Set Operations

Course Code: CSC 1204 Course Title: Discrete Mathematics

Dept. of Computer Science
Faculty of Science and Technology

Lecture No:	**8**	**Week No:**	**4**	**Semester:**	**Summer 2021-2022**
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Lecture Outline

2.2 Set Operations

- Union ($\cup$)
- Intersection ($\cap$)
- Disjoint Sets
- Principle of inclusion-exclusion
- Difference ("$-$")
- Complement("$\overline{}$")
- Symmetric Difference ($\oplus$)
- Set Identities
- Computer Representation of Sets

Objectives and Outcomes

- **Objectives**: To understand different types of set operations, to understand various ways of representing sets using a computer.

- **Outcomes**: Students are expected to be able to perform different types of set operations, be able to represent a set with a computer.

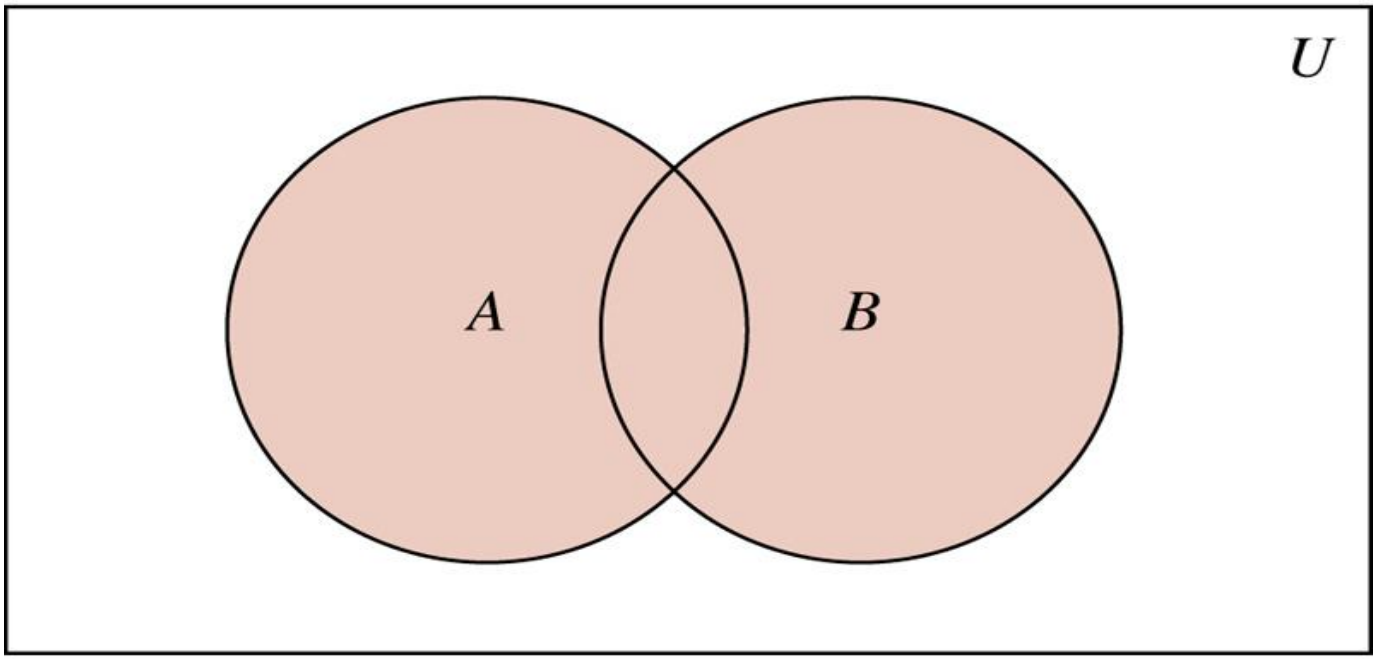
Union of Sets

- ***Definition***: Let *A* and *B* be sets. The union of the sets *A* and *B*, denoted by *A* U *B*, is the set that contains those elements that are either in *A* or in *B*, or in both.
- An element *x* belongs to the union of the sets *A* and *B* if and only if **x belongs to A or x belongs to B**. This tells us that

$$\mathbf{A \cup B = \{ x \mid x \in A \vee x \in B \}}$$

- Example 1: The **union** of the sets {1, 3, 5} and {1, 2, 3} is the set {1, 2, 3, 5}; that is, {1,3, 5} U {1, 2, 3} = {1, 2, 3, 5}
- Example 2: If A = {Cyndi, Faisal, Abdullah}, and B ={Laila, Rahim}, then, A ∪ B = {Cyndi, Faisal, Abdullah, Laila, Rahim}

FIGURE 1 Venn Diagram Representing the Union of *A* and *B*.



$A \cup B$ is shaded.

Intersection of Sets

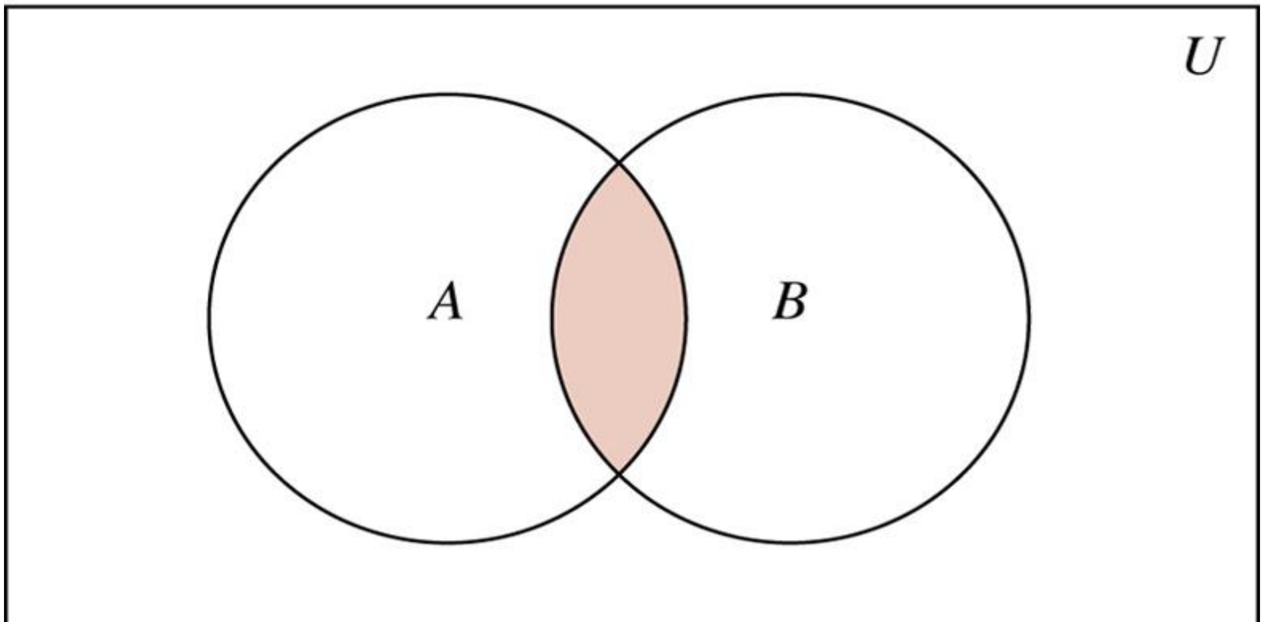
- ***Definition*** : Let ***A*** and ***B*** be sets. The *intersection* of the sets *A* and *B*, denoted by $A \cap B$, is the set containing those elements in both *A* and *B*.
- An element x belongs to the *intersection* of the sets *A* and *B* if and only if **x belongs to A and x belongs to B**

$$A \cap B = \{ x \mid x \in A \wedge x \in B \}$$

- Example 3: The *intersection* of the sets {1, 3, 5} and {1, 2, 3} is the set {1, 3}; that is, $\{1,3,5\} \cap \{1,2,3\} = \{1,3\}$.

- Example : If $B = \{x \mid x \text{ is an odd positive integer and } x \le 5\}$, and $C = \{x \mid x \text{ is a prime number less than } 11\}$, then $B \cap C = \{3, 5\}$

FIGURE 2: Venn Diagram Representing the Intersection of *A* and *B*

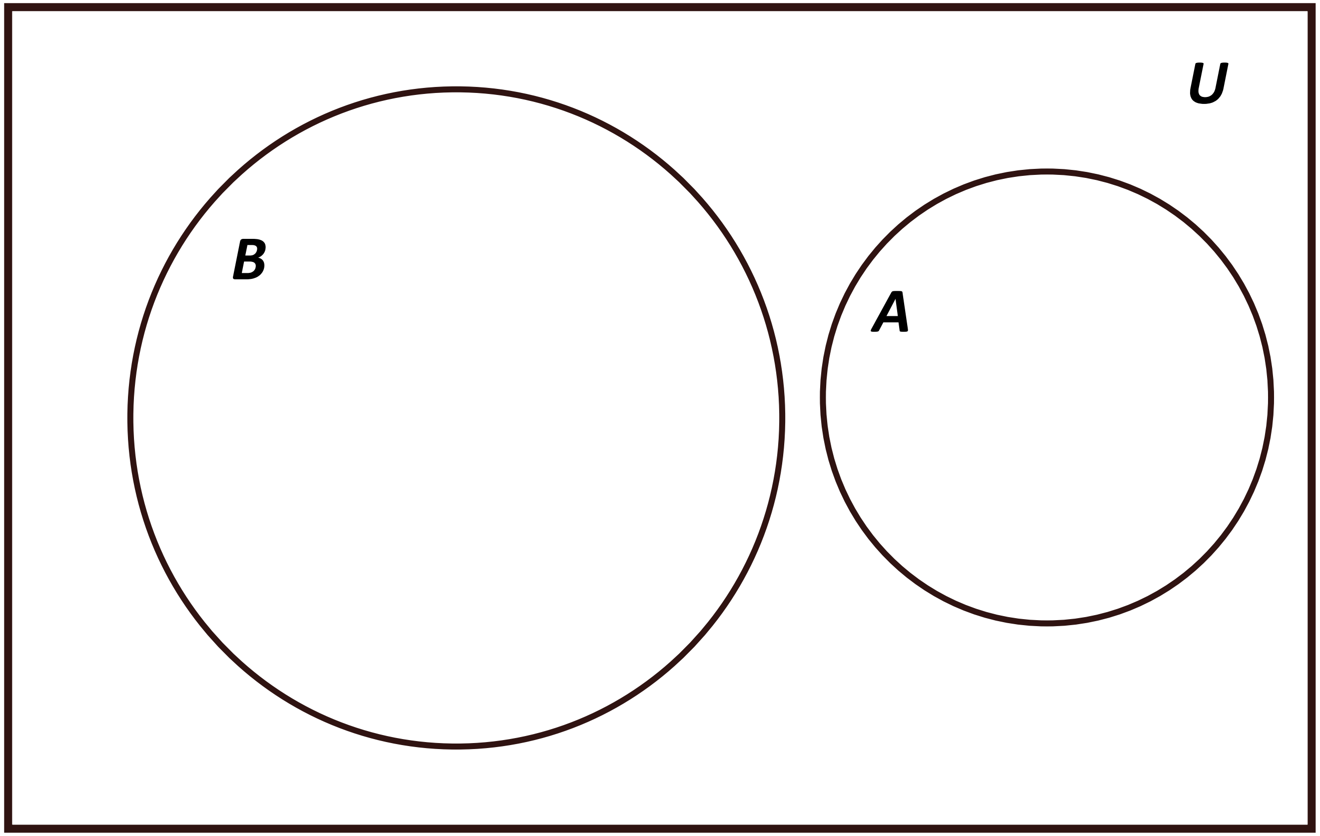


$A \cap B$ is shaded.

Disjoint Sets

- ***Definition***: Two sets are ***disjoint*** if their intersection is the empty set.
- Let A and B be sets. **If $A \cap B = \varnothing$** , that is if *A* and *B* do not have any element(s) in common, then *A* and *B* are said to be disjoint or non-intersecting.
- *Disjoint ==> Non-intersecting*
- **Example 5:** Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$. Because $A \cap B = \varnothing$, A and B are ***disjoint***.

Venn Diagram showing Disjoint of the two sets *A* and *B*



Principle of inclusion-exclusion

- Question: How many elements are there in the union of two sets *A* and *B*? *Or,* What is the ***cardinality*** of a union of two sets *A* and *B*?
- **Answer**: The number of elements in the union of the two sets is the sum of the number of elements in the sets minus the number of elements in their intersection.

  $$|A \cup B| = |A| + |B| - |A \cap B|$$

- The generalization of this result to unions of an arbitrary number of sets is called the **Principle of inclusion-exclusion**

Difference of Two Sets

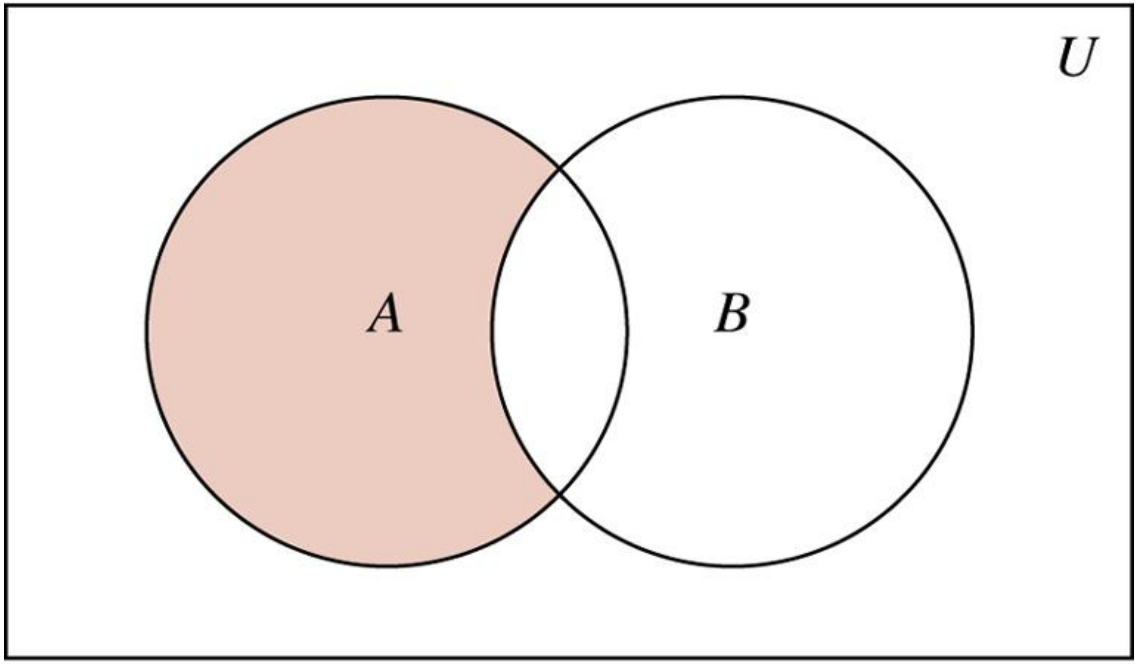
- ***Definition*** : The ***difference*** of the sets *A* and *B*, denoted by *A* **–** *B*, is the set containing those elements that are in *A* but not in *B*.
- An element *x* belongs to the difference of *A* and *B* if and only if $x \in A$ **and** $x \notin B$. This tell us that

$$A - B = \{ x \mid x \in A \wedge x \notin B \}$$

- The difference of A and B is also called the *complement of B with respect to A.*

$$A - B = A \cap \overline{B}$$

FIGURE 3: Venn Diagram for the Difference of *A* and *B*



$A - B$ is shaded.

Difference of **Two Sets: Examples**

- Example 6: The difference of {1, 3, 5} and {1, 2, 3} is the set {5}; that is, {1, 3, 5} – {1, 2, 3} = {5}.
- Note: This is the different from the difference of {1, 2, 3} and {1, 3, 5} which is the set {2}.
  - {1, 2, 3} – {1, 3, 5} = {2}.
- Example : Let $A = \{ a, b, c\}$ and $B = \{ b, c, d, e \}$.

$$A - B = \{ a \}$$

$$B - A = \{ d, e \}$$

Complement of a Set

- ***Definition***: Let U be the universal set. The ***complement*** of the set A, denoted by $\bar{A}$, is the complement of A with respect to U.
- An element x belongs to $\bar{A}$ if and only if $x \notin A$

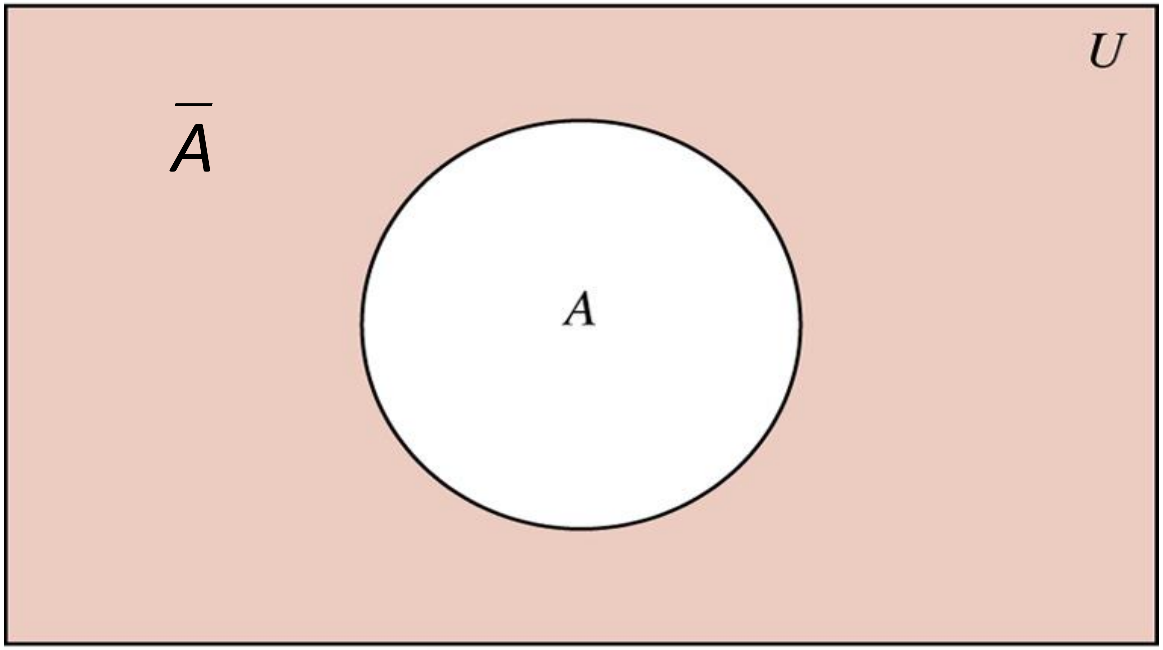
  $$\bar{A} = \{ x \mid x \notin A \}$$

- In other words, the complement of the set A is $U - A$,

  $$\bar{A} = U - A$$

- Practice yourself: *Example 8 & Example 9*

FIGURE 4: Venn Diagram for the Complement of the Set *A*



$\overline{A}$ is shaded.

Symmetric difference of Two Sets

- ***Definition***: The symmetric difference of two sets *A* and *B*, denoted by $A \oplus B$, is the set of elements that belongs to *A* or to *B*, but not to both *A* and *B*.

  A ⊕ B = The set containing those elements in exactly one of A and B.

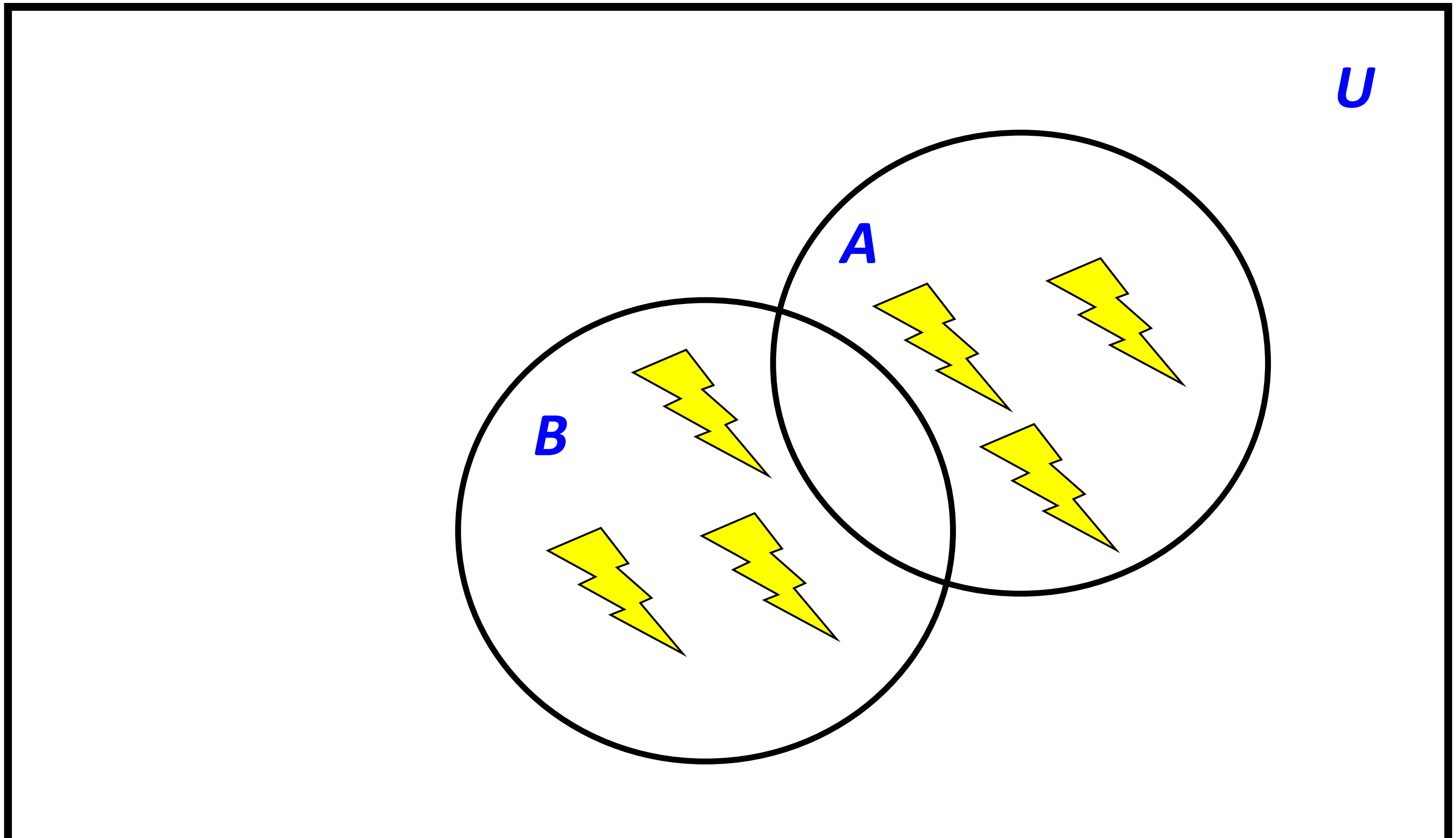
  $$\mathbf{A \oplus B = \{ x \mid (x \in A \land x \notin B) \lor (x \in B \land x \notin A) \}}$$

  $$= \mathbf{(A - B) \cup (B - A)}$$

- Example: If A = { -3, 0, 1, 2} and B = { 1, 2, 3, 4 }, then A ⊕ B = { -3, 0, 3, 4}
- *Alternate Solution: A – B = {–3, 0 }* and *B – A = { 3, 4 }.*

  Therefore, **A ⊕ B** = *(A – B)* U *(B – A)* = {–3, 0, 3, 4 }

Figure: Venn Diagram for The *Symmetric Difference of two sets A and B*, A ⊕ B



Set Identities

- To prove set identities:
- Show that **each is a subset of the other**
  - Show that $A \subseteq B$ and that $A \supseteq B$
- Using **membership tables**
  - Like truth tables
- Use logical equivalences to prove equivalent set definitions

Set Identities

TABLE 1 Set Identities.

Identity	*Name*
$A \cup \varnothing = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \varnothing$	Complement laws

Set Identities

- This table is gotten from the previous table(TABLE 6, page 24) of logical identities by rewriting as follows:

  - *Disjunction* "$\vee$" becomes *union* "$\cup$"
  - *Conjunction* "$\wedge$" becomes *intersection* "$\cap$"
  - *Negation* "$\neg$" becomes *complementation* "$-$"
  - ***False*** "**F**" becomes *the* ***empty set*** $\varnothing$
  - ***True*** "**T**" becomes *the* ***universal set*** **U**

Membership Table

- Membership Table: A table displaying the membership of elements in sets.

- <u>Example 13</u>: Use membership table Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution of Example 13

TABLE 2 A Membership Table for the Distributive Property.

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Set Identities

Prove that $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ using a membership table.

A	B	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$	$A \cup B$	$\overline{A \cup B}$
1	1	0	0	0	1	0
1	0	0	1	0	1	0
0	1	1	0	0	1	0
0	0	1	1	1	0	1

Generalized Unions and Intersections

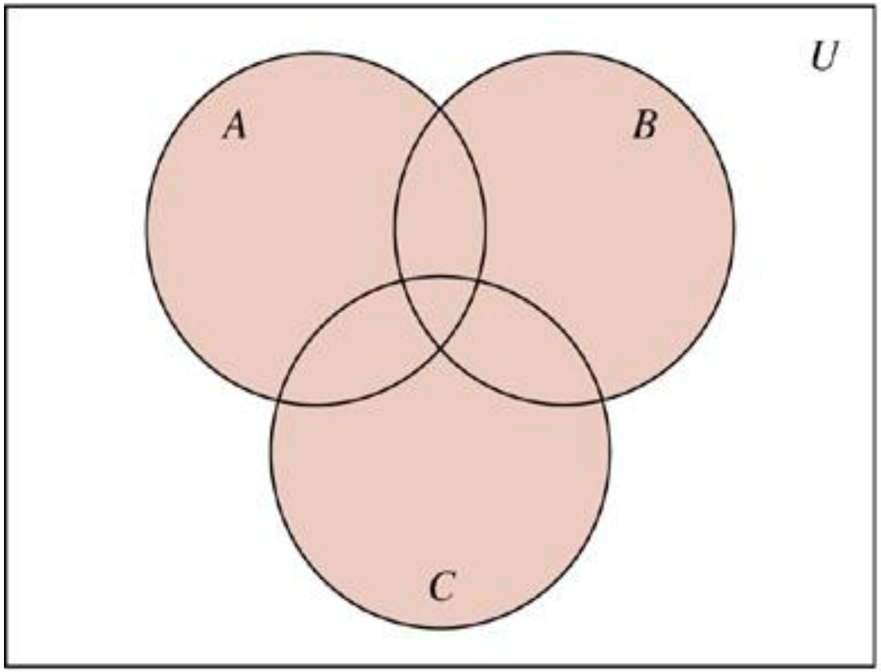
- **Definition** : The ***union*** of a collection of sets is the set containing those elements that are members of at least one set in the collection.

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

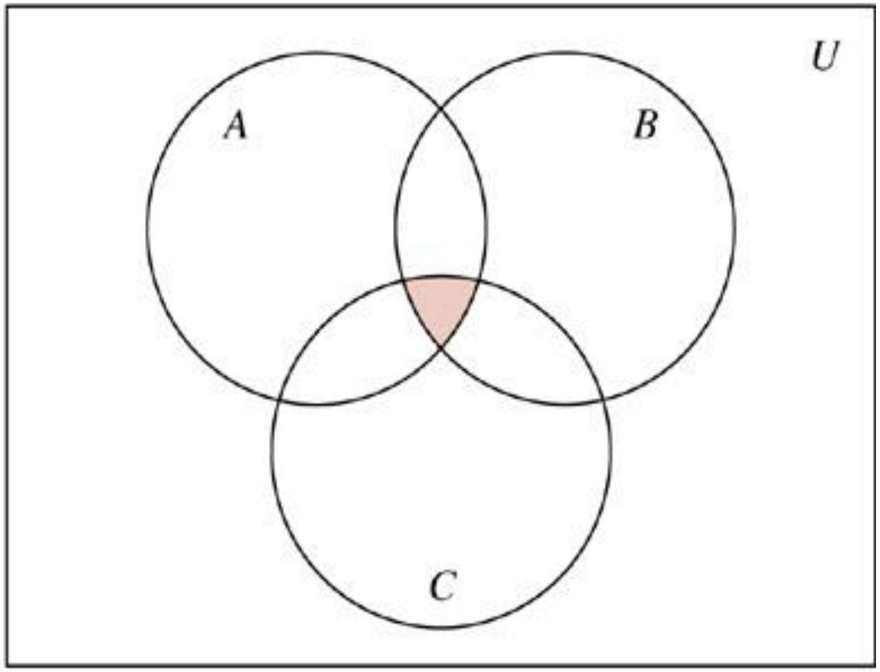
- **Definition**: The ***intersection*** of a collection of sets is the set containing those elements that are members of all the sets in the collection.

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

FIGURE 5: The **Union** and **Intersection** of *A*, *B*, and *C*



(a) $A \cup B \cup C$ is shaded.



(b) $A \cap B \cap C$ is shaded.

Example: Union and Intersection of three sets

- **Example 15 :** Let A = {0, 2, 4, 6, 8}, B = {0, 1, 2, 3, 4}, and C = {0, 3, 6, 9}. What are **A U B U C** and **A ∩ B ∩ C** ?

Solution:

A U B U C = { 0, 1, 2, 3, 4, 6, 8, 9}

A ∩ B ∩ C = { 0 }

- Class Work: Show the above sets in a Venn Diagram. Assume universal set U.

Computer Representation of Sets

- There are various ways to represent sets using a computer. One method is to store the elements of the set in an unordered fashion. However, if this is done, the operations of computing the union, intersection, or difference of two sets would be time-consuming, because each of these operations would require a large amount of searching for elements.

- We use a method for storing elements using an arbitrary ordering of the elements of the universal set. This method of representing sets makes computing combinations of sets easy.

Example 18

- Let U = { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 }

What bit string represents the subset of all odd integers in *U*?

What bit string represents the subset of all even integers in *U*?

What bit string represents the subset of integers not exceeding 5 in *U*?

- **Solution**:

{ 1, 3, 5, 7, 9 } ==> 1010101010

{ 2, 4, 6, 8, 10} ==> 0101010101

{ 1, 2, 3, 4, 5} ==> 1111100000

Example 19

- What is the bit string for the complement of the set { 1, 3, 5, 7, 9}, where U = { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 }?
- **Solution:**

  The bit string for the set { 1, 3, 5, 7, 9} is 1010101010.

  The bit string for the complement of this set is obtained by replacing 0s with 1s and vice versa.

  This yields the string 0101010101 which corresponds to the set {2, 4, 6, 8, 10}

Practice @ Home

- **Relevant odd-numbered Exercises** from your text book

Books

1. *Discrete Mathematics and its applications with combinatorics and graph theory (7th edition)* by Kenneth H. Rosen [Indian Adaptation by KAMALA KRITHIVASAN], published by McGraw-Hill

References

1. Discrete Mathematics, *Richard Johnsonbaugh*, Pearson education, Inc.
2. Discrete Mathematical Structures, *Bernard Kolman*, *Robert C. Busby*, *Sharon Ross,* Prentice-Hall, Inc.
3. *SCHAUM'S outlines Discrete Mathematics(2nd edition)*, by *Seymour Lipschutz*, *Marc Lipson*