CSE400 – Fundamentals of Probability in Computing

Kensi Patel - AU2440102

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Lecture 7: Expectation, CDFs, PDFs, and Problem Solving
Instructor: Dhaval Patel, PhD
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Lecture Flow and Logical Dependencies

1. Random Variables (assumed known from previous lectures)
2. Cumulative Distribution Function (CDF)
3. Probability Density Function (PDF)
4. Expectation of Random Variables
5. Expectation of a function of a RV
6. Linearity of Expectation
7. Moments and Central Moments
   - Variance
   - Skewness
   - Kurtosis

1. Cumulative Distribution Function (CDF)

Definition

For a random variable X,

$$F_X(x) = P(X \leq x)$$

This definition applies to both discrete and continuous random variables.

Properties of CDF

For any random variable X:

1. $0 \leq F_X(x) \leq 1$
2. $F_X(x)$ is non-decreasing
3. $\lim_{x \to -\infty} F_X(x) = 0$
4. $\lim_{x \to +\infty} F_X(x) = 1$
5. Right-continuous:

$$F_X(x) = \lim_{h \to 0^+} F_X(x + h)$$

Probability Using CDF

For any $a < b$:

$$P(a < X \leq b) = F_X(b) - F_X(a)$$

2. Probability Density Function (PDF)

Definition

For a continuous random variable X, the PDF $f_X(x)$ is defined as:

$$f_X(x) = \frac{d}{dx} F_X(x)$$

Relationship Between PDF and CDF

$$F_X(x) = \int_{\infty}^{x} f_X(t)\, d$$

Properties of PDF

1. $f_X(x) \geq 0$ for all x
2. Total area under the curve equals 1:
$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$
3. Probability over an interval:
$$P(a \leq X \leq b) = \int_{a}^{b} f_X(x)\, dx$$
4. For continuous RVs:
$$P(X = x) = 0$$

3. Expectation of Random Variables

Definition (Discrete RV)

If X takes values x_i with probabilities $p(x_i)$:

$$E[X] = \sum_i x_i\, p(x_i)$$

Definition (Continuous RV)

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

Interpretation

Expectation represents the mean (average) value of the random variable.

4. Expectation of a Function of a Random Variable

Let $Y = g(X)$

Discrete Case

$$E[g(X)] = \sum_i g(x_i)\, p(x_i)$$

Continuous Case

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

Important: No need to find the PDF of Y explicitly.

5. Linearity of Expectation

For random variables X, Y and constants a, b:

$$E[aX + bY] = aE[X] + bE[Y]$$

Key Exam Note

- Linearity holds regardless of independence
- Applies to any number of RVs

6. Moments of a Random Variable

n-th Moment (About Origin)

$$E[X^n]$$

Central Moments
Defined about the mean $\mu = E[X]$:

$$E[(X-\mu)^n]$$

7. Variance

Definition

$$\mathrm{Var}(X) = E[(X-\mu)^2]$$

Computational Formula

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

Standard Deviation

$$\sigma = \sqrt{\mathrm{Var}(X)}$$

8. Skewness

Definition
Third central moment (normalized):

$$\text{Skewness} = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$$

Measures asymmetry of distribution.

9. Kurtosis

Definition
Fourth central moment (normalized):

$$\text{Kurtosis} = E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right]$$

Measures peakedness / tail heaviness.

Exam-Focused Summary Table

Concept	Formula
CDF	$F_X(x) = P(X \leq x)$
PDF	$f_X(x) = \frac{d}{dx} F_X(x)$
Expectation	$E[X] = \int x f_X(x)\, dx$
Linearity	$E[aX + bY] = aE[X] + bE[Y]$
Variance	$E[X^2] - (E[X])^2$
Skewness	$E[((X-\mu)/\sigma)^3]$
Kurtosis	$E[((X-\mu)/\sigma)^4]$