$

Second Case:

The second case is almost identical to the first case. Here we are going to determine the area between $x = f(y)$ and $x = g(y)$ on the interval $[c, d]$ with $f(y) \geq g(y)$.



So the Area is,

$$A = \int_c^d (f(y) - g(y))dy$$

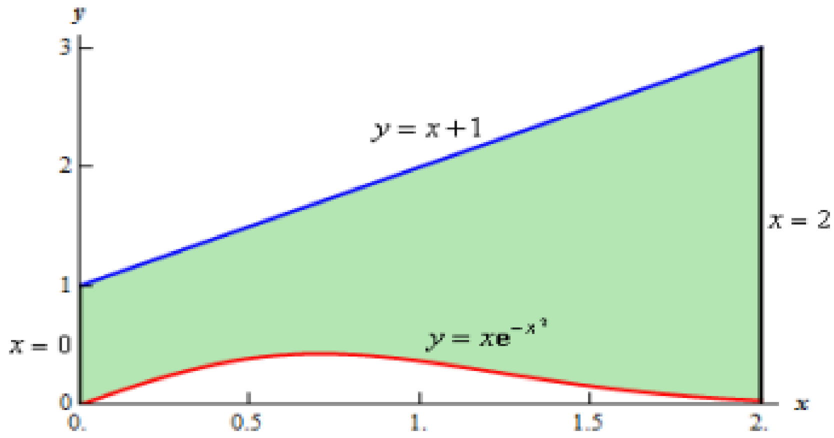
In other words,

$$A = \int_c^d \begin{pmatrix} \text{right} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{left} \\ \text{function} \end{pmatrix} dy, \qquad c \leq y \leq d$$

Example set-2.1

1. Write down the area in integral form and hence evaluate it

(a)



Solution:

Here, $y = x + 1$(Upper function)

$y = xe^{-x^2}$(Lower function)

$x = 0$ and $x = 2$

So the area is,

$$A = \int_a^b \begin{pmatrix} \text{upper} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{lower} \\ \text{function} \end{pmatrix} dx$$

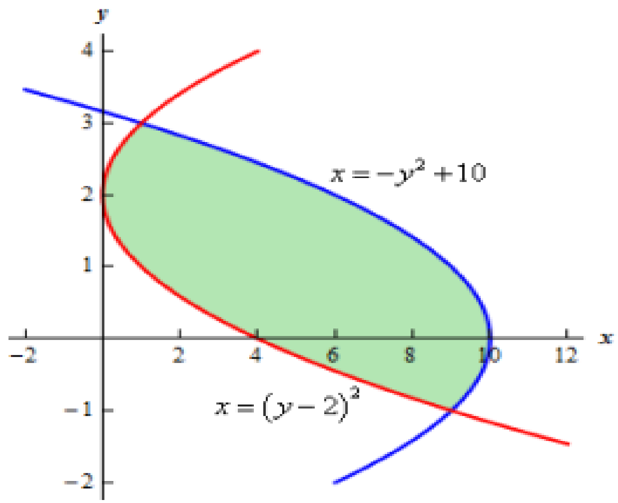
$$= \int_0^2 x + 1 - x\mathbf{e}^{-x^2} \, dx$$

$$= \left(\frac{1}{2}x^2 + x + \frac{1}{2}\mathbf{e}^{-x^2} \right) \Bigg|_0^2$$

$$= \frac{7}{2} + \frac{\mathbf{e}^{-4}}{2} = 3.5092$$

Example set-2.1

(b)



Solution:

Here, $x = -y^2 + 10$(Right function)
$x = (y-2)^2$ (Left function)
$y = -1$ and y $= 3$

So the area is,

$$A = \int_{-1}^{3} (-y^2 + 10 - (y-2)^2) dy$$

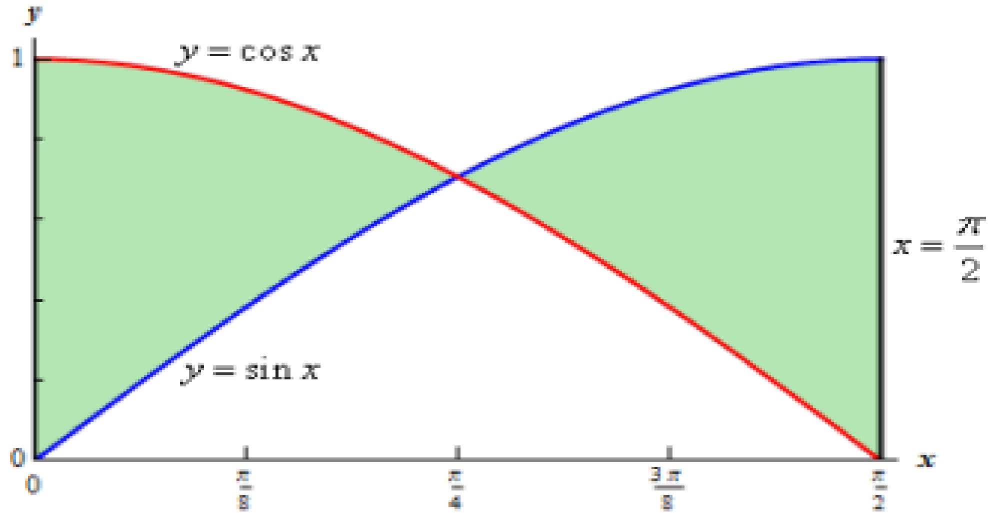
$$= \int_{-1}^{3} (-y^2 + 10 - y^2 + 4y - 4)) dy$$

$$= \int_{-1}^{3} (-2y^2 + 4y + 6) dy$$

?

Example set-2.1

(c)



Solution: The area is,

$$A = \int_0^{\frac{\pi}{4}} \cos x - \sin x \, dx + \int_{\pi/4}^{\pi/2} \sin x - \cos x \, dx$$

$$= \left(\sin x + \cos x\right)\Big|_0^{\frac{\pi}{4}} + \left(-\cos x - \sin x\right)\Big|_{\pi/4}^{\pi/2}$$

$$= \sqrt{2} - 1 + \left(\sqrt{2} - 1\right)$$

$$= 2\sqrt{2} - 2 = 0.828427$$

Example set-2.1

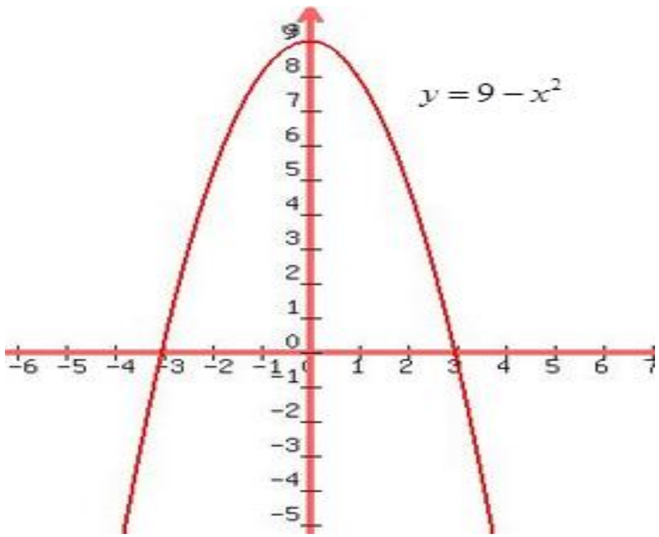
2. Sketch the region enclosed by $y = 9 - x^2$ and the x −axis. Hence find its area.

Solution: The region is shown in the Figure given below,

$$y = 9 - x^2, y = 0 \ (x \ \text{−axis})$$

$$9 - x^2 = 0$$

$$x^2 = 9 \quad \therefore x = \pm 3$$



So the area is,

$$A = \int_{-3}^{3} (9 - x^2 - 0)dx$$

$$= 2\int_{0}^{3} (9 - x^2)dx$$

$$= 36$$

Example set-2.1

3. Sketch the region enclosed by the parabolas $y = x^2$ and $x = y^2$. Hence find its area.

Solution: The region is shown in the Figure given below,

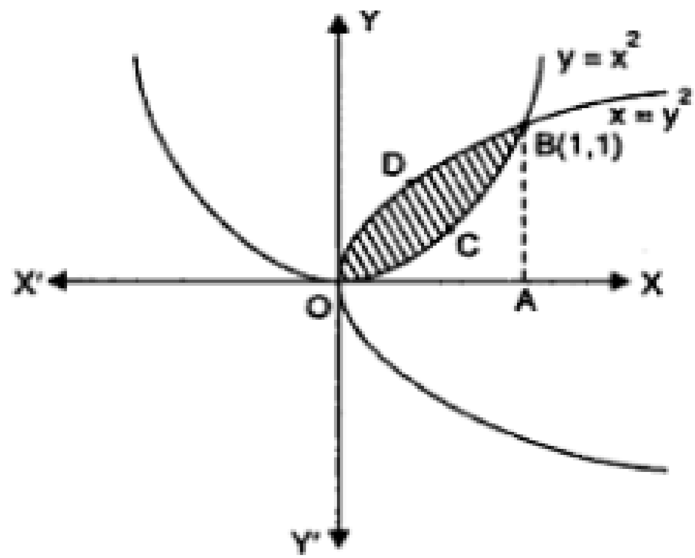
$$y = x^2, x = y^2$$

$$y = x^2, y = \sqrt{x}$$

$$x^2 = \sqrt{x}$$

$$x^4 = x$$

$$x(x^3 - 1) = 0 \qquad \therefore x = 0,1$$



So the area is,

$$= \int_0^1 (\sqrt{x} - x^2)dx$$

$$= \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right)\Bigg|_0^1$$

$$= \frac{1}{3}$$

Example set-2.1

4. Sketch the region enclosed by the parabolas $y = 2x^2 + 10$ and $y = 4x + 16$. Hence find its area.

Solution: The region is shown in the Figure given below,

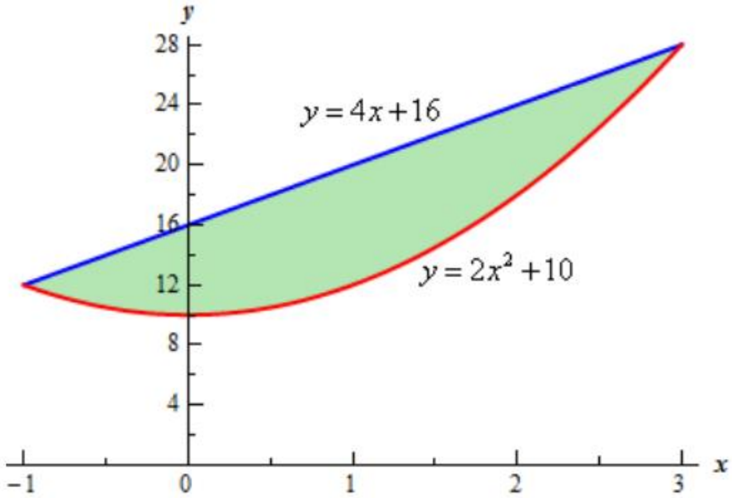
$$y = 2x^2 + 10\ ,\ y = 4x + 16$$

$$2x^2 + 10 = 4x + 16$$

$$2x^2 - 4x - 6 = 0$$

$$2(x+1)(x-3) = 0$$

$$\therefore x = -1,3$$



So the area is,

$$A = \int_a^b \begin{pmatrix} \text{upper} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{lower} \\ \text{function} \end{pmatrix} dx$$

$$= \int_{-1}^{3} 4x + 16 - \left(2x^2 + 10\right) dx$$

$$= \int_{-1}^{3} -2x^2 + 4x + 6\, dx$$

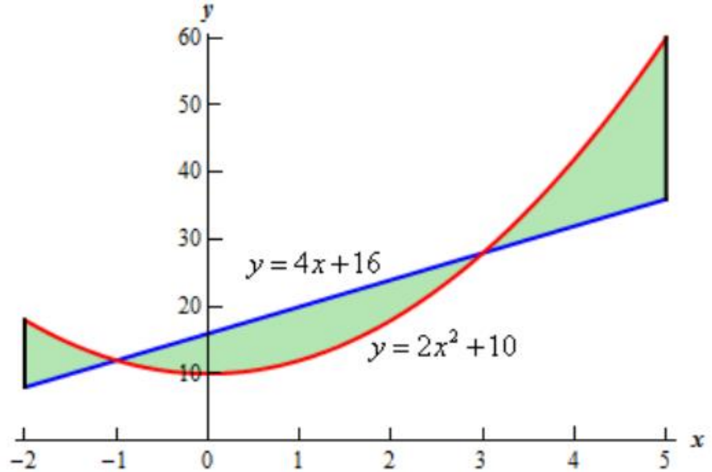
$$= \left(-\frac{2}{3}x^3 + 2x^2 + 6x\right)\Bigg|_{-1}^{3}$$

$$= \frac{64}{3}$$

Example set-2.1

5. Sketch the region enclosed by the parabolas $y = 2x^2 + 10$ and $y = 4x + 16$, $x = -2$ and $x = 5$ Hence find its area.

Solution: The region is shown in the Figure given below



$$y = 2x^2 + 10\ ,\ y = 4x + 16$$

$$2x^2 + 10 = 4x + 16$$

$$2x^2 - 4x - 6 = 0$$

$$2(x+1)(x-3) = 0$$

$$\therefore x = -1,3$$

So the area is,

$$A = \int_{-2}^{-1} 2x^2 + 10 - (4x+16)\,dx + \int_{-1}^{3} 4x + 16 - (2x^2+10)\,dx + \int_{3}^{5} 2x^2 + 10 - (4x+16)\,dx$$

$$= \int_{-2}^{-1} 2x^2 - 4x - 6\,dx + \int_{-1}^{3} -2x^2 + 4x + 6\,dx + \int_{3}^{5} 2x^2 - 4x - 6\,dx$$

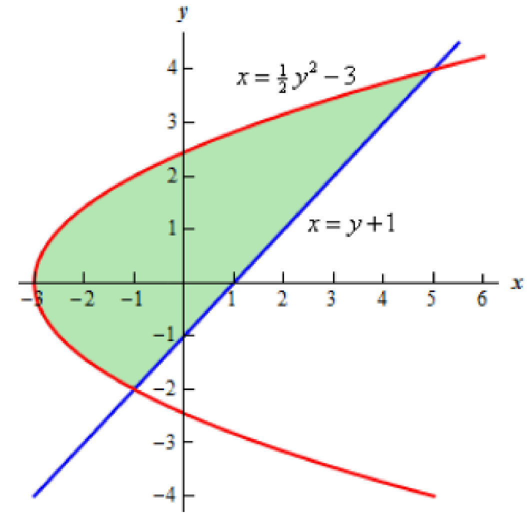
$$= \left(\frac{2}{3}x^3 - 2x^2 - 6x\right)\Bigg|_{-2}^{-1} + \left(-\frac{2}{3}x^3 + 2x^2 + 6x\right)\Bigg|_{-1}^{3} + \left(\frac{2}{3}x^3 - 2x^2 - 6x\right)\Bigg|_{3}^{5}$$

$$= \frac{14}{3} + \frac{64}{3} + \frac{64}{3}$$

$$= \frac{142}{3}$$

Example set-2.1

6. Determine the area enclosed by $x = \frac{1}{2}y^2 - 3$ and $y = x - 1$



Solution:

$x = \frac{1}{2}y^2 - 3$ and $y = x - 1$

So,

$x = \frac{1}{2}y^2 - 3$ and $x = y + 1$

$$y+1 = \frac{1}{2}y^2 - 3$$

$$2y+2 = y^2 - 6$$

$$0 = y^2 - 2y - 8$$

$$0 = (y-4)(y+2)$$

$\therefore y = -2,4$

So the area is,

$$A = \int_c^d \begin{pmatrix}\text{right}\\ \text{function}\end{pmatrix} - \begin{pmatrix}\text{left}\\ \text{function}\end{pmatrix} dy$$

$$= \int_{-2}^{4} (y+1) - \left(\frac{1}{2}y^2 - 3\right) dy$$

$$= \int_{-2}^{4} -\frac{1}{2}y^2 + y + 4\, dy$$

$$= \left(-\frac{1}{6}y^3 + \frac{1}{2}y^2 + 4y\right)\Bigg|_{-2}^{4}$$

$$= 18$$

Sample MCQ

1.If $f(x)$ and $g(x)$is defined in the interval [a,b] where$f(x) \geq g(x)$ then area between them could be found by

(a) ..... (b) $\int_a^b f(x) - g(x)\, dx$ (c) ....

2. What is the the area bounded by $y = 1 - x^2$ and the x $-$axis.

(a) 17 (b) .... (c)...

3.What is the the area bounded by $x = \frac{1}{2}y^2 - 3$ and $y = x - 1$

(a) ... (b) .... (c) 18

Exercise set-2.1

1. Sketch the region enclosed by the following curves and then find its area.

(a) $y = f(x) = x,\ 1 \le x \le 3$ and the x-axis.

(b) $y = f(x) = x^3,\ 1 \le x \le 2$ and the x-axis.

(c) $y = f(x) = x^2 + x + 4,\ 1 \le x \le 3.$

(d) $y = f(x) = \sin x\,, 0 \le x \le \frac{3\pi}{2}$ and the x-axis.

(e) $y = x^2 + 2$, the x-axis and the lines $x = 1$ and $x = 2$.

(f) $y = x^2 - 4$ and the x-axis.

(g) $x = 1 - y^2$ and the y-axis.

(h) $y = f(x) = x(1 - x)(2 - x)$ and the x-axis.

2. Sketch the region enclosed by the following curves and then find its area.

(a) $y = x^2$ and $y = x$

(b) $y = x(x-3)$ and the ordinates $x = 0$, $x = 5$

(c) $y = x^2$ and $y = 2 - x, x = 0, x \geq 0$

(d) $y = 3x - x^2$ and $y = x$

(e) $x = y^2$ and $y = x - 2$

(g) $y^2 = 4x + 4$ and $4x - y = 16$

3. **Calculus– James Stewart - 8th edition**

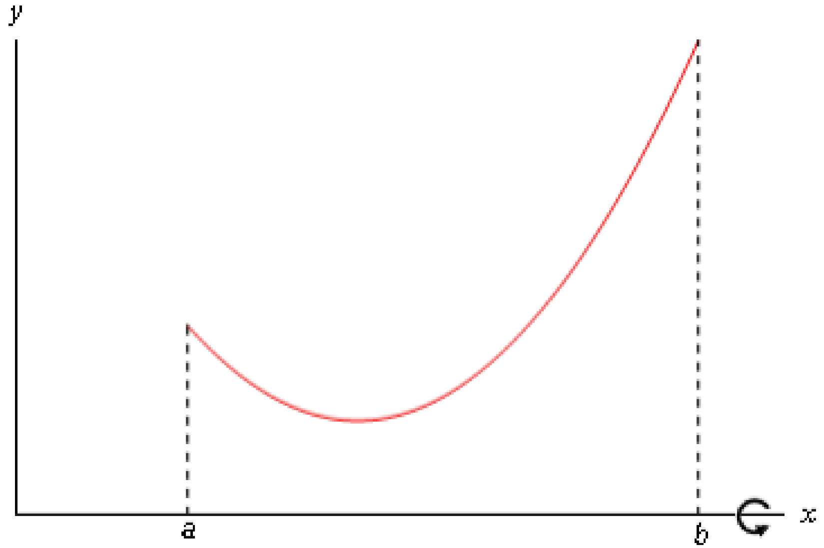
P- 434 Ex # 1, 3, 5 – 9, 13, 14, 17, 18, 22

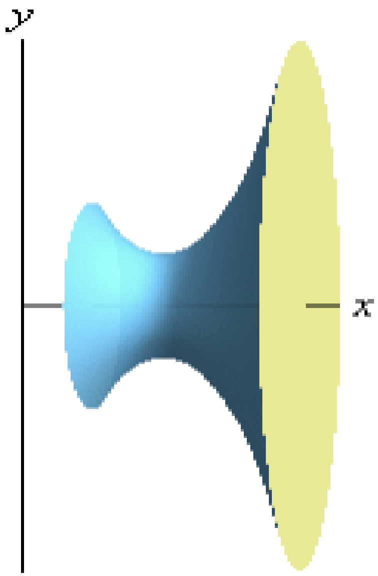
2.2 Volumes of Solids of Revolution

What is Solids of Revolution

If a region is rotated completely (i.e. through 2π radians) about a straight line, the solid formed is a solid of revolution. Any cross section perpendicular to the axis of rotation is circular.

To get a solid of revolution let's start with a function $y = f(x)$, on an interval $[a, b]$(Left side graph). Let's rotate the curve about x −axis(although it could be any vertical or horizontal axis) so that we get the following(right-side graph) three dimensional region.



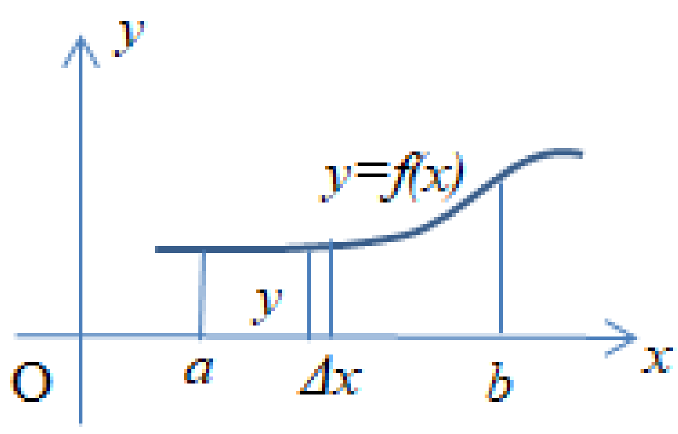


Now we are going to find the volume of the object

Volume of Solids of Revolution

Let us consider a solid generated by revolving about the *x*-axis of a region *R* bounded by a curve $y = f(x)$, the *x*-axis and the lines $x = a, x = b$.

The region *R* can be divided into small strips. When a typical strip of length y and width Δx is rotated completely about the *x*-axis, it forms a circular disc.



Volume of Solids of Revolution

The volume ΔV of the disc is , $\Delta V \approx \pi y^2 \Delta x$

The volume of the solid can be divided into small discs. Summing all the discs as $\Delta x \to 0$we have the volume of revolution V_x, **about the x −axis**

$$V_x = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \Delta x = \int_a^b \pi y^2 dx$$

In the same way, when a region bounded by the curve $x = f(y)$, , the *y*-axis and the lines $y = c, y = d$ is rotated **about the *y*-axis**, the solid formed has volume

$$V_y = \int_c^d \pi x^2 dy$$

This method is often called **method of disks** or the **method of rings**

Volume of Solids of Revolution

If we have two function $y = f(x)$ and $y = g(x)$ where $f(x) > g(x)$ and bounded by $x = a, x = \mathrm{b}$ then volume solid of revolution is **about $\boldsymbol{x}$ −axis** is given by

$$\mathrm{V}_x = \int_a^b \pi\left(\left(f(x)\right)^2 - \left(g(x)\right)^2\right) dx$$

$$\mathrm{V}_x = \int_a^b \pi\left((\text{outer radius})^2 - (\text{inner radius})^2\right) dx$$

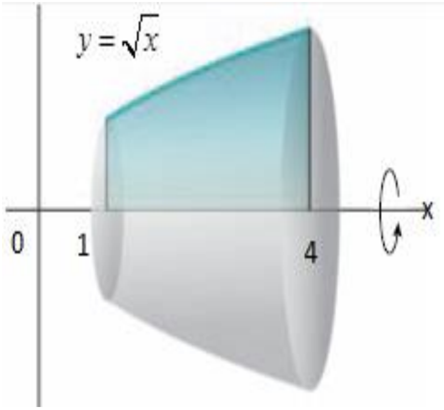
Similarly for the volume of solid of revolution about **y-axis** is,

$$\mathrm{V}_\mathrm{y} = \int_a^b \pi\left(\left(f(y)\right)^2 - \left(g(y)\right)^2\right) dy$$

Example set-2.2.1

1. Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval $[1,4]$ is revolved about the ***x*-axis**.

Solution:

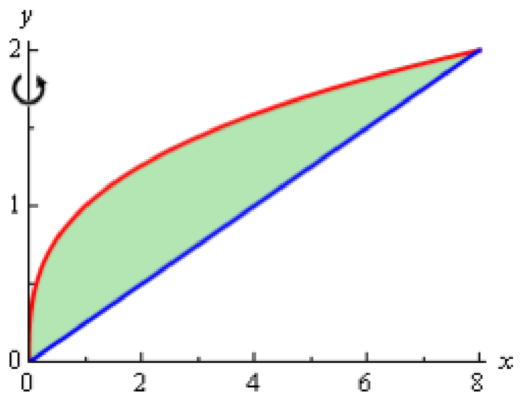


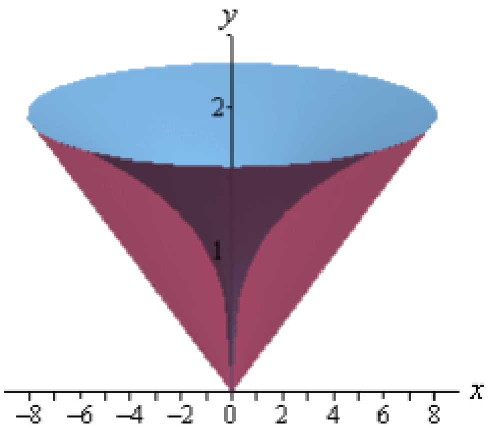
The volume is ,

$$V_x = \int_a^b \pi y^2 dx = \int_a^b \pi (f(x))^2 dx = \int_1^4 \pi (\sqrt{x})^2 dx = \pi \int_1^4 x\, dx = \frac{15\pi}{2}$$

2. Find the volume of the solid that is obtained when the region under the curve $y = \sqrt[3]{x}$ and $y = \frac{x}{4}$ that lies in the first quadrant and is revolved about the ***y*-axis**.

Solution:





$$y = \sqrt[3]{x} \quad \Rightarrow \quad x = y^3 \qquad y^3 = 4y$$

$$y = \frac{x}{4} \quad \Rightarrow \quad x = 4y \qquad y(y^2 - 4) = 0$$

$$\therefore\ y = 0, 2, -2$$

So, the volume is ,

$$V_y = \int_a^b \pi((right)^2 - (left)^2)dy = \pi \int_0^2 (16y^2 - y^6)dy = \frac{512\pi}{21}$$

Sample MCQ

1.If we have two function $y = f(x)$ and $y = g(x)$ where $f(x) > g(x)$ and bounded by $x = a, x =$ b then volume solid of revolution is **about** $\boldsymbol{x}$ **−axis** is given by

(a) ..... (b) $\int_a^b \pi\,((f(x))^2 - (g(x))^2)dx$ (c) ....

2. Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval $[1,4]$ is revolved about the ***x*-axis**.

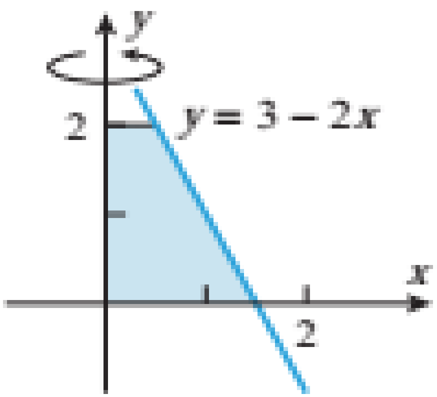
(a) $\frac{15\pi}{2}$ (b) .... (c)...

3.Find the volume of the solid that is obtained when the region under the curve $y = \sqrt[3]{x}$ and $y = \frac{x}{4}$ that lies in the first quadrant and is revolved about the ***y*-axis**.

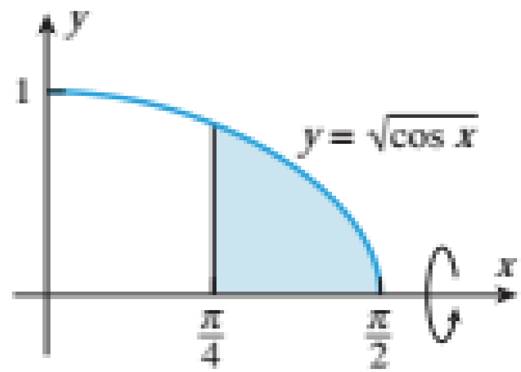
(a) ... (b) .... (c) $\frac{512\pi}{21}$

Exercise set-2.2.1

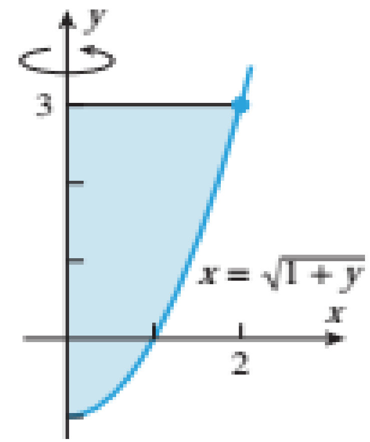
1. Find the volume of the solid that results when the shaded region is revolved about the indicated axis:



(a)



(b)



(c)

2. Find the volume of the solid when the region enclosed by the given curves is revolved about the ***x*-axis.**

(a) $y = \sqrt{x},\ x = 9.$
(b) $y = x^2, x = 0, x = 2.$
(c) $y = x^2 - 4x + 5, x = 1, x = 4.$
(d) $y = x, y = 1, x = 0.$

3. Find the volume of the solid when the region enclosed by the given curves is revolved about the ***y*-axis.**

(a) $y = \sqrt{x}, x = 0, y = 3.$
(b) $x = 1 - y^2,\ x = 0.$
(c) $y = \frac{1}{x}, y = 1, y = 2.$

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P- 446 Ex # 1-10