Fuzzy Qualitative Simulation

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Abstract— **The theory of fuzzy sets and the development of qualitative reasoning have had similar motivations: coping with complexity in reasoning about the properties of physical systems. An approach is described that utilizes fuzzy sets to develop a fuzzy qualitative simulation algorithm that allows a semiquantitative extension to qualitative simulation, providing three significant advantages over existing techniques. Firstly, it allows a more detailed description of physical variables, through an arbitrary, but finite, discretisation of the quantity space. The adoption of fuzzy sets also allows common-sense knowledge to be represented in defining values through the use of graded membership, enabling the subjective element in system modelling to be incorporated and reasoned with in a formal way. Secondly, the fuzzy quantity space allows more detailed description of functional relationships in that both strength and sign information can be represented by fuzzy relations holding against two or multivariables. Thirdly, the quantity space allows ordering information on rates of change to be used to compute temporal durations of the state and the possible transitions. Thus, an ordering of the evolution of the states and the associated temporal durations are obtained. This knowledge is used to develop an effective temporal filter that significantly reduces the number of spurious behaviors. Experimental results with the algorithm are presented and comparison with other recently proposed methods is made.**

I. Introduction

THE LAST DECADE has seen significant progress towards the development of formal methods for qualitative reasoning about the behavior of physical systems. Such reasoners offer the prospect of coping with the inherent complexity found in today's technological systems and of providing the basis for automated solutions to an extended range of application problems. In particular, qualitative modelling forms the heart of model-based reasoning techniques for automated diagnosis, control and industrial training systems. However, a number of important technical problems remain to be solved in order to make qualitative techniques viable for full-scale industrial applications [30].

At present these limitations fall into two non-distinct classes:

(1) qualitative ambiguity; and
(2) lack of temporal information.

The first occurs because of the inherent ambiguity of qualitative calculi [49]. In qualitative simulation, for example QSim [22], this appears as the generation of numerous spurious behaviors that tend to obscure the real behavior, making the results impracticable except for relatively small and (over) simplified problems. Reducing these spurious behaviors is a main occupation of this particular research area. Two approaches are being followed: the development of filtering techniques that utilise either system theoretic properties, e.g., the nonintersecting constraint [26], [48] and the energy constraint [18] or additional knowledge from some other source, e.g., experimental, and the development of more descriptive qualitative representations that allow more detailed information on the system to be incorporated, thereby reducing the ambiguity at source. In principle, the ambiguity cannot be eliminated due to the inherent essence of qualitative representation, therefore, a combination of these methods will be required.

The second limitation occurs because current techniques have no explicit information on temporal durations [45], [56]. Again, two approaches to this problem are being developed. The first associates an explicit temporal duration with each qualitative state [56]. Temporal constraint propagation methods are then used to propagate both the qualitative values and their durations [55]. The propagation of durations realises a form of temporal reasoning independently of the qualitative values. The second approach extends the representation of the qualitative state to allow ordering information on the relative rates of change of variables [8], [52], [53]. This allows a temporal duration, called the persistence time, to be calculated.

This paper, based on the earlier work reported by the authors [42], [43], presents a Fuzzy Simulation System that utilises the theory of fuzzy sets [63] to give an arbitrary, but finite, discretisation of the representation of system variables. The adoption of fuzzy sets allows common-sense knowledge to be incorporated in the interpretation of values through the use of graded membership. Such a system also allows both strength and sign information on the functional relationships holding against two or multi-variables to be represented, resulting in a considerable reduction of the inherent ambiguity of qualitative calculi. Further, this simulation system allows ordering information on the rates of change of variables to be modelled, producing a measure of how long the system remains within a particular state and/or when a state transition occurs. Thus, an ordering of the evolution of system states *and* the associated temporal durations are obtained. This makes qualitative simulation much more suitable for use within application systems, such as diagnosis [44], [45], control, and training. In so doing, the fuzzy simulation system combines the previously disparate fields of qualitative simulation and fuzzy sets, both of which were introduced to cope with the complexities found in reasoning about the behavior of physical systems.

The use of graded membership to represent imprecision within fuzzy sets allows the subjective element of common-sense knowledge to be incorporated within a formal algorithm for generating behavior from a structural model, thereby, com-

Manuscript received March 3, 1991; revised November 6, 1992 and January 4, 1993.

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IEEE Log Number 9209628.

bining theoretical knowledge [30], in the form of modelling constraints, and empirical knowledge, in the interpretation of system values. Furthermore, the use of fuzzy sets allows a rigorous approach to capturing uncertainty and provides a unifying framework from which other recently developed techniques can be generated as specializations [46].

The following section describes the background and motivation for this work and relates the developments presented herein to other recent work on qualitative simulation. This section also serves as a motivation for the use of fuzzy sets in qualitative reasoning. There then follows a short section giving the essential aspects of the theory of fuzzy sets that are fundamental to the development of the algorithm. The next three sections are organised by describing the use of fuzzy sets to represent the system variables—i.e., the fuzzy quantity space, the extended modelling primitives, that are allowed through the use of a fuzzy quantity space; and, the simulation algorithm that results from assuming continuity to determine state transitions within the quantity space. Section VII gives the results of applying the algorithm to two commonly used verification examples. Finally, the paper is concluded with a summary of the major results of the algorithm, a discussion on the planned extensions and a brief description of its potential usage within application systems.

II. Background

One research strategy for developing qualitative representations is to search for a qualitative mathematics capable of yielding significant results from a minimum of information. However, in practice there is usually some quantitative knowledge about the system being modelled, though rarely enough to specify a complete numerical calculation. Typical methods of qualitative simulation for the description of a physical system are too restrictive to use this knowledge because of the fundamental limitations in the representations of the physical structure as follows:

(1) A weak representation of functional dependencies between variables; typically, this captures sign information but has no measure of strength of dependency;
(2) Only sign information on the rate-of-change of the variables is represented, lacking ordering information amongst the rates of change;
(3) No explicit representation of time.

It is, therefore, desirable to develop qualitative simulation techniques which permit a more detailed description of quantities and functional relationships than existing qualitative reasoners, and yet requires neither the precision of real numbers nor exact and complex relations between variables so as to allow effective and efficient model-based reasoning tasks to be performed. Also, a technique which can qualitatively generate a physical system's behavior in terms of its state sequence with associated temporal durations will significantly enhance the viability for real application of such approaches.

A. A Brief Review of Recent Approaches

Recently, important work [8], [33], [35], [52], [53] has been done to develop abstractions of the real numbers that reduce the amount of qualitative ambiguity. As a result, less spurious behaviors are generated, though the reduction is, of course, determined by the kind of task to which these techniques are applied. Although there are many existing qualitative reasoning techniques that are not covered by this subsection, the work briefly reviewed herein is directly linked to the particular qualitative simulation technique to be developed later and a much more detailed version of this review is reported in [46]. A good collection of recent important contributions to qualitative reasoning, however, can be found in [54].

The method given in [35], named *FOG,* is concerned with reasoning with relative magnitudes, i.e., with the analysis of physical systems in which one quantity is much greater than another, or in the comparison of two systems of the same structure, but which have corresponding quantities of very different magnitudes. By this, some secondary effects of physical systems can be eliminated whilst the main properties of the systems are maintained. However, this method is developed for algebraic equations only and does not allow the simulation of dynamic behaviors modelled by differential equations. Also, the set of *FOG*'s inference rules, defined on the basis of the theory of nonstandard analysis [36], is somewhat arbitrary and, therefore, it has to rely on other control techniques to govern the application of the rules.

An extended approach to reasoning with orders of magnitude was presented in [33], based on seven primitive relations holding amongst system variables and/or the values of the variables such as "much smaller than" and "moderately larger than." Each of these relations is interpreted in terms of the location of the quotient of the two compared quantities within some interval, with all such intervals (which are disjoint) being defined with respect to a unique parameter chosen according to domain knowledge. This provides a concrete semantics for the relations and allows "exact" inferences to be performed. However, as pointed out in [33], such an interpretation is too strict compared to human reasoning. Actually, it is very difficult to determine the unique boundary between two disjoint intervals used to describe two different relations. Henceforth, a heuristic interpretation is further given by replacing the boundary points of the intervals with (crisp) regions, thereby allowing inferences more aggressive and human-like. Nevertheless, this approach does not remove the boundary problem but shifts the difficulty in choosing the boundary between the two intervals indicating two different relations onto that between an interval and a region which substitutes the (point) boundary. In addition, this approach remains a method for performing algebraic reasoning only and, is, therefore, restricted to static models.

To reason about the qualitative behavior of dynamic systems, a technique called *CHEPACHET* was established that makes use of the information provided by order of magnitude in solving qualitative constraints by representing quantities and derivatives in terms of distinguished subsets of the nonstandard real line [36]. This approach combines order of magnitude reasoning with qualitative simulation techniques. However, it allows only a fixed quantity space of orders of magnitude of the system variables and, therefore, the algebraic operation system has to be changed whenever it is used to incorporate different sets of orders of magnitude. A similar approach, called *HR-QSIM*, was presented in [52], [53], which extends the quantity space of *QSim* [22] to the hyperreals that includes infinitesimal and infinite values. Although it is more expressive in the magnitude values of system variables than *CHEPACHET*, the qualitative derivatives are still restricted to four orders of magnitude, namely, 0, *negl* (infinitesimal), *fin* (ordinary real number), and ∞ (infinite). Being a subroutine of a technique developed for solving comparative analysis problems, *HR-QSIM* performs qualitative calculations within the hyperreals. Unfortunately, as with previously mentioned other order-of-magnitude reasoners, it cannot be directly applied for simulating real physical systems without encounting difficulties of choosing boundaries between adjacent qualitative values [46]. In addition, another limitation of *HR-QSIM* is that it does not indicate how to calculate the arrival time, a key notion in *HR-QSIM*, for transitions where the predecessor and successor states of a variable have different rates of change.

It is important, however, to notice that both *CHEPACHET* and *HR-QSIM* provide a significant extension to the traditional representation of qualitative derivatives, where only three abstract symbolic values $\{+, 0, -\}$ are possible. This allows ordering information on the rates of change of system variables to be used to calculate a temporal duration associated with each qualitative state, though the resulting durations are in a very exaggerated form. In so doing, powerful temporal filtering methods can be developed that eliminate many of the spurious behaviors which cannot be ruled out by conventional filtering techniques. Such a significant advantage enables considerable progress towards effective qualitative reasoning about continuous dynamic physical systems. Nevertheless, in practice, it is difficult to choose the (sharp) boundaries of the key intervals required by the use of Non-Standard Analysis upon which these approaches are built (including *FOG*), resulting in non-intuitive interpretations of the set of behaviors [46]. A further common limitation is that the information on the strength of the functional dependencies between variables is still rather weak. In many cases partial quantitative information is available and should be utilised to improve the predictive power of these qualitative simulation algorithms.

There has also been considerable work relevant to the integration of quantitative and qualitative knowledge [16], [21], [23], [38], [39], [57] to reduce ambiguities, though each has utilized a different methodology to handle the quantitative information. For example, Kuipers and Berleant use incomplete quantitative information to augment the qualitative descriptions so that, by propagating upper and lower numerical bounds across constraints, such information provides knowledge about variables whose exact (landmark) values are not known [23]. Another example is the approach developed by Williams [57], where traditional three-valued symbolic algebra, i.e., the algebraic system $(\{-, 0, +\}, \hat{+}, \hat{-}, \hat{\times}, \hat{/})$, is merged with the ordinary real algebra. As such quantities may first be operated on by conventional arithmetic operators, while the results of these operations are then abstracted to the traditional symbolic values and classical symbolic operations are further evaluated. This produces a result that is less ambiguous than that produced by the traditional qualitative operations alone.

In spite of these significant developments, the area of qualitative reasoning is still embryonic and much remains to be done. It is clear that an extension to current qualitative simulation methods, to make them capable of capturing and using both sign and strength information on variable values and functional dependencies and extracting explicit temporal information, based on well-developed mathematical techniques, would greatly enhance the effectiveness of qualitative reasoning for approaches where such information is available.

B. Motivations for the Use of Fuzzy Sets

An alternative, and distinct, approach to coping with complexity in physical system modelling by using fuzzy sets has been proposed [64]. The theory of fuzzy sets has as its main aim the development of a methodology for the formulation and solution of problems that are too complex or ill-defined to be susceptible to analysis by conventional techniques. It deals with a subset of a universe of discourse, where the transition between full membership of a set and no membership is gradual rather than abrupt. Such subsets—called fuzzy sets—arise, for instance, when descriptions of ambiguity, vagueness, and ambivalence in the mathematical models of physical systems are needed. In the real world, the attributes of the system variables often emerge from an elusive vagueness or fuzziness, a readjustment to context, or an effect of human imprecision. The use of the "soft" boundaries of fuzzy sets, i.e., the graded memberships, allows subjective knowledge to be utilised in defining these attributes. With the accumulation of knowledge the subjectively assigned memberships can, of course, be modified [11]. Even in some cases where precise models are available, fuzziness may be a concomitant of complexity.

Advantages resulting from the adoption of fuzzy sets ease the requirement for encoding the knowledge about physical systems. As a simple example, a fuzzy set "*approximately 4*", within the real line R, can be depicted by Fig. 1, where $\mu_{Four}(x)$ denotes the degree of membership that an $x, x \in R$, belongs to the set "approximately 4." It is natural to appeal that 4 belongs to this set with a full membership $\mu_{Four}(4) = 1$ and 3.8 belongs to it with a membership, say, 0.8, whereas 10 does not belong to the set, i.e., $\mu_{Four}(10) = 0$. Considering another example, although the labels *small*, *medium*, and *large* have an intuitive appeal, if we try to use these labels to represent some values of physical variables by interpreting them with crisp intervals, such as *small* $= \{x \mid x > 0, x \ll 1\}$ and *medium* $= \{x \mid x > 0, x \sim 1\}$ as in *CHEPACHET* [8], it would then result in a non-intuitive representation. This is because, in practice, it is not realistic to draw an exact boundary between these sets [46]. Actually, when encoding a particular real number there may be difficulty in deciding which of the two sets the number should definitely belong to. It may well be the case that we can only say this number belongs to the *small* set with possibility A and to the *medium* one with possibility B. This results in the commonly shared boundary interpretation problem within recent approaches to

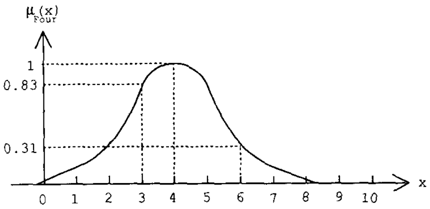

Fig. 1. Approximately four.

order-of-magnitude reasoning as indicated in the preceding subsection and reflects exactly the situation that requires gradual membership and no excluded-middle law; both of which are fundamental motivations for the development of fuzzy mathematics [46].

It is important, here, to notice that the concept of a fuzzy set is not merely a disguised form of subjective probability, although both the theory of fuzzy sets and the theory of probability are developed to attain realistic solutions to problems in decision analysis under uncertainty. In essence, fuzzy set theory is aimed at dealing with sources of uncertainty or imprecision that are inherently vague and nonstatistical in nature. For example, the proposition "*X is a large number,*" in which *large number* is a label of a fuzzy subset of nonnegative integers, defines a *possibility distribution* rather than the *probability distribution* of X. This implies that, if the degree to which an integer I fits our subjective perception of *large number* is $\mu(I)$, then the possibility that X may take I as its value is numerically equal to $\mu(I)$. Consequently, such a proposition conveys no information about the probability distribution of the values of X. Thus, probability theory does not provide an adequate tool for the analysis of problems in which the available information, like this proposition, is incomplete, imprecise, or unreliable. Attempts to utilise probability theory in qualitative reasoners have been made [12]. However, as argued above this does not fulfil one of the basic motivations of qualitative modelling: that of capturing human vagueness.

Since the middle of the seventies, fuzzy mathematics has been applied to many different fields of information processing, such as pattern recognition, signal processing, and process control [19], [25], [32], [37], [40]. It is also being incorporated within systems based on Artificial Intelligence techniques, and in particular, within Knowledge Based Systems, as a method of representing and handling knowledge [4], [65], [68]. These applications demonstrate that fuzzy set theory provides a conceptual framework for the solution of imprecisely formulated problems.

Considering the motivation for developing qualitative modelling techniques and the introduction of fuzzy sets, it can be seen that both cases are rooted in a common mainstay—handling physical world qualitatively instead of quantitatively and explicitly coping with the essential uncertainty or incompleteness of practical models. We assert that qualitative reasoning based on current qualitative modelling techniques and approximate reasoning based on fuzzy set theory are two strands of Artificial Intelligence research that are essential to developing and capturing common-sense reasoning. Unfortunately, these two approaches have been developed by separate communities, and very little cross-fertilization has occurred; only two basic attempts exist to combine qualitative reasoning and fuzzy sets [5], [6], [14].

Developed within the qualitative process theory (QPT) framework [17], the technique reported in [5], [6] adds one (fuzzy) strength label (termed the sensitivity) to a functional relationship that is originally represented by an unbound monotonic function and then propagates such labels through given cross relationships amongst physical quantities by a fuzzy relation algorithm. In so doing, it resolves some ambiguities resulting from reasoning with the original *QPT* but, as indicated in [5], requires explicit domain and/or system specific knowledge about the relationships between system variables. One important point is that, as with the basic *QPT* and other conventional qualitative reasoners, this technique does not generate state durations, although it should be possible to do so. *Dubois* and *Prade* developed a method for order of magnitude reasoning based on the use of fuzzy relations [14], by noticing that the extent to which two numerical values satisfy the basic relational operators of FOG [35], say, a quantity is negligible to another, is often a matter of degree. Modified relational operators allow reasoning about closeness and negligibility to be made in a rigorous way without being obliged to introduce arbitrary limitations on the chaining of the inference rules. In fact, when reasoning with relative orders of magnitude, conclusions can be obtained by composing the fuzzy relations representing the order of magnitude information together. More importantly, the rules thus derived have a precise semantics in terms of fuzzy sets and, therefore, overcome the boundary problem between relative orders of magnitude, avoiding interface problems with real numbers. However, as with *FOG*, this method is fundamentally restricted to algebraic relations and hence to static systems.

Viewing all this, it is of great interest and potential benefit to attempt to synthesize the above indicated two strands to form a formal qualitative reasoning technique for dynamic physical systems. From this basic motivation, this paper presents a semi-quantitative extension to qualitative simulation by means of fuzzy mathematics. Through this extension, it can be shown that the ambiguities and limitations pointed out earlier may be significantly reduced or resolved, and that fuzzy set theory can serve as a common mathematical tool for developing a formal foundation for the qualitative simulation of complex systems. For simplicity, this method of qualitative simulation is called Fuzzy Qualitative Simulation and abbreviated to *FuSim* hereinafter.

In general, *FuSim* is, at first sight, a constraint-centered approach like *QSim* [22], it views the structure of a physical system as a set of abstract equations derived from physics, this represents an explicit model of the system. Conversely, fuzzy sets are concerned with representing common sense knowledge, for example, empirical judgements from the designers or operators of the system. Thus, *FuSim* can be considered as a system that utilises an explicit model based on scientific presumptions *and* empirical judgements using fuzzy logic to combine the merits of both approaches.

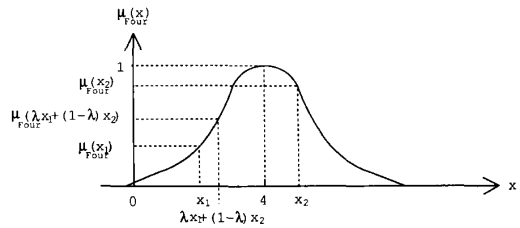

Fig. 2. A normal convex fuzzy cumber.

III. A Brief Review of Relevant Fuzzy Set Theory

This section presents a brief review of the relevant aspects of fuzzy mathematics which forms the basis of our fuzzy qualitative simulation. A more extensive treatment of fuzzy mathematics can be found in [13], [20], [60].

Let X be a classical set of objects, called the universe, whose generic elements are denoted x. Then a *fuzzy set* is a set of pairs

$$A = \{(x, \mu_A(x)) \mid \mu_A(x) \in [0,1], x \in X\},$$

where $\mu_A(x)$ is called the grade of membership of x in A, or sometimes, the membership distribution [20], [61]. Clearly, the closer the value of $\mu_A(x)$ is to 1, the more x belongs to A. When $\mu_A(x)$ is restricted to the values 0 and 1, the fuzzy set A degenerates to an ordinary (crisp) set and its membership distribution becomes the characteristic function of the classical set. If X is the real line R, A is called a *fuzzy number*. Furthermore, if A satisfies the two conditions below

1) $x \in X, \mu_A(x) = 1;$
2) $x_1, x_2 \in X, \lambda \in [0,1],$
   $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2));$

A is a *normal and convex fuzzy number* [13], [61]. Namely, A is unimodal with the maximum value of its membership distribution being 1. The fuzzy set "approximately 4," given in the previous section, is such a fuzzy number, as illustrated in Fig. 2.

When we want to exhibit an element $x \in X$ that *typically* belongs to a fuzzy set A, we may demand its membership value to be greater than some threshold $\alpha \in [0,1]$. The ordinary set of such elements is called the *α-cut* of A and denoted A_α [13]:

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}.$$

For example, two characteristic functions of the α-cuts of "approximately 4" are shown in Fig. 3, where α is equal to 0.5 and 0.9, respectively.

Using the concept of α-cuts, a fuzzy set A may be decomposed into its associated α-cut A_α through the *resolution identity* [20]:

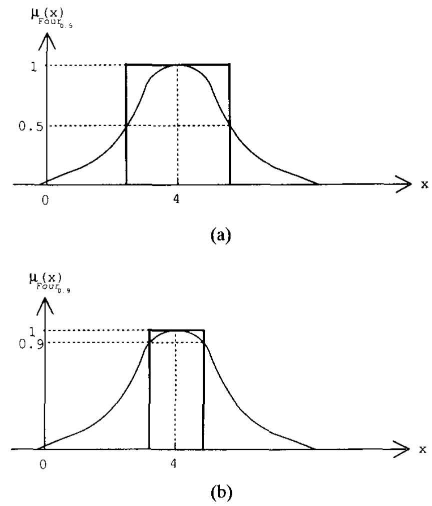

Fig. 3. The α-cuts of "approximately four."

$$A = \int_0^1 \alpha A_\alpha,$$

where $\alpha A_\alpha = \{(x, \alpha) \mid x \in A_\alpha\}$ is a fuzzy set representing the product of a scalar α with the set A_α and $\int_0^1$ is the union operator on αA_α, with α ranging from 0 to 1. Here, the union of two fuzzy sets is defined by

$$A \cup B = \{(x, \mu_{A\cup B}(x)) \mid \mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x)), x \in X\}.$$

That is, the resolution indentity can be viewed as the result of combining those elements in A that fall into the same level set.

One of the most basic ideas of fuzzy set theory, which provides a general extension of classical mathematical concepts to fuzzy environments, is the *extension principle*. This can be stated as follows [61]: If an n-ary function f maps the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ onto a universe Y such that $y = f(x_1, x_2, \cdots, x_n)$, and $A_1, A_2, \cdots, A_n$ are n fuzzy sets in $X_1, X_2, \cdots, X_n$, respectively, characterized by membership functions $\{\mu_{A_i}(x_i), i = 1, 2, \cdots, n\}$, a fuzzy set on Y can then be induced such that the (1) results (shown at the bottom of the page) where Φ is the empty set.

The extension principle can be used to define a set of algebraic operations, O, on a family of fuzzy sets U in a universe X. If U is finite, the algebraic system $< U, O >$ may not be closed. In order to maintain U unchanged after algebraic operations, an *approximation principle* [2], [61], [62] is used such that for an n-ary operator $o \in O$, and n fuzzy

$$\mu_B(y) = \begin{cases} \underset{\substack{x_1,\cdots,x_n \\ y=f(x_1,\cdots,x_n)}}{Sup}, & \min(\mu_{A_1}(x_1), \cdots, \mu_{A_n}(x_n)), f^{-1}(Y) \neq \Phi; \\ 0, & f^{-1}(Y) = \Phi, \end{cases} \tag{1}$$

sets $A_1, A_2, \cdots, A_n \in U$, if $\hat{A} = o(A_1, A_2, \cdots, A_n)$ and

$$d(\mu_{\hat{A}}(x), \mu_A(x)) = \min_{B \in U} d(\mu_{\hat{A}}(x), \mu_B(x)), \quad A \in U$$

then the approximation of $\hat{A}$ is A. Where $d(.,.)$ is any distance which satisfies the axioms of a metric [2].

It is clear that the selection of a distance metric plays a critical role in the application of the approximation principle. However, no matter which kind of distance metric is used, it is usually not efficient to directly calculate the distance between the membership distributions. In order to save computation time and memory storage, a metric based on appropriate features of the membership distributions is usually used. Two important features of a fuzzy number are its *power* and *center* which are closely related to the interval span and centerpoint of the fuzzy number, respectively. Formally speaking, the power of a fuzzy set A is defined as the integral of the membership distribution $\mu_A(x)$, i.e.,

$$Power(A) = \int_R \mu_A(x)dx.$$

The center of A is taken as the central element among all those elements whose degree of membership within A are equal to the maximum value of the overall membership distribution.

Another crucial concept in fuzzy set theory is the *fuzzy relation*, which is a generalization of the normal crisp relation. An n-ary fuzzy relation in $X_1 \times X_2 \times \cdots \times X_n$ is, in fact, a fuzzy set on $X_1 \times X_2 \times \cdots \times X_n$. Fuzzy relations can be composed, and this composition is closely related to the extension principle. For example, if R is a relation from X_1 to X_2 (or, equivalently, a relation in $X_1 \times X_2$), and S is a relation from X_2 to X_3, then the composition of R and S is a fuzzy relation from X_1 to X_3 denoted by $R_o S$ and defined by [13], [66]

$$\mu_{R_o S}(x_1, x_3) = \mathop{Sup}_{x_2 \in X_2} \min(\mu_R(x_1, x_2), \mu_S(x_2, x_3)),$$
$$(x_1, x_3) \in X_1 \times X_3.$$

Fuzzy relations and fuzzy relation composition form the basis of approximate (or fuzzy) reasoning [13], [61], [62]. Informally, approximate reasoning means a process by which a possibly imprecise conclusion is deduced from a collection of imprecise premises. For example, a rule like

$$\text{if } x \text{ is } A_i, \text{ then if } y \text{ is } B_i, \text{ then } z \text{ is } C_i,$$

which governs the relationship between the premises A_i, B_i and the conclusion C_i, can be translated into a fuzzy relation

$$\mu_{R_i}(x, y, z) = \min(\mu_{A_i}(x), \mu_{B_i}(y), \mu_{C_i}(z)),$$

where A_i is a fuzzy set on the universe of a variable x; similarly, B_i and C_i are fuzzy sets, but not on the same universe.

When a set of such rules is available, a synthesized R can be obtained by composing each single fuzzy relation into

$$\mu_R(x, y, z) = \max_{i \in \{1, 2, \cdots, n\}} \min(\mu_{A_i}(x), \mu_{B_i}(y), \mu_{C_i}(z)).$$

If the premises are known, e.g., variables x and y in the premises take values A' and B', respectively, then the value of the conclusion variable z is obtained by applying the compositional rule of inference

$$C' = (A' \times B')_o R.$$

In the above, we have presented some important basic aspects of the theory of fuzzy sets, which are necessary and fundamental for the development of our fuzzy qualitative simulation algorithm to be presented in the following sections.

IV. Fuzzy Quantity Space

The choice of representation of physical quantities plays a critical role in qualitative modelling. All qualitative simulation techniques describe quantities with a small set of symbols, called *qualitative values*, which are abstracted from the underlying field that the variables of a physical system take values from, sometimes called the *support set*. Intuitively, this set of qualitative values should be *finite* with an adequate varying *granularity* and *cover* the whole field of interest. These three important properties are called *finiteness*, *granularity*, and *coverage*, respectively, and described as follows:

Finiteness: each variable has a finite number of associated qualitative values.

Granularity: if $x_1, x_2 \in R$ characterize "similar things" or stand for "similar properties" of a variable x, then the relevant qualitative values of x_1 and x_2 are equal to each other.

Coverage: all numerical values that variables may take are mapped onto their associated qualitative values with respect to the assumed granularity such that the complete set of the qualitative values covers the underlying field of interest.

We exploit fuzzy qualitative values to provide a semi-quantitative extension to the quantity representation of both magnitude and derivative of a system variable. A *fuzzy qualitative value* of a system variable is a fuzzy number chosen from a subset of normal convex fuzzy numbers. This subset is generated by an arbitrary but finite discretization of the underlying numeric range of the variable. A set consisting of all the elements of such subsets, for all the variables in the system, is called a *fuzzy quantity space* and written as Q_F. The real number zero is required to belong to Q_F.

It is important to notice that, with this definition, a system variable takes values from a subset of a quantity space Q_F. This subset can be rather different from the other subsets of the Q_F from which other system variables take values. Moreover, the magnitude and the rate of change of a variable can also have different sets of qualitative values. From one aspect, this enables a flexible representation of knowledge about systems since, if necessary, we can model a physical system with different detailed abstractions of its variables in response to the extent of which we know about the variables.

As a fuzzy quantity space Q_F is generated by a finite discretization of the underlying range of each variable of a system being modelled the Q_F will have the desirable properties of finiteness and coverage, as long as the system contains a finite number of variables. Granularity in the Q_F is obtained by the arbitrariness of the discretization of the numeric ranges of system variables that are assumed to be of interest. Hence, we can translate a subset of a numeric range to one qualitative value according to what is needed in a particular modelling process, such that the extensions of a single qualitative intension may be rather different. The adoption of fuzzy subsets has a direct distinct advantage over the traditional crisp representations when considering granularity. In fact, if we intend to describe the qualitative values of system variables only in terms of the crisp subsets of the underlying real range of the variables, the mapping from the real range to a quantity space will result in the search for the limits of the real numbers served as the boundaries between (disjointly) adjacent qualitative values within the quantity space. This usually incurs severe difficulties in determining these limits [46]. The fuzzy representation of qualitative values is more general than ordinary (crisp) interval representations, since it can represent not only the information stated by a well-determined real interval but also the knowledge embedded in the soft boundaries of the interval. Thus, fuzzy quantity space removes, or largely weakens (if not completely resolving), the boundary interpretation problem, achieved through the description of a gradual rather than an abrupt change in the degree of membership of which a physical quantity is mapped onto a particular qualitative value. It is, therefore, closer to our common sense intuition of the description of a qualitative value.

Let us investigate a simple example. By fuzzy representation, the underlying real range [0, 10], from which a physical variable takes values, can be mapped onto a set of qualitative values, say, $\{$*zero, small, medium, large*$\}$, as shown in Fig. 4(a), where each qualitative value A, actually a normal convex fuzzy number, has an associated linguistic term so that it corresponds to the perceived meaning. Within this quantity space, 3 belongs to *medium* with a strength or membership equal to 1.0 and 2.8 belongs to *medium* with a membership 0.93. Notice, however, that 2.8 also belongs to the set *small* with a membership 0.4. This non-exclusivity of values is an important aspect of fuzzy sets and, again, is important in capturing our common sense intuition. However, using the crisp intervals, with the characteristic functions being $\mu_B(x), B \in$ $\{$*zero, small, medium, large*$\}$, as shown in Fig. 4(b), 3 belongs to [3, 7] named *medium* while 2.8 does not but fully belongs to the interval defined as *small*. Clearly, such a crisp representation would often result in non-intuitive interpretations in practice.

This definition on a fuzzy quantity space is given in a general form. Operations performed within such a quantity space, consisting of normal and convex fuzzy numbers with arbitrary forms of distribution, however, usually entail various types of computational difficulties. As a matter of fact, operations on fuzzy qualitative values are based upon the extension principle outlined in the previous section. This principle is invoked every time an arithmetic operation is performed and requires expensive calculation. Also, the computational implementation of the calculation with arbitrary membership distributions of fuzzy numbers can only be done in a discrete domain obtained by sampling the original continuous distribution. The use of the extension principle with sampled membership

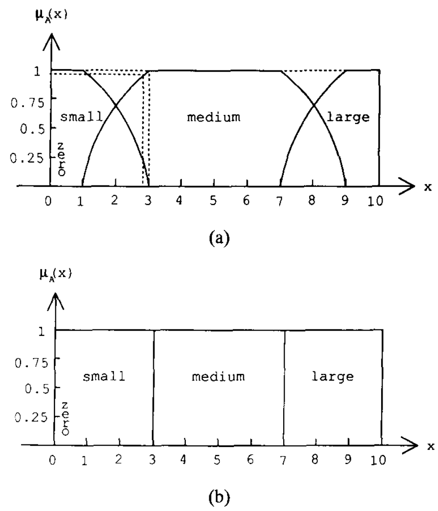

Fig. 4. Comparison between fuzzy and crisp quantity spaces.

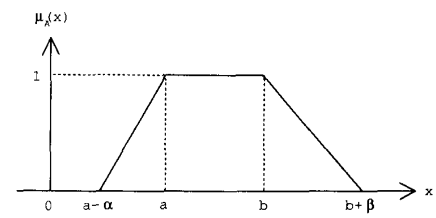

Fig. 5. Parametric representation of a normal convex fuzzy number.

distributions generates a considerable increase in the discrete samples of the result, and furthermore, only some of the resulting samples are correct. Fortunately, computationally more efficient ways to characterize fuzzy numbers have been developed [4]. This utilises a parametric approximation of the membership function. Actually, the membership distribution of a normal convex fuzzy number can be approximated by the 4-tuple, $[a, b, \alpha, \beta]$, as shown in Fig. 5, and defined as

$$\mu_A(x) = \begin{cases} 0 & x < a - \alpha \\ \alpha^{-1}(x - a + \alpha) & x \in [a - \alpha, a] \\ 1 & x \in [a, b] \\ \beta^{-1}(b + \beta - x) & x \in [b, b + \beta] \\ 0 & x > b + \beta \end{cases} .$$

The arithmetic operations on these fuzzy numbers are well-developed [4], and for the preceding reasons, we adopt such a representation to form the fuzzy quantity space for ***FuSim***. A fuzzy quantity space formed in this way makes it possible to build a bridge between "sets" and "values" because this representation allows a real number, a real interval, a fuzzy number, and a fuzzy interval to be uniformly described. Thus, the qualitative category representation and the ordinal representation can be combined in a natural way. For example, the real number 4 can be denoted by a real interval [4, 4], which, in turn, can be represented by a 4-tuple fuzzy number [4, 4, 0, 0], whilst this fuzzy number is a special fuzzy subset of the real line. Similarly, the real interval [3.8, 4] can be represented by the fuzzy description [3.8, 4, 0, 0], and the strict fuzzy number "*approximately 4*" may be expressed by [4, 4, 3, 3]. In this way, when there does exist a precise qualitatively distinct "landmark" value (as those used in *QSim* [22]), this value can also be represented in the form of a 4-tuple number. Furthermore, even if the landmarks are only partially known, say, in terms of the lower and upper (exact) boundaries of the range within which a landmark value falls, such knowledge can still be encoded by the 4-tuple version of a real interval as shown above.

In common with conventional qualitative simulation, three classes of operations can be performed, i.e., algebraic, derivative, and function relational, on the quantity space Q_F. However, the semi-quantitative description of quantity presented by the fuzzy numbers allows a much more flexible method for capturing functional information, allowing strength as well as sign information to be represented if indeed such information is available (and necessary). These three kinds of operation form the basis of our modelling primitives given in the next section.

V. Fuzzy Representational Primitives

Our fuzzy simulation system adopts a constraint-centered ontology in that the model is derived either from an underlying differential equation representation or from direct application of first order energy storage mechanisms [31]. The sets of possible values which system variables can take are restricted by either algebraic, derivative, or function relational constraints amongst the variables.

An expression of the form

$$Q(z) = f(Q(x), Q(y)), \quad Q(x), Q(y), Q(z) \in Q_F,$$

is called a *relevant constraint* on the system variables x, y, and z; where x and y are called *constraining variables* and z is called a *constrained variable.* Both constraining and constrained variables take values from the fuzzy quantity space Q_F, and these values are denoted by $A(x)$, $B(y)$, and $C(z)$, respectively. With respect to such a distinction amongst system variables, it should be emphasised that causal implications will not be used in the generation of the behaviors in the simulation algorithm to be presented in the next section. Rather, the notions of constrained and constraining variables are only used to distinguish the position at which a particular variable appears in a given constraint for the utilisation of the approximation principle (see later).

When f is an algebraic operator, the basic arithmetic operations on Q_F are those in the set of fuzzy numbers. As discussed in the previous section, to reduce the computational difficulties, we use the 4-tuple parametric representation of fuzzy numbers shown in Fig. 5 to form fuzzy quantity spaces. The formulae for the associated arithmetic operations are given in Table I, where $<_0$ is the partial order $<_\alpha$ when $\alpha = 0$. This partial order $<_\alpha$, within the set of 4-tuple fuzzy numbers, is defined such that for $A, B \in Q_F, A \neq B$, we say A is α-less than $B, A <_\alpha B$, iff $a < b, a \in A_\alpha, b \in B_\alpha$ with A_α and B_α being the α-cuts of A and B, respectively.

TABLE I
FORMULAE FOR ARITHMETIC OPERATIONS WITH FUZZY NUMBERS

Operation	Result	Conditions
$-n$	$(-d, -c, \delta, \gamma)$	all n
$\frac{1}{n}$	$\left[\frac{1}{d}, \frac{1}{c}, \frac{\delta}{d}(d+\delta), \frac{\gamma}{c}(c-\gamma)\right]$	$n >_0 0, n <_0 0$
$m + n$	$(a+c, b+d, \tau+\gamma, \beta+\delta)$	all m, n
$m - n$	$(a-d, b-c, \tau+\delta, \beta+\gamma)$	all m, n
$m \times n$	$(ac, bd, a\gamma+c\tau-\tau\gamma, b\delta+d\beta+\beta\delta)$	$m >_0 0, n >_0 0$
	$(ad, bc, d\tau - a\delta + \tau\delta, -b\gamma + c\beta - \beta\gamma)$	$m <_0 0, n >_0 0$
	$(bc. ad, b\gamma - c\beta + \beta\gamma, -d\tau + a\delta - \tau\delta)$	$m >_0 0, n <_0 0$
	$(bd, ac, -b\delta - d\beta - \beta\delta, -a\gamma - c\tau + \tau\gamma)$	$m <_0 0, n <_0 0$
	$m = [a, b, \tau, \beta], \quad n = [c, d, \gamma, \delta]$	

It is important to realise that, for different fuzzy quantity spaces constructed by different subsets of the 4-tuple parametric fuzzy numbers the fundamental arithmetic operations performed within these spaces are of an identical form. That is to say, the basic algebraic operation system is fixed for any particular fuzzy qualitative simulation and, henceforth, does not depend on a particular quantity space.

In common with the normal quantity spaces of other qualitative simulation techniques there exists no inverse under the addition and multiplication operations. In general, this presents difficulties in solving even simple fuzzy equations [34]. Actually, if the values of the constraining variables are known, the value which the constrained variable may take can be obtained by performing the operation on the values of the constraining variables (of course, such computed value does not necessarily belong to the set of those predefined qualitative values used to describe the constrained variable). However, in general, an unknown value of a constraining variable cannot be found by solving the equation as it can be done in solving numerical equations (e.g., via transposition) due to the lack of inverse operations. Fortunately, this problem is avoided within a qualitative simulation process, by following the seminal work of *QSim*, where each variable always has an associated set of known values (generated by the use of continuity) and constraints are only used to check for the consistency amongst values taken by different variables instead of propagating them to find unknowns. Nevertheless, it should be noticed that spurious behaviors often remain after the consistence-checking because of the inherent ambiguity of qualitative representation. Seeing this, we use the approximation principle previously introduced within our simulation algorithm to select the "best" next state amongst those predicted by using continuity for future generation of the behavior.

Clearly, when using the approximation principle, distance measures play a central role in the calculation of the degree of closeness between two fuzzy sets each of which belongs to a different subset of the same universe of discourse. In the present application of this principle, a quantity space Q_F is one of the two subsets, whilst another one, denoted $\hat{Q}_F$, is the collection of all the results from operations applied to the elements among Q_F. It is apparent from Table I that the $\hat{Q}_F$ is also a subset of 4-tuple parametric fuzzy numbers as Q_F.

Hence, we may choose the following as a distance measure [2]:

$$d(A, \hat{A}) = [(Power(A) - Power(\hat{A}))^2 + (Centre(A) - Centre(\hat{A}))^2]^{\frac{1}{2}}, \quad A \in Q_F, \quad \hat{A} \in \hat{Q_F}$$

where, for 4-tuple parametric fuzzy numbers,

$$Power([a, b, \alpha, \beta]) = \frac{1}{2}[2(b - a) + \alpha + \beta],$$

$$Centre([a, b, \alpha, \beta]) = \frac{1}{2}[a + b].$$

The common coefficient, $\frac{1}{2}$, on the right hand side of the above two expressions can be omitted when substituting these expressions into the distance expression.

Once we have a desirable distance measure, the approximation of a fuzzy number $\hat{A}, \hat{A} \in \hat{Q}_F$, to a qualitative value A in Q_F can be determined by choosing A such that the distance between $\hat{A}$ and A is the smallest among all the distances between the fuzzy number $\hat{A}$ and all elements in Q_F. To reduce unnecessary calculation we first check if $\hat{A}$ intersects with any A. If so, the distance between $\hat{A}$ and A is calculated, otherwise A cannot be an approximation to $\hat{A}$. In the case when there are more than one value in Q_F which have the same shortest distance from $\hat{A}$, all such values are treated as the approximation results of the original calculation.

The use of the approximation principle greatly improves efficiency in performing qualitative simulation since it allows the states of system variables to take those values that are the most likely to be correct ones and, therefore, effectively limit the number of spurious states. Whilst this is extremely significant, the desirable property of soundness [22] of a simulation algorithm that relies upon this principle will be lost. That is, it is no longer guaranteed that the behaviors generated by such an algorithm always cover the underlying real behavior of the physical system being simulated. However, the utilisation of the approximation principle allows the most *likely* state, based on minimising the distance measure, to be propagated. This allows the properties of efficiency and certainty to be traded. We have recently extended this technique to associate degrees of commitment (or belief) to the possible behaviors produced by the simulation algorithm, thereby forming a basis for performing progressive reasoning [29]. In this manner, the most likely solution is generated first and the subsequent behaviors are only generated if higher priority behaviors fail. Ultimately all possible behaviors are generated, as in current algorithms, and the eventual soundness of qualitative predictions is guaranteed.

Simple examples can illustrate how arithmetic constraints serve to limit a set of the possible values that a variable can take. Suppose that a system has three variables, x, y, z, and that they satisfy

$$Q(z) = Q(x) + Q(y).$$

For simplicity, it is assumed that these variables are normalised to have the same underlying numeric range $[-1, 1]$ and take values from a fuzzy quantity space, $Q_F = \{-large, -medium, -small, zero, small, medium, large\}$, where each qualitative

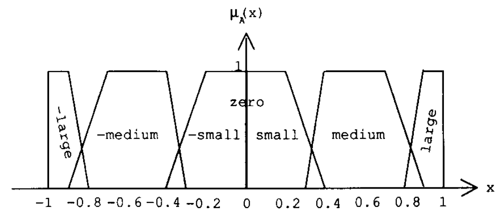

Fig. 6. A fuzzy quantity space.

value is defined by a 4-tuple parametric, normal and convex, fuzzy number $[a, b, \alpha, \beta]$ shown in Fig. 6 and

$$zero = [0, 0, 0, 0],$$
$$small = [0, 0.2, 0, 0.2],$$
$$medium = [0.4, 0.7, 0.1, 0.2],$$
$$large = [0.9, 1, 0.1, 0]$$
$$\text{and} \quad -[a, b, \alpha, \beta] = [-b, -a, \beta, \alpha].$$

If we know, at the beginning, that the values of the constraining variables x and y are $Q(x) = small$ and $Q(y) = large$, but the constrained variable z might be any value $Q(z)$ in the Q_F, then, through the relevant constraint, we shall have

$$Q(z) = [0, 0.2, 0, 0.2] + [0.9, 1, 0.1, 0] = [0.9, 1.2, 0.1, 0.2].$$

By checking if an element in Q_F intersects with $\hat{A} = [0.9, 1.2, 0.1, 0.2]$, the following set is generated,

$${}_{\hat{A}}Q_F = \{[0.4, 0.7, 0.1, 0.2], [0.9, 1, 0.1, 0]\}.$$

This implies that both $[0.4, 0.7, 0.1, 0.2]$ and $[0.9, 1, 0.1, 0]$ are possible values that z may take. However, to which degree of possibility that z takes either of these two values, or both, as its actual value is not distinguished within conventional qualitative simulation algorithms (if the use of continuity does predict these two values as the possible states of z). Fortunately, the utilisation of the approximation principle allows such a distinction to be made between the possible values of a variable. In fact, distances between $\hat{A}$ and $A, A \in {}_{\hat{A}}Q_F$, can be evaluated and the results are:

$$D = \{1, 0.63\}.$$

Thus, the approximation of $[0.8, 1.2, 0.1, 0.2]$ is deemed to be $[0.9, 1, 0.1, 0]$, based on the smaller distance 0.32. This results in

$$Q(z) = [0.9, 1, 0.1, 0] = large.$$

From the result above, the constrained variable z, which originally might be anything in the Q_F, now takes a single value by applying the addition constraint. Since the three variables have the same underlying numeric range, the result:

$$small + large = large$$

is generated, and is well-suited to our common sense calculus, putting the problem of order of magnitude reasoning on a firmer basis. Thus, when reasoning with order of magnitude information, conclusions can be obtained by checking the qualitative values for consistency against given algebraic constraints.

If the variable x takes a qualitative value $-small$ instead of *small*, then

$$Q(z) = -small + large = [0.7, 1, 0.3, 0,].$$

As with the case where $Q(z) = small + large$, a set ${}_{\hat{A}}Q_F$, is generated, by collecting all the elements in the Q_F which intersect with $\hat{A} = [0.7, 1, 0.3, 0]$, such that

$${}_{\hat{A}}Q_F = \{small, medium, large\}.$$

The distances between $\hat{A}$ and $A, A \in {}_{\hat{A}}Q_F$, are:

$$D = \{1.53, 0.6, 0.63\}.$$

It follows that

$$-small + large = medium.$$

This simple example clearly shows that the ambiguity with regard to conventional sign algebra is significantly reduced with an extended quantity space. Even in the case where we could not tell the difference between the two distances 0.6 and 0.63, only two values can be the result of the calculation $-small$ + *large*, namely, *medium* and *large*. Again, this result is suited to our common sense calculus and, from one aspect, reflects the fact that the problem of order of magnitude reasoning can be automatically solved by filtering qualitative values through fuzzy algebraic constraints.

As with any simulation language for dynamic systems a differential operator is essential for determining the transient behavior of such systems. Within the simulation, it provides a memory operation that accounts for energy storage in a physical system. In *FuSim* a derivative constraint is of the form

$$Q(y) = deriv Q(x), \quad Q(x), Q(y) \in Q_F.$$

Since we cope with qualitative magnitudes and qualitative rates of change in the same way, the above constraint simply indicates that the qualitative value of the magnitude of a variable y must be the same as that of the rate of change of a variable x. Importantly, as with other kinds of constraints, e.g., arithmetic, we do not use a derivative constraint to find an unknown but only to check for consistency between given values during the execution of *FuSim*.

Using the same representation for both magnitudes and rates of change has important implications and advantages over the simulation systems that use only sign information for the derivatives, for example *QSim*. Within the fuzzy qualitative model of a dynamic physical system the derivative of a variable can take any value on a pre-specified subset of the Q_F, thereby, allowing ordering information on the rates of change to be described. This information is utilised in the fuzzy simulation algorithm to derive temporal durations of system states and the possible transitions between the states and, further, to develop an effective temporal filtering technique.

Current qualitative simulation techniques model functional dependencies as monotonically increasing or decreasing functions, such that the sign of the rate of change of a variable y is the same as, or contrary to, the sign of the rate of change of another variable x. These functions allow partial knowledge of the relationship between variables to be represented. However, for many applications, these are often too weak as they do not allow any information other than very specific corresponding values (or a set of landmarks which are respectively taken by a set of variables linked by a single constraint at the same time point) on the strength of a relationship that is known to be utilised. Further, with this representation two functional relationships cannot be compared. The use of a fuzzy quantity space allows qualitative function constraints to be represented as fuzzy relations. This enables partial numerical information to be utilised within functional dependencies, thereby providing stronger descriptions of the influence of a variable on another than the simple monotonic operator when such knowledge is available, but without requiring a full analytic function.

A relation between a constrained variable and constraining variables may be viewed as a set of logical rules. When this relation is only qualitatively known, fuzzy logical rules can be stated to implement an approximate reasoning [13], [61], [62]. An example of such a rule is:

if pressure error is positive large or positive medium, and if change in pressure error is negative small, then heat input change is negative medium

where *positive large, positive medium,* and *negative small,* the values of the constraining variables (pressure error and change in pressure error), and *negative medium*, the value of the constrained variable (heat input change), are all fuzzy qualitative values of a fuzzy quantity space used for modelling a particular system. Such rules are of the form:

$$\text{if } x \text{ is } A_i, \text{ then if } y \text{ is } B_i, \text{ then } z \text{ is } C_i, \quad A_i, B_i, C_i \in Q_F,$$

which is a conditional proposition and, by translating it through the min operation, is equivalent to the following fuzzy relation:

$$\mu_{L_i}(x, y, z) = \min(\mu_{A_i}(x), \mu_{B_i}(y), \mu_{C_i}(z)).$$

When a set of n fuzzy rules is available, the resulting relation L is the union of the n elementary fuzzy relations $L_i, i = 1, 2, \cdots, n$:

$$\mu_L(x, y, z) = \max_{i \in \{1,2,\cdots,n\}} \min(\mu_{A_i}(x), \mu_{B_i}(y), \mu_{C_i}(z)).$$

This method of translation and aggregation of the rules is intuitively justified as follows: given the two consistent and nonredundant rules,

$$\text{if } x \text{ is } A, \textit{then } y \text{ is } B,$$
$$\text{if } x \textit{ is } \overline{A}, \textit{ then } y \textit{ is unrestricted;}$$

where $\overline{A}$ means the complement of A and is defined by $\mu_{\overline{A}}(x) = 1 - \mu_A(x)$, we have

$$\mu_L(x, y) = \max[1 - \mu_A(x), \min(\mu_A(x), \mu_B(y))],$$

which is exactly the same logical implication as the one discussed in [64].

In general, if the constraining variables x and y take fuzzy values A' and B', respectively, the fuzzy value C' of

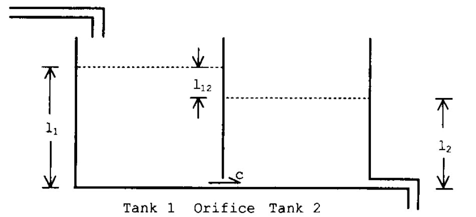

Fig. 7. Two coupled tanks.

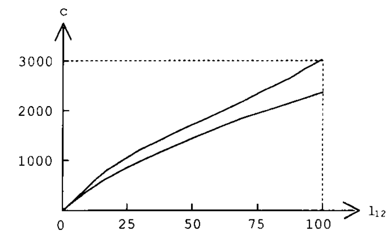

Fig. 8. Two characteristic curves of the orifice.

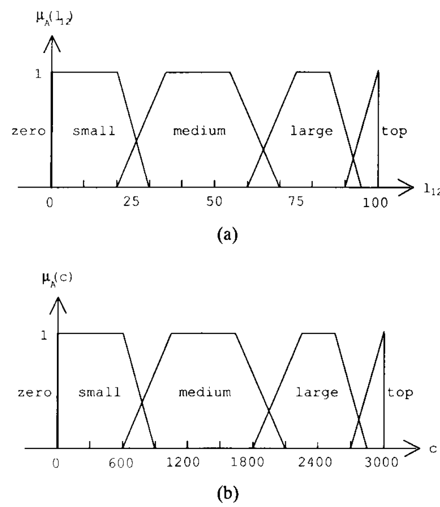

Fig. 9. Two subsets of fuzzy qualitative values.

the constrained variable z can be obtained by applying the compositional rule of inference [13]

$$C' = (A' \times B')_o L$$

or

$$\mu_{C'}(z) = \max_{x,y} \min(\mu_{A'}(x), \mu_{B'}(y), \mu_L(x, y, z)).$$

Within our simulation algorithm (as with other qualitative simulation methods), since each variable is associated with a set of known, possible values at each system state, this rule is not used to find an unknown but as an ordinary constraint (like the arithmetic ones) over those variables linked by a fuzzy relation so as to check for consistency amongst their given values. This kind of qualitative functional relationship reflects one of the distinct advantages of *FuSim*. It allows constraining variables or constrained variable to be mapped onto each other in a semi-quantitative way, making effective use, if needed, of as much information about the functional dependencies as is available.

As an example, consider a basic experimental apparatus consisting of two hold-up tanks which are coupled by an orifice, as depicted in Fig. 7. It is known that the characteristic equation for the orifice is of the form

$$c = c_d a \sqrt{2gl_{12}}$$

where c_d is the discharge coefficient for the orifice, a is the cross-sectional area of the orifice, g is the gravitational constant, l_{12} is the level difference of the fluids in the two tanks, and c is the flow rate of fluid from tank 1 to tank 2. Such a model describes a non-linear functional dependency between c and l_{12}. Also, the discharge coefficient c_d is subject to change because the coefficient is, in general, a function of the level difference l_{12} and the shape and length of the orifice. It is often difficult to get a precise numerical description for the characteristic equation, particularly in industrial settings. Actually, this is one of the basic reasons to consider qualitative modelling for this experimental apparatus. Nevertheless, some typical characteristic curves can be obtained from experiments under assumed conditions, e.g., a sharp orifice, as shown in Fig. 8.

In conventional qualitative simulation systems, say, *QSim*, information on the characteristic equation for the orifice and the experiment curves can only be reflected by a monotonically increasing function $M_0^+(c, l_{12})$, or alternatively, modelled by $sign(dc/dt) = sign(dl_{12}/dt)$ with a corresponding value (0, 0). However, if we use a fuzzy relevant constraint to qualitatively describe the functional dependency between c and l_{12}, on the basis of a fuzzy quantity space in which two subsets of qualitative values for the variables c and l_{12} are defined as in Fig. 9, we can have a set of logical rules reflecting the relationship such as

if l_{12} is *zero* then c is *zero*,
if l_{12} is ***medium*** then c is either ***medium*** or ***large***,
if l_{12} is ***large*** then c is either ***large*** or ***top (maximum)***, etc.

These rules can be interpreted as a fuzzy relation according to the previously given approach. However, since in this example we model the fuzzy quantity space by the linguistic terms which are used in the rules, the fuzzy relation degenerates to a conventional binary relation, within the ranges of the magnitudes of l_{12} and c:

$$\begin{array}{ll} c^{\sim}l12 & \begin{matrix} zero & small & medium & large & top \end{matrix} \\ \begin{matrix} zero \\ small \\ medium \\ large \\ top \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

Henceforth, when computing the fuzzy qualitative simulation, we can use this relation to rule out the impossible combinations of the qualitative values of the variables c and l_{12} by matching them with the matrix. More generally, if the rules are translated and aggregated by performing the min and max operations on the qualitative values of the related variables, a particular combination of the values of the variables is then checked by treating the compositional rule of inference as a constraint against the variables.

By this method, a fuzzy functional constraint can significantly extend the unbounded representation of an ordinary qualitative functional constraint, $M^+(.,.)$ or $M^-(.,.)$, through representing strength information on a functional dependency. In [23], Kuipers and Berleant have a similar motivation but use a different representational framework, which uses two numeric functional relationships, the upper and lower envelopes, to bound the qualitative functional dependency. However, this can only cope with a relation between two variables, whereas, a fuzzy relation can be used to denote a relation held among any finite number of variables, though, in *FuSim*, we do not use a relation to deal with more than three variables.

Based on the previously defined fuzzy quantity space and the above given fuzzy representational primitives, i.e, the algebraic, derivative, and functional constraints, we have developed a fuzzy simulation algorithm as now presented.

VI. Fuzzy Simulation Algorithm

This section describes the simulation algorithm developed for use within *FuSim*. It adopts the general approach taken by *QSim* [22] but is significantly extended to the fuzzy quantity space and the benefits that this representation offers. We first give the notion of a qualitative state of a system variable, in terms of a pair of qualitative magnitude and qualitative derivative, and then discuss the transition between two qualitative states of a variable. Next, we describe three filtering methods to restrict the value sets of the variables, namely, constraint filtering, temporal filtering, and global filtering. The section concludes with an overall summary of the execution of the simulation algorithm.

Like previous methods, the fuzzy qualitative simulation algorithm is based on the fundamental assumption that the variables of the physical system being simulated are continuously differentiable functions, usually of time. We call this basic assumption the *continuity and differentiability assumption*. Based on this, *FuSim* takes as input a set of system variables, a set of constraints relating the variables, and a set of initial values for the variables, and produces a tree of states with each path representing a possible behavior of the system as output. This, of course, is similar to *QSim*, however, *FuSim* considerably reduces the set of spurious behaviors through the use of the fuzzy quantity space and associated filters. In addition, with a degree of freedom, *FuSim* generates a sequence of temporal durations coupled with the system's behavior and, thus, gives an estimate of how long the system remains within a particular state and/or when a state transition occurs.

A. State Description

In this subsection, the notion of the fuzzy quantity space, as a collection of qualitative values, is used to define the *qualitative state* of a system variable at a given time (interval).

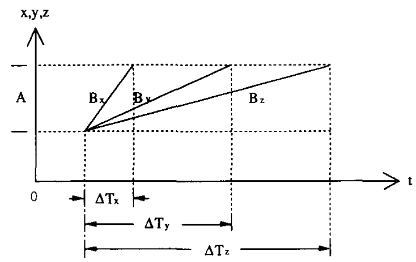

Fig. 10. . The persistence time of an interval-valued state.

For a given system variable x, the fuzzy qualitative state of x within a duration $\Delta T_p, QS(x, \Delta T_p)$ is a pair $< A, B >$, $A, B \in Q_F$, where A denotes the fuzzy magnitude and B the fuzzy rate of change of x. Following Davis [8] and Weld [52], [53] ΔT_p is called the *persistence time* of x in this fuzzy qualitative state. The persistence time records the amount of time a variable remains within a given qualitative state. Such a persistence time is inherently determined by the extent of the (fuzzy) magnitude and the (fuzzy) rate of change of the state. Notice that the rate of change B takes its values from a subset of Q_F rather than the set of three symbolic values $\{+, 0, -\}$. As such it permits ordering information on the rates of change and hence a measure of its variation with time. Fig. 10 shows a simplified case in which each of the variables x, y, and z has the same interval-valued magnitude A, but their derivatives are real numbers B_x, B_y, and B_z with $B_x < B_y < B_z$, respectively. The durations for x, y, and z to persist within A can be depicted and the relation $\Delta T_x > \Delta T_y > \Delta T_z$ holds among them. This clearly shows the importance of having ordering information on rates of change to be able to determine a persistence time for a given state. Unfortunately, such a graphical representation is not possible when $A, B_x, B_y, B_z \in Q_F$.

Strictly speaking, the persistence time should also be a fuzzy number since it expresses the time within which a fuzzy state exists [50]. However, using the *resolution identity* a (crisp) interval, within which a persistence time may lie, can be calculated from the fuzzy magnitude A and fuzzy rate of change B by the following rules, where $W(.)$ expresses a plausible representation of the width of a fuzzy qualitative value and B_α denotes the α-cut of B with α being a degree of freedom:

(1) If $0 \overline{\in} B_\alpha$, then $\Delta T_p \in \alpha \left[\dfrac{W(A)}{|B|} \right]_\alpha$,

$$\text{where } |B| = \begin{cases} B, & B >_\alpha 0 \\ -B, & B <_\alpha 0 \end{cases}$$

(2) If $0 \in B_\alpha$, then ΔT_p is not well-determined, any length of time may elapse. We use the length of the α-cut of a fuzzy number as a measure of its width, as shown in Fig. 11. The α-cut, A_α, of a qualitative value $A = [p_1, p_2, p_3, p_4]$ is

$$[p_1 + p_3(\alpha - 1), p_2 + p_4(1 - \alpha), 0, 0],$$

namely, a crisp interval

$$[p_1 + p_3(\alpha - 1), p_2 + p_4(1 - \alpha)].$$

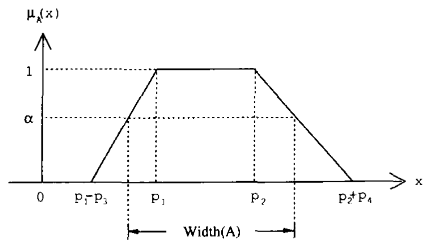

Fig. 11. The width of a fuzzy qualitative value.

Thus

$$W(A) = length(A_\alpha) = p_2 - p_1 + (1-\alpha)(p_3 + p_4).$$

In the course of a particular simulation the value of α used to calculate the width of fuzzy qualitative values is assumed to remain constant. Therefore, the coefficient α in rule (1), introduced by the application of the resolution identity, can be omitted, and the rule is simplified to:

$$\text{If } 0 \,\bar{\in}\, B_\alpha, \text{ then } \Delta T_p \in \frac{p_2 - p_1 + (1-\alpha)(p_3+p_4)}{|B|_\alpha}$$

where, if $B = [q_1, q_2, q_3, q_4]$, the analytic form of the $|B|_\alpha$ is decided by

$$|B|_\alpha = \begin{cases} [q_1 + q_3(\alpha - 1), q_2 + q_4(1-\alpha)], & B >_\alpha 0, \\ [-q_2 + q_4(\alpha - 1), -q_1 + q_3(1-\alpha)], & B <_\alpha 0. \end{cases}$$

Suppose that we have a quantity space $Q_F = \{$*zero, small, medium, large*$\}$, described by Fig. 12, and choose $\alpha = 0.5$ to calculate the width of qualitative values. If we know a variable x stays in the state $<$ *small, medium* $>$, and variable y in state $<$ *medium, zero* $>$; then, the persistence time of x staying at its current state lies in

$$\left[\frac{W(small)}{medium}\right]_\alpha = \frac{0.25}{[0.35, 0.75]} = [0.33, 0.71].$$

For variable y, since its rate of change is *zero* it will, of course, remain within its current state for ever unless there is some relevant constraint to force it to change.

Clearly, the persistence time obtained in this way presents a description of the amount of time within which a variable may remain in a particular state, although usually giving only a possible range. In the event that a variable is initially at an unspecified real value consistent with a qualitative state with a particular fuzzy magnitude (i.e., the given value falls within the α-cut of the fuzzy magnitude with respect to a chosen α), the above computational method for persistence time still provides a worst-case estimate of the duration within which the variable will stay within that state. This is a major distinguishing aspect of ***FuSim***, compared with traditional qualitative simulation approaches.

B. State Transitions

By a *state transition* we mean that a system variable x changes from state $QS(x, \Delta T_{p1}) = \langle A_1, B_1\rangle$ to $QS(x, \Delta T_{p2}) = \langle A_2, B_2\rangle, A_1, A_2, B_1, B_2 \in Q_F$.

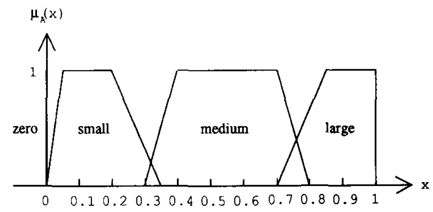

Fig. 12. A fuzzy quantity space.

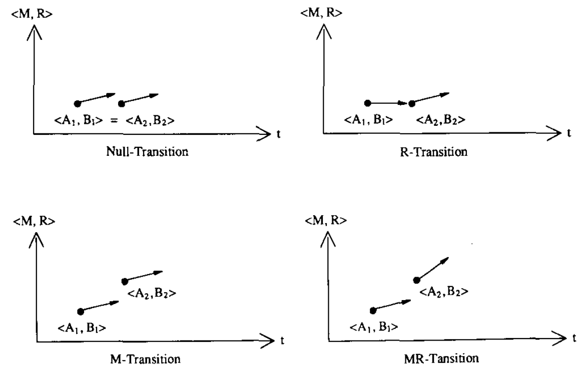

Fig. 13. Four kinds of transitions.

Four kinds of transitions are possible.

1) $A_1 = A_2, B_1 = B_2$—*Null-transition.*
2) $A_1 \neq A_2, B_1 = B_2$— *M-transition.*
3) $A_1 = A_2, B_1 \neq B_2$—*R-transition.*
4) $A_1 \neq A_2, B_1 \neq B_2$— *MR-transition.*

These transitions can be symbolically shown as in Fig. 13, where a solid circle indicates the fuzzy qualitative value of the magnitude of a variable, while an arrow indicates the fuzzy qualitative value of the rate of change of a variable with the changing direction. It is worth pointing out, however, that when the qualitative magnitude of a variable takes a real number as its value an R-transition is not possible.

On the basis of the continuity and differentiability assumption, possible state transitions, say, for a variable x to change from $QS(x, \Delta T_{p1}) = \langle A_1, B_1\rangle$ to $QS(x, \Delta T_{p2}) = \langle A_2, B_2\rangle$, can be represented by a set of rules. In common with other approaches these are called the *possible state transition rules.* Within these rules, $<_\alpha$ is the partial order defined in Section V, but now, holding within the value set of the magnitude or rate-of-change of every variable rather than the complete set of all the 4-tuple parametric fuzzy numbers. We call A and B the α *adjacent qualitative values* within a particular value set if and only if there does not exist a C, which belongs to that value set, such that $A <_\alpha C <_\alpha B$ if $A <_\alpha B$.

The following are the possible state transition rules, derived from versions of the intermediate value and mean value theorems [42], [43], [59], where A_1 and A_2 (or B_1 and B_2) are the α adjacent qualitative values of each other:

(1) If $B_1 >_\alpha 0 (B_1 <_\alpha 0)$, then
if $A_1 \in R$, then $A_2 >_\alpha A_1 (A_2 <_\alpha A_1)$,
else $A_2 \geq_\alpha A_1 (A_2 \leq_\alpha A_1)$.

2) If $B_1 = 0$, then
   if $A_1 \in R$, then
      if $A_2 >_\alpha A_1 (A_2 <_\alpha A_1$, or $A_2 = A_1)$,
      then $B_2 >_\alpha 0 (B_2 <_\alpha 0$, or $B_2 = 0)$;
   else
      if $A_2 \geq_\alpha A_1 (A_2 \leq_\alpha A_1)$,
         then $B_2 >_\alpha 0 (B_2 <_\alpha 0)$,
      if $A_2 = A_1$, then $B_2 \in \{0, X, Y\}$, where X and Y are the α adjacent qualitative values of 0.

From these transition rules, a set of transitions from one given qualitative state description to its possible successors (or next states) can be generated. Repeating such a process by treating a newly produced state as the given one and then generating further successor states from it, a sequence of possible future states can therefore be obtained for each system variable.

The time that a variable x takes to transition from one qualitative state to another has been called the *arrival time* [52], [53], written as ΔT_a. As with the persistence time, the arrival time, in *FuSim*, should also be a fuzzy number. Consequently, to determine the arrival time we use a similar formula to that previously used for calculating the persistence time. For simplicity, the following only discusses the computational rules for the transitions between two states $\langle A_1, B_1\rangle$ and $\langle A_2, B_2\rangle$, where A_1 and A_2 satisfy that $A_2 \geq_\alpha A_1, \alpha \in [0,1]$. As for the transitions between $\langle A_1, B_1\rangle$ and $< A_2, B_2 >$ with $A_2 \leq_\alpha A_1$, the computational rules can easily be determined from symmetry.

First, it is necessary to find a key point, called the *crossing-point*, which is shared by the membership distribution of the current qualitative magnitude A_1 of a variable and that of the qualitative magnitude A_2 in the next state of the variable. For a Null-transition or an R-transition, there does not exist such a crossing-point since $A_1 = A_2$. For an M-transition or MR-transition, where $A_1 = [p_1, p_2, p_3, p_4]$ and $A_2 = [q_1, q_2, q_3, q_4]$, a crossing-point (u, v) is defined as shown in Fig. 14. The value of u is the underlying real point where the membership distribution of A_1 intersects with the distribution of A_2 and v is the degree of membership of u within A_1 or A_2:

$$\textit{crossing-point}(u,v): \quad \begin{cases} u = \dfrac{1}{q_3 + p_4}(q_1 \times p_4 + p_2 \times q_3), \\ v = -\dfrac{1}{q_3 + p_4}(q_1 - p_2 - q_3 - p_4). \end{cases}$$

To be concise, the membership degree v of the crossing-point (u, v) is called the *crossing degree* and written as *c-degree* hereafter. Clearly, when a quantity space is fixed for a particular application, the set of all possible crossing-points can be pre-determined.

The crossing-point is used to develop two general rules for determining the arrival times:

1) For a Null-transition or an R-transition i.e., $< A, B_1 >$ to $< A, B_2 >$ with $B_1 = B_2$ or $B_1 \neq B_2$ respectively, $\Delta T_\alpha = 0$;
2) For an M-transition or an MR-transition, if the value of α used in the possible transition rules, is less than or equal to the c-degree of the crossing-point between $A_1 = [p_1, p_2, p_3, p_4]$ and $A_2 = [q_1, q_2, q_3, q_4]$, then $\Delta T_a = 0$; otherwise, use the following subrules to find the arrival time, where $(\Delta(A_1, A_2))\alpha$ is the α distance between A_1 and A_2. An α distance between two fuzzy numbers is defined as the shortest distance between any two underlying real numbers, each of which belongs to a different fuzzy number with the highest degree of membership being α. The subrules are:

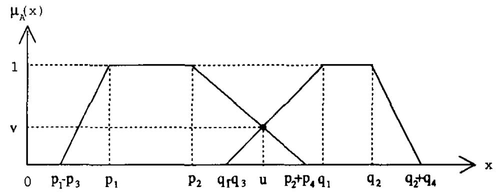

Fig. 14. The crossing-point of two fuzzy numbers.

2.1) For an M-transition, $< A_1, B >$ to $\langle A_2, B\rangle$ and $A_1 \neq A_2$, the rules to calculate ΔT_α are
   (a) If $0 \overline{\in} B_\alpha$, then $\Delta T_a \in \frac{(\Delta(A_1, A_2))_\alpha}{|B|_\alpha}$, where
   $$|B|_\alpha = \begin{cases} [q_1 + q_3(\alpha - 1), q_2 + q_4(1 - \alpha)], \\ \qquad B >_\alpha 0, \\ [-q_2 + q_4(\alpha - 1), -q_1 + q_3(1 - \alpha)] \\ \qquad B <_\alpha 0; \end{cases}$$
   and $(\Delta(A_1, A_2))_\alpha = q_1 - p_2 + (\alpha - 1)(q_3 + p_4)$.
   (b) If $0 \in B_\alpha$, then ΔT_a is not well-determined, any length of time may elapse.
2.2) For an MR-transition, $< A_1, B_1 >$ to $< A_2, B_2 >$, and $A_1 \neq A_2, B_1 \neq B_2$, let $B = B_2 - B_1$, then, the rules to calculate ΔT_a are the same as those in case (2.1).

Formally speaking, in each case (a) of the rule (2.1) and (2.2), ΔT_a should be written as

$$\Delta T_a \in \alpha \left\{ \frac{W(A_{2L} - A_{1R})_{c\text{-}degree} - W(A_{2L} - A_{1R})}{|B|} \right\}_\alpha .$$

However, for the same reason as given in the explanation of the relevant rule for calculating the persistence times, we use a simplified formula to determine the arrival times. By substituting $(\Delta(A_1, A_2))_\alpha = q_1 - p_2 + (\alpha - 1)(q_3 + p_4)$ into case (a) we have

$$\Delta T_a \in \frac{q_1 - p_2 + (\alpha - 1)(q_3 + p_4)}{|B|_\alpha}.$$

As a simple example, Fig. 15 shows the arrival time for a variable x to change from $QS(x, \Delta T_{p1}) =< A_1, B >$ to $QS(x, \Delta T_{p2}) =< A_2, B >$, ($M$-transition), where $A_1 = [p_1, p_2, p_3, p_4], A_2 = [q_1, q_2, q_3, q_4], B \in R, B > 0$, together with two persistence times for x to exist within the two states, respectively; and

$$W(A_1) = p_2 - p_1 + (1 - \alpha)(p_3 + p_4),$$
$$W(A_2) = q_2 - q_1 + (1 - \alpha)(q_3 + q_4),$$
$$(\Delta(A_1, A_2))_\alpha = q_1 - p_2 + (\alpha - 1)(q_3 + p_4).$$

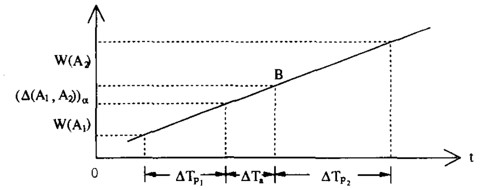

Fig. 15. The arrival time for an M-Transition.

Obviously, if $p_1 = p_2 = p, q_1 = q_2 = q, p_3 = p_4 = q_3 = q_4 = 0$, the fuzzy qualitative values degenerate into real numbers, then $\Delta T_{p1} = \Delta_{p2} = 0$; whilst ΔT_a now represents the sampling interval between real-valued states $< p, B >$ and $< q, B >$ in a discrete-time model.

Notice that in the real case where the states are represented by adjacent crisp intervals there is no arrival time cost for a state to transition when dealing with continuous variables. Actually, arrival times result from the "soft" boundary representation of fuzzy qualitative values. The rules about the arrival time reflect this fact. For example, if a variable x transitions from $< A_1, B_1 >$, to $< A_2, B_2 >$, where $A_1 = [p_1, p_2]$, $A_2 = [q_1, q_2]$, $p_1 < p_2 = q_1 < q_2$, and B_1, B_2 are crisp intervals, $0 \overline{\in} B_1 \cup B_2$; then, $(\Delta(A_1, A_2))_\alpha = 0$. Thus, $\Delta T_a = 0$.

C. Filtering Techniques

Possible state transition rules determine a set of plausible successor states from a given initial state. Basically, this is only decided upon the use of continuity of system variables. Further restrictions on these possible successor states are imposed by utilising the constraint relationships between the variables—called *constraint filtering,* and information on the persistence and arrival times—called *temporal filtering.* In addition, other knowledge about the system being modelled may be used to produce so-called *global filtering* methods. This subsection describes these three different filtering techniques.

Constraint Filtering As *FuSim* adopts a constraint-centered approach, the *(fuzzy) constraint filtering* criteria are essentially similar to those in *QSim.* Actually, for each argument of each constraint, its associated set of possible qualitative values, generated from the use of the possible state transition rules, is filtered for consistency with the definition of the constraints and the consistency between constraints which share the argument. The result of the fuzzy constraint filtering is a reduced set of possible transitions associated with each variable.

In general, for a given relevant constraint, $Q(z) = f(Q(x), Q(y))$, where variable y may be the same as variable x, each argument in the constraint will have a set of possible qualitative values as either the qualitative magnitudes or the qualitative derivatives of a variable in the next state. The actual form of the constraint determines whether this set of qualitative values is the magnitudes or the derivatives of the variables. Such three sets are denoted as S_x, S_y, and S_z, respectively. Then, a *constraint consistency filtering criterion* local to the constraint simply says:

for any $(Q_0(x), Q_0(y), Q_0(z)) \in S_x \times S_y \times S_z$,
if $\{Q_0(z)\} \cap \{f(Q_0(x), Q_0(y))\} = \Phi$,
filter$(Q_o(x), Q_0(y), Q_0(z))$ from $S_x \times S_y \times S_z$.

After each constraint, $Q(z) = f(Q(x), Q(y))$, has been checked by the filtering criterion above, it is then associated with sets of the qualitative values of its arguments $Q(x), Q(y)$, and $Q(z)$. If there are two or more relevant constraints sharing an argument, then the sets of the qualitative values of this argument in those constraints must be identical to each other. We call this rule the *pairwise consistency filtering criterion.* For example, if two constraints that share a common argument, say, the qualitative magnitude of a variable z, are $Q(u) = f(Q(y), Q(z))$ and $Q(z) = deriv(Q(x))$, and if the sets of possible qualitative values of the argument in the two constraints are S_{z_1} and S_{z_2} respectively; then, the filtered set of the possible qualitative magnitudes of z in the next state is $S_{z_1} \cap S_{z_2}$.

It is important to notice that, once the set of possible qualitative values of an argument within a constraint has been checked for either constraint or pairwise consistency and some spurious values are indeed removed from this set, all the other constraints which share an argument with this constraint should then be rechecked. For this reason, the Waltz filtering algorithm [51] is used for efficiency within *FuSim.* The algorithm entails an operation, called *refine,* on each relevant constraint and each argument of the constraint iteratively until the filtering rule produces no more changes [7]. Let $C : C(Q(x_i)), i = 1, 2, 3$, be a relevant constraint among three arguments: $Q(x_i)$, though $Q(x_i)$ may be equal to $Q(x_j), i \neq j$. And, let S_i be the set of qualitative values for the argument $Q(x_i)$. The refine operation is then defined by the following, where $C(A_1, A_2, A_3)$ indicates that A_1, A_2, and A_3 are consistent with the constraint C:

$$\textit{refine}(C, Q(x_i)) = \{A_i \in S_i \mid (A_j \in S_j, j = 1, 2, 3, j \neq i), C(A_1, A_2, A_3)\}.$$

A possible transition of a variable survives the constraint filtering if both the qualitative magnitude and the qualitative derivative of its next state remain after the filtering operation. Thereby, for each variable, the result of the fuzzy constraint filtering is a reduced set of its possible transitions.

Temporal Filtering In addition to constraint filtering, an effective temporal filtering method to further eliminate spurious successor states can be developed through utilising the estimates of the persistence time and the arrival time. This is based on an observation which indicates that, before entering the current state, a variable has spent a certain period of time persisting in the last state plus a duration to reach this state, except in the case where the current state is the initial state. Such an observation is generally true for all the system variables and, therefore, results in the following *temporal filtering* rule:

For any two system variables, x and y, if their persistence times within the present state, $\Delta T_p(x)$ and $\Delta T_p(y)$, lie in $[p_{1x}, p_{2x}]$ and $[p_{1y}, p_{2y}]$, and the arrival times, $\Delta T_a(x)$ and

$\Delta T_a(y)$, for them to transition to a possible successor state lie in $[a_{1x}, a_{2x}]$ and $[a_{1y}, a_{2y}]$ respectively, then, they must satisfy the following temporal constraint:

$$\begin{array}{ll} \text{If} & [p_{1x}, p_{2x}] = [p_{1y}, p_{2y}] (\text{or } [a_{1x}, a_{2x}] \\ & \quad = [a_{1y}, a_{2y}]), \\ \text{then} & [a_{1x}, a_{2x}] \cap [a_{1y}, a_{2y}] \neq \Phi (\text{or } [p_{1x}, p_{2x}] \\ & \quad \cap [p_{1y}, p_{2y}] \neq \Phi) \\ \text{else} & [p_{1x}, p_{2x}] + [a_{1x}, a_{2x}] \cap [p_{1y}, p_{2y}] \\ & \quad + [a_{1y}, a_{2y}] \neq \Phi. \end{array}$$

This temporal filter is rather powerful because it utilizes ordering information on the rates of change of system variables. Actually, the order relationships among fuzzy qualitative rates of change reflect additional knowledge about the higher-order (≥ 2) derivatives of the variables. The fact that higher-order derivative information can be used to eliminate certain impossible behaviors has previously been recognized [9], [24], [58], [59]. For instance, the *curvature constraints* in [24], namely the second-order derivative constraints, can be automatically derived by algebraically manipulating the ordinary qualitative model of a system, when the first-order derivative of a variable is zero, to constrain the possible transitions. This is done based on the assumption about the smoothness of partially known functional relationships. In *FuSim*, information on the differences in strength amongst rates of change (or the first-order derivatives) is reflected in the computational methods for the persistence and arrival times. This is possible since *FuSim* uses more than three values to describe the rate of change of a system variable. In contrast, the basic representation of the qualitative derivative within conventional qualitative simulation algorithms is restricted to the three values $\{-, 0, +\}$ [1]. In which case, no qualification on the rate of change (e.g., increasing slowly) is possible and hence no calculations of the temporal durations may be made. Noticing this, temporal filtering is regarded as one of the most important extensions to current qualitative simulation techniques, although such a filter does not require that a fuzzy quantity space be used.

Global Filtering: After constraint and temporal filtering, complete state descriptions are generated. A complete state description is only a mathematically plausible successor to the current state of the system, i.e., an assignment of a possible transition to each variable in the system without conflicting with the constraint and temporal restrictions. In order to eliminate as many spurious behaviors as possible, *global filtering* techniques are then applied. Such techniques are based on knowledge of system theoretic properties of the real behavior or other, often heuristic, information from external sources.

Currently implemented global filtering methods in *FuSim* are checking for *no-change* and *repeating*. A no-change means that the new state is identical to its immediate predecessor and, therefore, can be deleted in the simulation. By a repeating state we mean that the new state is identical to a state of its predecessors but not the immediate previous one, or identical to one of the other successors of its predecessors, which is not in the branch ended with it. When such a state is met the behavior is marked as repeating and no further next state is generated from it.

A powerful global filter for all second-order systems, called the nonintersection constraint has been independently developed [26], [48]. It is a general constraint based on the requirement that a trajectory of a system state variable in a phase space cannot intersect itself unless the trajectory is a closed curve. This principle is deduced from the existence and uniqueness theorems for simultaneous first-order differential equations. In *FuSim*, however, a variable x takes its qualitative values and qualitative derivatives from a fuzzy quantity space and the excluded-middle law is generally invalid inside this space. As such the non-intersection constraint cannot be directly applied in a plane formed on the basis of fuzzy values, called a fuzzy phase plane, for a second-order system. Nevertheless, by taking advantage of the membership distribution of a fuzzy qualitative value, which describes the degree of possibility that a variable takes an underlying real number value, we can deduce the *degree of possibility* that an underlying real trajectory intersects itself in terms of a fuzzy phase trajectory [3]. However, having considered that *FuSim* has already been able to significantly reduce the number of spurious behaviors that current qualitative simulations may produce, without using the non-intersection principle, and that the deduction about such degrees of possibilities would encounter significant computational complexity, we will realise this additional global filter only when it appears to be necessary.

In general, we conjecture that the extra information derived from the use of a fuzzy quantity space:

(1) ordering information on rates of change, and
(2) strength information on functional dependencies,

eliminates many spurious behaviors at source, and, therefore, obviates the need for extensive global filtering beyond the no-change and repeating filters when considering second-order systems. So far, our empirical results have justified this assertion.

D. Summary of the Simulation Algorithm

As with *QSim*, the fuzzy qualitative simulation algorithm starts with a description of a set of variables with an associated quantity space, a known structural model (in terms of a set of fuzzy constraints), and an initial state of a physical system and produces a set of possible behaviors of the system by generating and filtering the set of possible transitions from one qualitative state description to its successors. Different from other approaches to qualitative simulation, *FuSim* also generates a sequence of temporal durations associated with each possible sequence of system states or each possible behavior of the system. As a summary of this section, the skeleton of the *FuSim* algorithm is given as follows, where steps 2 and 3 are combined by the Waltz filtering algorithm for efficiency reasons. Within the simulation algorithm, constraint filters are followed by the temporal filter. This is so arranged in order to avoid unnecessary computation on temporal information required by the temporal filter.

Step 1: From the current system state, determine a set of possible transitions for each variable through the possible state transition rules, a set of qualitative magnitude and qualitative derivative pairs is then attained; meanwhile, compute the persistence times of the variables at current state.

Step 2: For each constraint, filter the qualitative value sets of all its arguments so that they satisfy the constraint.

Step 3: Filter the qualitative value set of each argument for consistency between constraints which share the argument.

Step 4: Compute the arrival times of the variables; then, use the temporal filtering rule to filter the qualitative value set for each variable which is in conflict with the rule.

Step 5: Generate all possible global interpretations from the value sets of all variables and use the simple global filters (no-change and repeating) to further filter these sets; then, mark each remaining interpretation a successor of the current state.

Step 6: Repeat steps 1–5 until no more changes occur or a resource limit is exceeded.

VII. Experimental Results

In this section, two examples are given to illustrate the operation and advantages of *FuSim*. For this purpose, the section is divided into two subsections, one presenting the fuzzy qualitative simulation of a system consisting of two coupled tanks, and the other showing the fuzzy qualitative simulation of a system composed of a mass on a spring. Both systems are commonly used to exhibit the capability and/or problems of a particular reasoner in the area of qualitative reasoning. These examples demonstrate that *FuSim* has a number of significant advantages over previous qualitative simulation techniques. In particular, it permits a more detailed description of quantities and functional relationships than existing qualitative simulation algorithms, and yet requires neither the precision of real numbers nor the exact and complex relationships amongst variables. Further, *FuSim* produces a unique behavior, in terms of a sequence of states associated with temporal durations, for each of the systems used in the examples without resorting to other supplementary filtering methods and hence added knowledge from other sources.

The *FuSim* algorithm has been implemented in the Quintus Prolog language on a SUN 3 workstation with 16M bytes of RAM, the human-computer interface is realised by using the Prowindows graphics extension. The results given in the following two examples are those obtained by directly running the *FuSim* programme. Within these experimental simulations the α, required by the resolution identity, is chosen to be 0.5. It should be noticed that the choice of this α value is made almost arbitrarily. Actually, the only restriction over choosing such a coefficient is that it must be larger than the maximum membership value amongst those membership values of all the crossing-points within the quantity space used. This is due to the fact that, if there is a crossing-point whose crossing degree is larger than the α, then, two originally intended adjacent values will degrade to a single value. In which case, two otherwise distinguishable states, with each taking a different qualitative value from those two adjacent values, now become a single state. Of course, different choices of the coefficient α will, in general, result in different persistence and arrival times computed during a simulation process. Fortunately, this only affects the precision of such estimations of the time scales without altering the fundamental simulation procedures of the *FuSim* algorithm. Indeed, the α value particularly chosen within a simulation process is used to exhibit the underlying real numbers that belong to some fuzzy number with their memberships being at least this value. The simple reason for us to choose 0.5 as its value within the two examples here is that, in the field of fuzzy sets and systems, it is common to use this value, reflecting the cases where the worst uncertain knowledge is present.

It is worth pointing out that a systematic investigation into different modelling dimensions within the fuzzy qualitative simulation has also been carried out and significant results are presented in [28], [41]. From one aspect, this investigation shows the clear advantages that *FuSim* brings to the qualitative simulation task through identifying and modifying the following four crucial modelling dimensions:

1) The level of *abstraction*, defined as the cardinality of the quantity space with associated underlying semantics;
2) The level of *commitment*, this implies the modification of the membership functions of the values in the quantity space;
3) The *resolution* of the model, this involves the increase or reduction of the number of variables within the model;
4) The *strength* of the relationships within the model, regarded as varying the "gain" between related variables.

Along with such modelling dimensions, multiple models of a single physical system can be developed [47]. This has a very important implication for the use of *FuSim* within model-based reasoning in general and model-based diagnosis in particular [44], [45]. Further details, however, are beyond the scope of this paper.

A. Example 1: Two Coupled Tanks

Consider the basic coupled-tanks apparatus, as depicted in the first subwindow within Fig. 16 (the screendump of the major interface window, automatically produced at the first stage of the fuzzy qualitative simulation of this system). The system consists of eight variables:

l_1, l_2 — the levels of tank 1 and tank 2,
l_{12} — the difference between the levels of tank 1 and tank 2,
i, o, c — the input flow rate, the output flow rate, and the cross-flow rate, respectively, and
n_1, n_2 — the inner flow rates of tank 1 and tank 2.

Each variable has its own underlying numeric range of values that it may take. For simplicity, however, we use a normalized range $[-1, 1]$ to form the basis on which the fuzzy quantity space is discretized. The following fuzzy quantity space was adopted:

$$Q_F = \{n\_top, n\_large, n\_medium, n\_small, zero, p\_small, p\_medium, p\_large, p\_top\}$$

with the qualitative values being represented by nine 4-tuple parametric fuzzy numbers as given below and shown in

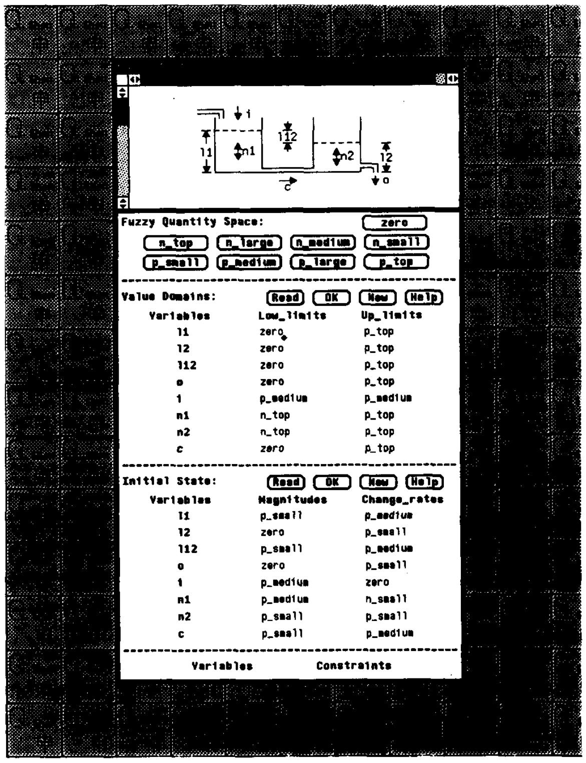

Fig. 16. Major interface window of the fuzzy simulation of two coupled tanks.

Fig. 17.

$$
\begin{aligned}
Q_F = \{&[-1, -1, 0, 0, 0.1], [-0.9, -0.75, 0.05, 0.15],\\
&[-0.6, -0.4, 0.1, 0.1], [-0.25, -0.15, 0.1, 0.15],\\
&[0, 0, 0, 0], [0.15, 0.25, 0.15, 0.1],\\
&[0.4, 0.6, 0.1, 0.1], [0.75, 0.9, 0.15, 0.05],\\
&[1, 1, 0.1, 0]\}.
\end{aligned}
$$

Thus, each variable takes its fuzzy qualitative values from a subset of the Q_F. The ranges of the magnitudes of all system variables are shown in Fig. 16 under the heading "value domains." For example, variable l_1 takes qualitative values from

$$\{zero, p\_small, p\_medium, p\_large, p\_top\}$$

while the rate of change of variable n_1 may take qualitative values from the whole Q_F. If we fix the input flow rate i to be a given numeric value, say, $i = 0.5$, we may use *medium* to qualitatively describe it. This implies that, during the simulation, i can only take this single quantitative value that is qualitatively named *medium* and, therefore, its rate-of-change is always equal to *zero*.

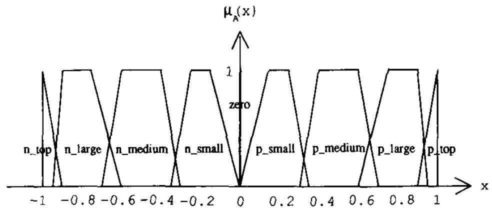

Fig. 17. The representation of the fuzzy quantity space in Example 1.

The system can be modelled by the following six algebraic and derivative constraint equations:

$$
\begin{gathered}
l_{12} = l_1 - l_2, \quad n_1 = i - c, \quad n_2 = c - o,\\
n_1 = deriv\ l_1, \quad n_2 = deriv\ l_2,\\
deriv\ l_{12} = deriv\ l_1 - deriv\ l_2.
\end{gathered}
$$

Notice that, in conventional qualitative simulation algorithms, where the values of the rates of change are modelled by three symbols $\{-, 0, +\}$, if the last algebraic constraint (amongst three rates of change) is used severe ambiguity will often result [10], [49]. However, this problem is considerably reduced in *FuSim* as the rate of change of variables takes values from a more detailed quantity space rather than the simple three symbolic values. In addition to the algebraic and derivative constraints the functional dependency between the cross-flow rate c and the level difference l_{12} may be modelled as shown in Section V by

$$
\begin{bmatrix}
c \sim l_{12} & zero & p\_small & p\_medium & p\_large & p\_top\\
zero & 1 & 0 & 0 & 0 & 0\\
p\_small & 0 & 1 & 0 & 0 & 0\\
p\_medium & 0 & 0 & 1 & 1 & 0\\
p\_large & 0 & 0 & 1 & 1 & 1\\
p\_top & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
$$

Similarly, there exists another functional relationship between the output flow rate o and the level of tank 2, l_2:

$$
\begin{bmatrix}
o \sim l_{12} & zero & p\_small & p\_medium & p\_large & p\_top\\
zero & 1 & 0 & 0 & 0 & 0\\
p\_small & 0 & 1 & 1 & 0 & 0\\
p\_medium & 0 & 0 & 1 & 0 & 0\\
p\_large & 0 & 0 & 1 & 1 & 0\\
p\_top & 0 & 0 & 1 & 1 & 1
\end{bmatrix},
$$

Clearly, the fuzzy relations obtained from the interpretation of the logic rules may contain some redundant information. However, the current implementation of *FuSim* does not check for this. Nevertheless, it will not affect the efficiency of the overall simulation because the translation from the rules to the relation matrices is done off-line.

The set of values under the heading "Initial State" in Fig. 16 is taken as the initial condition presented to the system variables. Now that a description of the fuzzy structure and an initial state of the coupled tank system have been given, the fuzzy qualitative simulation of the system is initiated and used to predict the possible behaviors of the system. Here, we give some typical samples in one cycle of the simulation to show how *FuSim* works and produces a single next state without ambiguity.

From the possible state transition rules, each variable will have a respective set of possible next states as follows:

l_1, l_{12}, c — (<*p_small, p_small*>, < *p_medium, p_small*>, < *p_small, p_medium*>, <*p_medium, p_medium*>, <*p_small, p_large*>, <*p_medium, p_large*>),

l_2, o — (< *zero, p_small*>, <*p_small, p_small*>, <*p_small, p_medium*>),

i — (<*p_medium, zero*>),

n_1 — (<*p_small, n_medium*>, <*p_small, n_small*>, <*p_small, zero*>, <*p_medium, n_medium*>, <*p_medium, n_small*>, <*p_medium, zero*>),

n_2 — (< *p_small, zero*>, <*p_small, p_small*>, <*p_small,p_medium*>, <*p_medium, zero*>, <*p_medium, p_small*>, <*p_medium, p_medium*>).

Fortunately, many of these possible next states are removed by the fuzzy constraint filter. For example, it is obvious that the state with variable l_1 being at < *p_small, p_large¿* and n_1 at $< p\_medium, n\_small >$ is a spurious one, since it invalidates the constraint $n_1 = deriv\ l_1$. Thereby, any conjunction (or combination) of the individual states of the variables, which includes < *p_small, p_large* > and < *p_medium, n_small* > as the states of variables l_1 and n_1 can be eliminated. This is efficiently accomplished by the Waltz algorithm. Likewise, state < *p_small, p_small* > of the variable c and < *p_medium, p_small* > of l_{12} cannot co-exist within a system state because they are inconsistent with the fuzzy relation $c \sim l_{12}$. Although any combination, with < *p_small, p_small* > and < *p_medium, p_small* > being the possible successors of c and l_{12}, respectively, could also be eliminated by the temporal filter, matching the values of c and l_{12} with the degenerated fuzzy relation avoids unnecessary calculation on temporal information.

Although the qualitative state < *p_small, p_small* > of variable c and < *p_medium, p_small* > of l_{12} survive all of the fuzzy algebraic and derivative constraint filtering, such a conjunction of states is removed by the fuzzy relational constraint $c \sim l_{12}$. It is important, however, to notice that this conjunction of state transitions of the variables c and l_{12} from the initial state does satisfy the condition that *sign(deriv c)* = *sign(deriv* l_{12}*)*. Thus, using conventional monotonic increasing function $M^+(.,.)$ as a constraint could not rule out this spurious case. This, from one aspect, shows the capability of fuzzy functional constraint filtering.

As previously pointed out, the direct use of the constraint

$$deriv\ l_{12} = deriv\ l_1 - deriv\ l_2$$

in conventional qualitative simulation methods (where the rate-of-change of a variable takes either of the three symbolic values $\{+, 0, -\}$) would often result in severe ambiguities. In *FuSim*, however, such a constraint is able to resolve ambiguities due to the fact that rates of change take more detailed values. As a matter of fact, the following pseudo successor state of the system,

$$\begin{aligned} l_1 &: < p\_medium,\ p\_small > \\ l_2 &: < p\_small,\ p\_medium > \\ l_{12} &: < p\_medium,\ p\_large > \\ o &: < p\_small,\ p\_small > \\ i &: < p\_medium,\ zero > \\ n_1 &: < p\_small,\ n\_small > \\ n_2 &: < p\_medium,\ p\_small > \\ c &: < p\_medium,\ p\_small > \end{aligned}$$

generated from the initial state, survives all the constraints in the system model other than this constraint. Furthermore, even the temporal filters cannot eliminate it. Fortunately, this state is removed since

$$deriv\ l_{12} = p\_large \neq p\_small - p\_medium = deriv\ l_1 - deriv\ l_2,$$

which invalidates the constraint. It is interesting, however, to indicate that the signs of the values of *deriv* l_{12}, *deriv* l_1, and *deriv* l_2, within this system state are consistent with respect to the constraint. This implies that the present spurious state would not be filtered using the same basic constraint within conventional methods.

After constraint filtering, the following conjunction of the states of the system variables still remains as a possible state transition from the initial one:

$$\begin{aligned} l_1 &: < p\_medium,\ p\_medium > \\ l_2 &: < p\_small,\ p\_small > \\ l_{12} &: < p\_small,\ p\_small > \\ o &: < p\_small,\ p\_small > \\ i &: < p\_medium,\ zero > \\ n_1 &: < p\_medium,\ n\_small > \\ n_2 &: < p\_small,\ p\_small > \\ c &: < p\_small,\ p\_small > . \end{aligned}$$

That is, this system state satisfies all the given constraints. However, we know that, on the one hand, within the initial state the persistence time of the variable l_1 takes value from $[0.077, 0.143]$ and the arrival time for it to transition to this state falls within $[0.346, 0.643]$; on the other hand, the variable n_1 persists within the initial state for a duration belonging to $[1, 4]$ and takes no time to arrive at this state (the arrival time cost by a Null-transition is *zero*). This is inconsistent with the temporal filtering rule since

$$[0.077, 0.143] + [0.346, 0.643] = [0.423, 0.786],$$

while

$$[0.423, 0.786] \cap [1, 4] = \Phi.$$

Thus, this state is eliminated from the set of possible successor states of the initial one. Alternatively, this can also be checked, and therefore ruled out, by comparing the time interval of the variable l_{12} against that of n_1. Actually, the persistence time of l_{12} at the initial state falls within $[0.346, 0.643]$ and the arrival time for the R-transition that l_{12} takes is *zero*. Thereby, the total time interval taken by l_{12} does not intersect with that of the variable n_1 and hence the state is deleted.

The only successor state that passes all the filters is the following:

$$\begin{aligned} l_1 &: < \mathit{p\_small}, \mathit{p\_medium} > \\ l_2 &: < \mathit{p\_small}, \mathit{p\_small} > \\ l_{12} &: < \mathit{p\_small}, \mathit{p\_medium} > \\ o &: < \mathit{p\_small}, \mathit{p\_small} > \\ i &: < \mathit{p\_medium}, \mathit{zero} > \\ n_1 &: < \mathit{p\_medium}, \mathit{n\_small} > \\ n_2 &: < \mathit{p\_small}, \mathit{p\_small} > \\ c &: < \mathit{p\_small}, \mathit{p\_medium} > . \end{aligned}$$

Graphically, this unique immediate successor of the initial state is shown in Fig. 18 together with those spurious successor states generated by the possible state transition rules. Within this figure, the solid circles (representing the actual magnitudes) with the real arrows (representing the actual rates of change) denote the initial state and the remaining successor state, whilst both the empty circles (representing pseudo magnitudes) with the dotted arrows (representing pseudo rates of change) and the solid circles with the dotted arrows denote the spurious states. The detailed implications of the arrows are given beside the diagram for variable n_2. Also, for simplicity, within the figure, *pt, pl, pm,* and *ps* indicate the positive qualitative values of the magnitudes: *p_top*, *p_large*, *p_medium*, and *p_small*, respectively.

B. Example 2: A Mass on a Spring

Let us now consider another example: the system consisting of a mass hung on a spring, as shown in Fig. 19.

This system includes three variables, each of which is associated with a normalized numerical range $[-1, 1]$ from which the variables take values:

x — the displacement of the mass from its rest point,
v — the velocity of the mass, and
a — the acceleration of the mass.

The following set of 4-tuple parametric fuzzy numbers was chosen to form the fuzzy quantity space:

$$\begin{aligned} Q_F = \{&[-1, -0.7, 0, 0.1], [-0.6, -0.6, 0, 0], \\ &[-0.5, -0.1, 0.1, 0.1], [0, 0, 0, 0], \\ &[0.1, 0.5, 0.1, 0.1], [0.6, 0.6, 0, 0], \\ &[0.7, 1, 0.1, 0]\}, \end{aligned}$$

with respective names which correspond to the perceived meaning:

$$\begin{aligned} Q_F = \{&\mathit{n\_top}, \mathit{n\_medium}, \mathit{n\_small}, \mathit{zero}, \\ &\mathit{p\_small}, \mathit{p\_medium}, \mathit{p\_top}.\} \end{aligned}$$

Notice that, here, the real number 0.6 is deliberately chosen to denote the concept *medium*, just to reflect the flexibility of the 4-tuple parametric representation of qualitative values within

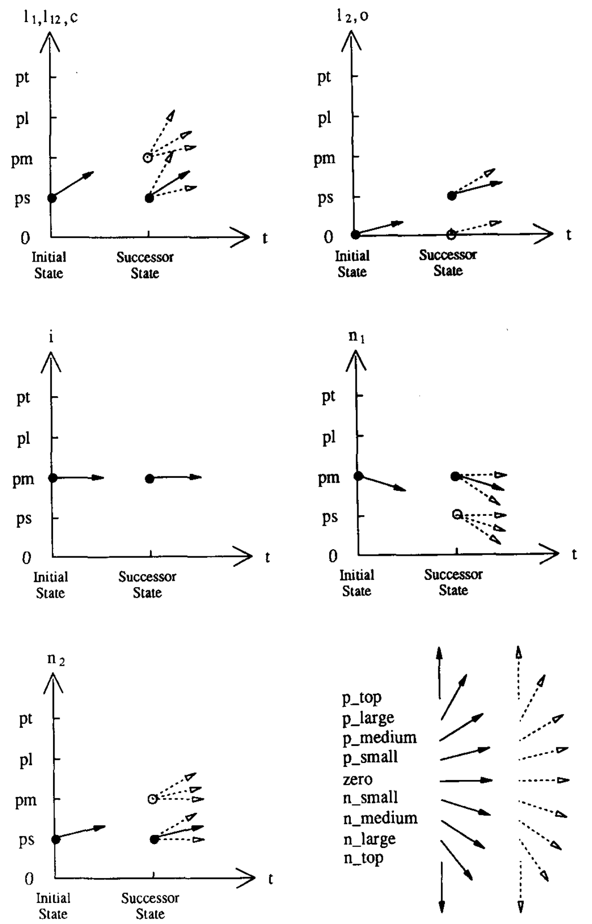

Fig. 18. A step-ahead simulation of two coupled tanks.

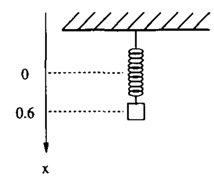

Fig. 19. A mass on a spring.

FuSim. For sake of notational simplicity, the Q_F is hereafter represented by

$$Q_F = \{-t, -0.6, -s, 0, s, 0.6, t\}.$$

The physical constraints in the system can be characterized by the following:

$$\begin{aligned} \mathit{deriv}\ x &= v, \\ \mathit{deriv}\ v &= a, \end{aligned}$$

$$\begin{array}{c|ccccccc} a \sim x & -t & -0.6 & -s & 0 & s & 0.6 & t \\ \hline -t & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ -0.6 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ -s & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0.6 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ s & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ t & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}$$

The first and second equations establish the ordinary derivative relationships holding amongst the distance, velocity, and acceleration of the mass. The third functional relation between a and x is a weak, but stronger than monotonic operator, form of Hooke's law represented in a fuzzy relation.

Suppose that the initial states of the variables x, v, and a are $< 0.6, 0 >, < 0, -0.6 >$, and $< -0.6, 0 >$, respectively. That is, the mass is moved away from the equilibrium point, $x = 0$, to $x = 0.6 > 0$, and then let go. The Waltz algorithm is used to check that such an initial state is indeed valid with respect to the above description of the fuzzy system model. The ***FuSim*** algorithm then takes this model and the initial condition as the input, and starts the simulation of the system. For this particular system, we are not concerned about the possible behavior of the rate of change of the variable a. Thus, behaviors only different in the possible values of the rate of change of a can be treated as a single behavior. By this, ***FuSim*** produces a unique behavior of the system as follows, given in the form of the combination of the states of the system variables (x, v, a):

$$\begin{aligned}
&(< 0.6, 0 >, < 0, -0.6 >, < -0.6, 0 >) \\
&\rightarrow (< s, -s >, < -s, -s >, < -s, s >) \\
&\rightarrow \left\{ \begin{array}{l} (< 0, -0.6 >, < -0.6, 0 >, < 0, 0 >) \\ (< 0, -0.6 >, < -0.6, 0 >, < 0, s >) \\ (< 0, -0.6 >, < -0.6, 0 >, < 0, 0.6 >) \end{array} \right\} \\
&\rightarrow (< -s, -s >, < -s, s >, < s, s >) \\
&\rightarrow (< -0.6, 0 >, < 0, 0.6 >, < 0.6, 0 >) \\
&\rightarrow (< -s, s >, < s, s >, < s, -s >) \\
&\rightarrow \left\{ \begin{array}{l} (< 0, 0.6 >, < 0.6, 0 >, < 0, -0.6 >) \\ (< 0, 0.6 >, < 0.6, 0 >, < 0, -s >) \\ (< 0, 0.6 >, < 0.6, 0 >, < 0, 0 >) \end{array} \right\} \\
&\rightarrow (< s, s >, < s, -s >, < -s, -s >) \\
&\rightarrow (< 0.6, 0 >, < 0, -0.6 >, < -0.6, 0 >)
\end{aligned}$$

and then, it indicates that the above cycle repeats. This is shown graphically in Fig. 20 (one cycle only).

It is very important to notice that, in this figure, the temporal points $(t_0, t_1, \cdots, t_8)$ satisfy the following expressions, calculated based on the sum of the persistence and arrival times experienced by the current state so far since the initial state:

$$\begin{aligned}
&t_0 = 0, \quad t_1 \in [1, 11], \quad t_2 \in [1.09, 12], \\
&t_3 \in [2.09, 23], \quad t_4 \in [2.18, 24], \quad t_5 \in [3.18, 35], \\
&t_6 \in [3.27, 36], \quad t_7 \in [4.27, 47], \quad t_8 \in [4.36, 48].
\end{aligned}$$

Thus, with ***FuSim***, we can determine the time when system state stays within a new state, even though not in an exact form. In fact, a sequence, $\{\Delta T_{p_0}, \Delta T_{a_1}, \Delta T_{p_1}, \Delta T_{a_2},$

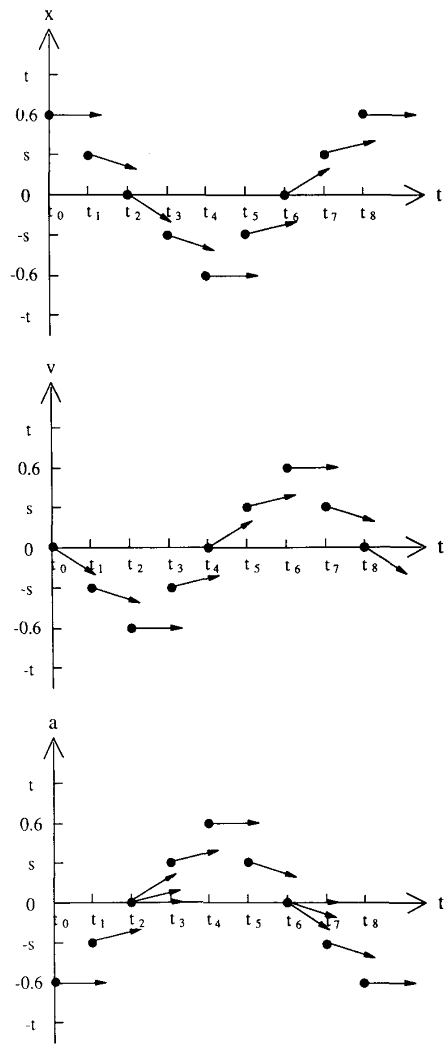

Fig. 20. A cycle of the behavior of "a mass on a spring" system.

$\Delta T_{p_2}, \cdots\}$, of the persistence times and arrival times of the system states can be computed as follows:

$$\{0, [0.09, 1], [0.91, 10], [0.09, 1]\}_i, \quad i = 1, 2, \cdots.$$

From this, we have that

$$\begin{aligned} t_1 &= t_0 + \Delta T_{a_1} + \Delta T_{p_1} \\ &\in [0, 0] + [0.09, 1] + [0.91, 10] = [1, 11], \end{aligned}$$

and that

$$\begin{aligned} t_2 &= t_1 + \Delta T_{a_2} + \Delta T_{p_2} \\ &\in [1, 11] + [0.09, 1] + [0, 0] = [1.09, 12]. \end{aligned}$$

Other time indices $t_n, n > 2$, are similarly determined.

Let us consider two detailed samples for obtaining the above persistence and arrival times. Firstly, following the rule to calculate the arrival time, the time cost for the system to transition from the initial state to its successor $(< s, -s >, <$

$-s, -s >, < -s, s >$) lies in the range $[0.09, 1]$. For instance, the variable a changes from $< -0.6, 0 >$ to $< -s, s >$, taking the arrival time

$$\Delta T_{a_1}(v) \in \frac{-0.5 - (-0.6) + (0.5 - 1)(0.1 + 0)}{||[0.1, 0.5, 0.1, 0.1]||_{0.5}} = [0.09, 1].$$

Secondly, the persistence time for the system (x, v, a) to remain within state $(< s, -s >, < -s, -s >, < -s, s >)$ lies in the range $[0.91, 10]$. For example, the variable x stays within the state $< s, -s >$ for the following duration, determined by the rule of calculating persistence times:

$$\Delta T_{p_1}(x) \in \frac{0.5 - 0.1 + (1 - 0.5)(0.1 + 0.1)}{||[-0.5, -0.1, 0.1, 0.1]||_{0.5}} = [0.091, 10].$$

As a complete example, Fig. 21 shows how *FuSim* produces the behavior of the system from the initial state to its first two successors, i.e., the first $1/4$ period of the complete cycle of the behavior illustrated in Fig. 20. The one step-ahead behavior from the initial state is depicted in Fig. 21(a) and the two step-ahead simulation results are given in Fig. 21(b). Within these diagrams, those states marked with either of the labels *C, T,* and *G* denote the spurious states eliminated by either fuzzy constraint filtering or temporal filtering or simple global filtering (no-change and repeating), respectively. In addition, the solid or empty circles and the real or dotted arrows represent the same implications as those presented in Fig. 18.

The elimination of the spurious states within Fig. 21(a) is explained as follows. The pseudo new state $< t, s >$ of the variable x is filtered by the constraint: *deriv* $x = v$, since there does not exist a new state of the variable v that takes s as its qualitative magnitude. Having removed the $< t, s >$ of x, the spurious successor state $< -t, -s >$ of the variable a can be deleted by the function relational constraint. After this, the $< -s, -t >$ of v can be deleted based on the pairwise filtering criterion. It is important to notice that the conventional monotonic decreasing constraint over the variables x and a cannot eliminate this state (even if the corresponding value $(0.6, -0.6)$ is known) due to the fact that, for variables (x, a), the pair $(< t, s >, < -t, -s >)$ transitioned from ($< 0.6, 0 >, < -0.6, 0 >$) satisfies $M^-_{(0.6,-0.6)}(x, a)$. Another system state $(< 0.6, 0 >, < 0, -0.6 >, < -0.6, 0 >)$ is removed by using the no-change filter and marked with G within the diagram for each variable. The only immediate successor state of the initial state is, therefore, the following:

$$(< s, -s >, < -s, -s >, < -s, s >).$$

By chance, no temporal filtering is required for removing those pseudo states at this stage.

Temporal filtering, however, plays a significant role in the reduction of the spurious transitions from the above first successor of the initial state. In fact, it can be seen from Fig. 21(b) that, after using constraint filters, some spurious states still remain. Fortunately, most of them are eliminated by the use of temporal filters. For instance, the system state

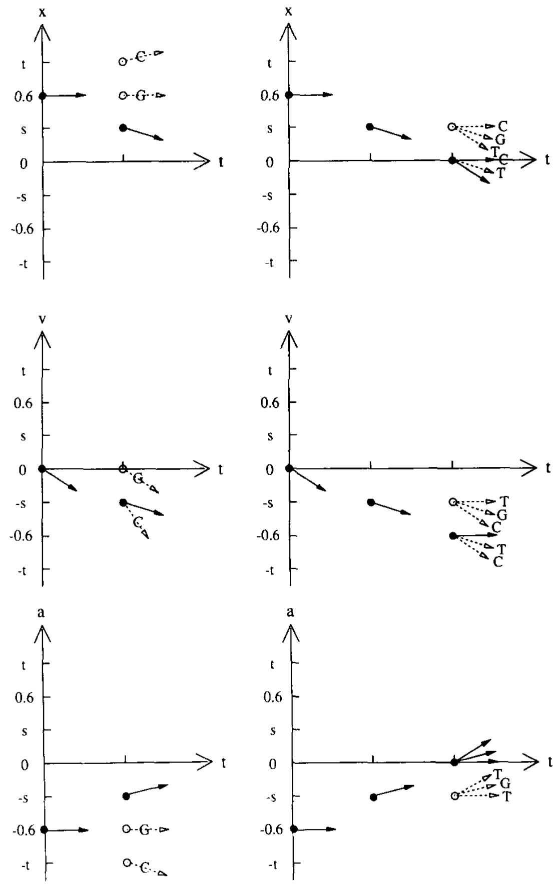

Fig. 21 (a) One step ahead. (b) Two step ahead generation and filtering of the first 1/4 period of the behavior.

$(< 0, -s >, < -s, 0 >, < 0, 0 >)$ is a pseudo next state since, within the present state $(< s, -s >, < -s, -s >, < -s, s >)$, the persistence time of each variable lies in $[0.91, 10]$ (see the detailed calculation given previously for example); while the arrival time for v to reach this state is 0 (an R-transition), but the time taken for x and a to transition is

$$\frac{0.1 - 0 + (0.5 - 1)(0.1 + 0)}{[0.1 + 0.1(0.5 - 1), 0.5 + 0.1(1 - 0.5)]} = [0.09, 1].$$

Hence, such a transition invalidates the temporal filtering rule and is deleted. As shown in the figure, the remaining few pseudo states are removed by using two simple global filters. The only surviving successor states of the system state $(< s, -s >, < -s, -s >, < -s, s >)$ that pass all three kinds of filters are the following:

$$(< 0, -0.6 >, < -0.6, 0 >, < 0, 0 >),$$
$$(< 0, -0.6 >, < -0.6, 0 >, < 0, s >),$$
$$(< 0, -0.6 >, < -0.6, 0 >, < 0, 0.6 >).$$

For this "mass on a spring" system, we are not concerned about the rate of change of the variable a, these three "different" states can, therefore, be treated as a single one. The path from the initial state, through $(< s, -s >, < -s, -s >, < -s, s >)$, to one of these three states forms the first $1/4$ period of the behavior cycle given in Fig. 20. Other $3/4$ period of the complete cycle shown in Fig. 20 was obtained in a similar way.

From the behavior obtained through *FuSim*, it is clear that the mass will oscillate between the maximum amplitudes 0.6 and -0.6 forever, with a period belonging to the interval $[4.36, 48]$.

VIII. Conclusion

The work presented herein synthesizes two previously disparate research areas of qualitative reasoning and fuzzy sets to produce an effective algorithm for the simulation of dynamic systems. The resulting algorithm shows three distinct advantages over previous methods and, in addition, provides a unifying framework from which other recent work on qualitative simulation can be interpreted. Firstly, it allows a more detailed description of system variables, through an arbitrary, but finite, discretisation of the quantity space. The use of graded membership within a fuzzy quantity space, albeit in a parametricised form, allows the subjective element of common-sense knowledge to be incorporated in the basic description of the quantity space. Secondly, the use of fuzzy relations and the fuzzy compositional rule of inference allows semiquantitative information about the strength, as well as the sign, of functional relationships to be represented and reasoned about. This is an important practical advantage, in that, imprecise and partial numerical information about functional relationships is often known, although not in enough detail to develop a well-posed numerical model. Current techniques do not make use of this knowledge or do so in an *ad hoc* manner. The third advantage of the fuzzy simulation algorithm is that, by allowing the rate-of-change to be defined on the full fuzzy quantity space, relative rates of change can be represented and temporal durations computed. This results in a powerful temporal filter that significantly reduces the number of spurious behaviors generated and enables a system's behavior to be explicitly expressed by a state sequence associated with a temporal sequence. Additionally, the use of the approximation principle allows the most likely next states to be propagated, thereby providing a technique for trading efficiency and certainty of the behavior generation. This establishes the basis for an extremely powerful progressive reasoning extension to qualitative simulation [29].

Work is on-going in two directions. The first is to refine and extend the algorithm through application to industrial size case studies with, perhaps, different modelling dimensions [28], [41]. The second, to utilise the *FuSim* as the predictive algorithm for the system model within model-based reasoners [27]. In particular, the use of *FuSim* within a model-based diagnostic mechanism for continuous dynamic systems, based on the comparison of the predicted behaviors with the observed behavior of an evolving system, shows considerable promise for the early detection and diagnosis of faults as they are occurring [44], [45].

Acknowledgment

The paper has benefited considerably from the through review and insightful comments made by the anonymous referees. This has allowed a number of the important properties of *FuSim* to be clarified and errors corrected. The authors are very grateful. We would also like to acknowledge the support of our colleagues, Mike Chantler, George Coghill, Chai Quek, and Mark Wiegand and to thank them for many interesting and challenging discussions.

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