Functions

Course Code: CSC 1204 Course Title: Discrete Mathematics

Dept. of Computer Science
Faculty of Science and Technology

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Lecture Outline

2.3 Functions

- Definition of Function
- Domain, Codomain, Range, Image, Preimage,
- One-to-one function
- Onto function
- One-to-one correspondence
- Inverse Functions
- Compositions of Functions
- Floor function
- Ceiling Function

Objectives and Outcomes

- **Objectives**: To understand what is function, domain, codomain, range, image, preimage; to understand different types of functions.
- **Outcomes**: Students are expected to be able to explain different types of functions with examples, be able to determine whether a function is one-to-one, onto, and/or one-to-one correspondence, be able to determine whether a function is invertible and find out the inverse of a function, be able to apply floor and ceiling functions.

Functions

- Definition 1: Let A and B be nonempty sets.

  A function f from A to B is an assignment of exactly one element of B to each element of A.

  We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.
- If f is a function from A to B, we write $f: A \to B$
- *Note*: Functions are sometimes called ***mappings*** or ***transformations***.

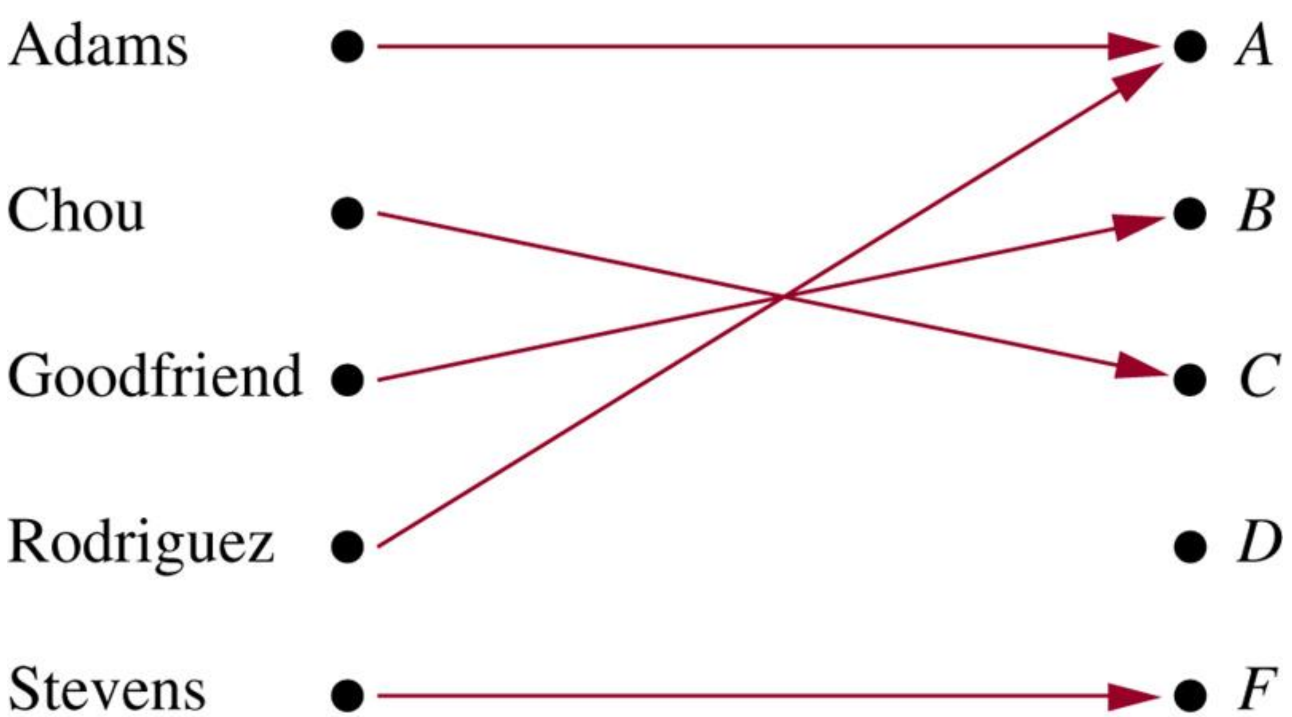
Functions

- Functions are specified in many different ways.
- Sometimes we explicitly state the assignments, as in Figure 1.
- Often we give a formula, such as $f(x) = x + 1$, to define a function.
- Other times we use a computer program to specify a function.

Functions

- A function $f: A \rightarrow B$ can also be defined in terms of a **relation** from A to B. [we will cover **Relation** in final term]
- A relation from A to B is just a **subset** of $\boldsymbol{A X B}$
- A relation from A to B that contains one, and only one, ordered pair (a, b) for every element $a \in A$, defines a function f from A to B. This function is defined by the assignment $f(a)=b$, where (a, b) is the unique ordered pair in the relation that has a as its first element.

FIGURE 1: Assignment of Grades in a Discrete Mathematics Class



Some Function Terminology

Definition 2: If f is a function from A to B, we say that ***A*** is the **domain** of f and ***B*** is the **codomain** of f.

- If $f(a) = b$, a is the ***preimage*** of b and b is the ***image*** of a.
- ***Range*** of f *is* the set of all images of elements of A.
- Also, if f is a function from A to B, we say that f maps from A to B.

Some Function Terminology

- If f is a function from A to B, we write $f: A \rightarrow B$
  - A is the *domain* of f
  - B is the *codomain* of f
  - If $f(a) = b$,
    - a is called the preimage of b
    - b is called the image of a
- ***Range of f*** : the set of all images of elements of A

Range versus **Codomain**

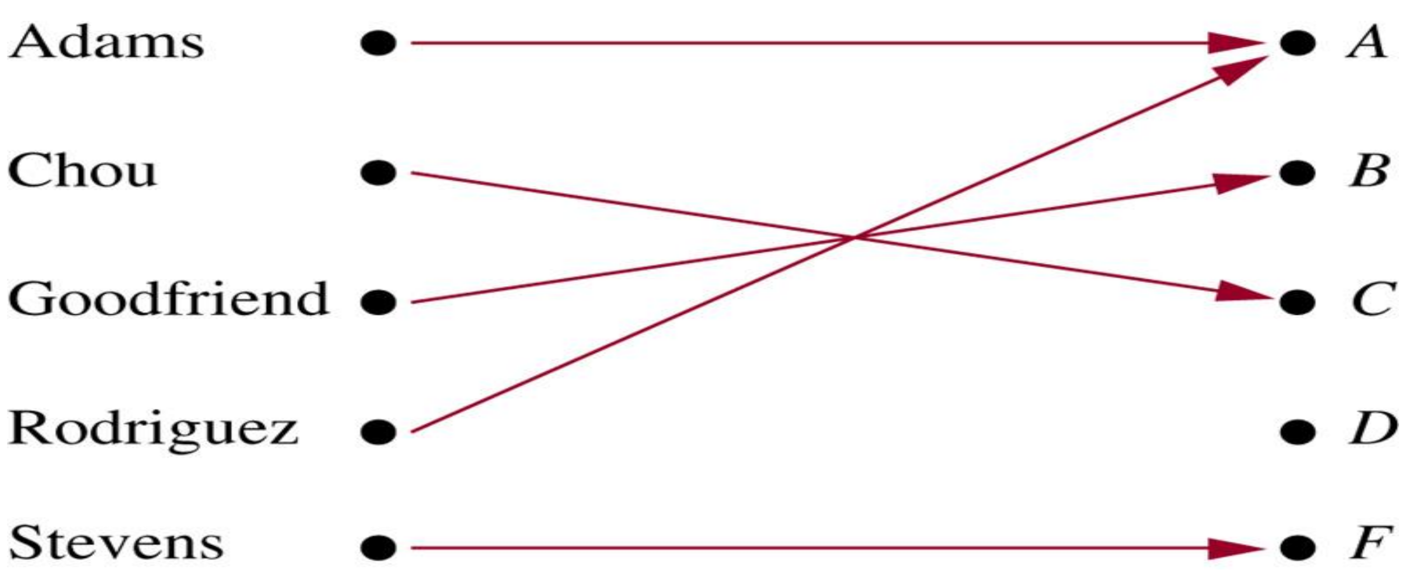
- The range of a function might ***not*** be its whole codomain.
- The codomain is the set that the function is ***declared*** to map all domain values into.
- The range is the particular set of values in the codomain the function ***actually*** maps elements of the domain to.

Range versus Codomain: *Example* (*See the* FIGURE 1 in the previous slide)

- Suppose I declare to you that: "*f* is a function mapping students in this class to the set of grades {*A, B, C,D,F*}".
- At this point, you know *f*'s codomain is: {*A, B, C,D,F*}, and it's range is unknown!
- Suppose the grades turn out all *A*s and *B*s.
- Then the range of *f* is {*A, B*}, but it's codomain is still {*A, B, C,D,F*}.

Example 1

- **What** are the **domain**, **codomain**, and **range** of the function that assigns grades to students of Discrete Math class as follows?



Solution of Example 1

- **Solution:**
- Let G be the function that assigns grade to a student of Discrete Mathematics class.
- The **domain of G** is the set {Adams, Chou, Goodfriend, Rodriguez, Stevens}
- The **codomain of G** is the set { A, B, C, D, F}
- The **range of G** is the set { A, B, C, F}
  - Because each grade except D is assigned to some student

Example 2

- Let R be the relation consisting of ordered pairs (Abdul, 22), (Brenda,24), (Carla,21), (Desire,22), (Eddie,24), and (Felicia,22), where each pair consists of a graduate student and the age of this student. *What is the function that this relation determines?*

- **Solution**: This relation defines the function f, where with f(Abdul)= 22, f(Brenda)=24, f(Carla)=21, f(Desire)=22, f(Eddie)= 24, and f(Felicia)=22.
- **Here, domain** is the set { Abdul, Brenda, Carla, Desire, Eddie, Felicia }
- To define the function f, we need to specify a codomain. Here, we can take the **codomain** to be the set of positive integers
- **Range** is the set {21,22,24}

Functions

Definition 3: Let f_1 and f_2 be functions from A to **R**. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to **R** defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$(f_1 f_2)(x) = f_1(x)\, f_2(x)$$

Example 6

- Let f_1 and f_2 be functions from **R** to **R** such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $\boldsymbol{f_1 + f_2}$ and $\boldsymbol{f_1 f_2}$?

- ***Solution*:**

  $(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$

  $(f_1 f_2)(x) = x^2(x - x^2) = x^3 - x^4$

Functions

- **Definition 4**: Let f be a function from the set A to the set B, and let S is a subset of A. The image of S under the function f is the subset of B that consists of the images of the elements of S.
- We denote the image of ***S*** by $f(S)$.

$$f(S) = \{t \mid \exists s \in S(t=f(s))\}$$

Example 7

- Let A = {a, b, c, d, e} and B = {1, 2, 3, 4} with f(a) = 2, f(b) = 1, f(c) = 4, f(d) = 1, f(e) = 1.
  What is the image of the subset S = {b, c, d}?

- **Solution**:
  The image of the subset S = {b, c, d} is the set f(S) = {1, 4}

One-to-One Functions

- **Definition 5**: A function f is *one-to-one* or *injective*, iff $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.
- A function ***f*: *A*→*B*** is said to be one-to-one if **all the elements** in the domain A have **distinct images**.

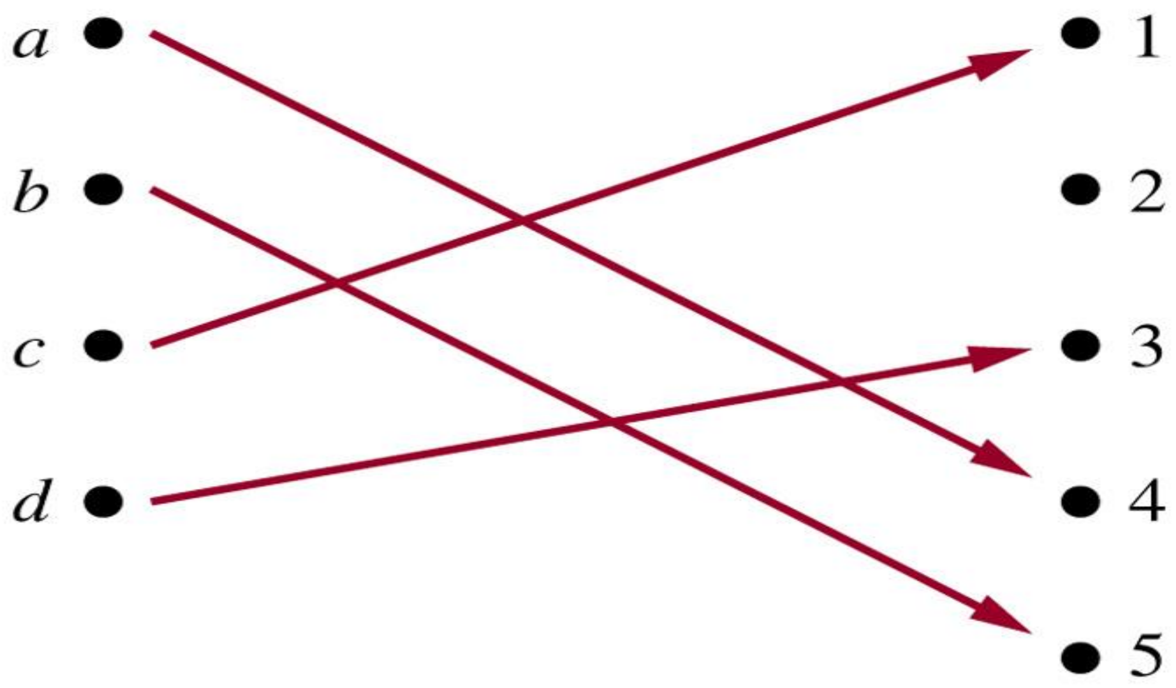
- We can express that f is one-to-one using quantifiers as $\forall a \forall b\ (f(a) = f(b) \rightarrow a = b\ )$, *or equivalently,* $\forall a \forall b\ (a \neq b \rightarrow f(a) \neq f(b)\ )$, where the universe of discourse is the domain of the function f

Example 8

- Determine whether the function f from {a, b, c, d} to {1, 2, 3, 4, 5} with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one.

- **Solution**: The function is one-to-one because every element of domain has a distinct image.
  - The function f is one-to-one because f takes on different values at the four elements of its domain.

FIGURE for Example 8 :
A *One-to-One* Function



Example 9

Example 9: Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

Solution: The function $f(x) = x^2$ is ***not*** one-to-one because, for instance, $f(1) = f(-1) = 1$, but $1 = -1$

(*i.e.* 1 and –1 have same image 1)

Class Work

Determine whether the function $f(x) = x^2$ from the set of **positive integers** to the set of **positive integers** is one-to-one.

Example 10

- Determine whether the function $f(x) = x + 1$ from the set of real numbers to the set of real numbers is one-to-one.

- **Solution**: The function $f(x) = x + 1$ is a one-to-one function. Since $x + 1 = y + 1$, when $x = y$

  For any real number x, there is a distinct image, just 1 bigger than x; so, the function is one-to-one.

Example : One-to-one function

- Let A = {1, 2, 3} and B = {a, b, c, d}, and let f(1) = a, f(2) = b, f(3) = d. Then *f* is injective, since the different elements 1, 2, 3 in A are assigned to the different elements a, c, d respectively in B
- Note: Every element of domain has a distinct image. So, the function is one-to-one.

Onto Function

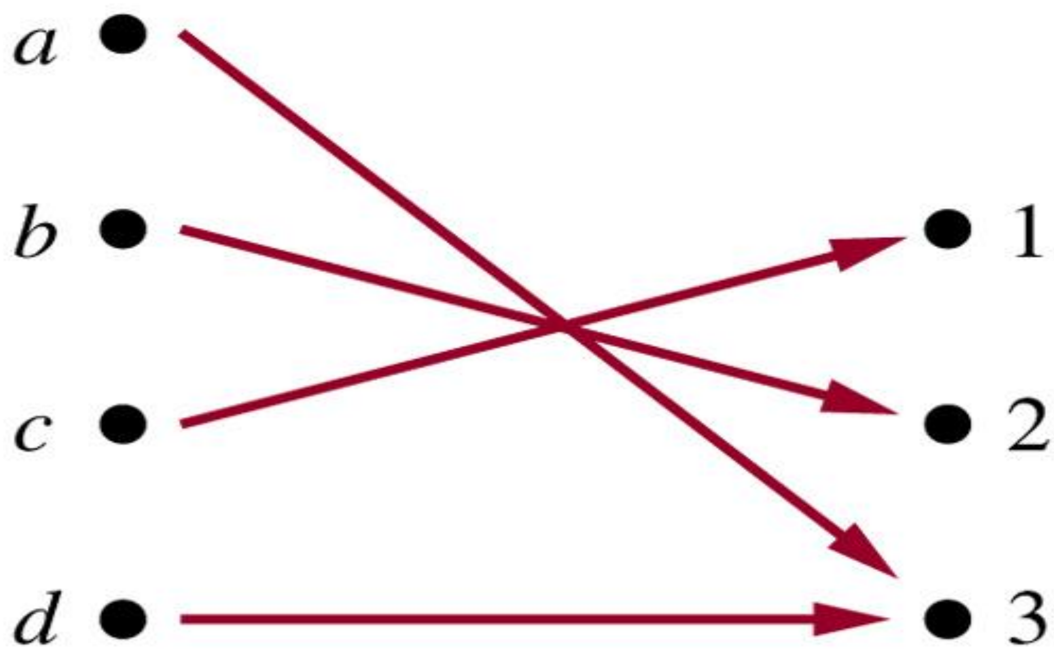
- **Definition 7**: A function f from A to B is ***onto*** or ***surjective***, iff for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.
- A function $f: A \rightarrow B$ is said to be an ***onto*** function if **each element of B** is the **image** of ***some*** element of A.
  - i.e., if B = range of f
- **Note:** A function is **onto** if every element of codomain has preimage(s).

Example 11

- Let f be the function from {a, b, c, d} to {1, 2, 3} defined by f(a)=3, f(b)=2, f(c)=1, and f(d)=3. Is f an onto?
  [see the Figure on next slide]

- **Solution**: Because all three elements of the codomain are images of elements in the domain, *f* is onto.

FIGURE for Example 11:
An *Onto* Function



Example 12

- Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?
- **Solution**: The function f is not onto, because there is no integer x with $x^2 = -1$, for instance.

- Note: The elements of the codomain that are negative integers (– 1, – 2, – 3 etc.) do not have any preimage.

Class Work

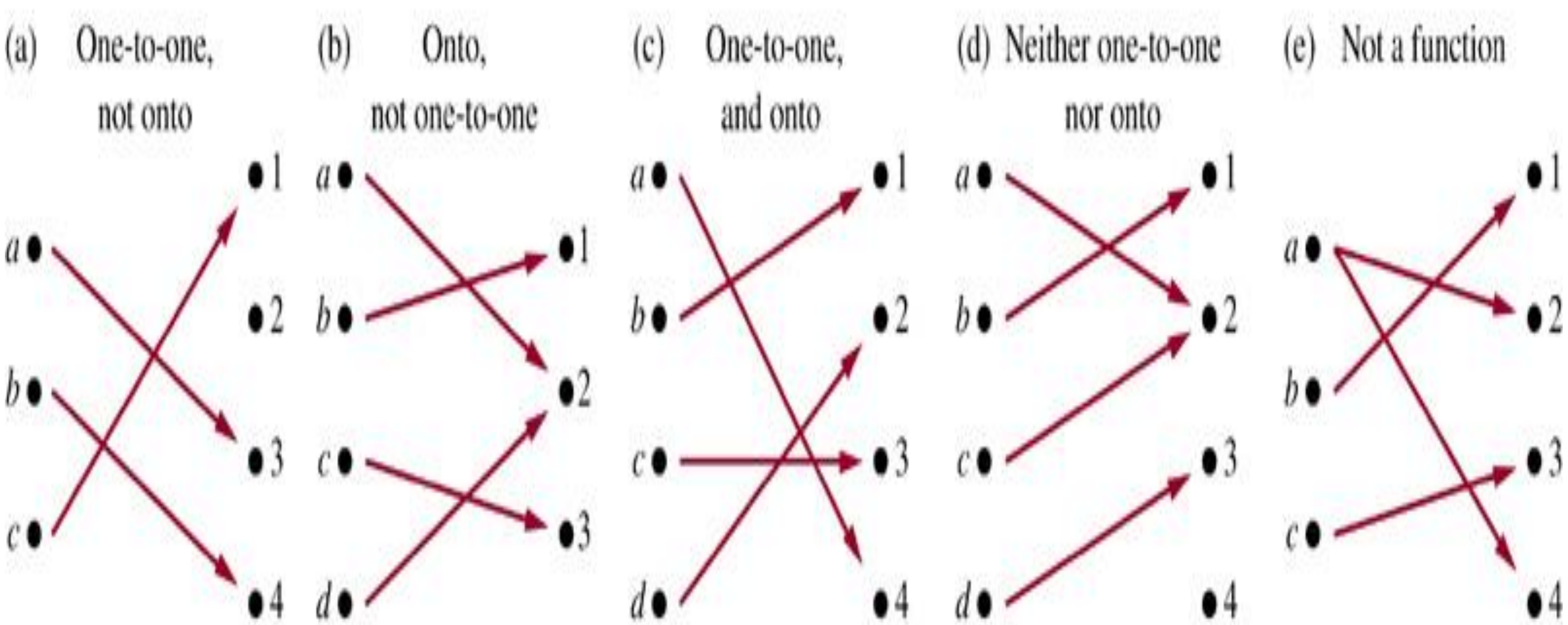
Is the function $f(x) = x^2$ from the set of positive integers to the set of positive integers onto?

One-to-one correspondence (bijection)

- **Definition 8**: A function *f* is a ***one-to-one correspondence*** or a ***bijection*** if it is both one-to-one and onto.

- **Example**: Let ***f*** be the function from A to B where A={1, 2, 3, 4} and B = {a, b, c, d} with *f*(1)=d, *f*(2)=b, *f*(3)=c, and *f*(4)=a, then *f* is bijective function.
  - *f* is one-to-one since the every element of domain has a distinct image
  - *f* is onto since every element of B is the image of some element in A.
  - Hence *f* is a *bijective function (or, one-to-one correspondence)*
- Practice yourself: Example 14

FIGURE: Examples of Different Types of Correspondences



Books

1. *Discrete Mathematics and its applications with combinatorics and graph theory (7th edition)* by Kenneth H. Rosen [Indian Adaptation by KAMALA KRITHIVASAN], published by McGraw-Hill

References

1. Discrete Mathematics, *Richard Johnsonbaugh*, Pearson education, Inc.
2. Discrete Mathematical Structures, *Bernard Kolman*, *Robert C. Busby*, *Sharon Ross,* Prentice-Hall, Inc.
3. *SCHAUM'S outlines Discrete Mathematics(2nd edition)*, by *Seymour Lipschutz*, *Marc Lipson*