CSE400 – Fundamentals of Probability in Computing

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Lecture 11

Lecture 11: Transformation of Random Variables
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1. Problem Setting and Notation

- Given a continuous random variable X with known PDF $f_X(x)$.
- Define a new random variable:
$$Y = g(X)$$
- Goal: Find $F_Y(y)$ and $f_Y(y)$.

2. Lecture Flow and Logical Dependencies

1. Transformation of one random variable
2. Function of two random variables
3. Example: $Z = X + Y$

Logical progression:
$$\text{Single RV} \to \text{Joint RV} \to \text{Example}$$

3. Transformation of One Random Variable

3.1 Assumptions

- Transformation:
$$Y = g(X)$$
- Assume $g(\cdot)$ is monotonic (increasing or decreasing), so the inverse exists:
$$X = g^{-1}(Y)$$

3.2 Step 1: CDF Method

- Definition:
$$F_Y(y) = P(Y \leq y)$$
- Substitute transformation:
$$F_Y(y) = P(g(X) \leq y)$$
- If g is increasing:
$$F_Y(y) = P\left(X \leq g^{-1}(y)\right)$$
- Hence:
$$F_Y(y) = F_X\left(g^{-1}(y)\right)$$

3.3 Step 2: PDF from CDF

- Differentiate:

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

- Substitute:

$$f_Y(y) = \frac{d}{dy} F_X\big(g^{-1}(y)\big)$$

- Using the chain rule:

$$f_Y(y) = f_X\big(g^{-1}(y)\big) \cdot \frac{d}{dy} g^{-1}(y)$$

- Since

$$\frac{d}{dy} g^{-1}(y) = \frac{dx}{dy},$$

- Final transformation formula:

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|_{x=g^{-1}(y)}$$

3.4 Decreasing Transformation Case

- If $g(\cdot)$ is decreasing:

$$F_Y(y) = P(X \geq g^{-1}(y)) = 1 - F_X\big(g^{-1}(y)\big)$$

- Differentiate:

$$f_Y(y) = -f_X(x) \frac{dx}{dy}$$

- Taking magnitude gives a unified result:

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

4. Worked Example

- Given:

$$X \sim U(-1, 1)$$

- PDF:

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

- Transformation:

$$Y = \sin\left(\frac{\pi X}{2}\right)$$

4.1 Inverse Mapping

$$x = \frac{2}{\pi} \sin^{-1}(y)$$

4.2 Differentiation

$$\frac{dx}{dy} = \frac{2}{\pi} \frac{1}{\sqrt{1 - y^2}}$$

4.3 PDF of Y

- Apply transformation rule:

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

- Substitute $f_X(x) = \frac{1}{2}$:

$$f_Y(y) = \frac{1}{2} \cdot \frac{2}{\pi} \frac{1}{\sqrt{1-y^2}}$$

- Final result:

$$f_Y(y) = \frac{1}{\pi\sqrt{1-y^2}}, \quad -1 < y < 1$$

- Limits from mapping:

$$x = -1 \Rightarrow y = -1, \qquad x = 1 \Rightarrow y = 1$$

5. Function of Two Random Variables

- Define:

$$Z = X + Y$$

Goal: Find $f_Z(z)$.
Examples mentioned:

- If $X, Y \sim N(0,1)$ then $Z \sim N(0,2)$
- Exponential RV case discussed

6. Distribution of $Z = X + Y$

6.1 Start from CDF

$$F_Z(z) = P(Z \leq z) = P(X + Y \leq z)$$

6.2 Convert to Double Integral

$$F_Z(z) = \iint_{x+y\leq z} f_{XY}(x,y)\,dx\,dy$$

Region defined by boundary:

$$x + y = z$$

6.3 Change Order of Integration

Equivalent expression:

$$F_Z(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_{XY}(x,y)\,dy\,dx$$

7. Logical Flow Summary

- Start from CDF definition
- Use inverse mapping
- Differentiate to obtain PDF
- Extend idea to joint RVs
- Apply to $Z = X + Y$

8. Final Key Formulas

Single RV Transformation

CDF Transformation

$$F_Y(y) = F_X\big(g^{-1}(y)\big)$$

PDF Transformation

$$f_Y(y) = f_X\big(g^{-1}(y)\big)\left|\frac{dx}{dy}\right|$$

Sum of Two Random Variables

Sum CDF

$$F_Z(z) = \iint_{x+y\leq z} f_{XY}(x,y)\,dx\,dy$$