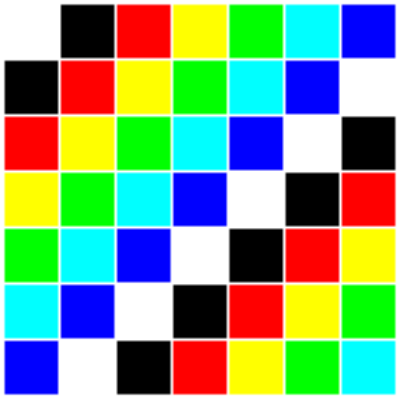
VISUALISING GROUPS

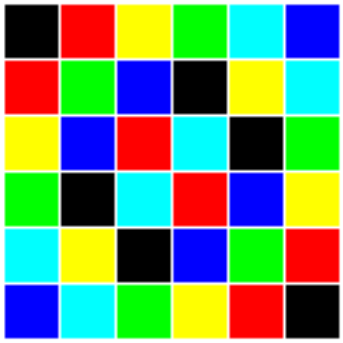
```
n = 7
def action(a, b):
    return (a + b)%n
```

Addition modulo 7



```
n = 7
def action(a, b):
    return (a * b)%n
```

Multiplication modulo 7



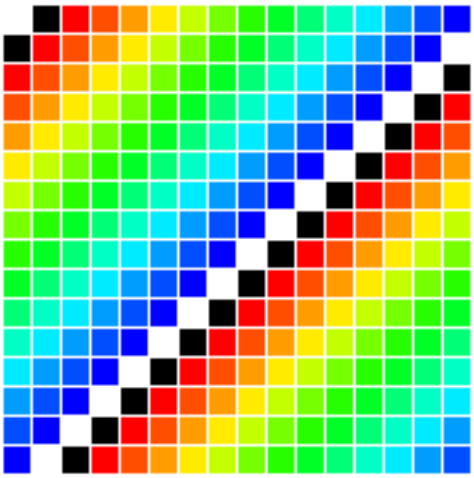
ADDITION MOD N

What do your addition grids tell you about modular addition?

What do the white squares represent?

```
n = 16
def action(a, b):
    return (a + b)%n
```

```
Addition modulo 16
```



Under addition, the white squares represent pairs of numbers that add to zero. And zero is the *identity* for addition.

$$13 + 3 = 0 \mod 16$$

In other words, 3 is the *inverse* of 13 under addition mod 16.

So the numbers $\{0, 1, 2, 3, \ldots, 15\}$ under addition mod 16 are *closed*, have an *identity* 0, and have *inverses*. This is a group!

Any other examples?

This is true for any modulus n.

The numbers $\{0, 1, \ldots, n-1\}$ are a group under addition modulo n.

MULTIPLICATION MOD N

Can we find a group under multiplication mod n?

This situation is a bit more complicated. And interesting.

What have you noticed?

Can you have 0 in a multiplication group?

No, 0 doesn't have an inverse under multiplication.

Suppose Z is the inverse of 0, then

$$Z \times 0 = 1 \mod n$$

because 1 is the identity.

But

$$Z \times 0 = 0$$

So

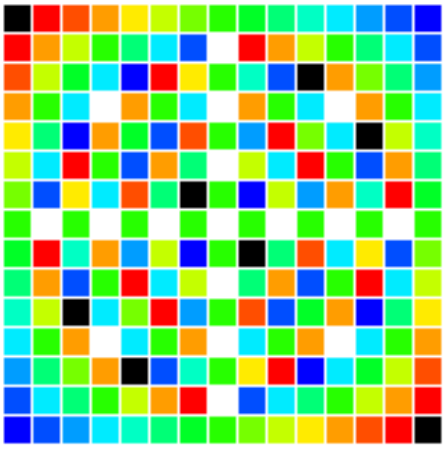
$$0 \stackrel{?}{=} 1$$

If we are going to find a group, it will just be the numbers $\{1, 2, \ldots, n-1\}$, ignoring the white squares along the top and the left of the grid.

Do they form a group?

```
n=16
def action(a, b):
    return (a * b)%n
```

Multiplication modulo 16



$\{1, 2, \ldots, 15\}$ **do not** form a group under multiplication mod 16.

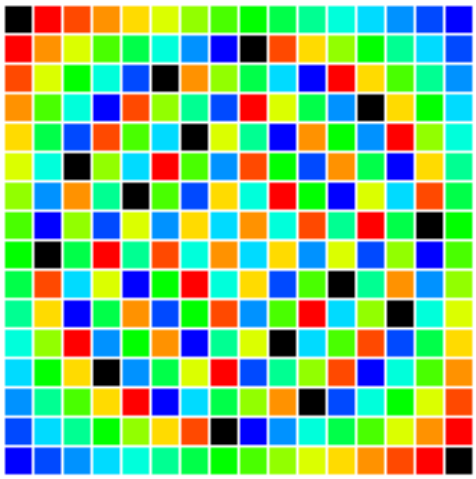
You can tell because

- There isn't a black square in every row or column. Some numbers don't have inverses.
- There are white squares, zeroes. But we excluded zero, so it's not closed.

Can you find any numbers which do give a group?

```
n=17
def action(a, b):
    return (a * b)%n
```

Multiplication modulo 17



$\{1, 2, \ldots, 16\}$ **do** form a group under multiplication mod 17.

You can tell because

- There is a black square in every row or column. All numbers have inverses.
- There are no white squares.

Do you notice anything about the moduli that do give a group, and the ones that don't?

THEOREM

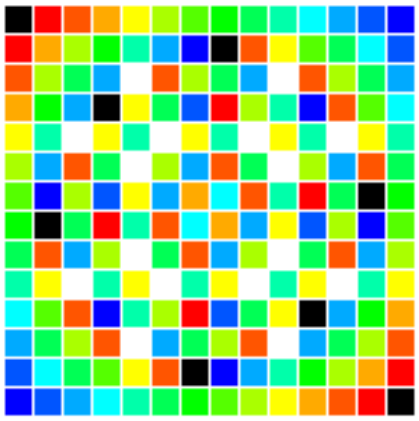
The numbers $\{1, 2, \ldots, p-1\}$ form a group under multiplication mod p if and only if p is prime.

We won't prove this theorem fully today, but think about those white squares.

Make a multiplication grid mod 15. What's the connection between 15, and the pairs that multiply to 0?

```
n=15
def action(a, b):
    return (a * b)%n


Multiplication modulo 15
```



For example,

$$10 \times 6 = 0 \mod 15$$

Because

$$10 \times 6 = 2 \times 5 \times 2 \times 3 = 4 \times 15 = 0 \mod 15$$

The factors of 15 were hidden in the 10 and 6

That couldn't happen with a prime like 17.

THE PROBLEM WITH ZERO DIVISORS

Usually if both sides of an equation have a common factor, you can divide both sides by it

$$\begin{aligned} 5x &= 15 \\ \cancel{5}x &= \cancel{5} \times 3 \\ x &= 3 \end{aligned}$$

Products that equal zero cause problems with this.

For example, 3 is a zero divisor mod 6 and

$$\begin{aligned} 15 &= 3 && \mod 6 \\ 3 \times 5 &= 3 \times 1 && \mod 6 \\ \text{But } 5 &\neq 1 && \mod 6 \end{aligned}$$

Cancelling doesn't work mod 6. But it does work mod 7.

This is because **zero divisors don't have inverses**.

There are no white squares on the same row or column as a black square.

Zero divisors don't have inverses

PROOF

Let a and b be zero divisors mod n.

In other words, neither a nor b is zero, but $ab = 0 \mod n$.

Suppose a has an inverse A, so that $A \times a = 1$

Then if

$$\begin{aligned} ab &= 0 && \mod n \\ Aab &= A \times 0 && \mod n \\ b &= 0 && \mod n \end{aligned}$$

But we said b wasn't zero, so this is a contradiction.

If you have an inverse, you're not a zero divisor. If you're a zero divisor, you don't have an inverse.

If you're multiplying modulo a prime number

- you don't get zero divisors
- so every number has an inverse
- and that means you've got a group