Laplace Transformation

Definition: Let the function $f(t)$ be defined for all positive values of t, then multiply $f(t)$bye^{-st} and integrate it with respect to t from zero to infinity. If the resulting integral exists (i.e., has some finite value), it is a function of s, s may be real or complex, say $F(s)$.

$$F(s) = \int_0^\infty e^{-st} f(t) dt.$$

This function $F(s)$ of variable s is called Laplace Transformation of the original function $f(t)$ and will be denoted by $\mathcal{L}\{f(t)\}$, where $\mathcal{L}$ denotes the Laplace transform operator. Thus

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt.$$

The original function $f(t)$ is called the inverse transform or inverse of $F(s)$ and will be denoted by$\mathcal{L}^{-1}\{F(s)\}$.

$$\therefore f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

Important formulae:

1. $\mathcal{L}\{c\} = \frac{c}{s}$, c is any constant, $(s > 0)$	**Some important formulae**
2. $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, when $n = 0,1,2,3,\ldots$	$\sin at = \frac{e^{iat} - e^{-iat}}{2i}$
3. $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, $(s > a)$	$\cos at = \frac{e^{iat} + e^{-iat}}{2}$
4. $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}, \lvert s\rvert > a$	$\sinh at = \frac{e^{at} - e^{-at}}{2}$
5. $\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}, \lvert s\rvert > a$	$\cosh at = \frac{e^{at} + e^{-at}}{2}$
6. $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}, \lvert s\rvert > a$	$\int e^{at} \sin bt \ dt = \frac{e^{at}(a\sin bt - b\cos bt)}{a^2+b^2}$
7. $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}, \lvert s\rvert > a$	$\int e^{at} \cos bt \ dt = \frac{e^{at}(a\cos bt + b\sin bt)}{a^2+b^2}$

Properties of Laplace transformation:

1. $\mathcal{L}\{af_1(t) + bf_2(t)\} = a\mathcal{L}\{f_1(t)\} + b\mathcal{L}\{f_2(t)\}$**(linearity)**, where a &b are constants.

2. If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$**(first shifting or translation)**

3. If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$**(multiplication by t^n)**

Proof of some selective formulae:

1. $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}, (s > a)$

Proof: From the definition of Laplace transformation we know that,

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}e^{at}dt = \lim_{p\to\infty}\int_0^p e^{-(s-a)t}dt = \lim_{p\to\infty}\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]\Bigg|_0^p = \lim_{p\to\infty}\frac{1-e^{-(s-a)p}}{(s-a)}$$

$= \frac{1}{s-a}$ if $s > a$.

2. $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$ if $|s| > a$ and 3. $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$ if $|s| > a$.

Proof:$\mathcal{L}\{e^{iat}\} = \frac{1}{s-ia} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$ - - - - - (1)

But, $e^{iat} = \cos at + i \sin at$

So, $\mathcal{L}\{e^{iat}\} = \int_0^\infty e^{-st}e^{iat}dt = \int_0^\infty e^{-st}(\cos at + i\sin at)dt$

$$= \int_0^\infty e^{-st}\cos at\, dt + i\int_0^\infty e^{-st}\sin at\, dt$$

$= \mathcal{L}\{\cos at\} + i \quad \mathcal{L}\{\sin at\}$ - - - - - - (2)

Comparing (1) and (2), we have on equating real and imaginary parts,

$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$ if $|s| > a$ and $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$ if $|s| > a$.

Some workout examples on Laplace transformation:

Example: 1	**Example: 2**
$\mathcal{L}\{(t^2+1)^2\}$ $= \mathcal{L}\{t^4 + 2t^2 + 1\}$ $= \mathcal{L}\{t^4\} + 2\mathcal{L}\{t^2\} + \mathcal{L}\{1\}$ $= \dfrac{4!}{s^{4+1}} + 2\dfrac{2!}{s^{2+1}} + \dfrac{1}{s}$ $= \dfrac{24}{s^5} + \dfrac{4}{s^3} + \dfrac{1}{s}$	$\mathcal{L}\{\mathrm{e}^{-3t} + 5\cosh t\}$ $= \mathcal{L}\{\mathrm{e}^{-3t}\} + 5\,\mathcal{L}\{\cosh\ t\}$ $= \dfrac{1}{\mathrm{s}-(-3)} + 5\dfrac{\mathrm{s}}{\mathrm{s}^2 - 1^2}$ $= \dfrac{1}{\mathrm{s}+3} + \dfrac{5\mathrm{s}}{\mathrm{s}^2-1}$

Example: 3

$\mathcal{L}\{(\sin t-\cos t)^2+\cos^2 3t\}$

$=\mathcal{L}\left\{\sin^2 t-2\sin t\cos t+\cos^2 t+\frac{1}{2}(1+\cos 6t)\right\}$

$=\mathcal{L}\left\{1-\sin 2t+\frac{1}{2}+\frac{1}{2}\cos 6t)\right\}$

$=\mathcal{L}\left\{\frac{3}{2}\right\}-\mathcal{L}\{\sin 2t\}+\frac{1}{2}\mathcal{L}\{\cos 6t\}$

$=\frac{3}{2}\frac{1}{s}-\frac{2}{s^2+2^2}+\frac{1}{2}\frac{s}{s^2+6^2}$

$=\frac{3}{2s}-\frac{2}{s^2+4}+\frac{1}{2}\left(\frac{s}{s^2+36}\right).$

Example: 4

$\mathcal{L}\{3-\mathrm{e}^{-t}\cos 2t+t\ \sinh 3t\}$

$=\mathcal{L}\{3\}-\mathcal{L}\{\mathrm{e}^{-t}\cos 2t\}+\mathcal{L}\{t\sinh 3t\}$

$=3\frac{1}{\mathrm{s}}-\frac{\mathrm{s}+1}{(\mathrm{s}+1)^2+2^2}+(-1)^1\frac{\mathrm{d}}{\mathrm{ds}}\left(\frac{3}{\mathrm{s}^2-3^2}\right)$

$=\frac{3}{\mathrm{s}}-\frac{\mathrm{s}+1}{\mathrm{s}^2+2\mathrm{s}+5}+\frac{3.2\mathrm{s}}{(\mathrm{s}^2-9)^2}$

$=\frac{3}{\mathrm{s}}-\frac{\mathrm{s}+1}{\mathrm{s}^2+2\mathrm{s}+5}+\frac{6\mathrm{s}}{(\mathrm{s}^2-9)^2}$

Example: 5

$\mathcal{L}\{t\cos t-\mathrm{e}^{-3t}\sin 2t+t^7\mathrm{e}^{5t}\}$

$=(-1)^1\frac{\mathrm{d}}{\mathrm{ds}}\frac{\mathrm{s}}{\mathrm{s}^2+1}-\frac{2}{(\mathrm{s}+3)^2+2^2}+\frac{7!}{(\mathrm{s}-5)^8}$

$=-\frac{1-\mathrm{s}^2}{(\mathrm{s}^2+1)^2}-\frac{2}{(\mathrm{s}+3)^2+4}+\frac{5040}{(\mathrm{s}-5)^8}$

Example: 6

$\mathcal{L}\{\mathrm{e}^{2t}t+\mathrm{e}^{t}\cosh 3t\}$

$=\frac{1}{(\mathrm{s}-2)^2}+\frac{\mathrm{s}-1}{(\mathrm{s}-1)^2-3^2}$

$=\frac{1}{(\mathrm{s}-2)^2}+\frac{\mathrm{s}-1}{\mathrm{s}^2-2\mathrm{s}-8}.$

Example: 7

$\mathcal{L}\{t\sin at\}$

$=\mathcal{L}\left\{t\frac{e^{iat}-e^{-iat}}{2i}\right\}$

$=\frac{1}{2i}\mathcal{L}\{te^{iat}-te^{-iat}\}$

$=\frac{1}{2i}\left[\frac{1!}{(s-ia)^2}-\frac{1!}{(s+ia)^2}\right]$(first Shifting Property)

$=\frac{2as}{(s^2+a^2)^2}$

Or

$\mathcal{L}\{t\sin at\}$

$=(-1)^1\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$

$=-\frac{(s^2+a^2).0-a.2s}{(s^2+a^2)^2}$

$=-\frac{-2as}{(s^2+a^2)^2}$

$=\frac{2as}{(s^2+a^2)^2}$

Example: 8

$\mathcal{L}\{t\cos at\}$

$=\mathcal{L}\left\{t\frac{e^{iat}+e^{-iat}}{2}\right\}$

$=\frac{1}{2}\mathcal{L}\{e^{iat}t+e^{-iat}t\}$

$=\frac{1}{2}\left[\frac{1!}{(s-ia)^2}+\frac{1!}{(s+ia)^2}\right]$(first shifting property)

$=\frac{a^2-s^2}{(s^2+a^2)^2}$

Or

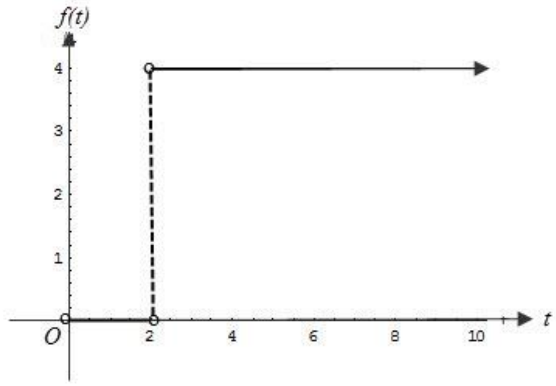
$\mathcal{L}\{t\cos at\}$

$=(-1)^1\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$

$=\frac{(s^2+a^2).1-s.2s}{(s^2+a^2)^2}$

$=\frac{a^2-s^2}{(s^2+a^2)^2}$

<table>
<tr><td>Piecewise function</td><td></td></tr>
<tr><td>Example:
Sketch $f(t) = \begin{cases} 0, & 0 < t < 2 \\ 4, t > 2 \end{cases}$ and find$\mathcal{L}\{f(t)\}$.

Solution:
$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$
$$= \int_0^2 e^{-st}.0\,dt + \int_2^\infty e^{-st}\,4\,dt$$
$= 4\left[\frac{e^{-st}}{-s}\right]\Big|_2^\infty = 4\frac{e^{-2s}}{s}.$</td><td></td></tr>
</table>

Reference Books:

1. Advanced Engineering Mathematics- Erwin Kreyszig.(10th Edition)
2. Differential Equations- Paul Blanchard, Robert L. Devaney, Glen R. Hall (4th Edition)

Problem Set 1

Find the Laplace Transforms and also sketch (if free hand sketching is getting complex then use MATLAB) the following functions (1-20):

Using direct formula

1. $f(t) = 3t + 12$, **Ans:** $F(s) = \frac{12}{s} + \frac{3}{s^2}$.
2. $f(t) = e^{5t}$, **Ans:** $F(s) = \frac{1}{s-5}$.
3. $f(t) = e^{-2t}$, **Ans:** $F(s) = \frac{1}{s+2}$.
4. $f(t) = (a - bt)^2$, **Ans:** $F(s) = \frac{a^2\,s^2 - 2abs + 2\,b^2}{s^3}$.
5. $f(t) = \cos \pi t$, **Ans:** $F(s) = \frac{s}{s^2+\pi^2}$.
6. $f(t) = \cos^2 \omega t$, **Ans:** $F(\boldsymbol{s}) = \frac{2\,\omega^2+s^2}{s\,(4\,\omega^2 + s^2)}$.
7. $f(t) = \sin(\omega t + \theta)$, **Ans:** $F(s) = \frac{\omega \cos\theta + s\,\sin\theta}{\omega^2 + s^2}$.
8. $f(t) = 1.5 \sin\left(3t - \frac{\pi}{2}\right)$, **Ans:** $F(s) = \frac{-3s}{2(s^2+9)}$.

First shifting or Translation property

If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{e^{at}\, f(t)\} = F(s-a)$.

9. $f(t) = e^{2t}\sinh t$, **Ans:** $F(s) = \frac{1}{(s-2)^2-1}$.
10. $f(t) = e^{-t}\sinh 4t$, **Ans:** $F(s) = \frac{4}{(s+1)^2-16}$.
11. $f(t) = e^{2t}\cos 3t$, **Ans:** $F(s) = \frac{s-2}{(s-2)^2\, +\, 9}$.
12. $f(t) = t^{10}\, e^{-5t}$, **Ans:** $F(s) = \frac{10\,!}{(s+5)^{11}}$.
13. $f(t) = \cosh 5t\;\; e^{3t}$, **Ans:** $F(s) = \frac{s-3}{(s-3)^2\, -25}$.

Property of multiplication by t^n

If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{t^n\, f(t)\} = (-1)^n\, \frac{d^n}{ds^n}\,[F(s)]$.

14. $f(t) = t\sin 2t$, **Ans:** $F(s) = \frac{4s}{(s^2+4)^2}$.
15. $f(t) = t\cos bt$, **Ans:** $F(s) = \frac{2\, s^2}{(b^2+s^2)^2} - \frac{1}{b^2+s^2}$.

Piece-wise Function

If $f(t)$ is a piece-wise function then $\boldsymbol{\mathcal{L}}\{f(t)\} = \int_{\mathbf{0}}^{\infty} e^{-st}\; f(t)\; dt$

16. $f(t) = \begin{cases} 1-t & ;0<t<1 \\ 0\,; & t>1 \end{cases}$
Ans: $F(s) = \frac{s+e^{-s}-1}{s^2}$.

17. $f(t) = \begin{cases} k\,; & 0<t<c \\ 0\,; & t>c \end{cases}$
Ans: $F(s) = -\,\frac{k\,(e^{-c\,s}-1)}{s}$.

18. $f(t) = \begin{cases} b-t\,; & 0<t<b \\ 0\,; & t>b \end{cases}$

Ans: $F(s) = \frac{e^{-bs}+\, bs\, -1}{s^2}$.

19. $f(t) = \begin{cases} 1\,; & 0<t<1 \\ -1\,; & 1<t<2 \\ 0; & t>2 \end{cases}$

Ans: $F(s) = \frac{e^{-2s}\,(e^s-1)^2}{s}$.

20. $f(t) = \begin{cases} t\,; & 0\le t\le 4 \\ 1\,; & t>4 \end{cases}$
Also sketch $f(t)$.
Ans: $F(s) = -\,\frac{e^{-4s}\, +\, 3s\, e^{-4s}\, -1}{s^2}$.

The Unit step function (Heaviside function):

In engineering applications, we frequently encounter functions whose values change abruptly at specified values of time *t*. One common example is when a voltage is switched on or off in an electrical circuit at a specified value of time *t*.

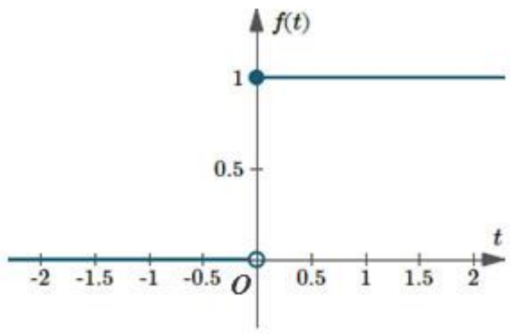
The value of $t = 0$ is usually taken as a convenient time to switch on or off the given voltage.

The switching process can be described mathematically by the function called the **Unit Step Function** (otherwise known as the **Heaviside function** after Oliver Heaviside).

The Unit step function or Heaviside's unit step function, $u(t)$, is defined as follows:

$$f(t) = u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases},$$

That is, u is a function of time t, and u has value **zero** when time is negative (before we flip the switch); and value **one** when time is positive or zero (from when we flip the switch).



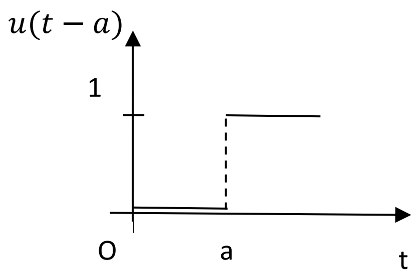
Graph of $f(t) = u(t)$, the unit step function.

Shifted (Right) Unit step function:

In many circuits, waveforms are applied at specified intervals other than t=0. Such a function may be described using the **shifted** (aka **delayed**) unit step function.

A function which has value0 up to the time $t = a$ and thereafter has value 1 is written:

$$u_a(t) = u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases},$$



Rectangular pulse:

A common situation in a circuit is for a voltage $V(t)$ to be applied at a particular time (say $t = a$) and removed later, at $t = b$ (say). We write such a situation using unit step functions as:

$$V(t) = u(t-a) - u(t-b)$$

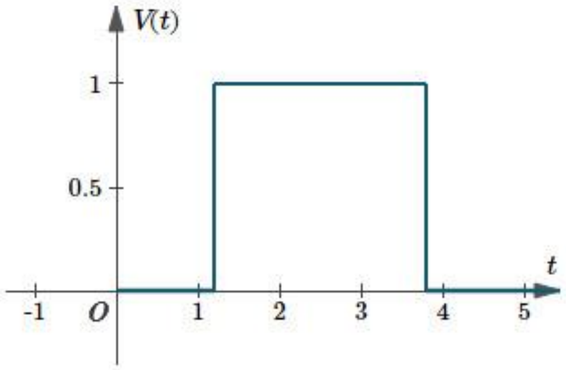
This voltage has strength 1, duration $(b-a)$.

Alternatively, $V(t)$ may be constructed using top hat function $u(t-a) - u(t-b)$ as follows:

$$V(t) = u(t-a) - u(t-b) = \begin{cases} 1, & a \le t < b \\ 0, & \text{otherwise} \end{cases},$$

Example01 :

The graph of $V(t) = u(t-1.2) - u(t-3.8)$ is as follows. Here, the duration is 3.8−1.2=2.6 .



Graph of $V(t) = u(t-1.2) - u(t-3.8)$ is an example of a rectangular pulse.

Example 02:

Write the following functions in terms of **unit step** function(s). Sketch each waveform.

(i) A 12-V source is switched on at t = 4 s,

(ii) $V(t) = \begin{cases} 1, & 0 < t < a \\ 0, & t > a \end{cases}$ and,

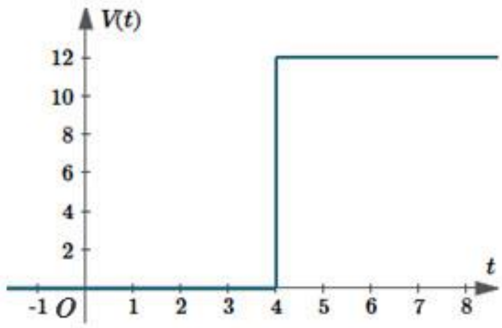
(iii) $V(t) = \begin{cases} 0, & t < 3 \\ 2t+8, & 3 < t < 5 \\ 0, & t > 5 \end{cases}$.

Solution:

(i) Since the voltage is turned on at t = 4, we need to use $\mathrm{u}(\mathrm{t}-4)$. We multiply by 12 since that is the voltage.

We write the function as follows: $V(t) = 12 \cdot u(t-4)$

Here's the graph:



Graph of $V(t) = 12 \cdot u(t-4)$

(ii) $V(t) = \begin{cases} 1, & 0 < t < a \\ 0, & t > a' \end{cases}$

In words, the voltage has value 1 until time $t = a$. Then it is turned off.

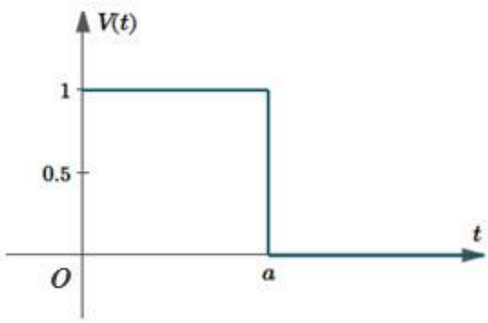
We have a "rectangular pulse" situation and need to use this formula:

$$V(t) = u(t-a) - u(t-b)$$

In our example, the pulse starts at $t = 0$, so we use $u(t)$ and finishes at $t = a$ so we use $u(t-a)$. So the required function is:

$$V(t) = 1 \cdot [u(t) - u(t-a)]$$

Here is the graph



Graph of $V(t) = 1 \cdot [u(t) - u(t-a)]$, a shifted unit step function

(iii) $V(t) = \begin{cases} 0, & t < 3 \\ 2t+8, & 3 < t < 5 \\ 0, & t > 5 \end{cases}$.

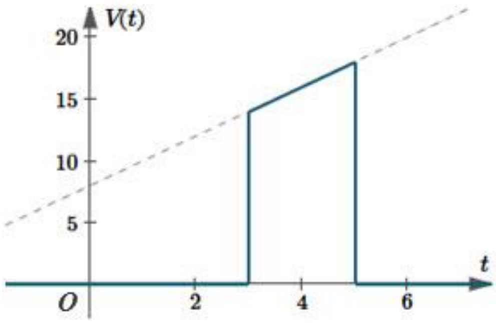
In this example, our function is $V(t) = 2t + 8$ which has slope 2 and V-intercept 8.

The signal is only turned on between $t = 3$ and $t = 5$. The rest of the time it is off.

So our voltage function will be:

$$V(t) = (2t + 8) \cdot [u(t - 3) - u(t - 5)]$$

The graph is as follows:



Graph of $V(t) = (2t + 8) \cdot [u(t - 3) - u(t - 5)]$. The dashed line is $V(t) = (2t + 8)$.

Now, the Laplace transform of **unit step function** is,

$$\mathcal{L}\{1. u(t - a)\} = \frac{e^{-as}}{s}$$

$$\mathcal{L}\{f(t). u(t - a)\} = e^{-as}\mathcal{L}\{f(t + a)\}$$

Example: 01

Find the Laplace transformation of $t^2 u(t - 3)$.

Solution: $\mathcal{L}\{t^2 u(t - 3)\} = e^{-3s}\mathcal{L}\{(t + 3)^2\}$

$$= e^{-3s}\mathcal{L}\{t^2 + 6t + 9\}$$

$$= e^{-3s}\left[\frac{2}{s^3} + 6\frac{1}{s^2} + 9\frac{1}{s}\right].$$

Example: 02

Find the Laplace transformation of $e^{-2t}u_\pi(t)$.

Solution: $u_\pi(t) = \begin{cases} 0, & t < \pi \\ 1, & t > \pi \end{cases}$.

$$= u(t - \pi)$$

$$\mathcal{L}\{e^{-2t}u_\pi(t)\} = e^{-\pi s}\mathcal{L}\{f(t + \pi)\} \quad = e^{-\pi s}\mathcal{L}\{e^{-2(t+\pi)}\}$$

$$= e^{-\pi s}\mathcal{L}\{e^{-2t}e^{-2\pi}\} = e^{-\pi s}e^{-2\pi}\mathcal{L}\{e^{-2t}\} = e^{-\pi(s+2)}\frac{1}{s + 2}.$$

Example: 03

Find the Laplace transformation of $t\, u_2(t)$.

Solution: $\mathcal{L}\{t\, u_2(t)\} = \mathcal{L}\{t\, u(t - 2)\}$

$$= e^{-2s}\mathcal{L}\{t + 2\}$$

$$= e^{-2s}\left(\frac{1}{s^2} + 2\frac{1}{s}\right)$$

$$= e^{-2s}\left(\frac{2}{s} + \frac{1}{s^2}\right).$$

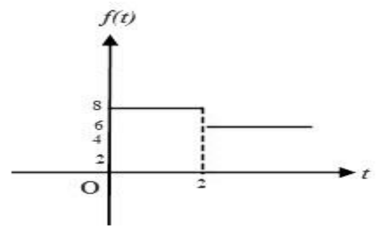
Example: 04 Given$f(t) = \begin{cases} 8, & 0 < t < 2 \\ 6, & t > 2 \end{cases}$’

(i) sketch $f(t)$,

(ii) convert $f(t)$ to unit step function and,

(iii) find the Laplace transformation of $f(t)$.

Solution:

(i) 	(ii) $f(t) = 8[u(t-0) - u(t-2)] + 6[u(t-2)]$ $= 8[u(t-0)] - 2[u(t-2)]$ (iii) $\mathcal{L}\{f(t)\} = \mathcal{L}\{8\} - 2\mathcal{L}\{u(t-2)\}$ $= \frac{8}{s} - 2.\frac{e^{-2s}}{s}.$

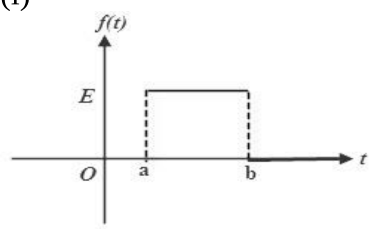
Example:05 Given$f(t) = \begin{cases} E, & a < t < b \\ 0, & t > b \end{cases}$, where E, a and b are positive constants.

(i) sketch $f(t)$,

(ii) convert $f(t)$ to unit step function and,

(iii) find the Laplace transformation of $f(t)$.

Solution:

(i) 	(ii) $f(t) = E\begin{cases} 1 & a < t < b \\ 0 & t > b \end{cases}$ $= E[u(t-a) - u(t-b)]$ (iii) $\mathcal{L}\{f(t)\} = E\mathcal{L}\{u(t-a) - u(t-b)\}$ $= E\left[\frac{e^{-as}}{s} - \frac{e^{-bs}}{s}\right].$

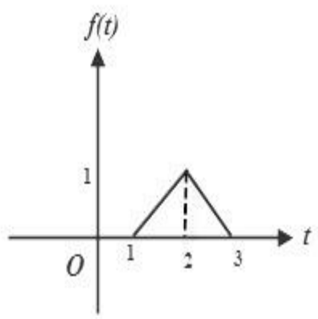
Example: 06 Given$f(t) = \begin{cases} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \end{cases}$

(i) sketch $f(t)$,

(ii) convert $f(t)$ to unit step function and,

(iii) find the Laplace transformation of $f(t)$.

Solution:

(i) 	(ii) $f(t) = (t-1)[u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)]$ $= (t-1)u(t-1) - (t-1)u(t-2) + (3-t)u(t-2) - (3-t)u(t-3)$ $= (t-1)u(t-1) - 2(t-2)u(t-2) + (t-3)u(t-3)$ (iii) $\mathcal{L}\{f(t)\} = e^{-s}\mathcal{L}\{t+1-1\} - 2\,e^{-2s}\mathcal{L}\{t+2-23\} + e^{-3s}\mathcal{L}\{t+3-3\}$ $= e^{-s}\frac{1}{s^2} - 2\,e^{-2s}\frac{1}{s^2} + e^{-3s}\frac{1}{s^2}$

	$= \frac{1}{s^2}(e^{-s} - 2\,e^{-2s} + e^{-3s}).$

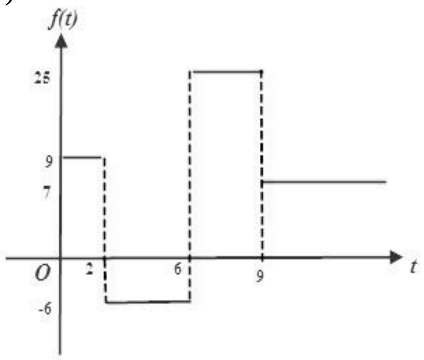
Example: 07 Given $f(t) = \begin{cases} 9, & 0 < t < 2 \\ -6, & 2 \le t < 6 \\ 25, & 6 \le t < 9 \\ 7, & t > 9 \end{cases}$,

(i) sketch $f(t)$,

(ii) convert $f(t)$ to unit step function and,

(iii) find the Laplace transformation of $f(t)$.

Solution:

(i) 	(ii) $f(t) = 9[u(t-0) - u(t-2)] + (-6)[u(t-2) - u(t-6)] + 25[u(t-6) - u(t-9)] + 7[u(t-9)]$ $= 9u(t-0) - 15u(t-2) + 31u(t-6) - 18u(t-9)$ (iii) $\mathcal{L}\{f(t)\} = \mathcal{L}\{9u(t-0)\} - \mathcal{L}\{15u(t-2)\} + \mathcal{L}\{31u(t-6)\} - \mathcal{L}\{18u(t-9)\}$ $= 9e^{0s}\frac{1}{s} - 15e^{-2s}\frac{1}{s} + 31e^{-6s}\frac{1}{s} - 18e^{-9s}\frac{1}{s}$ $= (9 - 15e^{-2s} + 31e^{-6s} - 18e^{-9s})\frac{1}{s}.$

6.4 Short Impulses. Dirac's Delta Function. Partial Fractions

An airplane making a "hard" landing, a mechanical system being hit by a hammerblow, a ship being hit by a single high wave, a tennis ball being hit by a racket, and many other similar examples appear in everyday life. They are phenomena of an impulsive nature where actions of forces mechanical, electrical, etc. are applied over short intervals of time.

We can model such phenomena and problems by "Dirac's delta function," and solve them very effecively by the Laplace transform.

To model situations of that type, we consider the function

(1) $$f_k(t-a) = \begin{cases} 1/k & \text{if } a \leqq t \leqq a+k \\ 0 & \text{otherwise} \end{cases}$$ (Fig. 132)

(and later its limit as $k \to 0$). This function represents, for instance, a force of magnitude $1/k$ acting from $t = a$ to $t = a + k$, where k is positive and small. In mechanics, the integral of a force acting over a time interval $a \leqq t \leqq a + k$ is called the **impulse** of the force; similarly for electromotive forces $E(t)$ acting on circuits. Since the blue rectangle in Fig. 132 has area 1, the impulse of f_k in (1) is

(2) $$I_k = \int_0^\infty f_k(t-a)\,dt = \int_a^{a+k} \frac{1}{k}\,dt = 1.$$

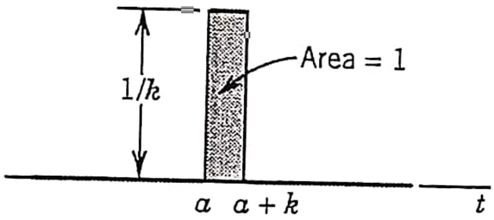


Fig. 132. The function $f_k(t-a)$ in (1)

To find out what will happen if k becomes smaller and smaller, we take the limit of f_k as $k \to 0$ $(k > 0)$. This limit is denoted by $\delta(t-a)$, that is,

$$\delta(t-a) = \lim_{k \to 0} f_k(t-a).$$

$\delta(t-a)$ is called the **Dirac delta function**[2] or the **unit impulse function.**

$\delta(t-a)$ is not a function in the ordinary sense as used in calculus, but a so-called *generalized function.*[2] To see this, we note that the impulse I_k of f_k is 1, so that from (1) and (2) by taking the limit as $k \to 0$ we obtain

(3) $$\delta(t-a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \int_0^\infty \delta(t-a)\,dt = 1,$$

but from calculus we know that a function which is everywhere 0 except at a single point must have the integral equal to 0. Nevertheless, in impulse problems, it is convenient to operate on $\delta(t-a)$ as though it were an ordinary function. In particular, for a *continuous* function $g(t)$ one uses the property often called the **sifting property** of $\delta(t-a)$, not to be confused with *shifting* but from calculus we know that a function which is everywhere 0 except at a single point must have the integral equal to 0. Nevertheless, in impulse problems, it is convenient to operate on $\delta(t-a)$ as though it were an ordinary function. In particular, for a *continuous* function $g(t)$ one uses the property often called the **sifting property** of $\delta(t-a)$, not to be confused with *shifting*

(4)
$$\int_0^\infty g(t)\delta(t-a)\,dt = g(a)$$

which is plausible by (2).

To obtain the Laplace transform of $\delta(t-a)$, we write

$$f_k(t-a) = \frac{1}{k}[u(t-a) - u(t-(a+k))]$$

and take the transform see (2)

$$\mathcal{L}\, f_k(t-a) = \frac{1}{ks}[e^{-as} - e^{-(a+k)s}] = e^{-as}\frac{1-e^{-ks}}{ks}.$$

We now take the limit as $k \to 0$. By l'H pital's rule the quotient on the right has the limit 1 (differentiate the numerator and the denominator separately with respect to k, obtaining se^{-ks} and s, respectively, and use $se^{-ks}/s \to 1$ as $k \to 0$). Hence the right side has the limit e^{-as}. This suggests defining the transform of $\delta(t-a)$ by this limit, that is,

(5)
$$\mathcal{L}\,\delta(t-a) = e^{-as}.$$

Problem Set 2

Unit step function

Sketch the following functions and find their Laplace transforms (21-25):

$$\mathcal{L}\{f(t)\,u(t-a)\} = e^{-as}\,\mathcal{L}\{f(t+a)\}.$$

21. $f(t) = t\ u(t-1)$, **Ans:** $F(s) = \frac{e^{-s}}{s} + \frac{e^{-s}}{s^2}$.

22. $f(t) = (t-1)\,u(t-1)$, **Ans:** $F(s) = \frac{e^{-s}}{s^2}$.

23. $f(t) = (t-1)^2\,u(t-1)$, **Ans:** $F(s) = \frac{2\ e^{-s}}{s^3}$.

24. $f(t) = e^{-2t}\,u(t-3)$, **Ans**: $F(s) = \frac{e^{-6}\ e^{-3s}}{s+2}$.

25. $f(t) = 4\cos t\ u(t-\pi)$, **Ans:** $F(s) = -\frac{4s\ e^{-\pi s}}{s^2+1}$.

Sketch the following functions. Write $f(t)$ in terms of unit step function and hence find their Laplace transforms: (26-27)

26. $f(t) = \begin{cases} t\,; 0 < t < 1 \\ 2; \quad\quad t > 1 \end{cases}$

Ans: $f(t) = t\,u(t) + (2-t)\,u(t-1).$

$F(s) = \frac{1}{s^2} + e^{-s}\left(\frac{1}{s} - \frac{1}{s^2}\right).$

27. $f(t) = \begin{cases} t^2\,, & 0 \le t < 1 \\ t-3, & t \ge 1 \end{cases}$’

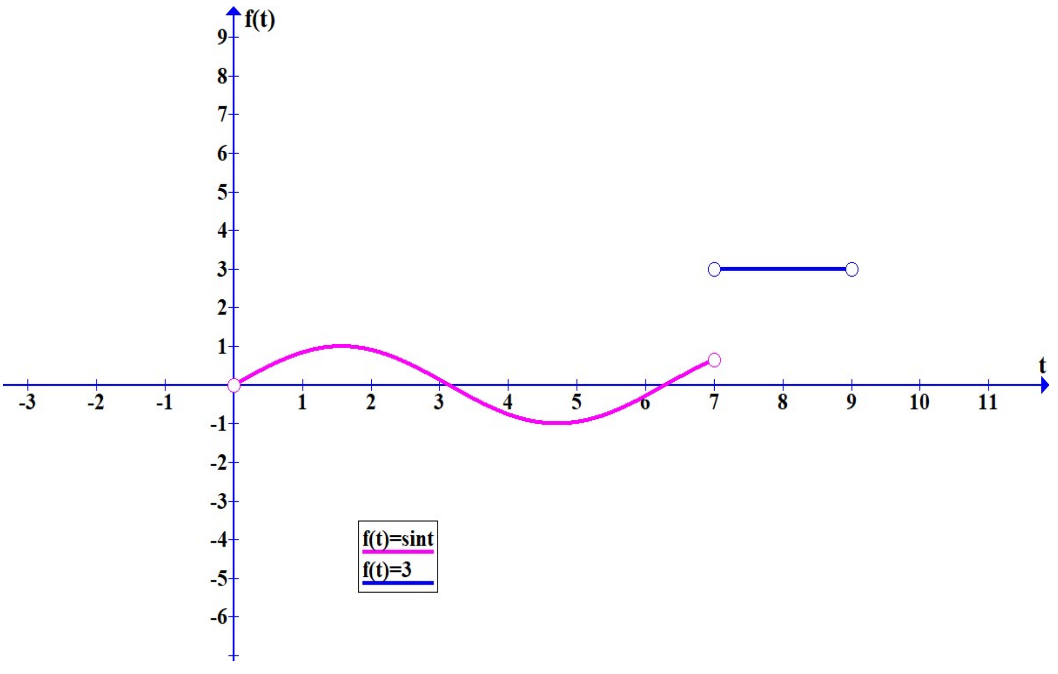
Ans: $f(t) = t^2\,u(t) + (t - 3 - t^2)\,u(t-1).$

$F(s) = \frac{2}{s^3} - e^{-s}\left[\frac{2}{s} - \frac{1}{s^2} + e^{s}\left(\frac{e^{-s}(s^2+2s+2)-2}{s^3} + \frac{2}{s^3}\right)\right] \rightarrow$ MATLAB answer.

$F(s) = \frac{2}{s^3} - e^{-s}\left(\frac{3}{s} + \frac{1}{s^2} + \frac{2}{s^3}\right) \rightarrow$ By hand calculation. But both expressions are same.

Using the following graphs write $f(t)$ in terms of unit step function and hence find the Laplace transforms of $f(t)$: (28-29)

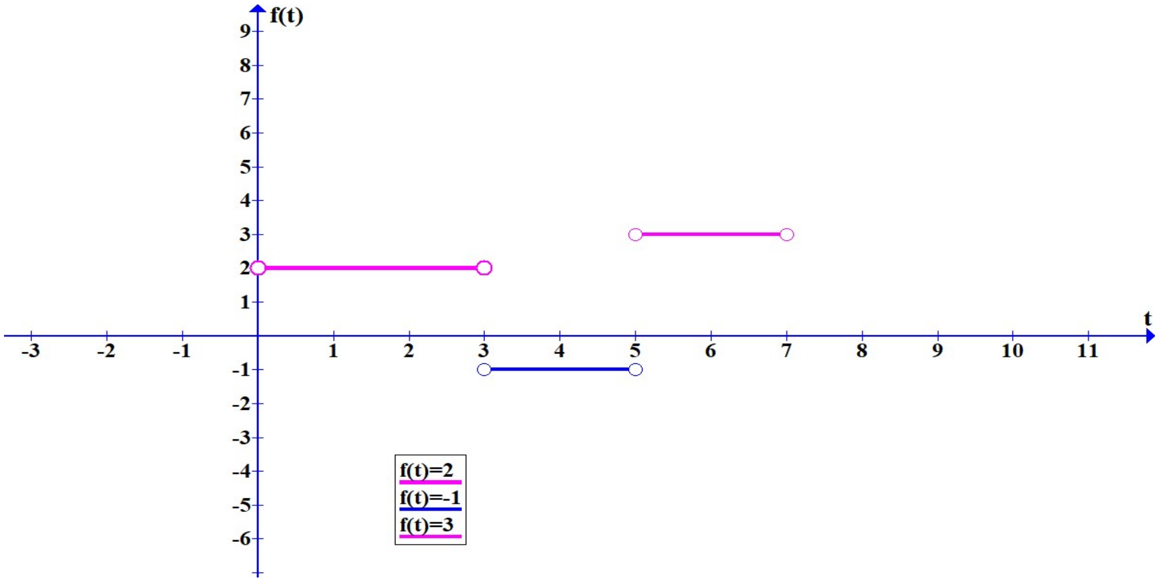
28.



Ans: $f(t) = \sin t\;\; u(t) + (3 - \sin t)\;\; u(t-7) - 3\,u(t-9)$.

$F(s) = e^{-7s}\;\left(\frac{3}{s} - \frac{\cos 7 + s\;\sin 7}{s^2+1}\right) - \frac{3\,e^{-9s}}{s} + \frac{1}{s^2+1}\,.$

29.



Ans: $f(t) = 2\,u(t) \; - \; 3\,u(t-3) \; + \; 4\; u(t-5) \; -3\; u(t-7)$.

$$F(s) = \frac{4\;e^{-5s}}{s} - \frac{3\;e^{-3s}}{s} - \frac{3\;e^{-7s}}{s} + \frac{2}{s}.$$

Dirac's delta (Unit impulse) function

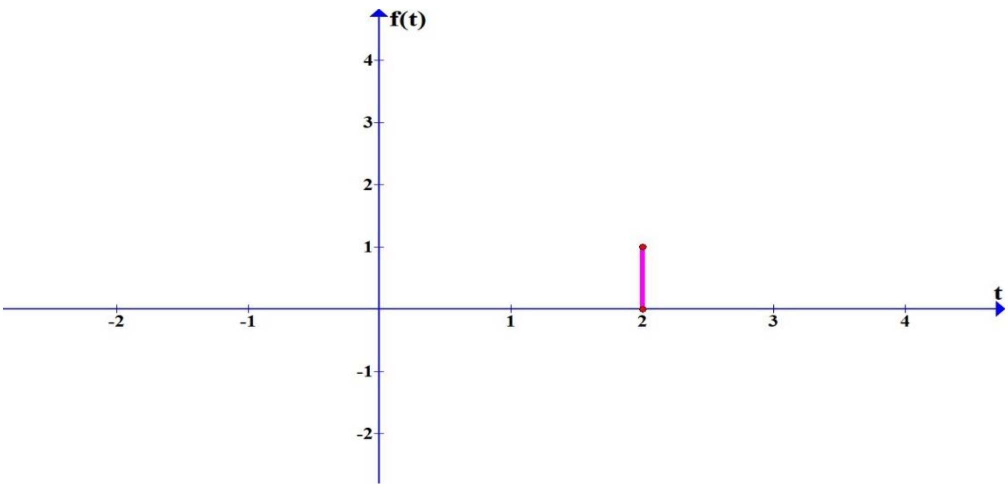
Sketch the following functions and find their Laplace transforms: (30-32)

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}.$$

$$\mathcal{L}\{f(t)\;\delta(t-a)\} = e^{-as}\,f(a).$$

30. $f(t) = \delta(t-2)$,

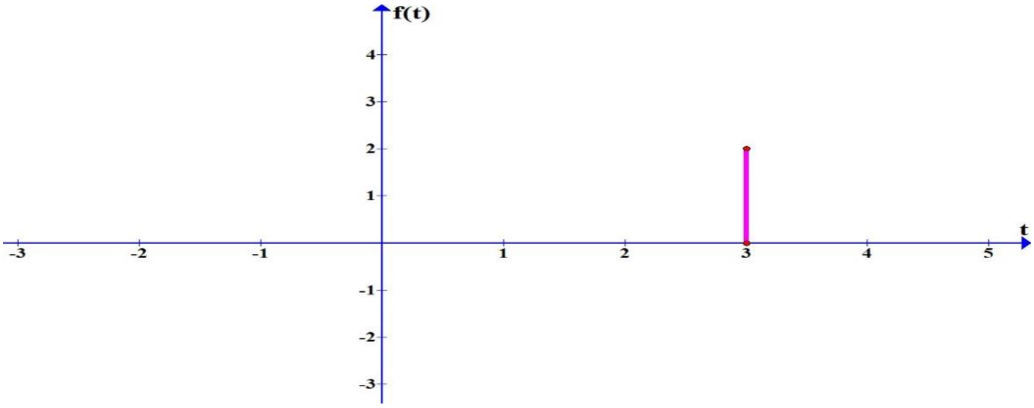
Ans:



$F(s) = e^{-2s}$.

31. $f(t) = 2\,\delta(t-3)$,

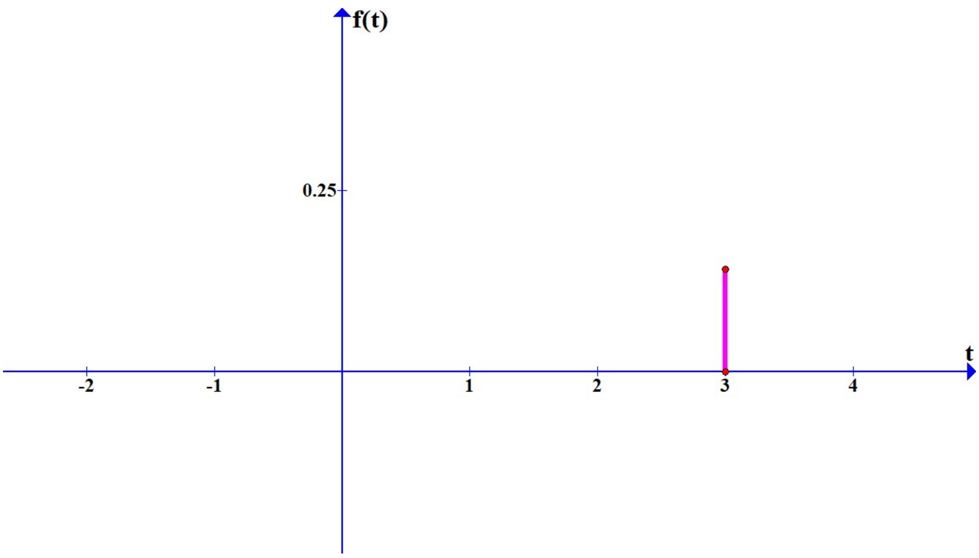
Ans:



$F(s) = 2\,e^{-3s}$.

32. $f(t) = \sin t\ \ \delta(t-3)$,

Ans:



$F(s) = \sin 3\ \ e^{-3s}$.