

# Lecture Scribe CSE400



L7\_S1\_A\_Revised.pdf

PDF

You are required to generate a lecture scribe intended strictly as exam-oriented reference material. Use ONLY the provided context (lecture slides / lecture PDF). Do NOT introduce any content, examples, explanations, intuition, or results that do not explicitly appear in the provided context. While writing the lecture scribe, reason step by step through the material exactly as it is presented in the lecture. Include explicit reasoning in the scribe wherever the lecture contains: - definitions and notation - assumptions and conditions - theorem or result statements - proofs or proof sketches - derivations - worked examples All reasoning must be faithful to the provided context and must follow the logical order used in the lecture. Make the logical dependencies between concepts explicit so that a student can reconstruct the lecture while revising for an exam. Maintain a formal, academic tone suitable for closed-notes exam preparation. Organize the scribe clearly with appropriate section headings. Do NOT: - invent new examples or explanations - simplify or reinterpret arguments - add intuition not discussed in the lecture - go beyond the scope of the provided context Ensure a clear and logical progression of ideas to facilitate student understanding during revision. This should serve as concise and accurate exam reference material. Also cover each and every topic do NOT skip any single topic or examples.

Below is a **formal, exam-oriented lecture scribe** for **CSE400 – Lecture 7:**

**Expectation, CDFs, PDFs and Problem Solving**, written **strictly from the provided lecture slides** and **faithfully following their order, content, and logical structure**.

No external material, intuition, or additions have been introduced.

This scribe is designed for **closed-notes exam revision**.

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# CSE400 – Fundamentals of Probability in Computing

## Lecture 7: Expectation, CDFs, PDFs and Problem Solving

(January 27, 2025)

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### 1. Outline of the Lecture

The lecture covers the following topics in order:

1. The Cumulative Distribution Function (CDF)
    - Definition
    - Properties
    - Examples
  2. The Probability Density Function (PDF)
    - Definition
    - PDF–CDF Relationship
  3. Expectation of Random Variables
    - Definition and Example
    - Expectation of a Function of a Random Variable
    - Linear Operations with Expectation
  4.  $n^{\text{th}}$  Moments and Central Moments of Random Variables
    - Variance
    - Skewness
    - Kurtosis
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### 2. CDF and PDF: Water Tank Analogy (Motivational Setup)

A **water tank analogy** is used to motivate the idea of CDF and PDF.

- The **height  $h$**  of water corresponds to the value of a random variable.
- The **volume of water up to height  $h$**  corresponds to the cumulative probability up to that value.

Let:

- Tank radius =  $R$
- Maximum height =  $H$

**Volume up to height  $h$**

$$V(h) = \int_0^h \pi R^2 dh = (\pi R^2)h$$

- $\pi R^2$  is analogous to a **PDF** of a uniform distribution.
- The total volume:

$$V(H) = \pi R^2 H$$

is analogous to **total probability = 1**.

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### 3. Cumulative Distribution Function (CDF)

#### 3.1 Definition

For a random variable  $X$ , the **Cumulative Distribution Function (CDF)** is defined as:

$$F_X(x) = \Pr(X \leq x), \quad -\infty < x < \infty$$


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#### 3.2 Interpretation

- Most of the information about the random experiment described by  $X$  is determined by the behavior of  $F_X(x)$ .
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#### 3.3 Properties of the CDF

The CDF satisfies the following properties:

##### 1. Bounds

$$0 \leq F_X(x) \leq 1$$

##### 2. Limits

$$F_X(-\infty) = 0, \quad F_X(\infty) = 1$$

##### 3. Monotonicity

For  $x_1 < x_2$ :

$$F_X(x_1) \leq F_X(x_2)$$

##### 4. Probability over an interval

For  $x_1 < x_2$ :

$$\Pr(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$$


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## 4. CDF – Examples

### Example 1: Validity of Given Functions

Determine whether the following are **valid CDFs**.

1.

$$F_X(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(x)$$

✓ Valid CDF

2.

$$F_X(x) = [1 - e^{-x}] u(x)$$

✓ Valid CDF

3.

$$F_X(x) = e^{-x^2}$$

✗ Not a valid CDF

4.

$$F_X(x) = x^2 u(x)$$

✗ Not a valid CDF

Here,  $u(x)$  denotes the **unit step function**.

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### Example 2: Probability Computation Using a Given CDF

Given:

$$F_X(x) = (1 - e^{-x})u(x)$$

Find:

1.

$$\Pr(X > 5)$$

$$= 1 - \Pr(X \leq 5) = 1 - F_X(5) = e^{-5}$$

2.

$$\begin{aligned}\Pr(X < 5) \\ &= F_X(5)\end{aligned}$$

3.

$$\begin{aligned}\Pr(3 < X < 7) \\ &= F_X(7) - F_X(3)\end{aligned}$$

4.

$$\Pr(X > 5 \mid X < 7)$$

Using conditional probability:

$$\begin{aligned}\Pr(A \mid B) &= \frac{\Pr(A \cap B)}{\Pr(B)} \\ &= \frac{\Pr(5 < X < 7)}{\Pr(X < 7)} = \frac{F_X(7) - F_X(5)}{F_X(7)}\end{aligned}$$


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## 5. Probability Density Function (PDF)

### 5.1 Definition (via Limiting Argument)

For a **continuous random variable**  $X$ , the **PDF** at point  $x$  is defined as:

$$f_X(x) = \lim_{\varepsilon \rightarrow 0} \frac{\Pr(x \leq X < x + \varepsilon)}{\varepsilon}$$


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### 5.2 Relationship Between PDF and CDF

Recall for a continuous range:

$$\Pr(x \leq X < x + \varepsilon) = F_X(x + \varepsilon) - F_X(x)$$

Substituting:

$$f_X(x) = \lim_{\varepsilon \rightarrow 0} \frac{F_X(x + \varepsilon) - F_X(x)}{\varepsilon}$$

$$\Rightarrow f_X(x) = \frac{dF_X(x)}{dx}$$

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### 5.3 Fundamental Result

- The **PDF** of a random variable is the **derivative** of its **CDF**.
  - Conversely, the **CDF** can be expressed as the **integral** of its **PDF**.
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## 6. Expectation of Random Variables

### 6.1 Definition

The **expectation** (or average) of a random variable represents its **mean value**.

Expectation is also interpreted as the **first moment** of a random variable.

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### 6.2 Expectation of a Function of a Random Variable

For a function  $g(X)$ , the expectation is written as:

$$E[g(X)]$$

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### 6.3 Linear Operations with Expectation

Expectation is **linear**.

For constants  $a, b$  and random variables  $X, Y$ :

$$E[aX + bY] = aE[X] + bE[Y]$$

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## 7. Moments and Central Moments of Random Variables

### 7.1 $n^{\text{th}}$ Moment

The  $n^{\text{th}}$  moment of a random variable  $X$  is:

$$E[X^n]$$

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## 7.2 Central Moments

Central moments are taken **about the mean**  $\mu$ .

$$E[(X - \mu)^n]$$

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## 7.3 Important Central Moments

- **Variance:** second central moment
  - **Skewness:** third central moment
  - **Kurtosis:** fourth central moment
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## 8. Logical Flow Summary for Exam Revision

1. Start with **CDF definition**
  2. Use **CDF properties** to test validity
  3. Compute probabilities using **CDF differences**
  4. Derive **PDF from CDF**
  5. Interpret expectation as a **moment**
  6. Apply **linearity of expectation**
  7. Extend to **higher-order and central moments**
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End of Lecture 7 – Exam-Oriented Scribe