$$\frac{dp_{brian}}{dt} = -\frac{p_{brian}}{\tau_d}$$
$$\frac{dq_{brian}}{dt} = -\frac{q_{brian}}{\tau_d} - \frac{q_{brian}}{\tau_r}$$
$$g_{brian} = p - q$$

$$\left.\begin{array}{l} q_{brian} \leftarrow q_{brian} + 1 \\ p_{brian} \leftarrow p_{brian} + 1 \end{array}\right\} \text{ upon spike at time } t_i$$

We want to show that this model is equivalent to the explicit formula

$$g(t) = \sum_{i=1}^{k} \mathrm{e}^{-\frac{t-t_i}{\tau_d}} (1 - \mathrm{e}^{-\frac{t-t_i}{\tau_r}}) \Theta(t - t_i)$$

For this, we realize that the solutions p(t) and q(t) of our model can be seen as a combination of multiple solutions for the same differenital equation. These solutions only differ in their starting value.

$$p_{brian}(t) = \begin{cases} 0 \cdot \mathrm{e}^{-\frac{t}{\tau_d}} & \mathit{for}\ \ t \leq t_1 \\ \Big(p_{brian}(t_i) + 1\Big) \cdot \mathrm{e}^{-\frac{t-t_i}{\tau_d}} & \mathit{for}\ \ t_i < t \leq t_{i+1} \end{cases}$$

Since $c \cdot \mathrm{e}^{-\frac{t}{\tau_d}}$ is the general solution for $p' = -\dfrac{p}{\tau_d}$ where $c = p(0)$.

$$q_{brian}(t) = \begin{cases} 0 \cdot \mathrm{e}^{-\frac{t}{\tau_d} - \frac{t}{\tau_r}} & \mathit{for}\ \ t \leq t_1 \\ \Big(q_{brian}(t_i) + 1\Big) \cdot \mathrm{e}^{-\frac{t-t_i}{\tau_d} - \frac{t-t_i}{\tau_r}} & \mathit{for}\ \ t_i < t \leq t_{i+1} \end{cases}$$

Since $c \cdot \mathrm{e}^{-\frac{t}{\tau_d} - \frac{t}{\tau_r}}$ is the general solution for $q' = -\dfrac{q}{\tau_d} - \dfrac{q}{\tau_r}$ where $c = q(0)$.

Let $1 \leq i \leq k$. Then
$$\begin{aligned} p_{brian}(t_i) &= \Big(p_{brian}(t_{i-1}) + 1\Big) \cdot \mathrm{e}^{-\frac{t_i - t_{i-1}}{\tau_d}} \\ &= \bigg(\Big(p_{brian}(t_{i-2}) + 1\Big) \cdot \mathrm{e}^{-\frac{t_{i-1} - t_{i-2}}{\tau_d}} + 1\bigg) \cdot \mathrm{e}^{-\frac{t_i - t_{i-1}}{\tau_d}} \\ &= \bigg(\bigg(\bigg(\Big(p_{brian}(t_1) + 1\Big) \cdot \mathrm{e}^{-\frac{t_2 - t_1}{\tau_d}} + 1\bigg) \cdot \mathrm{e}^{-\frac{t_3 - t_2}{\tau_d}} + 1\bigg)\ \ldots\ \bigg) \cdot \mathrm{e}^{-\frac{t_i - t_{i-1}}{\tau_d}} \\ &= \bigg(\bigg(\Big(\mathrm{e}^{-\frac{t_2 - t_1}{\tau_d}} + 1\Big) \cdot \mathrm{e}^{-\frac{t_3 - t_2}{\tau_d}} + 1\bigg)\ \ldots\ \bigg) \cdot \mathrm{e}^{-\frac{t_i - t_{i-1}}{\tau_d}} \\ &= \sum_{r=i-1}^{1} \mathrm{e}^{\sum_{j=i-1}^{r} \frac{t_j - t_{j+1}}{\tau_d}} \\ &= \sum_{r=1}^{i-1} \exp\Big(\sum_{j=r}^{i-1} \frac{t_j - t_{j+1}}{\tau_d}\Big) \\ &= \sum_{r=1}^{i-1} \exp\Big(\frac{t_r - t_{i-1+1}}{\tau_d}\Big) = \sum_{r=1}^{i-1} \mathrm{e}^{\frac{t_r - t_i}{\tau_d}} \end{aligned}$$

Similarly $q_{brian}(t_i) = \sum_{r=1}^{i-1} \mathrm{e}^{\frac{t_r - t_i}{\tau_d} - \frac{t_r - t_i}{\tau_r}}$

Now let $1 \leq h \leq k$ and $t_h < t \leq t_{h+1}$. Let $\tilde{t} = t - t_h$.

$$
\begin{aligned}
g(t) &= \sum_{i=1}^{k} \mathrm{e}^{-\frac{t-t_i}{\tau_d}} (1 - \mathrm{e}^{-\frac{t-t_i}{\tau_r}}) \Theta(t - t_i) \\
&= \sum_{i=1}^{k} \mathrm{e}^{-\frac{t-t_i}{\tau_d}} \Theta(t - t_i) \;-\; \sum_{i=1}^{k} \mathrm{e}^{-\frac{t-t_i}{\tau_d} - \frac{t-t_i}{\tau_r}} \Theta(t - t_i) \\
&= \sum_{i=1}^{h} \mathrm{e}^{-\frac{t-t_i}{\tau_d}} \;-\; \sum_{i=1}^{h} \mathrm{e}^{-\frac{t-t_i}{\tau_d} - \frac{t-t_i}{\tau_r}} \\
&= \mathrm{e}^{-\frac{\tilde{t}}{\tau_d}} \left(\sum_{i=1}^{h} \mathrm{e}^{-\frac{t_i - t_h}{\tau_d}} \right) \;-\; \mathrm{e}^{-\frac{\tilde{t}}{\tau_d} - \frac{\tilde{t}}{\tau_r}} \left(\sum_{i=1}^{h} \mathrm{e}^{-\frac{t_i - t_h}{\tau_d} - \frac{t_i - t_h}{\tau_r}} \right) \\
&= \mathrm{e}^{-\frac{\tilde{t}}{\tau_d}} \left(\left(\sum_{i=1}^{h-1} \mathrm{e}^{-\frac{t_i - t_h}{\tau_d}} \right) + 1 \right) \;-\; \mathrm{e}^{-\frac{\tilde{t}}{\tau_d} - \frac{\tilde{t}}{\tau_r}} \left(\left(\sum_{i=1}^{h-1} \mathrm{e}^{-\frac{t_i - t_h}{\tau_d} - \frac{t_i - t_h}{\tau_r}} \right) + 1 \right) \\
&= \mathrm{e}^{-\frac{\tilde{t}}{\tau_d}} \left(p_{brian}(t_h) + 1 \right) \;-\; \mathrm{e}^{-\frac{\tilde{t}}{\tau_d} - \frac{\tilde{t}}{\tau_r}} \left(q_{brian}(t_h) + 1 \right) \\
&= \mathrm{e}^{-\frac{t-t_h}{\tau_d}} \left(p_{brian}(t_h) + 1 \right) \;-\; \mathrm{e}^{-\frac{t-t_h}{\tau_d} - \frac{t-t_h}{\tau_r}} \left(q_{brian}(t_h) + 1 \right) \\
&= p_{brian}(t) - q_{brian}(t) = g_{brian}(t) \qquad \square
\end{aligned}
$$

Finally we rewrite the brian model as follows:

$$
\begin{aligned}
\frac{dp}{dt} &= -\frac{p}{\tau_d} \\
\frac{dq}{dt} &= -\frac{q}{\tau_d} - \frac{q}{\tau_r} \\
g &= p - q
\end{aligned}
$$

$$
\begin{aligned}
\frac{dg}{dt} &= \frac{d(p-q)}{dt} = -\frac{p}{\tau_d} - \left(-\frac{q}{\tau_d} - \frac{q}{\tau_r} \right) \\
&= -\frac{p}{\tau_d} + \frac{q}{\tau_d} + \frac{q}{\tau_r} \\
&= \frac{q-p}{\tau_d} + \frac{q}{\tau_r} = -\frac{p-q}{\tau_d} + \frac{q}{\tau_r} \\
&= -\frac{g}{\tau_d} + \frac{q}{\tau_r}
\end{aligned}
$$