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Comparison of fuzzy numbers using a fuzzy distance measure

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Abstract

A new approach for ranking fuzzy numbers based on a distance measure is introduced. A new class of distance measures for interval numbers that takes into account all the points in both intervals is developed first, and then it is used to formulate the distance measure for fuzzy numbers. The approach is illustrated by numerical examples, showing that it overcomes several shortcomings such as the indiscriminative and counterintuitive behavior of several existing fuzzy ranking approaches.

Keywords: Fuzzy numbers; Fuzzy ranking; Distance measure

1. Introduction

The purpose of this paper is to introduce a new method for ranking fuzzy numbers (FNs). Fuzzy ranking is a topic that has been studied by many researchers (see e.g., [6–8,14,19,27] for an overview and comparison of various methods). In a more recent review, Wang and Kerre [22,23] proposed several axioms as reasonable properties to determine the rationality of a fuzzy ordering or ranking method and systematically compared a wide array of fuzzy ranking methods. Almost each method, however, has pitfalls in some respect, such as inconsistency with human intuition, indiscrimination and difficulty of interpretation [6,8,27]. The fuzzy ranking method suggested in this paper overcomes many of the problems inherent to existing methods. To formulate this ranking method, a new class of distance measures for interval numbers (INs), that takes into account all the points in both intervals, is developed first. Then it is used to form the distance measure for FNs. The method for ranking FNs suggested in this paper is based on comparison of distance from FNs to predetermined crisp ideals of the best and the worst, a concept which is very common to decision makers faced with multi-criteria decision making problems. The paper is organized as follows — The new class of distance measure for INs is introduced in Section 2. Then the new class of distance measure for FNs is developed in Section 3,

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leading to the fuzzy ranking in Section 4 and numerical examples in Section 5. Section 6 is devoted to discussion and conclusions.

2. Distance measure for interval numbers

A new distance measure for two INs is presented in this section. It will be used later to construct a distance measure for FNs.

Let $\boldsymbol{F}(\mathbf{R})$ be the set of INs in $\mathbf{R}$, and the distance between two INs $A(a_1, a_2)$ and $B(b_1, b_2)$ be defined as

$$D^2(A,B) = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \times \left\{ \left[\left(\frac{a_1 + a_2}{2} \right) + x(a_2 - a_1) \right] - \left[\left(\frac{b_1 + b_2}{2} \right) + y(b_2 - b_1) \right] \right\}^2 \mathrm{d}x \, \mathrm{d}y \tag{1}$$

$$= \left[\left(\frac{a_1 + a_2}{2} \right) - \left(\frac{b_1 + b_2}{2} \right) \right]^2 + \frac{1}{3} \left[\left(\frac{a_2 - a_1}{2} \right)^2 + \left(\frac{b_2 - b_1}{2} \right)^2 \right]. \tag{2}$$

It can be proved that $D(A,B) = \sqrt{D^2(A,B)}$ is a distance on $\boldsymbol{F}(\mathbf{R})$. First, $D(A,B) \geqslant 0$. Symmetry is transparent. If $D(A,B) = 0$ then $A = B$. The triangle inequality follows from the fact that the function to be integrated in (1) is the square of Euclidean distance.

Although only the lower and upper bound values of the two INs appear in the Eq. (2), which is derived from Eq. (1) for operational purpose, the integral in (1) shows that this distance takes into account every point in both intervals when computing the distance between those two INs. It is different from most existing distance measures for interval numbers which often use only the lower and upper bound values (e.g., those used in Bárdossy et al. [3], Diamond [9], Diamond and Körner [10], and Diamond and Tanaka [11]). Bertoluzza et al. [5] proposed a distance measure for intervals which also considers every point of both intervals. Its general form, however, is too complicated and the authors later restricted the measure to a particular case with a finite number of considered values for operational purpose.

As an example, consider the crisp number $A(0,0)$ and two INs $B(-1,3)$ and $C(1,3)$. Using the distance measure applied in Diamond [9] and Bárdossy et al. [3], it is found that $d^2(A,B) = d^2(A,C) = 10$. With our distance measure, $D^2(A,B) = 7/3 < D^2(A,C) = \frac{13}{3}$. This result makes more sense as A is inside of B and outside of C, leading to the expectation that the distance from A to B should be smaller than the distance from A to C. In case of crisp numbers, this new distance measure becomes the Euclidean distance. Hence it may be considered as a generalization of the usual Euclidean distance, a feature not seen in the common Hausdorff distance or the "dissemblance index" distance of Kaufman and Gupta [17]. It should be mentioned that the distance measure proposed in this paper has similar characteristics but different formulation than the one suggested in Tran and Duckstein [21].

3. Distance measure for fuzzy numbers

To be able to deal with curvilinear membership functions, generalized left right fuzzy numbers (GLRFN) of Dubois and Prade [12] as described by Bárdossy and Duckstein [2] are used in this section. A fuzzy set $A = (a_1, a_2, a_3, a_4)$ is called a GLRFN if its membership function satisfy the following:

$$\mu(x) = \begin{cases} L\left(\dfrac{a_2 - x}{a_2 - a_1}\right) & \text{for } a_1 \leqslant x \leqslant a_2, \\ 1 & \text{for } a_2 \leqslant x \leqslant a_3, \\ R\left(\dfrac{x - a_3}{a_4 - a_3}\right) & \text{for } a_3 \leqslant x \leqslant a_4, \\ 0 & \text{else} \end{cases} \tag{3}$$

where L and R are strictly decreasing functions defined on $[0,1]$ and satisfying the conditions:

$$L(x) = R(x) = 1 \quad \text{if } x \leqslant 0,$$

$$L(x) = R(x) = 0 \quad \text{if } x \geqslant 1.$$

For $a_2 = a_3$, we have the classical definition of left right fuzzy numbers (LRFN) of Dubois and Prade [12]. Trapezoidal fuzzy numbers (TrFN) are special cases of GLRFN with $L(x) = R(x) = 1 - x$. Triangular fuzzy numbers (TFN) are also special cases of GLRFN with $L(x) = R(x) = 1 - x$ and $a_2 = a_3$.

A GLRFN A is denoted as

$$A = (a_1, a_2, a_3, a_4)_{L^A - R^A} \tag{4}$$

and an α-level interval of fuzzy number A as

$$\begin{aligned} A(\alpha) &= (A_L(\alpha), A_U(\alpha)) \\ &= (a_2 - (a_2 - a_1)L_A^{-1}(\alpha), \\ &\quad a_3 + (a_4 - a_3)R_A^{-1}(\alpha)) \end{aligned} \tag{5}$$

Let $\boldsymbol{F}(\mathbf{R})$ be the set of GLRFNs in $\mathbf{R}$. Using the distance measure for interval numbers defined above, a distance between two GLRFNs A and B can be defined as

$$\begin{aligned} &D^2(A,B,f) \\ &= \left\langle \int_0^1 \left\{ \left[\left(\frac{A_L(\alpha) + A_U(\alpha)}{2} \right) - \left(\frac{B_L(\alpha) + B_U(\alpha)}{2} \right) \right]^2 \right. \right. \\ &\quad + \frac{1}{3} \left[\left(\frac{A_U(\alpha) - A_L(\alpha)}{2} \right)^2 + \left(\frac{B_U(\alpha) - B_L(\alpha)}{2} \right)^2 \right] \Bigg\} \\ &\quad \times f(\alpha)\,\mathrm{d}\alpha \Big\rangle \Big/ \int_0^1 f(\alpha)\,\mathrm{d}\alpha. \end{aligned} \tag{6}$$

Here f, which serves as a weighting function, is a continuous positive function defined on $[0,1]$. The distance is a weighted sum (integral) of the distances between two intervals at all α levels from 0 to 1. It is reasonable to choose f as an increasing function, indicating greater weight assigned to the distance between two intervals at a higher α level.

It can be proved that $D(A,B) = \sqrt{D^2(A,B,f)}$ is a distance on $\boldsymbol{F}(\mathbf{R})$. First, $D(A,B) \geqslant 0$. Symmetry is transparent. If $D(A,B) = 0$ then $A = B$ by the continuity of f, L, and R. The triangle inequality follows from the fact that the function to be integrated in (6) is the square of the distance between two INs presented in the previous section. Since $D(A,B) = |a - b|$ if A and B are crisp numbers, this distance may be considered as a generalization of the usual Euclidean distance.

The calculation of the distance can be reformulated using the following derivation:

$$\begin{aligned} &D^2(A,B,f) \\ &= \int_0^1 \left\{ \left(\frac{a_2 + a_3}{2} - \frac{b_2 + b_3}{2} \right)^2 \right. \\ &\quad + \left(\frac{a_2 + a_3}{2} - \frac{b_2 + b_3}{2} \right) [(a_4 - a_3)R_A^{-1}(\alpha) \\ &\quad - (a_2 - a_1)L_A^{-1}(\alpha) - (b_4 - b_3)R_B^{-1}(\alpha) \\ &\quad + (b_2 - b_1)L_B^{-1}(\alpha)] \\ &\quad + \frac{1}{3} \left(\frac{a_3 - a_2}{2} \right)^2 + \frac{1}{3} \left(\frac{a_3 - a_2}{2} \right) \\ &\quad \times [(a_4 - a_3)R_A^{-1}(\alpha) + (a_2 - a_1)L_A^{-1}(\alpha)] \\ &\quad + \frac{1}{3} \left(\frac{b_3 - b_2}{2} \right)^2 + \frac{1}{3} \left(\frac{b_3 - b_2}{2} \right) \\ &\quad \times (b_4 - b_3)R_B^{-1}(\alpha) + (b_2 - b_1)L_B^{-1}(\alpha)] \\ &\quad + \frac{1}{3}[(a_4 - a_3)^2 (R_A^{-1}(\alpha))^2 + (a_2 - a_1)^2 \\ &\quad \times (L_A^{-1}(\alpha))^2 + (b_4 - b_3)^2 (R_B^{-1}(\alpha))^2 \\ &\quad + (b_2 - b_1)^2 (L_B^{-1}(\alpha))^2] \\ &\quad - \frac{1}{3}[(a_2 - a_1)(a_4 - a_3)L_A^{-1}(\alpha)R_A^{-1}(\alpha) \\ &\quad + (b_2 - b_1)(b_4 - b_3)L_B^{-1}(\alpha)R_B^{-1}(\alpha)] \\ &\quad + \frac{1}{2}[(a_4 - a_3)(b_2 - b_1)R_A^{-1}(\alpha)L_B^{-1}(\alpha) \end{aligned}$$

$$+(a_2-a_1)(b_4-b_3)L_A^{-1}(\alpha)R_B^{-1}(\alpha)$$
$$-(a_4-a_3)(b_4-b_3)R_A^{-1}(\alpha)R_B^{-1}(\alpha)$$
$$\left.-(a_2-a_1)(b_2-b_1)L_A^{-1}(\alpha)L_B^{-1}(\alpha)]\right\}$$
$$\times f(\alpha)\,\mathrm{d}\alpha\Big/\int_0^1 f(\alpha)\,\mathrm{d}\alpha. \tag{7}$$

By introducing:

$$\int_0^1 f(\alpha)\,\mathrm{d}\alpha = S, \tag{8}$$

$$\int_0^1 (L_A^{-1}(\alpha))^n f(\alpha)\,\mathrm{d}\alpha = L_A^{n*}, \tag{9}$$

$$\int_0^1 (R_A^{-1}(\alpha))^n f(\alpha)\,\mathrm{d}\alpha = R_A^{n*}, \tag{10}$$

$$\int_0^1 L_A^{-1}(\alpha)L_B^{-1}(\alpha)f(\alpha)\,\mathrm{d}\alpha = LL_{AB}^{*}, \tag{11}$$

$$\int_0^1 R_A^{-1}(\alpha)R_B^{-1}(\alpha)f(\alpha)\,\mathrm{d}\alpha = RR_{AB}^{*}, \tag{12}$$

$$\int_0^1 L_A^{-1}(\alpha)R_B^{-1}(\alpha)f(\alpha)\,\mathrm{d}\alpha = LR_{AB}^{*}, \tag{13}$$

$$\int_0^1 R_A^{-1}(\alpha)L_B^{-1}(\alpha)f(\alpha)\,\mathrm{d}\alpha = RL_{AB}^{*}, \tag{14}$$

(7) can be rewritten as

$$D^2(A,B)$$
$$=\left(\frac{a_2+a_3}{2}-\frac{b_2+b_3}{2}\right)^2$$
$$+\frac{1}{\mathrm{S}}\left(\frac{a_2+a_3}{2}-\frac{b_2+b_3}{2}\right)[(a_4-a_3)R_A^*$$
$$-(a_2-a_1)L_A^*-(b_4-b_3)R_B^*$$
$$+(b_2-b_1)L_B^*]$$
$$+\frac{1}{3\mathrm{S}}\left(\frac{a_3-a_2}{2}\right)^2+\frac{1}{3\mathrm{S}}\left(\frac{a_3-a_2}{2}\right)$$
$$\times[(a_4-a_3)R_A^*+(a_2-a_1)L_A^*]$$
$$+\frac{1}{3\mathrm{S}}\left(\frac{b_3-b_2}{2}\right)^2+\frac{1}{3\mathrm{S}}\left(\frac{b_3-b_2}{2}\right)$$
$$\times[(b_4-b_3)R_B^*+(b_2-b_1)L_B^*]$$
$$+\frac{1}{3\mathrm{S}}[(a_4-a_3)^2R_A^{**}+(a_2-a_1)^2L_A^{**}$$
$$+(b_4-b_3)^2R_B^{**}+(b_2-b_1)^2L_B^{**}]$$
$$-\frac{1}{3\mathrm{S}}[(a_2-a_1)(a_4-a_3)L_A^*R_A^*$$
$$+(b_2-b_1)(b_4-b_3)L_B^*R_B^*]$$
$$+\frac{1}{2\mathrm{S}}[(a_4-a_3)(b_2-b_1)R_A^*L_B^*$$
$$+(a_2-a_1)(b_4-b_3)L_A^*R_B^*$$
$$-(a_4-a_3)(b_4-b_3)R_A^*R_B^*$$
$$-(a_2-a_1)(b_2-b_1)L_A^*L_B^*]. \tag{15}$$

Table 1 gives the equations to compute distance for some of the commonly used fuzzy numbers with two different weighting functions: $f(\alpha)=1$ representing equal weights for intervals at different α levels, and $f(\alpha)=\alpha$ indicating more weight given to intervals at higher α level. As a numerical example, consider crisp number $A(1)$, TFN $\mathrm{B}(2,2,4)_{\mathrm{T}}$, and TrFN $\mathrm{C}(0,0,2,4)_{\mathrm{Tr}}$ (Fig. 1). Let $f(\alpha)=\alpha$. Using the Hagaman distance (D_{H}) measure applied in Bárdossy et al. [3], it is found that $D_{\mathrm{H}}^2(A,B)=D_{\mathrm{H}}^2(A,C)=2$. With our distance measure, $D^2(A,B)=17/9>D^2(A,C)=7/9$. This result makes more sense as A is outside of B and inside of C, leading to the expectation that the distance from A to B should be greater than the distance from A to C.

Table 1
Distance functions for some commonly used FNs

Fuzzy numbers	$f(\alpha)$	$D_T^2(A,B,f)$
Trapezoidal fuzzy numbers $A=(a_1,a_2,a_3,a_4)_{\mathrm{Tr}}$ $B=(b_1,b_2,b_3,b_4)_{\mathrm{Tr}}$	α	$\left(\frac{a_2+a_3}{2}-\frac{b_2+b_3}{2}\right)^2+\frac{1}{3}\left(\frac{a_2+a_3}{2}-\frac{b_2+b_3}{2}\right)[(a_4-a_3)-(a_2-a_1)-(b_4-b_3)+(b_2-b_1)]$ $\frac{2}{3}\left(\frac{a_3-a_2}{2}\right)^2+\frac{1}{9}\left(\frac{a_3-a_2}{2}\right)[(a_4-a_3)+(a_2-a_1)]+\frac{2}{3}\left(\frac{b_3-b_2}{2}\right)^2+\frac{1}{9}\left(\frac{b_3-b_2}{2}\right)[(b_4-b_3)+(b_2-b_1)]$ $+\frac{1}{18}[(a_4-a_3)^2+(a_2-a_1)^2+(b_4-b_3)^2+(b_2-b_1)^2]-\frac{1}{18}[(a_2-a_1)(a_4-a_3)+(b_2-b_1)(b_4-b_3)]$ $+\frac{1}{12}[(a_4-a_3)(b_2-b_1)+(a_2-a_1)(b_4-b_3)-(a_4-a_3)(b_4-b_3)-(a_2-a_1)(b_2-b_1)]$
	1	$\left(\frac{a_2+a_3}{2}-\frac{b_2+b_3}{2}\right)^2+\frac{1}{2}\left(\frac{a_2+a_3}{2}-\frac{b_2+b_3}{2}\right)[(a_4-a_3)-(a_2-a_1)-(b_4-b_3)+(b_2-b_1)]$ $\frac{1}{3}\left(\frac{a_3-a_2}{2}\right)^2+\frac{1}{6}\left(\frac{a_3-a_2}{2}\right)[(a_4-a_3)+(a_2-a_1)]+\frac{1}{3}\left(\frac{b_3-b_2}{2}\right)^2+\frac{1}{6}\left(\frac{b_3-b_2}{2}\right)[(b_4-b_3)+(b_2-b_1)]$ $+\frac{1}{9}[(a_4-a_3)^2+(a_2-a_1)^2+(b_4-b_3)^2+(b_2-b_1)^2]-\frac{1}{9}[(a_2-a_1)(a_4-a_3)+(b_2-b_1)(b_4-b_3)]$ $+\frac{1}{6}[(a_4-a_3)(b_2-b_1)+(a_2-a_1)(b_4-b_3)-(a_4-a_3)(b_4-b_3)-(a_2-a_1)(b_2-b_1)]$
Triangular fuzzy numbers $A=(a_1,a_2,a_3)_{\mathrm{T}}$ $B=(b_1,b_2,b_3)_{\mathrm{T}}$	α	$(a_2-b_2)^2+\frac{1}{3}(a_2-b_2)[(a_3+a_1)-(b_3+b_1)]+\frac{1}{18}[(a_3-a_2)^2+(a_2-a_1)^2+(b_3-b_2)^2+(b_2-b_1)^2]$ $-\frac{1}{18}[(a_2-a_1)(a_3-a_2)+(b_2-b_1)(b_3-b_2)]-\frac{1}{12}(2a_2-a_1-a_3)(2b_2-b_1-b_3)$
	1	$(a_2-b_2)^2+\frac{1}{2}(a_2-b_2)[(a_3+a_1)-(b_3+b_1)]+\frac{1}{9}[(a_3-a_2)^2+(a_2-a_1)^2+(b_3-b_2)^2+(b_2-b_1)^2]$ $-\frac{1}{9}[(a_2-a_1)(a_3-a_2)+(b_2-b_1)(b_3-b_2)]+\frac{1}{6}(2a_2-a_1-a_3)(2b_2-b_1-b_3)$

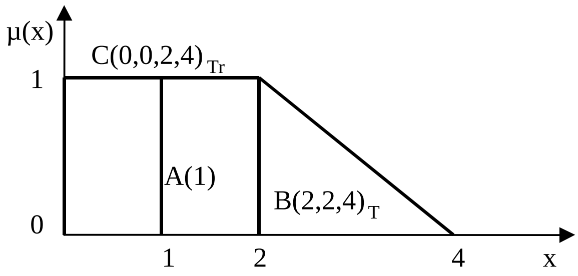

Fig. 1. Illustration of the distance measure for fuzzy numbers.

4. Ranking based on distance measure for fuzzy numbers

The method for ranking FNs suggested in this paper is based on comparison of distance from FNs to some predetermined targets: the crisp maximum (Max) and the crisp minimum (Min). The idea is that a FN is ranked first if its distance to the crisp maximum ($D_{\max}$) is the smallest but its distance to the crisp minimum ($D_{\min}$) is the greatest. If only one of these conditions is satisfied, a FN might be outranked the others depending upon context of the problem (for example, the attitude of the decision-maker in a decision situation). This point will be discussed further in the next section.

The Max and Min are chosen as follows:

$$\mathrm{Max}(\boldsymbol{I}) \geqslant \sup\left(\bigcup_{i=1}^{I} s(A_i)\right),$$

$$\mathrm{Min}(\boldsymbol{I}) \leqslant \inf\left(\bigcup_{i=1}^{I} s(A_i)\right),$$

where $s(A_i)$ is the support of FNs A_i, $i=1,\dots,I$. Then $D_{\max}$ and $D_{\min}$ of FN A can be computed as follows:

$$\begin{aligned}
D^2(A,\mathrm{M}) = {} & \left(\frac{a_2+a_3}{2}-\mathrm{M}\right)^2 + \frac{1}{\mathrm{S}}\left(\frac{a_2+a_3}{2}-\mathrm{M}\right) \\
& \times[(a_4-a_3)R_A^* - (a_2-a_1)L_A^*] \\
& + \frac{1}{3\mathrm{S}}\left(\frac{a_3-a_2}{2}\right)^2 + \frac{1}{3\mathrm{S}}\left(\frac{a_3-a_2}{2}\right) \\
& \times[(a_4-a_3)R_A^* + (a_2-a_1)L_A^*] \\
& + \frac{1}{3\mathrm{S}}[(a_4-a_3)^2R_A^{**} + (a_2-a_1)^2L_A^{**}] \\
& - \frac{1}{3\mathrm{S}}[(a_2-a_1)(a_4-a_3)L_A^*R_A^*] \qquad (16)
\end{aligned}$$

where M is either Max or Min. Hence, $D_{\max} = \sqrt{D^2(A,\mathrm{Max})}$ and $D_{\min} = \sqrt{D^2(A,\mathrm{Min})}$. Table 2 gives the equations to compute $D_{\max}$ and $D_{\min}$ for some of the commonly used fuzzy numbers with two different weighting functions: $f(\alpha)=1$ and $f(\alpha)=\alpha$.

5. Numerical examples

Fig. 2 and Table 3 show several typical examples to illustrate the current method and compare it with some other ranking methods. Most of results for other methods are adapted from the data in Bortolan and Degani [6], Lee and Li [19], and Chen and Hwang [8].

For example (a), a fair ranking is given by all methods, complying with human intuition. Such agreeable ranking however is not seen in example (b), where two symmetrical FNs have the same mode but different supports. In this case, a FN may be preferable or equal to the other, depending in a decision situation for example, on the attitude of the decision-maker (e.g., risk-prone or risk-averse). Hence a sufficient ranking method should provide meaningful indices for all three alternatives (e.g., $A_1=A_2, A_1>A_2$, or $A_1<A_2$). All methods except Dubois and Prade's [9] and the current method give only one answer, either non-discrimination of two FNs or in favor of one over another. Note that the Fortemps and Roubens's method of "area of compensation" [14], which is considered robust with compensation, linearity, and additivity properties [20], also gives a non-discriminative result for this example. Considering that the FNs represent some measure of risk or gain. To explain the result from their method, Lee and Li [19] suggest that human intuition would favor the FN with smaller spread. This explanation however might not be true for a risk-prone decision-maker, who may favor the FN with larger spread. On the other hand, four indices in Dubois and Prade's provide sufficient information for the decision maker to choose one FN over the other, de-

Table 2
Ranking functions for some commonly used FNs

Fuzzy numbers	$f(\alpha)$	$D^2(A,M^*,f)$
Trapezoidal fuzzy numbers $A=(a_1,a_2,a_3,a_4)_{\mathrm{Tr}}$	α	$\left(\frac{a_2+a_3}{2}-M\right)^2+\frac{1}{3}\left(\frac{a_2+a_3}{2}-M\right)[(a_4-a_3)-(a_2-a_1)]$ $\frac{2}{3}\left(\frac{a_3-a_2}{2}\right)^2+\frac{1}{9}\left(\frac{a_3-a_2}{2}\right)[(a_4-a_3)+(a_2-a_1)]+\frac{1}{18}[(a_4-a_3)^2+(a_2-a_1)^2]-\frac{1}{18}[(a_2-a_1)(a_4-a_3)]$
	1	$\left(\frac{a_2+a_3}{2}-M\right)^2+\frac{1}{2}\left(\frac{a_2+a_3}{2}-M\right)[(a_4-a_3)-(a_2-a_1)]$ $\frac{1}{3}\left(\frac{a_3-a_2}{2}\right)^2+\frac{1}{6}\left(\frac{a_3-a_2}{2}\right)[(a_4-a_3)+(a_2-a_1)]+\frac{1}{9}[(a_4-a_3)^2+(a_2-a_1)^2]-\frac{1}{9}[(a_2-a_1)(a_4-a_3)]$
Triangular fuzzy numbers $A=(a_1,a_2,a_3)_{\mathrm{T}}$	α	$(a_2-M)^2+\frac{1}{3}(a_2-M)[(a_3+a_1)-2M]+\frac{1}{18}[(a_3-a_2)^2+(a_2-a_1)^2]-\frac{1}{18}[(a_2-a_1)(a_3-a_2)]$
	1	$(a_2-M)^2+\frac{1}{2}(a_2-M)[(a_3+a_1)-2M]+\frac{1}{9}[(a_3-a_2)^2+(a_2-a_1)^2]-\frac{1}{9}[(a_2-a_1)(a_3-a_2)]$

*M is either Max or Min.

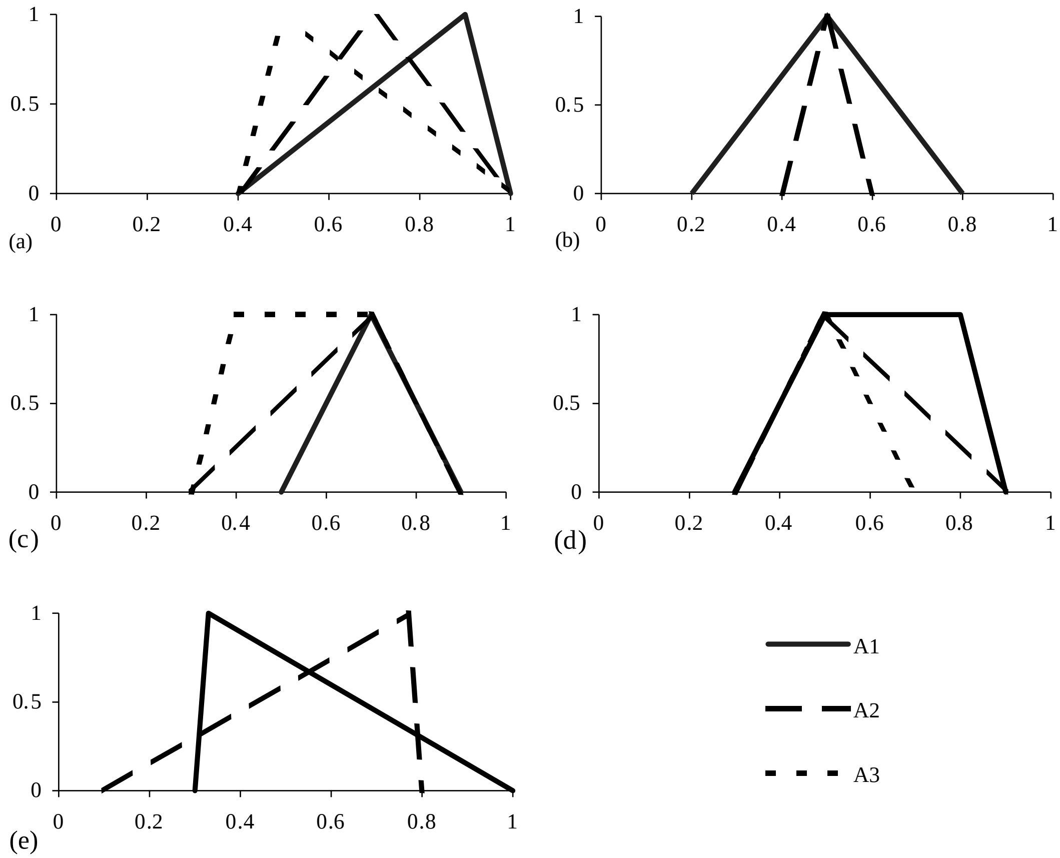

Fig. 2. Examples of FN comparison.

pending on what index is used dominantly. For the current method, if D_{max} is used, A_2 is preferable since $D_{max}(A_2) < D_{max}(A_1)$. Conversely, if D_{min} is used, A_1 is preferable as $D_{min}(A_1) > D_{min}(A_2)$. This conflict is understandable because A_1 may be preferable or less preferable due to its larger right and left dispersion, respectively. On this account, a risk-prone decision maker may want to use D_{min} in selecting an alternative, while a risk-averse person may prefer D_{max}. If both D_{max} and D_{min} are used in conjunction, A_1 and A_2 may be ranked equally. Hence the current method provides an appropriate three-way explanation for ranking in this case. This feature is not seen in other methods except that of Dubois and Prade [13].

For example (c), three FNs A_1, A_2, and A_3 are different on the left side. Although intuition would yield $A_1 > A_2 > A_3$, non-discriminative result is seen using several methods. Example (d) is a mirror case of example (c), where three FNs A_1, A_2, and A_3 are only different on the right side. Different from result in previous example, all methods but the Bass and Kwakernaak's [4] give the same result here. This shows that several methods only consider a partial set of attributes of FNs being compared, especially those on the right-hand side.

Example (e) is very typical in comparing fuzzy numbers where intuition is not as obvious as in other examples. Some methods favor A_1 over A_2 while others give a reverse result. For the Dubois and Prade's method, as four indices are not identical, the decision-maker needs to select which index (or indices) to use in deriving ranking order. With the Lee and Li's method, there is conflict in results when two different distributions — uniform and proportional — are

Table 3
Examples of FN comparison

Methods		(a)			(b)		(c)			(d)			(e)	
		A_1	A_2	A_3	A_1	A_2	A_1	A_2	A_3	A_1	A_2	A_3	A_1	A_2
Yager [16–18]	F_1	0.760	0.700	0.630	0.500	0.500*	0.700	0.630	0.570	0.620	0.560	0.500	0.610	0.530
	F_2	0.900	0.760	0.660	0.610	0.540	0.750	0.750	0.750*	0.810	0.640	0.580	0.660	0.690
	F_3	0.800	0.700	0.600	0.600	0.500	0.700	0.650	0.570	0.620	0.540	0.500	0.580	0.560
Bass & Kwakernaak [4]		1.000	0.740	0.600	1.000	1.000*	1.000	1.000	1.000*	1.000	1.000	1.000*	0.840	1.000
Baldwin & Guild [1]	$1:p$	0.420	0.330	0.300	0.270	0.270*	0.370	0.270	0.270*	0.450	0.370	0.270	0.420	0.330
	g	0.550	0.400	0.340	0.300	0.240	0.420	0.350	0.350*	0.530	0.400	0.280	0.440	0.370
	$r:a$	0.280	0.230	0.220	0.200	0.230	0.270	0.190	0.190*	0.310	0.280	0.210	0.340	0.240
Kerre [13]		1.000	0.860	0.760	0.910	0.910*	1.000	0.910	0.750	1.000	0.850	0.750	0.960	0.890
Jain [11,12]	$k=1$	0.900	0.760	0.660	0.730	0.670	0.820	0.820	0.820*	0.900	0.690	0.640	0.660	0.690
	$k=2$	0.840	0.650	0.540	0.600	0.480	0.710	0.710	0.710	0.820	0.560	0.450	0.530	0.510
	$k=1/2$	0.950	0.860	0.780	0.830	0.800	0.890	0.890	0.890	0.940	0.800	0.770	0.780	0.810
Dubois & Prade [9]	PD	1.000	0.740	0.600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.840	1.000
	PSD	0.740	0.230	0.160	0.730	0.240	0.500	0.500	0.500	0.800	0.200	0.000	0.540	0.460
	ND	0.630	0.380	0.180	0.270	0.760	0.670	0.350	0.000	0.500	0.500	0.500	0.540	0.460
	NSD	0.260	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.160
Lee & Li [14]	$U \cdot m$	0.760	0.700	0.630	0.500	0.500	0.700	0.630	0.570	0.620	0.560	0.500	0.610	0.530
	$U \cdot G$	—	—	—	0.120	0.040	—	—	—	—	—	—	—	—
	$P \cdot m$	0.800	0.700	0.600	0.500	0.500	0.700	0.650	0.580	0.630	0.550	0.500	0.530	0.580
	$P \cdot G$	—	—	—	0.090	0.030	—	—	—	—	—	—	—	—
Fortemps & Roubens [10]	F_0	0.800	0.700	0.600	0.500	0.500*	0.700	0.650	0.575	0.625	0.550	0.500	0.490	0.610
Tran & Duckstein	$D_{\max}; f(x)=x$	0.187	0.308	0.442	0.505	0.501	0.304	0.342	0.457	0.395	0.473	0.502	0.574	0.355
	$D_{\min}; f(x)=x$	0.838	0.704	0.573	0.505	0.501	0.702	0.671	0.585	0.650	0.539	0.502	0.451	0.673
	$D_{\max}; f(x)=1$	0.231	0.316	0.416	0.510	0.501	0.307	0.365	0.445	0.398	0.462	0.504	0.531	0.417
	$D_{\min}; f(x)=1$	0.808	0.707	0.611	0.510	0.501	0.703	0.658	0.590	0.639	0.560	0.504	0.512	0.628

*Cases of indiscrimination.

used. Lee and Li [19] suggest that when conflict occurs as in this example, the proportional distribution seems more reasonable. However, a difficulty is that the choice of either distribution is arbitrary [8], without a clear motivation for either choice. On the other hand, the current approach prefers A_2 to A_1 with an agreeing result of $D_{\max}$ and $D_{\min}$ for both cases of $f(\alpha)=\alpha$ and $f(\alpha)=1$. As $D_{\max}$ and $D_{\min}$ reveal the distance of FNs to the reference points, this result is understandable and plausible.

6. Discussion and conclusions

The criteria suggested by Zhu and Lee [27] are used to evaluate the current ranking method. They include complexity, ease of interpretation, robustness, flexibility, and transitivity.

- Complexity can be judged via the amount of computation to accommodate the ranking. Eq. (16) and those in Table 2 show that the computation of $D_{\max}$ and $D_{\min}$ in the current method is straightforward and can be programmed easily. This is a very good practical aspect which is not seen in several other methods. Furthermore, this is an absolute ranking and no pairwise comparison of FNs is necessary, making the computation process simple and transparent.
- Ease of interpretation is one of the most crucial criteria for the decision-maker. With the current method, the distance to the reference (crisp) points is a very common concept and can be easily understood by decision maker(s). On the other hand, some concepts in other methods, such as concepts of possibility and necessity in Dubois and Prade's [13], or probability measure in Lee and Li [19], are not easy to perceive by decision maker(s).
- Robustness refers to the ability of consistent ranking for a diversity of cases. Among the methods in Table 3, only the Dubois and Prade's [13] and the current method show their ability of distinguish different FNs robustly and consistently. Note that a mixed comparison of FNs and crisp numbers can be done using the current method.
- Flexibility, according to Zhu and Lee [27], is the ability of providing more than one index and/or allowing the participation of decision makers. Regarding this criterion, the $D_{\max}$ and $D_{\min}$ indices of the current method are meaningful and efficient for ranking purpose. In general, these two indices are in agreement, providing a consistent ranking order. If conflict does occur, they provide purposeful information for the decision-maker to reach a decision, depending upon his/her attitude towards "risk". On the other hand, the presence of the weighting function f allows the participation of the decision-maker in a flexible way. For example, when the decision-maker is risk-neutral, $f(\alpha)=\alpha$ seems to be reasonable. A risk-averse decision-maker might want to put more weight on information at higher α level by using other functions, such as $f(\alpha)=\alpha^2$ or a higher power of α. For a risk-prone decision-maker, a constant ($f(\alpha)=1$), or even a decreasing function f can be utilized.
- Transitivity refers to the ability of giving a consistent conclusion in the comparison of more than two FNs. As the current method uses a distance function to map FNs to real numbers which can be ordered linearly, the transitive property of this method is satisfied.

In general, the current ranking method possesses several good characteristics and advantages as compared to other existing ranking methods.

In conclusion, the distance measures for INs and FNs introduced in this paper are considered to be at least as reasonable as other existing distance measures. They can be used in other applications, such as fuzzy regression or fuzzy goal programming. The current ranking method can be a valuable tool in a variety of problems (e.g., fuzzy scheduling, fuzzy control) due to its simplicity, ease of interpretation, and effectiveness in ranking FNs.

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