Sets

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Dept. of Computer Science
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Lecture Outline

2.1 Sets

- Definition of Set
- Representation of a Set
- Different Types of Sets
- Standard Numerical Sets
- $\in$-Notation
- Venn diagram
- Equal Set, Subset, Proper Subset, Cardinality of a Set, Super Set, Power Set, Ordered *n*-tuples, Cartesian Product of Sets

Objectives and Outcomes

- **Objectives**: To understand Set, element/member of a Set, Representation of Set, different types of Sets, Set notations, Venn diagram, Cardinality of a set, Power set, Ordered *n*-tuples, Cartesian Product of Sets.
- **Outcomes**: Students are expected to be able to explain different types of sets with examples, be able to understand different set notations, be able to draw Venn diagram, be able to find the cardinality and power set of a given set, be able to find the Cartesian product of sets.

Sets

- Definition1: A ***set*** is an unordered collection of objects.
- Definition2: Objects in a set are called ***elements***, or ***members*** of the set. A Set is said to ***contain*** its elements.
- **Capital letters** (e.g. A, B, C, O, N) are ordinarily used to denote **sets** and **lowercase letters** (e.g., *a, b, c, e*) are used to denote **elements of a sets**.
- Elements of a Set may NOT be the same type always!

Sets

- A set is defined only by the elements which it contains. Thus repeating an element, or changing the ordering of elements in the description of the set, does NOT change the set itself.
  - A = {1, 2, 4} = { 1, 1, 1, 2, 2, 4, 4, 4, 4} = {2, 4,1}

- Sets can have other sets as member.
  - B = { a, {a, b}, {{x}}, y }

Representation of a Set

- Basically there are two ways of representing a set:

1) **List notation/form**: All elements of the set are listed, the elements being separated by commas(**,**) and are enclosed by curly braces “**{**“ and “**}**”

   Example: B = { a, e, i, o, u}

   A = { 2, 4, 6, 8}

2) **Set builder notation/form, or, Rule method,** : A set is defined by specifying a property that elements of the set have in common.

   Example: B = { x| x is a vowel in the English alphabets}

   A = { x| x is an even integer between 1 and 8}

Different Types of Sets

- **Empty set/ Null set**: A set with no member is called empty/null set. Empty set is denoted by $\varnothing$. Empty set is also denoted by { }.

  Examples: B = { }, A = { x is a multiple of 4 | x is odd }

  Note: $\varnothing \neq \{\varnothing\}$

- **Singleton set**: A set with one element is called a singleton set.

  Examples: B = {4} , S = {a}

- **Finite set** : A set with finite number of elements in it, is called a finite set.
  Example: A = { 1, 3, 4,77}

- **Infinite set** : An infinite set is a set which contains infinite number of elements.

  Examples: N = { 0, 1, 2, 3, ……….}

  B = { 1, 1/3, 1/9, 1/27, ……….}

Some Examples of Sets

- {1, 2, 3} is the set containing elements "1" and "2" and "3."
- {1, 1, 2, 3, 3} = {1, 2, 3} since repetition is irrelevant.
- {1, 2, 3} = {3, 2, 1} since sets are unordered *(order of elements does NOT matter)*
- {0,1, 2, 3, …} is a way we denote an infinite set (in this case, the natural numbers).
- $\varnothing$ = {} is the empty set, or the set containing no elements.
- A = { a, 2, Fred, New York } ==> Although elements are usually same type, but they may be of different types

Sets: More examples

- Example 1 : The set V of all vowels in the English alphabet cab be written as V = { a, e, i, o, u }.

- Example 2: The set O of odd positive integers less than 10 can be denoted by { 1, 3, 5, 7, 9 }.

- Example 4 : The set of positive integers less than 100 can be denoted by { 1, 2, 3, ……….,99 }.

Standard Numerical Sets

N = {0, 1, 2, 3, …}, natural numbers

Z = {…,-2, -1, 0, 1, 2, …}, integers

Z$^+$ = {1, 2, 3, …}, positive integers

Q = $\{p/q \mid p \in \mathbf{Z},\ q \in \mathbf{Z}, \text{ and } q \neq 0\}$, rational numbers

R = real numbers

- The real numbers: **R ==>** contains any decimal number of arbitrary precision
- The rational numbers: **Q ==>** these are decimal numbers whose decimal expansion repeats

$\in$-Notation

- The Greek letter **"$\in$"** (**epsilon**) is used to denote that an object is an *element* of a set. When crossed out **"$\notin$"** denotes that the object is *not an element*."

  Example: $3 \in S$ reads: "3 is an element of the set *S* ".

  $3 \notin S$ reads: "3 is an not element of the set *S* ".

- **Q: Which of the following are true:**

  1. $3 \in \mathbf{R}$
  2. $-3 \in \mathbf{N}$
  3. $-3 \in \mathbf{R}$
  4. $0 \notin \mathbf{Z}^+$
  5. $\exists\, x,\ \ x \in \mathbf{R} \wedge x^2 = -5$

$\in$-Notation

Answers:

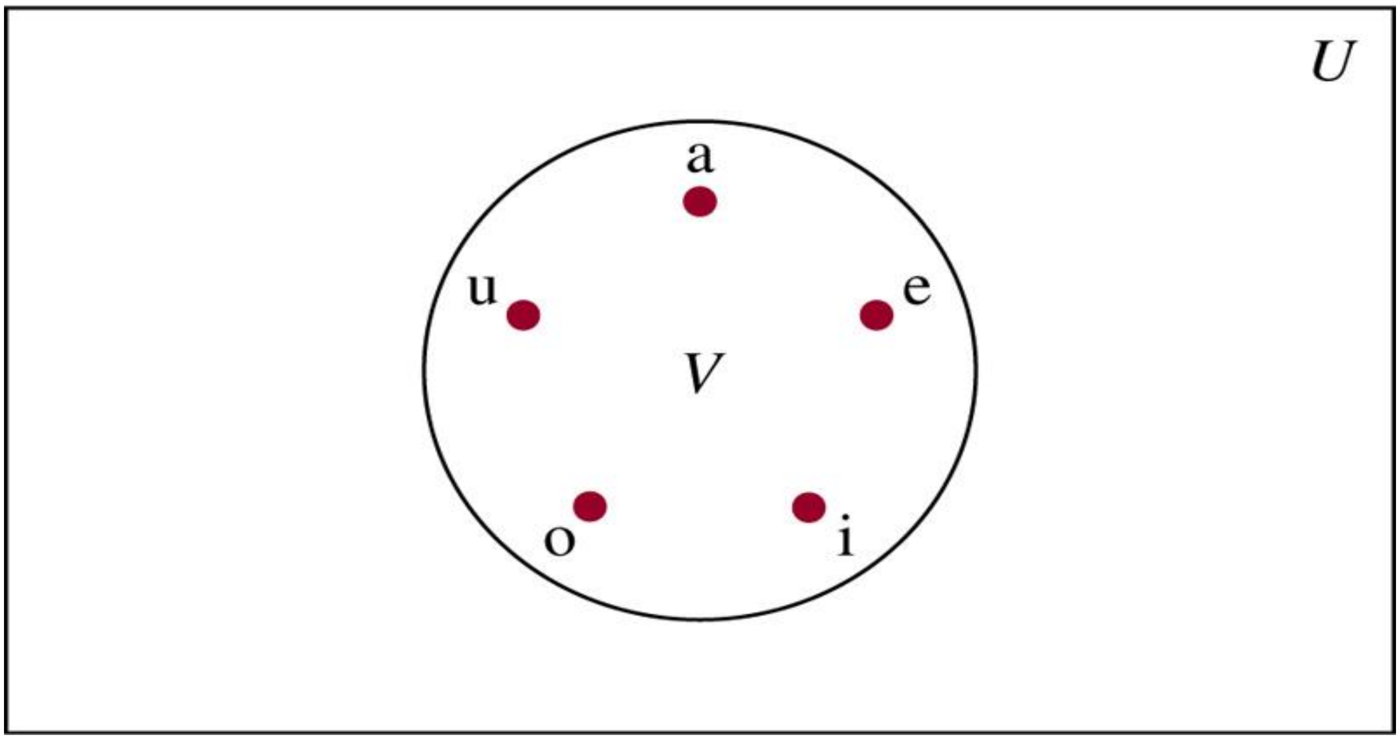
1. $3 \in \mathbf{R}$. True: 3 is a real number.
2. $-3 \in \mathbf{N}$. False: natural numbers don't contain negatives.
3. $-3 \in \mathbf{R}$. True: -3 is a real number.
4. $0 \notin \mathbf{Z}^+$. True: 0 is NOT a positive integer.
5. $\exists x\ x \in \mathbf{R} \wedge x^2 = -5$. False: square of a real number is non-negative, so can't be -5.

Venn diagram

- Sets can be represented graphically using **Venn diagrams**.
- In Venn diagrams, the **universal set U**, which contains all the objects under consideration, is represented by a **rectangle**.

  Note: the universal set varies depending on which objects are of interest
- Inside the rectangle,
  - **Circles** or other geometrical figures are used to represent **sets**.
  - Sometimes **points** are used to represent the particular **elements** of the set.
- Venn diagrams are often used to indicate the relationships between sets.

FIGURE 1
Venn Diagram for the Set of Vowels



Equal Set

- *Definition 3*: **Two sets are *equal* if and only if they have the same elements.**

  ***A = B* iff $\forall x(x \in A \leftrightarrow x \in B)$**

- Two sets A and B are said to be equal if and only if every element of A is an element of B and consequently every element of B is an element of A; that is $A \subseteq B$ and $B \subseteq A$ and it is written as $A = B$
- Example 6 (p.113) : The sets { 1, 3, 5} & { 3, 5, 1 } are equal because they have the same elements.

Subset

- Definition 4: **The set A is a subset of B if and only if every element of A is also an element of B.**
- We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.
- $A \subseteq B$ if and only if the quantification

  $\forall x(x \in A \rightarrow x \in B)$ is true
- **Note**: Every non-empty set S is guaranteed to have at least two subsets, the empty set and the set S itself, that is $\varnothing \subseteq \mathbf{S}$ and $\mathbf{S} \subseteq \mathbf{S}$

Subset

- **Theorem 1**: For every non-empty set S,

  (1) $\varnothing \subseteq S$, and

  (2) $S \subseteq S$

- ***Note***: If $A \subseteq B$ and $B \subseteq A$, then $A=B$
- Sets may have other sets as members

  $A = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$

  $B = \{ x \mid x \text{ is a subset of the set } \{a, b\}\}$

  Note: These two sets above are equal, that is, $A = B$

- **Note** : In the above example, **$\{a\} \in A$, but $a \notin A$**

Proper Subset

- **Proper subset:** Any subset A is said to be proper subset of another set B if A is a subset of B, but there is at least one element of B which does not belong to A,

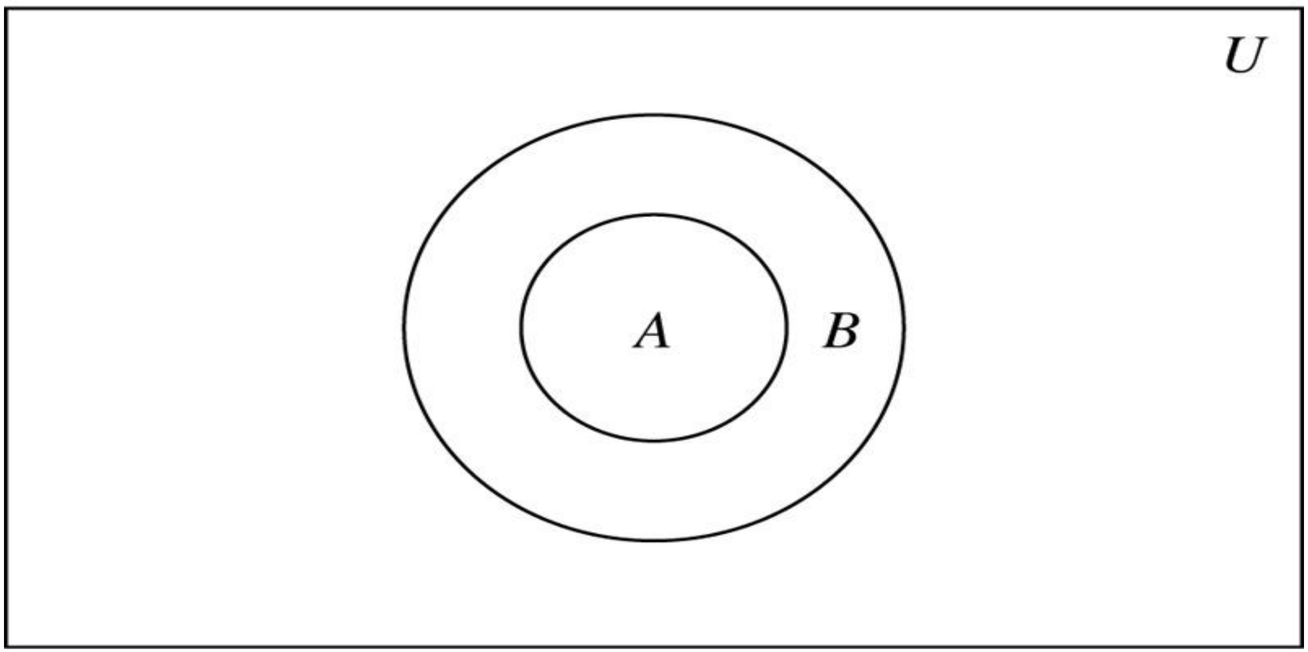
  i.e., **if $A \subseteq B$ but $A \neq B$.**

  $\forall x(x \in A \rightarrow x \in B) \wedge \exists x (x \in B \wedge x \notin A)$

- **$A \subset B$** means "A is a proper subset of B."
- **Example**: A = { 1, 5 }, B = { 1, 5, 6}

  Here, A is a proper subset of B, i.e., **$A \subset B$**

FIGURE 2 Venn Diagram Showing that *A* is a Subset of *B*



Cardinality of a Set

- **Definition**: If there are exactly n distinct members in the set S (n is a nonnegative integer), we say that S is a *finite set* and that n is the *cardinality* of S.
- The ***cardinality*** of a set is the **number of distinct elements** in the set.
- $|S|$ denotes the cardinality of set S that has n number of elements; i.e., $|S| = n$
- *Examples:*

  $|\varnothing| = 0$

  $|\{1, 5, 7, 8\}| = 4$

Cardinality of a Set

- Question: Compute cardinality of each of the sets.
  1. {1, -13, 4, -13, 1}
  2. {3, {1,2,3,4}, ∅}
  3. {}
  4. { {}, {{}}, {{{}}} }
- ***Hint:*** After eliminating the repetitions/redundancies just look at the number of top level commas and add 1 (except for the empty set).

Answers

1. $|\{1, -13, 4, -13, 1\}| = |\{1, -13, 4\}| = 3$
2. $|\{3, \{1,2,3,4\}, \varnothing\}| = 3$
3. $|\{\}| = |\varnothing| = 0$
4. $|\{ \{\}, \{\{\}\}, \{\{\{\}\}\} \}| = |\{ \varnothing, \{\varnothing\}, \{\{\varnothing\}\}| = 3$

Cardinality of a Set: More examples

- Example 9: Let A be the set of odd positive integers less than 10. Then $|A| = 5$
- Example 10: Let A be the set of letters in the English alphabet. Then $|A| = 26$
- Example 11: Because null set has no elements, it follows that, $|\varnothing| = 0$
- Examples:
  - The cardinality of the set $\{\varnothing\}$ is 1, i.e., $|\{\varnothing\}| = 1$
  - If $B = \{3,3,3,3,3\}$, $|B| = 1$
  - If $C = \{ \varnothing, \{\varnothing\}, \{\varnothing,\{\varnothing\}\} \}$, $| C | = 3$.
  - If $S = \{0,1,2,3,\ldots\}$, **$|S|$ is infinite**

Super Set

- If A is a subset of B, then B is called the super set of A and written as **$B \supseteq A$** which is read as **"B is a super set of A"**.

- **$A \supseteq V$** means "**A is a superset of V**."

- Example: { 1, 4, 7, 8} is a superset of the set { 4, 7 }

Quick Examples

- $\{1,2,3\} \subseteq \{1,2,3,4,5\}$ Yes/No?
- $\{1,2,3\} \subset \{1,2,3,4,5\}$ Yes/No?

- Is $\emptyset \subseteq \{1,2,3\}$? Yes!
- Is $\emptyset \in \{1,2,3\}$? **No!**
- Is $\emptyset \subseteq \{\emptyset,1,2,3\}$? Yes!
- Is $\emptyset \in \{\emptyset,1,2,3\}$? Yes!

Quiz: Yes/No?

1. Is $\{x\} \subseteq \{x\}$?

2. Is $\{x\} \in \{x,\{x\}\}$?

3. Is $\{x\} \subseteq \{x,\{x\}\}$?

4. Is $\{x\} \in \{x\}$?

Answer: 1. Yes 2. Yes 3. Yes 4. No

Power Set

- **Definition : The *power set* of S is the set of all subsets of the set S.**
- We say, "P(S) is the set of all subsets of S"
- The power set of S is denoted by ***P(S)***

- **Note**: If a set has n elements, then its power set has 2^n elements.

Power Set: Examples

- Example 13: What is the power set of the set {0,1,2 } ?
- Solution: The power set P ({ 0, 1, 2 }) is the set of all subsets of { 0, 1, 2 } . Hence,

  $P(\{0, 1, 2\}) = \{\varnothing, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$
- Example 14: What is the power set of empty set? What is the power set of the set {∅} ?
- Solution:

  $P(\varnothing) = \{\varnothing\}$

  $P(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$

Ordered *n*-tuples

- The order of elements in a collection is often important. Because sets are unordered, a different structure is needed to represent ordered collections. This is provided by **ordered *n*-tuples**.
- Definition : The ***Ordered n-tuple*** $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, ……….and a_n as its nth element.

Ordered *n*-tuples

- Two ordered *n*-tuples are equal if and only if each corresponding pair of their elements is equal.

  In other words, $(a_1, a_2, \ldots.., a_n) = (b_1, b_2, \ldots\ldots.., b_n)$

  if and only if $a_i = b_i$ for $i = 1, 2, \ldots\ldots, n$

- 2-tuples are called ***ordered pairs***. The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$
- **Note**: (a, b) and (b, a) are not equal unless $a = b$

Ordered *n*-tuples

- Notationally, *n*-tuples look like sets except that **curly braces** are replaced by **parentheses.**
- As opposed to sets, **repetition** and **ordering DO MATTER** with *n*-tuples.

  $(11, 11, 11, 12, 13) \neq (11, 12, 13)$

  But, $\{11, 11, 11, 12, 13\} = \{11, 12, 13\}$
- **Note:**
  - For **set**, we use **curly braces { }**
  - For ***n*-tuples**, or **ordered pairs**, we use **parentheses ( )**

Cartesian Product of Sets

- Definition : Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.

  $$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

Cartesian Product of Sets: Example

- Example 16: What is the Cartesian product of A = { 1, 2 } and B = { a, b, c} ?
- **Solution**:

  A X B = { (1,a), (1,b), (1,c), (2,a), (2,b), (2,c) }

- **Note** : $\boldsymbol{A \times B}$ and $\boldsymbol{B \times A}$ are **not equal**, unless A=∅ or B=∅ or A=B
- **Note**: Cartesian product of more than two sets can be defined
- **Note**: A **subset** *R* of the Cartesian product *A* X *B* is called a **Relation** from the set A to the set B (Relation will be covered in the final term)
- Practice @ Home: Example 17

Example:
Cartesian Product of Three Sets

- **Question**: If $A = \{1,2\}$, $B = \{3,4\}$, $C = \{5,6,7\}$, what is $A \times B \times C$?
- **Solution**:

$A \times B \times C = \{ (1,3,5), (1,3,6), (1,3,7), (1,4,5), (1,4,6), (1,4,7), (2,3,5), (2,3,6), (2,3,7), (2,4,5), (2,4,6), (2,4,7) \}$

- **Note:** $|A \times B \times C| = |A|.|B|.|C| = 2.2.3 = 12$

- Practice @ Home: Example 18

Practice @ Home

- **Relevant odd-numbered Exercises** from your text book
- **Exercises:** 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 23, 27, 29, 33, 35

Books

1. *Discrete Mathematics and its applications with combinatorics and graph theory (7th edition)* by Kenneth H. Rosen [Indian Adaptation by KAMALA KRITHIVASAN], published by McGraw-Hill

References

1. Discrete Mathematics, *Richard Johnsonbaugh*, Pearson education, Inc.
2. Discrete Mathematical Structures, *Bernard Kolman*, *Robert C. Busby*, *Sharon Ross,* Prentice-Hall, Inc.
3. *SCHAUM'S outlines Discrete Mathematics(2nd edition)*, by *Seymour Lipschutz*, *Marc Lipson*