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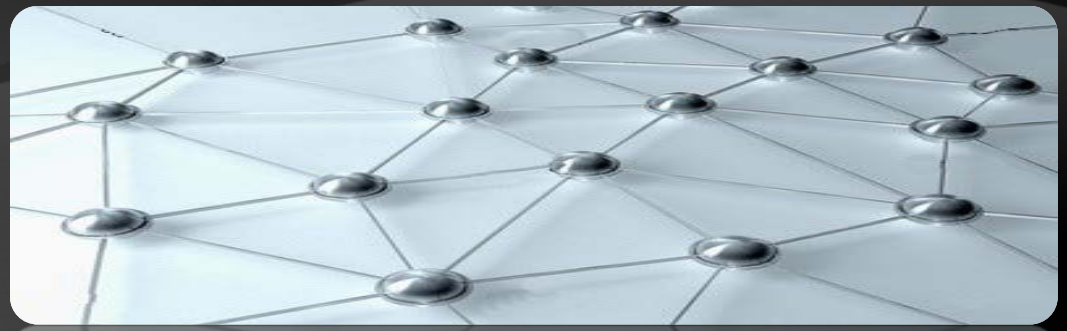
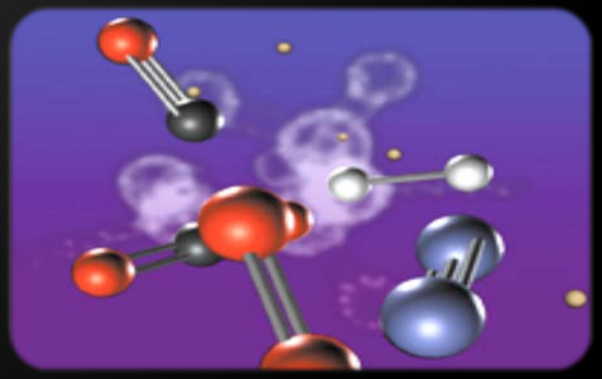


Graphs

Fundamentals, Terminology, Traversal, Algorithms



Data Structures and Algorithms

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2. Representing Graphs
3. Graph Traversal Algorithms
4. Connectivity
5. Dijkstra's Algorithm
6. Topological sorting
7. Prim and Kruskal

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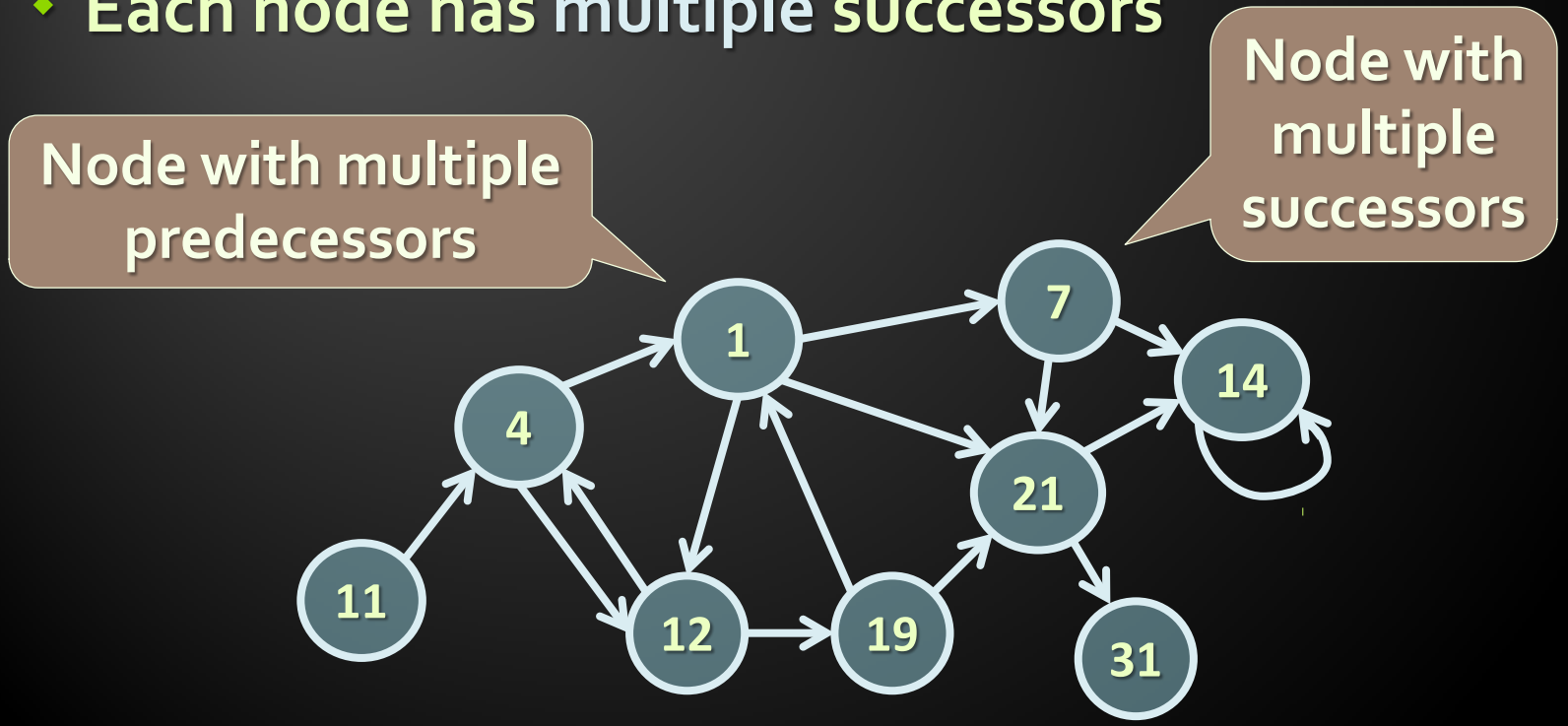
Graphs

Definitions and Terminology

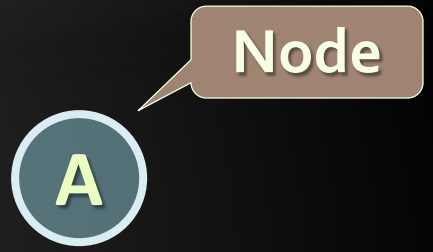
Graph Data Structure

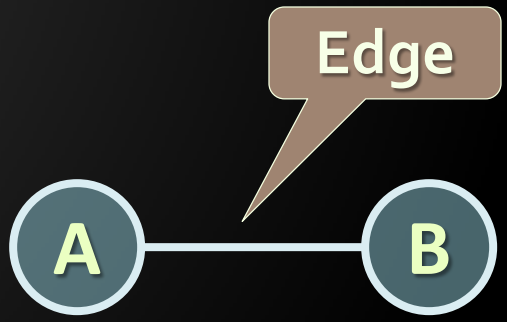
- Set of nodes with many-to-many relationship between them is called graph
  - Each node has multiple predecessors
  - Each node has multiple successors



Graph Definitions

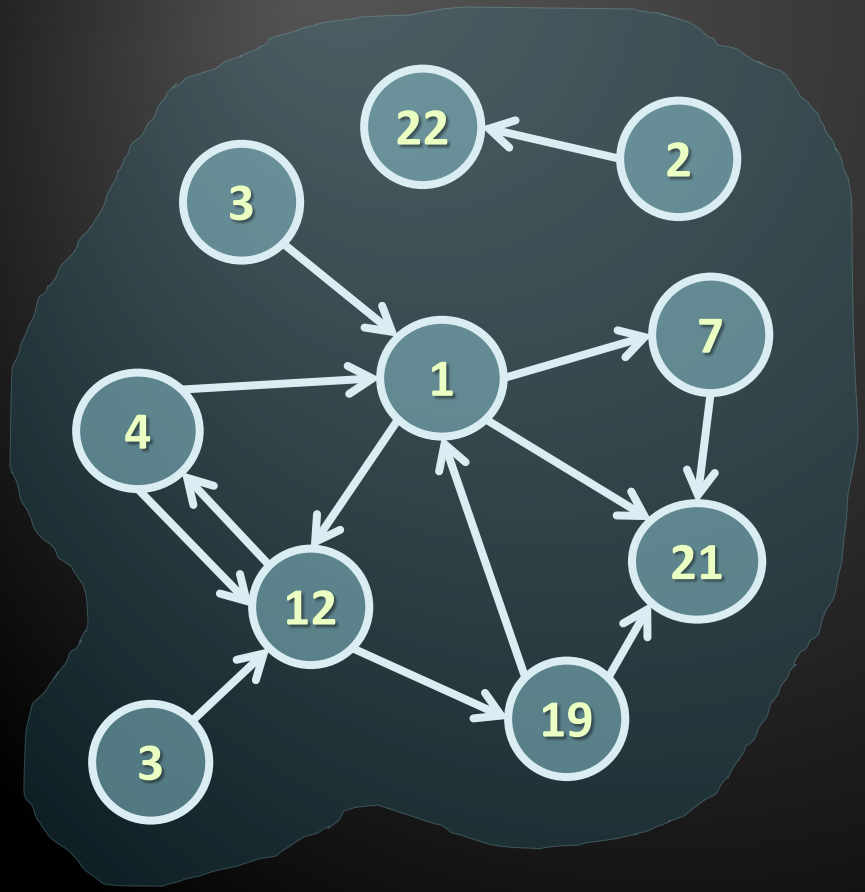
- Node (vertex)
  - Element of graph
  - Can have name or value
  - Keeps a list of adjacent nodes
- Edge
  - Connection between two nodes
  - Can be directed / undirected
  - Can be weighted / unweighted
  - Can have name / value



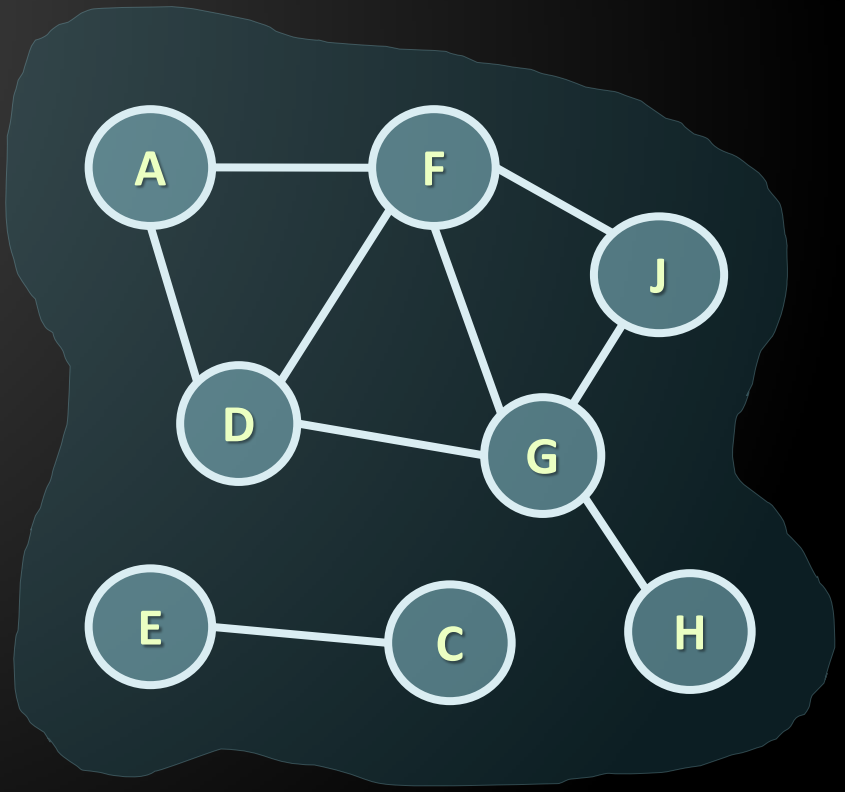


Graph Definitions (2)

- Directed graph
  - Edges have direction

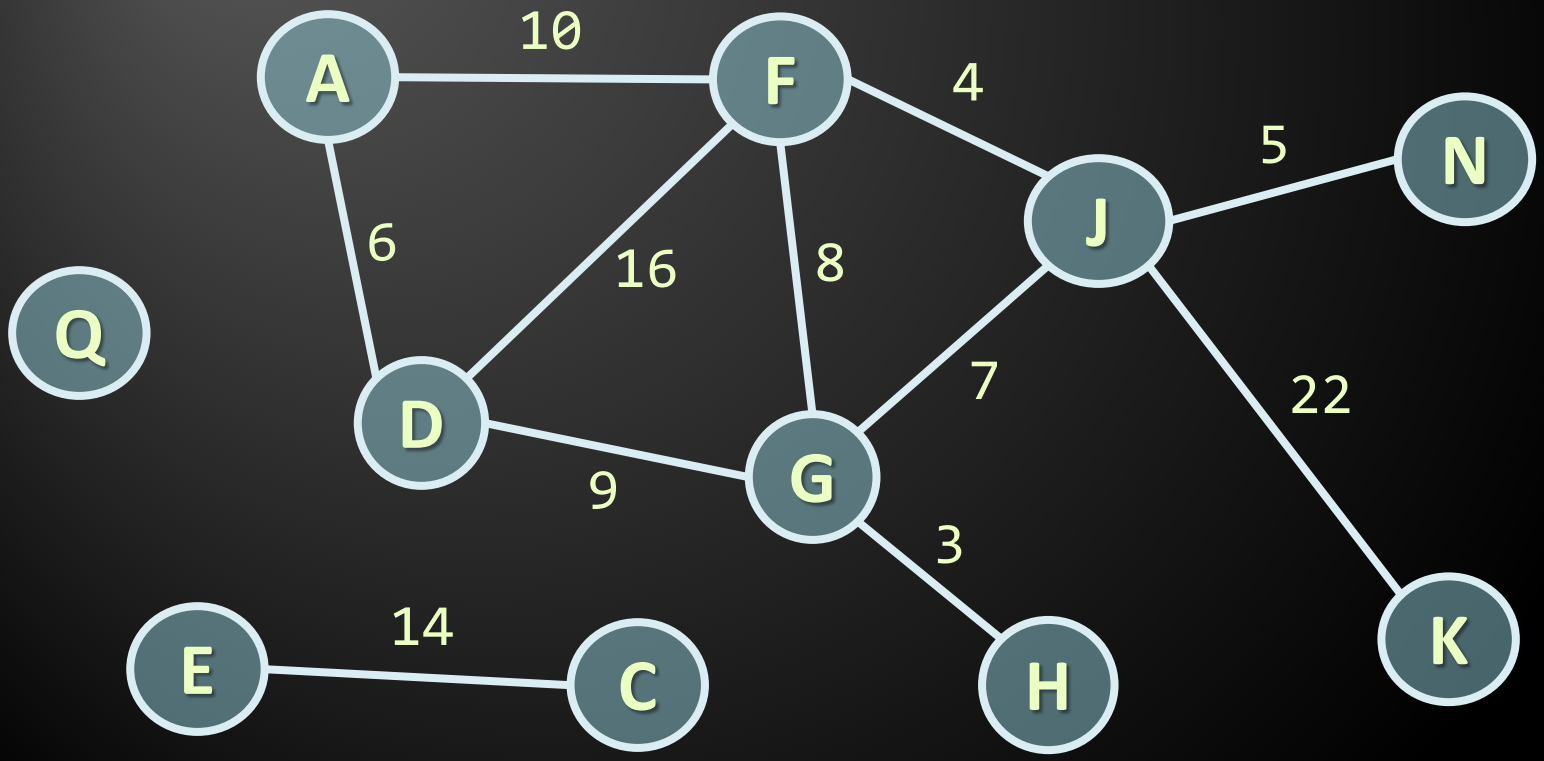


- Undirected graph
  - Undirected edges



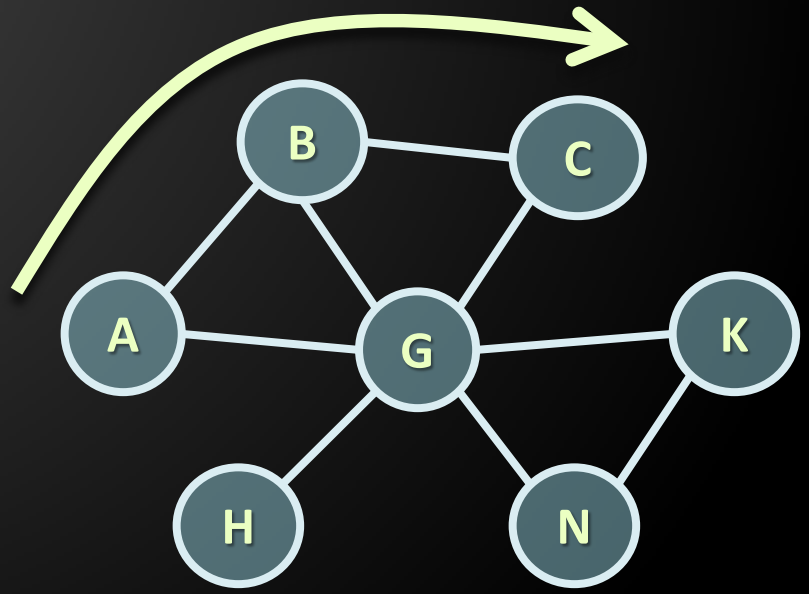
Graph Definitions (3)

- Weighted graph
  - Weight (cost) is associated with each edge



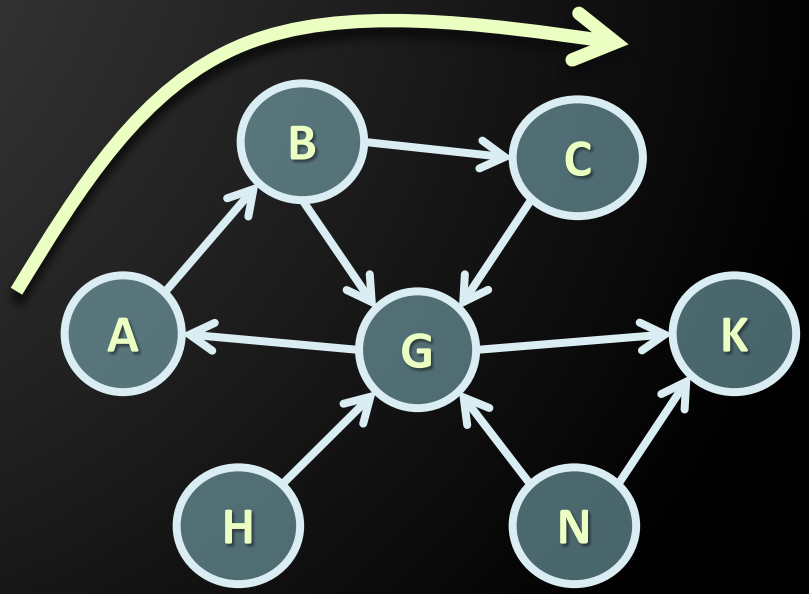
Graph Definitions (4)

- Path (in undirected graph)
  - Sequence of nodes n_1, n_2, … n_k
  - Edge exists between each pair of nodes n_i, n_{i+1}
  - Examples:
    - A, B, C is a path
    - H, K, C is not a path



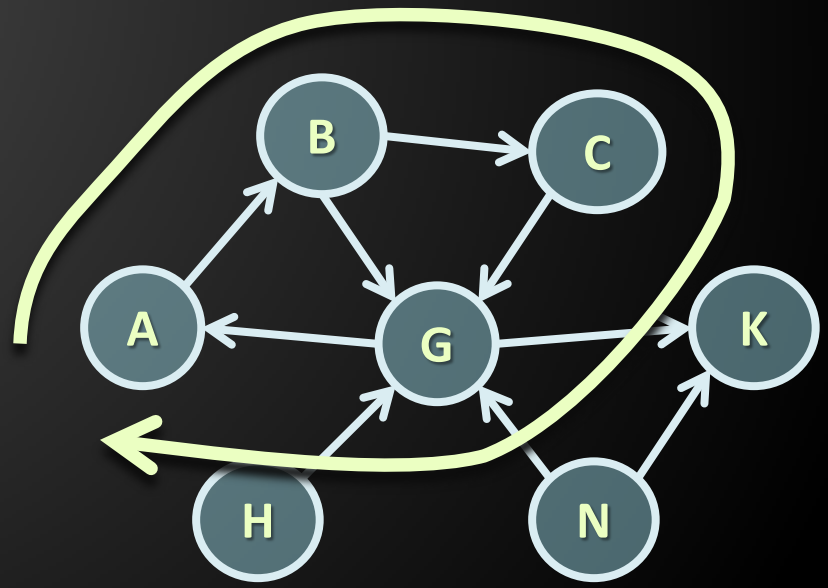
Graph Definitions (5)

- Path (in directed graph)
  - Sequence of nodes n_1, n_2, … n_k
  - Directed edge exists between each pair of nodes n_i, n_{i+1}
  - Examples:
    - A, B, C is a path
    - A, G, K is not a path



Graph Definitions (6)

- Cycle
  - Path that ends back at the starting node
  - Example:
    - A, B, C, G, A
- Simple path
  - No cycles in path
- Acyclic graph
  - Graph with no cycles
  - Acyclic undirected graphs are trees

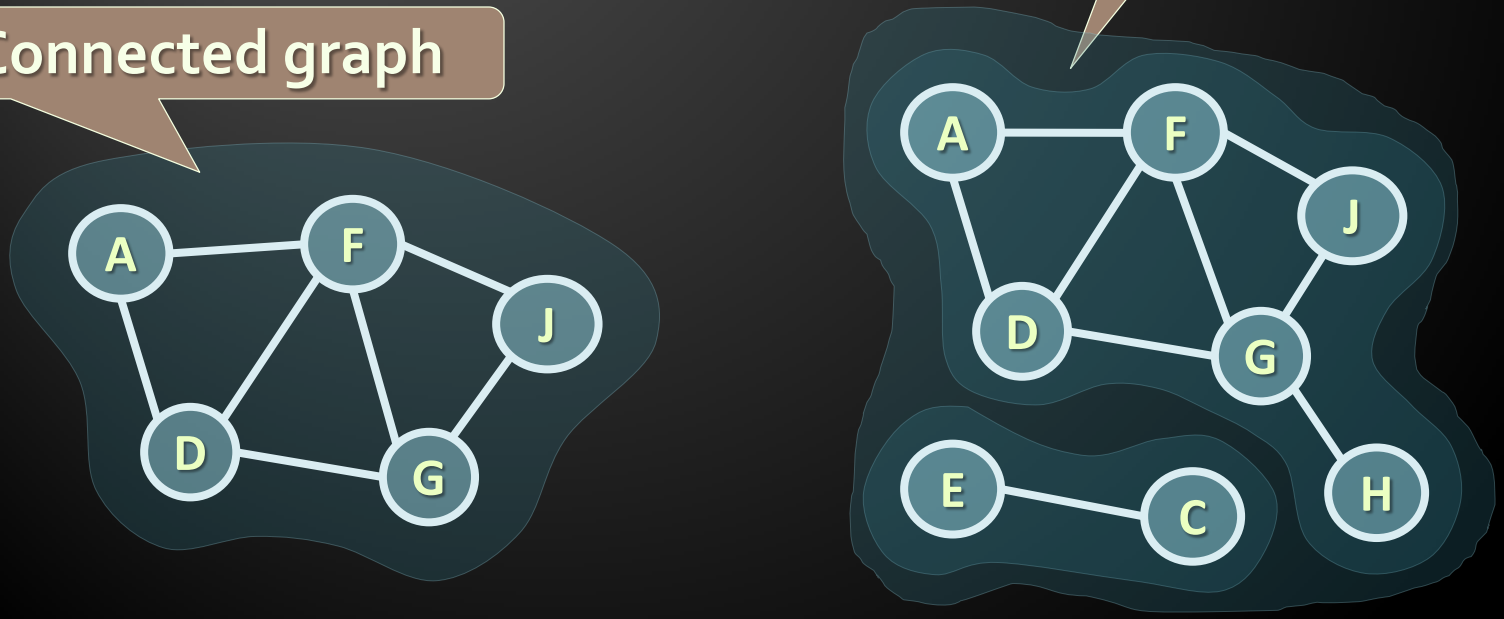


Graph Definitions (7)

- Two nodes are reachable if
  - Path exists between them
- Connected graph
  - Every node is reachable from any other node

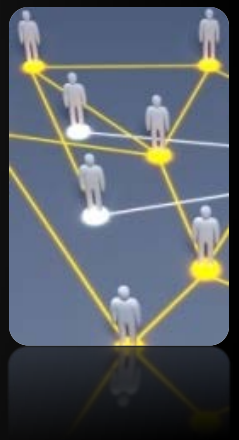
Unconnected graph with two connected components

Connected graph



Graphs and Their Applications

- Graphs have many real-world applications
  - Modeling a computer network like Internet
    - Routes are simple paths in the network
  - Modeling a city map
    - Streets are edges, crossings are vertices
  - Social networks
    - People are nodes and their connections are edges
  - State machines
    - States are nodes, transitions are edges



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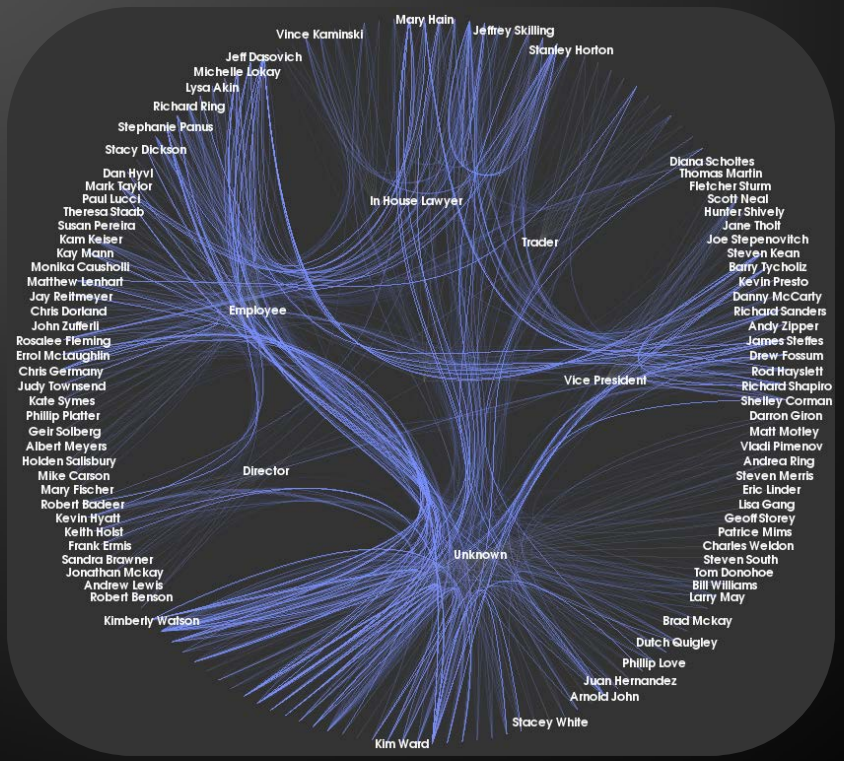
Representing Graphs

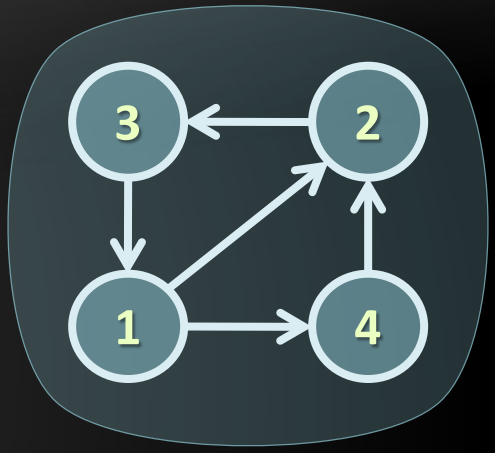
Classic and OOP Ways



Representing Graphs

- Adjacency list
  - Each node holds a list of its neighbors

```
1 → {2, 4}
2 → {3}
3 → {1}
4 → {2}
```



- Adjacency matrix
  - Each cell keeps whether and how two nodes are connected

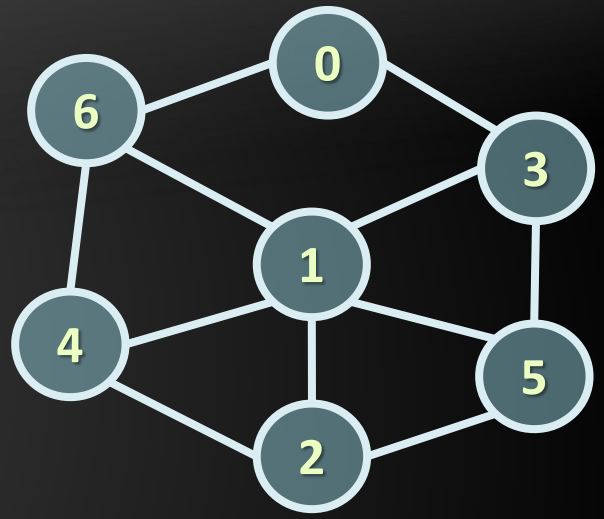
	1	2	3	4
1	0	1	0	1
2	0	0	1	0
3	1	0	0	0
4	0	1	0	0

- Set of edges

```
{1,2} {1,4} {2,3} {3,1} {4,2}
```

Simple C# Representation

```
public class Graph
{
  List<int>[] childNodes;
  public Graph(List<int>[] nodes)
  {
    this.childNodes = nodes;
  }
}
```




```
Graph g = new Graph(new List<int>[] {
  new List<int> {3, 6}, // successors of vertice 0
  new List<int> {2, 3, 4, 5, 6},// successors of vertice 1
  new List<int> {1, 4, 5}, // successors of vertice 2
  new List<int> {0, 1, 5}, // successors of vertice 3
  new List<int> {1, 2, 6}, // successors of vertice 4
  new List<int> {1, 2, 3}, // successors of vertice 5
  new List<int> {0, 1, 4}  // successors of vertice 6
});
```

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Advanced C# Representation

- Using OOP:
  - Class Node
  - Class Connection (Edge)
  - Class Graph
  - Optional classes
- Using external library:
  - QuickGraph - http://quickgraph.codeplex.com/

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Representing Graphs

Live Demo

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Traversing Graphs

Good old DFS and BFS

Graph Traversal Algorithms

- Depth-First Search (DFS) and Breadth-First Search (BFS) can traverse graphs
  - Each vertex should be visited at most once

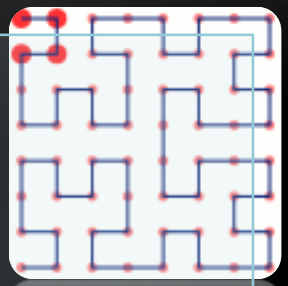
```
BFS(node)
{
  queue ← node
  visited[node] = true
  while queue not empty
    v ← queue
    print v
    for each child c of v
      if not visited[c]
        queue ← c
        visited[c] = true
}
```

```
DFS(node)
{
  stack ← node
  visited[node] = true
  while stack not empty
    v ← stack
    print v
    for each child c of v
      if not visited[c]
        stack ← c
        visited[c] = true
}
```

Recursive DFS Graph Traversal

```
void TraverseDFSRecursive(node)
{
  if (not visited[node])
  {
    visited[node] = true;
    print node;
    foreach child node c of node
    {
       TraverseDFSRecursive(c);
    }
  }
}

vois Main()
{
  TraverseDFS(firstNode);
}
```



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Graphs and Traversals

Live Demo

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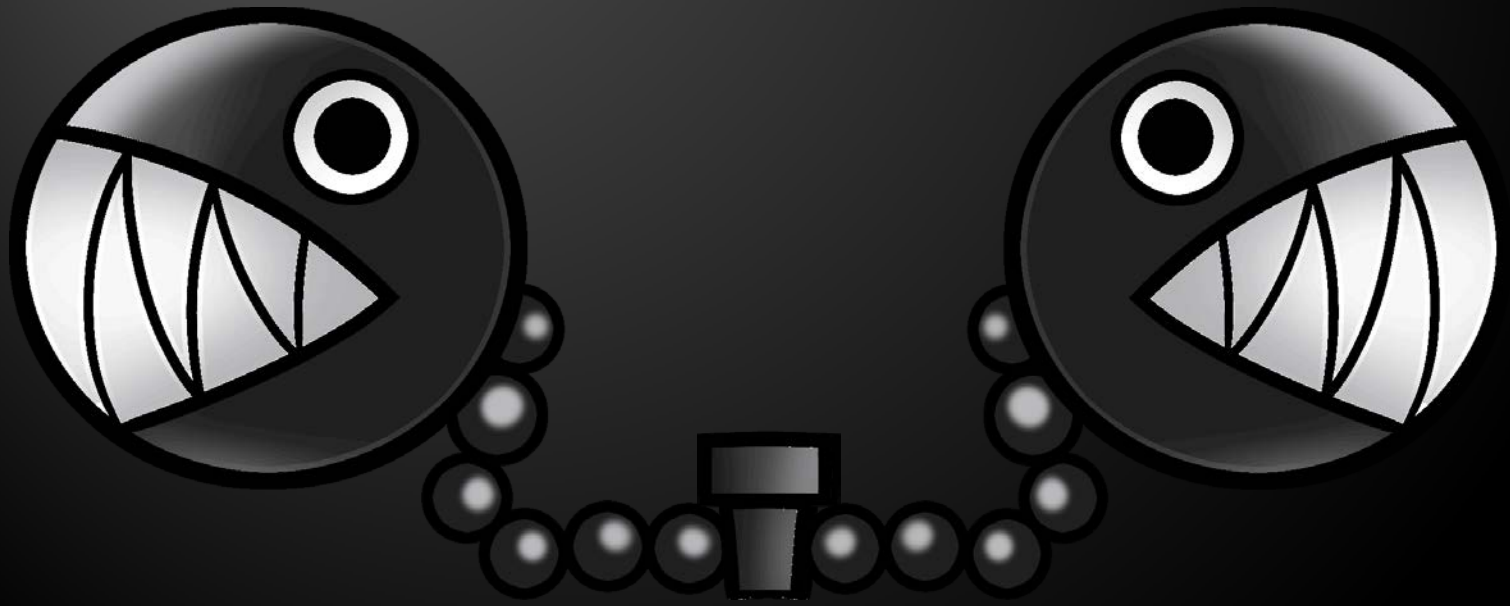
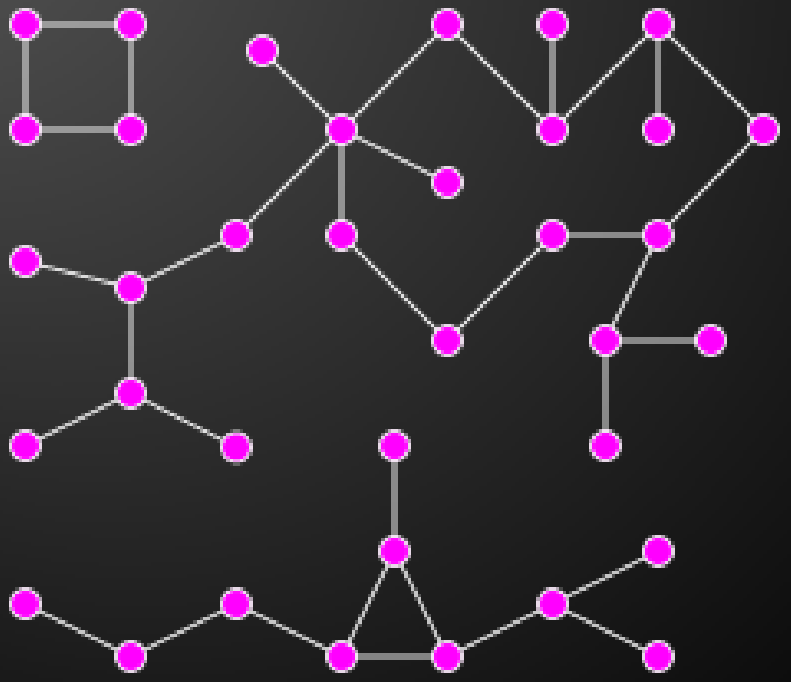
Connectivity

Connecting the chain



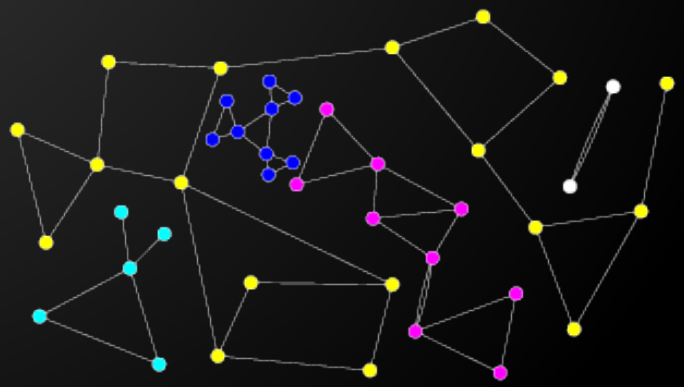
Connectivity

- Connected component of undirected graph
  - A sub-graph in which any two nodes are connected to each other by paths



Connectivity (2)

- A simple way to find number of connected components
  - A loop through all nodes and start a DFS or BFS traversing from any unvisited node
- Each time you start a new traversing
  - You find a new connected component!

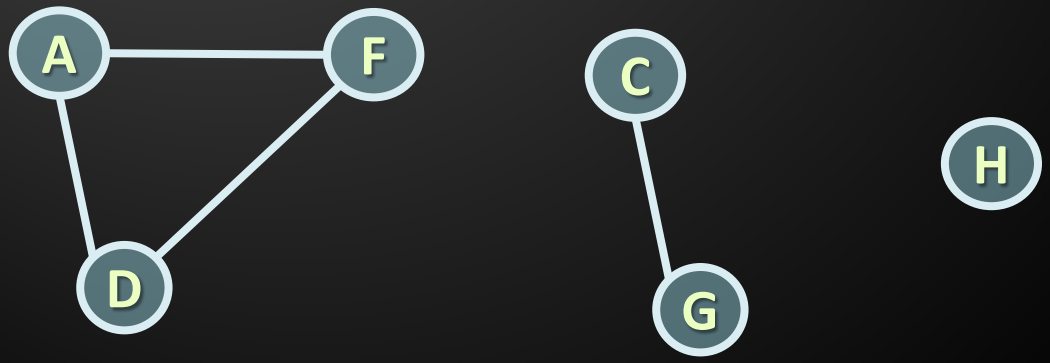


Connectivity (3)

- Algorithm:

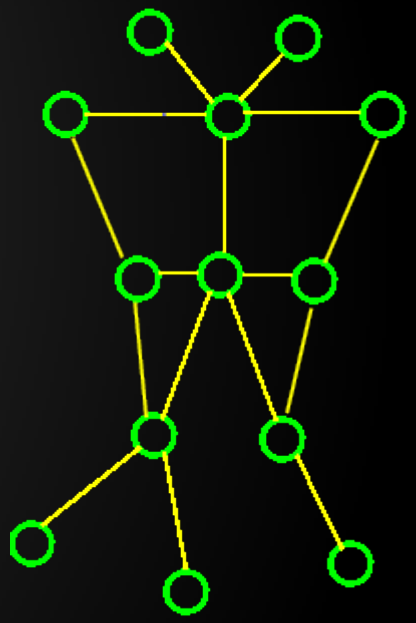
```
foreach node from graph G
{
    if node is unvisited
    {
        DFS(node);
        countOfComponents++;
    }
}
```

*Note: Do not forget to mark each node in the DFS as visited!



Connectivity (4)

- Connected graph
  - A graph with only one connected component
- In every connected graph a path exists between any two nodes
- Checking whether a graph is connected
  - If DFS / BFS passes through all vertices → graph is connected!



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Connectivity

Live Demo

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Dijkstra's Algorithm

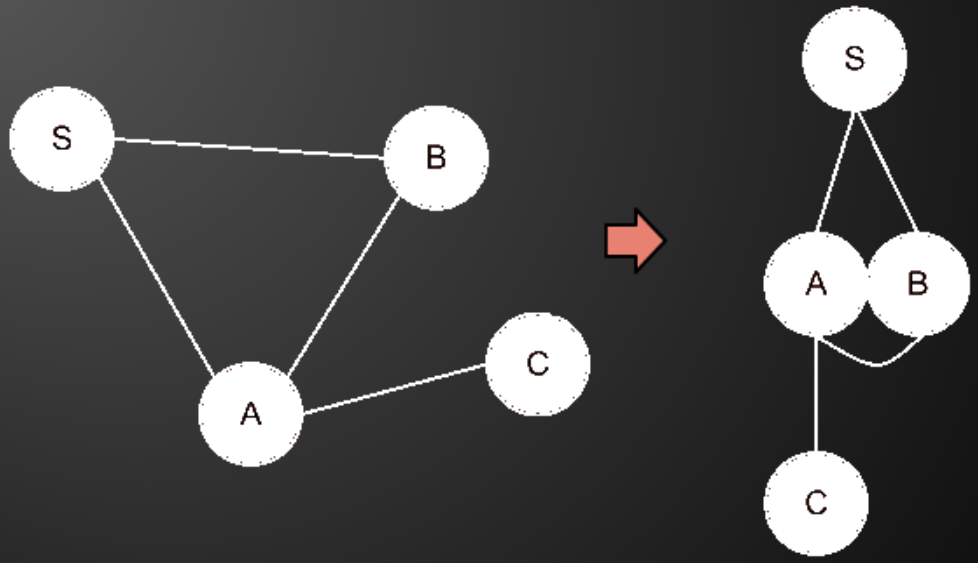
Shortest path in graph

Dijkstra's Algorithm

- Find the *shortest path* from vertex A to vertex B - a directed path between them such that no other path has a lower weight.
- Assumptions
  - Edges can be directed or not
  - Weight does not have to be distance
  - Weights are positive or zero
  - Shortest path is not necessary unique
  - Not all edges need to be reachable

Dijkstra's Algorithm (2)

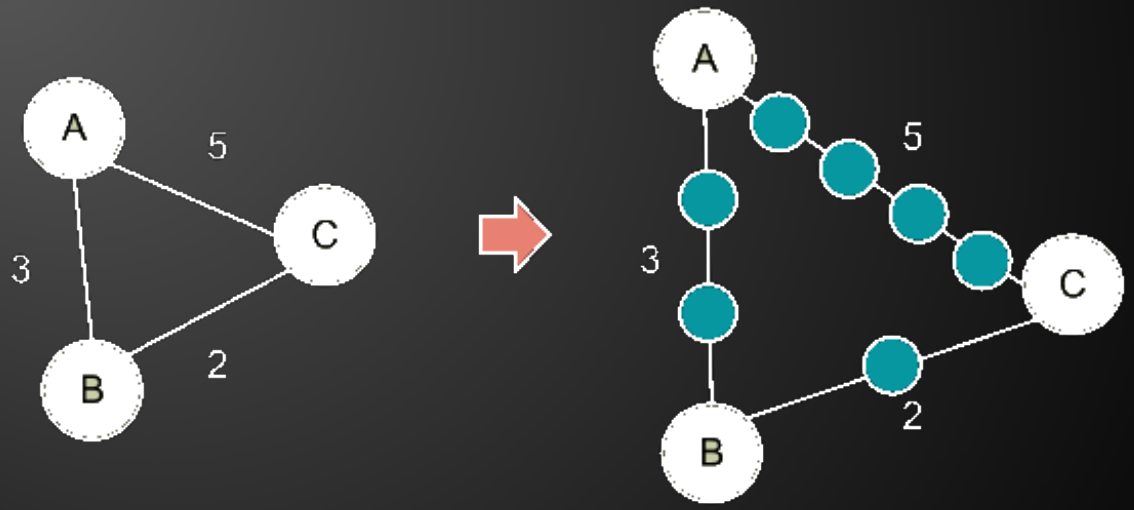
- In non-weighted graphs or edges with same weight finding shortest path can be done with BFS



*Note: Path from A to B does not matter – triangle inequality

Dijkstra's Algorithm (3)

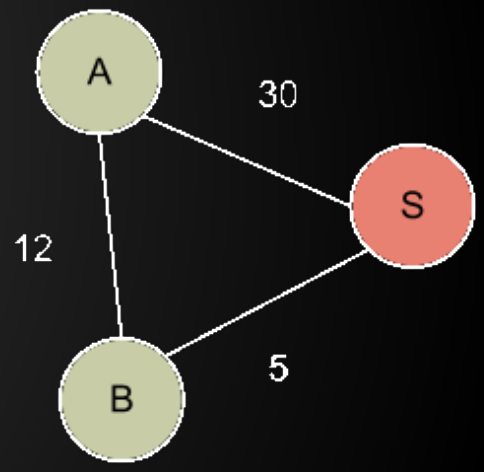
- In weighted graphs – simple solution can be done with breaking the edges in sub-vertexes



*Too much memory usage even for smaller graphs!

Dijkstra's Algorithm (4)

- Solution to this problem – priority queue instead of queue + keeping information about the shortest distance so far
- Steps:
  - Enqueue all distances from S
  - Get the lowest in priority - B
  - If edge B-A exists, check (S-B) + (B-A) and save the lower one
  - Overcome the triangle inequality miss



Dijkstra's Algorithm (5)

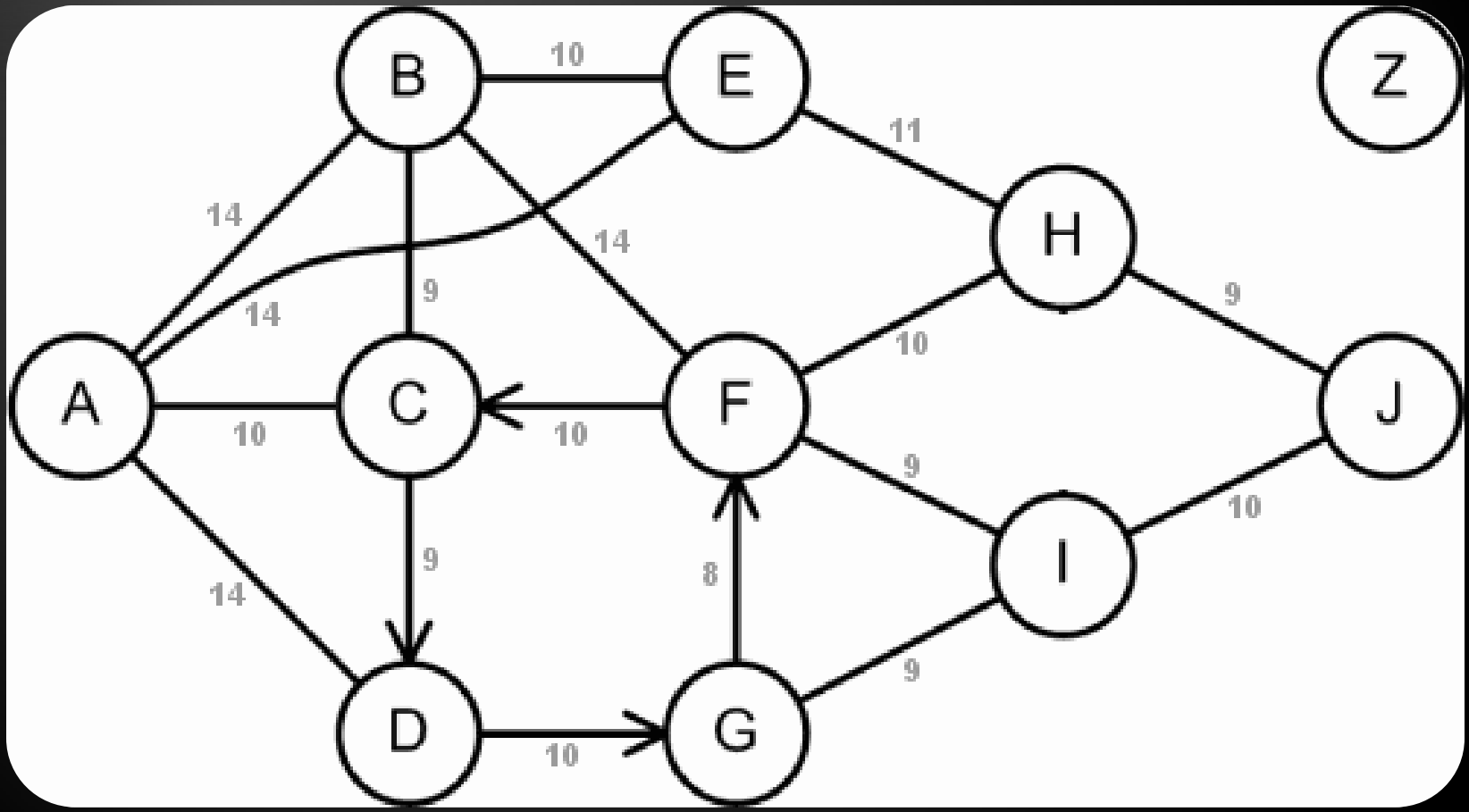
- Dijkstra's algorithm:
  1. Set the distance to every node to Infinity except the source node – must be zero
  2. Mark every node as unprocessed
  3. Choose the first unprocessed node with smallest non-infinity distance as current. If such does not exist, the algorithm has finished
  4. At first we set the current node our Source

Dijkstra's Algorithm (6)

- Dijkstra's algorithm:
  5. Calculate the distance for all unprocessed neighbors by adding the current distance to the already calculated one
  6. If the new distance is smaller than the previous one – set the new value
  7. Mark the current node as processed
  8. Repeat step 3.

Dijkstra's Algorithm (7)

- Example graph:



Dijkstra's Algorithm (8)

- Pseudo code

```
set all nodes DIST = INFINITY;
set current node the source and distance = 0;
Q -> all nodes from graph, ordered by distance;
while (Q is not empty)
{
  a = dequeue the smallest element (first in PriorityQueue);
     if (distance of a == INFINITY) break;
     foreach neighbour v of a
        {
            potDistance = distance of a + distance of (a-v);
            if (potDistance < distance of v)
            {
                distance of v = potDistance;
                reorder Q;
            }
        }
}
```

Dijkstra's Algorithm (9)

- Modifications
  - Saving the route
  - Having a target node
  - Array implementation, Queue, Priority Queue
  - A*
- Complexity
  - O((|V| + |E|).log(|V|))
- Applications –GPS, Networks, Air travels, etc.

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Dijkstra's Algorithm

Live Demo

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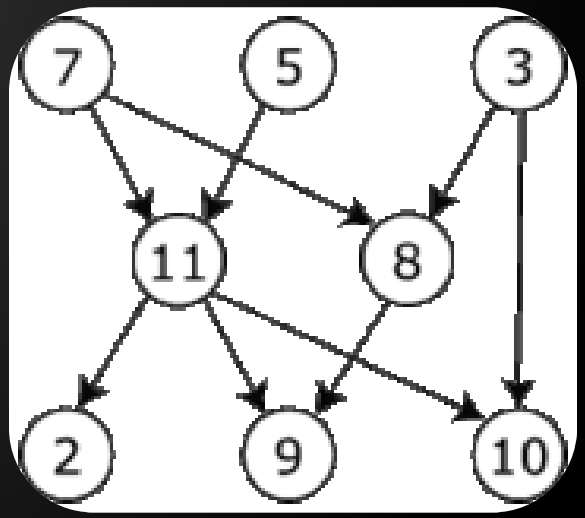
Topological Sorting

Order it!

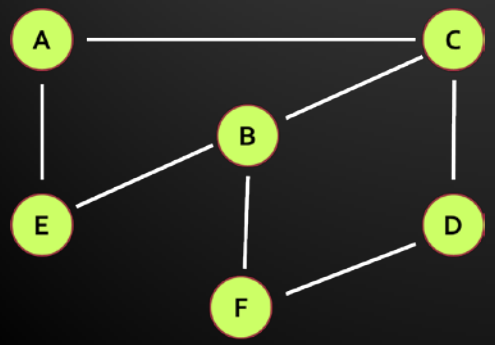
Topological Sorting

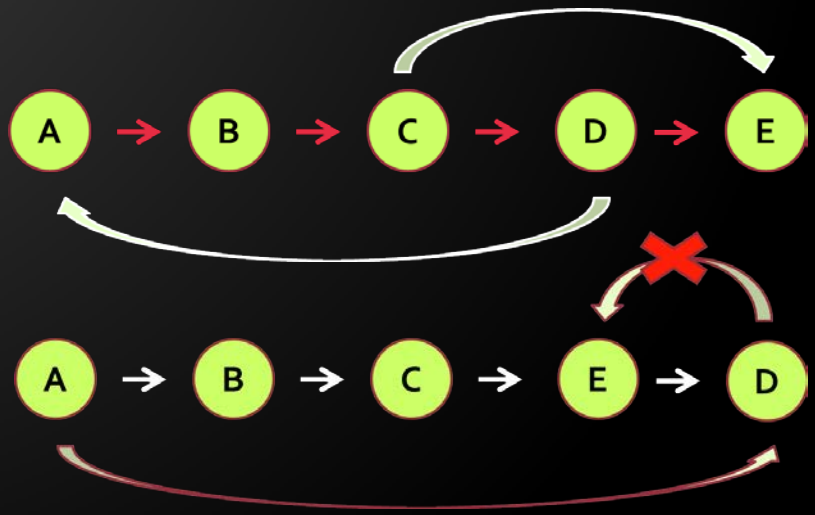
- Topological ordering of a directed graph
  - linear ordering of its vertices
  - for every directed edge from vertex u to vertex v, u comes before v in the ordering
- Example:
  - 7, 5, 3, 11, 8, 2, 9, 10
  - 3, 5, 7, 8, 11, 2, 9, 10
  - 5, 7, 3, 8, 11, 10, 9, 2



Topological Sorting (2)

- Rules
  - Undirected graph cannot be sorted
  - Directed graphs with cycles cannot be sorted
  - Sorting is not unique
  - Various sorting algorithms exists and they give different results





Topological Sorting (3)

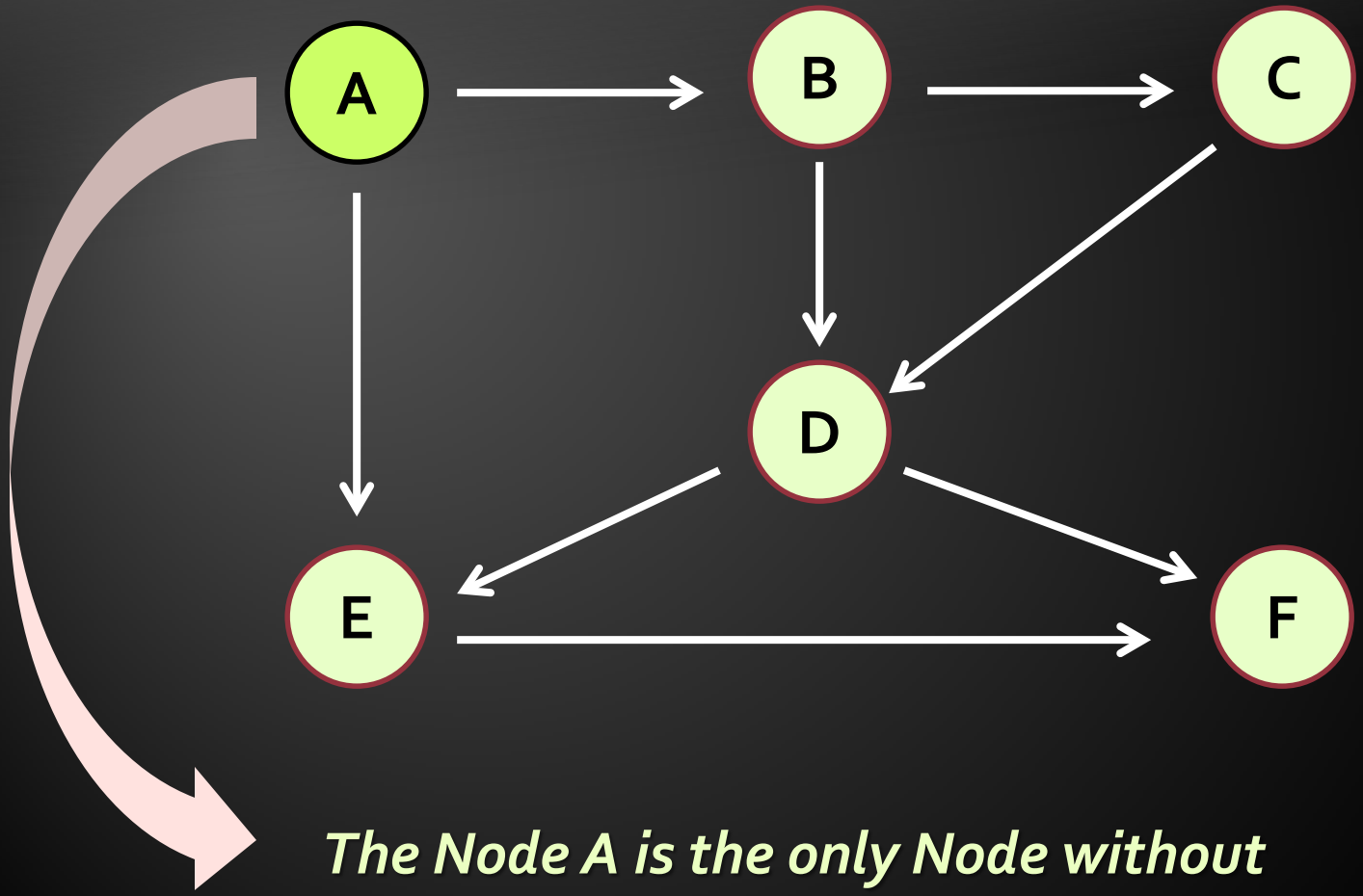
- Source removal algorithm
  - Create an Empty List
  - Find a Node without incoming Edges
  - Add this Node to the end of the List
  - Remove the Edge from the Graph
  - Repeat until the Graph is empty

Topological Sorting (4)

- Pseudo code

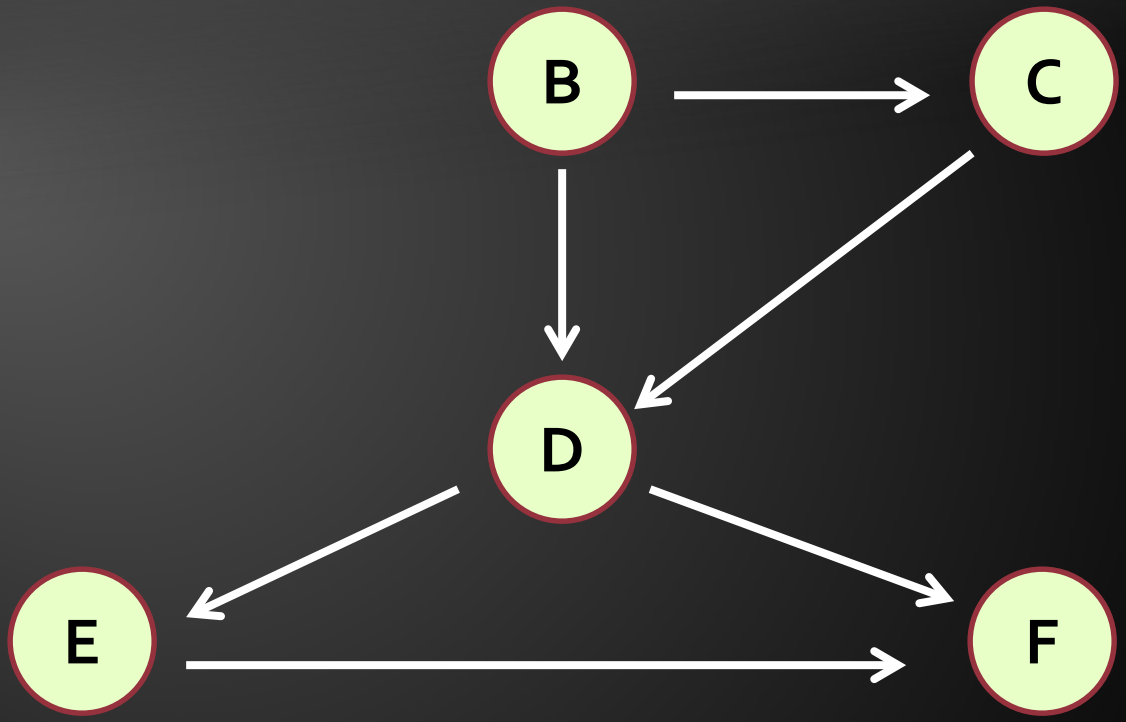
```
L ← Empty list that will contain the sorted elements
S ← Set of all nodes with no incoming edges
while S is non-empty do
    remove a node n from S
    insert n into L
    for each node m with an edge e from n to m do
        remove edge e from the graph
        if m has no other incoming edges then
            insert m into S
if graph has edges then
    return error (graph has at least one cycle)
else
    return L (a topologically sorted order)
```

Step #1-2



The Node A is the only Node without Incoming Edges

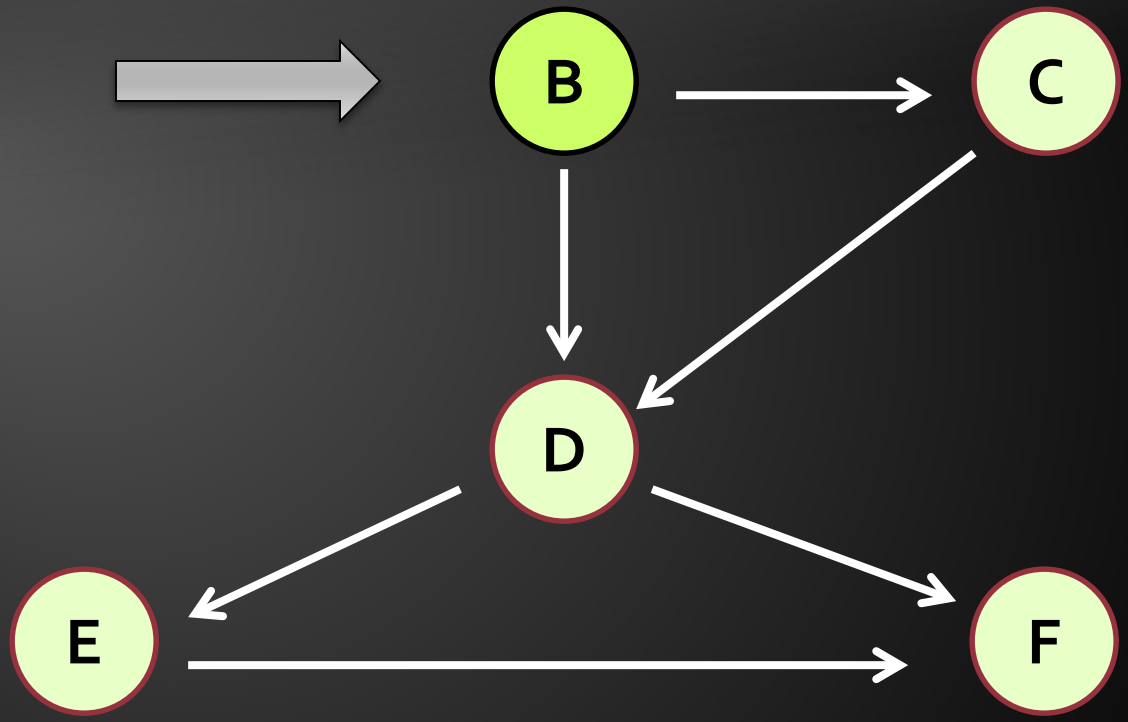
Step #3-4



L

A

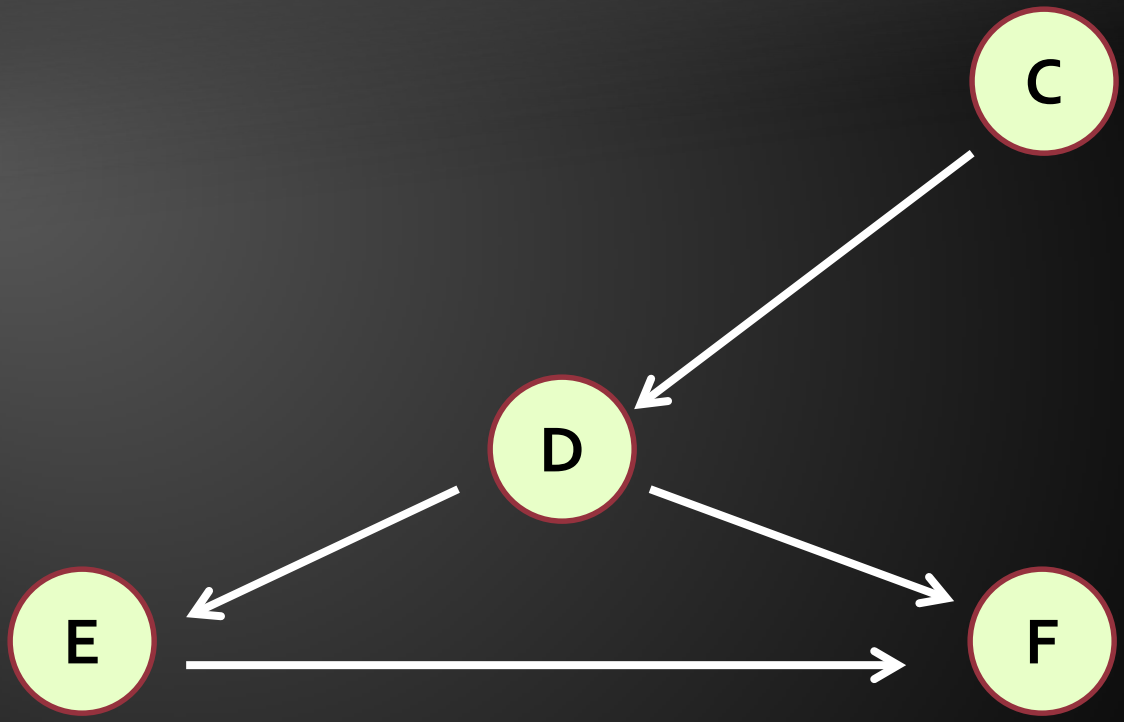
Step #2-3



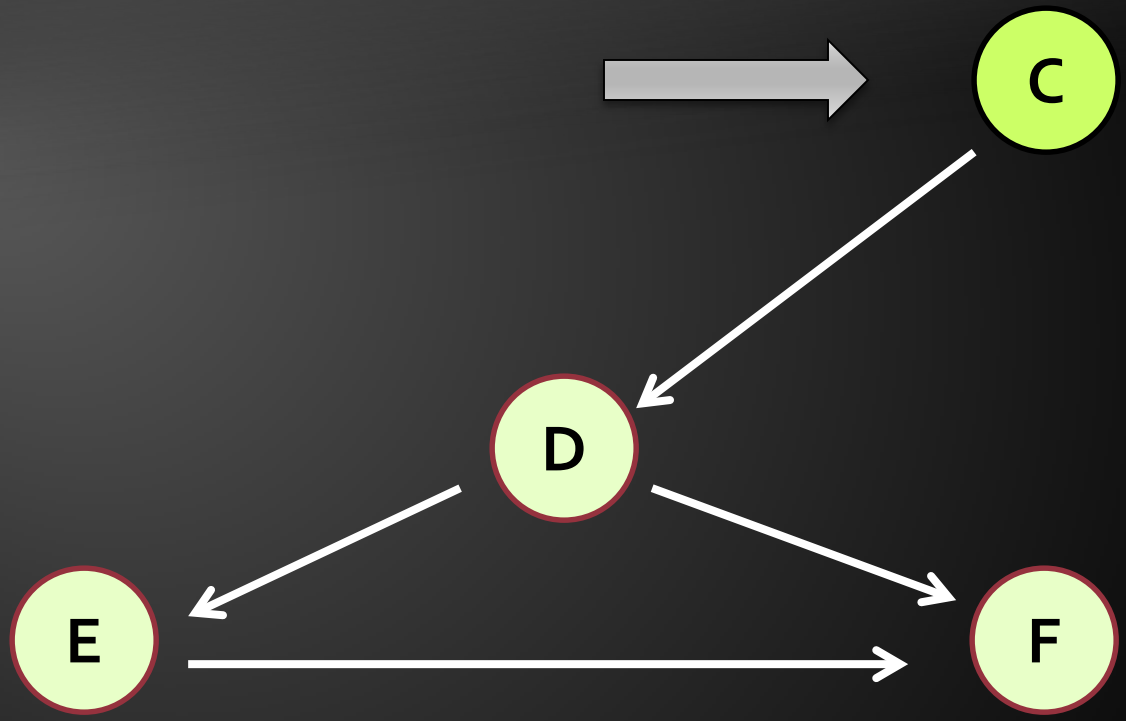
L

A

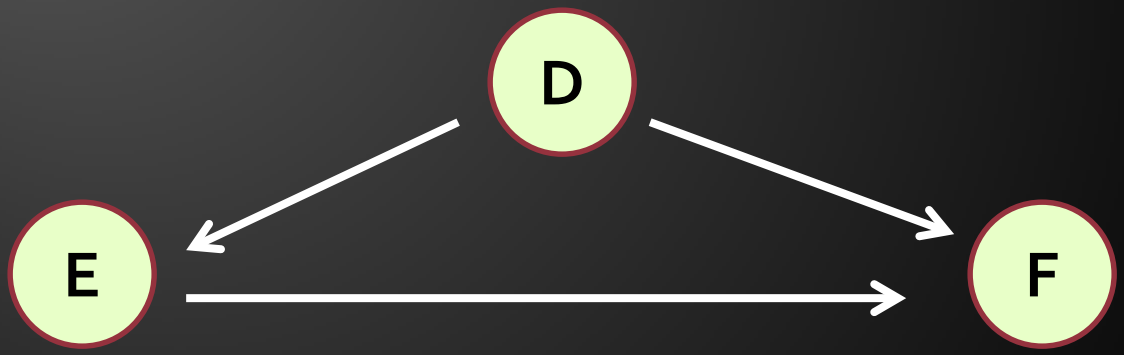
Step #2-3

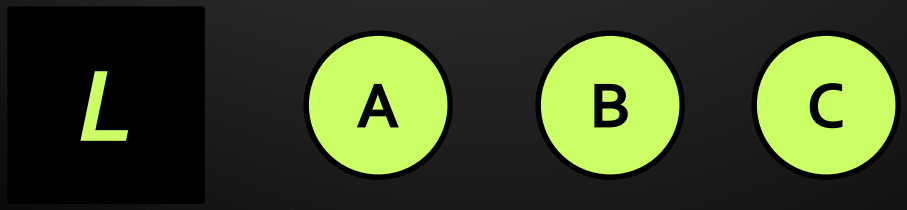


Step #2-3

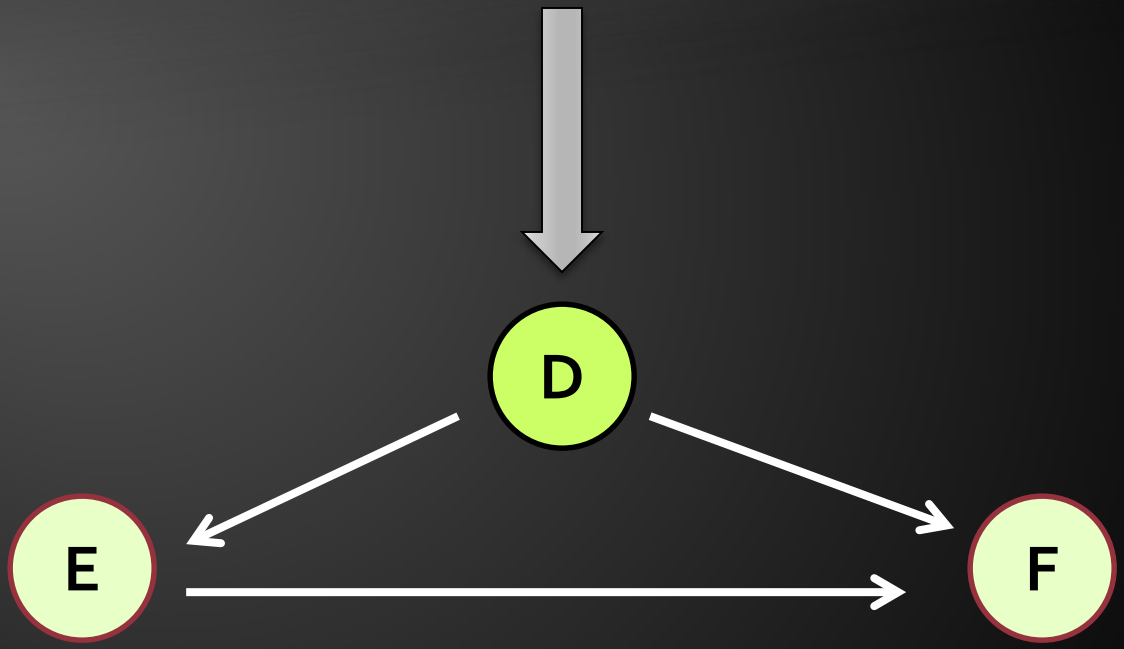


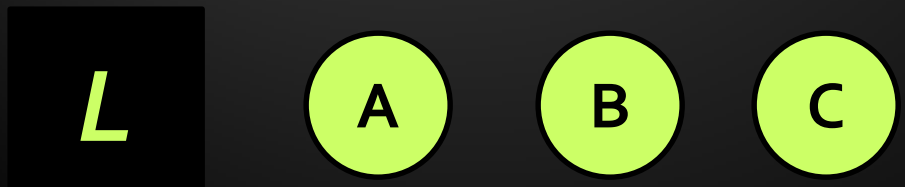
Step #2-3



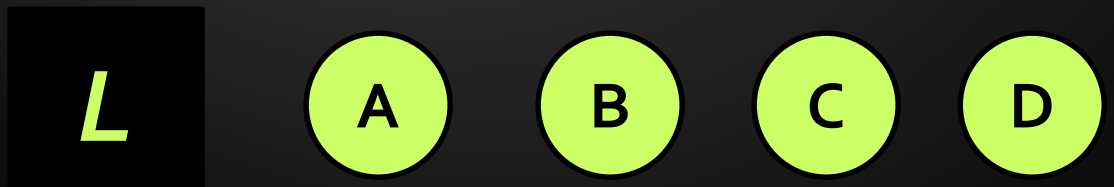


Step #2-3

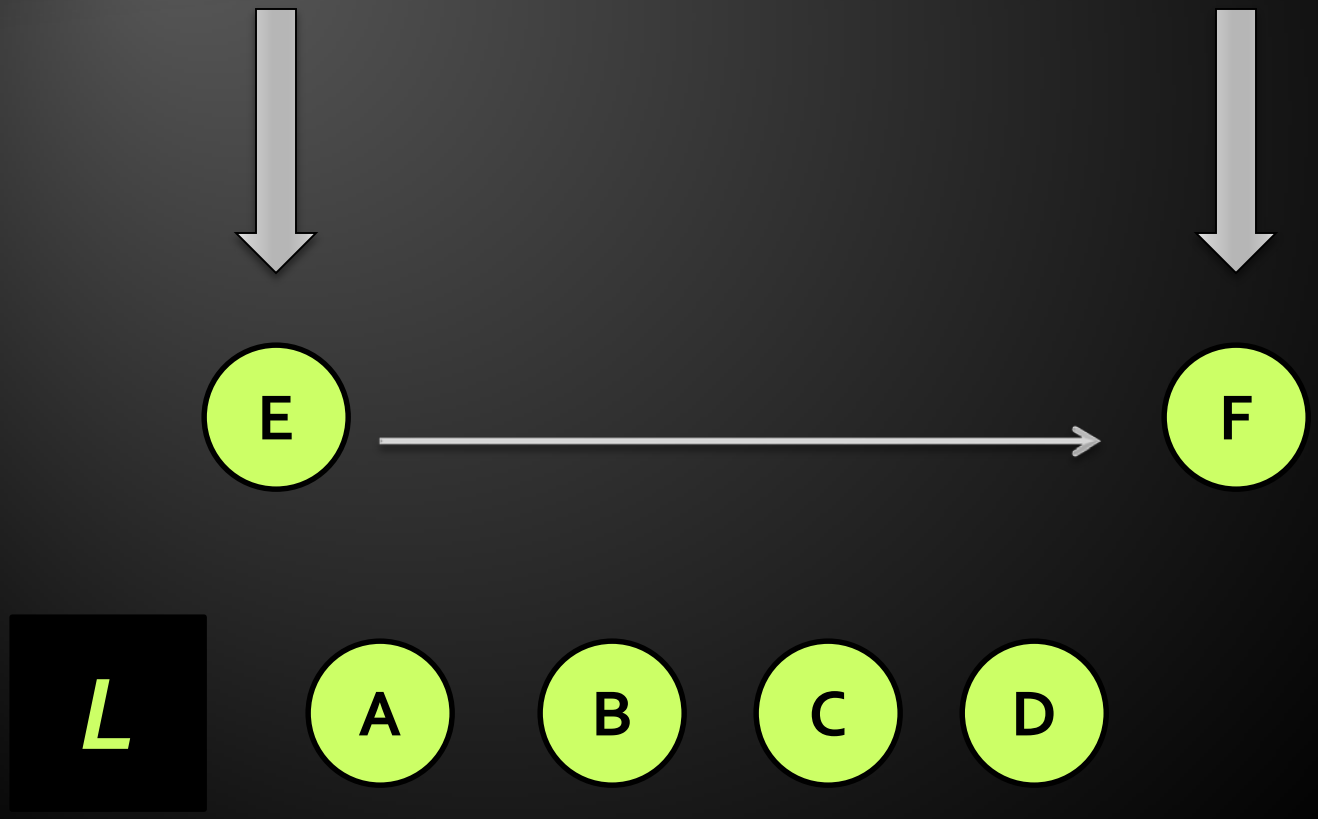




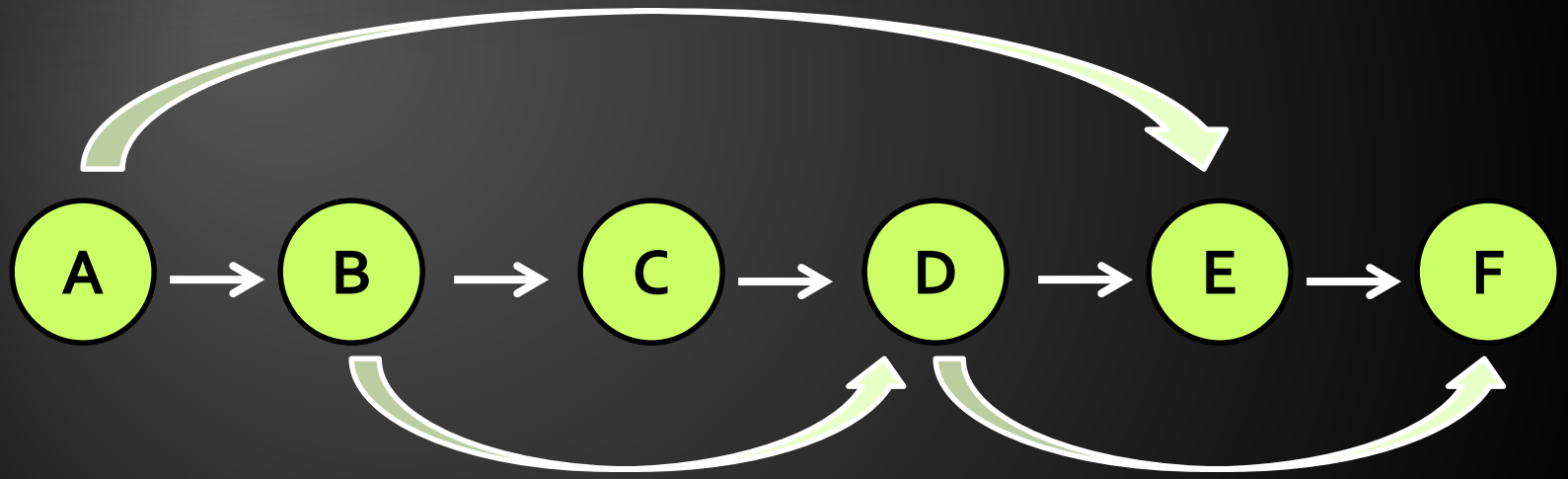
Step #2-3



Step #2-3



Result:TS



Topological Sorting (5)

- DFS algorithm
  - Create an empty List
  - Find a Node without Outgoing Edges
  - Mark the Node as visited
  - Add the node to the List
  - Stop when reach visited node
  - Reverse the List and get the TS of the Elements

Topological Sorting (6)

- Pseudo code

```
L ← Empty list that will contain the sorted nodes
while there are unmarked nodes do
    select an unmarked node n
    visit(n)
function visit(node n)
    if n has a temporary mark then stop (not a DAG)
    if n is not marked (i.e. has not been visited yet) then
        mark n temporarily
        for each node m with an edge from n to m do
            visit(m)
        mark n permanently
        add n to head of L
```

- http://www.geeksforgeeks.org/topological-sorting/

Telerik Academy

TS Using DFS

Live Demo

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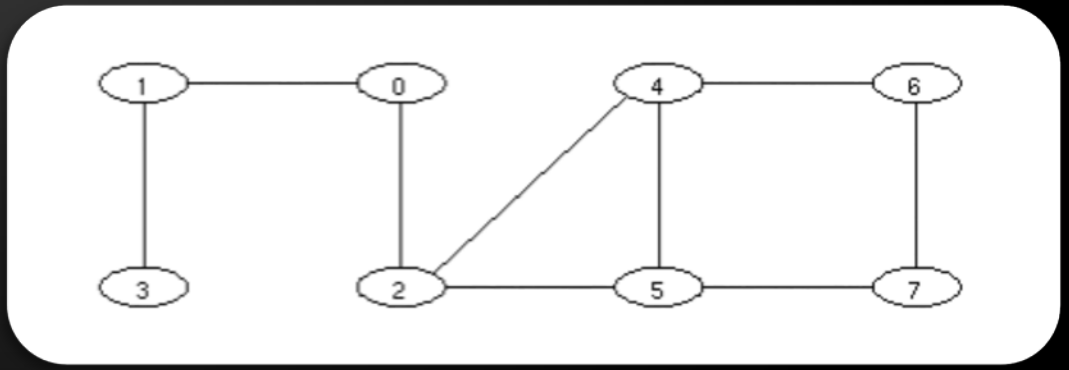
Minimum Spanning Tree

Minimum Spanning Tree

- Spanning Tree
  - Subgraph (Tree)
  - Connects all vertices together
- All connected graphs have spanning tree

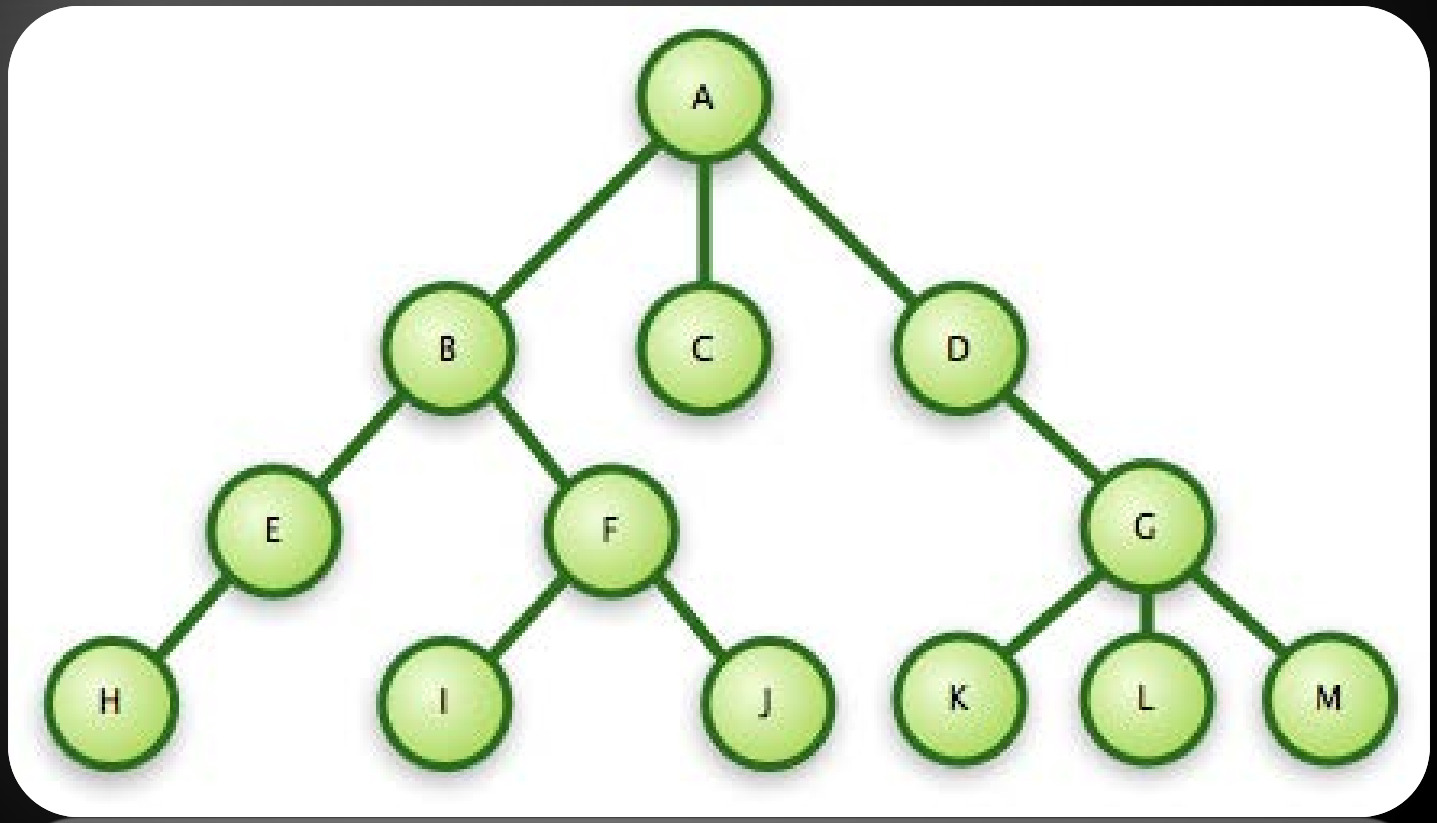


Minimum Spanning Tree

- Minimum Spanning Tree
  - weight <=

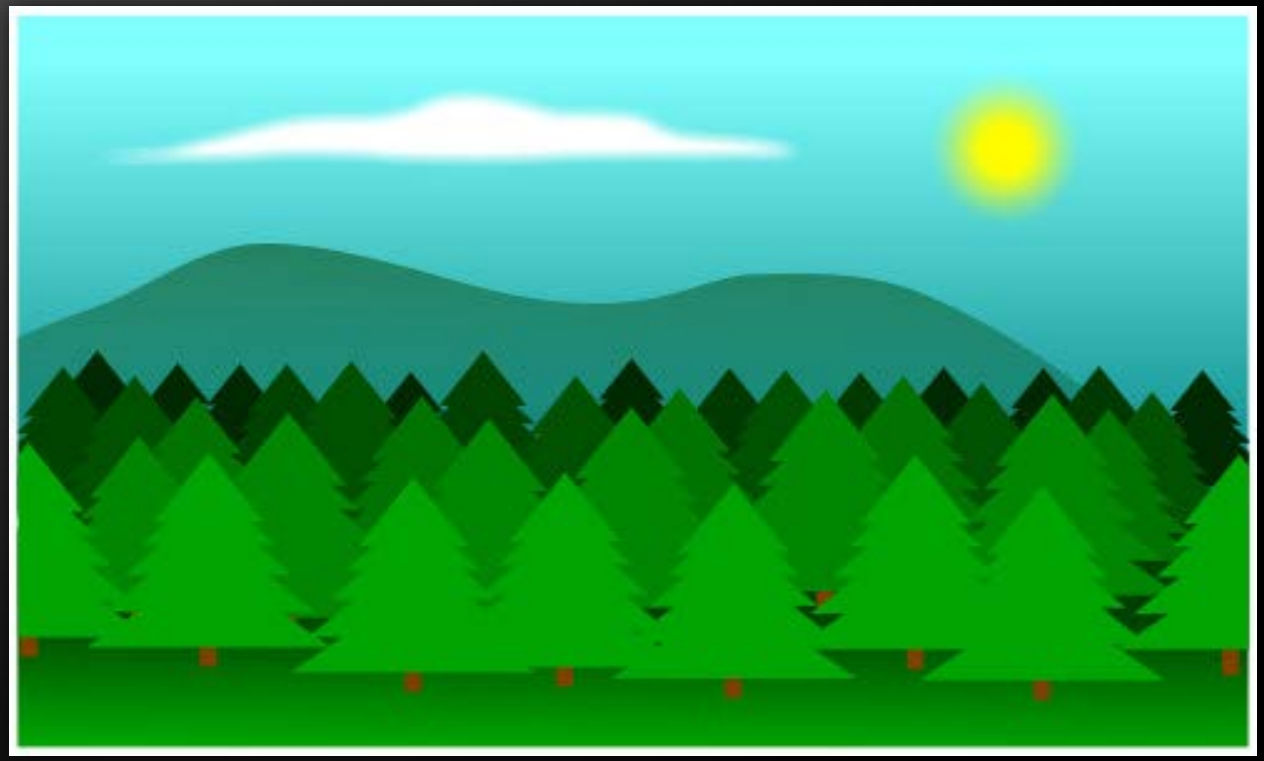
    weight(all other spanning trees)
- First used in electrical network
  - Minimal cost of wiring

Minimum Spanning Forest

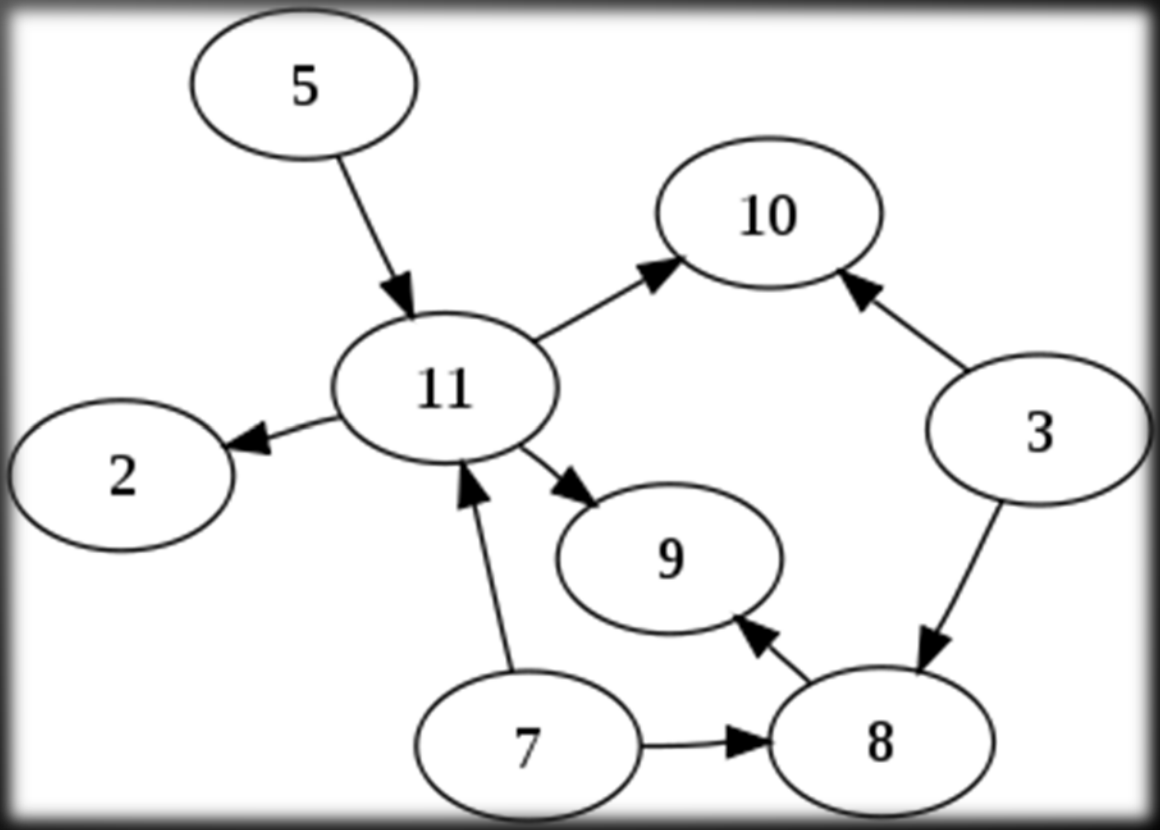


Minimum Spanning Forest

- Minimum Spanning Forest – set of all minimum spanning trees (when the graph is not connected)



Prim's Algorithm

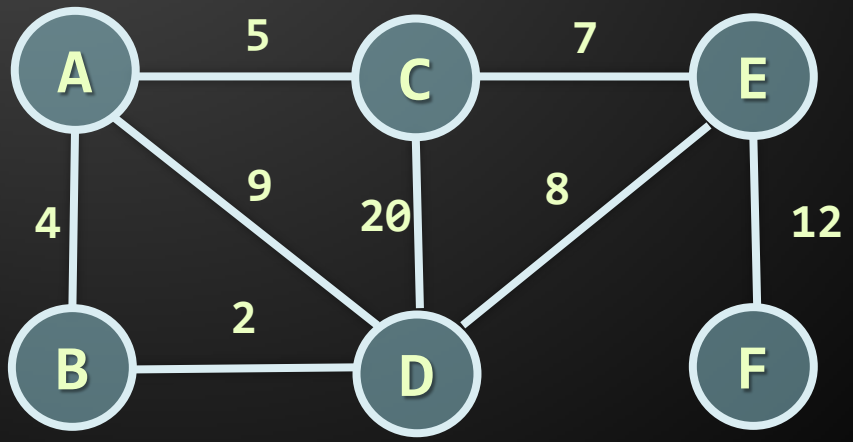


Prim's Algorithm

- Create a tree T containing a single vertex (chosen randomly)
- Create a set S from all the edges in the graph
- Loop until every edge in the set connects two vertices in the tree
  - Remove from the set an edge with minimum weight that connects a vertex in the tree with a vertex not in the tree
  - Add that edge to the tree
- Note: the graph must be connected

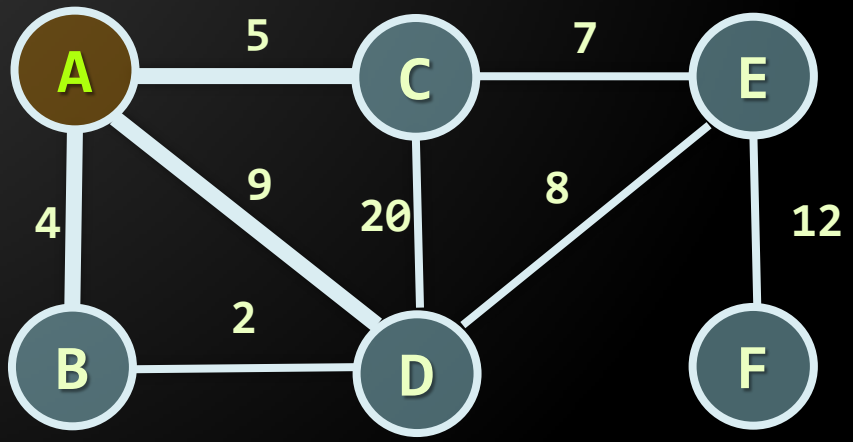
Prim's Algorithm

- Note: at every step before adding an edge to the tree we check if it makes a cycle in the tree or if it is already in the queue
- When we add a vertex we check if it is the last which is not visited



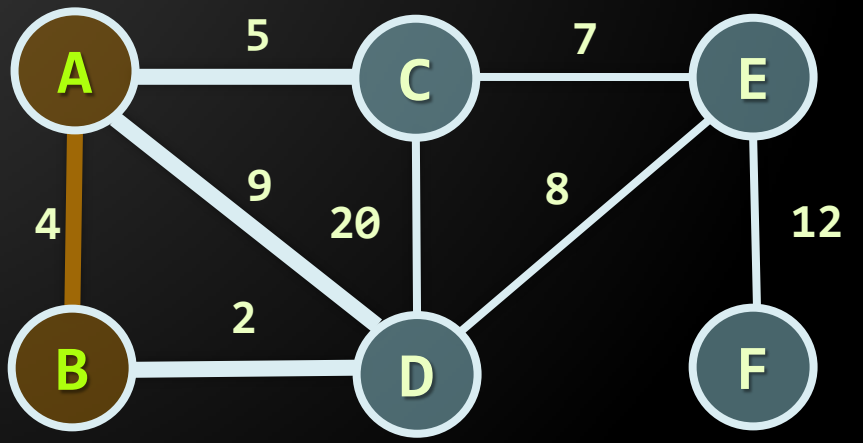
Prim's Algorithm

- We build a tree with the single vertex A
- Priority queue which contains all edges that connect A with the other nodes (AB, AC, AD)



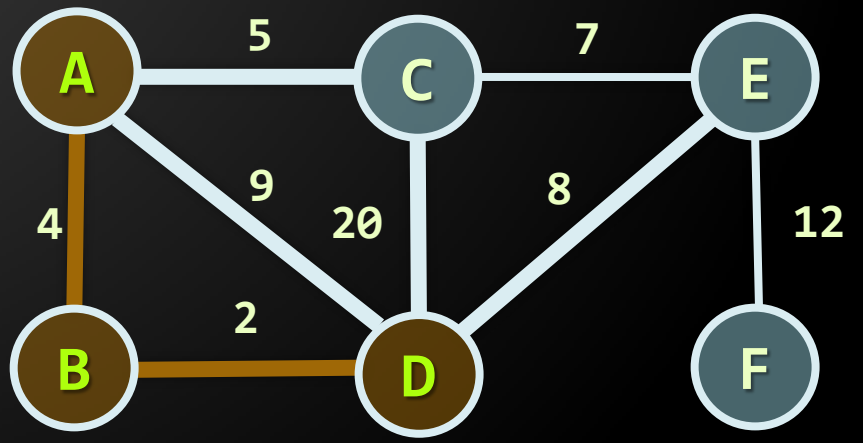
Prim's Algorithm

- The tree still contains the only vertex A
- We dequeue the first edge from the priority queue (4) and we add the edge and the other vertex (B) form that edge to the tree
- We push all edges that connect B with other nodes in the queue
  - Note that the edges 5 and 9 are still in the queue



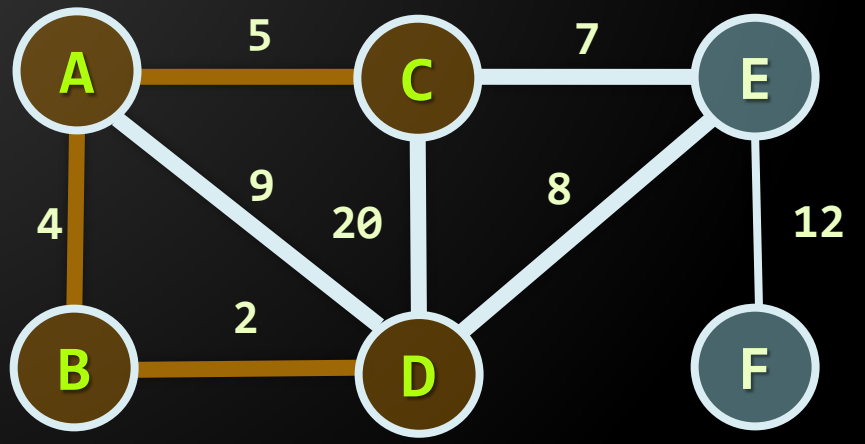
Prim's Algorithm

- Now the tree contains vertices A and B and the edge between them
- We dequeue the first edge from the priority queue (2) and we add the edge and the other vertex (D) from that edge to the tree
- We push all edges that connect D with other nodes in the queue



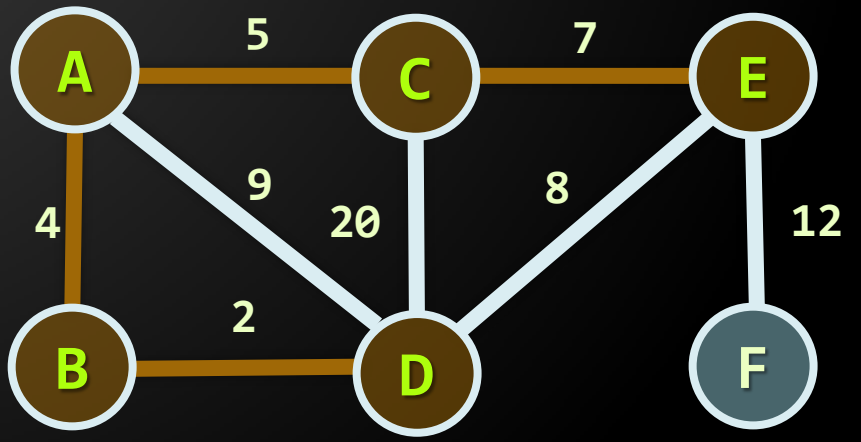
Prim's Algorithm

- Now the tree contains vertices A, B and D and the edges (4, 2) between them
- We dequeue the first edge from the priority queue (5) and we add the edge and the other vertex (C) from that edge to the tree
- We push all edges that connect C with other nodes in the queue



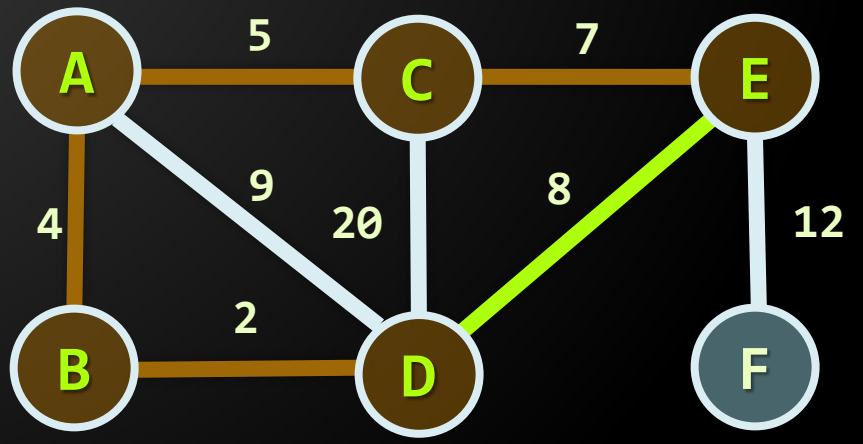
Prim's Algorithm

- Now the tree contains vertices A, B, D and C and the edges (4, 2, 5) between them
- We dequeue the first edge from the priority queue (7) and we add the edge and the other vertex (E) from that edge to the tree
- We push all edges that connect C with other nodes in the queue



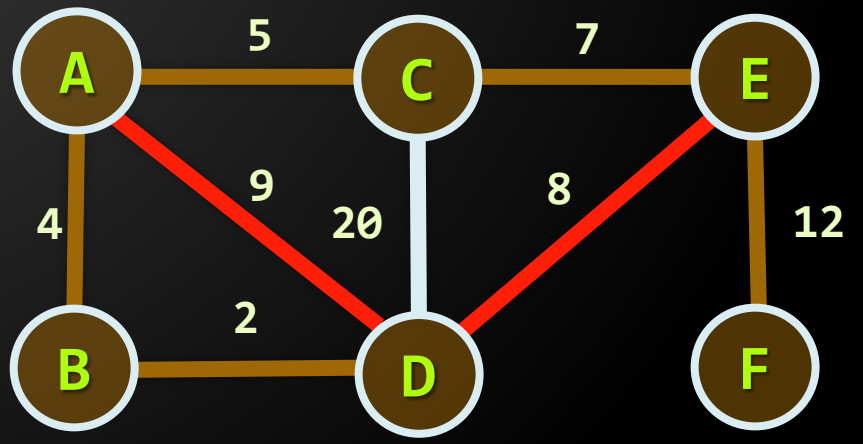
Prim's Algorithm

- Now the tree contains vertices A, B, D, C and E and the edges (4, 2, 5, 7) between them
- We dequeue the first edge from the priority queue (8)



Prim's Algorithm

- This edge will cost a cycle
- So we get the next one – 9
  - This edge will also cost a cycle
- So we get the next one – 12
  - We add it to the tree
  - We add the vertex F to the tree



Prim

Live Demo

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Kruskal's Algorithm

Kruskal's Algorithm

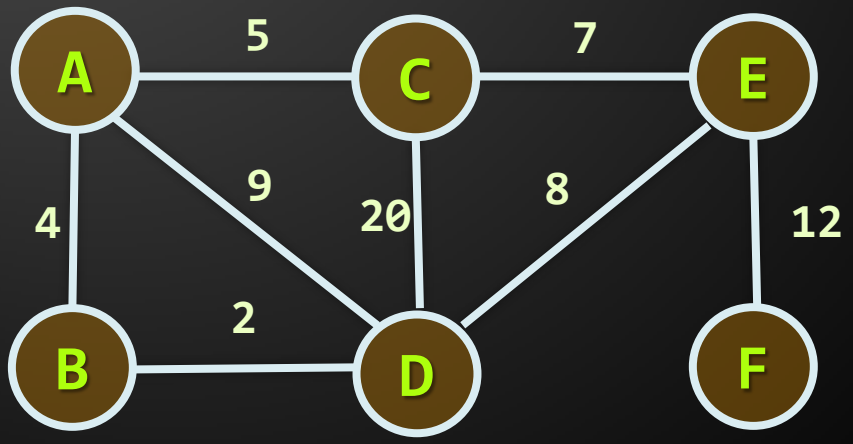
- The graph may not be connected
  - If the graph is not connected – minimum spanning forest

Kruskal's Algorithm

- Create forest F (each tree is a vertex)
- Set S – all edges in the graph
- While S if nonempty and F is not spanning
  - Remove edge with min cost from S
  - If that edge connects two different trees – add it to the forest (these two trees are now a single tree)
  - Else discard the edge
- The graph may not be connected

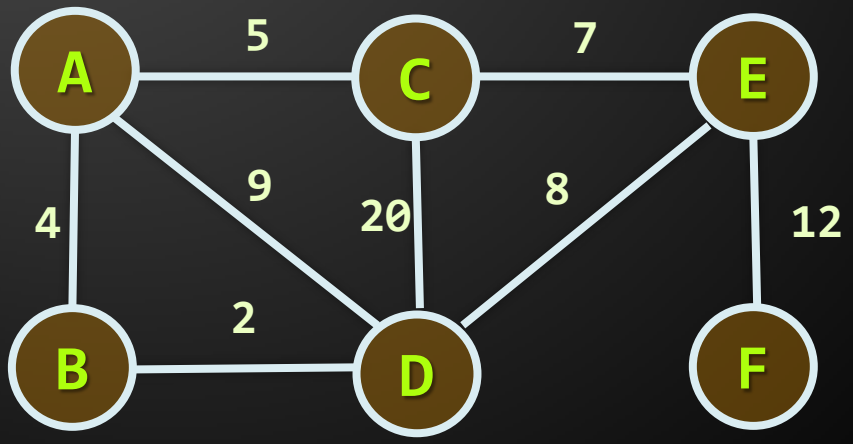
Kruskal's Algorithm

- We build a forest containing all vertices from the graph
- We sort all edges
- Edges are – 2, 4, 5, 7, 8, 9, 12, 20



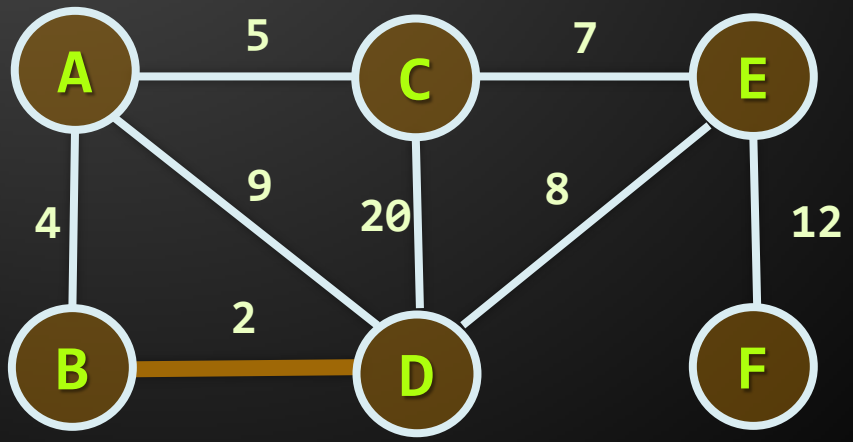
Kruskal's Algorithm

- At every step we select the edge with the smallest weight and remove it from the list with edges
- If it connects two different trees from the forest we add it and connect these trees



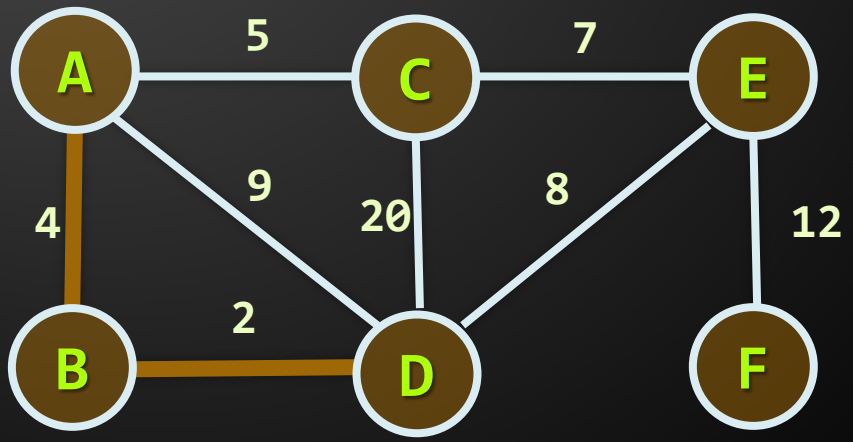
Kruskal's Algorithm

- We select the edge 2
- This edge connects the vertices B and D (they are in different trees)
- We add it



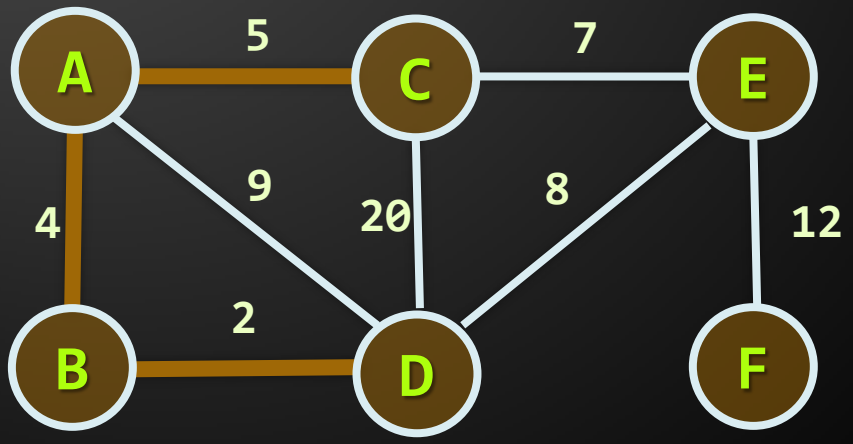
Kruskal's Algorithm

- We select the edge 4
- This edge connects the vertices A and B (they are different trees)
- We add it



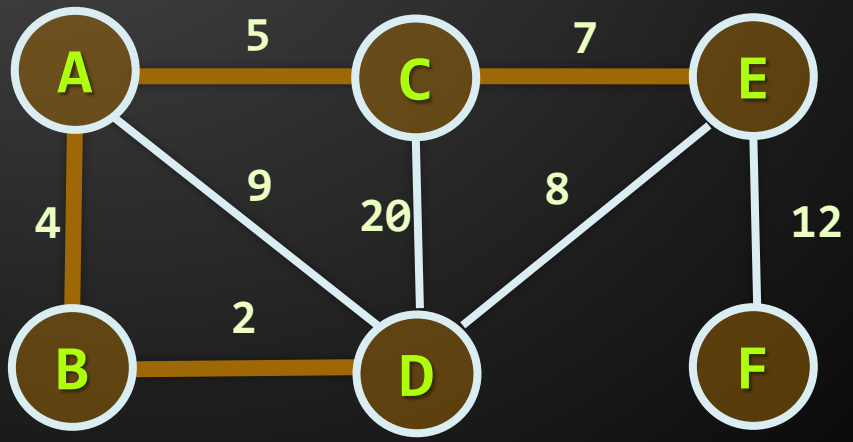
Kruskal's Algorithm

- We select the edge 5
- This edge connects the vertices A and C (they are different trees)
- We add it



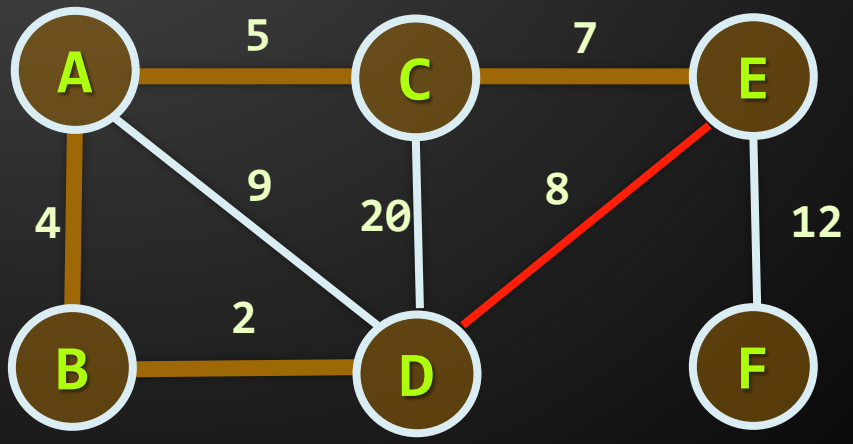
Kruskal's Algorithm

- We select the edge 7
- This edge connects the vertices C and E (they are different trees)
- We add it



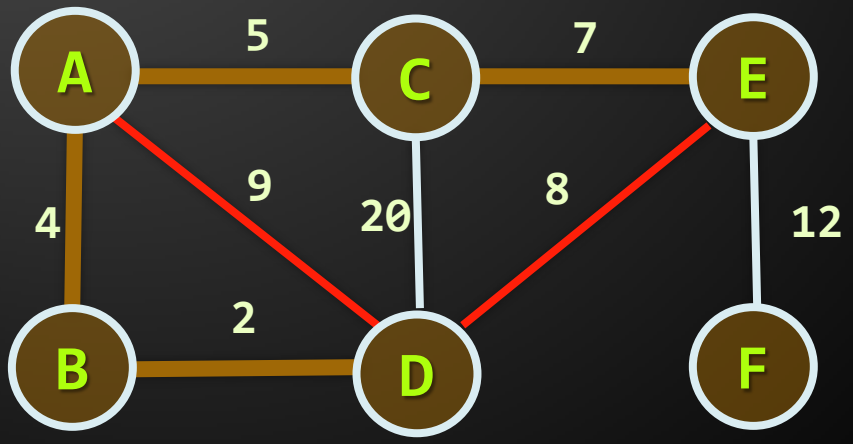
Kruskal's Algorithm

- We select the edge 8
- This edge connects the vertices E and D (they are not different trees)
- We don't add it



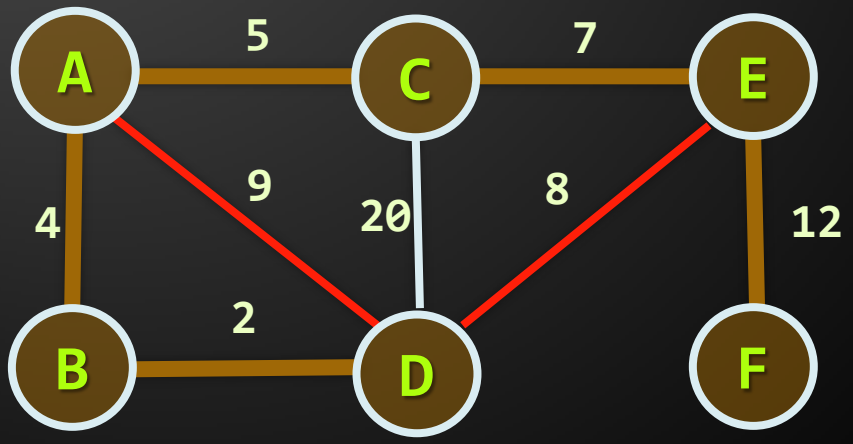
Kruskal's Algorithm

- We select the edge 9
- This edge connects the vertices A and D (they are not different trees)
- We don't add it



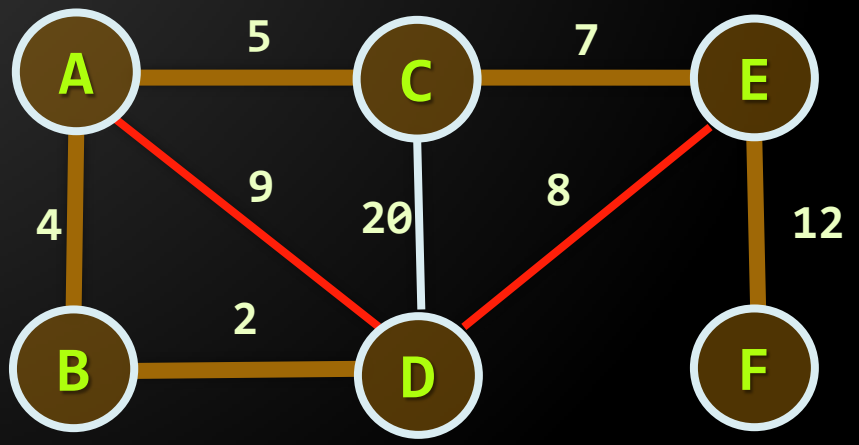
Kruskal's Algorithm

- We select the edge 12
- This edge connects the vertices E and F (they are not different trees)
- We add it



Kruskal's Algorithm

- We can have a function that checks at every step if all vertices are connected and the tree that we build is spanning
- If we have such function we stop
- Otherwise we check for the other edges
  - We just won't add them to the tree



Telerik Academy

Kruskal

Live Demo

Exercises

1. Solve these problems from BGCoder:
   1. Algo Academy March 2012 – Problem 05 – Friends of Pesho
   2. Algo Academy February 2013 – Problem 04 – Salaries
2. You are given a cable TV company. The company needs to lay cable to a new neighborhood (for every house). If it is constrained to bury the cable only along certain paths, then there would be a graph representing which points are connected by those paths. But the cost of some of the paths is more expensive because they are longer. If every house is a node and every path from house to house is an edge, find a way to minimize the cost for cables.

Trees and Graphs

Questions?