Lecture Note-03
Complex Integration

Line integral in the complex plane

Complex definite integrals are called (complex) **line integrals**. They are written as

$$\int_C f(z)\,dz.$$

Here the **integrand** $f(z)$ is integrated over a given curve *C*. This curve *C* in the complex plane is called the **path of integration**.

If *C* is a **closed path** (one whose terminal point coincides with its initial point),

then it is denoted by $\oint_C f(z)dz \cdot$

Partitioning of path *C*: If *C* is a combination of $C1$ and $C2$ then, $\int_C f(z)\,dz = \int_{C1} f(z)\,dz + \int_{C2} f(z)\,dz.$

We may represent C by a parametric representation $z(t) = x(t) + i\,y(t) \quad a \le t \le b$. That is,

$\int_C f(z)\,dz = \int_C f(z(t))\,z'(t)dt$. The sense of increasing *t* is called the **positive sense** on *C*.

Note: Parametric representation of any curve is not unique.

Example 1: Find and sketch the path whose orientation is given by $z(t) = (1+3i)t\ (1 \le t \le 2)$.

Solution:

$z(t) = (1+3i)t\ (1 \le t \le 2)$

$x(t) + i\,y(t) = t + i\,3t$

Comparing real and imaginary part, we get

$x(t) = t\,,\ y(t) = 3t\ \ (1 \le t \le 2)$.

t	x	y	(x,y)
1	1	3	(1,3)
2	2	6	(2,6)

So, $z(t) = (1+3i)t\ (1 \le t \le 2)$ represents the line segment from (1,3) to (2,6) in complex plane.

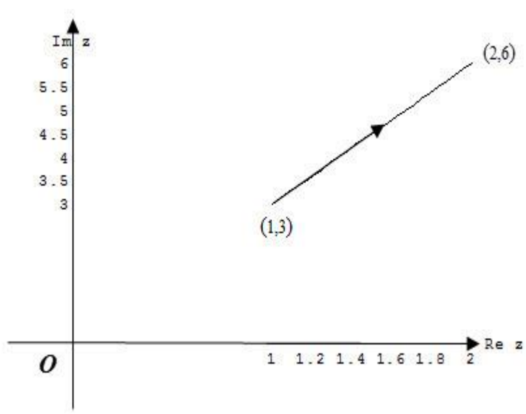

Fig: 1

Example 2: Find and sketch the path whose orientation is given by $z(t) = 2e^{it}$ $(0 \leq t < \pi)$.

Solution:

$z(t) = 2e^{it}$ $(0 \leq t < \pi)$

$x(t) + i\,y(t) = 2\cos(\mathrm{t}) + i\,2\sin(t)$

Comparing real and imaginary part,

we get $x(t) = 2\cos(t)$, $y(t) = 2\sin(t)$ $(0 \leq t < \pi)$.

So, $z(t) = 2e^{it}$ $(0 \leq t < \pi)$ represents upper semicircle of radius 2 with center (0,0).

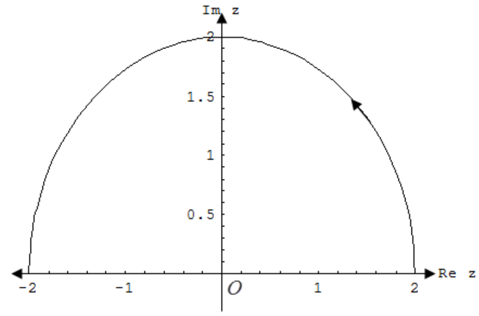

Fig: 2

Example 3: Sketch and represent the line segment from $1 + i$ to $4 - 2i$ parametrically.

Solution:

The equation of straight line passing through the points (1,1) to (4,-2) is, $y - 1 = \left(\frac{-2-1}{4-1}\right)(x - 1)$

That is, $y = -x + 2$

Let, $x = t$ then $y = -t + 2$ where t varies from $t = 1$ to $t = 4$.

So, the parametric equation of line segment from $1 + i$ to $4 - 2i$ is,

$x(t) = \mathrm{t}$, $y(t) = -\mathrm{t} + 2$ $(1 \leq t \leq 4)$.

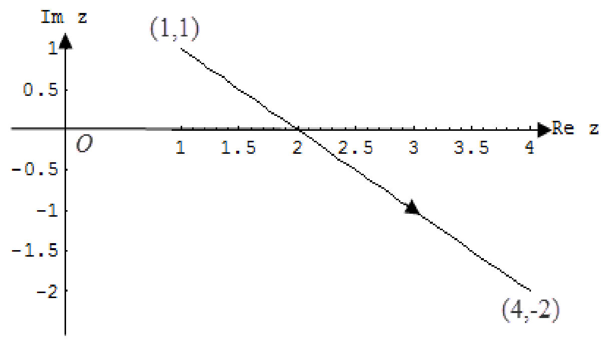

Fig: 3

Example 4: Sketch and represent unit circle (counterclockwise) parametrically.

Solution:

unit circle (counterclockwise)

That is, $|z| = 1$ (counterclockwise)

Or, $|x + i\,y| = 1 \Rightarrow \sqrt{x^2 + y^2} = 1 \Rightarrow x^2 + y^2 = 1$

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Let, $x = \cos\,t$ and $y = \sin\,t$,

Then $(\cos t)^2 + (\sin t)^2 = 1$ where t varies from $t = 0$ to $t = 2\pi$.

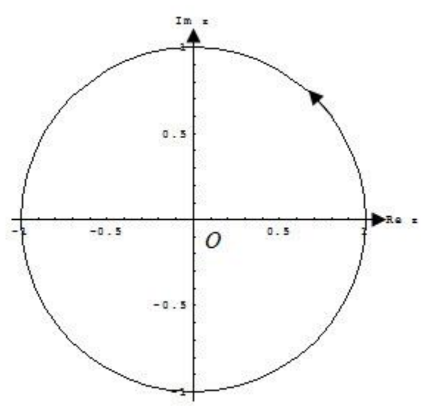

Fig: 4

So, the parametric equation of unit circle (counterclockwise) is,

$x(t) = \cos t$, $y(t) = \sin t$ $(0 \leq t < 2\pi)$.

Example 5: Sketch the path C consisting of two line segments, one from $z = 0$ to $z = 2$ and other from $z = 2$ to $z = 3+i$, hence evaluate $\int_C f(z)\,dz$, if $f(z) = z^2$.

Solution:

Given, C consists of two line segments, one from $z = 0$ to $z = 2$ and other from $z = 2$ to $z = 3+i$.

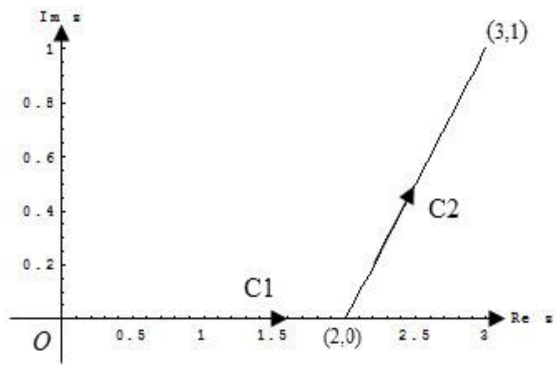

Fig: 5

Along C1:

Equation of the line, which passes through (0,0) and (2,0), is $y = 0$

$f(z) = z^2 = (x+iy)^2 = x^2$ [using $y = 0$]

We know, $z = x + iy = x$, $dz = dx$

and x varies from 0 to 2

$\int_{C1} f(z)\,dz = \int_0^2 x^2 dx = \frac{8}{3}$.

Along C2:

Equation of the line, which passes through (2,0) and (3,1) is,

$$y - 0 = \left(\frac{1-0}{3-2}\right)(x-2) \Rightarrow y = x - 2.$$

$f(z) = z^2 = (x+iy)^2 = [(y+2)+iy]^2$ [using $x = y+2$]

We know, $z = x + iy = y + 2 + iy$, $dz = (1+i)dy$ and y varies from 0 to 1.

$$\int_{C2} f(z)\,dz = \int_0^1 [(y+2)+iy]^2(1+i)\,dy = i\int_0^1 (4 + 4i - 2y^2 + 2iy^2 + 8iy)dy = \frac{10}{3} + \frac{26}{3}i$$

Now, $\int_C f(z)\,dz = \int_{C1} f(z)\,dz + \int_{C2} f(z)\,dz = 6 + \frac{26}{3}i$.

Example 6. Sketch the path C from $z = 0$ to $z = 4 + 2i$ along the curve $z = t^2 + it$ and hence evaluate $\int_C f(z)\,dz$, where $f(z) = \bar{z}$.

Solution:

Given, $z = 0$ to $z = 4 + 2i$ and

$z = t^2 + it \Rightarrow x + iy = t^2 + it$.

$\therefore\ x = t^2$ and $y = t$

Now, $f(z) = \bar{z} = x - iy = y^2 - iy$

and $z = x + iy = y^2 + iy$

$\Rightarrow dz = 2ydy + idy = (2y + i)dy$

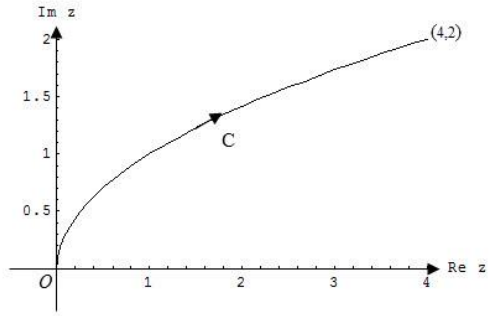

Fig: 6

Therefore,

$$\int_C f(z)\,dz = \int_0^2 (y^2 - iy)(2y + i)dy \quad = \int_0^2 (2y^3 - iy^2 + y)dy \quad = \left[\frac{2y^4}{4} - i\frac{y^3}{3} + \frac{y^2}{2}\right]_0^2 = 10 - \frac{8}{3}i.$$

Example 7: Sketch the path C from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^3$ and hence evaluate $\int_C f(z)\,dz$,where $f(z) = \begin{cases} y, & \text{when} \quad y > 0 \\ 2, & \text{when} \quad y < 0 \end{cases}$.

Solution:

Given, C is the arc from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^3$.

$$f(z) = \begin{cases} y, & \text{when} \quad y > 0 \\ 2, & \text{when} \quad y < 0 \end{cases} \quad = \begin{cases} x^3, & \text{when} \quad x > 0 \\ 2, & \text{when} \quad x < 0 \end{cases}$$

and, $z = x + iy = x + i\,x^3$, $dz = (1 + 3x^2 i)dx$

Now, $\int_C f(z)\,dz = \int_{C1} f(z)\,dz + \int_{C2} f(z)\,dz$

$= \int_{-1}^{0} 2.(1 + 3x^2 i)dx + \int_0^1 x^3(1 + 3x^2 i)dx$

$= \frac{9}{4} + \frac{5}{2}i.$

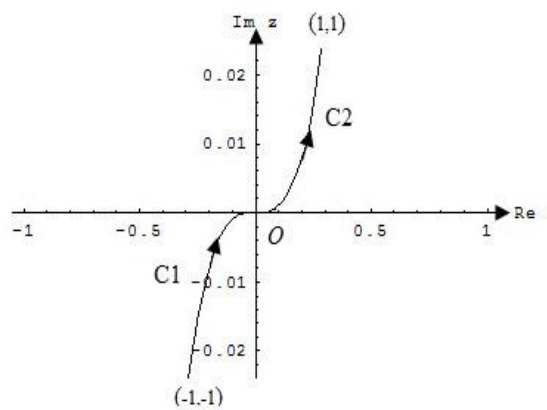

Fig: 7

Example 8: Sketch the path C from $z = -1$ to $z = 1$ along the upper half of the circle $|z| = 1$ and hence evaluate $\int_C f(z)\,dz$, where $f(z) = \bar{z}$.

Solution:

Given, C is the upper half of the circle $|z| = 1$ from z = -1 to z = 1.

$|z| = 1, z = 1.e^{i\Theta}, dz = ie^{i\Theta}d\theta$, where θ varies from π to 0, and $f(z) = \bar{z} = e^{-i\theta}$

Now, $\int_C f(z)\,dz = \int_{\pi}^{0} e^{-i\theta}\, ie^{i\theta}\, d\theta = -\pi i.$

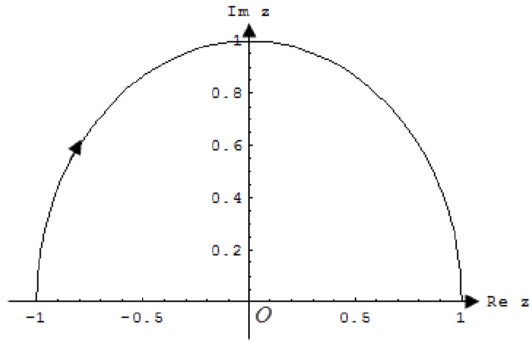

Fig: 8

Matlab command to evaluate line integrals:

1. Evaluate $\int_C \mathrm{Re}\,(z)dz$, where C is the shortest path from 0 to 1+$2i$ along $z(t) = t + 2it, 0 \le t \le 1$.

```
>> fun=@(z) real(z);
>> q=integral(fun,0,1+2i)
q =    0.5000 + 1.0000i
```

2. Evaluate $\int_C \mathrm{Re}\,(z)dz$, where C consists of the shortest path from z= 0 to z=1 and then to z=1+$2i$.

```
>> fun=@(z) real(z);
>> q=integral(fun, 0,1+2i,'Waypoints',1)
q =    0.5000 + 2.0000i
```

3. Evaluate $\int_C \bar{z}\,dz$, where C is the line segment from z=2 to z=2+3i .

```
>> fun=@(z) conj(z);
>> q=integral(fun,2,2+3i)
q =    4.5000 + 6.0000i
```

Sample Exercise Set on Line Integral: 3

Sample Exercise

1. Find and sketch the path and its orientation. Also classify whether the indicated points are interior, exterior or boundary of the following curves-(vi):

(i) $z(t) = (1 + 3i)t \ (1 \le t \le 4)$

(iii) $z(t) = 3e^{it} \ (0 \le t \le \pi)$

(ii) $z(t) = (2 - i)t \ (-2 \le t \le 2)$

(iv) $z(t) = 5e^{-it} \left(0 \le t \le \frac{\pi}{2}\right)$

(v) $z(t) = 6\sin(t) + i\,4\cos(t) \ (0 \le t \le 2\pi)$; (5,1)

(vi) $z(t) = 2\cos(t) + i\ \sin(t) \quad (0 \le t \le 2\pi); \ \ (6, 5)$

(vii) $z(t) = 1 + i + e^{-\pi i t} \ (0 \le t \le 2)$

(viii) $z(t) = 3 + 4i + (5\cosh t + 2\,i\sinh t)$.

2. Sketch and represent them parametrically. Also classify whether the indicated points are interior, exterior or boundary of the following curves-(iii & iv):

(i) Line segment from $-1 + 2i$ to $4 - 2i$,

(ii) unit circle: $|z| = 1$ (clockwise)

(iii) $|z - 4i| = 3$ (counter clockwise) ; (1,6)

(iv) $|z - 5 + i| = 4$ (counter clock wise) ; (1,2)

3. Sketch the path C from $z = 0$ to $z = 3i$ and hence evaluate $\int_C z^2\, dz$.

4. Sketch the path C from $z = 0$ to $z = 3$ and hence evaluate $\int_C \bar{z}\, dz$.

5. Sketch the path C from $z = 1 + i$ to $z = 3 + 3i$ and hence evaluate $\int_C Re\, z\, dz$.

6. $\int_C \ln(z) dz$, C is the shortest path from i to $2i$.

7. Sketch the path C, which is the circle $|z| = 2$ and hence evaluate $\int_C (z + z^{-1})dz$.

- Sketch the corresponding paths and hence evaluate them (8-11):

8. $\int_C (e^{2z} + \cos z)\, dz$, C is the shortest path from $z = 2$ to $z = 4$.

9. $\int_C (z\,.\,\bar{z})\, dz$, C is the path around the square with vertices $0, 1, 1+i, i$.

10. $\int_C \left(\frac{5}{z-2i} - \frac{6}{(z-2i)^2}\right) dz$, C is the circle $|z - 2i| = 4$, clockwise

Reference Book: Advanced Engineering Mathematics (10th edition) by Erwin Kreyszig, Herbert Kreyszig, Edward J. Norminton, published by John Wiley & Sons, Inc