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L11_S1_A.pdf
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You are required to generate a lecture scribe intended strictly as exam-oriented reference material. Use ONLY the provided context (lecture slides / lecture PDF). Do NOT introduce any content, examples, explanations, intuition, or results that do not explicitly appear in the provided context. While writing the lecture scribe, reason step by step through the material exactly as it is presented in the lecture. Include explicit reasoning in the scribe wherever the lecture contains: - definitions and notation - assumptions and conditions - theorem or result statements - proofs or proof sketches - derivations - worked examples All reasoning must be faithful to the provided context and must follow the logical order used in the lecture. Make the logical dependencies between concepts explicit so that a student can reconstruct the lecture while revising for an exam. Maintain a formal, academic tone suitable for open-notes exam preparation. Organize the scribe clearly with appropriate section headings. Do NOT: - invent new examples or explanations - simplify or reinterpret arguments - add intuition not discussed in the lecture - go beyond the scope of the provided context Ensure a clear and logical progression of ideas to facilitate student understanding during revision. This should serve as concise and accurate exam reference material. Also cover each and every topic do NOT skip any single topic or examples. Get notes for handwritten content too.

Lecture Scribe: Transformation of Random Variables

Course: CSE400 – Fundamentals of Probability in Computing

Lecture 11: Transformation of Random Variables

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Source: L11_S1_A

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1. Introduction and Problem Statement

1.1 Known and Unknown Quantities

From the lecture introduction (page 1):

- Let **X** be a continuous random variable (CRV).
- The **PDF** of X is known:

$$f_X(x)$$

- The **CDF** of X is known:

$$F_X(x)$$

This is explicitly stated in the handwritten note:

> "$f_X(x)$: PDF of X is known a priori." (page 1)

1.2 Definition of Transformation

A new random variable is defined as a function of X:

$$Y = g(X)$$

This defines **Y as a transformation of X**, producing a **new random variable** (page 1).

The lecture explicitly raises the objective:

> "How to find $f_Y(y)$ and $F_Y(y)$?" (page 1)

Thus, the main goal of transformation of random variables is:

Given:

$$f_X(x)$$

Find:

$$f_Y(y), \quad F_Y(y)$$

for:

$$Y = g(X)$$

1.3 Extension to Functions of Two Random Variables

The lecture also introduces transformations involving two random variables (page 1):

Examples of transformations:

$$Z = g(X, Y)$$

Specific examples listed:

$$Z = X + Y$$

$$Z = X - Y$$

$$Z = \frac{X}{Y}$$

$$Z = X \cdot Y$$

This establishes the broader goal of finding distributions of functions of one or more random variables.

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2. Outline of Lecture Topics

As specified in the outline (page 2):

Topic 1: Transformation of Random Variables

Objective:

> Learning transformation techniques for random variables.

Topic 2: Function of Two Random Variables

Objective:

> Joint transformations and derived distributions.

Topic 3: Illustrative Example

Specific case:

$$Z = X + Y$$

Goal:

Derive distribution of Z from distributions of X and Y.

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3. Transformation of a Single Random Variable

3.1 Assumption: Invertible Transformation

From handwritten notes on page 2 and page 3:

Assumption:

$$Y = g(X)$$

where:

- g(.) is monotonic
- Thus inverse exists:

$$X = g^{-1}(Y)$$

This is explicitly shown graphically (page 3):

- Increasing monotonic case
- Decreasing monotonic case

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4. Step-by-Step Derivation Using CDF Method

Step 1: Definition of CDF of Y

From page 3:

By definition:

$$F_Y(y) = Pr(Y \leq y)$$

Substitute transformation:

$$Y = g(X)$$

Thus:

$$F_Y(y) = Pr(g(X) \leq y)$$

Since inverse exists:

$$= Pr(X \leq g^{-1}(y))$$

Thus:

$$F_Y(y) = F_X(g^{-1}(y))$$

This gives the CDF of Y in terms of CDF of X.

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5. Finding PDF of Y

PDF is obtained by differentiation of CDF.

From page 3:

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

Substitute CDF expression:

$$f_Y(y) = \frac{d}{dy} F_X(g^{-1}(y))$$

Using chain rule:

$$= f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

Thus,

Final result:

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)}$$

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6. Case: Decreasing Transformation

From page 4:

If transformation is decreasing:

$$F_Y(y) = Pr(Y \leq y)$$

$$= Pr(X \geq g^{-1}(y))$$

Thus:

$$= 1 - F_X(g^{-1}(y))$$

Differentiating:

$$f_Y(y) = -f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

Thus final expression written in magnitude form:

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|}$$

This is explicitly shown on page 4.

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7. Example: Uniform Random Variable Transformation

7.1 Given

From page 5:

X is uniformly distributed:

$$X \sim Uniform(-1, 1)$$

Thus PDF:

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0, & otherwise \end{cases}$$

7.2 Transformation

Given:

$$Y = \sin\left(\frac{\pi X}{2}\right)$$

7.3 Finding Inverse Transformation

Solve for X:

$$X = \frac{2}{\pi}\sin^{-1}(Y)$$

7.4 Compute derivative

Differentiate inverse function:

$$\frac{dx}{dy} = \frac{2}{\pi}\frac{1}{\sqrt{1-y^2}}$$

7.5 Apply Transformation Formula

Use formula:

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

Substitute values:

$$= \frac{1}{2}\cdot\frac{2}{\pi}\frac{1}{\sqrt{1-y^2}}$$

Thus final result:

$$\boxed{f_Y(y) = \frac{1}{\pi\sqrt{1-y^2}}}$$

Valid range from lecture diagram:

$$-1 < y < 1$$

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8. Transformation of Two Random Variables

Now consider transformation:

$$Z = X + Y$$

From page 6:

Goal:

Find:

(i)

$$f_Z(z)$$

(ii)

$$f_Z(z)$$

if X and Y independent

(iii)
If:

$$X \sim N(0,1)$$

$$Y \sim N(0,1)$$

Then prove:

$$Z \sim N(0,2)$$

(iv)

If X and Y exponential RVs with parameter λ, find:

$$f_Z(z)$$

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9. Derivation of PDF of Z Using Joint PDF

From page 7:

Start with definition of CDF:

$$F_Z(z) = Pr(Z \leq z)$$

Substitute:

$$Z = X + Y$$

Thus:

$$F_Z(z) = Pr(X + Y \leq z)$$

Integration using Joint PDF

The probability is computed using joint PDF:

$$F_Z(z) = \iint f_{XY}(x, y) dx dy$$

From lecture diagram (page 7):

Region defined by:

$$x + y \leq z$$

Equivalent integral representation shown:

$$F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{XY}(x, y) dx dy$$

Alternate order of integration:

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{XY}(x, y) dy dx$$

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10. Summary of Key Results

Single Variable Transformation

If:

$$Y = g(X)$$

Then:

$$F_Y(y) = F_X(g^{-1}(y))$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|$$

Two Variable Transformation

If:

$$Z = X + Y$$

Then:

$$F_Z(z) = Pr(X + Y \leq z)$$

$$= \int \int_{x+y \leq z} f_{XY}(x, y) dx dy$$

End of Lecture Scribe

This scribe strictly follows lecture slides, handwritten notes, derivations, and examples as presented in Lecture 11.

Sources