CSE400 – Fundamentals of Probability in Computing
Lecture 7: Expectation, CDFs, PDFs and Problem Solving

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1 Outline of the Lecture

The lecture covers the following topics in order:

- The Cumulative Distribution Function (CDF)

  - Definition
  - Properties
  - Examples

- The Probability Density Function (PDF)

  - Definition
  - PDF–CDF relationship

- Expectation of random variables

  - Definition and example
  - Expectation of a function of a random variable
  - Linear operations with expectation

- n^{th} moments and central moments of random variables

  - Variance
  - Skewness
  - Kurtosis

2 CDF and PDF: Water Tank Analogy

A water tank analogy is used to motivate the idea of CDF and PDF.

- The height h of water corresponds to the value of a random variable.

- The volume of water up to height h corresponds to the cumulative probability.

  Let

- Tank radius $= R$

- Maximum height $= H$

The volume up to height h is

$$V(h) = \int_0^h \pi R^2 \, dh = (\pi R^2)h.$$

Here, πR^2 is analogous to the PDF of a uniform distribution.
The total volume is

$$V(H) = \pi R^2 H,$$

which is analogous to total probability $= 1$.

3 Cumulative Distribution Function (CDF)

3.1 Definition

For a random variable X, the cumulative distribution function is defined as

$$F_X(x) = \Pr(X \le x), \qquad -\infty < x < \infty.$$

3.2 Interpretation

Most of the information about the random experiment described by X is determined by the behavior of $F_X(x)$.

3.3 Properties of the CDF

- **Bounds**
$$0 \le F_X(x) \le 1$$

- **Limits**
$$F_X(-\infty) = 0, \qquad F_X(\infty) = 1$$

- **Monotonicity**
For $x_1 < x_2$,
$$F_X(x_1) \le F_X(x_2)$$

- **Probability over an interval**
For $x_1 < x_2$,
$$\Pr(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1).$$

4 CDF – Examples

4.1 Example 1: Validity of Given Functions

Determine whether the following are valid CDFs.

$$F_X(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(x) \qquad \text{(Valid CDF)}$$

$$F_X(x) = [1 - e^{-x}] \, u(x) \qquad \text{(Valid CDF)}$$

$$F_X(x) = e^{-x^2} \qquad \text{(Not a valid CDF)}$$

$$F_X(x) = x^2 u(x) \qquad \text{(Not a valid CDF)}$$

Here, $u(x)$ denotes the unit step function.

4.2 Example 2: Probability Computation Using a Given CDF

Given

$$F_X(x) = (1 - e^{-x})u(x),$$

$$\begin{aligned}
\Pr(X > 5) &= 1 - \Pr(X \le 5) \\
&= 1 - F_X(5) = e^{-5}.
\end{aligned}$$

$$\Pr(X < 5) = F_X(5).$$

$$\Pr(3 < X < 7) = F_X(7) - F_X(3).$$

4.3 Conditional Probability

$$\Pr(X > 5 \,|\, X < 7) = \frac{\Pr(5 < X < 7)}{\Pr(X < 7)} = \frac{F_X(7) - F_X(5)}{F_X(7)}.$$

5 Probability Density Function (PDF)

5.1 Definition (via Limiting Argument)

For a continuous random variable X, the PDF is defined as

$$f_X(x) = \lim_{\varepsilon \to 0} \frac{\Pr(x \le X < x + \varepsilon)}{\varepsilon}.$$

5.2 Relationship Between PDF and CDF

$$\Pr(x \le X < x + \varepsilon) = F_X(x + \varepsilon) - F_X(x).$$

Substituting,

$$f_X(x) = \lim_{\varepsilon \to 0} \frac{F_X(x + \varepsilon) - F_X(x)}{\varepsilon}.$$

Hence,

$$f_X(x) = \frac{d}{dx} F_X(x).$$

5.3 Fundamental Result

The PDF of a random variable is the derivative of its CDF.
    Conversely, the CDF can be obtained by integrating the PDF.

6 Expectation of Random Variables

6.1 Definition

The expectation of a random variable represents its mean value.
    Expectation is also interpreted as the first moment of a random variable.

6.2 Expectation of a Function of a Random Variable

For a function $g(X)$, the expectation is written as

$$E[g(X)].$$

6.3 Linear Operations with Expectation

Expectation is linear. For constants a, b and random variables X, Y,

$$E[aX + bY] = aE[X] + bE[Y].$$

7 Moments and Central Moments

7.1 n^{th} Moment

The nth moment of a random variable X is

$$E[X^n].$$

7.2 Central Moments

Central moments are taken about the mean μ,

$$E[(X - \mu)^n].$$

7.3 Important Central Moments

- Variance: second central moment

- Skewness: third central moment

- Kurtosis: fourth central moment

8 Logical Flow Summary for Exam Revision

- Start with the definition of the CDF

- Use CDF properties to test validity

- Compute probabilities using CDF differences

- Derive the PDF from the CDF

- Interpret expectation as a moment

- Apply linearity of expectation

- Extend to higher-order and central moments