Lecture-3

Application of Laplace transformation

The Laplace transformation is useful in solving differential equations. There are four steps to follow, such as

Differential equation

↓

Apply Laplace Transformation

↓

Use the Initial Values

↓

Solve the algebraic equation for $Y(s)$

↓

Apply Inverse Laplace Transformation.

Important formulae

$$\mathcal{L}\{\dot{f}(t)\} = \mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0).$$

$\mathcal{L}\{\ddot{f}(t)\} = \mathcal{L}\left\{\frac{d^2f(t)}{dt^2}\right\} = s^2F(s) - sf(0) - \dot{f}(0)$ where $f(0)$, and $\dot{f}(0)$are the initial values of f and$\dot{f}$.

$$\mathcal{L}\{\dddot{f}(t)\} = \mathcal{L}\left\{\frac{d^3f(t)}{dt^3}\right\} = s^3F(s) - s^2f(0) - s\dot{f}(0) - \ddot{f}(0).$$

The general case for the Laplace transform of an n^{th} derivative is

$$\mathcal{L}\{f^n(t)\} = \mathcal{L}\left\{\frac{d^nf(t)}{dt^n}\right\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).$$

Solving Ordinary Differential equations with constant coefficients:

The Laplace transform is useful in solving linear ordinary differential equations with constant coefficients. Having obtained expressions for the Laplace transforms of derivatives, we are now in a position to use Laplace transform and also inverse Laplace transform methods to solve ordinary differential equations with constant coefficients. To illustrate this, consider the general second-order differential equation

$$\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \beta y = f(t) \quad \text{or} \quad \ddot{y}(t) + \alpha\dot{y}(t) + \beta y(t) = f(t) \qquad (1)$$

Where, $\propto$ and β are constants, subject to initial conditions

$$y(0) = A, \dot{y}(0) = B \qquad (2)$$

where A and B are given constants. On taking the Laplace transform of both sides and using condition (2), we obtain the algebraic equation for determination of $\mathcal{L}\{y(t)\} = Y(s)$. The required solution is then obtained by finding the inverse Laplace transform of $Y(s)$. The method is easily extended for the higher order differential equations.

Example:
Solve the differential equation or the initial value problem
$\ddot{y}(t) + y(t) = e^t, \qquad y(0) = 1, \dot{y}(0) = -2.$
Solution:
Given,
$\ddot{y}(t) + y(t) = e^t,$
$\Rightarrow \mathcal{L}\{\ddot{y}(t)\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{e^t\}$ [applying Laplace transformation]
$\Rightarrow s^2Y(s) - sy(0) - \dot{y}(0) + Y(s) = \frac{1}{s-1}$ [let, $\mathcal{L}\{y(t)\} = Y(s)$]
$\Rightarrow s^2Y(s) - s(1) - (-2) + Y(s) = \frac{1}{s-1}$ [using the initial values]
$\Rightarrow Y(s) = \frac{s^2-3s+3}{(s^2+1)\,(s-1)}$ [solving the equation]
$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{A}{s-1} + \frac{Bs+C}{s^2+1}\right\}$ [applying inverse Laplace transformation and using partial fraction]

$$\frac{s^2 - 3s + 3}{(s^2 + 1)\,(s - 1)} \equiv \frac{A}{s - 1} + \frac{Bs + C}{s^2 + 1}$$
$$\Rightarrow s^2 - 3s + 3 \equiv A(s^2 + 1) + (Bs + C)(s - 1)$$
$$\Rightarrow s^2 - 3s + 3 \equiv (A + B)s^2 + (C - B)s + A - C$$

Equating coefficients

$$A + B = 1$$
$$C - B = -3$$
$$A - C = 3$$

Solving we get, $A = \frac{1}{2}$, $B = \frac{1}{2}$, $C = -\frac{5}{2}$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{2}}{s-1} + \frac{\frac{s}{2}}{s^2+1} - \frac{\frac{5}{2}}{s^2+1}\right\}$$

$$\Rightarrow y(t) = \frac{e^t}{2} + \frac{1}{2}\cos t - \frac{5}{2}\sin t.$$

Therefore the solution of the differential equation is $y(t) = \frac{e^t}{2} + \frac{1}{2}\cos t - \frac{5}{2}\sin t$.

Example: A resistance R in series with inductance L is connected with e.m.f $E(t) = t$. The current $i(t)$ is given by

$$L\frac{di}{dt} + Ri = t; \qquad i(0) = 0$$

Use Laplace transform to find the current $i(t)$.

Solution:

Given,

$$L\frac{di}{dt} + Ri = t$$

$\Rightarrow L\mathcal{L}\left\{\frac{di}{dt}\right\} + R\mathcal{L}\{i\} = \mathcal{L}\{t\}$ [applying Laplace transformation]

$\Rightarrow LsI(s) - L.i(0) + RI(s) = \frac{1}{s^2}$ [let, $\mathcal{L}\{i(t)\} = I(s)$]

$\Rightarrow LsI(s) - L(0) + RI(s) = \frac{1}{s^2}$ [using the initial values]

$\Rightarrow I(s) = \frac{1}{(Ls+\mathrm{R})s^2}$ [Solving the equation]

$\Rightarrow \mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(Ls+\mathrm{R})s^2}\right\}$ [applying inverse Laplace transformation]

$\Rightarrow i(t) = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s^2} + \frac{C}{(Ls+\mathrm{R})}\right\}$[using partial fraction]

$$\frac{1}{(Ls+\mathrm{R})s^2} \equiv \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(Ls+\mathrm{R})}$$

$$\Rightarrow 1 \equiv As(Ls+R) + B(Ls+R) + Cs^2$$

$$\Rightarrow 1 \equiv (LA+C)s^2 + (AR+BL)s + BR$$

Equating coefficients

$$LA + C = 0$$

$$AR + BL = 0$$

$$BR = 1$$

Solving we get, $A = -\frac{L}{R^2}, B = \frac{1}{R}, C = \frac{L^2}{R^2}$

Hence,

$$i(t) = \mathcal{L}^{-1}\left\{-\frac{L}{R^2}\frac{1}{s} + \frac{1}{R}\frac{1}{s^2} + \frac{L^2}{R^2}\frac{1}{(Ls+\mathrm{R})}\right\}$$

$$\Rightarrow i(t) = \mathcal{L}^{-1}\left\{-\frac{L}{R^2}\frac{1}{s} + \frac{1}{R}\frac{1}{s^2} + \frac{L^2}{R^2 L}\frac{1}{\left(s+\frac{R}{L}\right)}\right\}$$

$$\Rightarrow i(t) = -\frac{L}{R^2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{R}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{L}{R^2}\mathcal{L}^{-1}\left\{\frac{1}{\left(s+\frac{R}{L}\right)}\right\}$$

$$\Rightarrow i(t) = -\frac{L}{R^2} + \frac{t}{R} + \frac{L}{R^2}e^{-\frac{R}{L}t}.$$

Example:

An inductor of 2 henrys, a resistor of 16 ohms and a capacitor of .02 farads are connected in series with an e.m.f. of E volts. At $t = 0$ the charge on the capacitor and current in the circuit are zero. Find the charge and current at any time $t > 0$ if E= 300 (volts).

Let $q(t)$ and $i(t)$ be the instantaneous charge and current respectively at time t. By Kirchhoff's law's, we have

$$2\frac{di}{dt} + 16i + \frac{q}{0.02} = E$$

$$\Rightarrow 2\frac{d^2q}{dt^2} + 16\frac{dq}{dt} + 50q = E \quad \text{[since } i = \frac{dq}{dt}\text{]}$$

$$\Rightarrow 2\frac{d^2q}{dt^2} + 16\frac{dq}{dt} + 50q = E \ldots\ldots (1)$$

With the initial conditions

$$q(0) = 0, i(0) = \dot{q}(0) = 0.$$

Solution:

If $E = 300$, then equation (1) becomes

$$\frac{d^2q}{dt^2} + 8\frac{dq}{dt} + 25q = 150$$

$\Rightarrow \mathcal{L}\left\{\frac{d^2q}{dt^2}\right\} + 8\mathcal{L}\left\{\frac{dq}{dt}\right\} + 25\mathcal{L}\{q\} = \mathcal{L}\{150\}$ [applying Laplace transform]

$\Rightarrow \{s^2Q(s) - s\,q(0) - \dot{q}(0)\} + 8\{sQ(s) - q(0)\} + 25Q(s) = \frac{150}{s}$ [let, $\mathcal{L}\{q(t)\} = Q(s)$]

$\Rightarrow \{s^2Q(s) - s\,.0 - 0\} + 8\{sQ(s) - 0\} + 25Q(s) = \frac{150}{s}$ [using the initial values]

$$\Rightarrow s^2Q(s) + 8sQ(s) + 25Q(s) = \frac{150}{s}$$

$$\Rightarrow (s^2 + 8s + 25)Q = \frac{150}{s}$$

$\Rightarrow Q(s) = \frac{150}{s(s^2+8s+25)}$ [solving the equation]

$$\Rightarrow \mathcal{L}^{-1}\{Q(s)\} = \mathcal{L}^{-1}\left\{\frac{150}{s(s^2 + 8s + 25)}\right\}$$

$\Rightarrow q(t) = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{Bs+C}{(s^2+8s+25)}\right\}$ [using partial fraction]

$$\frac{150}{s(s^2 + 8s + 25)} \equiv \frac{A}{s} + \frac{Bs + C}{(s^2 + 8s + 25)}$$

$$\Rightarrow 150 \equiv A(s^2 + 8s + 25) + (Bs + C)s$$

$$\Rightarrow 150 \equiv (A + B)s^2 + (8A + C)s + 25A$$

Equating coefficients

$$A + B = 0$$

$$8A + C = 0$$

$$25A = 150$$

Solving we get, $A = 6, B = -6, C = -48$

$$q(t) = \mathcal{L}^{-1}\left\{\frac{6}{s} - \frac{6s+48}{(s^2+8s+25)}\right\}$$

$$\Rightarrow q(t) = \mathcal{L}^{-1}\left\{\frac{6}{s} - \frac{6(s+4)+24}{(s+4)^2+9}\right\}$$

$$\Rightarrow q(t) = \mathcal{L}^{-1}\left\{\frac{6}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{6(s+4)}{(s+4)^2+9}\right\} - \mathcal{L}^{-1}\left\{\frac{24}{(s+4)^2+9}\right\}$$

$q(t) = 6 - 6e^{-4t}Cos\, t - 8\, e^{-4t}\sin 3t$ [applying inverse Laplace transformation]

$i(t) = \frac{dq}{dt} = 50\, e^{-4t}\sin 3t.$

Example:

The current $i(t)$ in an electrical circuit is given by the DE, $\frac{d^2i}{dt^2} + 2\frac{di}{dt} = \begin{cases} 0, & 0 < t < 10 \\ 1, & 10 < t < 20 \\ 0, & t > 20 \end{cases}$

$i(0) = \frac{di}{dt}(0) = 0.$ Determine current $i(t)$.

Solution: Using unit step function the DE becomes,

$$\frac{d^2i}{dt^2} + 2\frac{di}{dt} = u(t-10) - u(t-20) \;\ldots\ldots..(1).$$

$$\Rightarrow \mathcal{L}\left\{\frac{d^2i}{dt^2}\right\} + 2\mathcal{L}\left\{\frac{di}{dt}\right\} = \mathcal{L}\{u(t-10)\} - \mathcal{L}\{u(t-20)\}$$ [applying Laplace transform]

$$\Rightarrow \left\{s^2I(s) - si(0) - \frac{di}{dt}(0)\right\} + 2\{sI(s) - i(0)\} = \frac{e^{-10s}}{s} - \frac{e^{-20s}}{s}$$

$[\because \mathcal{L}\{f(t).u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$

$$\Rightarrow I(s)\big(s(s+2)\big) = \frac{e^{-10s}}{s} - \frac{e^{-20s}}{s}$$

$$\Rightarrow I(s) = \frac{1}{s(s+2)}\left(\frac{e^{-10s}}{s} - \frac{e^{-20s}}{s}\right)$$

Applying inverse Laplace transform, we get

$$\mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}\left(\frac{e^{-10s}}{s} - \frac{e^{-20s}}{s}\right)\right\} \ldots\ldots\ldots\ldots(2).$$

We know $\mathcal{L}^{-1}\{e^{-as}G(s)\} = u(t-a).g(t-a)$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+2)}\right\} = \frac{t}{2} + \frac{e^{-2t}}{4} - \frac{1}{4}.$

$$\therefore i(t) = \left[\frac{(t-10)}{2} + \frac{e^{-2(t-10)}}{4} - \frac{1}{4}\right]u(t-10) - \left[\frac{(t-20)}{2} + \frac{e^{-2(t-20)}}{4} - \frac{1}{4}\right]u(t-20).$$

$$= \begin{cases} 0, & 0 < t < 10 \\ \dfrac{(t-10)}{2} + \dfrac{e^{-2(t-10)}}{4} - \dfrac{1}{4}, & 10 < t < 20\,. \\ 5 + \dfrac{e^{-2(t-10)}}{4} - \dfrac{e^{-2(t-20)}}{4}, & t > 20 \end{cases}$$

Problem set 3.1

Apply Laplace transform to solve the following ordinary differential equations and hence justify your answer, where $\dot{y} \equiv \frac{dy(t)}{dt}$ and $\ddot{y} \equiv \frac{d^2y(t)}{dt^2}$: (1-12)

1. $\dot{y}(t) = 3; \quad y(0) = 2.$ **Ans:** $y(t) = 3t + 2.$
2. $\dot{y}(t) = 4t; \quad y(0) = 1.$ **Ans:** $y(t) = 2\,t^2 + 1.$
3. $\dot{y}(t) = 2t - 1; \quad y(0) = 3.$ **Ans:** $y(t) = t\,(t-1) + 3.$
4. $\dot{y}(t) = t^2; \quad y(0) = 4.$ **Ans:** $y(t) = \frac{t^3}{3} + 4.$
5. $\dot{y}(t) = e^{2t}; \quad y(0) = 2.$ **Ans:** $y(t) = \frac{e^{2t}}{2} + \frac{3}{2}.$
6. $\dot{y}(t) + y(t) = 2\,; \quad y(0) = 0.$ **Ans:** $y(t) = 2 - 2\,e^{-t}.$
7. $\ddot{y}(t) = 5\,; \quad y(0) = 1, \dot{y}(0) = 2.$ Ans: $y(t) = \frac{t\,(5t+4)}{2} + 1.$
8. $\ddot{y}(t) - 2\,\dot{y}(t) = \cos t\,; \quad y(0) = 0, \dot{y}(0) = 1.$

   Ans: $f(t) = \frac{7}{10}\,e^{2t} - \frac{4}{5}\left(\sin t + \frac{1}{2}\cos t\right) - \frac{1}{2}.$
9. $\ddot{y}(t) + 3\,\dot{y}(t) - y(t) = e^{t}; \quad y(0) = \dot{y}(0) = 0.$

   Ans:

   $f(t) = \frac{1}{3}\,e^{t} - \frac{1}{3}\,e^{\frac{-3t}{2}}\left[\cosh\left(\frac{\sqrt{13}}{2}\,t\right) + \frac{5}{\sqrt{13}}\sinh\left(\frac{\sqrt{13}}{2}\,t\right)\right].$
10. $\ddot{y}(t) - 7\dot{y}(t) + 12y(t) = 0, y(0) = 2, \dot{y}(0) = 1.$

    Ans: $y(t) = -5e^{4t} + 7e^{3t}.$
11. $\ddot{y}(t) + y(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}, y(0) = 0, \dot{y}(0) = 0.$

    Ans: $y(t) = t - \sin t$ if $0 < t < 1$ and $\cos(t-1) + \sin(t-1) - \sin t$ if $t > 1.$

Shifted data problems:

This is a short name for initial value problem with initial conditions referring to some later instant $t = t_0$ instead of $t = 0$. In this case, the conditions $y(0)$ and $y'(0)$ occurring in the Laplace transform approach cannot be used immediately.

12. $y'' + y = 2t, \quad y\left(\frac{\pi}{4}\right) = \frac{\pi}{2}, \quad y'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}$.

Solution: set $t = \bar{t} + \frac{\pi}{4}$ so that $t = \frac{\pi}{4}$ gives $\bar{t} = 0$ and then Laplace transform becomes applicable throughout.

Now, the shifted problem is

$$\bar{y}'' + \bar{y} = 2\left(\bar{t} + \frac{\pi}{4}\right), \quad \bar{y}(0) = \frac{\pi}{2} \text{ and } \bar{y}'(0) = 2 - \sqrt{2}.$$

Using Laplace transform on both sides we obtain

$$s^2\,\bar{Y}(s) - s\bar{y}(0) - \bar{y}'(0) = \frac{2}{s^2} + \frac{\pi}{2}\,\frac{1}{s}$$

$$\Rightarrow \bar{Y}(s) = \frac{2}{s^2\,(s^2+1)} + \frac{\pi}{2}\,\frac{1}{s\,(s^2+1)} + \frac{\pi}{2}\,\frac{s}{s^2+1} + \left(2 - \sqrt{2}\right)\frac{1}{s^2+1}$$

Applying inverse Laplace on both sides,

$$\Rightarrow \bar{y}(\bar{t}) = 2\,(\bar{t} - \sin\bar{t}) + \frac{\pi}{2}\,(1 - \cos\bar{t}) + \frac{\pi}{2}\cos\bar{t} + \left(2 - \sqrt{2}\right)\sin\bar{t}$$

Substituting $\bar{t} = t - \frac{\pi}{4}$ we obtain the solution

$$y(t) = 2t - \sin t + \cos t.$$

Solving Simultaneous Ordinary Differential Equations by Laplace Transform

Example:

$$\begin{cases} \frac{dx(t)}{dt} = 2x(t) - 3y(t) \\ \frac{dy(t)}{dt} = y(t) - 2x(t) \end{cases} \text{ subject to } x(0) = 8, y(0) = 3.$$

Solution:

Taking the Laplace transforms of both equations

$$\Rightarrow \mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = 2\mathcal{L}\{x(t)\} - 3\mathcal{L}\{y(t)\}$$

$$\mathcal{L}\left\{\frac{dy(t)}{dt}\right\} = \mathcal{L}\{y(t)\} - 2\mathcal{L}\{x(t)\}$$

$$\Rightarrow sX(s) - x(0) = 2X(s) - 3Y(s)$$

$$sY(s) - y(0) = Y(s) - 2X(s)$$

$$\Rightarrow (s-2)X(s) + 3Y(s) = 8$$

$2X(s) + (s-1)Y(s) = 3$ [using initial condition and rearranging]

Now solving this two equations simultaneously using **Cramer's rule** and partial fraction we get,

$$X(s) = \frac{\begin{vmatrix} 8 & 3 \\ 3 & s-1 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{8s-17}{s^2-3s-4} = \frac{8s-17}{(s+1)(s-4)} = \frac{5}{s+1} + \frac{3}{s-4}$$

$$Y(s) = \frac{\begin{vmatrix} s-2 & 8 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{3s-22}{s^2-3s-4} = \frac{3s-22}{(s+1)(s-4)} = \frac{5}{s+1} - \frac{2}{s-4}$$

Now taking inverse Laplace transform we get,

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{5}{s+1} + \frac{3}{s-4}\right\}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{5}{s+1} - \frac{2}{s-4}\right\}$$

$$\Rightarrow y(t) = 5e^{-t} + 3e^{4t}$$

$$y(t) = 5e^{-t} - 2e^{4t}$$

Problem set 3.2

Solve the following system of differential equations where $x(t) \equiv x$, $y(t) \equiv y$, $\dot{y} \equiv \frac{dy(t)}{dt}$ and $\dot{x} \equiv \frac{dx(t)}{dt}$, using Laplace transformation. Also justify your answers. (13-16)

13. $\dot{x} = y$
    $\dot{y} = 16x;\ \ x(0) = 0, y(0) = 4.$
    Answer: $x(t) = \sinh 4t,\ \ y(t) = 4\cosh 4t$
14. $\dot{x} = -4y$
    $\dot{y} = x;\ \ x(0) = 2, y(0) = 0.$
    Answer: $x(t) = 2\cos 2t,\ \ y(t) = \sin 2t$
15. $\dot{x} = 2x + y$
    $\dot{y} = 4x + 2y;\ \ x(0) = 1, y(0) = 6.$
    Answer: $x(t) = e^{2t}(\cosh 2t + 3\sinh 2t), y(t) = e^{2t}(6\cosh 2t + 2\sinh 2t).$
16. $\dot{x} = 3x + y$
    $\dot{y} = 4x + 3y;\ \ x(0) = 3, y(0) = 2.$
    Answer: $x(t) = e^{3t}(3\cosh 2t + \sinh 2t),\ \ y(t) = e^{3t}(2\cosh 2t + 6\sinh 2t)$

Problem set 3.3 (Application)

General talk:

The Laplace transform is widely used in the following science and engineering field** .

1. Analysis of electronic circuits.

2. System modeling.

3. Digital signal processing.

4. Nuclear physics.

5. Process control.

The following examples highlights the importance of laplace transform in different engineering fields.

Problem-1:

The following example based on the concepts from nuclear physics. Consider the following first

order linear differential equation

$$\frac{dN}{dt} = -\lambda N \dots\dots\dots (1)$$

This equation is the fundamental relationship describing radioactive decay, where $N = N(t)$

represents the number of undecayed atoms remaining in a sample of a radioactive isotope

at time t and λ is the decay constant.

We can use laplace transform to solve this equation (1).

Rearranging the above equation (1) we get,

$$\frac{dN}{dt} + \lambda N = 0 \dots\dots\dots (2)$$

Taking laplace transform on both sides of (2)

$$s\,L(N) - N(0) + \lambda\,L(N) = 0$$
$$\Rightarrow s\,\overline{N} - N_0 + \lambda\,\overline{N} = 0$$

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Where $L(N) = \overline{N}$ and $N(0) = N_0$

$$\Rightarrow \overline{N} = \frac{N_0}{s + \lambda}$$

Now taking inverse laplace transform on both sides we get,

$$N(t) = N_0\, e^{-\lambda t}.$$

Which is indeed the correct form for radioactive decay.

Problem-2:

An apple pie with an initial temperature of 170^o C is removed from the oven and left to cool in a room with an air temperature of 20^o C. Given that the temperature of the pie initially decreases at a rate of $3.\,0^o$ C/min. How long will it take for the pie to cool to a temperature of 30^o C?

[Hints. Assuming that the pie obeys Newton's Law of cooling and

$\frac{\mathrm{dT}}{\mathrm{dt}} = -\mathrm{k}(\mathrm{T} - 20), T(0) = 170\,, T'(0) = -3.0$]

Where, T is the temperature of the pie in degree Celsius ,t is the time in minutes and k is an unknown constants.

EXAMPLE 3 Four-Terminal *RLC*-Network

Find the output voltage response in Fig. 135 if $= 20\ \Omega$, $L = 1$ H, $C = 10^{-4}$ F, the input is $\delta(t)$ (a unit impulse at time $t = 0$), and current and charge are zero at time $t = 0$.

Solution. To understand what is going on, note that the network is an LC-circuit to which two wires at A and B are attached for recording the voltage $v(t)$ on the capacitor. Recalling from Sec. 2.9 that current $i(t)$ and charge $q(t)$ are related by $i = q' = dq/dt$, we obtain the model

$$Li' + i + \frac{q}{C} = Lq'' + q' + \frac{q}{C} = q'' + 20q' + 10{,}000q = \delta(t).$$

From (1) and (2) in Sec. 6.2 and (5) in this section we obtain the subsidiary equation for $Q(s) = \mathcal{L}(q)$

$$(s^2 + 20s + 10{,}000)Q = 1. \qquad \text{Solution} \qquad Q = \frac{1}{(s+10)^2 + 9900}.$$

By the first shifting theorem in Sec. 6.1 we obtain from Q damped oscillations for q and v; rounding $9900 \approx 99.50^2$, we get (Fig. 135)

$$q = \mathcal{L}^{-1}(Q) = \frac{1}{99.50}e^{-10t}\sin 99.50t \qquad \text{and} \qquad v = \frac{q}{C} = 100.5e^{-10t}\sin 99.50t. \quad \blacksquare$$

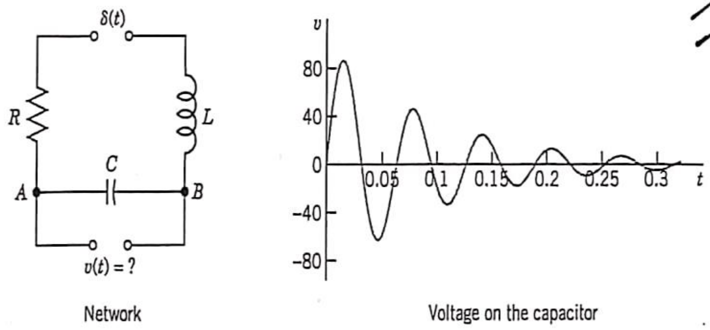


Fig. 135. Network and output voltage in Example 3

EXAMPLE 4 **Unrepeated Complex Factors. Damped Forced Vibrations**

Solve the initial value problem for a damped mass–spring system acted upon by a sinusoidal force for some time interval (Fig. 136),

$$y'' + 2y' + 2y = r(t), \quad r(t) = 10 \sin 2t \text{ if } 0 < t < \pi \text{ and } 0 \text{ if } t > \pi; \qquad y(0) = 1, \quad y'(0) = -5.$$

Solution. From Table 6.1, (1), (2) in Sec. 6.2, and the second shifting theorem in Sec. 6.3, we obtain the subsidiary equation

$$(s^2 \quad - s + 5) + 2(s \quad - 1) + 2 \quad = 10 \frac{2}{s^2 + 4}(1 - e^{-\pi s}).$$

We collect the -terms, $(s^2 + 2s + 2)$, take $-s + 5 - 2 = -s + 3$ to the right, and solve,

$$= \frac{20}{(s^2 + 4)(s^2 + 2s + 2)} - \frac{20e^{-\pi s}}{(s^2 + 4)(s^2 + 2s + 2)} + \frac{s - 3}{s^2 + 2s + 2}. \tag{6}$$

For the last fraction we get from Table 6.1 and the first shifting theorem

$$\mathcal{L}^{-1}\left\{\frac{s + 1 - 4}{(s + 1)^2 + 1}\right\} = e^{-t}(\cos t - 4 \sin t). \tag{7}$$

In the first fraction in (6) we have unrepeated complex roots, hence a partial fraction representation

$$\frac{20}{(s^2 + 4)(s^2 + 2s + 2)} = \frac{As + B}{s^2 + 4} + \frac{Ms + N}{s^2 + 2s + 2}.$$

Multiplication by the common denominator gives

$$20 = (As + B)(s^2 + 2s + 2) + (Ms + N)(s^2 + 4).$$

We determine A, B, M, N. Equating the coefficients of each power of s on both sides gives the four equations

(a) $[s^3]: \quad 0 = A + M$ (b) $[s^2]: \quad 0 = 2A + B + N$

(c) $[s]: \quad 0 = 2A + 2B + 4M$ (d) $[s^0]: \quad 20 = 2B + 4N.$

We can solve this, for instance, obtaining $M = -A$ from (a), then $A = B$ from (c), then $N = -3A$ from (b), and finally $A = -2$ from (d). Hence $A = -2$, $B = -2$, $M = 2$, $N = 6$, and the first fraction in (6) has the representation

$$\frac{-2s - 2}{s^2 + 4} + \frac{2(s + 1) + 6 - 2}{(s + 1)^2 + 1}. \quad \text{Inverse transform:} \quad -2 \cos 2t - \sin 2t + e^{-t}(2 \cos t + 4 \sin t). \tag{8}$$

The sum of this inverse and (7) is the solution of the problem for $0 < t < \pi$, namely (the sines cancel),

$$y(t) = 3e^{-t}\cos t - 2\cos 2t - \sin 2t \qquad \text{if } 0 < t < \pi. \tag{9}$$

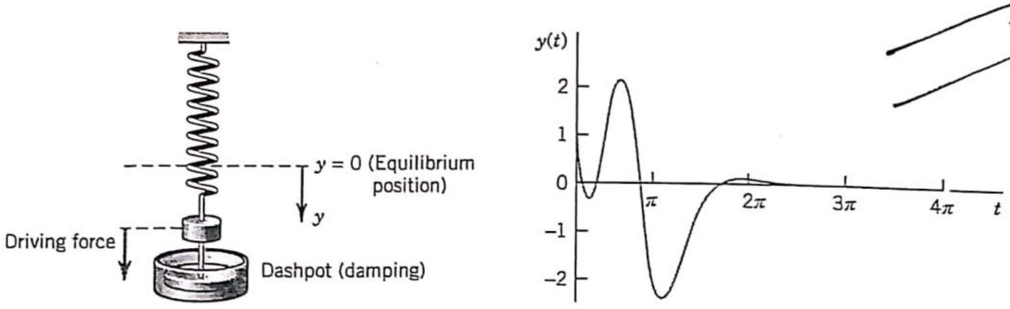
In the second fraction in (6), taken with the minus sign, we have the factor $e^{-\pi s}$, so that from (8) and the second shifting theorem (Sec. 6.3) we get the inverse transform of this fraction for $t > 0$ in the form

$$+2\cos(2t - 2\pi) + \sin(2t - 2\pi) - e^{-(t-\pi)}[2\cos(t-\pi) + 4\sin(t-\pi)]$$
$$= 2\cos 2t + \sin 2t + e^{-(t-\pi)}(2\cos t + 4\sin t).$$

The sum of this and (9) is the solution for $t > \pi$,

$$y(t) = e^{-t}[(3 + 2e^{\pi})\cos t + 4e^{\pi}\sin t] \qquad \text{if } t > \pi. \tag{10}$$

Figure 136 shows (9) (for $0 < t < \pi$) and (10) (for $t > \pi$), a beginning vibration, which goes to zero rapidly because of the damping and the absence of a driving force after $t = \pi$. ■



Mechanical system Output (solution)

Fig. 136. Example 4

EXAMPLE 5 Response of a Damped Vibrating System to a Single Square Wave

Using convolution, determine the response of the damped mass–spring system modeled by

$$y'' + 3y' + 2y = r(t), \qquad r(t) = 1 \text{ if } 1 < t < 2 \text{ and } 0 \text{ otherwise}, \qquad y(0) = y'(0) = 0.$$

This system with an **input** (a driving force) *that acts for some time only* (Fig. 143) has been solved by partial fraction reduction in Sec. 6.4 (Example 1).

Solution y onvolution. The transfer function and its inverse are

$$Q(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}, \qquad \text{hence} \qquad q(t) = e^{-t} - e^{-2t}.$$

Hence the convolution integral (3) is (except for the limits of integration)

$$y(t) = \int q(t-\tau)\cdot 1\, d\tau = \int [e^{-(t-\tau)} - e^{-2(t-\tau)}]\, d\tau = e^{-(t-\tau)} - \frac{1}{2}e^{-2(t-\tau)}.$$

Now comes an important point in handling convolution. $r(\tau) = 1$ if $1 < \tau < 2$ only. Hence if $t < 1$, the integral is zero. If $1 < t < 2$, we have to integrate from $\tau = 1$ (not 0) to t. This gives (with the first two terms from the upper limit)

$$y(t) = e^{-0} - \tfrac{1}{2}e^{-0} - (e^{-(t-1)} - \tfrac{1}{2}e^{-2(t-1)}) = \tfrac{1}{2} - e^{-(t-1)} + \tfrac{1}{2}e^{-2(t-1)}.$$

If $t > 2$, we have to integrate from $\tau = 1$ to 2 (not to t). This gives

$$y(t) = e^{-(t-2)} - \tfrac{1}{2}e^{-2(t-2)} - (e^{-(t-1)} - \tfrac{1}{2}e^{-2(t-1)}).$$

Figure 143 shows the input (the square wave) and the interesting output, which is zero from 0 to 1, then increases, reaches a maximum (near 2.6) after the input has become zero (why), and finally decreases to zero in a monotone fashion. ■

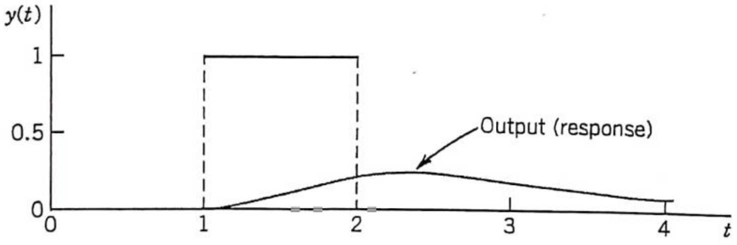


Fig. 143. Square wave and response in Example 5

XAMPLE 6 A Volterra Integral Equation of the Second Kind

Solve the Volterra integral equation of the second kind[3]

$$y(t) - \int_0^t y(\tau) \sin (t - \tau)\, d\tau = t.$$

Solution. From (1) we see that the given equation can be written as a convolution, $y - y \quad \sin t = t$. Writing $Y = \mathcal{L}(y)$ and applying the convolution theorem, we obtain

$$Y(s) - Y(s)\frac{1}{s^2+1} = Y(s)\frac{s^2}{s^2+1} = \frac{1}{s^2}.$$

The solution is

$$Y(s) = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4} \qquad \text{and gives the answer} \qquad y(t) = t + \frac{t^3}{6}.$$

Find the solution of following DE's using Laplace Transformation:
Ref: Differential Equations by Paul Blanchard, Robert L. Devaney & Glen R. Hall , Fourth Edition -Page-577(15-24),Page-600(27-34), Page 608(2-6).

In Exercises 15–24,

(a) compute the Laplace transform of both sides of the equation;

(b) substitute the initial conditions and solve for the Laplace transform of the solution;

(c) find a function whose Laplace transform is the same as the solution; and

(d) check that you have found the solution of the initial-value problem.

15. $\frac{dy}{dt} = -y + e^{-2t}, \quad y(0) = 2$

16. $\frac{dy}{dt} + 5y = e^{-t}, \quad y(0) = 2$

17. $\frac{dy}{dt} + 7y = 1, \quad y(0) = 3$

18. $\frac{dy}{dt} + 4y = 6, \quad y(0) = 0$

19. $\frac{dy}{dt} + 9y = 2, \quad y(0) = -2$

20. $\frac{dy}{dt} = -y + 2, \quad y(0) = 4$

21. $\frac{dy}{dt} = -y + e^{-2t}, \quad y(0) = 1$

22. $\frac{dy}{dt} = 2y + t, \quad y(0) = 0$

23. $\frac{dy}{dt} = -y + t^2, \quad y(0) = 1$

24. $\frac{dy}{dt} + 4y = 2 + 3t, \quad y(0) = 1$

In Exercises 27–34,

(a) compute the Laplace transform of both sides of the differential equation,

(b) substitute in the initial conditions and simplify to obtain the Laplace transform of the solution, and

(c) find the solution by taking the inverse Laplace transform.

27. $\dfrac{d^2y}{dt^2} + 4y = 8, \quad y(0) = 11, \quad y'(0) = 5$

28. $\dfrac{d^2y}{dt^2} - y = e^{2t}, \quad y(0) = 1, \quad y'(0) = -1$

29. $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 5y = 2e^t, \quad y(0) = 3, \quad y'(0) = 1$

30. $\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 13y = 13u_4(t), \quad y(0) = 3, \quad y'(0) = 1$

31. $\dfrac{d^2y}{dt^2} + 4y = \cos 2t, \quad y(0) = -2, \quad y'(0) = 0$

32. $\dfrac{d^2y}{dt^2} + 3y = u_4(t)\cos(5(t-4)), \quad y(0) = 0, \quad y'(0) = -2$

33. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 9y = 20u_2(t)\sin(t-2), \quad y(0) = 1, \quad y'(0) = 2$

34. $\dfrac{d^2y}{dt^2} + 3y = w(t), \quad y(0) = 2, \quad y'(0) = 0,$ where

$$w(t) = \begin{cases} t, & \text{if } 0 \le t < 1; \\ 1, & \text{if } t \ge 1. \end{cases}$$

In Exercises 2–5, solve the given initial-value problem.

2. $\dfrac{d^2y}{dt^2} + 3y = 5\delta_2(t), \quad y(0) = 0, \quad y'(0) = 0$

3. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 5y = \delta_3(t), \quad y(0) = 1, \quad y'(0) = 1$

4. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 2y = -2\delta_2(t), \quad y(0) = 2, \quad y'(0) = 0$

5. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + 3y = \delta_1(t) - 3\delta_4(t), \quad y(0) = 0, \quad y'(0) = 0$

6. **(a)** Discuss the qualitative behavior of the solution of the initial-value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y = \delta_4(t), \quad y(0) = 1, \quad y'(0) = 0.$$

(b) Compute the solution of this initial-value problem.