CSE 400 — Lecture 11 Scribe

Fundamentals of Probability in Computing

Lecture 11: Transformation of Random Variables

1 Transformation of Random Variables

1.1 Problem Setting

Let X be a continuous random variable (CRV) with known:

PDF: $f_X(x)$

CDF: $F_X(x)$

These are assumed known a priori.

Define a new random variable:

$$Y = g(X)$$

The objective is to determine:

$$F_Y(y), \qquad f_Y(y)$$

The transformation is treated under the assumption of monotonicity of $g(\cdot)$.

Two cases arise:

- Monotonically increasing
- Monotonically decreasing

1.2 Step 1: CDF Method

$$F_Y(y) = P(Y \leq y)$$

Since $Y = g(X)$,

$$F_Y(y) = P(g(X) \leq y)$$

The next step depends on monotonicity of $g(\cdot)$.

Case 1: Monotonically Increasing

If g is increasing:

$$g(X) \leq y \iff X \leq g^{-1}(y)$$

Thus,

$$F_Y(y) = P(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$$

Differentiate to obtain PDF:

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X(g^{-1}(y))$$

Using chain rule:

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{d}{dy}g^{-1}(y)$$

Equivalently written:

$$f_Y(y) = f_X(x)\frac{dx}{dy} \quad \text{evaluated at } x = g^{-1}(y)$$

Case 2: Monotonically Decreasing

If g is decreasing:

$$g(X) \leq y \iff X \geq g^{-1}(y)$$

Thus,

$$F_Y(y) = P(X \geq g^{-1}(y)) = 1 - F_X(g^{-1}(y))$$

Differentiate:

$$f_Y(y) = \frac{d}{dy}\left[1 - F_X(g^{-1}(y))\right]$$

$$= -f_X(g^{-1}(y)) \cdot \frac{d}{dy}g^{-1}(y)$$

Since $\frac{d}{dy}g^{-1}(y) < 0$,

$$f_Y(y) = f_X(x)\frac{dx}{dy} \quad \text{evaluated at } x = g^{-1}(y)$$

1.3 Change of Limits

Adjust limits according to the mapping $y = g(x)$.

The support of Y is determined by transforming the support of X.

2 Worked Example

Given:

$$X \sim \text{Uniform}(-1, 1)$$

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Define:

$$Y = \sin\left(\frac{\pi X}{2}\right)$$

Find $f_Y(y)$.

Step 1: Inverse Transformation

$$X = \frac{2}{\pi}\sin^{-1}(Y)$$

Step 2: Derivative

$$\frac{dx}{dy} = \frac{2}{\pi} \cdot \frac{1}{\sqrt{1-y^2}}$$

Step 3: Apply Transformation Formula

$$f_Y(y) = f_X(x)\frac{dx}{dy}$$

Substitute $f_X(x) = \frac{1}{2}$:

$$f_Y(y) = \frac{1}{2} \cdot \frac{2}{\pi} \cdot \frac{1}{\sqrt{1-y^2}}$$

$$f_Y(y) = \frac{1}{\pi\sqrt{1-y^2}}$$

Support:
Since $x \in (-1, 1)$,

$$y \in (-1, 1)$$

Final answer:

$$f_Y(y) = \begin{cases} \dfrac{1}{\pi\sqrt{1-y^2}}, & -1 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

3 Function of Two Random Variables

Let:

$$Z = X + Y$$

Goal:

1. Find $f_Z(z)$.
2. If X and Y are independent, find $f_Z(z)$.
3. If $X, Y \sim N(0, 1)$, prove $Z \sim N(0, 2)$.
4. If exponential with parameter λ, find $f_Z(z)$.

4 Derivation of $f_Z(z)$

Step 1: CDF Method

$$F_Z(z) = P(Z \leq z) = P(X + Y \leq z)$$

Region:

$$\{(x, y) : x + y \leq z\}$$

Line:

$$x + y = z$$

Region below the line.

Step 2: Double Integral Form

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{XY}(x, y)\, dx\, dy$$

Equivalently (order switched):

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{XY}(x, y)\, dy\, dx$$

Step 3: If X and Y Independent

$$f_{XY}(x, y) = f_X(x) f_Y(y)$$

Thus,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx$$

This is the convolution form for the PDF of $Z = X + Y$.

5 Summary of Core Results

Single Variable Transformation

If $Y = g(X)$ and g is monotone:

$$f_Y(y) = f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)$$

Sum of Two Random Variables

$$F_Z(z) = P(X + Y \leq z)$$

If independent:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx$$