ROUGH SET METHOD BASED ON MULTI-GRANULATIONS

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Abstract

*The original rough set model is concerned primarily with the approximation of sets described by single binary relation on universe. In the view of granular computing, classical rough set theory is researched by single granulation (static granulation). The article extends the Pawlak rough set model to rough set model based on multi-granulations **MGRS**, where the set approximations are defined by using multi-equivalences on the universe. Mathematical properties of **MGRS** are investigated. It is shown that some properties of Pawlak rough set are special instances of **MGRS**, approximation measure of set described by using multi-granulations is always better than by using single granulation, which is suitable for describing more accurately the concept and solving problem according to user requirement.*

Keywords: *Rough set; Multi-granulations; Approximation measure.*

1. INTRODUCTION

Rough set theory, proposed by Z. Pawlak [1], has become well established as a mechanism for uncertainty management in a wide variety of applications related to artificial intelligence. Several extensions of rough set model have been proposed in the past, such as variable precision rough set (VPRS) model (see [2]), rough set model based on tolerance relation (see [3]), Bayesian rough set model (see [4]), fuzzy rough set model and rough fuzzy set model (see [5]), etc. In the view of granular computing, however, a general concept described by a set is always characterized via the so-called upper and lower approximations under static granulation, i.e., the concept is depicted by known knowledge induced by single relation on the universe. In practice, we often need to describe the concept through multi-relations on the universe according to user requirement or the target of solving problem. In the view of granular computing (proposed by L. A. Zadeh [6]), an equivalence relation on the universe can be regarded as a granulation, and a partition on the universe can be regarded as a granulation space [7, 8]. Several measures in knowledge base closely associated with granular computing, such as knowledge granulation, granulation measure, information entropy and rough entropy, were discussed by J. Y. Liang and Z. Z. Shi in [9, 10]. The main objective of this paper is an extension of classical rough set under static granulation, rough set model based on multi-granulations (**MGRS**).

The paper is organized as follows: in section 2, the basic concepts of rough set theory are reviewed. Rough set method based multi-granulations is proposed, its some useful properties are obtained in section 3. In section 4, we conclude the present research.

2. ROUGH SET CONCEPTS

Rough set theory [1] has become well established as a mechanism for uncertainty management in a wide variety of applications related to artificial intelligence.

Let $K=(U,R)$ be an approximation space, where U is a non-empty, finite set called the universe; R is a partition of U, or an equivalence relation on U. $[x]_R$ $(x\in U)$ denotes the equivalence class containing x.

An approximation space $K=(U,R)$ can be regarded as a knowledge base about U. Equivalence class of R is also called elementary set. The equivalence relation R partition the universe U into disjoint subsets. This partition of the universe U induced by R is denoted by U/R.

Given an equivalence relation R on U, and a subset X, we can define a lower approximation of X in U and a upper approximation of X in U by the following

$$\underline{R}X=\bigcup\{x\in U \mid [x]_R\subseteq X\}, \qquad (1)$$

and

$$\overline{R}X = \bigcup\{x \in U \mid [x]_R \cap X \neq \varnothing\}. \qquad (2)$$

The R-positive region of X is $POS_R(X) = \underline{R}X$, the R-negative region of X is $NEG_R(X) = U - \overline{R}X$, and the boundary or R-borderline region of X is $BN_R(X) = \overline{R}X - \underline{R}X$. X is called R-definable if and only if $\overline{R}X = \underline{R}X$. Otherwise $\overline{R}X \neq \underline{R}X$ and X is rough with respect to R.

We call X is R-definable if and only if $\underline{R}X = \overline{R}X$, and X is rough with respect to R if and only if $\underline{R}X \neq \overline{R}X$.

Let $K = (U, R)$ be an approximation space, $X \in U$ a subset on U. The approximation measure $\alpha_R(X)$ is defined as

$$\alpha_R(X) = \frac{|\underline{R}X|}{|\overline{R}X|}, \qquad (3)$$

where $X \neq \varnothing$, $|X|$ denotes the cardinality of set X.

Let $K = (U, \mathbf{R})$ be a knowledge base, $P, Q \in \mathbf{R}$ two equivalence relations, $U/P = \{X_1, X_2, \cdots, X_m\}$ and $U/Q = \{Y_1, Y_2, \cdots, Y_n\}$ are partitions induced by P and Q. We define a partial relation $\preceq$ on $\mathbf{R}$ as follows: $P \preceq Q$, if and only if, for every $X \in U/P$, there exists $Y \in U/Q$ such that $X \subseteq Y$. That is to say, Q is coarser than P, or P is finer than Q. If $P \preceq Q$ and $P \neq Q$, we say Q is strictly coarser than P (or P is strictly finer than Q), and denoted by $P \prec Q$. In fact, $P \prec Q \Leftrightarrow$ for every $X \in U/P$, there exist $Y \in U/Q$ such that $X \subseteq Y$, and exist $X_0 \in U/P$, $Y_0 \in U/Q$ such that $X_0 \subset Y_0$.

3. ROUGH SET APPROXIMATION BASED ON MULTI-GRANUALTIONS

In the view of granular computing, the approximation of a set is described by using a single equivalence relation (granulation) on the universe. Simply, we discuss firstly the approximation of set by using two equivalence relations on the universe, i.e., the target concept is described by two granulation spaces.

Definition 1. Let $K = (U, \mathbf{R})$ be a knowledge base, $\mathbf{R}$ a family of equivalence relations, $X \subseteq U$, $P, Q \in \mathbf{R}$, we define a lower approximation of X and a upper approximation of X in U by the following

$$\underline{P+Q}X = \bigcup\{x \mid [x]_P \subseteq X \; or \; [x]_Q \subseteq X\}, \qquad (4)$$

and

$$\overline{P+Q}X = \sim \underline{P+Q}(\sim X). \qquad (5)$$

We will illuminate the rough set approximation based on multi-granulations and the deference between the rough sets method and Pawlak rough sets by the following example.

Example 1 Let $K = (U, \mathbf{R})$ be a knowledge base, $X \subseteq U$, $R_1, R_2 \in \mathbf{R}$, where $U = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$, $X = \{e_1, e_2, e_6, e_8\}$.

$$U/R_1 = \{\{e_1, e_7\}, \{e_2, e_3, e_4, e_5, e_6\}, \{e_8\}\},$$

$$U/R_2 = \{\{e_1, e_2\}, \{e_3, e_4, e_5\}, \{e_6, e_7, e_8\}\},$$

$$U/R_1 \cap R_2 = \{\{e_1\}, \{e_2\}, \{e_3, e_4, e_5\}, \{e_6\}, \{e_7\}, \{e_8\}\}.$$

By computing, we have that

$$\underline{R_1 + R_2}X = \bigcup\{x \mid [x]_{R_2} \subseteq X \; or \; [x]_{R_2} \subseteq X\}$$
$$= \{e_8\} \cup \{e_1, e_2\}$$
$$= \{e_1, e_2, e_8\},$$

$$\overline{R_1 + R_2}X = \sim \underline{R_1 + R_2}(\sim X).$$
$$= \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\} \cap \{e_1, e_2, e_6, e_7, e_8\}$$
$$= \{e_1, e_2, e_6, e_7\}.$$

$$\underline{R_1 \cap R_2}X = \{e_1, e_2, e_6, e_8\},$$

$$\overline{R_1 \cap R_2}X = \{e_1, e_2, e_6, e_8\}.$$

Obviously, it follows from above computation that

$$\underline{R_1 + R_2}X \neq \underline{R_1 \cap R_2}X,$$

$$\overline{R_1 + R_2}X \neq \overline{R_1 \cap R_2}.X.$$

Directly from the definition of approximations we can get the following properties of the lower and the upper approximations.

Proposition 1. Let $K = (U, \mathbf{R})$ be a knowledge base, $X \subseteq U$, $P, Q \in \mathbf{R}$, the following properties hold

1) $\underline{P+Q}X \subseteq X \subseteq \overline{P+Q}X$;

2) $\underline{P+Q}\varnothing = \overline{P+Q}\varnothing = \varnothing$;

$\underline{P+Q}U = \overline{P+Q}U = U$;

3) $\underline{P+Q}(\sim X) = \sim \overline{P+Q}X$;

$\overline{P+Q}(\sim X) = \sim \underline{P+Q}X$;

4) $\underline{P+Q}(\underline{P+Q}X)=\overline{P+Q}(\underline{P+Q}X)=\underline{P+Q}X$;

5) $\overline{P+Q}(\overline{P+Q}X)=\underline{P+Q}(\overline{P+Q}X)=\overline{P+Q}X$;

6) $\underline{P+Q}X=\underline{P}X\cup\underline{Q}X$;

7) $\overline{P+Q}X=\overline{P}X\cap\overline{Q}X$;

8) $\underline{P+Q}X=\underline{Q+P}X, \overline{P+Q}X=\overline{Q+P}X$.

Proof.

If $P=Q(P,Q\in \mathbf{R})$, then (4) degenerates into $\underline{P}X=\bigcup\{x\mid[x]_P\subseteq X\}$, (5) degenerates into $\overline{P}X=\sim\underline{P}(\sim X)=\bigcup\{x\mid[x]_P\cap X\neq\varnothing\}$. Obviously, they are the same as upper approximation and lower approximation of Pawlak rough sets [1], and hence 1)-8) hold.

If $P\neq Q(P,Q\in\mathbf{R})$ we give their proving as follows.

1a) Let $x,y\in\underline{P+Q}X(x,y\in U)$, then $[x]_P\subseteq X$ and $[y]_Q\subseteq X$. But $x\in[x]_P$ and $y\in[y]_Q$, hence $x,y\in X$ and $\underline{P+Q}X\subseteq X$.

1b) Let $x,y\in X$, then $x\in[x]_P\cap X$ and $y\in[y]_Q\cap X$, i.e., $[x]_P\cap X\neq\varnothing$ and $[y]_Q\cap X\neq\varnothing$. Hence, $x,y\in\overline{P+Q}X$, and $X\subseteq\overline{P+Q}X$.

2a) From 1), we can know that $\underline{P+Q}\varnothing\subseteq\varnothing$ and $\varnothing\subseteq\underline{P+Q}\varnothing$ (because the empty set is included in every set), therefore $\underline{P+Q}\varnothing=\varnothing$.

2b) Suppose $\overline{P+Q}\varnothing\neq\varnothing$. Then there exists x such that $x\in\overline{P+Q}\varnothing$. Hence $[x]\cap\varnothing\neq\varnothing$, but $[x]\cap\varnothing=\varnothing$, what contradicts the assumption. So, $\overline{P+Q}\varnothing=\varnothing$.

2c) From 1), we can know that $\underline{P+Q}U\subseteq U$. And if $x\in U$, then $[x]\subseteq U$. Hence $x\in\underline{P+Q}U$ and $U\subseteq\underline{P+Q}U$, thus $\underline{P+Q}U=U$.

2d) From 1), we can get that $U\subseteq\overline{P+Q}U$. And $\overline{P+Q}U\subseteq U$ hold clearly. Thus, $\overline{P+Q}U=U$.

3) From (5), $\underline{P+Q}(\sim X)=\sim\overline{P+Q}X$ is obvious. And assume $X=\sim X$, then $\overline{P+Q}(\sim X)=\sim\underline{P+Q}(\sim(\sim X))=\sim\underline{P+Q}X$.

4a) From 1), we know that $\underline{P+Q}(\underline{P+Q}X)\subseteq\underline{P+Q}X$. If $x\in\underline{P+Q}X$, then $[x]_P$, $[x]_Q\subseteq X$, hence $\underline{P+Q}[x]_P\subseteq\underline{P+Q}X$ and $\underline{P+Q}[x]_Q\subseteq\underline{P+Q}X$. But $\underline{P+Q}[x]_P=[x]_P$ and $\underline{P+Q}[x]_Q=[x]_Q$, so $[x]_P$, $[x]_Q\subseteq\underline{P+Q}X$ and $x\in\underline{P+Q}(\underline{P+Q}X)$. Hence, one have that $\underline{P+Q}X=\underline{P+Q}(\underline{P+Q}X)$. Therefore, $\underline{P+Q}(\underline{P+Q}X)=\underline{P+Q}X$ hold.

4b) From 1), $\underline{P+Q}X\subseteq\overline{P+Q}(\underline{P+Q}X)$ hold. If $x\in\overline{P+Q}(\underline{P+Q}X)$, then $[x]_P\cap\underline{P+Q}X\neq\varnothing$, $[x]_Q\cap\underline{P+Q}X\neq\varnothing$, i.e., there exist $y\in[x]_P$ and $z\in[x]_Q$ such that $y\in\underline{P+Q}X$ and $z\in\underline{P+Q}X$. Hence $[y]_P\subseteq X$, $[z]_Q\subseteq X$. But $[x]_P=[y]_P$, $[x]_Q=[z]_Q$, thus $[x]_P\subseteq X$, $[x]_Q\subseteq X$, and $x\in\underline{P+Q}X$. Hence, We have that $\underline{P+Q}X\supseteq\overline{P+Q}(\underline{P+Q}X)$. Therefore, $\overline{P+Q}(\underline{P+Q}X)=\underline{P+Q}X$ hold.

5a) From 1), $\overline{P+Q}X\subseteq\overline{P+Q}(\overline{P+Q}X)$ hold. If $x\in\overline{P+Q}(\overline{P+Q}X)$, then $[x]_P\cap\overline{P+Q}X\neq\varnothing$ and $[x]_Q\cap\overline{P+Q}X\neq\varnothing$. For some $y\in[x]_P$, $y\in\overline{P+Q}X$, and some $z\in[x]_Q$, $z\in\overline{P+Q}X$, we have that $[y]_P\cap X\neq\varnothing$, $[z]_Q\cap X\neq\varnothing$. But $[x]_P=[y]_P$, $[x]_Q=[z]_Q$, thus $[x]_P\cap X\neq\varnothing$, $[x]_Q\cap X\neq\varnothing$, that is to say, $x\in\overline{P+Q}X$ hold, which yields $\overline{P+Q}X\supseteq\overline{P+Q}(\overline{P+Q}X)$. Therefore, we have that $\overline{P+Q}(\overline{P+Q}X)=\overline{P+Q}X$.

5b) From 1), one know that $\underline{P+Q}(\overline{P+Q}X)\subseteq\overline{P+Q}X$. If $x,y\in\overline{P+Q}X$, then $[x]_P\cap X\neq\varnothing$, $[y]_Q\cap X\neq\varnothing$. Thus, $[x]_P\subseteq\overline{P+Q}X$, $[y]_Q\subseteq\overline{P+Q}X$ (because if $x'\in[x]_P$, then $[x']\cap X=[x]_P\cap X\neq\varnothing$, i.e., $x'\in\overline{P+Q}X$). And $x\in\underline{P+Q}(\overline{P+Q}X)$, we have that $\underline{P+Q}(\overline{P+Q}X\supseteq\overline{P+Q}X)$. Therefore, we get that $\underline{P+Q}(\overline{P+Q}X)=\overline{P+Q}X$.

6) From (4), we can know easily that for $\forall x \in U$, if $[x]_P \subseteq X$ then $x \in \underline{P+Q}X$, and if $[x]_Q \subseteq X$ then $x \in \underline{P+Q}X$, that is, $\underline{P}X \subseteq \underline{P+Q}X$, $\underline{Q}X \subseteq \underline{P+Q}X$. And, if there exists $y \in X$ with $y \in \underline{P+Q}X - \bigcup\limits_{x\in U}[x]_P - \bigcup\limits_{x\in U}[x]_Q = \varnothing$, then $[y] = \varnothing$. Therefore, we have that $\underline{P+Q}X = \underline{P}X \cup \underline{Q}X$.

7) From (5) and 6), we can obtain that $\overline{P+Q}X = \sim \underline{P+Q}(\sim X) = \sim (\underline{P}(\sim X) \cup \underline{Q}(\sim X)) = \sim (\sim \overline{P}X \cup \sim \overline{Q}X) = \overline{P}X \cap \overline{Q}X$.

8) They are straightforward from Definition 1.

This completes the proof.

In order to discover the relationship the approximation of a single set and the approximation of two sets described by using two granulations, the following properties are given.

Proposition 2. Let $K = (U, \mathbf{R})$ be a knowledge base, $X, Y \subseteq U$, $P, Q \in \mathbf{R}$, the following properties hold

1) $\underline{P+Q}(X \cap Y) = (\underline{P}X \cap \underline{P}Y) \cup (\underline{Q}X \cap \underline{Q}Y)$;

2) $\overline{P+Q}(X \cup Y) = (\overline{P}X \cup \overline{P}Y) \cap (\overline{Q}X \cup \overline{Q}Y)$;

3) $\underline{P+Q}(X \cap Y) \subseteq \underline{P+Q}(X) \cap \underline{P+Q}(Y)$;

4) $\overline{P+Q}(X \cup Y) \supseteq \overline{P+Q}(X) \cup \overline{P+Q}(Y)$;

5) $X \subseteq Y \Rightarrow \underline{P+Q}X \subseteq \underline{P+Q}Y$;

6) $X \subseteq Y \Rightarrow \overline{P+Q}X \subseteq \overline{P+Q}Y$;

7) $\underline{P+Q}(X \cup Y) \supseteq \underline{P+Q}X \cup \underline{P+Q}Y$;

8) $\overline{P+Q}(X \cap Y) \subseteq \overline{P+Q}X \cap \overline{P+Q}Y$.

Proof.

If $P = Q(P, Q \in \mathbf{R})$, then (4) degenerates into $\underline{P}X = \bigcup\{x \mid [x]_P \subseteq X\}$, (5) degenerates into $\overline{P}X = \sim \underline{P}(\sim X) = \bigcup\{x \mid [x]_P \cap X \neq \varnothing\}$. Obviously, they are the same as upper approximation and lower approximation of Pawlak rough sets [1], and hence 1)-8) hold.

If $P \neq Q(P, Q \in \mathbf{R})$, we give their proving as follows.

1) $\underline{P+Q}(X \cap Y) = \underline{P}(X \cap Y) \cup \underline{Q}(X \cap Y)$
$$= (\underline{P}X \cap \underline{P}Y) \cup (\underline{Q}X \cap \underline{Q}Y).$$

2) $\overline{P+Q}(X \cup Y) = \overline{P}(X \cup Y) \cap \overline{Q}(X \cup Y)$
$$= (\overline{P}X \cup \overline{P}Y) \cap (\overline{Q}X \cup \overline{Q}Y).$$

3) It follows from 1) that
$$\begin{aligned}
&\underline{P+Q}(X \cap Y) = (\underline{P}X \cap \underline{P}Y) \cup (\underline{Q}X \cap \underline{Q}Y)\\
&= ((\underline{P}X \cap \underline{P}Y) \cup \underline{Q}X) \cap ((\underline{P}X \cap \underline{P}Y) \cup \underline{Q}Y)\\
&= ((\underline{P}X \cup \underline{Q}X) \cap (\underline{P}Y \cup \underline{Q}X))\\
&\quad \cap ((\underline{P}X \cup \underline{Q}Y) \cap (\underline{P}Y \cup \underline{Q}Y))\\
&= \underline{P+Q}X \cap \underline{P+Q}Y\\
&\quad \cap ((\underline{P}X \cup \underline{Q}Y) \cap (\underline{P}Y \cup \underline{Q}X))\\
&\subseteq \underline{P+Q}X \cap \underline{P+Q}Y.
\end{aligned}$$

4) It follows from 2) that
$$\begin{aligned}
&\overline{P+Q}(X \cup Y) = (\overline{P}X \cup \overline{P}Y) \cap (\overline{Q}X \cup \overline{Q}Y)\\
&= ((\overline{P}X \cup \overline{P}Y) \cap \overline{Q}X) \cup ((\overline{P}X \cup \overline{P}Y) \cap \overline{Q}Y)\\
&= ((\overline{P}X \cap \overline{Q}X) \cup (\overline{P}Y \cap \overline{Q}X))\\
&\quad \cup ((\overline{P}X \cap \overline{Q}Y) \cup (\overline{P}Y \cap \overline{Q}Y))\\
&= \overline{P+Q}X \cup \overline{P+Q}Y\\
&\quad \cup ((\overline{P}X \cap \overline{Q}Y) \cup (\overline{P}Y \cap \overline{Q}X))\\
&\supseteq \overline{P+Q}X \cup \overline{P+Q}Y.
\end{aligned}$$

5) If $X \subseteq Y$, then $X \cap Y = X$. It follows from 3) that
$$\begin{aligned}
&\underline{P+Q}(X \cap Y) = \underline{P+Q}X \subseteq \underline{P+Q}X \cap \underline{P+Q}Y\\
&\Rightarrow \underline{P+Q}X = \underline{P+Q}X \cap \underline{P+Q}Y\\
&\Rightarrow \underline{P+Q}X \subseteq \underline{P+Q}Y.
\end{aligned}$$

6) If $X \subseteq Y$, then $X \cup Y = Y$. It follows from 4) that
$$\begin{aligned}
&\overline{P+Q}(X \cup Y) = \overline{P+Q}Y \supseteq \overline{P+Q}X \cup \overline{P+Q}Y\\
&\Rightarrow \overline{P+Q}Y = \overline{P+Q}X \cup \overline{P+Q}Y\\
&\Rightarrow \overline{P+Q}X \subseteq \overline{P+Q}Y.
\end{aligned}$$

7) It is clear that $X \subseteq X \cup Y$, $Y \subseteq X \cup Y$. It follows that
$$\underline{P+Q}X \subseteq \underline{P+Q}(X \cup Y),$$
$$\underline{P+Q}Y \subseteq \underline{P+Q}(X \cup Y).$$

Hence $\underline{P+Q}X \cup \underline{P+Q}Y \subseteq \underline{P+Q}(X \cup Y)$;

8) It is clear that $X \cap Y \subseteq X$, $X \cap Y \subseteq Y$. It follows that
$$\overline{P+Q}(X \cap Y) \subseteq \overline{P+Q}X,$$
$$\overline{P+Q}(X \cap Y \subseteq \overline{P+Q}Y).$$

Hence $\overline{P+Q}(X \cap Y \subseteq \overline{P+Q}X \cap \overline{P+Q}Y)$.

This completes the proof.

Based on above conclusions, we here extend to classical rough set method to rough set model based on multi-granulations, where the set approximations are defined by using multi-equivalences on the universe.

Definition 2. Let $K=(U,\mathbf{R})$ be a knowledge base, $X \subseteq U$, $P_1, P_2, \cdots, P_m \in \mathbf{R}$, we can define a lower approximation of X and a upper approximation of X related to $P_1, P_2, \cdots, P_m$ by the following

$$\underline{\sum\nolimits_{i=1}^{m} P_i} X = \bigcup \{x \mid [x]_{P_i} \subseteq X, i \le m\}, \tag{6}$$

and

$$\overline{\sum\nolimits_{i=1}^{m} P_i} X = \sim \underline{\sum\nolimits_{i=1}^{m} P_i} (\sim X). \tag{7}$$

Directly from the definition of approximations we can get the following properties of the lower and the upper approximations.

Proposition 3. Let $K=(U,\mathbf{R})$ be a knowledge base, $X \subseteq U$, the following properties hold

1) $\underline{\sum_{i=1}^{m} P_i} X = \bigcup_{i=1}^{m} \underline{P_i} X;$

2) $\overline{\sum_{i=1}^{m} P_i} X = \bigcap_{i=1}^{m} \overline{P_i} X;$

3) $\overline{\sum_{i=1}^{m} P_i} (\sim X) = \sim \underline{\sum_{i=1}^{m} P_i} X;$

4) $\underline{\sum_{i=1}^{m} P_i} (\sim X) = \sim \overline{\sum_{i=1}^{m} P_i} X.$

Proof.

1) It follows from (4) that for $\forall x \in U$, if $[x]_{P_i} \subseteq X$, ($P_i \in \mathbf{R}$), then $x \in \underline{\sum_{i=1}^{m} P_i} X$, that is to say, $\underline{P_i} X \subseteq \underline{\sum_{i=1}^{m} P_i} X$. And, if there exists $y \in X$ such that $y \in \underline{\sum_{i=1}^{m} P_i} X - \bigcup_{i=1}^{m} [x]_{P_i} = \varnothing$, then $y = \varnothing$. Thus, we have that $\underline{\sum_{i=1}^{m} P_i} X = \bigcup_{i=1}^{m} \underline{P_i} X$.

2) From (7) and 1), we have that

$$\overline{\sum\nolimits_{i=1}^{m} P_i} X = \sim \underline{\sum\nolimits_{i=1}^{m} P_i} (\sim X) = \sim \bigcup\nolimits_{i=1}^{m} \underline{P_i} (\sim X)$$
$$= \sim \bigcup\nolimits_{i=1}^{m} (\sim \overline{P_i} X) = \bigcap\nolimits_{i=1}^{m} \overline{P_i} X.$$

3) It is straightforward from (7).

4) Let $X = \sim X$ in (7), we have that

$$\underline{\sum\nolimits_{i=1}^{m} P_i} (\sim X) = \sim \overline{\sum\nolimits_{i=1}^{m} P_i} X.$$

This completes the proof.

Proposition 4. Let $K=(U,\mathbf{R})$ be a knowledge base, $X_1, X_2, \cdots, X_n \subseteq U$ be n subsets on U, $P_1, P_2, \cdots, P_m \in \mathbf{R}$, the following properties hold

1) $\underline{\sum_{i=1}^{m} P_i} (\bigcap_{j=1}^{n} X_j) = \bigcup_{i=1}^{m} (\bigcap_{j=1}^{n} \underline{P_i} X_j)$;

2) $\overline{\sum_{i=1}^{m} P_i} (\bigcup_{j=1}^{n} X_j) = \bigcap_{i=1}^{m} (\bigcup_{j=1}^{n} \overline{P_i} X_j)$;

3) $\underline{\sum_{i=1}^{m} P_i} (\bigcap_{j=1}^{n} X_j) \subseteq \bigcap_{j=1}^{n} (\underline{\sum_{i=1}^{m} P_i} X_j)$;

4) $\overline{\sum_{i=1}^{m} P_i} (\bigcup_{j=1}^{n} X_j) \supseteq \bigcup_{j=1}^{n} (\overline{\sum_{i=1}^{m} P_i} X_j)$;

5) $\underline{\sum_{i=1}^{m} P_i} (\bigcup_{j=1}^{n} X_j) \supseteq \bigcup_{j=1}^{n} (\underline{\sum_{i=1}^{m} P_i} X_j)$;

6) $\overline{\sum_{i=1}^{m} P_i} (\bigcap_{j=1}^{n} X_j) \subseteq \bigcap_{j=1}^{n} (\overline{\sum_{i=1}^{m} P_i} X_j)$.

Proof.

Similar to proposition 2, we can prove the following properties.

1)
$$\begin{aligned}\underline{\sum\nolimits_{i=1}^{m} P_i} (\bigcap\nolimits_{j=1}^{n} X_j) &= \bigcup\nolimits_{i=1}^{m} \underline{P_i} (\bigcap\nolimits_{j=1}^{n} X_j) \\ &= \bigcup\nolimits_{i=1}^{m} (\bigcap\nolimits_{j=1}^{n} \underline{P_i} X_j).\end{aligned}$$

2)
$$\begin{aligned}\overline{\sum\nolimits_{i=1}^{m} P_i} (\bigcup\nolimits_{j=1}^{n} X_j) &= \bigcap\nolimits_{i=1}^{m} \overline{P_i} (\bigcup\nolimits_{j=1}^{n} X_j) \\ &= \bigcap\nolimits_{i=1}^{m} (\bigcup\nolimits_{j-1}^{n} \overline{P_i} X_j).\end{aligned}$$

3)
$$\begin{aligned}\underline{\sum\nolimits_{i=1}^{m} P_i} (\bigcap\nolimits_{j=1}^{n} X_j) &= \bigcup\nolimits_{i=1}^{m} (\bigcap\nolimits_{j=1}^{n} \underline{P_i} X_j) \\ &= \bigcap\nolimits_{j=1}^{n} (\bigcup\nolimits_{i=1}^{m} \underline{P_i} X_j) \cap \cdots \\ &= \bigcap\nolimits_{j=1}^{n} (\underline{\sum\nolimits_{i=1}^{m} P_i} X_j) \cap \cdots \\ &\subseteq \bigcap\nolimits_{j=1}^{n} (\underline{\sum\nolimits_{i=1}^{m} P_i} X_j).\end{aligned}$$

4)
$$\begin{aligned}\overline{\sum\nolimits_{i=1}^{m} P_i} (\bigcup\nolimits_{j=1}^{n} X_j) &= \bigcap\nolimits_{i=1}^{m} (\bigcup\nolimits_{j=1}^{n} \overline{P_i} X_j) \\ &= \bigcup\nolimits_{j=1}^{n} (\bigcap\nolimits_{i=1}^{m} \overline{P_i} X_j) \cup \cdots \\ &= \bigcup\nolimits_{j=1}^{n} (\overline{\sum\nolimits_{i=1}^{m} P_i} X_j) \cup \cdots \\ &\supseteq \bigcup\nolimits_{j=1}^{n} (\overline{\sum\nolimits_{i=1}^{m} P_i} X_j).\end{aligned}$$

5) It follows from $\forall X_j \subseteq \bigcup_{j=1}^{n} X_j$ that

$$\underline{\sum_{i=1}^{m} P_i} X_j \subseteq \underline{\sum_{i=1}^{m} P_i}(\bigcup_{j=1}^{n} X_j).$$

Hence, we have that

$$\underline{\sum_{i=1}^{m} P_i}(\bigcup_{j=1}^{n} X_j) \supseteq \bigcup_{j=1}^{n}(\underline{\sum_{i=1}^{m} P_i} X_j).$$

6) It follows from $\bigcap_{j=1}^{n} X_j \subseteq X_j \, (j \in \{1,2,\cdots,n\})$ that $\overline{\sum_{i=1}^{m} P_i} X_j \supseteq \overline{\sum_{i=1}^{m} P_i}(\bigcap_{j=1}^{n} X_j)$. Hence, we have

$$\overline{\sum_{i=1}^{m} P_i}(\bigcap_{j=1}^{n} X_j) \subseteq \bigcap_{j=1}^{n}(\overline{\sum_{i=1}^{m} P_i} X_j).$$

This completes the proof.

Proposition 5. Let $K=(U,\mathbf{R})$ be a knowledge base, $X_1, X_2, \cdots, X_n \subseteq U$ with $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n$ be n subsets on U, $P_1, P_2, \cdots, P_m \in \mathbf{R}$, the following properties hold

1) $\underline{\sum_{i=1}^{m} P_i} X_1 \subseteq \underline{\sum_{i=1}^{m} P_i} X_2 \subseteq \cdots \subseteq \underline{\sum_{i=1}^{m} P_i} X_n$;

2) $\overline{\sum_{i=1}^{m} P_i} X_1 \subseteq \overline{\sum_{i=1}^{m} P_i} X_2 \subseteq \cdots \subseteq \overline{\sum_{i=1}^{n} P_i} X_n$.

Proof.

Suppose $1 \le i \le j \le n$ then $X_i \subseteq X_j$ holds.

1) Clearly, $X_i \cap X_j = X_i$. Hence, it follows from 3) in proposition 4 that

$$\underline{\sum_{i=1}^{m} P_i} X_i = \underline{\sum_{i=1}^{m} P_i}(X_i \cap X_j) \subseteq \underline{\sum_{i=1}^{m} P_i} X_i \cap \underline{\sum_{i=1}^{m} P_i} X_j)$$

$$\Rightarrow \underline{\sum_{i=1}^{m} P_i} X_i = \underline{\sum_{i=1}^{m} P_i} X_i \cap \underline{\sum_{i=1}^{m} P_i} X_j$$

$$\Rightarrow \underline{\sum_{i=1}^{m} P_i} X_i \subseteq \underline{\sum_{i=1}^{m} P_i} X_j.$$

Therefore, we have that

$$\underline{\sum_{i=1}^{m} P_i} X_1 \subseteq \underline{\sum_{i=1}^{m} P_i} X_2 \subseteq \cdots \subseteq \underline{\sum_{i=1}^{m} P_i} X_n.$$

2) Clearly, $X_i \cup X_j = X_j$. Hence, it follows from 4) in proposition 4 that

$$\overline{\sum_{i=1}^{m} P_i} X_j = \overline{\sum_{i=1}^{m} P_i}(X_i \cup X_j \supseteq \overline{\sum_{i=1}^{m} P_i} X_i \cup \overline{\sum_{i=1}^{m} P_i} X_j)$$

$$\Rightarrow \overline{\sum_{i=1}^{m} P_i} X_j = \overline{\sum_{i=1}^{m} P_i} X_i \cup \overline{\sum_{i=1}^{m} P_i} X_j$$

$$\Rightarrow \overline{\sum_{i=1}^{m} P_i} X_i \subseteq \overline{\sum_{i=1}^{m} P_i} X_j.$$

Therefore, we have

$$\overline{\sum_{i=1}^{m} P_i} X_1 \subseteq \overline{\sum_{i=1}^{m} P_i} X_2 \subseteq \cdots \overline{\sum_{i=1}^{m} P_i} X_n.$$

This completes the proof.

Uncertainty of a set (category) is due to the existence of a borderline region. The greater the borderline region of a set, the lower is the accuracy of the set. Similar to $\alpha_R(X)$, in order to express this idea more precisely we introduce the accuracy measure as follows.

Definition 3. Let $K=(U,\mathbf{R})$ be a knowledge base, $X \subseteq U$, $P=\{P_1, P_2, \cdots, P_m\}$, $\forall P_i \in \mathbf{R} \, (i \le m)$, the approximation measure is defined by

$$\alpha_P(X) = \frac{\left|\underline{\sum_{i=1}^{m} P_i} X\right|}{\left|\overline{\sum_{i=1}^{m} P_i} X\right|}, \tag{8}$$

where $X \ne \varnothing$, $|X|$ denotes the cardinality of set X.

Proposition 6. Let $K=(U,\mathbf{R})$ be a knowledge base, $X \subseteq U$, $P=\{P_1, P_2, \cdots, P_m\}$, $P' \subseteq P$ a subset of P, $\forall P_i \in \mathbf{R} \, (i \le m)$, then

$$\alpha_P(X) \ge \alpha_{P'} \ge \alpha_{P_i}(X)(i \le m).$$

Proof.

Considering that $P' \subseteq P$ is a subset of P, we follow from definition 2 that

$$\bigcup_{i=1}^{m} \underline{P_i} X \supseteq \bigcup_{P_i \in P, P_i \notin P'} \underline{P_i} X,$$

$$\bigcap_{i=1}^{m} \overline{P_i} X \subseteq \bigcap_{P_i \in P, P_i \notin P'} \overline{P_i} X.$$

Then, it is clear that $|\bigcup_{i=1}^{m} \underline{P_i} X| \ge |\bigcup_{P_i \in P, P_i \notin P} \underline{P_i} X|$, $|\bigcap_{i=1}^{m} \overline{P_i} X| \le |\bigcap_{P_i \in P, P_i \notin P} \overline{P_i} X|$.

Hence, we have that

$$\alpha_P(X) = \frac{\left|\underline{\sum_{i=1}^{m} P_i} X\right|}{\left|\overline{\sum_{i=1}^{m} P_i} X\right|} = \frac{\left|\bigcup_{i=1}^{m} \underline{P_i} X\right|}{\left|\bigcap_{i=1}^{m} \overline{P_i} X\right|}$$

$$\ge \frac{\left|\bigcup_{P_i \in P, P_i \notin P'} \underline{P_i} X\right|}{\left|\bigcap_{P_i \in P, P_i \notin P'} \overline{P_i} X\right|}$$

$$= \frac{\left|\underline{\sum_{P_i \in P, P_i \notin P'} P_i} X\right|}{\left|\overline{\sum_{P_i \in P, P_i \notin P'} P_i} X\right|}$$

$$= \alpha_{P'}(X).$$

Similarly, we have $\alpha_{P'}(X) \ge \alpha_{P_i}(X)(i \le m)$.

Thus, inequality $\alpha_P(X) \ge \alpha_{P'}(X) \ge \alpha_{P_i}(X)(i \le m)$ hold for arbitrary $P' \subseteq P$ and every $P_i \in P$.

This completes the proof.

Proposition 6 shows that approximation measure of set the concept enlarges as the number of granulations for describing the concept become increases.

Example 2 Continue from example 1.

Suppose $P=\{R_1,R_2\}$. By computing, we have that

$$\alpha_P(X)=\frac{\left|\underline{R_1+R_2}X\right|}{\left|\overline{R_1+R_2}X\right|}=\frac{3}{4},$$

$$\alpha_{R_1}(X)=\frac{\left|\underline{R_1}X\right|}{\left|\overline{R_1}X\right|}=\frac{1}{8},$$

$$\alpha_{R_2}(X)=\frac{\left|\underline{R_2}X\right|}{\left|\overline{R_2}X\right|}=\frac{2}{5},$$

Obviously, it follows from above computation that,

$$\alpha_P(X)>\alpha_{R_1}(X),$$

$$\alpha_P(X)>\alpha_{R_2}(X).$$

In particular, if $R_1\preceq R_2$, then $\alpha_P(X)=\alpha_{R_1}(X)$ holds. It can be understood by the following proposition.

Proposition 7. Let $K=(U,\mathbf{R})$ be a knowledge base, $X\subseteq U$, $P=\{P_1,P_2,\cdots,P_m\}$ with $P_1\preceq P_2\preceq\cdots\preceq P_m$, $\forall P_i\in\mathbf{R}\,(i\leq m)$, then

$$\underline{\sum\nolimits_{i=1}^{m}P_i}X=\underline{P_1}X, \tag{9}$$

$$\overline{\sum\nolimits_{i=1}^{m}P_i}X=\overline{P_1}X. \tag{10}$$

Proof.

Suppose $1\leq j\leq k\leq m$, $P_j\preceq P_k$. From the definition of $\preceq$, we know that for every $[x]_{P_j}\in U/P_j$, there exists $[x]_{P_k}\in U/P_k$ such that $[x]_{P_j}\subseteq[x]_{P_k}$. Therefore, we have that

$$\underline{P_j}X=\bigcup\{[x]_{P_j}\mid[x]_{P_j}\subseteq X\}$$
$$\supseteq\underline{P_k}X=\bigcup\{[x]_{P_k}\mid[x]_{P_k}\subseteq X\},$$

i.e.,

$$\underline{P_j+P_k}X=\underline{P_j}X\cup\underline{P_k}X=\underline{P_j}X.$$

Since $P_1\preceq P_2\preceq\cdots\preceq P_m$, one have that

$$\underline{\sum\nolimits_{i=1}^{m}P_i}X=\underline{\bigcup\nolimits_{i=1}^{m}P_i}X=\underline{P_1}X.$$

Similarly, we also have that

$$\overline{P_j}X=\bigcup\{[x]_{P_j}\mid[x]_{P_j}\cap X\neq\varnothing\}$$
$$\subseteq\overline{P_k}X=\bigcup\{[x]_{P_k}\mid[x]_{P_k}\cap X\neq\varnothing\},$$

i.e.,

$$\overline{P_j+P_k}X=\overline{P_j}X\cap\overline{P_k}X=\overline{P_j}X.$$

Thus, we have that

$$\overline{\sum\nolimits_{i=1}^{m}P_i}X=\bigcap\nolimits_{i=1}^{m}\overline{P_i}X=\overline{P_1}X.$$

This completes the proof.

6. CONCLUSIONS

The main objective of this paper is an extension of classical rough set under static granulation, rough set model based on multi-granulations (**MGRS**), where the approximations of sets are defined by using multi-equivalences on universe. These equivalences are chosen according to user requirement or target of solving problem. The method has some useful properties. In particular, some properties of Pawlak rough set are special instances of **MGRS**, approximation measure of set described by using multi-granulations is always better than by using single granulation.

Presented approach appears to be well suited for data mining applications where the acquisition of decision rules with high approximation measure, and further studying rough set theory. Further research is planned to evaluate the MGRS method in comparison to original Pawlak's approaches, and to extend other rough set methods.

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References

[1] Z. Pawlak. "Rough sets," *International Journal of Computer and Information Science*. Vol. 11, pp. 341-356, 1982

[2] W. Ziarko. "Variable precision rough sets model," *Journal of Computer System Science*. Vol. 46(1), pp. 39-59, 1993

[3] M. Kryszkiewicz. "Rough set approach to incomplete information systems," Information Sciences. Vol. 112,

pp. 39-49, 1998
[4] D. Ślęzak, W. Ziarko. "The investigation of the Bayesian rough set model," *International Journal of Approximate Reasoning*. Vol. 40, pp. 81-91, 2005
[5] D. Dubois, H. Prade. "Rough fuzzy sets and fuzzy rough sets," *International Journal of General Systems*. Vol. 17, pp. 191-209, 1990
[6] L. A. Zadeh. "Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic," *Fuzzy Sets and Systems*. Vol. 90. pp. 111-127, 1997
[7] Y. Y. Yao. "Granular computing: basic issues and possible solutions," *Proceedings of the Fifth International Conference on Computing and Information*. Vol. I, pp. 186-189, 2000
[8] Y. Y. Yao. "Perspectives of Granular Computing," *Proceedings of 2005 IEEE International Conference on Granular Computing*. Vol. 1, pp. 85-90, 2005
[9] J. Y. Liang, Z. Z. Shi. "The information entropy, rough entropy and knowledge granulation in rough set theory," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*. Vol. 12(1), pp. 37-46, 2004
[10] J. Y. Liang, D. Y. Li. Uncertainty and Knowledge Acquisition in Information Systems. Science Press, Beijing, China, 2005