4. In Section 1.1, we guessed solutions to the exponential growth model $dP/dt = kP$, where k is a constant (see page 6). Using the fact that this equation is separable, derive these solutions by separating variables.

In Exercises 5–24, find the general solution of the differential equation specified. (You may not be able to reach the ideal answer of an equation with only the dependent variable on the left and only the independent variable on the right, but get as far as you can.)

5 - 24

5. $\dfrac{dy}{dt} = (ty)^2$

6. $\dfrac{dy}{dt} = t^4 y$

7. $\dfrac{dy}{dt} = 2y + 1$

8. $\dfrac{dy}{dt} = 2 - y$

9. $\dfrac{dy}{dt} = e^{-y}$

10. $\dfrac{dx}{dt} = 1 + x^2$

11. $\dfrac{dy}{dt} = 2ty^2 + 3y^2$

12. $\dfrac{dy}{dt} = \dfrac{t}{y}$

13. $\dfrac{dy}{dt} = \dfrac{t}{t^2 y + y}$

14. $\dfrac{dy}{dt} = t\sqrt[3]{y}$

15. $\dfrac{dy}{dt} = \dfrac{1}{2y+1}$

16. $\dfrac{dy}{dt} = \dfrac{2y+1}{t}$

17. $\dfrac{dy}{dt} = y(1-y)$

18. $\dfrac{dy}{dt} = \dfrac{4t}{1+3y^2}$

19. $\dfrac{dv}{dt} = t^2 v - 2 - 2v + t^2$

20. $\dfrac{dy}{dt} = \dfrac{1}{ty + t + y + 1}$

21. $\dfrac{dy}{dt} = \dfrac{e^t y}{1+y^2}$

22. $\dfrac{dy}{dt} = y^2 - 4$

23. $\dfrac{dw}{dt} = \dfrac{w}{t}$

24. $\dfrac{dy}{dx} = \sec y$

In Exercises 25–38, solve the given initial-value problem.

25 - 38

25. $\dfrac{dx}{dt} = -xt, \quad x(0) = 1/\sqrt{\pi}$

26. $\dfrac{dy}{dt} = ty, \quad y(0) = 3$

27. $\dfrac{dy}{dt} = -y^2, \quad y(0) = 1/2$

28. $\dfrac{dy}{dt} = t^2 y^3, \quad y(0) = -1$

29. $\dfrac{dy}{dt} = -y^2, \quad y(0) = 0$

30. $\dfrac{dy}{dt} = \dfrac{t}{y - t^2 y}, \quad y(0) = 4$

31. $\dfrac{dy}{dt} = 2y + 1, \quad y(0) = 3$

32. $\dfrac{dy}{dt} = ty^2 + 2y^2, \quad y(0) = 1$

33. $\dfrac{dx}{dt} = \dfrac{t^2}{x + t^3 x}, \quad x(0) = -2$

34. $\dfrac{dy}{dt} = \dfrac{1-y^2}{y}, \quad y(0) = -2$

35. $\dfrac{dy}{dt} = (y^2+1)t, \quad y(0) = 1$

36. $\dfrac{dy}{dt} = \dfrac{1}{2y+3}, \quad y(0) = 1$

37. $\dfrac{dy}{dt} = 2ty^2 + 3t^2 y^2, \quad y(1) = -1$

38. $\dfrac{dy}{dt} = \dfrac{y^2+5}{y}, \quad y(0) = -2$

EXERCISES FOR SECTION 1.9

1 - 6

In Exercises 1–6, find the general solution of the differential equation specified.

1. $\dfrac{dy}{dt} = -\dfrac{y}{t} + 2$

2. $\dfrac{dy}{dt} = \dfrac{3}{t}y + t^5$

3. $\dfrac{dy}{dt} = -\dfrac{y}{1+t} + t^2$

4. $\dfrac{dy}{dt} = -2ty + 4e^{-t^2}$

5. $\dfrac{dy}{dt} - \dfrac{2t}{1+t^2}\,y = 3$

6. $\dfrac{dy}{dt} - \dfrac{2}{t}\,y = t^3e^t$

7 - 12

In Exercises 7–12, solve the given initial-value problem.

7. $\dfrac{dy}{dt} = -\dfrac{y}{1+t} + 2, \quad y(0) = 3$

8. $\dfrac{dy}{dt} = \dfrac{1}{t+1}y + 4t^2 + 4t, \quad y(1) = 10$

9. $\dfrac{dy}{dt} = -\dfrac{y}{t} + 2, \quad y(1) = 3$

10. $\dfrac{dy}{dt} = -2ty + 4e^{-t^2}, \quad y(0) = 3$

11. $\dfrac{dy}{dt} - \dfrac{2y}{t} = 2t^2, \quad y(-2) = 4$

12. $\dfrac{dy}{dt} - \dfrac{3}{t}\,y = 2t^3e^{2t}, \quad y(1) = 0$

In Exercises 13–18, the differential equation is linear, and in theory, we can find its general solution using the method of integrating factors. However, since this method involves computing two integrals, in practice it is frequently impossible to reach a formula for the solution that is free of integrals. For these exercises, determine the general solution to the equation and express it with as few integrals as possible.

13. $\dfrac{dy}{dt} = (\sin t)y + 4$

14. $\dfrac{dy}{dt} = t^2y + 4$

15. $\dfrac{dy}{dt} = \dfrac{y}{t^2} + 4\cos t$

16. $\dfrac{dy}{dt} = y + 4\cos t^2$

17. $\dfrac{dy}{dt} = -\dfrac{y}{e^{t^2}} + \cos t$

18. $\dfrac{dy}{dt} = \dfrac{y}{\sqrt{t^3 - 3}} + t$

19. For what value(s) of the parameter a is it possible to find explicit formulas (without integrals) for the solutions to
$$\frac{dy}{dt} = aty + 4e^{-t^2}?$$

20. For what value(s) of the parameter r is it possible to find explicit formulas (without integrals) for the solutions to
$$\frac{dy}{dt} = t^r y + 4?$$

18. Suppose that $f(y)$ is a continuous function for all y. The phase line for $dy/dt = f(y)$ must have the same number of sources as sinks.

19. Suppose that $f(y)$ is continuously differentiable for all y. Exactly one solution of $dy/dt = f(y)$ tends to ∞ as t increases.

20. Every solution of $dy/dt = y + e^{-t}$ tends to $+\infty$ or $-\infty$ as $t \to \infty$.

In Exercises 21–29,

(a) specify if the given equation is autonomous, linear and homogeneous, linear and nonhomogeneous, and/or separable, and

(b) find its general solution.

21 - 29

21. $\dfrac{dy}{dt} = 3 - 2y$

22. $\dfrac{dy}{dt} = ty$

23. $\dfrac{dy}{dt} = 3y + e^{7t}$

24. $\dfrac{dy}{dt} = \dfrac{ty}{1+t^2}$

25. $\dfrac{dy}{dt} = -5y + \sin 3t$

26. $\dfrac{dy}{dt} = t + \dfrac{2y}{1+t}$

27. $\dfrac{dy}{dt} = 3 + y^2$

28. $\dfrac{dy}{dt} = 2y - y^2$

29. $\dfrac{dy}{dt} = -3y + e^{-2t} + t^2$

In Exercises 30–39,

(a) specify if the given equation is autonomous, linear and homogeneous, linear and nonhomogeneous, and/or separable, and

(b) solve the initial-value problem.

30. $\dfrac{dx}{dt} = -2tx, \quad x(0) = e$

31. $\dfrac{dy}{dt} = 2y + \cos 4t, \quad y(0) = 1$

32. $\dfrac{dy}{dt} = 3y + 2e^{3t}, \quad y(0) = -1$

33. $\dfrac{dy}{dt} = t^2y^3 + y^3, \quad y(0) = -1/2$

34. $\dfrac{dy}{dt} + 5y = 3e^{-5t}, \quad y(0) = -2$

35. $\dfrac{dy}{dt} = 2ty + 3te^{t^2}, \quad y(0) = 1$

36. $\dfrac{dy}{dt} = \dfrac{(t+1)^2}{(y+1)^2}, \quad y(0) = 0$

37. $\dfrac{dy}{dt} = 2ty^2 + 3t^2y^2, \quad y(1) = -1$

38. $\dfrac{dy}{dt} = 1 - y^2, \quad y(0) = 1$

39. $\dfrac{dy}{dt} = \dfrac{t^2}{y + t^3y}, \quad y(0) = -2$

40. Consider the initial-value problem $dy/dt = y^2 - 2y + 1, \quad y(0) = 2$.

(a) Using Euler's method with $\Delta t = 0.5$, graph an approximate solution over the interval $0 \le t \le 2$.

(b) What happens when you try to repeat part (a) with $\Delta t = 0.05$?

(c) Solve this initial-value problem by separating variables, and use the result to explain your observations in parts (a) and (b).

How General is this Method?

This guess-and-test method leads to many questions. For example, what initial-value problems can we solve using this approach? What happens if the roots of the resulting quadratic equation are complex numbers rather than real numbers? In Chapter 3, we study linear systems (including the damped harmonic oscillator) in great detail. We will see that this method can be generalized so that we can always find an analytic expression for the general solution of any damped harmonic oscillator equation.

EXERCISES FOR SECTION 2.3

In Exercises 1–4, a harmonic oscillator equation for $y(t)$ is given.

(a) Using `HPGSystemSolver`, sketch the associated direction field.

(b) Using the guess-and-test method described in this section, find two nonzero solutions that are not multiples of one another.

(c) For each solution, plot both its solution curve in the yv-plane and its $y(t)$- and $v(t)$-graphs.

1 - 4

1. $\dfrac{d^2y}{dt^2} + 7\dfrac{dy}{dt} + 10y = 0$

2. $\dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt} + 6y = 0$

3. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + y = 0$

4. $\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 7y = 0$

In the damped harmonic oscillator, we assume that the coefficients m, b, and k are positive. However, the rationale underlying the guess-and-test method made no such assumption, and the same analytic technique can be used if some or all of the coefficients of the equation are negative. In Exercises 5 and 6, make the same graphs and perform the same calculations as were specified in Exercises 1–4. What is different in this case?

5. $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} - 10y = 0$

6. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} - 2y = 0$

7. Consider any damped harmonic oscillator equation

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$

(a) Show that a *constant* multiple of any solution is another solution.

(b) Illustrate this fact using the equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

discussed in the section.

(c) How many solutions to the equation do you get if you use this observation along with the guess-and-test method described in this section?

8. Consider any damped harmonic oscillator equation

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$

(a) Show that the sum of any two solutions is another solution.

(b) Using the result of part (a), solve the initial-value problem

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0, \quad y(0) = 2, \quad v(0) = -3.$$

(c) Using the result of part (a) in Exercise 7 along with the result of part (a) of this exercise, solve the initial-value problem

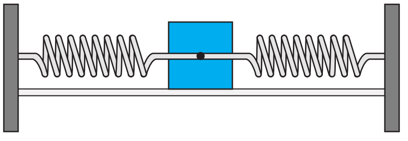
$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0, \quad y(0) = 3, \quad v(0) = -5.$$

(d) How many solutions to the equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

do you get if you use the results of Exercise 7 and this exercise along with the guess-and-test method described in this section?

In Exercises 9 and 10, we consider a mass sliding on a frictionless table between two walls that are 1 unit apart and connected to both walls with springs, as shown below.



Let k_1 and k_2 be the spring constants of the left and right spring, respectively, let m be the mass, and let b be the damping coefficient of the medium the spring is sliding through. Suppose L_1 and L_2 are the rest lengths of the left and right springs, respectively.

9. Write a second-order differential equation for the position of the mass at time t. [*Hint*: The first step is to pick an origin, that is, a point where the position is 0. The left-hand wall is a natural choice.]

10. **(a)** Convert the second-order equation of Exercise 9 into a first-order system.

(b) Find the equilibrium point of this system.

(c) Using your result from part (b), pick a new coordinate system and rewrite the system in terms of this new coordinate system.

(d) How does this new system compare to the system for a damped harmonic oscillator?

- If $b = 0$, the oscillator is undamped, and the equilibrium point at the origin in the phase plane is a center. All solutions are periodic, and the mass oscillates forever about its rest position. The (natural) period of the oscillations is $2\pi\sqrt{m/k}$.
- If $b > 0$ and $b^2 - 4km < 0$, the oscillator is underdamped. The origin in the phase plane is a spiral sink, and all other solutions spiral toward the origin. The mass oscillates back and forth as it tends to its rest position with period $4m\pi/\sqrt{4km - b^2}$.
- If $b > 0$ and $b^2 - 4km > 0$, the oscillator is overdamped. The origin in the phase plane is a real sink with two distinct eigenvalues. The mass tends to its rest position but does not oscillate.
- If $b > 0$ and $b^2 - 4km = 0$, the oscillator is critically damped. The system has exactly one eigenvalue, which is negative. All solutions tend to the origin tangent to the unique line of eigenvectors. As in the overdamped case, the mass tends to its rest position but does not oscillate.

The four cases just described completely classify the various long-term behaviors of all harmonic oscillators. In the next section we will derive a geometric way to classify these behaviors.

EXERCISES FOR SECTION 3.6

In Exercises 1–6, find the general solution (in scalar form) of the given second-order equation.

1- 6

1. $\dfrac{d^2y}{dt^2} - 6\dfrac{dy}{dt} - 7y = 0$

2. $\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} - 12y = 0$

3. $\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 9y = 0$

4. $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 4y = 0$

5. $\dfrac{d^2y}{dt^2} + 8\dfrac{dy}{dt} + 25y = 0$

6. $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 29y = 0$

In Exercises 7–12, find the solution of the given initial-value problem.

7,8, 11, 12

7. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} - 3y = 0$
$y(0) = 6,\ y'(0) = -2$

8. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} - 5y = 0$
$y(0) = 11,\ y'(0) = -7$

9. $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 13y = 0$
$y(0) = 1,\ y'(0) = -4$

10. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 20y = 0$
$y(0) = 2,\ y'(0) = -8$

11. $\dfrac{d^2y}{dt^2} - 8\dfrac{dy}{dt} + 16y = 0$
$y(0) = 3,\ y'(0) = 11$

12. $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 4y = 0$
$y(0) = 1,\ y'(0) = 1$

We can now form the general solution

$$v(t) = k_1 e^{-1250t} \sin(968t) + k_2 e^{-1250t} \cos(968t) + 2e^{-t/3},$$

where k_1 and k_2 are constants that we adjust according to the initial conditions. The first two terms of the general solution oscillate very quickly and decrease very quickly to zero. Hence for large t, every solution is close to the particular solution. The graph of the solution that satisfies the initial condition $(v(0), v'(0)) = (1, 0)$ is shown in Figure 4.7. To see the oscillations, we must look very close to $t = 0$ (see Figure 4.8). In the language of circuit theory we say that the natural response (the solution of the homogeneous equation) decays quickly and every solution approaches the forced response, which is $2e^{-t/3}$.

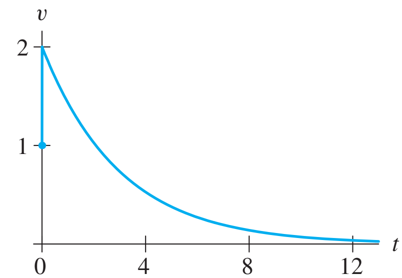


Figure 4.7
Solution of

$$\frac{d^2v}{dt^2} + 2500\frac{dv}{dt} + 2.5\times10^6 = 5\times10^6 e^{-t/3}$$

with initial condition $v(0) = 1$, $v'(0) = 0$.

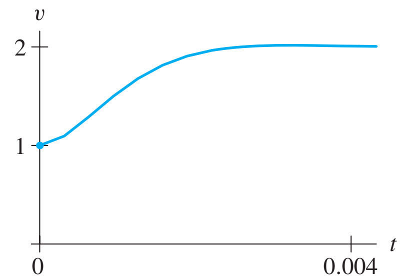


Figure 4.8
The same solution that is graphed in Figure 4.7. Here we show the graph only for values of t that are very close to 0 in order to illustrate the rapid increase from the initial value $v(0) = 1$.

EXERCISES FOR SECTION 4.1

In Exercises 1–8, find the general solution of the given differential equation.

1 - 8

1. $\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} - 6y = e^{4t}$

2. $\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 8y = 2e^{-3t}$

3. $\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} - 2y = 5e^{3t}$

4. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 13y = e^{-t}$

5. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 13y = -3e^{-2t}$

6. $\dfrac{d^2y}{dt^2} + 7\dfrac{dy}{dt} + 10y = e^{-2t}$

7. $\dfrac{d^2y}{dt^2} - 5\dfrac{dy}{dt} + 4y = e^{4t}$

8. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} - 6y = 4e^{-3t}$

9 - 12

In Exercises 9–12, find the solution of the given initial-value problem.

9. $\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 8y = e^{-t}$,
$y(0) = y'(0) = 0$

10. $\dfrac{d^2y}{dt^2} + 7\dfrac{dy}{dt} + 12y = 3e^{-t}$,
$y(0) = 2$, $y'(0) = 1$

11. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 13y = -3e^{-2t}$,
$y(0) = y'(0) = 0$

12. $\dfrac{d^2y}{dt^2} + 7\dfrac{dy}{dt} + 10y = e^{-2t}$,
$y(0) = y'(0) = 0$

In Exercises 13–18,

(a) compute the general solution,

(b) compute the solution with $y(0) = y'(0) = 0$, and

(c) describe the long-term behavior of solutions in a brief paragraph.

13. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 3y = e^{-t/2}$

14. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 3y = e^{-2t}$

15. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 3y = e^{-4t}$

16. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 20y = e^{-t/2}$

17. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 20y = e^{-2t}$

18. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 20y = e^{-4t}$

19. Find the general solution of

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t}.$$

[*Hint*: Just keep guessing.]

20. One of the most common forcing functions is constant forcing. The equation for a harmonic oscillator with constant forcing is

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = c,$$

where p, q, and c are constants. Since we can find the general solution of the unforced equation by our usual methods, we can find the general solution of the forced equation if we can find one particular solution. Find one particular solution of the equation

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = c.$$

[*Hint*: Guess a solution that looks like the forcing function.]

In Exercises 32–35,

(a) compute the general solution,

(b) compute the solution with $y(0) = y'(0) = 0$, and

(c) give a rough sketch and describe in a brief paragraph the long-term behavior of the solution in part (b).

32 - 35

32. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} = 3t + 2$

33. $\dfrac{d^2y}{dt^2} + 4y = 3t + 2$

34. $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = t^2$

35. $\dfrac{d^2y}{dt^2} + 4y = t - \dfrac{t^2}{20}$

36. We can extend the Method of Undetermined Coefficients in order to solve equations whose forcing functions are sums of several types of functions. More precisely, suppose that $y_1(t)$ is a solution of the equation

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t)$$

and that $y_2(t)$ is a solution of the equation

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = h(t).$$

Show that $y_1(t) + y_2(t)$ is a solution of the equation

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t) + h(t).$$

In Exercises 37–42,

(a) compute the general solution,

(b) compute the solution with $y(0) = y'(0) = 0$, and

(c) describe in a brief paragraph the long-term behavior of the solution in part (b).

37 - 42

37. $\dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt} + 6y = e^{-t} + 4$

38. $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = e^{-t} - 4$

39. $\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 8y = 2t + e^{-t}$

40. $\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 8y = 2t + e^{t}$

41. $\dfrac{d^2y}{dt^2} + 4y = t + e^{-t}$

42. $\dfrac{d^2y}{dt^2} + 4y = 6 + t^2 + e^{t}$

EXERCISES FOR SECTION 4.2

1 - 6
10,11,
12

In Exercises 1–10, find the general solution of the given equation.

1. $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = \cos t$

2. $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = 5\cos t$

3. $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = \sin t$

4. $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y = 2\sin t$

5. $\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 8y = \cos t$

6. $\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 8y = -4\cos 3t$

7. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 13y = 3\cos 2t$

8. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 20y = -\cos 5t$

9. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 20y = -3\sin 2t$

10. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = \cos 3t$

In Exercises 11–14, find the solution of the given initial-value problem.

11. $\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 8y = \cos t$
$y(0) = y'(0) = 0$

12. $\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 8y = 2\cos 3t$
$y(0) = y'(0) = 0$

13. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 20y = -3\sin 2t$
$y(0) = y'(0) = 0$

14. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = 2\cos 2t$
$y(0) = y'(0) = 0$

15. To find a particular solution of a forced equation with sine or cosine forcing, we solved the corresponding complexified equation. This method is particularly efficient, but there are other approaches. Find a particular solution via the Method of Undetermined Coefficients for the equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \cos 3t$$

(a) by using the guess $y_p(t) = a\cos 3t + b\sin 3t$ (with a and b as undetermined coefficients) and

(b) by using the guess $y_p(t) = A\cos(3t + \phi)$ (with A and ϕ as undetermined coefficients).

16. **(a)** Show that if $y_p(t)$ is a solution of

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t),$$

then $ky_p(t)$ is a solution of

Subsequent studies by the designers of the bridge revealed two important facts. First, the bridge has a large amplitude response to horizontal forcing at a frequency of one cycle per second (one Hz). Since people typically walk at a rate of two steps per second, the time between successive steps with the left foot is about one second. Of course, the right foot also pushes with the same frequency, so we should wonder about resonant forcing. Perhaps the slight forces that are caused by walking could provide a periodic forcing at precisely the resonant frequency.

Even if it had occurred to someone to consider this forcing term on the dynamics of the bridge, it probably would have been quickly dismissed as unimportant. The forces caused by a single pedestrian are tiny compared to the other horizontal forces on the bridge. While a large number of people marching together could provide a significant forcing term, tourists don't generally march in time.

A video of pedestrians crossing the bridge led to another surprising observation. They were, in fact, marching in time. In order to keep their balance on the bridge which was oscillating slightly left and right, they tended to time their steps with the oscillations. The result was a significant amount of forcing at the resonant frequency, which caused a larger forced response.

While the Millennium Bridge is a light and flexible structure, it is not entirely without damping. In the next section, we consider the interplay between the damping coefficient and the frequency of the forcing. We will see that the amplitude of the forced response does not tend to infinity if damping is present. However, it can be large if damping is small.

The analysis above is based on a purely linear model for the dynamics of the bridge. We must also remind ourselves that the linear model is only accurate in a certain regime. Outside this regime, when the loads are too great or the initial conditions too large, we must work with more accurate nonlinear models. We will review this lesson when we consider the Tacoma Narrows Bridge (see Section 4.5).

Moral

The key feature of resonance is that the amplitude of the forced response is very large *if the frequency of the forcing is very close to the natural frequency*. Resonance can be dangerous, as in the case of the Millennium Bridge. On the other hand, being able to selectively amplify a particular forcing frequency can be extremely useful. Amplifying a particular frequency of a radio signal is perhaps the most common example of the usefulness of resonance. In either case, it is clear we must be aware of the subtle dependence of the long-term behavior of a forced system on the specific values of the parameters.

EXERCISES FOR SECTION 4.3

1 - 4

In Exercises 1–8, compute the general solution of the given equation.

1. $\dfrac{d^2y}{dt^2} + 9y = \cos t$

2. $\dfrac{d^2y}{dt^2} + 9y = 5\sin 2t$

3. $\dfrac{d^2y}{dt^2} + 4y = -\cos\dfrac{t}{2}$

4. $\dfrac{d^2y}{dt^2} + 4y = 3\cos 2t$

5. $\dfrac{d^2y}{dt^2} + 9y = 2\cos 3t$

6. $\dfrac{d^2y}{dt^2} + 3y = 2\cos 9t$

7. $\dfrac{d^2y}{dt^2} + 3y = \cos 3t$

8. $\dfrac{d^2y}{dt^2} + 5y = 5\sin 5t$

In Exercises 9–14, compute the solution of the given initial-value problem.

9 - 14

9. $\dfrac{d^2y}{dt^2} + 9y = \cos t$
$y(0) = y'(0) = 0$

10. $\dfrac{d^2y}{dt^2} + 4y = 3\cos 2t$
$y(0) = y'(0) = 0$

11. $\dfrac{d^2y}{dt^2} + 5y = 3\cos 2t$
$y(0) = y'(0) = 0$

12. $\dfrac{d^2y}{dt^2} + 9y = \sin 3t$
$y(0) = 1,\ y'(0) = -1$

13. $\dfrac{d^2y}{dt^2} + 9y = 2\cos 3t$
$y(0) = 2,\ y'(0) = -9$

14. $\dfrac{d^2y}{dt^2} + 4y = \sin 3t$
$y(0) = 2,\ y'(0) = 0$

In Exercises 15–18, for the equation specified,

(a) determine the frequency of the beats,

(b) determine the frequency of the rapid oscillations, and

(c) use the information from parts (a) and (b) to give a rough sketch of a typical solution. [*Hint*: You should be able to do this sketch with no further calculation.]

15. $\dfrac{d^2y}{dt^2} + 4y = \cos\dfrac{9t}{4}$

16. $\dfrac{d^2y}{dt^2} + 11y = 2\cos 3t$

17. $\dfrac{d^2y}{dt^2} + 5y = 3\cos 2t$

18. $\dfrac{d^2y}{dt^2} + 6y = \cos 2t$

19. Consider the equation

$$\frac{d^2y}{dt^2} + 12y = 3\cos 4t + 2\sin t.$$

(a) Compute the general solution. [*Hint*: See Exercise 36 in Section 4.1.]

(b) Compute the solution with $y(0) = y'(0) = 0$.

(c) Sketch the graph of the solution of part (b).

(d) In a short essay, describe the behavior of the solution of part (b). How could you have predicted this behavior in advance?