CSE-400 Scribe - L11

L11_S1_A.pdf
PDF

Generate a lecture scribe intended strictly as exam-oriented reference material for CSE 400. Use only the provided context (lecture slides/pdfs and the explicitly relevant textbook parts). Treat this context as authoritative and complete. Do not use or introduce outside knowledge.

When processing handwritten notes, read them carefully and conserve their meanings. Interpret symbols, notation, and equations with precision. Resolve ambiguities strictly using consistency with the lecture slides and textbook content. Draw figures/diagrams when needed. Do not guess or hallucinate unclear material.

Write in a formal, academic tone suitable for closed-notes exam revision. Preserve the lecture's structure, ordering, notation, terminology, and scope exactly. While writing the scribe, reason step by step through the material exactly as presented in the context, and include that reasoning explicitly in the scribe.

Emphasize clarity of dependencies between assumptions, definitions, and results without introducing new explanations.

Focus more on: 1-State and explain all definitions, notation, and assumptions explicitly, 2-Present theorems, propositions, and results exactly as given, 3-Present proofs and derivations step by step, preserving all logical dependencies and, 4-Present worked examples step by step, showing all intermediate steps exactly as taught.

Content constraints: 1-Include only material present in the provided context, 2-Do not introduce new examples, interpretations, intuitions, alternative arguments, or additional explanations, 3-Do not omit conditions, assumptions, or logical steps present in the source material and, 4-Do not simplify, generalize, or paraphrase beyond faithful reconstruction.

Organization Note: Structure the scribe with clear sections and headings reflecting the lecture flow so that a student relying solely on this document can accurately reconstruct definitions, derivations, proofs, and example solutions for exam preparation.

CSE 400 – Fundamentals of Probability in Computing

Lecture 11: Transformation of Random Variables

Source:

---

1. Transformation of Random Variables

1.1 Problem Setting

Let X be a continuous random variable (CRV) with known:

- PDF: $f_X(x)$
- CDF: $F_X(x)$

These are assumed known *a priori* (as indicated on page 1).

Define a new random variable:

$$Y = g(X)$$

The objective is to determine:

- $F_Y(y)$
- $f_Y(y)$

The transformation $g(\cdot)$ is treated under the assumption of monotonicity (as indicated on page 2 and page 3 diagrams). Two cases arise:

- Monotonically increasing
- Monotonically decreasing

1.2 Step 1: CDF Method

From page 3, Step-1 is explicitly:

$$F_Y(y) = P(Y \leqslant y)$$

Since $Y = g(X)$,

$$F_Y(y) = P(g(X) \leqslant y)$$

The next step depends on monotonicity of $g(\cdot)$.

Case 1: $g(\cdot)$ Monotonically Increasing

Step 1: Express event in terms of X

If g is increasing:

$$g(X) \leqslant y \Leftrightarrow X \leqslant g^{-1}(y)$$

Thus,

$$F_Y(y) = P(X \leqslant g^{-1}(y))$$

$$F_Y(y) = F_X(g^{-1}(y))$$

Step 2: Differentiate to obtain PDF

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$f_Y(y) = \frac{d}{dy} F_X(g^{-1}(y))$$

Using chain rule (as written on page 3):

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

Equivalently written (page 3, page 4):

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|_{x=g^{-1}(y)}$$

Case 2: $g(\cdot)$ Monotonically Decreasing

From page 4:

$$F_Y(y) = P(Y \leqslant y)$$

$$= P(g(X) \leqslant y)$$

If g is decreasing:

$$g(X) \leqslant y \;\Leftrightarrow\; X \geqslant g^{-1}(y)$$

Thus,

$$F_Y(y) = P(X \geqslant g^{-1}(y))$$

$$F_Y(y) = 1 - F_X(g^{-1}(y))$$

Differentiate:

$$f_Y(y) = \frac{d}{dy}\left[1 - F_X(g^{-1}(y))\right]$$

$$= -f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

Because $\frac{d}{dy} g^{-1}(y) < 0$, the final form becomes (page 4):

$$f_Y(y) = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|} \quad \text{evaluated at } x = g^{-1}(y)$$

1.3 Change of Limits (S3)

As written on page 4:

- Adjust limits according to the mapping $y = g(x)$.
- The support of Y is determined by transforming the support of X.

---

2. Worked Example (Page 5)

Given

$$X \sim \text{Uniform}(-1, 1)$$

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Define:

$$Y = \sin\left(\frac{\pi X}{2}\right)$$

Find $f_Y(y)$.

Step 1: Inverse Transformation

$$Y = \sin\left(\frac{\pi X}{2}\right)$$

$$X = \frac{2}{\pi} \sin^{-1}(Y)$$

Step 2: Derivative

$$\frac{dx}{dy} = \frac{2}{\pi} \cdot \frac{1}{\sqrt{1 - y^2}}$$

Step 3: Apply Transformation Formula

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

Substitute $f_X(x) = \frac{1}{2}$:

$$f_Y(y) = \frac{1}{2} \cdot \frac{2}{\pi} \cdot \frac{1}{\sqrt{1 - y^2}}$$

$$f_Y(y) = \frac{1}{\pi\sqrt{1 - y^2}}$$

Support (from page 5):

Since $x \in (-1, 1)$,

$$y \in (-1, 1)$$

Final answer:

$$f_Y(y) = \begin{cases} \dfrac{1}{\pi\sqrt{1-y^2}}, & -1 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

---

3. Function of Two Random Variables

3.1 Definition

Let:

$$Z = X + Y$$

Goal (page 6):

1. Find $f_Z(z)$
2. If X and Y are independent, find $f_Z(z)$
3. If $X, Y \sim N(0,1)$ independent, prove $Z \sim N(0,2)$
4. If X, Y exponential with parameter λ, find $f_Z(z)$

---

4. Derivation of $f_Z(z)$

From page 7:

Step 1: CDF Method

$$F_Z(z) = P(Z \leqslant z)$$

$$= P(X + Y \leqslant z)$$

This corresponds to the region:

$$\{(x, y) : x + y \leqslant z\}$$

Graphically (page 7 diagram):

- Line: $x + y = z$
- Region below the line.

Step 2: Double Integral Form

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{XY}(x, y)\, dx\, dy$$

Equivalently (order switched as in page 7):

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{XY}(x, y)\, dy\, dx$$

Step 3: If X and Y Independent

$$f_{XY}(x, y) = f_X(x) f_Y(y)$$

Thus:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx$$

This is the convolution form for the PDF of $Z = X + Y$.

Summary of Core Results

1. Single Variable Transformation

If $Y = g(X)$ and g monotone:

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

---

2. Sum of Two RVs

$$F_Z(z) = P(X + Y \leqslant z)$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx \quad \text{(if independent)}$$

This completes the faithful reconstruction of Lecture 11 strictly following the structure and derivations of the provided material .