# INTEGRAL CALCULUS
# AND
# ORDINARY DIFFERENTIAL EQUATIOSNS

# METHODS OF INTEGRATION

**Integration by Parts**

$$\int u \cdot v \, dx = u \int v \, dx - \int \left( \frac{du}{dx} \int v \, dx \right) dx$$

Example: Evaluate $\int x^2 e^{3x} dx$

Here $u = x^2, v = e^{3x}$

$$\begin{aligned}
\int x^2 e^{3x} dx &= x^2 \int e^{3x} dx - \int \left( \tfrac{d}{dx} x^2 \int e^{3x} dx \right) dx \\
&= x^2 \tfrac{1}{3} e^{3x} - \int \left( 2x \tfrac{1}{3} e^{3x} \right) dx \\
&= \tfrac{1}{3} x^2 e^{3x} - \tfrac{2}{3} \int x \cdot \, e^{3x} \, dx \\
&= \tfrac{1}{3} x^2 e^{3x} - \tfrac{2}{3} \left[ x \int e^{3x} dx - \int \left( \tfrac{d}{dx} x \, \int e^{3x} dx \right) dx \right] \\
&= \tfrac{1}{3} x^2 e^{3x} - \tfrac{2}{3} \left[ x \tfrac{1}{3} e^{3x} - \int 1 \cdot \tfrac{1}{3} e^{3x} \, dx \right] \\
&= \tfrac{1}{3} x^2 e^{3x} - \tfrac{2}{9} x \, e^{3x} + \tfrac{2}{9} \int e^{3x} \, dx \\
&= \tfrac{1}{3} x^2 e^{3x} - \tfrac{2}{9} x \, e^{3x} + \tfrac{2}{9} \cdot \tfrac{1}{3} e^{3x} \\
&= \tfrac{1}{3} x^2 e^{3x} - \tfrac{2}{9} x \, e^{3x} + \tfrac{2}{27} e^{3x}
\end{aligned}$$

Example: Evaluate $\int x^2 e^{3x} dx$

Here $u = x^2, v = e^{3x}$

| | | |
|---|---|---|
| $+$ | $\boldsymbol{x^2}$ | $\boldsymbol{e^{3x}}$ |
| $-$ | $2x$ | $\frac{1}{3}e^{3x}$ |
| $+$ | $2$ | $\frac{1}{9}e^{3x}$ |
| $-$ | $0$ | $\frac{1}{27}e^{3x}$ |

$$\int x^2 e^{3x} dx = +x^2 \cdot \frac{1}{3}e^{3x} - 2x \cdot \frac{1}{9}e^{3x} + 2 \cdot \frac{1}{27}e^{3x}$$

$$= \frac{1}{3}x^2 e^{3x} - \frac{2}{9}x\, e^{3x} + \frac{2}{27}e^{3x}$$

$$\int u \cdot v \, dx = u \int v \, dx - \int \left( \frac{du}{dx} \int v \, dx \right) dx$$

**Example: Evaluate** $\int_1^2 x^2 \, ln \, x \, dx$

$$\int x^2 \, ln \, x \, dx = \ln x \int x^2 \, dx - \int \left[ \frac{d}{dx} \ln x \int x^2 \, dx \right] dx$$

$$= \ln x \frac{x^3}{3} - \int \left[ \frac{1}{x} \frac{x^3}{3} \right] dx$$

$$= \frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^2 \, dx$$

$$= \frac{1}{3} x^3 \ln x - \frac{1}{3} \frac{x^3}{3}$$

$$\therefore \int_1^2 x^2 \, ln \, x \, dx = \left[ \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 \right]_1^2$$

$$= \left[ \left( \frac{1}{3} 2^3 \ln 2 - \frac{1}{9} 2^3 \right) - \left( \frac{1}{3} 1^3 \ln 1 - \frac{1}{9} 1^3 \right) \right]$$

$$= \frac{8}{3} \ln 2 - \frac{8}{9} - 0 + \frac{1}{9}$$

$$= \frac{8}{3} \ln 2 - \frac{7}{9}$$



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## Class Practice:

## Evaluate

1. $\int x^2 e^{2x} dx$
2. $\int x^2 \sin 2x \, dx$
3. $\int x \sin(2x+1) \, dx$
4. $\int_0^{\pi} (2x^2+1) \cos 2x \, dx$

## Home Work

Integration by parts (P-472) Example # 1, 2, 3, 4, 5

Page – 476 Ex # 3, 5, 6, 8, 17

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## Integrals of the form $\int \sin Ax \cos Bx \, dx$, $\int \cos Ax \cos Bx \, dx$, $\int \sin Ax \sin Bx \, dx$

## Necessary Trigonometric Formulas

$$sin\,A\,cos\,B = \frac{1}{2}[sin(A+B) + sin(A-B)]$$

$$cos\,A\,cos\,B = \frac{1}{2}[cos(A+B) + cos(A-B)]$$

$$sin\,A\,sin\,B = \frac{1}{2}[cos(A-B) - cos(A+B)]$$

**Integrals of the form $\int \sin Ax \cos Bx \, dx$, $\int \cos Ax \cos Bx \, dx$, $\int \sin Ax \sin Bx \, dx$**

Example: Evaluate $\int sin\, 7\, x\, cos\, 3\, x\, dx$

$$\int sin\, 7\, x\, cos\, 3\, x\, dx = \frac{1}{2}\int [sin\, 1\, 0x + sin\, 4\, x]dx$$

$$= -\frac{1}{20} cos\, 1\, 0x - \frac{1}{8} cos\, 4\, x + C$$

Class Practice:

Evaluate the following:

1. $\int \sin 4x \cos 4x \, dx$
2. $\int \sin 3x \sin 2x \, dx$
3. $\int_0^{\pi/6} cos\, 4\, x\, sin\, 2\, xdx$
4. $\int_0^{\pi/4} \cos 4x \cos x \, dx$

Home Work

Page 485. Ex: 41, 42

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# Integration of irrational functions using trigonometric substitution

| Expression in the integrand | Substitution |
| --- | --- |
| $\sqrt{a^2 - x^2}$ | $x = a \sin \theta$ |
| $\sqrt{a^2 + x^2}$ | $x = a \tan \theta$ |
| $\sqrt{x^2 - a^2}$ | $x = a \sec \theta$ |

Example: Evaluate $\int \frac{dx}{x^2\sqrt{4-x^2}}$

Let, $x = 2\, sin\, \theta\, , dx = 2\, cos\, \theta\, d\theta$

So, $\int \frac{2\, cos\, \theta d\theta}{4\sin^2\theta\sqrt{4-4\sin^2\theta}} = \int \frac{2\cos\,\theta}{4\sin^2\theta\sqrt{4(1-\sin^2\theta)}} d\theta$

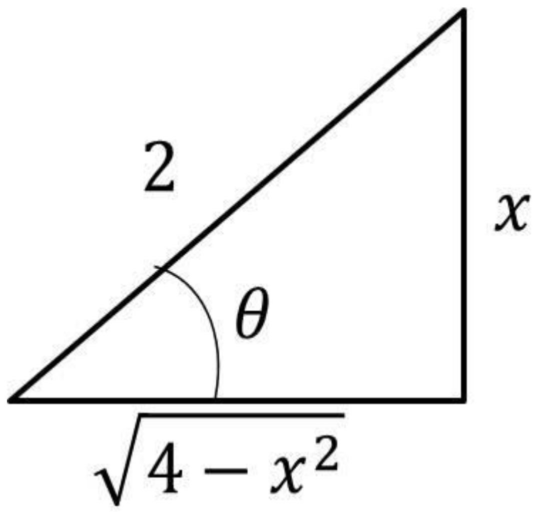
$$= \int \frac{2\cos\theta}{4\sin^2\theta\, 2\sqrt{\cos^2\theta}} d\theta$$

$$= \frac{1}{4}\int \frac{d\theta}{\sin^2\theta}$$

$$= \frac{1}{4}\int \csc^2\theta\, d\theta$$

$$= -\frac{1}{4}\cot\theta + C$$

$$= -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C.$$



Class Practice:

Evaluate the following:

1. $\int \frac{1}{\sqrt{4-x^2}}\,dx$
2. $\int \frac{dx}{x^2\sqrt{1-x^2}}$
3. $\int_0^1 x\sqrt{1-x^2}\,dx$
4. $\int \frac{\sqrt{x^2-4}}{x}\,dx$

Home Work

Trigonometric Substitution (P-486) Example # 1, 6, 7

P – 491 Ex # 1-6, 10-14, 21-24

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# Integration of the form $\int \sin^m x \cos^n x \, dx$

**Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$**

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \, (\cos^2 x)^k \cos x \, dx$$
$$= \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \sin x \, dx$$
$$= \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2} \sin 2x$$

**Integration of the form** $\int sin^m x\, cos^n x\, dx$

Evaluate $\int sin^4 x\, cos^5 x\, dx$

**Solution:** $\int sin^4 x\, cos^5 x\, dx = \int sin^4 x\, cos^4 x\, cos\, x\, dx$

$$= \int sin^4 x\, (1 - sin^2 x)^2\, cos\, x\, dx$$

Let, $u = sin\, x\, , \frac{du}{dx} = cos\, x\, , du = cos\, x\, dx$

$$\int sin^4 x\, (1 - sin^2 x)^2\, cos\, x\, dx = \int u^4 (1 - u^2)^2\, du$$

$$= \int\, (u^4 - 2u^6 + u^8) du$$

$$= \frac{1}{5} u^5 - \frac{2}{7} u^7 + \frac{1}{9} u^9 + C$$

$$= \frac{1}{5} sin^5 x - \frac{2}{7} sin^5 x + \frac{1}{9} sin^5 x + C$$

Class practice

1. Evaluate $\int sin^2 x\, cos^2 x\, dx$
2. Evaluate $\int_0^{\pi/2} sin^3 x\, cos^2 x\, dx$
3. Evaluate $\int sin^5 x\, cos^3 x\, dx$
4. Evaluate $\int_0^{\pi/6} sin^2 3\,x\, cos^3 3\,x\; dx$

Home Work

Page 484. Ex: 1, 2, 11, 17

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Partial Fraction

- **The Cover-up Rule - or how to make partial fractions easy**

**Example:**

Consider the partial fraction

$$\frac{3x}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

To obtain $A$ simply cover up the factor (x –1) with you finger tip in

$$\frac{3x}{(x-1)(x+2)} \qquad (1)$$

then you get :

$$\frac{3x}{x+2}$$

and substitute the value x = 1, giving

$$A = \frac{3(1)}{(1+2)} = 1$$

Likewise to obtain B cover up the factor (x + 2) in (1) and evaluate what is left at x = -2, giving

$$B = \frac{3(-2)}{(-2-1)} = 2$$

Finally you get:

$$\frac{3x}{(x-1)(x+2)} = \frac{1}{x-1} + \frac{2}{x+2}$$

Partial Fraction

- **The Cover-up Rule - or how to make partial fractions easy**

**Example:**

$$\frac{2}{(x-3)(x-4)(x-5)} = \frac{A}{x-3} + \frac{B}{x-4} + \frac{C}{x-5}$$

Then:

$$A = \frac{2}{(3-4)(3-5)} = 1,\quad B = \frac{2}{(4-3)(4-5)} = -2,\quad C = \frac{2}{(5-3)(5-4)} = 1$$

Therefore:

$$\frac{2}{(x-3)(x-4)(x-5)} = \frac{1}{x-3} - \frac{2}{x-4} + \frac{1}{x-5}$$

Partial Fraction

- **The Cover-up Rule - or how to make partial fractions easy**

**Example:**

The Keily's Method is also useful for repeated linear factors. This method is to use one factor at a time, keeping the rest outside the expression.

$$\frac{9}{(x+1)^2(x-2)} = \frac{1}{x+1}\left[\frac{9}{(x+1)(x-2)}\right]$$

$$= \frac{1}{x+1}\left[-\frac{3}{x+1} + \frac{3}{x-2}\right], \quad \text{by cover up rule}$$

$$= -\frac{3}{(x+1)^2} + \frac{3}{(x+1)(x-2)}$$

$$= -\frac{3}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{x-2}, \quad \text{by cover up rule again.}$$

**Point to note:** Keily's method should be used with care if the fraction is improper during the process, as in the following:

$$\frac{9x^2}{(x+1)^2(x-2)} = \frac{1}{x+1}\left[\frac{9x^2}{(x+1)(x-2)}\right]$$

Inside the bracket is improper and division is used before applying the Keily's Method.

## Integration of the form $\frac{p(x)}{q(x)}$ by Partial fraction

**Example:** Evaluate $\int \frac{x}{(x-2)(x+1)}\, dx$

**Solution:** Let $\frac{x}{(x-2)(x+1)} \equiv \frac{A}{x-2} + \frac{B}{x+1}$

Using cover –up rules $A = \frac{2}{(2+1)} = \frac{2}{3}$

Similarly $B = \frac{-1}{(-1-2)} = \frac{1}{3}$

Therefore $\int \frac{x}{(x-2)(x+1)}\, dx = \int \left[\frac{\frac{2}{3}}{x-2} + \frac{\frac{1}{3}}{x+1}\right] dx$

$$= \int \left[\frac{2}{3} \cdot \frac{1}{x-2} + \frac{1}{3} \cdot \frac{1}{x+1}\right] dx$$

$$= \frac{2}{3}\ln|x-2| + \frac{1}{3}\ln|x+1| + c$$

**Integration of the form $\frac{\boldsymbol{p(x)}}{\boldsymbol{q(x)}}$ by Partial fraction**

**Class practice:**

1. **Evaluate** $\int \frac{dx}{x^2+7x-18}$
2. Evaluate $\int_2^4 \frac{x-5}{(x-2)(x-3)} dx$
3. Evaluate $\int \frac{dx}{(x+1)(x+2)^2}$

**Home Work**

Partial Fraction (P-493) Example # 2, 3, 5

P-501 Ex # 9-12, 19-23

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# Sample MCQ

1. Evaluate $\int x \sin x \, dx$

(a)$-x \cos x + \sin x + c$ (b)………………….(c)………………..(d)……………

2. Evaluate $\int \sin 4x \cos 5x \; dx$

(a)……………..(b)$\frac{1}{2}\left(\cos x - \frac{1}{9}\cos 9x\right) + c$ (c)………………..(d)…………….

3. Evaluate $\int \frac{\sqrt{9-x^2}}{x^2} dx$

(a) $\frac{\sqrt{9-x^2}}{x^2} - \sin^{-1}\left(\frac{x}{3}\right) + c$ (b)……………… (c)……………….. (d)……………..

4. Evaluate$\int \sin^3 x \cos x \; dx$

$(a)$$-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x$ (b)………………. (c)……………….. (d)……………..