CSE400 – Fundamentals of Probability in Computing

Lecture 11:Transformation of random variables

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February 16, 2026

1 Introduction and Problem Statement

1.1 Known and Unknown Quantities

Let X be a continuous random variable (CRV).

The PDF of X is known:

$$f_X(x)$$

The CDF of X is known:

$$F_X(x)$$

Goal: Find PDF and CDF of a new random variable.

1.2 Definition of Transformation

Define a new random variable:

$$Y = g(X)$$

Goal:
Given:

$$f_X(x)$$

Find:

$$f_Y(y), \quad F_Y(y)$$

1.3 Extension to Two Random Variables

Transformation involving two random variables:

$$Z = g(X, Y)$$

Examples:

$$Z = X + Y$$

$$Z = X - Y$$

$$Z = \frac{X}{Y}$$

$$Z = XY$$

2 Transformation of a Single Random Variable

2.1 Invertible Transformation

Assume:

$$Y = g(X)$$

where g is monotonic.
Thus inverse exists:

$$X = g^{-1}(Y)$$

3 CDF Method

3.1 Step 1: Find CDF of Y

By definition:

$$F_Y(y) = P(Y \leq y)$$

Substitute transformation:

$$= P(g(X) \leq y)$$

Since inverse exists:

$$= P(X \leq g^{-1}(y))$$

Thus:

$$\boxed{F_Y(y) = F_X(g^{-1}(y))}$$

4 Finding PDF of Y

Differentiate CDF:

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

Substitute:

$$= \frac{d}{dy} F_X(g^{-1}(y))$$

Using chain rule:

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)}$$

5 Decreasing Transformation

If transformation is decreasing:

$$F_Y(y) = 1 - F_X(g^{-1}(y))$$

Thus:

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|}$$

6 Example: Uniform Random Variable

6.1 Given

$$X \sim Uniform(-1, 1)$$

PDF:

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0, & otherwise \end{cases}$$

6.2 Transformation

$$Y = \sin\left(\frac{\pi X}{2}\right)$$

6.3 Inverse Transformation

$$X = \frac{2}{\pi} \sin^{-1}(Y)$$

6.4 Derivative

$$\frac{dx}{dy} = \frac{2}{\pi} \frac{1}{\sqrt{1 - y^2}}$$

6.5 Apply Formula

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

$$= \frac{1}{2} \cdot \frac{2}{\pi} \cdot \frac{1}{\sqrt{1 - y^2}}$$

Final Result:

$$\boxed{f_Y(y) = \frac{1}{\pi\sqrt{1 - y^2}}}$$

Valid range:

$$-1 < y < 1$$

7 Transformation of Two Random Variables

Let:

$$Z = X + Y$$

Goal: Find $f_Z(z)$

8 CDF of Z

$$F_Z(z) = P(Z \leq z)$$

$$= P(X + Y \leq z)$$

Using joint PDF:

$$F_Z(z) = \int\int_{x+y\leq z} f_{XY}(x, y)dxdy$$

Alternate form:

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f_{XY}(x, y)dxdy$$

or

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f_{XY}(x, y)dydx$$

9 Summary

9.1 Single Variable Transformation

If:

$$Y = g(X)$$

Then:

$$F_Y(y) = F_X(g^{-1}(y))$$

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{dx}{dy}\right|$$

9.2 Two Variable Transformation

If:

$$Z = X + Y$$

Then:

$$F_Z(z) = P(X + Y \leq z)$$

$$= \int\int_{x+y\leq z} f_{XY}(x, y)dxdy$$