# CSE400: Fundamentals of Probability in Computing

Lecture 11: Transformation of Random Variables

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# 1 Lecture Outline

This lecture covers the following major topics:

1. Transformation of Random Variables
2. Function of Two Random Variables
3. Illustrative Example: Detailed derivation for the case $Z = X + Y$

# 2 Transformation of Random Variables

## 2.1 Basic Setup

Assume we have a continuous random variable $X$ whose distribution is known.
We assume:

$$f_X(x) \quad \text{(PDF of } X \text{ is known)}$$

and

$$F_X(x) \quad \text{(CDF of } X \text{ is known)}$$

Now define a new random variable $Y$ as a function of $X$:

$$Y = g(X)$$

The objective is to find:

$$F_Y(y) \quad \text{and} \quad f_Y(y)$$

### 2.2 Important Idea: Inverting the Transformation

If $Y = g(X)$ is invertible, then we can write:

$$X = g^{-1}(Y)$$

This inversion is the key step in computing the CDF and PDF of $Y$.

## 3 CDF Method for Transformation

### 3.1 Step 1: Derive the CDF of $Y$

The CDF of $Y$ is defined as:

$$F_Y(y) = P(Y \leq y)$$

Since $Y = g(X)$:

$$F_Y(y) = P(g(X) \leq y)$$

At this stage, the monotonic nature of $g(\cdot)$ matters.

### 3.2 Case 1: $g(\cdot)$ is Monotonically Increasing (+ve Increasing)

If $g(\cdot)$ is increasing, then:

$$g(X) \leq y \iff X \leq g^{-1}(y)$$

So:

$$F_Y(y) = P(X \leq g^{-1}(y))$$

Thus:

$$F_Y(y) = F_X(g^{-1}(y))$$

### 3.3 Step 2: Differentiate to Get PDF

Now:

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

Using chain rule:

$$f_Y(y) = \frac{d}{dy}\Big(F_X(g^{-1}(y))\Big)$$

So:

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{d}{dy}\big(g^{-1}(y)\big)$$

Let:

$$x = g^{-1}(y)$$

Then:

$$f_Y(y) = f_X(x) \cdot \frac{dx}{dy}$$

Since PDF must be non-negative, we write the final standard form:

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

### 3.4 Final PDF Transformation Formula (Monotone Case)

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|}$$

# 4 Transformation when $g(\cdot)$ is Monotonically Decreasing (-ve Decreasing)

### 4.1 Step 1: CDF of $Y$

Again:

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y)$$

If $g(\cdot)$ is decreasing, then:

$$g(X) \leq y \iff X \geq g^{-1}(y)$$

So:

$$F_Y(y) = P(X \geq g^{-1}(y))$$

Using complement rule:

$$P(X \geq a) = 1 - P(X < a)$$

Thus:

$$F_Y(y) = 1 - F_X(g^{-1}(y))$$

### 4.2 Step 2: Differentiate to Get PDF

Differentiate:

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

So:

$$f_Y(y) = \frac{d}{dy}\Big(1 - F_X(g^{-1}(y))\Big)$$

$$f_Y(y) = -f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

Since the derivative is negative for decreasing transformations, we again take absolute value:

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|}$$

### 4.3 Conclusion

Therefore, for both monotonic increasing and monotonic decreasing transformations, the same final PDF formula holds.

## 5 Example: $X \sim U(-1, 1)$ and $Y = \sin\left(\frac{\pi X}{2}\right)$

### 5.1 Given

$X$ is uniformly distributed over $(-1, 1)$:

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Transformation is:

$$Y = \sin\left(\frac{\pi X}{2}\right)$$

### 5.2 Step 1: Invert the Transformation

We solve for $x$:

$$y = \sin\left(\frac{\pi x}{2}\right)$$

Take inverse sine:

$$\sin^{-1}(y) = \frac{\pi x}{2}$$

Thus:

$$x = \frac{2}{\pi} \sin^{-1}(y)$$

So:

$$g^{-1}(y) = \frac{2}{\pi} \sin^{-1}(y)$$

### 5.3 Step 2: Compute Derivative

$$\frac{dx}{dy} = \frac{2}{\pi} \cdot \frac{d}{dy} \sin^{-1}(y)$$

We know:

$$\frac{d}{dy}\sin^{-1}(y) = \frac{1}{\sqrt{1-y^2}}$$

So:

$$\frac{dx}{dy} = \frac{2}{\pi} \cdot \frac{1}{\sqrt{1-y^2}}$$

### 5.4 Step 3: Apply Transformation Formula

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

Given $f_X(x) = \frac{1}{2}$ for $-1 < x < 1$.

Thus:

$$f_Y(y) = \frac{1}{2} \cdot \left(\frac{2}{\pi} \cdot \frac{1}{\sqrt{1-y^2}}\right)$$

$$f_Y(y) = \frac{1}{\pi\sqrt{1-y^2}}$$

### 5.5 Final Answer

$$\boxed{f_Y(y) = \frac{1}{\pi\sqrt{1-y^2}}, \quad -1 < y < 1}$$

## 6 Function of Two Random Variables

### 6.1 General Setup

Let $X$ and $Y$ be random variables.

We define a new random variable $Z$ as a function of two random variables:

$$Z = g(X, Y)$$

Common examples include:

$$Z = X + Y, \quad Z = X - Y, \quad Z = \frac{X}{Y}, \quad Z = XY$$

The objective is to find the distribution of $Z$, typically:

$$F_Z(z) \quad \text{or} \quad f_Z(z)$$

# 7 Illustrative Example: $Z = X + Y$

## 7.1 Goal

Define:

$$Z = X + Y$$

We want to compute:

$$F_Z(z) = P(Z \leq z)$$

That is:

$$F_Z(z) = P(X + Y \leq z)$$

## 7.2 CDF Derivation

$$F_Z(z) = P(X + Y \leq z)$$

This probability corresponds to a region in the $(x, y)$ plane:

$$x + y \leq z$$

Equivalently:

$$y \leq z - x$$

## 7.3 Region-Based Integration Form

To compute this probability using the joint PDF:

$$f_{X,Y}(x, y)$$

we integrate over the region:

$$\{(x, y) : x + y \leq z\}$$

Thus:

$$F_Z(z) = \int\int_{x+y\leq z} f_{X,Y}(x, y)\, dx\, dy$$

## 7.4 Integral Expression (As Written in Lecture)

The lecture shows the CDF computed as:

$$F_Z(z) = P(Z \leq z)$$

$$F_Z(z) = P(X + Y \leq z)$$

and written as a double integral:

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{X,Y}(x, y)\, dy\, dx$$

This represents:

- outer integral over $x$
- inner integral over $y$ from $-\infty$ to $z - x$

### 7.5 Alternative Order of Integration (As Shown)

The lecture also illustrates changing the integration order using region geometry.

From the graph, the boundary line is:

$$y = z - x$$

So the integration can also be expressed by integrating first over $x$ with bounds depending on $y$.

The lecture representation indicates rewriting into a second integral form by changing limits, written as:

$$F_Z(z) = \int_{y=-\infty}^{z} \int_{x=-\infty}^{z-y} f_{X,Y}(x, y)\, dx\, dy$$

This matches the region:

$$x \leq z - y$$

### 7.6 Key Conclusion

The derived CDF of $Z = X + Y$ is computed using region integration:

$$\boxed{F_Z(z) = \int \int_{x+y\leq z} f_{X,Y}(x, y)\, dx\, dy}$$

and the corresponding limit form:

$$\boxed{F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{X,Y}(x, y)\, dy\, dx}$$

# 8 Important Exam Notes from Lecture

- When transforming one random variable $Y = g(X)$, first find $F_Y(y)$ and then differentiate.

- The monotonic nature of $g(\cdot)$ determines whether inequality direction changes.
- Final PDF formula for monotone transformations is:

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

- For two random variables, distributions of expressions like $Z = X + Y$ are derived using joint probability region integration.

# 9 Highlighted Problems Mentioned in Lecture (For Practice)

The lecture lists the following tasks for $Z = X + Y$:

1. Find PDF of $Z$, $f_Z(z)$.
2. Find $f_Z(z)$ if $X$ and $Y$ are independent.
3. If $X \sim N(0, 1)$ and $Y \sim N(0, 1)$ are independent, prove $Z \sim N(0, 2)$.
4. If $X$ and $Y$ are exponential RVs with parameter $\lambda$, find $f_Z(z)$.

**Note:** The lecture slide only lists these as questions. No full derivation for these specific distributions is provided in the slides, hence they are not expanded here.

# 10 End of Lecture 11 Notes

This document strictly follows Lecture 11 slides and reconstructs the derivations as shown.