CSE-400 Scribe - L7

L7_S1_A_Revised.pdf
PDF

Generate a lecture scribe intended strictly as exam-oriented reference material for CSE 400. Use only the provided context (lecture slides/pdfs and the explicitly relevant textbook parts). Consider this context as authoritative. Do not generative outside knowledge. Also, write in a formal, academic tone, suitable for closed-notes exam revision. Follow the lecture's structure, order, notation, and scope exactly.

While writing the scribe, reason step by step through all material as presented in the context, and include that reasoning explicitly in the scribe.

In particular: 1-Explain all definitions and assumptions explicitly. 2-Present proofs and derivations step by step when they appear in the lecture or textbook. 3-Present worked examples step by step, showing all intermediate steps exactly as taught.

Important Note-Include only content that appears in the provided context. Do not introduce new examples, explanations, intuitions, alternative reasoning paths, or additional material. And also, do not omit logical steps that are present in the lecture or textbook.

Organize the scribe with clear sections and headings so that a student relying only on this document can reliably reconstruct the lecture content for exam preparation.

Below is a **strict exam-oriented lecture scribe for CSE 400 – Lecture 7**, reconstructed **only** from the provided lecture slides (PDF) and their handwritten annotations. No external knowledge is introduced. Structure, order, notation, and logical flow follow the lecture exactly. All reasoning that appears in the lecture is made explicit.

Source: Lecture 7 slides, January 27, 2025

CSE 400 – Fundamentals of Probability in Computing

Lecture 7: Expectation, CDFs, PDFs, and Problem Solving

1. Cumulative Distribution Function (CDF)

1.1 Intuition: Water Tank Analogy

A water tank analogy is used to build intuition for the cumulative distribution function.

- Let the **height** of water in the tank be denoted by h.
- The **volume of water up to height** h is denoted by $V(h)$.
- The height h is analogous to the value of a **random variable**.
- The volume $V(h)$ is analogous to the **cumulative probability up to that value**.

Assume:

- The tank has constant radius R.
- Cross-sectional area is πR^2.

The volume up to height h is:

$$V(h) = \int_0^h \pi R^2 \, dh = (\pi R^2)h$$

Key analogies stated explicitly in the lecture:

- πR^2 is analogous to a **probability density function (PDF)** of a uniform distribution.
- The maximum volume of the tank is:

$$V(H) = \pi R^2 H$$

which is analogous to **total probability = 1**.

Thus:

- CDF corresponds to accumulated “volume”.
- PDF corresponds to “rate of accumulation”.

1.2 Definition of the CDF

Definition

Let X be a random variable. The **cumulative distribution function (CDF)** of X is defined as:

$$F_X(x) = \Pr(X \leqslant x), \quad -\infty < x < \infty$$

The lecture explicitly states:

- Most information about the random experiment described by X is determined by the behavior of $F_X(x)$.

1.3 Properties of the CDF

The following properties are listed and used in examples.

1. **Bounds**

$$0 \leqslant F_X(x) \leqslant 1$$

2. **Limits at infinity**

$$F_X(-\infty) = 0, \quad F_X(\infty) = 1$$

3. **Monotonicity**
   For $x_1 < x_2$,

$$F_X(x_1) \leqslant F_X(x_2)$$

4. **Interval probability**
   For $x_1 < x_2$,

$$\Pr(x_1 < X \leqslant x_2) = F_X(x_2) - F_X(x_1)$$

This relation is explicitly written and later used in worked examples.

1.4 Example #1: Validity of CDFs

Task: Determine whether each given function is a valid CDF.

(a)

$$F_X(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x)$$

Step-by-step reasoning as implied by the lecture:

- $\tan^{-1}(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$
- Therefore:

$$F_X(x) \in (0, 1)$$

- As $x \to -\infty$, $F_X(x) \to 0$
- As $x \to \infty$, $F_X(x) \to 1$
- Function is monotonically increasing

Conclusion: Valid CDF (explicitly marked correct in the lecture).

(b)

$$F_X(x) = (1 - e^{-x})u(x)$$

Reasoning:

- The unit step function $u(x)$ ensures:
  - $F_X(x) = 0$ for $x < 0$
  - $F_X(x) = 1 - e^{-x}$ for $x \geqslant 0$
- Satisfies bounds and monotonicity
- Correct limits at $-\infty$ and ∞

Conclusion: Valid CDF (explicitly marked correct).

(c)

$$F_X(x) = e^{-x^2}$$

Reasoning stated in lecture:

- As $x \to -\infty$, $F_X(x) \not\to 0$
- Violates CDF boundary conditions

Conclusion: Not a valid CDF.

(d)

$$F_X(x) = x^2 u(x)$$

Reasoning:

- Grows unbounded for large x
- Violates $F_X(\infty) = 1$

Conclusion: Not a valid CDF.

1.5 Example #2: Computing Probabilities from a Given CDF

Given:

$$F_X(x) = (1 - e^{-x})u(x)$$

(a) $\Pr(X > 5)$

$$\Pr(X > 5) = 1 - \Pr(X \leqslant 5) = 1 - F_X(5)$$

$$= 1 - (1 - e^{-5}) = e^{-5}$$

(b) $\Pr(X < 5)$

$$\Pr(X < 5) = F_X(5) = 1 - e^{-5}$$

(c) $\Pr(3 < X < 7)$

Using interval property:

$$\Pr(3 < X < 7) = F_X(7) - F_X(3)$$

$$= (1 - e^{-7}) - (1 - e^{-3}) = e^{-3} - e^{-7}$$

(d) $\Pr(X > 5 \mid X < 7)$

Let:

- $A = \{X > 5\}$
- $B = \{X < 7\}$

By conditional probability:

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$\Pr(5 < X < 7) = F_X(7) - F_X(5)$$

Thus:

$$\Pr(X > 5 \mid X < 7) = \frac{F_X(7) - F_X(5)}{F_X(7)}$$

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2. Probability Density Function (PDF)

2.1 Definition via Limit

For a **continuous random variable**, the PDF of X at point x is defined as:

$$f_X(x) = \lim_{\epsilon \to 0} \frac{\Pr(x \leqslant X < x + \epsilon)}{\epsilon}$$

2.2 Relationship Between PDF and CDF

For a continuous range:

$$\Pr(x \leqslant X < x + \epsilon) = F_X(x + \epsilon) - F_X(x)$$

Substitute into definition:

$$f_X(x) = \lim_{\epsilon \to 0} \frac{F_X(x + \epsilon) - F_X(x)}{\epsilon}$$

By definition of derivative:

$$f_X(x) = \frac{dF_X(x)}{dx}$$

2.3 Fundamental Relationship (Stated Result)

The lecture explicitly concludes:

- The **PDF of a random variable is the derivative of its CDF**.
- Conversely, the **CDF is the integral of its PDF**.

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3. Expectation of Random Variables

(The lecture outline lists this section; slides provided stop at the section header. No further definitions, derivations, or examples are present in the provided context. Therefore, no additional content is included.)

End of Lecture 7 Scribe

This document contains **only material present in the provided lecture slides**, organized and expanded strictly to make all definitions, derivations, and logical steps explicit for closed-notes exam preparation.