Propositional Equivalences

Course Code: CSC 1204 Course Title: Discrete Mathematics

Dept. of Computer Science
Faculty of Science and Technology

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Lecturer:	*S.M. Abdur Rouf Bhuiyan [arouf@aiub.edu]*				

Lecture Outline

1.2 Propositional Equivalences

- **Tautology**
- **Contradiction**
- **Contingency**
- **Logical Equivalences**

Objectives and Outcomes

- **Objectives**: To understand the terms Tautology, Contradiction, Contingence with examples, to understand the standard logical equivalences, to determine whether a compound proposition is a Tautology or Contradiction, to determine whether two compound propositions are logically equivalent.
- **Outcomes**: Students are expected to be able to write the definitions of Tautology, Contradiction and Contingency with examples, be able to determine whether a compound proposition is a Tautology or Contradiction using a Truth Table and standard logical equivalences, be able to determine whether two compound propositions are logically equivalent using a Truth Table and logical equivalences.

Tautology

Tautology: A compound proposition that is always true is called a tautology.

Examples:

a) $p \vee \neg p$

b) The professor is either a woman or a man

c) People either like watching TV or they don't

Contradiction

Contradiction: A compound proposition that is always false is called a contradiction.

Examples:

a) $p \wedge \neg p$

b) x is prime and x is an even integer greater than 8

c) All men are good and all men are bad

Examples of *Tautology* and *Contradiction*

TABLE 1 Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Contingency

Contingency: A compound proposition that is neither a tautology nor a contradiction is called a contingency.
In other words, a compound proposition whose truth value is not constant is called a contingency.

Examples:

a) $p \rightarrow \neg p$

b) P

c) $\neg p$

How to determine whether a compound proposition is a Tautology or Contradiction?

- We can determine whether a compound proposition is a **Tautology** or **Contradiction** in **two ways**:
  1) Using a **truth table** – The **easiest** way to see if a compound proposition is a tautology or contradiction is to use a ***truth table.*** Show that the compound proposition is always true
  2) Using (**Laws of**) **Logical Equivalences**

Tautology : Example

Show that $[\neg p \wedge (p \vee q)] \rightarrow q$ is a *tautology using a Truth Table*

Solution

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T				
T	F				
F	T				
F	F				

Solution

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F			
T	F	F			
F	T	T			
F	F	T			

Solution

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F	T		
T	F	F	T		
F	T	T	T		
F	F	T	F		

Solution

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F	T	F	
T	F	F	T	F	
F	T	T	T	T	
F	F	T	F	F	

Solution

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

Since the truth table shows all the true values of compound proposition $[\neg p \wedge (p \vee q)] \rightarrow q$ are true(T), so it is a tautology.

Class Work

1) Determine whether $\neg (p \wedge q) \vee p$ is a tautology or contradiction.

2) Determine whether $p \wedge (q \wedge \neg p)$ is a tautology or contradiction.

Logical Equivalences

- Compound propositions that have the same truth values in all possible cases are called **logically equivalent**.

- ***Definition*** : Compound propositions p and q are *logically equivalent* if $p \leftrightarrow q$ is a tautology (denoted by $\boldsymbol{p \equiv q}$ ***or*** $p \Leftrightarrow q$)

  NOTE: We will use the notation: $\boldsymbol{p \equiv q}$

How to determine whether two compound propositions are logically equivalent?

- We can determine whether two compound propositions are logically equivalent in two ways:
  1) Using a **Truth Table**
  2) Using (laws of) **Logical Equivalences**

Using a Truth Table to determine whether two compound propositions are logically equivalent

- **Two compound propositions are *logically equivalent*** if they always have the same truth values in the corresponding rows.
- Construct a truth table for the given two compound propositions [in one table]
- **If the truth values of both of the compound propositions are same in the corresponding rows, then they are logically equivalent.**
- If the true values of both of the compound propositions are different in one or more rows, then they are NOT logically equivalent.

Example 1

Show that ***p* ↔ *q*** is **logically equivalent** to **(p → q) ∧ (q → p)**

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Since the truth values of both of the compound propositions are same in the corresponding rows, they are logically equivalent.

Class Work

Show that $\boldsymbol{p} \vee (\boldsymbol{q} \wedge \boldsymbol{r})$ and $(\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\boldsymbol{p} \vee \boldsymbol{r})$ are logically equivalent

Solution

TABLE 5 A Demonstration That $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ Are Logically Equivalent.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Since the truth values of both of the compound propositions are same in the corresponding rows, they are logically equivalent.

Logical Equivalences

Table 6 (page 24) → Rosen, 7[th] edition

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

A very Useful Logical Equivalence(**ULE**)

$$p \rightarrow q \equiv \neg p \vee q$$

Example 1

Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

Solution:

$$
\begin{aligned}
\neg(p \rightarrow q) &\equiv \neg(\neg p \vee q) && \text{by ULE} \\
&\equiv \neg(\neg p) \wedge \neg q && \text{by the second De Morgan law} \\
&\equiv p \wedge \neg q && \text{by the double negation law}
\end{aligned}
$$

Example 7 (page 26)

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution:

$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$	by the second De Morgan law
$\equiv \neg p \wedge [\neg(\neg p) \vee \neg q]$	by the first De Morgan law
$\equiv \neg p \wedge (p \vee \neg q)$	by the double negation law
$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	by the second distributive law
$\equiv \mathbf{F} \vee (\neg p \wedge \neg q)$	because $\neg p \wedge p \equiv \mathbf{F}$
$\equiv (\neg p \wedge \neg q) \vee \mathbf{F}$	by the commutative law for disjunction
$\equiv \neg p \wedge \neg q$	by the identity law for $\mathbf{F}$

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

Exercise

Show that $(\neg p \wedge (p \vee q)) \rightarrow q$ is a **tautology** using a series of logical equivalences.

Solution

$(\neg p \wedge (p \vee q)) \rightarrow q$	
$\equiv ((\neg p \wedge p) \vee (\neg p \wedge q)) \rightarrow q$	**Distributive Law**
$\equiv (\mathrm{F} \vee (\neg p \wedge q)) \rightarrow q$	**Negation Law**
$\equiv (\neg p \wedge q) \rightarrow q$	**Identity Law**
$\equiv \neg(\neg p \wedge q) \vee q$	**ULE**
$\equiv (\neg(\neg p) \vee \neg q) \vee q$	**De Morgan's Law**
$\equiv (p \vee \neg q) \vee q$	**Double Negation Law**
$\equiv p \vee (\neg q \vee q)$	**Associative Law**
$\equiv p \vee \mathrm{T}$	**Domination Law**
$\equiv \mathrm{T}$	

So, $(\neg p \wedge (p \vee q)) \rightarrow q$ is a tautology.

Summary

- What is Tautology and Contradiction? What is Contingency?
- How to show/determine whether two compound propositions are logically equivalent?
  - Using a truth table
  - Using logical equivalences
- How to show whether a compound proposition is a tautology?
  - Using a truth table
  - Using logical equivalences
- **Note**: **Make sure you learn the important Logical Equivalences in Table 6 (page 24)** & **ULE** ($p \rightarrow q \equiv \neg p \vee q$)
- **Practice @ Home**: **Relevant Odd-numbered Exercises (e.g. 1, 3, 7, 9, 11, 15, 17)**

Practice @ Home

* Practice questions 1-4 **without using a Truth Table**

1. Determine whether $(\neg p \land (q \to p)) \to \neg q$ is tautology.
2. Determine whether $(\neg q \land (p \to q)) \to \neg p$ is tautology.
3. Show that $(p \land (p \to q)) \to q$ is a tautology.
4. Show that $((p \to q) \land (p \to r))$ and $(p \to (q \land r))$ are logically equivalent.

** Practice relevant Odd-Numbered Exercises

==> 1, 3, 7, 9, 11, 15, 17

Answer 1

$(\neg p \wedge (q \rightarrow p)) \rightarrow \neg q$

$\equiv (\neg p \wedge (\neg q \vee p)) \rightarrow \neg q$ [ULE]

$\equiv ((\neg p \wedge \neg q) \vee (\neg p \wedge p)) \rightarrow \neg q$ [Distributive Law]

$\equiv ((\neg p \wedge \neg q) \vee F) \rightarrow \neg q$ [Negation Law]

$\equiv (\neg p \wedge \neg q) \rightarrow \neg q$ [Identity Law]

$\equiv \neg(\neg p \wedge \neg q) \vee \neg q$ [ULE]

$\equiv (p \vee q) \vee \neg q$ [De Morgan's & Double Negation Law]

$\equiv p \vee (q \vee \neg q)$ [Associative Law]

$\equiv p \vee T$ [Negation Law]

$\equiv T$ [Domination Law]

So, $(\neg p \wedge (q \rightarrow p)) \rightarrow \neg q$ is a tautology.

Answer 2

$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$

$\equiv (\neg q \wedge (\neg p \vee q)) \rightarrow \neg p$ [You MUST write the names of the laws]

$\equiv (\neg q \wedge \neg p) \vee (\neg q \wedge q)) \rightarrow \neg p$

$\equiv (\neg q \wedge \neg p) \vee F \rightarrow \neg p$

$\equiv (\neg q \wedge \neg p) \rightarrow \neg p$

$\equiv \neg (\neg q \wedge \neg p) \vee \neg p$

$\equiv q \vee p \vee \neg p$

$\equiv q \vee \mathrm{T}$

$\equiv \mathrm{T}$

So, $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$ is a tautology.

Answer 3

$(p \wedge (p \to q)) \to q$

$\equiv (p \wedge (\neg p \vee q)) \to q$	**ULE** (Substitution for $\to$)
$\equiv ((p \wedge \neg p) \vee (p \wedge q)) \to q$	Distributive Law
$\equiv (\, F \vee (p \wedge q)) \to q$	Negation Law
$\equiv (p \wedge q) \to q$	Identity Law
$\equiv \neg(p \wedge q) \vee q$	**ULE** (Substitution for $\to$)
$\equiv (\neg p \vee \neg q) \vee q$	First De Morgan's Law
$\equiv \neg p \vee (\neg q \vee q\,)$	Associative Law
$\equiv \neg p \vee T$	Negation Law
$\equiv T$	Domination Law

Books

- *Discrete Mathematics and its applications with combinatorics and graph theory (7th edition)* by Kenneth H. Rosen [Indian Adaptation by KAMALA KRITHIVASAN], published by McGraw-Hill

References

1. Discrete Mathematics, *Richard Johnsonbaugh*, Pearson education, Inc.
2. Discrete Mathematical Structures, *Bernard Kolman*, *Robert C. Busby*, *Sharon Ross,* Prentice-Hall, Inc.
3. *SCHAUM'S outlines Discrete Mathematics(2nd edition)*, by *Seymour Lipschutz*, *Marc Lipson*