Lecture-01

Complex Numbers

A complex number is the form $a+ib$ where a and b are real numbers and i is called the imaginary unit, has the property that $i^2=-1$ or $i=\sqrt{-1}$. In general, if c is any positive number, we would write:

$$\sqrt{-c}=i\sqrt{c}$$

If $z=a+ib$, then a is called the real part of z and b is called the imaginary part of z and are denoted by $\mathrm{Re}\{z\}=a$ and $\mathrm{Im}\{z\}=b$ respectively. From this, it is obvious that two complex numbers $a+ib$ and $c+id$ are equal if and only if $a=c$ and $b=d$, that is, the real and imaginary components are equal. If $a=0$ the number $z=ib$ is said to be purely imaginary, if $b=0$ the number $z=a$ is real.

The standard rectangular form of a complex number is $z=a+ib$. The symbol z, which can stand for any of complex numbers, is called a complex variable.

Graphical Representation of Complex Number/ Argand Diagram:

Since a complex number $a+ib$ can be considered as an ordered pair of real numbers, we can represent such numbers by points in a xy plane called the **complex plane or Argand diagram**. Mathematician Argand represented a complex number in a diagram known as **Argand diagram**.

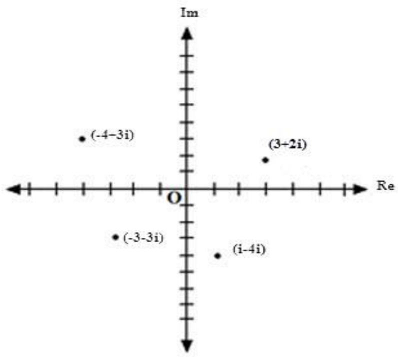

Figure:1

A complex number $a+ib$ can be represented by a point P whose co-ordinates are (a,b). The horizontal axis is called the **real axis** and the vertical axis is called the **imaginary** axis.

To each complex number there corresponds one and only one point in the plane, and conversely to each point in the plane there corresponds one and only one complex number. Because of this we often refer to the complex number z as the point. This is shown in **Figure: 1.**

Fundamental operations with complex number:

Addition and Subtraction:

The **sum** and **difference** of complex numbers is defined by adding or subtracting their real components where $a, b \in \mathbb{R}$ i.e.:

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

$$(a-bi)+(c-di)=(a-c)+(b-d)i$$

For instance: $(3+i)+(1-7i)=(3+1)+(1-7)\,i=(4-6i)$ (similar to vector addition)

Product:

The commutative and distributive properties hold for the **product** of complex numbers i.e.,

$$(a+bi)(c+di)=a(c+di)+bi(c+di)=ac+adi+bci+bdi^2$$

We know: $i^2=-1$

Therefore giving us: $(a+bi)(c+di)=(ac-bd)+i\,(ad+bc)$.

Division:

$$\frac{(a+bi)}{(c+di)}=\frac{(a+bi)}{(c+di)}\cdot\frac{(c-di)}{(c-di)}=\frac{(ac+bd)}{(c^2+d^2)}+i\,\frac{(bc-ad)}{(c^2+d^2)}.$$

Basically, when **dividing** two complex numbers we are rationalizing the denominator of a rational expression multiplying the numerator and denominator by the conjugate of the denominator.

Example: Express $\frac{-3+\mathrm{i}}{7-3\mathrm{i}}$ in the form $a+ib$.

Solution: We must multiply the numerator and denominator by the conjugate of $7-3i$ i.e., $7+3i$.

$$\frac{-3+\mathrm{i}}{7-3\mathrm{i}}=\frac{-3+\mathrm{i}}{7-3\mathrm{i}}\cdot\frac{7+3\mathrm{i}}{7+3\mathrm{i}}=\frac{(-21-3)}{(49+9)}+i\,\frac{(7-9)}{(49+9)}=-\frac{12}{29}-i\,\frac{1}{29}.$$

Conjugates

The complex conjugate, or briefly conjugate, of a complex number $z=a+ib$ is $\bar{z}=a-ib$. The complex conjugate of a complex number z is often indicated by $\bar{z}$. The geometric interpretation of a complex conjugate is the reflection along the real axis. This can be seen in the **Figure:2** below where $z=a+ib$ is a complex number. Listed below are also several properties of conjugates.

Some Properties:

1) $\bar{\bar{z}} = z$
2) $\overline{z+w} = \bar{z} + \bar{w}$
3) $\overline{zw} = \bar{z}\bar{w}$
4) $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}, w \neq 0.$

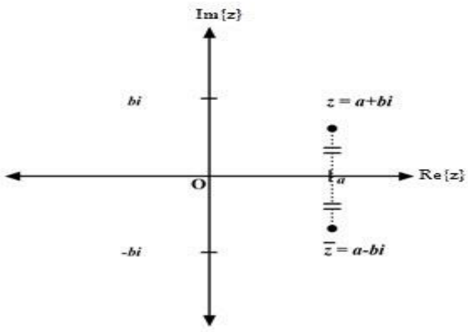

Figure:2

Absolute value/Modulus

The distance from the origin to any complex number is the **absolute value** or **modulus.** Looking at the **Figure: 3** below we can see that Pythagoras' Theorem gives us a formula to calculate the absolute value of a complex number $z = a + bi$ denoted by mod z or $|z|$.

i.e. mod $z = |z| = \sqrt{a^2 + b^2}$

Example:

If $z_1 = 1 - i$, $z_2 = -2 - 3i$ and $z_3 = 2i$, then evaluate $|3{z_1}^3 + 2\bar{z_2} - 5z_3|$.

Solution:

Given, $z_1 = 1 - i$, $z_2 = -2 - 3i$, $z_3 = 2i$

$$\begin{aligned}
\therefore |3{z_1}^3 + 2\bar{z_2} - 5z_3| &= |3(1-i)^3 + 2(-2+3i) - 5(2i)| \\
&= |(3-9-4) + i(-9+3+6-10)| \\
&= |(-10) + i(-10)| \\
&= \sqrt{(-10)^2 + (-10)^2} \\
&= 10\sqrt{2}.
\end{aligned}$$

Powers of imaginary unit i

Power of imaginary unit i are given below:

$i^0 = 1; i^1 = i;\ i^{-2} = -1;\ i^3 = i^2.i = -i;$

$i^4 = i^3.i = 1;\ i^5 = i^4.i = i;\ i^6 = i^5.i = -1;\ i^7 = i^6.i = -i$

One can prove by induction that for any positive integer n

$i^{4n} = 1; i^{4n+1} = i; i^{4n+2} = -1; i^{4n+3} = -i$

Hence $i^n \in \{-1, 1, -i, i\}$ for all integer $n \geq 0$. If n is a negative integer, we have

$$i^n = \left(i^{-1}\right)^{-n} = \left(\frac{1}{i}\right)^{-n} = (-i)^{-n}$$

Example: $i^{105} + i^{23} + i^{20} - i^{34} = i^{4\cdot 26+1} + i^{4\cdot 5+3} + i^{4\cdot 5} - i^{4\cdot 8+2} = i - i + 1 + 1 = 2$

MATLAB command for complex numbers

If $z_1 = 1 - i, z_2 = -2 + 4i, z_3 = \sqrt{3} - 2i,$ evaluate by MATLAB commands

(i) $z_1^{\,2} + 2z_1 - 3$ (ii) $|2z_2 - 3z_1|$ (iii) $\left|z_1\overline{z_2} - z_2\overline{z_1}\right|$ (iv) $\mathrm{Re}\left\{z_1^{\,3} + 3z_1^{\,3}\right\}$ (v) $\mathrm{Im}\left\{\dfrac{z_1^{\,2}}{z_3}\right\}$

>> clear **>> z1=complex(1,-1)** **>>z1 =1.0000 - 1.0000i** **>> z2=complex(-2, 4);** **>> z3=complex(sqrt(3),-2);** **>> e1=z1^2+2*z1-3** **e1 = -1.0000 - 4.0000i** **>> e2=abs(2*z2-3*z1)** **e2 = 13.0384** **>> e3=abs(z1*conj(z2)+z2*conj(z1))** **e3 = 12** **>> e4=real(2*z1^3+3*z2^2)** **e4 = -40** **>> e5=imag(z1^2/z3)** **e5 = -0.4949**	**>> clear** **>> z1=1-i;** **>> z2=-2+4*i;** **>> z3=sqrt(3)-2*i;** **>> e1=z1^2+2*z1-3** **e1 = -1.0000 - 4.0000i** **>> e2=abs(2*z2-3*z1)** **e2 = 13.0384** **>> e3=abs(z1*conj(z2)+z2*conj(z1))** **e3 = 12** **>> e4=real(2*z1^3+3*z2^2)** **e4 = -40** **>> e5=imag(z1^2/z3)** **e5 = -0.4949**

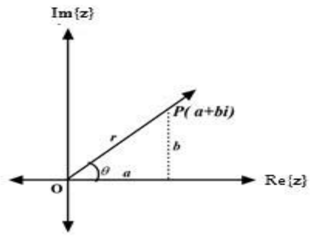

Figure:3

Polar form of Complex Number and Argument:

It is often useful to exchange Cartesian co-ordinates (x, y) to polar coordinates (r, θ).

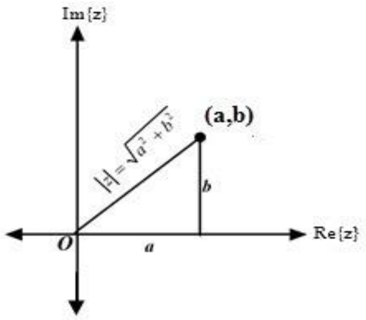

Figure: 4

If P is a point in the complex plane corresponding to the complex number (a, b) or $a + ib$, then we see from **Figure: 4** that

$$a = r\cos\theta,\ b = r\sin\theta \qquad (1)$$

where, $r = \sqrt{a^2 + b^2}$ is the distance of the point (a,b) from the origin and called the **modulus or absolute value** of $z = a + ib$, denoted by mod z or $|z|$.
On the other hand, if $z \neq 0$, then any number θ satisfying the equations (1) is called an **amplitude or argument** of z, and denoted by $arg\ z$. Hence, we can write z in polar form as

$$z = r(\cos\theta + i\sin\theta)$$

Note that, for a given complex number z, $arg\ z$ is not unique. Since adding or subtracting multiples of 2π from θ will result in the arm in Fig. 01 being in the same position, the argument can have many values.
Any particular choice of length 2π , decided upon in advance, is called the **principle range** and the value of θ is called its principle value.

In our study, we will consider the real number θ the principal argument of z if θ satisfies the equations $0 \le \theta < 2\pi$ and $-\pi < \theta \le \pi$. **The principal argument of z is usually denoted by Arg z.**
Generally, we use the formula for argument is

$$\arg z = \theta = \text{Arg } z + 2n\pi, \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

where $Arg\ z$ may be calculated by the following formula where the quadrant containing the point corresponding to z must be specified,

$$\text{Arg } z = \tan^{-1}\left(\frac{b}{a}\right).$$

N.B. Angles measured in an anticlockwise sense are regarded as positive while those measured in a clockwise sense are regarded as negative.
It follows that

$z = a + ib = r(\cos\theta + i\sin\theta)$, which is called the polar form of the complex number, and r and θ are called the polar coordinates.

From the Euler's formula we know, $e^{i\theta} = \cos\theta + i\sin\theta$.

Hence, the standard polar form of a complex number is $z = re^{i\theta}$.

Some important properties of modulus:

$$|z_1.z_2| = |z_1||z_2|$$

$$|z|^2 = z.\bar{z}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

Some important properties of argument

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

$$\arg(z^n) = n\arg(z).$$

Example:

Find $\operatorname{Im}\{z\}$, where $z = \frac{(1-i)^2}{1+i}$. Hence convert the number from rectangular form to polar form.

Solution:

Given, $z = \frac{(1-i)^2}{1+i} = \frac{(1-i)^2\,(1-i)}{(1+i)\,(1-i)} = \frac{(1-2i-1)(1-i)}{1-i^2}$

$= \frac{-2i+2i^2}{1+1} = -1 - i.$

$\therefore$ $\operatorname{Im}\{z\} = -1$ and $\operatorname{Re}\{z\} = -1$.

Now, $r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$.

$\alpha = \tan^{-1}\frac{1}{1} = \tan^{-1} 1 = \frac{\pi}{4}$

$\theta = \pi + \alpha = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$.

Therefore, the polar form of $z = \sqrt{2}\, e^{\frac{5\pi}{4} i}$.

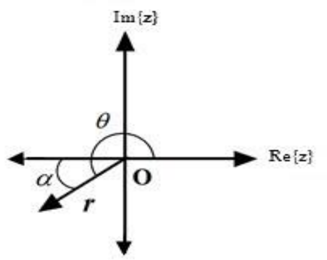

Figure: 5

Example:

Find the rectangular form of $z = \sqrt{2}e^{i\frac{\pi}{4}}$.

Solution:

Given, $z = \sqrt{2}e^{i\frac{\pi}{4}}$

Here $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$

We know that, $a = r\cos\theta = \sqrt{2}\cos\frac{\pi}{4} = 1$

And $b = r\sin\theta = \sqrt{2}\sin\frac{\pi}{4} = 1$

Hence $z = a + ib = 1 + i$.

MATLAB command for complex numbers

Express the following complex numbers in polar form and locate the points in complex plane: $2+2\sqrt{3}\,i$, $5-5\,i$, $-\sqrt{6}+\sqrt{2}\,i$, $-3\,i$, $1+i$, $1-i$, $-1-i$, $4i$, $-i\sqrt{3}$ **>> z1=2+2*sqrt(3)*i;** **>> R=abs(z1)** **R = 4.0000** **>> theta=angle(sym(z1))** **theta = pi/3** **>> z2=R*exp(i*theta)** **z2 =2 + 3^(1/2)*2i** **>> plot(z1, '*')**	Find rectangular form of $z = \sqrt{2}\, e^{\frac{i\pi}{4}}$ **>> clear** **>> z1=sqrt(2)*exp(i*pi/4)** **z1 = 1.0000 + 1.0000i**

Example:

Find the principal argument of $z = \frac{(-1-i)^{12}}{(3+\sqrt{3}\, i)^6\, (-2+2\sqrt{3}\, i)^3}$.

Solution:

Let, $z = \frac{{z_1}^{12}}{{z_2}^6\, {z_3}^3}$, where

$z_1 = -1 - i = \sqrt{2}\ e^{i\frac{5\pi}{4}}$ $\quad [r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$ and $\theta = \pi + \alpha = \pi + \tan^{-1}\frac{1}{1} = \pi + \frac{\pi}{4} = \frac{5\,\pi}{4}]$

$z_2 = 3 + \sqrt{3}\, i = 2\sqrt{3}\ e^{i\frac{\pi}{6}}$ $\quad [r = \sqrt{12} = 2\sqrt{3}$ and $\theta = \tan^{-1}\frac{\sqrt{3}}{3} = \frac{\pi}{6}]$

$z_3 = -2 + 2\sqrt{3}\, i = 2^2\ e^{i\frac{2\pi}{3}}$ $\quad [r = \sqrt{(-2)^2 + \left(2\sqrt{3}\right)^2} = 2^2$ and $\theta = \pi - \alpha = \pi - \frac{\pi}{3} = \frac{2\,\pi}{3}]$

Therefore, $z = \frac{\left(\sqrt{2}\ e^{i\frac{5\pi}{4}}\right)^{12}}{\left(2\sqrt{3}\ e^{i\frac{\pi}{6}}\right)^6 \left(2^2\ e^{i\frac{2\pi}{3}}\right)^3} = \frac{2^6}{2^6.3^3.2^6}\quad e^{i\,(15\pi-\pi-2\pi)} = \frac{1}{3^3.2^6}\quad e^{i\,12\pi}$.

$\arg z = 12\pi\ = 2\pi +\ 10\pi$

$\operatorname{Arg} z = 2\pi$ or $\operatorname{Arg} z = 0$

Example: Find the Principal argument of $z_1 = 2e^{-i\frac{\pi}{6}}$ and $z_2 = 15e^{i\frac{15\pi}{2}}$

Solution:

Given, $\arg z_1 = -\frac{\pi}{6}$ or, $-\frac{\pi}{6} + 2\pi = \frac{11\pi}{6}$

$\operatorname{Arg} z_1 = \frac{11\pi}{6}$.

$$\arg z_2 = \frac{15\pi}{2} = \frac{3\pi}{2} + 6\pi$$

$\operatorname{Arg} z_2 = \frac{3\pi}{2}$.

De`Moivre's Theorem

De`Moivre's Theorem is a generalized formula to compute powers of a complex number in it's polar form.

Looking at $z = r(\cos\theta + i\sin\theta)$ we can find $(z).(z)$ easily:

$$z^2 = (z).(z) = r^2(\cos 2\theta + i\sin 2\theta)$$

$$z^3 = (z^2).(z) = r^3(\cos 3\theta + i\sin 3\theta)$$

Which brings us to **De`Moivre's Theorem**:

If $z = r(\cos\theta + i\sin\theta)$ and n are positive integers then

$z^n = r^n(\cos n\theta + i\sin n\theta) = \left(re^{i\theta}\right)^n = r^n\ e^{in\theta}$

Basically, in order to find the nth power of a complex number we take the nth power of the absolute value or length and multiply the argument by n.

For finding m-th roots of a complex number:

$$z^m = (cos\theta + isin\theta)^m = \cos(m\theta) + i\sin(m\theta)$$

We can also write, $cos\theta + isin\theta = \cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)$,

$$z^m = (cos\theta + isin\theta)^m = [\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)]^m$$
$$= \cos\ \{\mathrm{m}(\theta + 2n\pi)\} + i\sin\ \{\mathrm{m}(\theta + 2n\pi)\}$$

Similarly,

$$z^{\frac{1}{m}} = (\mathrm{cis}\theta)^{\frac{1}{m}} = (cos\theta + i\sin\theta)^{\frac{1}{m}} = [\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)]^{\frac{1}{m}}$$
$$= \left[\cos\left(\frac{\theta + 2n\pi}{m}\right) + i\sin\left(\frac{\theta + 2n\pi}{m}\right)\right]$$

where, $n = 0, 1, 2, \ldots, m - 1$

Euler's Form:

We can also define this formula according to Euler's formulae as

$$z^m = r^m e^{m\theta i}$$
$$z^{\frac{1}{m}} = r^{\frac{1}{m}} e^{\frac{\theta}{m} i}.$$

Example: Find all values of z for which $z^3 + 2 - i2\sqrt{3} = 0$ and also locate these values in the complex plane.

Solution: Given, $z^3 + 2 - i2\sqrt{3} = 0$.

Here the numbers of roots are 3.

$z^3 + 2 - i2\sqrt{3} = 0$

or, $z = \left(-2 + i2\sqrt{3}\right)^{\frac{1}{3}}$

or, $z = \left(2^2\ e^{i\frac{2\pi}{3}}\right)^{\frac{1}{3}}$

or, $z = \left(2^2\ e^{i\left(\frac{2\pi}{3} + 2n\pi\right)}\right)^{\frac{1}{3}}$ [As $\theta = \theta + 2n\pi$]

or, $z_n = 2^{\frac{2}{3}}\ e^{i\left(\frac{2\pi + 6n\pi}{9}\right)}; n = 0,1,2$

when $n = 0$, $z_0 = 2^{\frac{2}{3}}\ e^{i\left(\frac{2\pi}{9}\right)}$

when $n = 1$, $z_1 = 2^{\frac{2}{3}}\ e^{i\left(\frac{8\pi}{9}\right)}$

when $n = 2$, $z_2 = 2^{\frac{2}{3}}\ e^{i\left(\frac{14\pi}{9}\right)}$

The distance of each root from the origin is same as $2^{\frac{2}{3}}$ and the angular distance $\frac{2\pi}{3}$ of two consecutive roots are same.

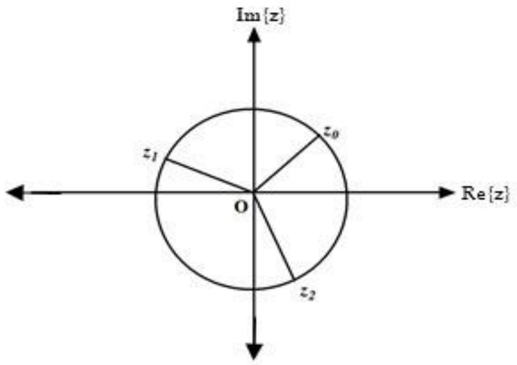

Figure: 6

MATLAB command for complex numbers :

Find all the values of z and locate the values in the complex plane

(i) $z^3 + i = 0$, (ii) $z^5 - 1 = 0$, (iii) $z^3 - 1 + i = 0$, (iv) $z^4 = -81$, (v) $z^3 - 64 = 0$.

$z^3 + i = 0$	$z^3 - 64 = 0$
>> p=[1 0 0 i]; **>> r=roots(p)** **r =** **-0.8660 - 0.5000i** **-0.0000 + 1.0000i** **0.8660 - 0.5000i** **>> plot(r, 'o')**	**>> p=[1 0 0 64];** **>> r=roots(p)** **r =** **-4.0000 + 0.0000i** **2.0000 + 3.4641i** **2.0000 - 3.4641i** **>> plot(r,'*')**

Example:

Describe and graph the locus represented by $1 < |z + i| \le 2$.

Solution:

Given, $1 < |z + i| \le 2$.

or, $1 < |x + iy + i| \le 2$

or, $1 < |x + i(y + 1)| \le 2$

or, $1 < \sqrt{x^2 + (y + 1)^2} \le 2$

or, $1 < (x - 0)^2 + (y - (-1))^2 \le 2^2$

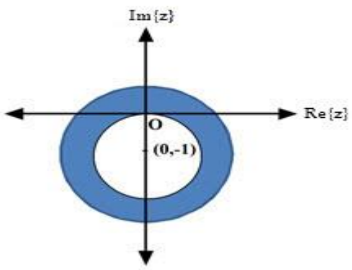

Figure: 7

Chapter 01

Exercise Set

1. Express $z = \frac{(1+i)^2}{1-i}$ in the form $a + ib$, where a,b$\in \mathbb{R}$

2. Evaluate each of the followings:

(a) $Re\left\{\frac{1+\sqrt{3}i}{1-i}\right\}$, (b) $\left|\frac{z}{\bar{z}}\right|$, (c) $Im\left\{\frac{z}{\bar{z}}\right\}$, (d) $Im\left\{\frac{1+\sqrt{3}i}{1-i}\right\}$

3. Convert the following numbers into polar form:

(a) $z = 1 + i, -1 - i, 1 - i$ (b) $z = -3 - \sqrt{3}i$, (c) $z = 2i, -2i$, and (d) $z = -4 + 4i$, (e) $z = -5$

4. Convert the following numbers into rectangular form:

(a) $z = \sqrt{3}e^{i\frac{\pi}{3}}$ (b) $z = 2e^{i\frac{\pi}{4}}$, and (c) $z = 2e^{-i\frac{\pi}{4}}$

5. Find the principal argument of the followings:

(a) $z = -1 + i$, (b) $z = -5$, (c) $z = -5 - i$ (d) $z = -5 + i$

6. Find all values of z for the following equations in terms of exponential functions and also locate these values in the complex plane:

(a) $z = \sqrt[3]{1+i}$ or $z^3 = 1 + i$

(b) $z = \sqrt[4]{-4}$ or $z^4 = -4$

(c) $z = \sqrt[4]{i}$ or $z^4 = i$

(d) $z = \sqrt[8]{1}$ or $z^8 = 1$

(e) $z = \sqrt[5]{-1}$ or $z^5 = -1$.

Reference Book: Advanced Engineering Mathematics (10th edition) by Erwin Kreyszig, Herbert Kreyszig, Edward J. Norminton, published by John Wiley & Sons, Inc