CSE400: Fundamentals of Probability in Computing

Lecture 11: Transformation of Random Variables

Lecture Information

- Course: CSE400 - Fundamentals of Probability in Computing
- Topic: Transformation of Random Variables
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Contents

1 Introduction

In this lecture, we study:

1. Transformation of Random Variables
2. Function of Two Random Variables
3. Illustrative Example: $Z = X + Y$

The goal is to learn systematic techniques to determine the probability distribution of a new random variable obtained as a function of one or more given random variables.

2 Transformation of Random Variables

2.1 Basic Definition

Let X be a random variable with known probability distribution.

Let a new random variable Y be defined as:

$$Y = g(X)$$

where $g(\cdot)$ is a function of X.

Our objective:

Find the distribution of Y

2.2 Method for Continuous Random Variables

Assume:

- X is continuous
- $g(\cdot)$ is differentiable and monotonic

We proceed step-by-step.

Step 1: CDF Approach

Define:

$$F_Y(y) = P(Y \leq y)$$

Since $Y = g(X)$,

$$F_Y(y) = P(g(X) \leq y)$$

If g is strictly increasing:

$$P(g(X) \leq y) = P(X \leq g^{-1}(y))$$

Thus,

$$F_Y(y) = F_X(g^{-1}(y))$$

Step 2: Differentiate to Get PDF

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

Using chain rule:

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

2.3 Final Formula (Single Variable Transformation)

$$\boxed{f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|}$$

where $x = g^{-1}(y)$.
This formula is fundamental for exams.

3 Function of Two Random Variables

Now consider two random variables:

$$Z = h(X, Y)$$

where (X, Y) has joint distribution $f_{X,Y}(x, y)$.
Our goal:

Find the distribution of Z

3.1 Joint Transformation Method

Let:

$$U = g_1(X, Y), \quad V = g_2(X, Y)$$

We compute the joint density of (U, V) using the Jacobian method.

3.2 Jacobian Definition

If the transformation is invertible:

$$(x, y) \to (u, v)$$

Then:

$$f_{U,V}(u, v) = f_{X,Y}(x, y)\,|J|$$

where Jacobian:

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

3.3 Procedure

1. Define transformation
2. Find inverse transformation
3. Compute Jacobian determinant
4. Substitute into joint density
5. Integrate out unwanted variable if needed

4 Illustrative Example: $Z = X + Y$

We now derive distribution of:

$$Z = X + Y$$

4.1 Assumptions

Let:

- X and Y be continuous random variables
- Joint density: $f_{X,Y}(x, y)$

4.2 CDF Derivation

Start from definition:

$$F_Z(z) = P(Z \leq z)$$

Since $Z = X + Y$,

$$F_Z(z) = P(X + Y \leq z)$$

4.3 Region Interpretation

The event:

$$X + Y \le z$$

represents a region in the xy-plane.
Thus,

$$F_Z(z) = \int\int_{x+y\le z} f_{X,Y}(x, y)\, dx\, dy$$

4.4 PDF of Z

Differentiate:

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

By differentiation under the integral sign:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z - x)\, dx$$

4.5 If X and Y Are Independent

If independent:

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)$$

Substitute:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx$$

4.6 Final Result (Convolution)

$$\boxed{f_Z(z) = (f_X * f_Y)(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx}$$

This is called the **convolution** of two densities.

5 Logical Understanding for Exams

5.1 Why Convolution Appears

Because:

- $Z = z$ occurs when $X = x$ and $Y = z - x$
- We integrate over all possible x

Thus probability accumulates across all decompositions of z.

5.2 Decision Logic

- If $Y = g(X) \to$ Use single-variable transformation formula.
- If $Z = X + Y$ and independent $\to$ Use convolution.
- If general transformation of two variables $\to$ Use Jacobian method.

6 Summary for Revision

Key Formulas

1. Single Variable Transformation:

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

2. Joint Transformation (Jacobian):

$$f_{U,V}(u, v) = f_{X,Y}(x, y)|J|$$

3. Sum of Independent Variables:

$$f_Z(z) = \int f_X(x) f_Y(z - x)\, dx$$

Exam Strategy

1. Identify transformation type.
2. Check independence.
3. Decide between:
   - Inverse + derivative method
   - Jacobian method

- Convolution method

4. Clearly define integration limits.
5. Always justify steps.

End of Lecture 11 Scribe