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Maintenance of approximations in incomplete ordered decision systems while attribute values coarsening or refining

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ABSTRACT

Approximations in rough sets theory are important operators to discover interesting patterns and dependencies in data mining. Both certain and uncertain rules are unraveled from different regions partitioned by approximations. In real-life applications, an information system may evolve with time by different factors such as attributes, objects, and attribute values. How to update approximations efficiently becomes vital in data mining related tasks. Dominance-based rough set approaches deal with the problem of ordinal classification with monotonicity constraints in multi-criteria decision analysis. Data missing frequently appears in the Incomplete Ordered Decision Systems (IODSs). Extended dominance characteristic relation-based rough set approaches process the IODS with two cases of missing data, *i.e.*, "lost value" and "do not care". This paper focuses on dynamically updating approximations of upward and downward unions while attribute values coarsening or refining in the IODS. Under the extended dominance characteristic relation based rough sets, it presents the principles of dynamically updating approximations w.r.t. attribute values' coarsening and refining in the IODS and algorithms for incremental updating approximations of an upward union and downward union of classes. Comparative experiments from datasets of UCI and empirical results show the proposed method is efficient and effective in maintenance of approximations.

1. Introduction

Granular computing (GrC) proposed by Zadeh [1,2] and Lin [3] has been widely used in image processing, pattern identification, knowledge discovery, and many other fields [4–7]. Granule is a clump of objects drawn together by indistinguishability, similarity or functionality [2]. Granules and their relationships are employed to find solutions to any desired problems. Different granularity levels are formed by decomposing and composing of granules. Different concept or rule levels are then unraveled. The frameworks, models, methodologies, and techniques of GrC were studied in [8–10]. Yao described basic issues and methods of GrC [11–13]. Pedrycz proposed the knowledge-based fuzzy clustering to form granules and further investigated GrC related work, *e.g.*, a granular-oriented neural network, a granular time series approach [14–18].

Rough sets theory is a mathematical tool to process inconsistent data in information systems [19–22]. It is seemed as an important sub branch of GrC. Skowron and Polkowski studied the rough sets theory based GrC in [23–25]. The equivalence relation between any of the two objects is formed by an equality of attribute values in Traditional Rough Sets (TRSs). The preference order is important to the multi-criteria decision analysis, *e.g.*, credit appraisal, risk evaluation, and feasibility study [26,27]. Greco et al. proposed a Dominance-based Rough Set Approach (DRSA) for decision making [28]. The equivalence relation in TRS is replaced by the dominating (dominated) relation in the DRSA. Furthermore, they proposed fuzzy dominance relation-based rough sets by introducing fuzzy logic into the dominance relation [29,30]. To deal with the inconsistency caused by errors in recording, measurement and observation, Inuiguchi et al. proposed a Variable-Precision Dominance relation-based Rough Set Approach (VP-DRSA) and studied the reduction under VP-DRSA [31]. Hu et al. proposed a method for extracting fuzzy preference relations from samples characterized by numerical criteria [32]. Qian et al. extended the dominance relation to interval information systems and set-valued information systems [33,34]. Kotłowski et al. introduced a probability model to deal with the problem of the ordinal classification and proposed a

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stochastic DRSA [35]. Huang introduced a graded dominance relation to dominance interval-valued fuzzy objective information systems and proposed graded dominance interval-based fuzzy objective information systems [36]. The DRSA has been used in multi-criteria classification and ordinal attribute reduction [37–39]. It has been also successfully applied in a variety of fields in decision-making, including bankrupt risk prediction [40], customer behavior prediction [41], Kansei data analysis in product development [42], and IT business value analysis [43].

Real-life applications face often data missing due to various uncertainties. Yang et al. studied three cases of data missing in Incomplete Interval-valued Information Systems (IIISs), *i.e.*, interval-valued data with known lower and unknown upper limits, interval-valued data with unknown lower and known upper limits, and interval-valued data with both unknown lower and unknown upper limits. They discussed the relative reduction in IIIS [44]. Shao and Zhang applied DRSA to reasoning in Incomplete Ordered Information Systems (IOISs) [45]. Yang et al. proposed a similarity dominance relation and studied the reduction under IOIS [46]. Hu and Liu proposed a generalized extended dominance relation model in which the proportion of common attributes whose values are not lost between two objects must greater than a given threshold [47]. Luo et al. proposed a Limited Extended Dominance Relation (LEDR) considering maximum and minimum values in the dominating relation or dominated relation when comparing to lost value [48]. Grzymala-Busse proposed a characteristic relation to process the two cases of data missing, that is, "do not know" and "do not care" [49]. Chen et al. proposed an Extended Dominance Characteristic Relation (EDCR) to process Incomplete Ordered Decision Systems (IODSs) both with these two cases of data missing and considering the proportion of common attribute whose values are not lost [50].

The information system evolves with time, namely, the attributes of the information system, the objects in the universe, or the attribute values of the objects may change. The attribute values may change due to different reasons, *i.e.*, revise the errors and the variation in the hierarchy level of attributes' value domain [51]. Attributes' values have a hierarchy structure are common in the real-life applications. In [52], Hong et al. proposed methods to mine cross-level rules under fuzzy rough sets model. Feng et al. proposed algorithms to mine decision rules from different levels of abstraction [53]. In [54], Chen et al. proposed algorithms for learning decision tree classifiers from data with hierarchical class labels. The attributes' values have preference order in DRSA, *e.g.*, scores of students, evaluation of products. There are few literatures about updating knowledge when the attributes' values vary in IODS. In DRSA, certain rules may be induced from lower approximation. The lower approximation of upper union means if an object x_i is at least as good as another object x_j on all of the considered criteria so x_i should belong to the decision classes not lower than which x_j belongs to. The lower approximation of downward union means if an object x_i is at most as good as another object x_j on all of the considered criteria so x_i should belong to the decision classes not higher than which x_j belongs to. When attributes' values evolve with time, the partial order between x_i and x_j may vary. Then the granule induced by the dominating (dominated) classes and the approximations may alter accordingly. Generally, the computation of approximations is a necessary step in knowledge representation and reduction based on rough sets. Approximations may further be applied to data mining related tasks. Considering the partial order in attributes' value set, we introduce the concept of Attribute Values Coarsening or Refining (AVCR) and multi-level AVCR in IODS. The principle for incrementally updating approximations of upward and downward unions of classes is further studied by analyzing the variation of granularity in IODS. The IF–THEN rules can be finally updated since certain/uncertain rules are induced from lower/upper approximations.

Incremental updating is an effective method in dynamic data processing [55–57] since previous data structure and knowledge may be used effectively to maintain knowledge when an information system varies. The efficiency of knowledge discovery can usually be improved because it does not need to recalculate the whole data. Much research has been focused on maintenance of knowledge. In the case of variation of objects, An et al. updated rules firstly in TRS, which requires new objects consistent to original decision table and new decision classes cannot be added to the table [58]. Shusaku and Hiroshi obtained uncertain rules from clinic database incrementally, but they did not mention how to obtain certain rules [59]. Bang and Zeungnam divided new samples into seven types and proposed an algorithm for updating rules, which is relative to conditional classes and decision classes [60]. Tong and An classified new samples into four types, *i.e.*, confirmative, entirely new, entirely contradiction, and partly contradiction. They presented an incremental learning rule method [61]. Liu et al. proposed a parallel algorithm based on an improved discernibility matrix to extract rules [62]. Guo et al. proposed a method to extract rules incrementally based on the search tree, which does not need to construct the discernibility matrix [63]. Zheng and Wang proposed an effective rule extraction method, RRIA, which updates rules incrementally by adding and pruning the rule tree [64]. Blaszczynski and Slowinski presented an incremental updating algorithm to induce rules based on the Apriori algorithm under variable consistency DRSA [65]. Fan et al. studied the different cases according to the effect to the rule set when the objects are added or deleted. They proposed an algorithm for updating rules through updating SI (Strength Index) in the different cases [66]. Skowron et al. proposed the definition of function approximation and gave the methods to induce the rule set through objects known [67]. Liu et al. constructed the accuracy matrix and the coverage matrix and presented the concept of interesting knowledge. They proposed a method to update interesting knowledge through updating the accuracy matrix and the coverage matrix [68]. In the case of variation of attributes, Li et al. updated approximations when multi-attributes are added or deleted simultaneously under rough sets based the characteristic relation by analyzing the change of lower boundary region [69]. Cheng proposed two incremental methods for fast computing the approximations in rough fuzzy sets, *i.e.*, one starts from the boundary set and the other is based on the cut sets of a fuzzy set [70]. In the case of variation of attributes' values, Chen et al. defined AVCR in TRS and proposed algorithms for updating approximations dynamically by analyzing the evolvement of boundary region and granularity [51]. To our best knowledge, an incremental method for updating approximations under IOIS has not yet been discussed so far. In this paper, we discuss methods for updating approximations of upward and downward unions in the EDCR based rough sets w.r.t. AVCR.

The paper is organized as follows. In Section 2, we review basic concepts of IOIS and the dominance relation and introduce the definition of an extended dominance relation. In Section 3, we propose the concepts of AVCR and multi-level AVCR and further investigate the properties, methods and algorithms w.r.t. AVCR. In Section 4, we employ an example to illustrate the proposed methods for incrementally updating approximations. In Section 5, we verify the algorithms by extensive experiments, and then we analyze and discuss the experimental results. We conclude the paper with the future research directions in Section 6.

2. Dominance-based rough set approach for IODS

In this section, we introduce basic concepts of decision systems, including IODS, dominate relation and an EDCR in IODS with missing values [28,50,71].

Definition 2.1. A decision system is a 4-tuple $S = (U, A, V, f)$. U is a finite non-empty set of objects, called the universe. A is a non-empty finite set of attributes, $A = C \cup D$, $C \cap D = \emptyset$, where C and D denote the sets of condition attributes and decision attributes, respectively. V is a domain of attributes. The domain of$C$ is ranked according to an increasing or decreasing preference. The attributes are criteria. $f{:}U \times A \to V$ is an information function, $f = \{f(x_i, q) | f(x_i, q){:}x_i \to v_q, q \in C, x_i \in U, 1 \leqslant i \leqslant |U|\}$. $f(x_i, a_l) = v_{il}$ $(i = 1, 2, \ldots, |U|, l = 1, 2, \ldots, |A|)$ denotes the attribute value of object x_i under a_l. If all attributes' values in the decision system are known, it is a complete ordered decision system. If there exists any missing data, it is an IODS. The missing data are classified into two cases that are "do not know" and "do not care". All the missing values are denoted by "?" or "$*$", where the lost value or "do not know" is denoted by "?", "do not care" is denoted by "$*$".

Definition 2.2. Let $x, y \in U, P \subseteq C$. If $\forall q \in P, f(y, q) \succeq f(x, q)$, then we denote yD_Px. The relation is called a dominate relation. Here, $f(y, q) \succeq f(x, q)$ means x at least as good as (outranks) y with respect to criteria q.

Definition 2.3. Let $S = (U, A, V, f)$ be an IODS, $A = C \cup D, P \subseteq C$, $x, y \in U$, P-dominating sets and P-dominated sets are defined as follows:

$$D_P^+(x) = \{y | yD_Px\} \tag{1}$$

$$D_P^-(x) = \{y | xD_Py\} \tag{2}$$

Considering two cases of missing data that are "do not know" and "do not care", we define an EDCR as follows. Here we only consider the case of which condition attributes' values may lose and decision attributes' values do not lose.

Definition 2.4. Let $S = (U, A, V, f)$, $P \subseteq C$, $B_P(x) = \{b | b \in P \wedge f(x, b) \neq * \wedge f(x, b) \neq ?\}$. A k-degree EDCR on $P(EDCR(P^\kappa))$ is defined as follows:

$$\begin{aligned} EDCR(P^\kappa) = \{(x, y) | x \in U \wedge y \in U \wedge ((|B_P(x) \cap B_P(y)|/|P|) \\ \geqslant \kappa \wedge (f(x, q) = * \vee (f(y, q) = *) \wedge f(x, q) \\ \neq ? \wedge f(y, q) \neq ? \to f(y, q) \succeq f(x, q))\} \end{aligned} \tag{3}$$

where $|\cdot|$ denotes the cardinality of a set, $0 < \kappa \leqslant 1$. $yD_P^\kappa x$ denotes y is k-degree extended dominate x.

Definition 2.5. For $P \subseteq C, x \subseteq U, D_P^{+\kappa}(x) = \{y \in U | yD_P^\kappa x\}$ is P dominating set of x, $D_P^{-\kappa}(x) = \{y \in U : xD_P^\kappa y\}$ is P dominated set of x.

Property 2.1. *Given an IODS $S = (U, A, V, f)$, the following properties hold.*

(1) *$EDCR(P^\kappa)$ is reflexive, transitive, and asymmetric.*
(2) *For any α, β, $0 \leqslant \alpha \leqslant \beta \leqslant 1$, $D_P^{+\beta}(x) \subseteq D_P^{+\alpha}(x)$, $D_P^{-\beta}(x) \subseteq D_P^{-\alpha}(x)$.*
(3) *For any $P \subseteq C$, $G = \{D_P^{+\kappa}(x) | x \in U\}$ constitutes a covering of U. $H = \{D_P^{-\kappa}(x) | x \in U\}$ also constitutes a covering of U.*
(4) *For any $P \subseteq C$, $x_i, x_j \in U$, if $x_j \in D_P^{+\kappa}(x_i)$, then $D_P^{+\kappa}(x_j) \subseteq D_P^{+\kappa}(x_i)$; if $x_j \in D_P^{-\kappa}(x_i)$, then $D_P^{-\kappa}(x_j) \subseteq D_P^{-\kappa}(x_i)$.*

From Property 2.1, dominating classes and dominated classes evolve with the variation of parameter k. The cardinality of dominating classes and dominated classes reach minimum when $k = 1$ and reach maximum when $k = 0$. In addition, for any $E \subseteq F \subseteq C$, $D_C^{+\kappa}(x) \subseteq D_F^{+\kappa}(x) \subseteq D_E^{+\kappa}(x)$, $D_C^{-\kappa}(x) \subseteq D_F^{-\kappa}(x) \subseteq D_E^{-\kappa}(x)$ are not always true.

Definition 2.6. Given an IODS $S = (U, A, V, f)$, let L be the amount of $f(x_i, a_i)$, which s.t. $f(x_i, a_i) = * \vee f(x_i, a_i) = ?$, $a_i \in C, x_i \in U$. Then a *default rate* $\lambda (0 \leqslant \lambda < 1)$ is defined as: $\lambda = \frac{L}{|U| * |A|}$.

Property 2.2. *For any $0 \leqslant \lambda_1 \leqslant \lambda_2 < 1$, let $D_C^{+\kappa}(x)^{\lambda_i}$ denote $D_C^{+\kappa}(x)$ of IODS having a default rate λ_i $(i = 1, 2)$. $D_C^{+\kappa}(x)^{\lambda_2} \subseteq D_C^{+\kappa}(x)^{\lambda_1}$, $D_C^{-\kappa}(x)^{\lambda_2} \subseteq D_C^{-\kappa}(x)^{\lambda_1}$ is not always true.*

The set of decision attributes D partitions U into a finite number of classes. Let $Cl = \{Cl_t | t \in \{1, \ldots, n\}\}$, where $Cl_n \succ \cdots \succ Cl_s \succ \cdots \succ Cl_1$. An upward union and a downward union of classes are defined respectively as follows. $Cl_t^{\geqslant} = \cup_{s \geqslant t} Cl_s$, $Cl_t^{\leqslant} = \cup_{s \leqslant t} Cl_s$, where $t, s \in \{1, \ldots, n\}$. $x \in Cl_t^{\geqslant}$ means x at least belongs to class Cl_t. $x \in Cl_t^{\leqslant}$ means x at most belongs to class Cl_t. The monotonicity in the IODS is that if an object x is at least as good as another object y on all condition attributes under the EDCR, then the object x should be assigned to a class not worse than y.

Definition 2.7. Let $S = (U, A, V, f)$ be an IODS, $P \subseteq C$, $x \in U$, $Cl_t^{\geqslant}$, $Cl_t^{\leqslant} \subseteq U$, $t = 1, 2, \ldots, n$. The upper and lower approximations of $Cl_t^{\geqslant}$ and $Cl_t^{\leqslant}$ under the EDCR are defined respectively as follows:

$$\underline{P}(Cl_t^{\geqslant})^\kappa = \{x | D_P^{+\kappa}(x) \subseteq Cl_t^{\geqslant}\}, \quad \overline{P}(Cl_t^{\geqslant})^\kappa = \cup_{x \in Cl_t^{\geqslant}} D_P^{+\kappa}(x) \tag{4}$$

$$\underline{P}\left(Cl_t^{\leqslant}\right)^\kappa = \left\{x | D_P^{-\kappa}(x) \subseteq Cl_t^{\leqslant}\right\}, \quad \overline{P}\left(Cl_t^{\leqslant}\right)^\kappa = \cup_{x \in Cl_t^{\leqslant}} D_P^{-\kappa}(x) \tag{5}$$

The positive region and boundary region are defined respectively as follows:

$$Pos_P(Cl_t^{\geqslant})^\kappa = \underline{P}(Cl_t^{\geqslant})^\kappa, \quad Pos_P\left(Cl_t^{\leqslant}\right)^\kappa = \underline{P}\left(Cl_t^{\leqslant}\right)^\kappa \tag{6}$$

$$Bn_P(Cl_t^{\geqslant})^\kappa = \overline{P}(Cl_t^{\geqslant})^\kappa - \underline{P}(Cl_t^{\geqslant})^\kappa, \quad Bn_P\left(Cl_t^{\leqslant}\right)^\kappa = \overline{P}\left(Cl_t^{\leqslant}\right)^\kappa - \underline{P}\left(Cl_t^{\leqslant}\right)^\kappa \tag{7}$$

The objects in the boundary region are inconsistent, which violate the monotone properties of the data in the IODS, *i.e.*, an object x is not worse than another object y in all condition attributes under the EDCR. However, x assigned to a class is worse than y.

Information granularity measure is the average measurement of the size of information granules under the attribute set. Liang and Shi defined the granularity of knowledge under the tolerance relation in IIS [72]. Furthermore, they discussed the property of dynamic granularity when attributes evolve with time. Xu et al. defined the granularity measure and rough entropy in a complete order information system [73]. Similarly, we extend the definition of an information granule to the EDCR as follows.

Let $D_P^{\triangle\kappa}(x_i)$ denote the dominating or dominated class in an IODS under the EDCR. The superscript $\triangle$ is substituted by + or $-$. When $\triangle$ is substituted by +, $D_P^{+\kappa}(x_i)$ is the dominating class of x_i. When $\triangle$ is substituted by $-$, $D_P^{-\kappa}(x_i)$ is the dominated class of x_i.

Definition 2.8. Let $D^\triangle(P) = \{D_P^{\triangle\kappa}(x_i) | x_i \in U, P \subseteq C\}$. Then, $\cup_{x_i \in U} D_P^{\triangle\kappa}(x_i) = U (1 \leqslant i \leqslant |U|)$, $D_P^{\triangle\kappa}(x_i) \neq \emptyset$, $D^\triangle(P)$ is a covering of S. Let

$$\widehat{D^\triangle}(P) = \{D_P^{\triangle\kappa}(x_i) | D_P^{\triangle\kappa}(x_i) = \{x_i\}, x_i \in U\} \tag{10}$$

$$\overset{\vee}{D^\triangle}(P) = \{D_P^{\triangle\kappa}(x_i) | D_P^{\triangle\kappa}(x_i) = U, x_i \in U\} \tag{11}$$

Then denote $\widehat{D^\triangle}(P)$ an identical relation under EDCR. $\overset{\vee}{D^\triangle}(P)$ is a universal relation under EDCR. When $\triangle$ is +, $\widetilde{D^+}(P)$ is a family of the dominating set. When $\triangle$ is $-$, $\widetilde{D^-}(P)$ is a family of the dominated set.

Definition 2.9. Let $S = (U, A, V, f)$ be an IODS, $K(P) = \{D_P^{\triangle\kappa}(x_i), x_i \in U, i = 1, 2, \ldots, |U|\}$, $\forall P \subseteq C \subseteq A$. The granularity measure of P is defined as follows:

$$GK^{\triangle\kappa}(P) = -\sum_{i=1}^{|U|} \frac{1}{|U|} \log_2 \frac{|D_P^{\triangle\kappa}(x_i)|}{|U|} \tag{12}$$

If $D_P^{\triangle\kappa}(x_i) = \widehat{D}(P)$, the information granularity measure of P can achieve the minimum value 0; If $D_P^{\triangle\kappa}(x_i) = \check{D}(P)$, the information granularity measure of P can achieve the maximum value $\log_2|U|$. When $\triangle$ is +, $GK^{+\kappa}(P)$ is the granularity measure of P induced by the dominating classes. When $\triangle$ is −, $GK^{-\kappa}(P)$ is the granularity measure of P induced by the dominated set.

Property 2.3. *Given an IODS $S = (U, A, V, f)$, the following properties hold.*

(1) *For any α, β, $0 \leqslant \alpha \leqslant \beta \leqslant 1$, $GK^{\triangle\alpha}(P) \geqslant GK^{\triangle\beta}(P)$.*
(2) *For any $E \subseteq F \subseteq C$, $0 \leqslant k \leqslant 1$, $GK^{\triangle\kappa}(C) < GK^{\triangle\kappa}(E) < GK^{\triangle\kappa}(F)$ is not always true.*

Remark 1. For any k, $0 \leqslant k \leqslant 1$, $GK^{+\kappa}(P) \neq GK^{-\kappa}(P)$, generally.

Definition 2.10. Let $S = (U, A, V, f)$ be an IODS, $K(P) = \{D_P^{\triangle\kappa}(x_i), x_i\ inU,\ i = 1, 2, \ldots, |U|\}$, $\forall P \subseteq C \subseteq A$. Then the rough entropy of P is defined as follows:

$$E_r^{\triangle\kappa}(P) = -\sum_{i=1}^{|U|} \frac{1}{|U|} \log_2 \frac{1}{|D_P^{\triangle\kappa}(x_i)|} \tag{13}$$

If $D_P^{\triangle\kappa}(x_i) = \widehat{D}(P)$, the rough entropy of P can reach the minimum value 0; If $D_P^{\triangle\kappa}(x_i) = \check{D}(P)$, the rough entropy of P can obtain the maximum value $\log_2|U|$. When $\triangle$ is +, $E_r^{+\kappa}(P)$ is the rough entropy of P induced by the dominating set. When $\triangle$ is −, $E_r^{-\kappa}(P)$ is the rough entropy of P induced by the dominated set.

Property 2.4. *Given an IODS $S = (U, A, V, f)$, the following properties hold.*

(1) *For any α, β, $0 \leqslant \alpha \leqslant \beta \leqslant 1$, $E_r^{\triangle\alpha}(P) \leqslant E_r^{\triangle\beta}(P)$.*
(2) *For any $E \subseteq F \subseteq C$, $0 \leqslant k \leqslant 1$, $E_r^{\triangle k}(E) < E_r^{\triangle k}(F) < E_r^{\triangle k}(C)$ is not always true.*

Remark 2. For any k, $0 \leqslant k \leqslant 1$, $E_r^{+\kappa}(P) \neq E_r^{-\kappa}(P)$, generally.

In a dynamic information system, we denote IS^t as the IODS at time t. $D_P^{\triangle\kappa}(x_i)^t$ denotes the dominating classes or dominated classes of the IODS at time t. $GK^{\triangle\kappa}(P)^t$ and $E_r^{\triangle\kappa}(P)^t$ denote the granularity of P at time t, respectively. Let $\underline{P}(Cl_t^{\geqslant})^{\kappa t}$ and $\overline{P}(Cl_t^{\geqslant})^{\kappa t}$ denote the lower and upper approximations of $Cl_t^{\geqslant}$, respectively.

Proposition 2.1. *For any $x_i \in U$,*

(1) *If $D_P^{\triangle\kappa}(x_i)^t \subseteq D_P^{\triangle\kappa}(x_i)^{t+1}$, then $GK^{\triangle\kappa}(P)^t \leqslant GK^{\triangle\kappa}(P)^{t+1}$, $E_r^{\triangle\kappa}(P)^t \geqslant E_r^{\triangle\kappa}(P)^{t+1}$.*
(2) *If $D_P^{\triangle\kappa}(x_i)^t \subseteq D_P^{\triangle\kappa}(x_i)^{t+1}$, then $\underline{P}(Cl_t^{\geqslant})^{\kappa t} \subseteq \underline{P}(Cl_t^{\geqslant})^{\kappa(t+1)}$, $\overline{P}(Cl_t^{\geqslant})^{\kappa(t+1)} \subseteq \overline{P}(Cl_t^{\geqslant})^{\kappa t}$.*

3. Maintenance of approximations dynamically in the IODS

The coarsening and refining of attributes' values are required under the changing requirement of applications or revising errors [33]. One example of the semantic level in real-life applications is given in Fig. 1. The higher semantic level in the Fig. 1 means the description is coarser and more general. When the semantic level moves downward, it means the description is refiner and more detailed.

In Fig. 2, we give an illustration of variation of an attribute value in the IODS. Points in Fig. 2 denote the objects described by two attributes a_1 and a_2. In Fig. 2a, plus points in the region R_b is the dominating class of the point (2, 2) and circle points in the region R_c are the dominated class of the point (2, 2). When the attribute value on a_2 of the point (2, 2) varies from 2 to 3.5, see Fig. 2b. The dominating and dominated relations between objects alter. In Fig. 2b, the points in the region R_{b2} are no longer including in the dominating class of the point (2, 3.5) and the points in the region R_{a2} are dominated by the point (2, 3.5). Similarly, the points in the region R_{c1} and R_{d1} are affected while the attribute value on R_{a2}

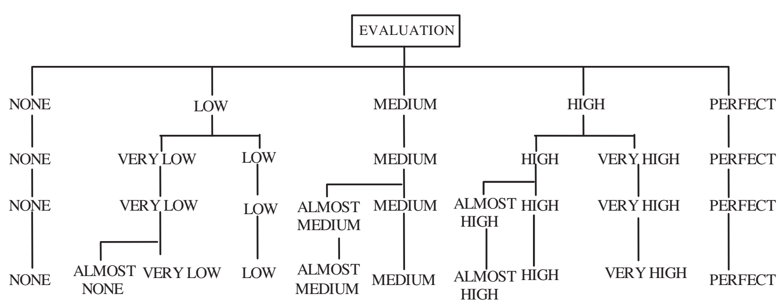

Fig. 1. The semantic level.

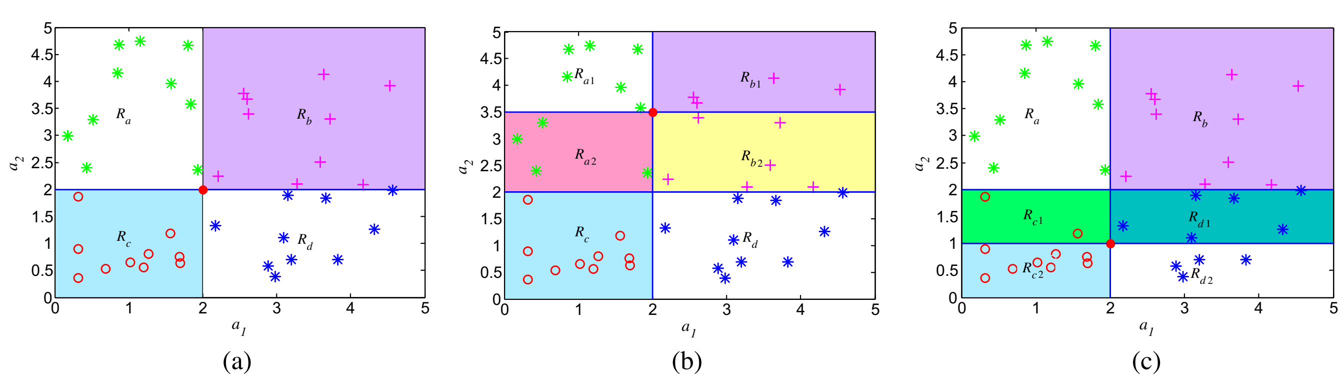

Fig. 2. An illustration of variation of an attribute value in the IODS.

of the point $(2,2)$ varies from 2 to 1 as shown in Fig. 2c. From this illustration, different directions and levels of variation will alter the dominating relation among objects.

In this following, we first give the definition of AVCR in the IODS, which is different from the definition in TRS due to the partial order exists in the attributes' value set. We then discuss the variation of granularity in term of AVCR and propose the incremental updating principle for granules. We further present algorithms for updating approximations of the upward and downward unions dynamically.

3.1. Incremental updating principle while refining attribute values

The definition of Attribute Values Refining (AVR) in the IODS is given as follows.

Definition 3.1. Let $S=(U,A,V,f)$ be an IODS, $B\subseteq A$, $a_l\in B$, $f(x_i,a_l)$ is the attribute value of object x_i under attribute a_l. Then $U_{a_l}=\{x_{i'}\in U|f(x_{i'},a_l)=f(x_i,a_l)\wedge f(x_i,a_l)\neq *\wedge f(x_i,a_l)\neq ?\}$. Let $f(x_{i'},a_l)=v,\ \exists x_{i'}\in U_{a_l},\ \exists v\notin V_l,\ v\neq *$ and $v\neq ?$. Then,

(1) If $f\left(x_{i'}^{\vee},a_l\right)\prec f(x_{i'},a_l)$, then we call $f(x_{i'},a_l)$ is refined downward. $x_{i'}^{\vee-}$ denotes object $x_{i'}$ after refining downward. Let $V^{\vee-}=\left\{v|f\left(x_{i'}^{\vee-},a_l\right)\prec v\prec f(x_{i'},a_l),v\in V_{a_l}\right\}$. If $V^{\vee-}=\emptyset$, then $f(x_{i'},a_l)$ is refined downward single level. If $V^{\vee-}\neq\emptyset$, then $f(x_{i'},a_l)$ is refined downward multi-level.

(2) If $f\left(x_{i'}^{\vee},a_l\right)\succ f(x_{i'},a_l)$, then we call $f(x_{i'},a_l)$ is refined upward. $x_{i'}^{\vee+}$ denotes the object $x_{i'}$ after refining upward. Let $V^{\vee+}=\left\{v|f(x_{i'},a_l)\prec v\prec f\left(x_{i'}^{\vee+},a_l\right),v\in V_{a_l}\right\}$. If $V^{\vee+}=\emptyset$, we call $f(x_{i'},a_l)$ is refined upward single level. If $V^{\vee+}\neq\emptyset$, we call $f(x_{i'},a_l)$ is refined upward multi-level. $V_{a_l}^{\vee+}$ denotes the domain of attribute $a_l^{\vee}$ after refining upward. $V_{a_l}^{\vee-}$ denotes the domain of attribute $a_l^{\vee}$ after refining downward.

Example 3.1. Let $S=(U,A,V,f)$ be an IODS, $V_{a_l}=\{VL,L,M,H,VH\}$, $\exists VL\prec L\prec M\prec H\prec VH$, $a_l\in C_1$, $\exists f(x_i,a_l)=M$. Then

(1) If $f(x_i^{\vee-},a_l)=AM$, $AM\prec M$, then the attribute value is refined downward. There are two cases of refining downward as follows:
  i. $VL\prec L\prec AM\prec M\prec H\prec VH$, $V_{a_l}^{\vee-}=\{VL,L,AM,M,H,VH\}$.
  ii. $VL\prec AM\prec L\prec M\prec H\prec VH$, $V_{a_l}^{\vee-}=\{VL,AM,L,M,H,VH\}$.

Case i belongs to single level refining due to the need in the variation of precisions. Case ii belongs to multi-level refining due to the need to revise errors or other special requirements.

(2) If $f(x_i^{\vee+},a_l)=AH$, $M\prec$ AH, then the attribute value is refined upward. There may be two cases of refining upward as follows:
  i. $VL\prec L\prec M\prec AH\prec H\prec VH$, $V_{a_l}^{\vee+}=\{VL,L,M,AH,H,VH\}$.
  ii. $VL\prec L\prec M\prec H\prec AH\prec VH$, $V_{a_l}^{\vee+}=\{VL,L,M,H,AH,VH\}$.

Case i belongs to single level refining upward. Case ii belongs to multi-level refining upward.

We define an EDCR in the IODS. Extended dominate sets form the covering of IODS. It is a realization of granules in GrC. The variation of granularities on different levels and different directions of AVCR are discussed as follows. For convenience, let $\overline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa}$ and $\underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa}$ denote upper and lower approximations of an upward union of classes after refining downward, respectively. Let $\overline{P}^{\vee+}(Cl_t^{\geqslant})^{\kappa}$ and $\underline{P}^{\vee+}(Cl_t^{\geqslant})^{\kappa}$ denote upper and lower approximations of an upward union of classes after refining upward, respectively. The upper and lower approximations of a downward union of classes after refining downward and upward are denoted as follows respectively: $\overline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa},\ \underline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa},\overline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa},\ \underline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa}$.

(1) Refining downward
If $f(x_i,a_l)=v_1$, let $f\left(x_i^{\vee-},a_l\right)=v_2$, then $v_2\prec v_1$. Let $V^{\vee-}=\{v|v_1\prec v_l\prec v_2,v_l\in V_{a_l}\}$, $v_{a_l}^{\max}=\max(V_{a_l})$, $v_{a_l}^{\min}=\min(V_{a_l})$>. We not only consider single level and multi-level refining but also consider the case when the value of refining upward is larger than the maximum value in the value domain or the case when the value of refining downward is smaller than the minimum value in the value domain to improve the efficiency of incremental updating. Because in these two cases the computation time of updating of dominating classes or dominated classes will be less than other cases.

- $V^{\vee-}\neq\emptyset$ (the attribute value is refined downward multi-level)

If the object x_i is refined downward multi-level, then it will not dominate the object x_j which *s.t.* $x_i\in D_P^{+\kappa}(x_j)$ and $v_2\prec f(x_j,a_l)\preceq v_1$. If x_j *s.t.* $x_i\notin D_P^{+\kappa}(x_j)$ and $v_2\prec f(x_j,a_l)\prec v_1$, then it may dominate x_i. Therefore, the granules induced by dominating classes and dominated classes may alter.

Property 3.1. *Let* $C_1=\{x_j|x_i\in D_P^{+\kappa}(x_j)\wedge v_2\prec f(x_j,a_l)\preceq v_1\wedge(f(x_j,a_l)\neq *\vee f(x_j,a_l)\neq ?)\}$, $C_2=\{x_j|x_i\notin D_P^{+\kappa}(x_j)\wedge v_2\prec f(x_j,a_l)\prec v_1\wedge f(x_j,a_l)\neq *\vee f(x_j,a_l)\neq ?\}$, $\forall x_k\in U$. *For* $D_P^{+\kappa}(x_k)$, *the following results hold:*

i. $\forall x_j\in C_1,\ D_P^{+\kappa}\left(x_j^{\vee-}\right)=D_P^{+\kappa}(x_j)-\{x_i^{\vee-}\}$.
ii. $\forall x_j\in C_2$, *if* $C_3=\{x_j\big|x_jD_P^{+\kappa}x_i^{\vee-},x_j\in C_2\}$, *then* $D_P^{+\kappa}\left(x_i^{\vee-}\right)=D_P^{+\kappa}(x_i)\cup C_3$.

Proof

i. $\forall x_j\in C_1$, $\because x_i\in D_P^{+\kappa}(x_j)$, $\therefore f(x_j,a_l)\preceq f(x_i,a_l)$. $\because f\left(x_i^{\vee-},a_l\right)=v_2\prec v_1$, $v_2\prec f(x_j,a_l)\preceq v_1$. $\therefore f(x_j,a_l)\succ f\left(x_i^{\vee-},a_l\right)$, $\therefore x_i^{\vee-}\notin D_P^{+\kappa}\left(x_j^{\vee-}\right)$, $D_P^{+\kappa}\left(x_j^{\vee-}\right)=D_P^{+\kappa}(x_j)-\{x_i^{\vee-}\}$.
ii. $\forall x_j\in C_2$, $\because x_i\notin D_P^{+\kappa}(x_j)$, $v_2\prec f(x_j,a_l)\prec f(x_i,a_l)=v_1$, $f\left(x_i^{\vee-},a_l\right)=v_2\prec f(x_j,a_l)$. $\therefore$ If $x_jD_P^{+\kappa}x_i^{\vee-}$, then $D_P^{+\kappa}\left(x_i^{\vee-}\right)=D_P^{+\kappa}(x_i)\cup\{x_j\}$. $\because C_3=\{x_j\big|x_jD_P^{+\kappa}x_i^{\vee-},x_j\in C_2\}$, $\therefore D_P^{+\kappa}\left(x_i^{\vee-}\right)=D_P^{+\kappa}(x_i)\cup C_3$. □

Considering the variation of dominating classes in Property 3.1, we propose an updating method for upward union $Cl_t^{\geqslant}$ as follows.

Proposition 3.1. *For* $\underline{P}(Cl_t^{\geqslant})^{\kappa}$ *and* $\overline{P}(Cl_t^{\geqslant})^{\kappa}$, *the following results hold:*

i. $\forall x_j\in C_1$
  (a) *If* $x_j\in Bn_P(Cl_t^{\geqslant})^{\kappa}$, $x_i^{\vee-}\notin Cl_t^{\geqslant}$ *and* $D_P^{+\kappa}\left(x_j^{\vee-}\right)\subseteq Cl_t^{\geqslant}$, *then* $\underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa}=\underline{P}(Cl_t^{\geqslant})^{\kappa}\cup\{x_j\}$.
  (b) *If* $x_i^{\vee-}\notin D_P^{+\kappa}(x_k)$, $\forall x_k\in Cl_t^{\geqslant}$, *then* $\overline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa}=\overline{P}(Cl_t^{\geqslant})^{\kappa}-\{x_i\}$.
  (c) *Otherwise,* $\overline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa}=\overline{P}(Cl_t^{\geqslant})^{\kappa}$, $\underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa}=\underline{P}(Cl_t^{\geqslant})^{\kappa}$.
ii. $\forall x_j\in C_2$.
  (a) *If* $x_i\in\underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa}$, $C_3\neq\emptyset$ *and* $\forall x_j\in C_3$, $\exists x_j\notin C_t^{\geqslant}$, *then* $\underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa}=\underline{P}(Cl_t^{\geqslant})^{\kappa}-\{x_i^{\vee-}\}$.
  (b) *If* $x_i^{\vee-}\in Cl_t^{\geqslant}$, *then* $\overline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa}=\overline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa}\cup C_3$.
  (c) *Otherwise,* $\overline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa}=\overline{P}(Cl_t^{\geqslant})^{\kappa}$, $\underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa}=\underline{P}(Cl_t^{\geqslant})^{\kappa}$.

Proof

i. $\forall x_j\in C_1$, by Definition 3.1, $D_P^{+\kappa}\left(x_j^{\vee-}\right)=D_P^{+\kappa}(x_j)-\{x_i^{\vee-}\}$. (a) If $x_j\in Bn_P(Cl_t^{\geqslant})^{\kappa}$, $x_i^{\vee-}\notin Cl_t^{\geqslant}$, then $D_P^{+\kappa}(x_j^{\vee-})\subset D_P^{+\kappa}(x_j)$. If

$D_P^{+\kappa}(x_j^{\vee-}) \subseteq Cl_t^{\geqslant}$, then $\underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_j\}$. (b) $\because$ $\overline{P}(Cl_t^{\geqslant})^{\kappa} = \cup_{x \in Cl_t^{\geqslant}} D_P^{+\kappa}(x)$, $D_P^{+\kappa}\left(x_j^{\vee-}\right) = D_P^{+\kappa}(x_j) - \{x_i^{\vee-}\}$. $\therefore$ If $x_i^{\vee-} \notin D_P^{+\kappa}(x_k)$, $\forall x_k \in Cl_t^{\geqslant}$, then $\overline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa} - \{x_i\}$. (c) $\because$ $D_P^{+\kappa}(x_j^{\vee-}) = D_P^{+\kappa}(x_j) - \{x_i^{\vee-}\} \subset D_P^{+\kappa}(x_j)$. $\therefore$ If $D_P^{+\kappa}(x_j) \subseteq Cl_t^{\geqslant}$, then $D_P^{+\kappa}\left(x_j^{\vee-}\right) \subseteq Cl_t^{\geqslant}$; If $D_P^{+\kappa}(x_j) \not\subset Cl_t^{\geqslant}$, then $D_P^{+\kappa}\left(x_j^{\vee-}\right) \not\subset Cl_t^{\geqslant}$; $\therefore \underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa}$. In addition, if $\exists x_i^{\vee-} \in D_P^{+\kappa}(x_k)$, $\forall x_k \in Cl_t^{\geqslant}$, then $\overline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa}$.

ii. $\forall x_j \in C_2$, (a) $\because D_P^{+\kappa}(x_i^{\vee-}) = D_P^{+\kappa}(x_i) \cup C_3$, $\therefore D_P^{+\kappa}(x_i^{\vee-}) \supseteq D_P^{+\kappa}(x_i)$. $\therefore$ If $x_i \in \underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa}$, then $D_P^{+\kappa}(x_i) \subseteq Cl_t^{\geqslant}$. If $C_3 \neq \varnothing$ and $\forall x_j \in C_3$, $\exists x_j \notin C_t^{\geqslant}$, then $D_P^{+\kappa}(x_i^{\vee-}) \not\subset Cl_t^{\geqslant}$; $\underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} = \underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} - \{x_i^{\vee-}\}$. (b) $\overline{P}(Cl_t^{\geqslant})^{\kappa} = \cup_{x \in Cl_t^{\geqslant}} D_P^{+\kappa}(x)$, If $x_i \in Cl_t^{\geqslant}$, then $\overline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} = \overline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} \cup C_3$. (c) $\overline{P}(Cl_t^{\geqslant})^{\kappa} = \cup_{x \in Cl_t^{\geqslant}} D_P^{+\kappa}(x)$, $\therefore$ If $x_i \notin Cl_t^{\geqslant}$, then $\overline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa} \because$ $D_P^{+\kappa}(x_i^{\vee-}) = D_P^{+\kappa}(x_i) \cup C_3$, $\therefore$ If $x_i \notin \underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa}$, then $D_P^{+\kappa}(x_i) \not\subset Cl_t^{\geqslant}$. $\therefore$ $D_P^{+\kappa}(x_i^{\vee-}) \not\subset Cl_t^{\geqslant}$. $\therefore$ If $x_i \in Cl_t^{\geqslant}$, then $\overline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} = \overline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} \cup C_3$. □

When x_i is coarsening downward multi-level, x_j *s.t.* $x_j \in D_P^{-k}(x_i)$ and $v_2 \prec f(x_j, a_l) \preceq v_1$ will not be dominated by x_i and x_j *s.t.* $x_j \notin D_P^{-k}(x_i)$ and $v_2 \prec f(x_j, a_l) \prec v_1$ will dominate x_i. We propose a method for updating dominated classes as follows.

Property 3.2. *Let* $C_1 = \left\{x_j \middle| x_j \in D_P^{-k}(x_i) \wedge v_2 \prec f(x_j, a_l) \preceq v_1 \wedge (f(x_j, a_l) \neq * \vee f(x_j, a_l) \neq ?)\right\}$, $C_2 = \left\{x_j | x_j \notin D_P^{-k}(x_i) \wedge v_2 \prec f(x_j, a_l) \prec v_1 \wedge (f(x_j, a_l) \neq * \vee f(x_j, a_l) \neq ?)\right\}$. *Then*

i. $D_P^{-\kappa}(x_i^{\vee-}) = D_P^{-\kappa}(x_i) - C_1$.
ii. $C_3 = \{x_j | x_i^{\vee-} D_P^{-\kappa} x_j, \forall x_j \in C_2\}$, $\forall x_j \in C_3, D_P^{-\kappa}\left(x_j^{\vee-}\right) = D_P^{-\kappa}(x_j) \cup \{x_i^{\vee-}\}$.

Proof. i. $\forall x_j \in C_1$, $\because x_j \in D_P^{-k}(x_i)$, $\therefore f(x_j, a_l) \prec f(x_i, a_l)$. $\because f(x_i^{\vee-}, a_l) = v_2$, $\therefore v_2 \prec f(x_j, a_l) \preceq v_1$, $x_j \notin D_P^{-k}(x_i)$; $\therefore$ $D_P^{-\kappa}(x_i^{\vee-}) = D_P^{-\kappa}(x_i) - C_1$. ii. $\forall x_j \in C_2$, $\because f(x_i^{\vee-}, a_l) = v_2 \prec f(x_j, a_l)$, $\therefore$ If $x_i^{\vee-} D_P^{-\kappa} x_j$, then $D_P^{-\kappa}\left(x_j^{\vee-}\right) = D_P^{-\kappa}(x_j) \cup \{x_i^{\vee-}\}$.

In the following, the principle for updating the approximations of downward union $Cl_t^{\leqslant}$ is presented when attribute values coarsening downward multi-level. □

Proposition 3.2. *For* $\underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $\overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$, *the following results hold.*

i.
(a) *If* $x_i \in Bn_P\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $D_P^{-\kappa}(x_i^{\vee-}) \subseteq \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$, *then* $\underline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_i^{\vee-}\}$.
(b) *If* $x_i \in Cl_t^{\geqslant}$ *and* $\exists C' = \{x_k | x_k \in C_1 \wedge x_k \notin D_P^{-\kappa}(x'),$ $\forall x' \in Cl_t^{\geqslant}\}$, *then* $\overline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - C'$.
(c) *Otherwise,* $\overline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$, $\underline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$.

ii. $\forall x_j \in C_3$
(a) *If* $x_j \in \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $x_i^{\vee-} \notin Cl_t^{\leqslant}$, *then* $\underline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_j\}$.
(b) *If* $x_j \in Cl_t^{\leqslant}$ *and* $x_i \notin D_P^{-\kappa}(x'), \forall x' \in Cl_t^{\leqslant}$, *then* $\overline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_i^{\vee-}\}$.
(c) *Otherwise,* $\overline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$, $\underline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$.

Proof

i.
(a) $\because D_P^{-\kappa}(x_i^{\vee-}) = D_P^{-\kappa}(x_i) - C_1$, $\therefore D_P^{-\kappa}(x_i^{\vee-}) \subseteq D_P^{-\kappa}(x_i)$. $\therefore$ If $x_i \in Bn_P\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $D_P^{-\kappa}(x_i^{\vee-}) \subseteq \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$, then $\underline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_i^{\vee-}\}$; (b) $\because$ $\overline{P}(Cl_t^{\geqslant})^{\kappa} = \cup_{x \in Cl_t^{\geqslant}} D_P^{+\kappa}(x)$, $D_P^{-\kappa}(x_i^{\vee-}) = D_P^{-\kappa}(x_i) - C_1$, $\therefore x_i \in Cl_t^{\geqslant}$ and $\exists C' = \{x_k | x_k \in C_1 \wedge x_k \notin D_P^{-\kappa}(x'), \forall x' \in Cl_t^{\geqslant}\}$. Then $\overline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - C'$; (c) It is obvious. □

- $V^{\vee-} = \varnothing$ (refining downward single level).

If x_i is refining downward single level, then it will not dominate x_j *s.t.* $x_i \in D_P^{+\kappa}(x_j) \wedge f(x_j, a_l) = v_1$. The incremental updating principle for dominating classes is illustrated as follows.

Property 3.3. *Let* $C_1 = \{x_j | x_i \in D_P^{+\kappa}(x_j) \wedge f(x_j, a_l) = v_1\}$. *Then* $\forall x_j \in C_1$, $D_P^{+\kappa}\left(x_j^{\vee-}\right) = D_P^{+\kappa}(x_j) - \{x_i^{\vee-}\}$.

Proof. $\forall x_j \in C_1$, $\because x_i \in D_P^{+\kappa}(x_j)$, $\therefore f(x_j, a_l) = v_1 \prec f(x_i, a_l)$. $\because f(x_j, a_l) = v_1 \succ f(x_i^{\vee-}, a_l) = v_2$, $\therefore D_P^{+\kappa}\left(x_j^{\vee-}\right) = D_P^{+\kappa}(x_j) - \{x_i^{\vee-}\}$. □

Considering the variation of dominating classes when attribute values coarsening downward single level, the updating principle for approximations of upward union $Cl_t^{\geqslant}$ is given as follows.

Proposition 3.3. *For* $\underline{P}(Cl_t^{\geqslant})^{\kappa}$ *and* $\overline{P}(Cl_t^{\geqslant})^{\kappa}$, $\forall x_j \in C_1$, *the following results hold:*

i. If $x_j \in Bn_P\left(Cl_t^{\leqslant}\right)^{\kappa}$, $x_i \notin Cl_t^{\geqslant}$ and $D_P^{+\kappa}\left(x_j^{\vee-}\right) \subseteq Cl_t^{\geqslant}$, then $\underline{P}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_j\}$.
ii. If $x_j \in Cl_t^{\geqslant}$ and $x_i \notin D_P^{-\kappa}(x')$, $x' \in Cl_t^{\geqslant}$, then $\overline{P}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa} - \{x_i^{\vee-}\}$.
iii. Otherwise, $\overline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa}$, $\underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa}$.

Proof. It is similar to the proof of Proposition 3.1(i).

If x_j *s.t.* $x_j \in D_P^{-\kappa}(x_i)$ and $f(x_j, a_l) = v_1$, then it will not be dominated by x_i. The principle for updating dominated classes is presented as follows. □

Property 3.4. *Let* $C_1 = \{x_j | x_j \in D_P^{-\kappa}(x_i) \wedge f(x_j, a_l) = v_1\}$. *Then* $D_P^{-\kappa}(x_i^{\vee-}) = D_P^{-\kappa}(x_i) - C_1$.

Proof. It is similar to the proof of Property 3.3. □

In the following, we propose the principle for updating approximations of downward union $Cl_t^{\leqslant}$.

Proposition 3.4. *For* $\underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $\overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$, *the following results hold:*

i. *If* $x_i \notin Bn_P\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $D_P^{-\kappa}(x_i^{\vee-}) \subseteq Cl_t^{\leqslant}$, *then* $\underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_i\}$.
ii. *If* $x_i \in Cl_t^{\leqslant}$ *and* $\exists C' = \left\{x_k | x_k \in C_1 \wedge x_k \notin D_P^{-\kappa}(x'), x' \in Cl_t^{\leqslant}\right\}$, *then* $\overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - C'$.
iii. *Otherwise,* $\overline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$, $\underline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$.

Proof. It is similar to the proof of Proposition 3.2(ii). □

- $v_2 \prec v_{a_l}^{\min}$.

In this case, no object in the universe will be dominated by x_i. If x_j s.t. $x_i \notin D_P^{+\kappa}(x_j)$, $f(x_j, a_l) \prec v_1$, then it may dominate x_i. The updating principle for dominating classes is given as follows.

Property 3.5. *Let* $C_1 = \{x_j | x_i \in D_P^{+\kappa}(x_j)\}$, $C_2 = \{x_j | x_i \notin D_P^{+\kappa}(x_j) \wedge f(x_j, a_l) \prec v_1\}$. *Then*

i. $\forall x_j \in C_1,\ D_P^{+\kappa}\left(x_j^{\vee-}\right) = D_P^{+\kappa}(x_j) - \{x_i\}$.
ii. $C_3 = \left\{x_j | x_j D_P^{+\kappa} x_i^{\vee-}, \forall x_j \in C_2\right\},\ D_P^{+\kappa}\left(x_i^{\vee-}\right) = D_P^{+\kappa}(x_i) \cup C_3$.

Proof. It is similar to the proof of Property 3.1. □

In the following, we present the principle for updating approximations of upward union $Cl_t^{\geqslant}$.

Proposition 3.5. *For* $\underline{P}(Cl_t^{\geqslant})^{\kappa}$ *and* $\overline{P}(Cl_t^{\geqslant})^{\kappa}$, *the following results hold:*

i. $\forall x_j \in C_1$
(a) *If* $x_j \in Bn_P\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $D_P^{+\kappa}\left(x_j^{\vee-}\right) \subseteq Cl_t^{\geqslant}$, *then* $\underline{P}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_j\}$.
(b) *If* $x_j \in Cl_t^{\leqslant}$ *and* $x_i \notin D_P^{+\kappa}(x'), x' \in Cl_t^{\leqslant}$, *then* $\overline{P}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa} - \{x_i\}$.
(c) *Otherwise*, $\overline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa}$, $\underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa}$.
ii. $\forall x_j \in C_2$
(a) *If* $x_i \in \underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa}$ *and* $C_3 \not\subset Cl_t^{\geqslant}$, *then* $\underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa} - \{x_i\}$.
(b) *If* $x_i \in Cl_t^{\leqslant}$, *then* $\overline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa} \cup C_3$.
(c) *Otherwise*, $\overline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa}$, $\underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa}$.

Proof. It is similar to the proof of Proposition 3.1. □

When $f(x_i^{\vee-}, a_l) \prec v_{a_l}^{\min}$, no object will be dominated by x_i. The object x_j s.t.$x_j \notin D_P^{-\kappa}(x_i)$ and $f(x_j, a_l) \prec v_1$ may dominate x_i. The updating principle for dominated classes is presented as follows.

Property 3.6. *Let* $C_2 = \{x_j | x_i \notin D_P^{-\kappa}(x_j) \wedge f(x_j, a_l) \prec v_1\}$. *Then*

i. $D_P^{-\kappa}(x_i^{\vee-}) = \{x_i\}$.
ii. $C_3 = \left\{x_j | x_i^{\vee-} D_P^{-\kappa} x_j, \forall x_j \in C_2\right\},\ \forall x_j \in C_3,\ D_P^{-\kappa}\left(x_j^{\vee-}\right) = D_P^{-\kappa}(x_j) \cup \{x_i^{\vee-}\}$.

Proof. It is similar to the proof of Property 3.2. □

Considering the different variation of dominated classes, the updating principle for the approximations of downward union is given as follows.

Proposition 3.6. *For* $\underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $\overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$, *the following results hold.*

i.
(a) *If* $x_i \notin \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $x_i \in Cl_t^{\leqslant}$, *then* $\underline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_i^{\vee-}\}$.
(b) *If* $x_i \in Bn_p\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $x_i \notin Cl_t^{\leqslant}$, *then* $\underline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_i^{\vee-}\}$.
(c) *If* $x_i \in Cl_t^{\leqslant}$ *and* $C' = \{x_k | x_k \in D_P^{-\kappa}(x_i) \wedge x_k \notin D_P^{-\kappa}(x'), x' \in Cl_t^{\leqslant}\}$, *then* $\overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - C'$.
(d) *Otherwise*, $\overline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$, $\underline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$.
ii. $\forall x_j \in C_3$.
(a) *If* $x_j \in \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $x_i \notin Cl_t^{\leqslant}$, *then* $\underline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_j\}$.
(b) *If* $x_j \in Cl_t^{\leqslant}$ *and* $x_i \notin \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$, *then* $\overline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_i^{\vee-}\}$.
(c) *Otherwise*, $\overline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$, $\underline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$.

Proof. It is similar to the proof of Proposition 3.2. □

(2) Refining upward.
If $f(x_i, a_l) = v_1$, $f(x_i^{\vee+}, a_l) = v_2$, $v_2 \succ v_1$, the attribute values is refined upward. Let $V^{\vee+} = \{v | v_1 \prec v \prec v_2, v \in V_{a_l}\}$, $v_{a_l}^{\max} = \max(V_{a_l})$.

- If $V^{\vee+} \neq \emptyset$ (refining upward multi-level)

When x_i is refined upward multi-level, x_j s.t. $x_j \in D_P^{+\kappa}(x_i)$ and $v_1 \preceq f(x_j, a_l) \prec v_2$ will not dominate $x_i$and x_j s.t. $x_j \notin D_P^{+\kappa}(x_i)$ and $v_1 \preceq f(x_j, a_l) \prec v_2$ may be dominated by x_i.

Property 3.7. *Let* $C_1 = \{x_j | x_j \in D_P^{+\kappa}(x_i) \wedge v_1 \preceq f(x_j, a_l) \prec v_2, i \neq j\}$, $C_2 = \{x_j | x_j \notin D_P^{+\kappa}(x_i) \wedge v_1 \preceq f(x_j, a_l) \prec v_2\}$. *Then*

i. $D_P^{+\kappa}(x_i^{\vee+}) = D_P^{+\kappa}(x_i) - C_1$.
ii. $C_3 = \{x_j | x_i^{\vee+} D_P^{+\kappa} x_j, \forall x_j \in C_2\},\ \forall x_j \in C_3,\ D_P^{+\kappa}(x_j^{\vee+}) = D_P^{+\kappa}(x_j) \cup \{x_i^{\vee+}\}$.

Proof

i. $\forall x_j \in C_1$, $\because x_j \in D_P^{+\kappa}(x_i)$, $\therefore f(x_j, a_l) \succeq f(x_i, a_l)$. $v_1 \preceq f(x_j, a_l) \prec v_2$, $f(x_i^{\vee+}, a_l) = v_2 \succ f(x_j, a_l)$. $\because$ $x_j \notin$ $D_P^{+\kappa}(x_i^{\vee+})$, $\therefore D_P^{+\kappa}(x_i^{\vee+}) = D_P^{+\kappa}(x_i) - C_1$. ii. $\forall x_j \in C_2$, $\because x_j \notin D_P^{+\kappa}(x_i)$, $v_1 \preceq f(x_k, a_l) \prec v_2$, $f(x_i^{\vee+}, a_l) = v_2 \succ f(x_j, a_l)$. $\therefore$ If $x_i^{\vee+} D_P^{+\kappa} x_j$, then $D_P^{+\kappa}(x_j^{\vee+}) = D_P^{+\kappa}(x_j) \cup \{x_i^{\vee+}\}$. $\therefore$ $C_3 = \{x_j | x_i^{\vee+} D_P^{+\kappa} x_j, \forall x_j \in C_2\}$, $\forall$ $x_j \in C_3$, $D_P^{+\kappa}(x_j^{\vee+}) = D_P^{+\kappa}(x_j) \cup \{x_i^{\vee+}\}$. □

The principle for updating approximations of upward union $Cl_t^{\geqslant}$ when attribute value is refined upward multi-level is given as follows.

Proposition 3.7. *For* $\underline{P}(Cl_t^{\geqslant})^{\kappa}$ *and* $\overline{P}(Cl_t^{\geqslant})^{\kappa}$, *the following results hold:*

i. *For* $x_i^{\vee+}$,
(a) *If* $x_i \in Bn_P(Cl_t^{\geqslant})^{\kappa}$ *and* $D_P^{+\kappa}(x_i^{\vee+}) \subseteq Cl_t^{\geqslant}$, *then* $\underline{P}^{\vee+}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_i^{\vee+}\}$.
(b) *If* $x_i \in Cl_t^{\geqslant}$ *and* $C' = \{x_k | x_k \in C_1 \wedge x_k \notin D_P^{+\kappa}(x'), x' \in Cl_t^{\geqslant}\}$, *then* $\overline{P}^{\vee+}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa} - C'$.
(c) *Otherwise*, $\overline{P}^{\vee+}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa}$, $\underline{P}^{\vee+}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa}$.
ii. $\forall x_j \in C_3$
(a) *If* $x_j \in \underline{P}(Cl_t^{\geqslant})^{\kappa}$ *and* $C_3 \not\subset Cl_t^{\geqslant}$, *then* $\underline{P}^{\vee+}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa} - \{x_j\}$.
(b) *If* $x_i \notin \overline{P}(Cl_t^{\geqslant})^{\kappa}$ *and* $x_j \in Cl_t^{\geqslant}$, *then* $\overline{P}^{\vee+}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_i\}$.
(c) *Otherwise*, $\overline{P}^{\vee+}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa}$, $\underline{P}^{\vee+}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa}$.

Proof. It is similar to the proof of Proposition 3.1. □

If x_js.t. $x_i \in D_P^{-\kappa}(x_j)$ and $v_1 \preceq f(x_j, a_l) \prec v_2$, then x_i will not be dominated by x_j when attribute value is refined upward multi-level. For x_js.t. $x_i \notin D_P^{-\kappa}(x_j)$ and $v_1 \prec f(x_j, a_l) \prec v_2$, it may be dominated by x_i. The updating principle of dominated classes is shown as follows.

Property 3.8. *Let* $C_1 = \{x_j | x_i \in D_P^{-\kappa}(x_j) \wedge v_1 \preceq f(x_j, a_l) \prec v_2, i \neq j\}$, $C_2 = \{x_j | x_i \notin D_P^{-\kappa}(x_j) \wedge v_1 \prec f(x_j, a_l) \prec v_2\}$. *Then*

i. $\forall x_j \in C_1,\ D_P^{-\kappa}(x_j^{\vee +}) = D_P^{-\kappa}(x_j) - \{x_i\}$.
ii. $C_3 = \{x_j | x_j D_P^{-\kappa} x_i, \forall x_j \in C_2\},\ D_P^{-\kappa}(x_i^{\vee +}) = D_P^{-\kappa}(x_i) \cup C_3$.

Proof

i. $\forall x_j \in C_1,\ \because x_i \in D_P^{-\kappa}(x_j),\ \therefore f(x_i, a_l) \preceq f(x_j, a_l).\quad \because\ f(x_i^{\vee +}, a_l) = v_{l2} \succ f(x_j, a_l),\ \therefore x_i \notin D_P^{-\kappa}(x_j),\ D_P^{-\kappa}(x_j^{\vee +}) = D_P^{-\kappa}(x_j) - \{x_i\}$. ii. $\because \forall x_j \in C_2,\ \therefore x_i \notin D_P^{-\kappa}(x_j),\ \therefore f(x_i, a_l) \succ f(x_j, a_l),\ \forall x_j \in C_2$. If $x_j D_P^{-\kappa} x_i$, then $D_P^{-\kappa}(x_i^{\vee +}) = D_P^{-\kappa}(x_i) \cup \{x_j\}$. $\therefore\ C_3 = \{x_j | x_j D_P^{-\kappa} x_i, \forall x_j \in C_2\}, D_P^{-\kappa}(x_i^{\vee +}) = D_P^{-\kappa}(x_i) \cup C_3$.

The principle of updating approximations of downward union $Cl_t^{\leqslant}$ is given as follows.

Proposition 3.8. *For* $\underline{P}^{\vee +}\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $\overline{P}^{\vee +}\left(Cl_t^{\leqslant}\right)^{\kappa}$, *the following result hold:*

i. $\forall x_j \in C_1$
 (a) *If* $x_i \notin Cl_t^{\leqslant},\ D_P^{-\kappa}(x_j) \in Bnp\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $D_P^{-\kappa}(x_j^{\vee +}) \subset Cl_t^{\leqslant}$, *then* $\underline{P}^{\vee +}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_j^{\vee +}\}$.
 (b) *If* $x_j \in Cl_t^{\leqslant},\ x_i \notin Cl_t^{\leqslant}$ *and* $x_i \notin D_P^{-\kappa}(x'), \forall x' \in Cl_t^{\leqslant}$, *then* $\overline{P}^{\vee +}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_i\}$.
 (c) *Otherwise,* $\overline{P}^{\vee +}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa},\ \underline{P}^{\vee +}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$.
ii.
 (a) *If* $x_i \in \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $C_3 \not\subset Cl_t^{\leqslant}$, *then* $\underline{P}^{\vee +}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_i\}$.
 (b) *If* $x_i \in Cl_t^{\leqslant}$, *then* $\overline{P}^{\vee +}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup C_3$.
 (c) *Otherwise,* $\overline{P}^{\vee +}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa},\ \underline{P}^{\vee +}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$.

Proof. It is similar to the proof of Proposition 3.2. □

- If $V^{\vee +} = \varnothing$ (refining upward single level)

When attribute value of x_i is refined upward single level, x_j*s.t.* $x_j \in D_P^{+\kappa}(x_i)$ and $f(x_j, a_l) = v_1$ will not dominate x_i.

Property 3.9. *Let* $C_1 = \{x_j | x_j \in D_P^{+\kappa}(x_i) \wedge f(x_j, a_l) = v_1\}$. *Then* $D_P^{+\kappa}(x_i^{\vee +}) = D_P^{+\kappa}(x_i) - C_1$.

Proof. $\because x_j \in D_P^{+\kappa}(x_i),\ \therefore f(x_j, a_l) \succeq f(x_i, a_l),\ \therefore f(x_i^{\vee +}, a_l) = v_{l2} \succ v_{l1},\ x_j \notin D_P^{+\kappa}(x_i),\ D_P^{+\kappa}(x_i^{\vee +}) = D_P^{+\kappa}(x_i) - \{x_j\}.\ \therefore$ If $C_1 = \{x_j | x_j \in D_P^{+\kappa}(x_i) \wedge f(x_j, a_l). = v_{l1}\}$, then $D_P^{+\kappa}(x_i^{\vee +}) = D_P^{+\kappa}(x_i) - C_1$.

In the single level upward refining, the approximations of upward union $Cl_t^{\geqslant}$ are only affected by the dominating class of x_i. □

Proposition 3.9. *For* $\underline{P}^{\vee +}(Cl_t^{\geqslant})^{\kappa}$ *and* $\overline{P}^{\vee +}(Cl_t^{\geqslant})^{\kappa}$, *the following results hold:*

i. *If* $x_i \in Bn_P(Cl_t^{\geqslant})^{\kappa}$ *and* $D_P^{+\kappa}(x_i^{\vee +}) \subseteq Cl_t^{\geqslant}$, *then* $\underline{P}^{\vee +}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_i^{\vee +}\}$.
ii. *If* $x_i \in Cl_t^{\geqslant}$ *and* $C' = \{x_k | x_k \in C_1 \wedge x_k \notin D_P^{+\kappa}(x'), x' \in Cl_t^{\geqslant}\} \neq \varnothing$, *then* $\overline{P}^{\vee +}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa} - C'$.
iii. *Otherwise,* $\overline{P}^{\vee +}(Cl_t^{\geqslant})^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa},\ \underline{P}^{\vee +}(Cl_t^{\geqslant})^{\kappa} = \underline{P}\left(Cl_t^{\geqslant}\right)^{\kappa}$.

Proof. It is similar to the proof of Proposition 3.1. □

For x_j *s.t.* $x_i \in D_P^{-\lambda}(x_j)$ and $f(x_j, a_l) = v_1$, x_i will not be dominated by x_j. The updating principle of dominated classes is given as follows.

Property 3.10. *Let* $C_1 = \left\{x_j \middle| x_i \in D_P^{-\lambda}(x_j) \wedge f(x_j, a_l) = v_1\right\}$. *Then* $\forall x_j \in C_1,\ D_P^{-\kappa}(x_j^{\vee +}) = D_P^{-\kappa}(x_j) - \{x_i\}$.

Proof. $\forall x_j \in C_1,\ \because x_i \in D_P^{-\lambda}(x_j),\ \therefore f(x_j, a_l) \succeq f(x_i, a_l),\ f(x_i, a_l) = v_{l2} \succ v_{l1}.\ \therefore x_i \notin D_P^{-\lambda}(x_j),\ D_P^{-\kappa}(x_j^{\vee +}) = D_P^{-\kappa}(x_j) - \{x_i\}$.

$\forall x_j \in C_1$, the dominated classes of x_j will vary. The following proposition for updating approximations of downward union $Cl_t^{\leqslant}$ holds. □

Proposition 3.10. *For* $\underline{P}^{\vee +}\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $\overline{P}^{\vee +}\left(Cl_t^{\leqslant}\right)^{\kappa}$, $\forall x_j \in C_1$, *the following results hold:*

i. *If* $x_i \notin Cl_t^{\leqslant},\ x_j \in Bn_P\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $D_P^{-\lambda}(x_j) \subseteq Cl_t^{\leqslant}$, *then* $\underline{P}^{\vee +}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_j\}$.
ii. *If* $x_j \in Cl_t^{\leqslant}$ *and* $x_i \notin D_P^{-\kappa}(x'), \forall x' \in Cl_t^{\leqslant}$, *then* $\overline{P}^{\vee +}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_i\}$.
iii. *Otherwise,* $\overline{P}^{\vee +}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa},\ \underline{P}^{\vee +}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$.

Proof. It is similar to the proof of Proposition 3.2(i). □

- If $v_2 \succ v_{a_l}^{\max}$.

If $v_2 \succ v_{a_l}^{\max}$, then no object except x_i will dominate x_i. For x_j *s.t.* $x_j \notin D_P^{+\kappa}(x_i)$ and $v_1 \prec f(x_j, a_l)$, it may be dominated by x_i.

Property 3.11. *Let* $C_2 = \{x_j | x_j \notin D_P^{+\kappa}(x_i) \wedge v_1 \prec f(x_j, a_l)\}$. *Then*

i. $D_P^{+\kappa}(x_i^{\vee +}) = \{x_i\}$.
ii. $C_3 = \{x_i^{\vee +} D_P^{+\kappa} x_j, \forall x_j \in C_2\},\ \forall x_j \in C_3,\ D_P^{+\kappa}(x_j^{\vee +}) = D_P^{+\kappa}(x_j) \cup \{x_i^{\vee +}\}$.

Proof. i. $\because v_2 \succ v_{a_l}^{\max} \therefore \neg\exists x_j$ *s.t.* $x_j D_P^{+\kappa} x_i \therefore D_P^{+\kappa}(x_i^{\vee +}) = \{x_i\}$. ii. $\forall x_j \in C_2$, if $x_i^{\vee +} D_P^{+\kappa} x_j$, then $D_P^{+\kappa}(x_j^{\vee +}) = D_P^{+\kappa}(x_j) \cup \{x_i^{\vee +}\}$. $\therefore\ C_3 = \{x_i^{\vee +} D_P^{+\kappa} x_j, \forall x_j \in C_2\},\ \forall x_j \in C_3,\ D_P^{+\kappa}(x_j^{\vee +}) = D_P^{+\kappa}(x_j) \cup \{x_i^{\vee +}\}$. □

The updating principle of upward union $Cl_t^{\geqslant}$ regarding the variation of dominating classes is given as follows.

Proposition 3.11. *For* $\underline{P}^{\vee +}(Cl_t^{\geqslant})^{\kappa}$ *and* $\overline{P}^{\vee +}(Cl_t^{\geqslant})^{\kappa}$, *the following results hold:*

i.
 (a) *If* $x_i \in Cl_t^{\geqslant}$, *then* $\underline{P}^{\vee +}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_i^{\vee +}\}$.
 (b) $C' = D_P^{+\kappa}(x_i) - \{x_i\}$, *If* $x_i \in Cl_t^{\geqslant}$ *and* $C'' = \{x_k | x_k \in C' \wedge x_k \notin D_P^{+\kappa}(x'), x' \in Cl_t^{\geqslant}\} \neq \varnothing$, *then* $\overline{P}^{\vee +}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa} - C''$.
 (c) *Otherwise,* $\overline{P}^{\vee +}(Cl_t^{\geqslant})^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa},\ \underline{P}^{\vee +}(Cl_t^{\geqslant})^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$.
ii. $\forall x_j \in C_3$.
 (a) *If* $x_i \notin Cl_t^{\geqslant}$ *and* $x_j \in \underline{P}(Cl_t^{\geqslant})^{\kappa}$, *then* $\underline{P}^{\vee +}(Cl_t^{\geqslant})^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_j\}$.
 (b) *If* $x_j \in Cl_t^{\geqslant}$ *and* $x_i \notin D_P^{+\kappa}(x'), x' \in Cl_t^{\geqslant}$, *then* $\overline{P}^{\vee +}(Cl_t^{\geqslant})^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_i^{\vee +}\}$.
 (c) *Otherwise,* $\overline{P}^{\vee +}(Cl_t^{\geqslant})^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa},\ \underline{P}^{\vee +}(Cl_t^{\geqslant})^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$.

Proof. It is similar to the proof of Proposition 3.1. □

x_i will not be dominated by other objects. For x_j s.t. $x_i \notin D_P^{-\kappa}(x_j)$ and $f(x_j, a_l) \succ v_1$, it may be dominated by x_i.

Property 3.12. *Let* $C_1 = \{x_j | x_i \in D_P^{-\kappa}(x_j)\}$, $C_2 = \{x_j | x_i \notin D_P^{-\kappa}(x_j) \wedge f(x_j, a_l) \succ v_1\}$. *Then*

i. $\forall x_j \in C_1,\ D_P^{-\kappa}(x_j^{\vee+}) = D_P^{-\kappa}(x_j) - \{x_i\}$.
ii. *Let* $C_3 = \{x_j | x_j D_P^{-\kappa} x_i, \forall x_j \in C_2\}$. *Then* $D_P^{-\kappa}(x_i^{\vee+}) = D_P^{-\kappa}(x_i) \cup C_3$.

Proof. It is similar to the proof of Property 3.11. □

In the following, we propose a proposition for updating approximations of downward union $Cl_t^{\leqslant}$ considering different variation of dominated classes.

Proposition 3.12. *For* $\underline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $\overline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa}$, *the following results hold:*

i. $\forall x_j \in C_1$.
  (a) *If* $x_i \notin Cl_t^{\leqslant}$, $x_j \in Bn_P\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $D_P^{-\kappa}(x_j^{\vee+}) \subseteq Cl_t^{\leqslant}$, *then* $\underline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_j^{\vee+}\}$.
  (b) *If* $x_j \in Cl_t^{\leqslant}$ *and* $x_i \notin D_P^{+\kappa}(x'), x' \in Cl_t^{\leqslant}$, *then* $\overline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_i\}$.
  (c) *Otherwise,* $\overline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$, $\underline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$.

ii.
  (a) *If* $x_i \in \underline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $C' = \{x_j | \exists x_j \notin Cl_t^{\leqslant} \wedge x_j \in C_3\}$, *then* $\underline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - C'$.
  (b) *If* $x_i \in Cl_t^{\leqslant}$ *and* $C'' = \{x_j | x_j \notin \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}, x \in C_3\}$, *then* $\overline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup C''$.

Proof. It is similar to the proof of Proposition 3.2. □

3.2. Principles for updating approximations incrementally when attribute values coarsening

In this section, we first introduce the definition of attribute values' coarsening in the IODS.

Definition 3.2. Let $S = (U, A, V, f)$ be an IODS. $B \subseteq A$, $a_l \in B$, $f(x_i, a_l)$ is the attribute value of object x_i on attribute a_l, $f(x_i, a_l) \neq *$, $f(x_i, a_l) \neq ?$. $f(x_k, a_l)$ is the attribute value of object $x_k (k \neq i)$ on attribute a_l, $f(x_k, a_l) \neq *$, $f(x_k, a_l) \neq ?$, $f(x_i, a_l) \neq f(x_k, a_l)$. Then, $U_{a_l} = \{x_{i'} \in U | f(x_{i'}, a_l) = f(x_i, a_l)\}$. Let $f(x_{i'}, a_l) = f(x_k, a_l)$, $\forall x_{i'} \in U_{a_l}$, then we call the attribute value $f(x_i, a_l)$ is coarsen to $f(x_k, a_l)$.

(1) If $\exists f(x_i, a_l) \prec f(x_k, a_l)$, then we call the attribute value $f(x_i, a_l)$ is coarsen upward. let $x_i^{\wedge+}$ denote the object x_i after coarsening upward. Let $V^{\wedge+} = \{v | f(x_i, a_l) \prec v \prec f(x_k, a_l), v \in V_{a_l}\}$. If $V^{\wedge+} = \emptyset$, then the attribute value $f(x_i, a_l)$ is coarsen upward single level. If $V^{\wedge+} \neq \emptyset$, then the attribute value $f(x_i, a_l)$ is coarsen upward multi-level.
(2) If $\exists f(x_i, a_l) \succ f(x_k, a_l)$, then we call the attribute value $f(x_i, a_l)$ is coarsen downward. Let $x_i^{\wedge-}$ denote object x_i after coarsening downward. Let $V^{\wedge-} = \{v | f(x_k, a_l) \prec v \prec f(x_i, a_l), v \in V_{a_l}\}$. We call the attribute value $f(x_i, a_l)$ is coarsen downward single level when $V^{\wedge-} = \emptyset$. If $V^{\wedge-} \neq \emptyset$, then we call the attribute value $f(x_i, a_l)$ is coarsen downward multi level.

Let $a_l^{\wedge+}$ be the attribute a_l after coarsening upward. $V_{a_l}^{\wedge+}$ denotes the domain of the attribute $a_l^{\wedge-}$ after coarsening upward. $a_l^{\wedge-}$ denotes the attribute a_l after coarsening downward. $V_{a_l}^{\wedge-}$ denotes the domain of the attribute $a_l^{\wedge-}$ after coarsening downward.

Example 3.2. Let $S = (U, A, V, f)$ be an IODS. $V_{a_l} = \{VL, L, M, H, VH\}$, $\exists VL \prec L \prec M \prec H \prec VH$, $U' = \{x_i | f(x_i, a_l) = M, a_l \in C\}$, then

(1) If $f(x_i^{\wedge}, a_l) = L$, $\forall x_i \in U'$, then the attribute value is coarsened downward single level, $V_{a_l}^{\wedge-} = \{VL, L, H, VH\}$.
(2) If $f(x_i^{\wedge}, a_l) = VL$, $\forall x_i \in U'$, then the attribute value is coarsened downward multi-level, $V_{a_l}^{\wedge-} = \{VL, L, H, VH\}$.
(3) If $f(x_i^{\wedge}, a_l) = H$, $\forall x_i \in U'$, then the attribute value is coarsened upward single level, $V_{a_l}^{\wedge+} = \{VL, L, H, VH\}$.
(4) If $f(x_i^{\wedge}, a_l) = VH$, $\forall x_i \in U'$, then the attribute value is coarsened upward multi-level, $V_{a_l}^{\wedge+} = \{VL, L, H, VH\}$.

Following the above example, the attribute domain is the same after different levels coarsening. But the changes of dominated classes are different in different cases of coarsening.

We present the principle of incremental updating for approximations of an upward union of classes as follows. Let $\overline{P}^{\wedge-}(Cl_t^{\geqslant})^{\kappa}$, $\underline{P}^{\wedge-}(Cl_t^{\geqslant})^{\kappa}$, $\overline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa}$ and $\underline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa}$ denote the approximations of an upward union of classes after coarsening downward and upward, respectively. The approximations of a downward union of classes after coarsening upward and downward are defined as $\overline{P}^{\wedge-}\left(Cl_t^{\leqslant}\right)^{\kappa}$, $\underline{P}^{\wedge-}\left(Cl_t^{\leqslant}\right)^{\kappa}$, $\overline{P}^{\wedge+}\left(Cl_t^{\leqslant}\right)^{\kappa}$, $\underline{P}^{\wedge+}\left(Cl_t^{\leqslant}\right)^{\kappa}$, respectively.

(1) Coarsening downward.

Let $f(x_i^{\wedge-}, a_l) = v_2$, where $\forall f(x_i, a_l) = v_1$, $\exists v_1, v_2 \in V_l$, $v_1 \succ v_2$. Let $V^{\wedge-} = \{v | v_2 \prec v \prec {}_1, v \in V_l\}$.

- If $V^{\wedge-} \neq \emptyset$ (Coarsening downward multi-level).

When the attribute values coarsening downward multi-level, x_j s.t. $x_i \in D_p^{+\kappa}(x_j)$ and $v_1 \succeq f(x_j, a_l) \succ v_2$ will not be dominated by x_i and x_j s.t. $x_i \notin D_p^{+\kappa}(x_j)$ and $v_1 \succ f(x_j, a_l) \succeq v_2$ may dominate x_i. The updating principle for dominating classes is given as follows.

Property 3.13. *For* $D_p^{+\kappa}(x_j)$, *let* $C^{\wedge-} = \{x_i | f(x_i, a_i) = v_1, x_i \in U\}$, $C_1 = \{x_j | x_i \in D_p^{+\kappa}(x_j) \wedge v_1 \succeq f(x_j, a_l) \succ v_2, x_i, x_j \in U, x_i \neq x_j, x_i \in C^{\wedge-}\}$, $C_2 = \{x_j | x_i \notin D_p^{+\kappa}(x_j) \wedge v_1 \succ f(x_j, a_l) \succeq v_2, x_i, x_j \in U, x_i \neq x_j, x_i \in C^{\wedge-}\}$. *Then*

i. $\forall x_j \in C_1,\ D_p^{+}\left(x_j^{\wedge-}\right) = D_p^{+}(x_j) - C^{\wedge-}$.
ii. *Let* $C_3 = \{x_j | x_j D_P^{+\kappa} x_i, \forall x_j \in C_2\}$. *Then* $D_p^{+}\left(x_i^{\wedge-}\right) = D_p^{+}(x_i) \cup C_3$.

Proof. It is similar to the proof of Property 3.1. □

In the following, the principle for updating approximations of upward union $Cl_t^{\geqslant}$ is presented.

Proposition 3.13. *For* $\underline{P}^{\wedge-}(Cl_t^{\geqslant})^{\kappa}$ *and* $\overline{P}^{\wedge-}(Cl_t^{\geqslant})^{\kappa}$, *the following results hold:*

i. $\forall x_j \in C_1$.
  (a) *If* $x_j \in Bn_P(Cl_t^{\geqslant})^{\kappa}$ *and* $D_p^{+}\left(x_j^{\wedge-}\right) \subseteq Cl_t^{\geqslant}$, *then* $\underline{P}^{\wedge-}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_j^{\wedge-}\}$.
  (b) *If* $x_j \in Cl_t^{\geqslant}$ *and* $C'' = \{x_k | x_k \in C^{\wedge-} \wedge x_k \notin D_P^{+\kappa}(x'), x' \in Cl_t^{\geqslant}\} \neq \emptyset$, *then* $\overline{P}^{\wedge-}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa} - C''$.
  (c) *Otherwise,* $\underline{P}^{\wedge-}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa}$, $\overline{P}^{\wedge-}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa}$.

ii.
  (a) *If* $x_i \in \underline{P}(Cl_t^{\geqslant})^{\kappa}$ *and* $C' = \{x_j | x_j \notin Cl_t^{\geqslant}, x_j \in C_3\} \neq \emptyset$, $\underline{P}^{\wedge-}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa} - \{x_i\}$.

(b) *If* $x_i \in Cl_t^{\geqslant}$ *and* $C'' = \{x_j | x_j \notin \overline{P}(Cl_t^{\geqslant})^{k}, x_j \in C_3\} \neq \emptyset$, *then* $\overline{P}^{\wedge-}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa} \cup C'$.
(c) *Otherwise*, $\underline{P}^{\wedge-}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa}$, $\overline{P}^{\wedge-}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa}$.

Proof. It is similar to the proof of Proposition 3.1. □

x_i will not dominate x_j *s.t.* $x_j \in D_p^{-\kappa}(x_i)$ and $v_2 \prec f(x_j, a_i) \preceq v_1$. x_j *s.t.*$x_j \notin D_p^{-\kappa}(x_i)$ and $v_2 \preceq f(x_j, a_i) \prec v_1$ may dominate x_i. The updating principle for dominated classes is presented as follows.

Property 3.14. *Let* $C^{\wedge-} = \{x_i | f(x_i, a_i) = v_1, x_i \in U\}, C_1 = \{x_j | x_j \in D_p^{-\kappa}(x_i) \wedge v_2 \prec f(x_j, a_i) \preceq v_1, x_j \in U, x_i \in C^{\wedge-}, i \neq j\}$, $C_2 = \{x_j | x_j \notin D_p^{-\kappa}(x_i) \wedge v_2 \preceq f(x_j, a_i) \prec v_1, x_j \in U, x_i \in C^{\wedge-}, i \neq j\}$. *Then*

i. $\forall x_i \in C^{\wedge-}$, $D_P^{-\kappa}(x_i^{\wedge-}) = D_P^{-\kappa}(x_i) - C_1$.
ii. $C_3 = \{x_j \big| x_i^{\wedge-} D_p^{-\kappa} x_j, \forall x_j \in C_2, x_i \in C^{\wedge-}\}$, $C_{4j} = \{x_i \big| x_i D_p^{-\kappa} x_j, \exists x_j \in C_3.\}, \forall x_j \in C_3$, $D_p^{-\kappa}\left(x_j^{\wedge-}\right) = D_p^{-\kappa}(x_j) \cup C_{4j}$.

Proof. It is similar to the proof of Property 3.2. □

Considering the variation of dominated classes, the principle for updating approximations of downward union $Cl_t^{\leqslant}$ is proposed as follows.

Proposition 3.14. *For* $\underline{P}^{\wedge-}\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $\overline{P}^{\wedge-}\left(Cl_t^{\leqslant}\right)^{\kappa}$, *the following results hold:*

i. $\forall x_i \in C^{\wedge-}$.
(a) *If* $x_i \in Bn_P\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $D_p^{-\kappa}(x_i^{\wedge-}) \subseteq Cl_t^{\leqslant}$, *then* $\underline{P}^{\wedge-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_i^{\wedge-}\}$.
(b) *If* $x_i \in Cl_t^{\leqslant}$ *and* $C'' = \{x_k | x_k \in C_1 \wedge x_k \notin D_p^{-\kappa}(x'), x' \in Cl_t^{\leqslant}\} \neq \emptyset$, *then* $\overline{P}^{\wedge-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - C''$.
(c) *Otherwise*, $\overline{P}^{\wedge-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$, $\underline{P}^{\wedge-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$.

ii. $\forall x_j \in C_3$.
(a) *If* $x_j \in \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $C_{4j} \not\subset Cl_t^{\leqslant}$, *then* $\underline{P}^{\wedge-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_j\}$.
(b) *If* $x_j \in Cl_t^{\leqslant}$, *then* $\overline{P}^{\wedge-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup C_{4j}$.
(c) *Otherwise*, $\overline{P}^{\wedge-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$, $\underline{P}^{\wedge-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$.

Proof. It is similar to the proof of Proposition 3.2. □

- $V^{\wedge-} = \emptyset$ (Coarsening downward single level).

When attribute values are coarsen downward single level, x_i may be dominated by x_j *s.t.* $f(x_j, a_i) = v_2$ and $x_i \notin D_p^{+\kappa}(x_j)$. In the following, we give the principle for updating dominating classes.

Property 3.15. *Let* $C^{\wedge-} = \{x_i | f(x_i, a_i) = v_1\}$, $C_2 = \{x_j | f(x_j, a_i) = v_2 \wedge x_i \notin D_p^{+\kappa}(x_j), x_i \in C^{\wedge-}\}$, $C_3 = \{x_j | x_j D_p^{+\kappa} x_i, \exists x_j \in C_2\}$. *Then,* $D_p^{+\kappa}(x_i^{\wedge-}) = D_p^{+\kappa}(x_i) \cup C_3$, $\forall x_i \in C^{\wedge-}$.

Proof. $\because \forall x_i \in C^{\wedge-}$, $x_i \notin D_p^{+\kappa}(x_j)$, $f(x_i, a_i) \succ f(x_j, a_i)$. $\because$ $\forall x_j \in C_2, f\left(x_i^{\wedge-}, a_i\right) = f(x_j, a_i)$. $\therefore$ If $x_j D_p^{+\kappa} x_i$, then $D_p^{+\kappa}\left(x_i^{\wedge-}\right) = D_p^{+\kappa}(x_i) \cup x_j$. $\therefore$ If $C_{3i} = \{x_j | x_j D_p^{+\kappa} x_i, \exists x_j \in C_2\}$, then, $D_p^{+\kappa}\left(x_i^{\wedge-}\right) = D_p^{+\kappa}(x_i) \cup C_{3i}$, $\forall x_i \in C^{\wedge-}$. □

Because dominating classes of x_i which belong to $C^{\wedge-}$ may alter, we only consider these dominating classes when updating approximations. The principle for updating approximations of upward union $Cl_t^{\geqslant}$ is given as follows.

Proposition 3.15. *For* $\underline{P}^{\wedge-}(Cl_t^{\geqslant})^{\kappa}$ *and* $\overline{P}^{\wedge-}(Cl_t^{\geqslant})^{\kappa}$, $\forall x_i \in C^{\wedge-}$, *the following results hold:*

i. *If* $x_i \in \underline{P}(Cl_t^{\geqslant})^{\kappa}$ *and* $C_3 \not\subset Cl_t^{\geqslant}$, *then* $\underline{P}^{\wedge-}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa} - \{x_i\}$.
ii. *If* $x_i \in Cl_t^{\geqslant}$, *then* $\overline{P}^{\wedge-}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa} \cup C_3$.
iii. *Otherwise*, $\underline{P}^{\wedge-}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa}$, $\overline{P}^{\wedge-}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa}$.

Proof. It is similar to the proof of Proposition 3.1. □

x_j *s.t.* $f(x_j, a_i) = v_2$ and $x_j \notin D_p^{-\kappa}(x_i)$ may dominate x_i when the attribute values of $x_i$is coarsen downward single level.

Property 3.16. *Let* $C^{\wedge-} = \{x_i | f(x_i, a_i) = v_1, x_i \in U\}, C_2 = \{x_j | f(x_j, a_i) = v_2 \wedge x_j \notin D_p^{-\kappa}(x_i), x_i \in C^{\wedge-}, x_j \in U\}$, $C_3 = \{x_i^{\wedge-} \big| x_i^{\wedge-} D_p^{-\kappa} x_j, \forall x_i \in C^{\wedge-}, x_j \in C_2.\}$. *Then,* $\forall x_j \in C_2$, $D_p^{-k}\left(x_j^{\wedge-}\right) = D_p^{-k}(x_j) \cup C_3$.

Proof. $\because \forall x_i \in C^{\wedge-}$, $x_j \in U$, $x_j \notin D_p^{-\kappa}(x_i)$, $f(x_j, a_i) = v_2 \prec f(x_i, a_i) = v_1$. $\because$ $f(x_i^{\wedge-}, a_i) = f(x_j, a_i) = v_2$, $\therefore$ If $x_i^{\wedge-} D_p^{-\kappa} x_j$. Then $D_p^{-\kappa}\left(x_j^{\wedge-}\right) = D_p^{-\kappa}(x_j) \cup x_i^{\wedge-}$. $\therefore$ Let $C_3 = \{x_i^{\wedge-} \big| x_i^{\wedge-} D_p^{-\kappa} x_j, \forall x_i \in C^{\wedge-}, x_j \in C_2\}$. Then $\forall x_j \in C_2$, $D_p^{-\kappa}(x_j^{\wedge-}) = D_p^{-\kappa}(x_j) \cup C_3$. □

In the following, the principle for updating approximations of downward union $Cl_t^{\leqslant}$ is given.

Proposition 3.16. *For* $\underline{P}^{\wedge-}\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $\overline{P}^{\wedge-}\left(Cl_t^{\leqslant}\right)^{\kappa}$, $\forall x_j \in C_2$, *the following results hold:*

i. *If* $x_j \in \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ *and* $C_3 \not\subset Cl_t^{\leqslant}$, *then* $\underline{P}^{\wedge-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_j\}$.
ii. *If* $x_j \in Cl_t^{\leqslant}$, *then* $\overline{P}^{\wedge-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup C_3$.
iii. *Otherwise*, $\underline{P}^{\wedge-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$, $\overline{P}^{\wedge-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$.

Proof. It is similar to the proof of Proposition 3.2. □

(2) Coarsening upward.

Let $f_l(x_i^{\wedge+}) = v_2$, $\forall f_l(x_i) = v_1$, $\exists v_1, v_2 \in V_{a_i}$, $v_2 \succ v_1$.

- If $V^{\wedge+} \neq \emptyset$ (Coarsening upward multi-level)

When attributes values coarsening upward multi-level, x_j *s.t.* $x_j \in D_p^{+\kappa}(x_i)$ and $v_1 \preceq f(x_j, a_i) \prec v_2$will not dominate x_i. x_j *s.t.* $x_j \notin D_p^{+\kappa}(x_i)$ and $v_1 \preceq f(x_j, a_i) \prec v_2$ will be dominated by x_i. The updating principle for dominating classes is given as follows.

Property 3.17. *For* $D_p^{+\kappa}(x_i)$, *let* $C^{\wedge+} = \{x_i | f(x_i, a_i) = v_1\}$, $C_1 = \{x_j | x_j \in D_p^{+\kappa}(x_i), v_1 \preceq f(x_j, a_i) \prec v_2, x_j \in U\}$, $C_2 = \{x_j | x_j \notin D_P^{+\kappa}(x_i) \wedge v_1 \preceq f(x_j, a_i) \prec v_2, x_j \in U, x_i \in C^{\wedge+}\}$, $C_3 = \{x_i | x_i^{\wedge+} D_P^{\kappa} x_j$, $\forall x_i \in C^{\wedge}, x_j \in C_2\}$. *Then*

i. $\forall x_i \in C^{\wedge+}$, $D_P^{+\kappa}(x_i^{\wedge+}) = D_P^{+\kappa}(x_i) - C_1$.
ii. $\forall x_j \in C_2$, $D_P^{+\kappa}(x_j^{\wedge+}) = D_P^{+\kappa}(x_j) \cup C_3$.

Proof. i. $\because$ $\forall x_i \in C^{\wedge+}$, $x_j \in U$, if $x_j \in D_p^{+\kappa}(x_i)$, then $f(x_j, a_i) \succ f(x_i, a_i)$. $\therefore$ $f\left(x_i^{\wedge+}, a_i\right) = v_2 \succ f(x_j, a_i) = v_1$, $\therefore x_j \notin D_p^{+\kappa}(x_i)$, $D_P^{+\kappa}(x_i^{\wedge+}) = D_P^{+\kappa}(x_i) - \{x_j\}$. $\therefore$ Let $C_1 = \{x_j | x_j \in D_p^{+\kappa}(x_i), v_1 \preceq f(x_j, a_i) \prec v_2, x_j \in U\}$. Then $\forall x_i \in C^{\wedge+}$, $D_P^{+\kappa}(x_i^{\wedge+}) = D_P^{+\kappa}(x_i) - C_1$. ii. $\because$ $x_j \in U$, $x_j \notin D_p^{+\kappa}(x_i)$, $f(x_j, a_i) \succeq f(x_i, a_i)$, $f(x_j, a_i) \prec f\left(x_i^{\wedge+}, a_i\right)$. $\therefore$ If $x_i^{\wedge+} D_P^{\kappa} x_j$, then $D_p^{+\kappa}(x_j^{\wedge+}) = D_P^{+\kappa}(x_j) \cup x_i^{\wedge+}$. $\therefore$ Let $C_2 = \{x_j | x_j \notin D_p^{+\kappa}(x_i), v_1 \preceq f(x_j, a_i) \prec v_2, x_j \in U, x_i \in C^{\wedge+}\}$, $C_3 = \{x_i | x_i^{\wedge+} D_P^{\kappa} x_j, \forall x_i \in C^{\wedge}, x_j \in C_2\}$. Then $\forall x_j \in C_2$, $D_P^{+\kappa}(x_j^{\wedge+}) = D_P^{+\kappa}(x_j) \cup C_3$. □

Considering different variations of dominating classes, the updating principle for upward union is given as follows.

Proposition 3.17. *For $\underline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa}$ and $\overline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa}$, the following results hold:*

i. $\forall x_i \in C^{\wedge+}$. (a) If $x_i \in Bn_P(Cl_t^{\geqslant})^{\kappa}$ and $D_P^+(x_i^{\wedge+}) \subseteq Cl_t^{\geqslant}$, then $\underline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_i^{\wedge+}\}$. (b) If $x_i \in Cl_t^{\geqslant}$ and $C'' = \{x_k | x_k \in C_1 \wedge x_k \notin D_P^+(x'), x' \in Cl_t^{\geqslant}\} \neq \varnothing$, then $\overline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa} - C''$. (c) Otherwise, $\underline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa}$, $\overline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa}$.
ii. $\forall x_j \in C_2$. (a) If $x_j \in \underline{P}(Cl_t^{\geqslant})^{\kappa}$ and $C_3 \not\subset Cl_t^{\geqslant}$, then $\underline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa} - \{x_j\}$. (b) If $x_j \in Cl_t^{\geqslant}$, then $\overline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa} \cup C_3$. (c) Otherwise, $\underline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa}$, $\overline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa}$.

Proof. It is similar to the proof of Proposition 3.1. □

Now we consider the variations of dominated classes. x_j *s.t.* $x_i \in D_p^{-\kappa}(x_j)$ and $v_1 \preceq f(x_j, a_i) \prec v_2$ will not dominate x_i. x_j *s.t.* $x_i \notin D_p^{-\kappa}(x_j)$ and $v_1 \preceq f(x_j, a_i) \prec v_2$ may be dominated by x_i. Then the principle for updating dominated classes is given as follows.

Property 3.18. *Let* $C^{\wedge+} = \{f(x_i, a_1) = v_1, \forall x_i \in U\}, C_1 = \{x_j | x_i \in D_p^{-\kappa}(x_j) \wedge v_1 \preceq f(x_j, a_i) \prec v_2, \forall x_i \in C^{\wedge+}, x_j \in U\}, C_2 = \{x_j | x_i \notin D_p^{-\kappa}(x_j) \wedge v_1 \preceq f(x_j, a_i) \prec v_2, x_i \in C^{\wedge+}, x_j \in U\}$. *Then*

i. $\forall x_j \in C_1,\ D_p^{-\kappa}(x_j^{\wedge+}) = D_p^{-\kappa}(x_j) - C^{\wedge+}$.
ii. $C_3 = \{x_j | x_j D_p^{-\kappa} x_i, x_i \in C^{\wedge+}, \forall x_j \in C_2\}$. $\forall x_i \in C^{\wedge+},\ D_p^{-\kappa}(x_i^{\wedge+}) = D_p^{-\kappa}(x_i) \cup C_3$.

Proof

i. $\because\ \forall x_j \in C_1,\ \therefore x_i \in D_p^{-\kappa}(x_j),\ f(x_i, a_i) \preceq f(x_j, a_i)$. $\because\ f(x_i^{\wedge+}, a_i) = v_{2i} \succ f(x_j, a_i),\ \therefore x_i^{\wedge+} \notin D_p^{-\kappa}(x_j),\ D_p^{-\kappa}(x_j^{\wedge+}) = D_p^{-\kappa}(x_j) - \{x_i\}$. $\therefore$ Let $C^{\wedge+} = \{f(x_i, a_1) = v_1, \forall x_i \in U\}$. Then $\forall x_j \in C_1,\ D_p^{-\kappa}(x_j^{\wedge+}) = D_p^{-\kappa}(x_j) - C^{\wedge+}$.
ii. $\because\ \forall x_j \in C_2,\ x_i \notin D_p^{-\kappa}(x_j),\ v_1 \preceq f(x_j, a_i) \prec v_2$. $\because\ f(x_i^{\wedge+}, a_i) = v_2 \succ f(x_j, a_i)$, $\therefore$ If $x_j \in D_p^{-\kappa}(x_i^{\wedge+})$, then $D_p^{-\kappa}(x_i^{\wedge+}) = D_p^{-\kappa}(x_i) \cup \{x_j\}$. $\therefore$ Let $C_3 = \{x_j | x_j D_p^{-\kappa} x_i, x_i \in C^{\wedge+}, \forall x_j \in C_2\}$. Then $\forall x_i \in C^{\wedge+},\ D_p^{-\kappa}(x_i^{\wedge+}) = D_p^{-\kappa}(x_i) \cup C_3$. □

The different variations of dominated classes are taken into consideration. Then we give the updating principle for approximations of downward union $Cl_t^{\leqslant}$ as follows.

Proposition 3.18. *For $\underline{P}^{\wedge+}\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $\overline{P}^{\wedge+}\left(Cl_t^{\leqslant}\right)^{\kappa}$, the following results hold:*

i. $\forall x_j \in C_1$.
   (a) If $x_j \in Bn_p\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $D_p^+(x_j^{\wedge+}) \subseteq Cl_t^{\leqslant}$, then $\underline{P}^{\wedge+}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_i^{\wedge+}\}$.
   (b) If $x_j \in Cl_t^{\leqslant}$ and $C'' = \{x_k | x_k \in C_1 \wedge x_k \notin D_p^+(x'), x' \in Cl_t^{\leqslant}\} \neq \varnothing$, then $\overline{P}^{\wedge+}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - C''$.
   (c) Otherwise, $\overline{P}^{\wedge+}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$, $\underline{P}^{\wedge+}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$.
ii. $\forall x_i \in C^{\wedge+}$.
   (a) If $x_i \in \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $C_3 \not\subset Cl_t^{\leqslant}$, then $\underline{P}^{\wedge+}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_i\}$.
   (b) If $x_i \in Cl_t^{\leqslant}$, then $\overline{P}^{\wedge+}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup C_3$.
   (c) Otherwise, $\overline{P}^{\wedge+}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$, $\underline{P}^{\wedge+}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$.

Proof. It is similar to the proof of Proposition 3.2. □

- If $V^{\wedge+} = \varnothing$ (Coarsening upward single level).

Let $\forall f(x_i^{\wedge+}, a_i) = v_2,\ \forall f(x_i, a_i) = v_1$, *s.t.* $\neg\exists v \in V_i \wedge v_1 \prec v \prec v_2$, *i.e.*, the attribute values is coarsening upward single level. x_j*s.t.* $f(x_j, a_i) = v_2$ and $x_j \notin D_p^{+\kappa}(x_i)$ may be dominated by x_i. In the following, we give the updating principle for dominating classes.

Property 3.19. $\forall x_j \in C_1$, *let* $C^{\wedge+} = \{x_i | f(x_i, a_i) = v_1\}, C_1 = \{x_j | f(x_j, a_i) = v_2 \wedge x_j \notin D_p^{+\kappa}(x_i), x_i \in C^{\wedge+}, x_j \in U\}$, $C_2 = \{x_i^{\wedge+} | x_i^{\wedge+} D_p^{+\kappa} x_j, \forall x_i \in C^{\wedge}, x_i \in C^{\wedge+}, x_j \in C_1\}$. *Then we have: If* $x_i^{\wedge+} D_p^{+\kappa} x_j$, *then* $D_p^{+\kappa}(x_j) = D_p^{+\kappa}(x_j) \cup C_2$.

Proof. $\because \forall x_j \in C_1,\ f(x_j, a_i) = v_2 \succ f(x_i, a_i) = v_1,\ x_j \notin D_p^{+\kappa}(x_i)$. $\because$ $f(x_i^{\wedge+}, a_i) = v_2$, $\therefore$ If $x_i^{\wedge+} D_p^{+\kappa} x_j$, then $D_p^{+\kappa}(x_j) = D_p^{+\kappa}(x_j) \cup \{x_i^{\wedge+}\}$. $\therefore$ Let $C_2 = \{x_i^{\wedge+} | x_i^{\wedge+} D_p^{+\kappa} x_j, \forall x_i \in C^{\wedge}, x_i \in C^{\wedge+}, x_j \in C_1\}$. Then $\forall x_j \in C_1$, $D_p^{+\kappa}(x_j) = D_p^{+\kappa}(x_j) \cup C_2$. □

In the following, the updating principle for approximations of upward union $Cl_t^{\geqslant}$ is proposed considering the variation of dominating classes.

Proposition 3.19. $\forall x_j \in C_1$, *for* $\underline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa}$ *and* $\overline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa}$, *the following results hold:*

i. If $x_j \in \underline{P}(Cl_t^{\geqslant})^{\kappa}$ and $C_2 \not\subset Cl_t^{\geqslant}$, then $\underline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa} - \{x_j\}$.
ii. If $x_j \in Cl_t^{\geqslant}$, then $\overline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa} \cup C_2$.
iii. Otherwise, $\underline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa}$, $\overline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa} = \overline{P}(Cl_t^{\geqslant})^{\kappa}$.

Proof. It is similar to the proof of Proposition 3.1. □

In this case, x_i may dominate x_j *s.t.* $f(x_j, a_i) = v_2$ and $x_i \notin D_p^{-\kappa}(x_j)$, *i.e.*, the cardinality of the dominated classes of $x_i$may increase. The updating principle for dominated classes is given as follows.

Property 3.20. *Let* $C^{\wedge+} = \{x_i | f(x_i, a_i) = v_1, x_i \in U, a_i \in A\}, C_2 = \{x_j | f(x_j, a_i) = v_2 \wedge x_i \notin D_p^{-\kappa}(x_j), x_i \in U, a_i \in A\}$, $C_3 = \{x_j | x_j D_p^{-\kappa} x_i, \forall x_i \in C^{\wedge+}, x_j \in C_2\}$. *Then,* $\forall x_i \in C^{\wedge+},\ D_p^{-\kappa}(x_i^{\wedge+}) = D_p^{-\kappa}(x_i) \cup C_3$.

Proof. $\because \forall x_i \in C^{\wedge+},\ x_j \in C_2,\ x_i \notin D_p^{-\kappa}(x_j),\ f(x_i, a_i) \prec f(x_j, a_i) = v_2$. $\therefore$ If $x_j D_p^{-\kappa} x_i$, then $D_p^{-\kappa}(x_i^{\wedge+}) = D_p^{-\kappa}(x_i) \cup \{x_j\}$. $\therefore$ Let $C_3 = \{x_j | x_j D_p^{-\kappa} x_i, \forall x_i \in C^{\wedge+}, x_j \in C_2\}$, $\forall x_i \in C^{\wedge+},\ D_p^{-\kappa}(x_i^{\wedge+}) = D_p^{-\kappa}(x_i) \cup C_3$. □

The updating principle for downward union $Cl_t^{\leqslant}$ while attribute values coarsening upward single level is given as follows.

Proposition 3.20. $\forall x_i \in C^{\wedge+}$.

i. If $x_i \in \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $C_3 \not\subset Cl_t^{\leqslant}$, then $\underline{P}^{\wedge-}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_i\}$.
ii. If $x_i \in Cl_t^{\leqslant}$, then $\overline{P}^{\wedge+}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup C_3$.
iii. Otherwise, $\overline{P}^{\wedge+}\left(Cl_t^{\leqslant}\right)^{\kappa} = \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$, $\underline{P}^{\wedge+}\left(Cl_t^{\leqslant}\right)^{\kappa} = \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$.

Proof. It is similar to the proof of Proposition 3.2. □

3.3. Algorithms for incremental updating approximations in the IODS

Basic granules are formed by the *EDCR* among the objects in the IODS. The cardinality of an granule may increase or decrease in the process of attribute coarsening and refining. The approximation of upward and downward unions may then alter. Based on the propositions, we design an algorithm for incremental updating approximations as follows. Let $\overline{P}^{\wedge *}(Cl_t^{\geqslant})^{\kappa}(\overline{P}^{\wedge *}\left(Cl_t^{\leqslant}\right)^{\kappa})$ and $\underline{P}^{\wedge *}(Cl_t^{\geqslant})^{\kappa}(\underline{P}^{\wedge *}\left(Cl_t^{\leqslant}\right)^{\kappa})$ denote the upper and lower approximations of $Cl_t^{\geqslant}\left(Cl_t^{\leqslant}\right)$ after attributes coarsening, respectively. The superscript * denotes all cases of coarsening, *i.e.*, it can be substituted by $^+$ and $^-$ according to the different variation of attributes values.

Algorithm 3.1. A Method for Incremental Updating Approximations while Attributes Coarsening in the IODS

INPUT: 1. The dominating and dominated classes of each object x_i: $D_P^{+k}(x_i),\ D_P^{-k}(x_i)(0 < k \leqslant 1,\ \forall x_i \in U)$;
2. Upward and downward unions: $Cl_t^{\geqslant}, Cl_t^{\leqslant}(0 \leqslant t \leqslant n)$;
3. Approximations and boundary of upward and downward unions: $\underline{P}(Cl_t^{\geqslant})^{\kappa}, \overline{P}(Cl_t^{\geqslant})^{\kappa}, \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}, \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$, $Bn_p(Cl_t^{\geqslant})^{\kappa}, Bn_p\left(Cl_t^{\leqslant}\right)^{\kappa}$;
4. The attribute values: v_1 and v_2 (Note: attribute value v_1 is coarsened to v_2);

OUTPUT: $\underline{P}^{\wedge *}(Cl_t^{\geqslant})^{\kappa}, \overline{P}^{\wedge *}(Cl_t^{\geqslant})^{\kappa}, \underline{P}^{\wedge *}\left(Cl_t^{\leqslant}\right)^{\kappa}, \overline{P}^{\wedge *}\left(Cl_t^{\leqslant}\right)^{\kappa}$.

BEGIN
1. For each x_i in U and $f(x_i, a_i) = v_1$ do
2. If $v_2 \prec v_1$ Then // Coarsening downward
1. If $\exists v_2 \prec v \prec v_1 (v \in V_l)$ Then // Coarsening downward multi-level
3. Call CoarseningDM();
4. Else // Coarsening downward single level
5. Call CoarseningDS();
6. End If
7. Else // Coarsening upward
8. If $\exists v_1 \prec v \prec v_2 (v \in V_l)$ Then // Coarsening upward multi-level
9. Call CoarseningUM();
10. Else // Coarsening upward single level
11. Call CoarseningUS();
12. End If
13. End If
14. End
15. Output $\underline{P}^{\wedge *}(Cl_t^{\geqslant})^{k},\ \overline{P}^{\wedge *}(Cl_t^{\geqslant})^{k},\ \underline{P}^{\wedge *}\left(Cl_t^{\leqslant}\right)^{k},\ \overline{P}^{\wedge *}\left(Cl_t^{\leqslant}\right)^{k}$.

END BEGIN

Function CoarseningDM() // Updating approximations while AVC downward multi-level
1. For each x_i in U and $f(x_i, a_i) = v_1$ Do
2. For each x_j in U and $v_1 \succeq f(x_j, a_l) \succeq v_2$ Do
3. If $x_i \in D_p^{+\kappa}(x_j)$ Then
4. $D_p^{+}\left(x_j^{\wedge -}\right) \leftarrow D_p^{+}(x_j) - C^{\wedge -}$;
5. If $x_j \in Bn_P(Cl_t^{\geqslant})^{\kappa}$ and $D_p^{+}\left(x_j^{\wedge -}\right) \subseteq Cl_t^{\geqslant}$ Then $\underline{P}^{\wedge -}(Cl_t^{\geqslant})^{\kappa} \leftarrow \underline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_j^{\wedge -}\}$; End If
6. If $x_j \in Cl_t^{\geqslant}$ and $C'' = \{x_k | x_k \in C^{\wedge -} \wedge x_k \notin D_P^{+\kappa}(x'), x' \in Cl_t^{\geqslant}\} \neq \varnothing$ Then $\overline{P}^{\wedge -}(Cl_t^{\geqslant})^{\kappa} \leftarrow \overline{P}(Cl_t^{\geqslant})^{\kappa} - C''$; End If
7. Else If $x_i \notin D_p^{+\kappa}(x_j)$ and $x_j D_P^{+\kappa} x_i$ Then
8. $D_p^{+}(x_i^{\wedge -}) \leftarrow D_p^{+}(x_i) \cup \{x_j\}$;
9. If $x_i \in \underline{P}(Cl_t^{\geqslant})^{\kappa}$ and $x_j \notin Cl_t^{\geqslant}$ Then $\underline{P}^{\wedge -}(Cl_t^{\geqslant})^{\kappa} \leftarrow \underline{P}(Cl_t^{\geqslant})^{\kappa} - \{x_i\}$; End If
10. If $x_i \in Cl_t^{\geqslant}$ and $x_j \notin \overline{P}(Cl_t^{\geqslant})^{k}$ Then $\overline{P}^{\wedge -}(Cl_t^{\geqslant})^{\kappa} \leftarrow \overline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_j\}$; End If
11. End If
12. If $x_j \in D_p^{-\kappa}(x_i)$ Then
13. $D_P^{-\kappa}(x_i^{\wedge -}) \leftarrow D_P^{-\kappa}(x_i) - \{x_j\}$;
14. If $x_i \in Bn_P\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $D_p^{-\kappa}(x_i^{\wedge -}) \subseteq Cl_t^{\leqslant}$ Then $\underline{P}^{\wedge -}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_i^{\wedge -}\}$; End If
15. If $x_i \in Cl_t^{\leqslant}$ and $x_j \notin D_p^{-\kappa}(x')\left(x' \in Cl_t^{\leqslant}\right)$ Then $\overline{P}^{\wedge -}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_j\}$; End If
16. Else If $x_j \notin D_p^{-\kappa}(x_i)$ and $x_i^{\wedge -} D_p^{-\kappa} x_j$ Then
17. $D_p^{-\kappa}(x_j^{\wedge -}) \leftarrow D_p^{-\kappa}(x_j) \cup \{x_i^{\wedge -}\}$;
18. If $x_j \in \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $x_i \notin Cl_t^{\leqslant}$ Then $\underline{P}^{\wedge -}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_j\}$; End If
19. If $x_j \in Cl_t^{\leqslant}$ Then $\overline{P}^{\wedge -}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_i^{\wedge -}\}$; End If
20. End If
21. End
22. End
23. Return $\underline{P}^{\wedge *}(Cl_t^{\geqslant})^{k},\ \overline{P}^{\wedge *}(Cl_t^{\geqslant})^{k},\ \underline{P}^{\wedge *}\left(Cl_t^{\leqslant}\right)^{k},\ \overline{P}^{\wedge *}\left(Cl_t^{\leqslant}\right)^{k}$.

Function CoarseningDS() // Updating approximations while AVC downward single level
1. For each x_j in U and $f(x_j, a_i) = v_2$ Do
2. For each x_i in U and $f(x_i, a_i) = v_1$ Do
3. If $x_i \notin D_p^{+\kappa}(x_j)$ and $x_j D_p^{+\kappa} x_i$ Then
4. $D_p^{+\kappa}(x_i^{\wedge -}) \leftarrow D_p^{+\kappa}(x_i) \cup \{x_j\}$;
5. If $x_i \in \underline{P}(Cl_t^{\geqslant})^{\kappa}$ and $x_j \notin Cl_t^{\geqslant}$ Then $\underline{P}^{\wedge -}(Cl_t^{\geqslant})^{\kappa} \leftarrow \underline{P}(Cl_t^{\geqslant})^{\kappa} - \{x_i\}$; End If
6. If $x_i \in Cl_t^{\geqslant}$ and $x_j \notin \overline{P}(Cl_t^{\geqslant})^{\kappa}$ Then $\overline{P}^{\wedge -}(Cl_t^{\geqslant})^{\kappa} \leftarrow \overline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_j\}$; End If
7. End If
8. If $x_j \notin D_p^{-\kappa}(x_i)$ and $x_i^{\wedge -} D_p^{-\kappa} x_j$ Then
9. $D_p^{-k}\left(x_j^{\wedge -}\right) \leftarrow D_p^{-k}(x_j) \cup \{x_i\}$;
10. If $x_j \in \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $x_i \notin Cl_t^{\leqslant}$ Then $\underline{P}^{\wedge -}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_j\}$; End If
11. If $x_j \in Cl_t^{\leqslant}$ and $x_i \notin \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ Then $\overline{P}^{\wedge -}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_i\}$; End If
12. End If
13. End
14. End
15. Return $\underline{P}^{\wedge *}(Cl_t^{\geqslant})^{k},\ \overline{P}^{\wedge *}(Cl_t^{\geqslant})^{k},\ \underline{P}^{\wedge *}\left(Cl_t^{\leqslant}\right)^{k},\ \overline{P}^{\wedge *}\left(Cl_t^{\leqslant}\right)^{k}$.

Function CoarseningUM() // Updating approximations while AVC upward multi-level
1. For each x_i in U and $f(x_i, a_i) = v_1$ Do
2. For each x_j in U and $v_1 \preceq f(x_j, a_i) \prec v_2$ Do
3. If $x_j \in D_P^{+\kappa}(x_i)$ Then
4. $D_P^{+\kappa}(x_i^{\wedge +}) \leftarrow D_P^{+\kappa}(x_i) - \{x_j\}$;

(*continued on next page*)

5. If $x_i \in Bn_P(Cl_t^{\geqslant})^{\kappa}$ and $x_j \notin Cl_t^{\geqslant}$ and $D_P^{+}(x_i^{\wedge+}) \subseteq Cl_t^{\geqslant}$ Then $\underline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa} \leftarrow \underline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_i^{\wedge+}\}$; End If
6. If $x_i \in Cl_t^{\geqslant}$ and $x_j \notin D_P^{+}(x')(x' \in Cl_t^{\geqslant})$ Then $\overline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa} \leftarrow \overline{P}(Cl_t^{\geqslant})^{\kappa} - \{x_j\}$; End If
7. Else If $x_j \notin D_P^{+\kappa}(x_i)$ and $x_i^{\wedge+} D_P^{\kappa} x_j$ Then
8. $D_P^{+\kappa}(x_j^{\wedge+}) \leftarrow D_P^{+\kappa}(x_j) \cup \{x_i\}$;
9. If $x_j \in \underline{P}(Cl_t^{\geqslant})^{\kappa}$ and $x_i \notin Cl_t^{\geqslant}$ Then $\underline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa} \leftarrow \underline{P}(Cl_t^{\geqslant})^{\kappa} - \{x_j\}$; End If
10. If $x_j \in Cl_t^{\geqslant}$ and $x_i \notin \overline{P}(Cl_t^{\geqslant})^{\kappa}$ Then $\overline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa} \leftarrow \overline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_j\}$; End If
11. End If
12. If $x_i \in D_p^{-\kappa}(x_j)$ Then
13. $D_p^{-\kappa}(x_j^{\wedge+}) \leftarrow D_p^{-\kappa}(x_j) - \{x_i\}$;
14. If $x_j \in Bn_p\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $x_i \notin Cl_t^{\leqslant}$ and $D_p^{+}(x_j^{\wedge+}) \subseteq Cl_t^{\geqslant}$ Then $\underline{P}^{\wedge+}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_i^{\wedge+}\}$; End If
15. If $x_j \in Cl_t^{\leqslant}$ and $x_i \notin D_P^{+}(x')(x' \in Cl_t^{\leqslant})$ Then $\overline{P}^{\wedge+}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_i\}$; End If
16. Else If $x_i \notin D_p^{-\kappa}(x_j)$ and $x_j D_p^{-\kappa} x_i$ Then
17. $D_p^{-\kappa}(x_i^{\wedge+}) \leftarrow D_p^{-\kappa}(x_i) \cup \{x_j\}$;
18. If $x_i \in \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $x_j \notin Cl_t^{\leqslant}$ Then $\underline{P}^{\wedge+}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_i\}$; End If
19. If $x_i \in Cl_t^{\leqslant}$ and $x_j \notin \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ Then $\overline{P}^{\wedge+}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_j\}$; End If
20. End If
21. End
22. End
23. Return $\underline{P}^{\wedge*}(Cl_t^{\geqslant})^{k}$, $\overline{P}^{\wedge*}(Cl_t^{\geqslant})^{k}$, $\underline{P}^{\wedge*}\left(Cl_t^{\leqslant}\right)^{k}$, $\overline{P}^{\wedge*}\left(Cl_t^{\leqslant}\right)^{k}$.

Function CoarseningUS() // Updating approximations while AVC upward single level
1. For each x_i in U and $f(x_j, a_l) = v_1$ Do
2. For each x_j in U and $f(x_j, a_l) = v_2$ Do
3. If $x_j \notin D_P^{+\kappa}(x_i)$ and $x_i^{\wedge+} D_p^{+\kappa} x_j$ Then
4. $D_p^{+\kappa}(x_j) \leftarrow D_p^{+\kappa}(x_j) \cup \{x_i\}$;
5. If $x_j \in \underline{P}(Cl_t^{\geqslant})^{\kappa}$ and $x_i \notin Cl_t^{\geqslant}$ Then $\underline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa} \leftarrow \underline{P}(Cl_t^{\geqslant})^{\kappa} - \{x_j\}$; End If
6. If $x_j \in Cl_t^{\geqslant}$ and $x_i \notin \overline{P}(Cl_t^{\geqslant})^{\kappa}$ Then $\overline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa} \leftarrow \overline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_i\}$; End If
7. End If
8. If $x_i \notin D_p^{-\kappa}(x_j)$ and $x_j D_p^{-\kappa} x_i$ Then
9. $D_p^{-\kappa}(x_i^{\wedge+}) \leftarrow D_p^{-\kappa}(x_i) \cup \{x_j\}$;
10. If $x_i \in \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $x_j \notin Cl_t^{\leqslant}$ Then $\underline{P}^{\wedge-}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_i\}$; End If;
11. If $x_i \in Cl_t^{\leqslant}$ and $x_j \notin \overline{P}(Cl_t^{\geqslant})^{\kappa}$ Then $\overline{P}^{\wedge+}(Cl_t^{\geqslant})^{\kappa} \leftarrow \overline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_j\}$; End If;;
12. End If
13. End
14. End
15. Return $\underline{P}^{\wedge*}(Cl_t^{\geqslant})^{k}$, $\overline{P}^{\wedge*}(Cl_t^{\geqslant})^{k}$, $\underline{P}^{\wedge*}\left(Cl_t^{\leqslant}\right)^{k}$, $\overline{P}^{\wedge*}\left(Cl_t^{\leqslant}\right)^{k}$.

Let $\overline{P}^{\vee*}(Cl_t^{\geqslant})^{\kappa}(\overline{P}^{\vee*}\left(Cl_t^{\leqslant}\right)^{\kappa})$ and $\underline{P}^{\vee*}(Cl_t^{\geqslant})^{\kappa}(\underline{P}^{\vee*}\left(Cl_t^{\leqslant}\right)^{\kappa})$ denote the upper and lower approximations of $Cl_t^{\geqslant}\left(Cl_t^{\leqslant}\right)$ after attributes refining, respectively. The superscript $*$ denotes all cases of coarsening, *i.e.*, it can be substituted by $^{+}$and $^{-}$ according to the different variation of attributes values. If $f(x_i, a_l) = v_1$, let $f(x_i, a_l) = v_2$, $v_2 \notin V_{a_l}$, *i.e.*, the attribute value $f(x_i, a_l)$ is refined to v_2. We design the algorithm for updating approximations of upward and downward unions as follows.

Algorithm 3.2. A Method for Incremental Updating Approximations while Attributes Refining in the IODS

INPUT: 1. The dominating and dominated classes of each object x_i: $D_P^{+k}(x_i)$, $D_P^{-k}(x_i)(0 < k \leqslant 1, \ \forall x_i \in U)$;
2. Upward and downward unions: $Cl_t^{\geqslant}$, $Cl_t^{\leqslant}(0 \leqslant t \leqslant n)$;
3. Approximations and boundary of upward and downward unions: $\underline{P}(Cl_t^{\geqslant})^{\kappa}, \overline{P}(Cl_t^{\geqslant})^{\kappa}, \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}, \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$, $Bn_p(Cl_t^{\geqslant})^{\kappa}, Bn_p\left(Cl_t^{\leqslant}\right)^{\kappa}$;
4. The attribute values: v_1 and v_2 (Note: attribute value v_1 is refined to v_2, $v_2 \notin V$);

OUTPUT: $\underline{P}^{\vee*}(Cl_t^{\geqslant})^{\kappa}$, $\overline{P}^{\vee*}(Cl_t^{\geqslant})^{\kappa}$, $\underline{P}^{\vee*}\left(Cl_t^{\leqslant}\right)^{\kappa}$, $\overline{P}^{\vee*}\left(Cl_t^{\leqslant}\right)^{\kappa}$.

BEGIN
1. If $v_2 \prec v_1$ Then // Refining downward
2. If $v_2 \prec v_{a_l}^{\min}$ Then // New attribute values v_2 is smaller than $v_{a_l}^{\min}$
3. Call RefiningMin();
4. Else
5. If $\exists v_2 \prec v \prec v_1$ Then // Refining downward multi-level
6. Call RefiningDM();
7. Else // Refining downward single level
8. Call RefiningDS();
9. End If
10. End If
11. Else // Refining Upward
12. If $v_2 \succ v_{\max}$ Then // New value v_2 is larger than $v_{\max}$
13. Call RefiningMax();
14. Else
15. If $\exists v_1 \prec v \prec v_2$ Then // Refining upward multi-level
16. Call RefiningUM();
17. Else // Refining upward single level
18. Call RefiningUS();
19. End If
20. End If
21. End If

Output $\underline{P}^{\wedge*}(Cl_t^{\geqslant})^{k}$, $\overline{P}^{\wedge*}(Cl_t^{\geqslant})^{k}$, $\underline{P}^{\wedge*}\left(Cl_t^{\leqslant}\right)^{k}$, $\overline{P}^{\wedge*}\left(Cl_t^{\leqslant}\right)^{k}$.

END BEGIN

Function RefiningMin() // Updating approximations while new attribute values v_2 is smaller than $v_{a_l}^{\min}$
1. For each x_j in U Do
2. If $x_i \in D_P^{+\kappa}(x_j)$ Then
3. $D_P^{+\kappa}\left(x_j^{\vee-}\right) \leftarrow D_P^{+\kappa}(x_j) - \{x_i\}$;

4. If $x_j \in Bn_P\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $D_P^{+\kappa}\left(x_j^{\vee-}\right) \subseteq Cl_t^{\geqslant}$ Then $\underline{P}(Cl_t^{\geqslant})^{\kappa} \leftarrow \underline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_j\}$; End If
5. If $x_j \in Cl_t^{\leqslant}$ and $x_i \notin Cl_t^{\leqslant}$ and $x_i \notin D_P^{+\kappa}(x')(x' \in Cl_t^{\leqslant})$ Then $\overline{P}(Cl_t^{\geqslant})^{\kappa} \leftarrow \overline{P}(Cl_t^{\geqslant})^{\kappa} - \{x_i\}$; End If
6. Else If $f(x_j, a_l) \prec v_1$ and $x_j D_P^{+\kappa} x_i^{\vee-}$ Then
7. $D_P^{+\kappa}(x_i^{\vee-}) \leftarrow D_P^{+\kappa}(x_i) \cup \{x_j\}$;
8. If $x_i \in \underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa}$ and $x_j \notin Cl_t^{\geqslant}$ Then $\underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} \leftarrow \underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} - \{x_i\}$; End If
9. If $x_i \in Cl_t^{\leqslant}$ Then $\overline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} \leftarrow \overline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_j\}$; End If
10. End If
11. If $x_i \notin D_P^{-\kappa}(x_j)$ and $f(x_j, a_l) \prec v_1$ and $x_i^{\vee-} D_P^{-\kappa} x_j$ Then
12. $D_P^{-\kappa}\left(x_j^{\vee-}\right) \leftarrow D_P^{-\kappa}(x_j) \cup \{x_i^{\vee-}\}$;
13. If $x_j \in \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $x_i \notin Cl_t^{\leqslant}$ Then $\underline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_j\}$; End If
14. If $x_j \in Cl_t^{\leqslant}$ and $x_i \notin \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ Then $\overline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_i^{\vee-}\}$; End If
15. End If
16. End
17. $D_P^{-\kappa}(x_i^{\vee-}) \leftarrow \{x_i\}$;
18. If $x_i \notin \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $x_i \in Cl_t^{\leqslant}$ Then $\underline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_i^{\vee-}\}$; End If
19. If $x_i \in Bn_P\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $x_i \notin Cl_t^{\leqslant}$ Then $\underline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_i^{\vee-}\}$; End If
20. If $x_i \in Cl_t^{\leqslant}$ and $C' = \{x_k | x_k \in D_P^{-\kappa}(x_i) \wedge x_k \notin D_P^{-\kappa}(x'), x' \in Cl_t^{\leqslant}\}$ Then $\overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - C'$; End If
21. Return $\underline{P}^{\wedge*}(Cl_t^{\geqslant})^{k}$, $\overline{P}^{\wedge*}(Cl_t^{\geqslant})^{k}$, $\underline{P}^{\wedge*}\left(Cl_t^{\leqslant}\right)^{k}$, $\overline{P}^{\wedge*}\left(Cl_t^{\leqslant}\right)^{k}$.

Function RefiningDS() // Updating approximations while AVR downward single-level
1. For each x_j in U Do
2. If $x_i \in D_P^{+\kappa}(x_j)$ and $f(x_j, a_l) = v_1$ Then
3. $D_P^{+\kappa}\left(x_j^{\vee-}\right) \leftarrow D_P^{+\kappa}(x_j) - \{x_i^{\vee-}\}$;
4. If $x_j \in Bn_P\left(Cl_t^{\geqslant}\right)^{\kappa}$, $x_i \notin Cl_t^{\geqslant}$ and $D_P^{+\kappa}\left(x_j^{\vee-}\right) \subseteq Cl_t^{\geqslant}$ Then $\underline{P}(Cl_t^{\geqslant})^{\kappa} \leftarrow \underline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_j\}$; End If
5. If $x_j \in Cl_t^{\geqslant}$ and $x_i \notin Cl_t^{\geqslant}$ and $x_i \notin D_P^{-\kappa}(x')(x' \in Cl_t^{\geqslant} - \{x_j\})$ Then $\overline{P}(Cl_t^{\geqslant})^{\kappa} \leftarrow \overline{P}(Cl_t^{\geqslant})^{\kappa} - \{x_i^{\vee-}\}$; End If
6. $D_P^{-\kappa}(x_i^{\vee-}) \leftarrow D_P^{-\kappa}(x_i) - \{x_j\}$;
7. If $x_i \notin Bn_P\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $D_P^{-\kappa}(x_i^{\vee-}) \subseteq Cl_t^{\leqslant}$ Then $\underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_i\}$; End If
8. If $x_i \in Cl_t^{\leqslant}$ and $x_j \notin Cl_t^{\leqslant}$ and $x_k \notin D_P^{-\kappa}(x')(x' \in Cl_t^{\leqslant})$ Then $\overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_j\}$; End If
9. End If
10. End
11. Return $\underline{P}^{\wedge*}(Cl_t^{\geqslant})^{k}$, $\overline{P}^{\wedge*}(Cl_t^{\geqslant})^{k}$, $\underline{P}^{\wedge*}\left(Cl_t^{\leqslant}\right)^{k}$, $\overline{P}^{\wedge*}\left(Cl_t^{\leqslant}\right)^{k}$.

Function RefiningDM() // Updating approximations while AVR downward multi-level
1. For each x_j in U Do
2. If $x_i \in D_P^{+\kappa}(x_j)$ and $v_2 \prec f(x_j, a_l) \preceq v_1$ Then
3. $D_P^{+\kappa}\left(x_j^{\vee-}\right) \leftarrow D_P^{+\kappa}(x_j) - \{x_i^{\vee-}\}$;
4. If $x_j \in Bn_P(Cl_t^{\geqslant})^{\kappa}$ and $x_i^{\vee-} \notin Cl_t^{\geqslant}$ and $D_P^{+\kappa}(x_j^{\vee-}) \subseteq Cl_t^{\geqslant}$ Then $\underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} \leftarrow \underline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_j\}$; End If
5. If $x_j \in Cl_t^{\geqslant}$, $x_i^{\vee-} \notin D_P^{+\kappa}(x_k)$, $\forall x_k \in Cl_t^{\geqslant} - \{x_j\}$ Then $\overline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} \leftarrow \overline{P}(Cl_t^{\geqslant})^{\kappa} - \{x_i\}$; End If
6. Else If $x_i \notin D_P^{+\kappa}(x_j)$ and $v_2 \prec f(x_j, a_l) \prec v_1$ and $x_j D_P^{+\kappa} x_i^{\vee-}$ Then
7. $D_P^{+\kappa}(x_i^{\vee-}) \leftarrow D_P^{+\kappa}(x_i) \cup \{x_j\}$;
8. If $x_i \in \underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa}$ and $x_j \notin C_t^{\geqslant}$ Then $\underline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} = \underline{P}(Cl_t^{\geqslant})^{\kappa} - \{x_i^{\vee-}\}$; End If
9. If $x_i^{\vee-} \in Cl_t^{\geqslant}$ Then $\overline{P}^{\vee-}(Cl_t^{\geqslant})^{\kappa} \leftarrow \overline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_j\}$; End If
10. End If
11. If $x_j \in D_P^{-k}(x_i)$ and $v_2 \prec f(x_j, a_l) \preceq v_1$ Then
12. $D_P^{-\kappa}(x_i^{\vee-}) \leftarrow D_P^{-\kappa}(x_i) - \{x_j\}$;
13. If $x_i \in Bn_P\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $D_P^{-\kappa}(x_i^{\vee-}) \subseteq \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ Then $\underline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_i^{\vee-}\}$; End If
14. If $x_i \in Cl_t^{\geqslant}$ and $x_k \notin D_P^{-\kappa}(x'), \forall x' \in Cl_t^{\geqslant}$ Then $\overline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_j\}$; End If
15. Else If $x_j \notin D_P^{-k}(x_i)$ and $v_2 \prec f(x_j, a_l) \prec v_1$ Then
16. $D_P^{-\kappa}\left(x_j^{\vee-}\right) \leftarrow D_P^{-\kappa}(x_j) \cup \{x_i^{\vee-}\}$;
17. If $x_j \in \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $x_i^{\vee-} \notin Cl_t^{\leqslant}$ Then $\underline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_j\}$; End If
18. If $x_j \in Cl_t^{\leqslant}$ and $x_i \notin D_P^{-\kappa}(x')(\forall x' \in Cl_t^{\leqslant})$ Then $\overline{P}^{\vee-}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_i^{\vee-}\}$; End If
19. End If
20. End
21. Return $\underline{P}^{\wedge*}(Cl_t^{\geqslant})^{k}$, $\overline{P}^{\wedge*}(Cl_t^{\geqslant})^{k}$, $\underline{P}^{\wedge*}\left(Cl_t^{\leqslant}\right)^{k}$, $\overline{P}^{\wedge*}\left(Cl_t^{\leqslant}\right)^{k}$.

Function RefiningMax() // Updating approximations while new value v_2 is larger than $v_{\max}$
1. $D_P^{+\kappa}(x_i^{\vee+}) \leftarrow \{x_i\}$;
2. If $x_i \in Cl_t^{\geqslant}$ Then $\underline{P}^{\vee+}(Cl_t^{\geqslant})^{\kappa} \leftarrow \underline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_i^{\vee+}\}$; End If
3. If $x_i \in Cl_t^{\geqslant}$ and $C'' = \{x_k | x_k \in D_P^{+\kappa}(x_i) - \{x_i\} \wedge x_k \notin D_P^{+\kappa}(x'), x' \in Cl_t^{\geqslant}\} \neq \varnothing$ Then
4. $\overline{P}^{\vee+}(Cl_t^{\geqslant})^{\kappa} \leftarrow \overline{P}(Cl_t^{\geqslant})^{\kappa} - C''$;
5. End If
6. For each x_j in U Do
7. If $x_j \notin D_P^{+\kappa}(x_i)$ and $x_i^{\vee+} D_P^{+\kappa} x_j$ Then
8. $D_P^{+\kappa}\left(x_j^{\vee+}\right) \leftarrow D_P^{+\kappa}(x_j) \cup \{x_i^{\vee+}\}$;
9. If $x_i \notin Cl_t^{\geqslant}$ and $x_j \in \underline{P}(Cl_t^{\geqslant})^{\kappa}$ Then $\underline{P}^{\vee+}(Cl_t^{\geqslant})^{\kappa} \leftarrow \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_j\}$; End If
10. If $x_j \in Cl_t^{\geqslant}$ and $x_i \notin D_P^{+\kappa}(x')(x' \in Cl_t^{\geqslant})$ Then $\overline{P}^{\vee+}(Cl_t^{\geqslant})^{\kappa} \leftarrow \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_i^{\vee+}\}$; End If
11. End If
12. If $x_i \in D_P^{-\kappa}(x_j)$ Then

(continued on next page)

13. $D_P^{-\kappa}\left(x_j^{\vee+}\right) \leftarrow D_P^{-\kappa}(x_j) - \{x_i\}$;
14. If $x_i \notin Cl_t^{\leqslant}$ and $x_j \in Bn_P\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $D_P^{-\kappa}(x_j^{\vee+}) \subseteq Cl_t^{\leqslant}$ Then $\underline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \underline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_j^{\vee+}\}$; End If
15. If $x_j \in Cl_t^{\leqslant}$ and $x_i \notin D_P^{+\kappa}(x')(x' \in Cl_t^{\leqslant})$ Then $\overline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_i\}$; End If
16. Else If $f(x_j, a_l) \succ \upsilon_1$ and $x_j D_P^{-\kappa} x_i$ Then
17. $D_P^{-\kappa}(x_i^{\vee+}) \leftarrow D_P^{-\kappa}(x_i) \cup \{x_j\}$;
18. If $x_i \in \underline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $x_j \notin Cl_t^{\leqslant}$ Then $\underline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \underline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_i\}$; End If
19. If $x_i \in Cl_t^{\leqslant}$ and $x_j \notin \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ Then $\overline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_j\}$; End If
20. End If
21. End
22. Return $\underline{P}^{\wedge*}(Cl_t^{\geqslant})^k$, $\overline{P}^{\wedge*}(Cl_t^{\geqslant})^k$, $\underline{P}^{\wedge*}\left(Cl_t^{\leqslant}\right)^k$, $\overline{P}^{\wedge*}\left(Cl_t^{\leqslant}\right)^k$.

Function RefiningUM() // Updating approximations while AVR upward multi-level
1. For each x_j in U Do
2. If $x_j \in D_P^{+\kappa}(x_i)$ and $\upsilon_1 \preceq f(x_j, a_l) \prec \upsilon_2$ Then
3. $D_P^{+\kappa}(x_i^{\vee+}) \leftarrow D_P^{+\kappa}(x_i) - \{x_j\}$;
4. If $x_i \in Bn_P(Cl_t^{\geqslant})^{\kappa}$ and $D_P^{+\kappa}(x_i^{\vee+}) \subseteq Cl_t^{\geqslant}$ Then $\underline{P}^{\vee+}(Cl_t^{\geqslant})^{\kappa} \leftarrow \underline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_i^{\vee+}\}$; End If
5. If $x_i \in Cl_t^{\geqslant}$ and $x_i \notin Cl_t^{\geqslant}$ and $x_k \notin D_P^{+\kappa}(x')(x' \in Cl_t^{\geqslant})$ Then $\overline{P}^{\vee+}(Cl_t^{\geqslant})^{\kappa} \leftarrow \overline{P}(Cl_t^{\geqslant})^{\kappa} - \{x_j\}$; End If
6. Else If $x_j \notin D_P^{+\kappa}(x_i)$ and $\upsilon_1 \preceq f(x_j, a_l) \prec \upsilon_2$ and $x_i^{\vee+} D_P^{+\kappa} x_j$ Then
7. $D_P^{+\kappa}\left(x_j^{\vee+}\right) \leftarrow D_P^{+\kappa}(x_j) \cup \{x_i^{\vee+}\}$;
8. If $x_j \in \underline{P}(Cl_t^{\geqslant})^{\kappa}$ and $x_i \not\subset Cl_t^{\geqslant}$ Then $\underline{P}^{\vee+}(Cl_t^{\geqslant})^{\kappa} \leftarrow \underline{P}(Cl_t^{\geqslant})^{\kappa} - \{x_j\}$; End If
9. If $x_i \notin \overline{P}(Cl_t^{\geqslant})^{\kappa}$ and $x_j \in Cl_t^{\geqslant}$ Then $\overline{P}^{\vee+}(Cl_t^{\geqslant})^{\kappa} \leftarrow \overline{P}(Cl_t^{\geqslant})^{\kappa} \cup \{x_i\}$; End IF
10. End If
11. If $x_i \in D_P^{-\kappa}(x_j)$ and $\upsilon_1 \preceq f(x_j, a_l) \preceq \upsilon_2$ Then
12. $D_P^{-\kappa}\left(x_j^{\vee+}\right) \leftarrow D_P^{-\kappa}(x_j) - \{x_i\}$;
13. If $x_i \notin Cl_t^{\leqslant}$ and $D_P^{-\kappa}(x_j) \in Bn_p\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $D_P^{-\kappa}\left(x_j^{\vee+}\right) \subset Cl_t^{\leqslant}$ Then $\underline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_j^{\vee+}\}$; End If
14. If $x_j \in Cl_t^{\leqslant}$ and $x_i \notin Cl_t^{\leqslant}$ and $x_i \notin D_P^{-\kappa}(x')(\forall x' \in Cl_t^{\leqslant})$ Then $\overline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \overline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_i\}$; End If
15. Else If $x_i \notin D_P^{-\kappa}(x_j)$ and $\upsilon_1 \prec f(x_j, a_l) \prec \upsilon_2$ and $x_j D_P^{-\kappa} x_i$ Then
16. $D_P^{-\kappa}(x_i^{\vee+}) \leftarrow D_P^{-\kappa}(x_i) \cup \{x_j\}$;
17. If $x_i \in \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $x_j \notin Cl_t^{\leqslant}$ Then $\underline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_i\}$; End If
18. If $x_i \in Cl_t^{\leqslant}$ Then $\overline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_j\}$; End If
19. End If
20. End
21. Return $\underline{P}^{\wedge*}(Cl_t^{\geqslant})^k$, $\overline{P}^{\wedge*}(Cl_t^{\geqslant})^k$, $\underline{P}^{\wedge*}\left(Cl_t^{\leqslant}\right)^k$, $\overline{P}^{\wedge*}\left(Cl_t^{\leqslant}\right)^k$.

Function RefiningUS() // Updating approximations while AVR upward single-level 1. For each x_j in U and $f(x_j, a_l) = \upsilon_1$ Do
2. If $x_j \in D_P^{+\kappa}(x_i)$ Then
3. $D_P^{+\kappa}(x_i^{\vee+}) \leftarrow D_P^{+\kappa}(x_i) - \{x_j\}$;
4. If $x_i \in Cl_t^{\geqslant}$ and $x_i \in Bn_P(Cl_t^{\geqslant})^{\kappa}$ and $D_P^{+\kappa}(x_i^{\vee+}) \subseteq Cl_t^{\geqslant}$ Then $\underline{P}^{\vee+}(Cl_t^{\geqslant})^{\kappa} \leftarrow \underline{P}^{\vee+}(Cl_t^{\geqslant})^{\kappa} \cup \{x_i^{\vee+}\}$; End If
5. If $x_i \in Cl_t^{\geqslant}$ and $x_j \notin Cl_t^{\geqslant}$ and $x_j \notin D_P^{+\kappa}(x')(x' \in Cl_t^{\geqslant})$ Then $\overline{P}^{\vee+}(Cl_t^{\geqslant})^{\kappa} \leftarrow \overline{P}(Cl_t^{\geqslant})^{\kappa} - \{x_j\}$; End If
6. End If
7. If $x_i \in D_P^{-\lambda}(x_j)$ Then
8. $D_P^{-\kappa}\left(x_j^{\vee+}\right) \leftarrow D_P^{-\kappa}(x_j) - \{x_i\}$;
9. If $x_i \notin Cl_t^{\leqslant}$ and $x_j \in Bn_P\left(Cl_t^{\leqslant}\right)^{\kappa}$ and $D_P^{-\lambda}(x_j) \subseteq Cl_t^{\leqslant}$ Then $\underline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \underline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} \cup \{x_j\}$; End If
10. If $x_j \in Cl_t^{\leqslant}$ and $x_i \notin Cl_t^{\leqslant}$ and $x_i \notin D_P^{-\kappa}(x')\left(\forall x' \in Cl_t^{\leqslant}\right)$ Then $\overline{P}^{\vee+}\left(Cl_t^{\leqslant}\right)^{\kappa} \leftarrow \overline{P}\left(Cl_t^{\leqslant}\right)^{\kappa} - \{x_i\}$; End If
11. End If
12. End
13. Return $\underline{P}^{\wedge*}(Cl_t^{\geqslant})^k$, $\overline{P}^{\wedge*}(Cl_t^{\geqslant})^k$, $\underline{P}^{\wedge*}\left(Cl_t^{\leqslant}\right)^k$, $\overline{P}^{\wedge*}\left(Cl_t^{\leqslant}\right)^k$.

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4. An illustration

In this section, we employ an example to illustrate the methods of updating approximations w.r.t. AVCR. Given an IODS $S = (U, A, V, f)$ in Table 1. $U = \{x_i, i = 1, 2, \ldots, 8\}$, $V_{a_1} = \{0, 1, 2\}, V_{a_2} = \{0, 1, 2, 4\}, V_{a_3} = \{0, 1\}$, $V_{a_4} = \{0, 1, 2, 3\}$, $V_d = \{0, 1, 2, 3\}$.

$D_P^{+0.5}(x_1) = \{x_1, x_4, x_6, x_7\}$, $D_P^{+0.5}(x_2) = \{x_1, x_2, x_4, x_6, x_7\}$,
$D_P^{+0.5}(x_3) = \{x_1, x_2, x_3, x_6, x_7, x_8\}$, $D_P^{+0.5}(x_4) = \{x_4, x_6, x_7\}$,

$D_P^{+0.5}(x_5) = \{x_5\}$, $D_P^{+0.5}(x_6) = \{x_6\}$,
$D_P^{+0.5}(x_7) = \{x_6, x_7\}$, $D_P^{+0.5}(x_8) = \{x_6, x_8\}$.

$U/d = \{C_0, C_1, C_2\} = \{\{x_1, x_2, x_5, x_7\}, \{x_3\}, \{x_4, x_8\}, \{x_6\}\}$,
$Cl_0^{\geqslant} = \{x_1, x_2, x_5, x_7, x_3, x_4, x_8, x_6\}$, $Cl_1^{\geqslant} = \{x_3, x_4, x_6, x_8\}$,

$Cl_2^{\geqslant} = \{x_4, x_6, x_8\}$, $Cl_3^{\geqslant} = \{x_6\}$. $Cl_0^{\leqslant} = \{x_1, x_2, x_5, x_7\}$,
$Cl_1^{\leqslant} = \{x_1, x_2, x_3, x_5, x_7\}$, $Cl_2^{\leqslant} = \{x_1, x_2, x_5, x_7, x_3, x_4, x_8\}$,

$Cl_3^{\leqslant} = \{x_1, x_2, x_5, x_7, x_3, x_6, x_4, x_8\}$.

$\underline{P}(Cl_1^{\geqslant})^{0.5} = \{x_6, x_8\}$, $\overline{P}(Cl_1^{\geqslant})^{0.5} = \{x_1, x_2, x_3, x_4, x_6, x_7, x_8\}$,
$Bn_p(Cl_1^{\geqslant})^{0.5} = \overline{P}(Cl_1^{\geqslant})^{0.5} - \underline{P}(Cl_1^{\geqslant})^{0.5} = \{x_1, x_2, x_3, x_4, x_7\}$.

Table 1
An incomplete order decision system.

U	a_1	a_2	a_3	a_4	d
x_1	1	2	1	2	0
x_2	0	1	1	?	0
x_3	*	0	0	1	1
x_4	1	3	?	*	2
x_5	*	2	?	*	0
x_6	2	4	1	*	3
x_7	1	4	?	3	0
x_8	2	0	0	3	2

Table 2
An incomplete order decision system after coarsening.

U	a_1	a_2	a_3	a_4	d
x_1	1	2	1	2	0
x_2	0	1	1	?	0
x_3	*	0	0	1	1
x_4	1	3	?	*	2
x_5	0	*	?	*	0
x_6	2	1	1	*	3
x_7	1	1	?	3	0
x_8	2	0	0	3	2

Table 3
A description of datasets.

ID	Dataset	Samples	Features	Classes	Default rate (%)
1	Lung cancer	32	57	3	0.27
2	Iris	150	6	3	9.56
3	Parkinsons	195	23	2	9.04
4	Sonar	208	61	2	5.79
5	Ionosphere	351	35	2	18.87
6	Movementlibras	360	91	15	7.85
7	Wdbc	569	32	2	28.13
8	Transfusion	748	5	2	16.42
9	Mammographic masses	961	6	2	2.41
10	Winered	1599	12	6	4.23

$D_P^{-0.5}(x_1) = \{x_1, x_2, x_3, x_4\}, D_P^{-0.5}(x_2) = \{x_2, x_3\},$
$D_P^{-0.5}(x_3) = \{x_3\}, \quad D_P^{-0.5}(x_4) = \{x_1, x_2, x_4\}, \quad D_P^{-0.5}(x_5) = \{x_5\},$

$D_P^{-0.5}(x_6) = \{x_1, x_2, x_3, x_4, x_6, x_7, x_8\},$
$D_P^{-0.5}(x_7) = \{x_1, x_2, x_3, x_4, x_7\}, \quad D_P^{-0.5}(x_8) = \{x_3, x_8\}.$
$Cl_1^{\leqslant} = \{x_1, x_2, x_3, x_5, x_7\}.$

$$\underline{P}\left(Cl_1^{\leqslant}\right)^{0.5} = \{x_2, x_3, x_5\}, \quad \overline{P}\left(Cl_1^{\leqslant}\right)^{0.5} = \{x_1, x_2, x_3, x_4, x_5, x_7\},$$

$$Bn_p\left(Cl_1^{\leqslant}\right)^{0.5} = \overline{P}(Cl_1^{\leqslant})^{0.5} - \underline{P}\left(Cl_1^{\leqslant}\right)^{0.5} = \{x_1, x_4, x_7\}.$$

(1) Let $f(x_i^{\wedge-}, a_2) = 1, \ \forall x_i \in U', \ U' = \{x_i | f(x_i, a_2) = 4, x_i \in U\}$, $V^{\wedge-} = \{v | 1 \prec v \prec 4, v \in V_2\} = \{2\} \neq \emptyset$. That is attribute value coarsening downward multi-level. The IODS after coarsening is shown in Table 2.

i.
(a) $C^{\wedge-} = \{x_6, x_7\}, C_1 = \{x_1, x_4\}, C_2 = \{x_5\}$.
(b) $\forall x_j \in C_1, \ D_p^+(x_1^{\wedge-}) = D_p^+(x_1) - C^{\wedge-} = \{x_1, x_4\}, \ D_p^+(x_4^{\wedge-}) = D_p^+(x_4) - C^{\wedge-} = \{x_4\}$.
(c) $C_3 = \emptyset$.
(d) $x_1 \in Bn_P(Cl_1^{\geqslant})^{0.5}$ and $D_p^+(x_1^{\wedge-}) \not\subset Cl_1^{\geqslant}, \ \underline{P}^{\wedge-}(Cl_1^{\geqslant})^{0.5} = \underline{P}(Cl_1^{\geqslant})^{0.5}$.
(e) $x_4 \in Bn_P(Cl_1^{\geqslant})^{0.5}$ and $D_p^+(x_4^{\wedge-}) \subseteq Cl_1^{\geqslant}, \ \underline{P}^{\wedge-}(Cl_1^{\geqslant})^{0.5} = \underline{P}(Cl_1^{\geqslant})^{0.5} \cup \{x_4\} = \{x_2, x_3, x_5, x_4\}$.
(f) $C'' = \emptyset, \ \overline{P}^{\wedge-}(Cl_1^{\geqslant})^{0.5} = \overline{P}(Cl_1^{\geqslant})^{0.5}$.

ii.
(a) $C_1 = \{x_1, x_4\}, C_2 = \{x_5\}$;
(b) $\forall x_i \in C^{\wedge-}, \ D_p^{-0.5}(x_6^{\wedge-}) = D_p^{-0.5}(x_6) - C_1 = \{x_2, x_3, x_6, x_7, x_8\}$, $D_p^{-\kappa}(x_7^{\wedge-}) = D_p^{-0.5}(x_7) - C_1 = \{x_2, x_3, x_7\}; C_3 = \emptyset$;
(c) $\forall x_7 \in C^{\wedge-}, x_7 \in Bn_P\left(Cl_1^{\leqslant}\right)^{0.5}, D_p^{-0.5}(x_7^{\wedge-}) \subseteq Cl_1^{\leqslant}, \underline{P}^{\wedge-}\left(Cl_1^{\leqslant}\right)^{0.5} = \underline{P}\left(Cl_1^{\leqslant}\right)^{0.5} \cup \{x_7^{\wedge-}\} = \{x_2, x_3, x_5, x_7\}; \quad x_7 \in Cl_1^{\leqslant}, \ C'' = \emptyset$, $\overline{P}^{\wedge-}\left(Cl_1^{\leqslant}\right)^{0.5} = \overline{P}\left(Cl_1^{\leqslant}\right)^{0.5}$.

5. Experimental evaluation

We have carried out several experiments to verify the effectiveness of the proposed methods while attribute values are coarsening and refining in the IODS. We downloaded 10 datasets from the UC Irvine Machine Learning Database Repository (www.ics.uci.edu/~mlearn/MLRepository.html) listed in Table 3. All data type is numeric. Note that some datasets do not have lost values and there is only one case of data missing in the incomplete datasets. In the experiments, we set lost values and two cases of data missing randomly in these datasets. We performed the experiments on a computer with Intel Core 2 Duo CPU T6500 2.10 GHz CPU, 4.0 GB of memory, running Microsoft Windows Vista Home Basic. We developed algorithms in C#.

5.1. Comparative experiments on 10 datasets

In this section, we compare the computation time of incremental updating with that of non-incremental updating while attributes values coarsening or refining on the 10 real world datasets shown in Table 3. We divide each of these 10 datasets into 10 parts of equal size. The first part is regarded as the first dataset, the combination of the first part and the second part is viewed as the second dataset, and so on. The experimental results of these 10 datasets are shown in Figs. 3 and 4. In each of these sub-figures, the x-coordinate pertains to the size of the dataset which starts from the smallest one of the 10 datasets and the y-coordinate concerns the computing time. The computation time is the arithmetic mean value of all steps of coarsening or refining.

In Figs. 3 and 4, square lines denote the computation time of non-incremental updating and point lines denote the computation time of incremental updating. The computation time of incremental updating is lower than that of non-incremental updating both in coarsening and refining.

5.2. Experiments on two datasets with different features

We have selected different attributes from dataset Movementlibras and Sonar. We compare the computation time of incremental updating with that of non-incremental updating when the number of attributes alter both in the case of attributes coarsening and refining. The results are depicted in Figs. 5 and 6.

The computation time of non-incremental updating increases with the increasing size of attributes. But the computation time of incremental updating decreases with the increasing size of attributes in the case of attributes coarsening or refining. Because for any $E \subseteq F \subseteq C, \ D_C^{+\kappa}(x) \subseteq D_F^{+\kappa}(x) \subseteq D_E^{+\kappa}(x), \ D_C^{-\kappa}(x) \subseteq D_F^{-\kappa}(x) \subseteq D_E^{-\kappa}(x)$ are not always true, *i.e.*, the cardinality of the dominating (dominated) classes may increase or decrease. In this test, the cardinality of the dominating (dominated) classes decreases when the number of attributes increases. The computation time of approximations of upward and downward unions decreases. Consequently, the computation time of incremental updating decreases is shown in Fig. 6.

5.3. Experiments on two datasets with different default rates

In this section, different default rates are set randomly to datasets, Winerred and Transfusion. We conduct incremental updating and non-incremental on these two datasets with different default rates. The results are depicted in Figs. 7 and 8. For the dataset Winered, the computation time increases in the first three points and then decreases in the rest points with the increase of default rate. It follows from Property 2.2, we know the cardinality of dominating (dominating) classes may increase or decrease with the increase of default rate. So, the granularity measure of the IODS may

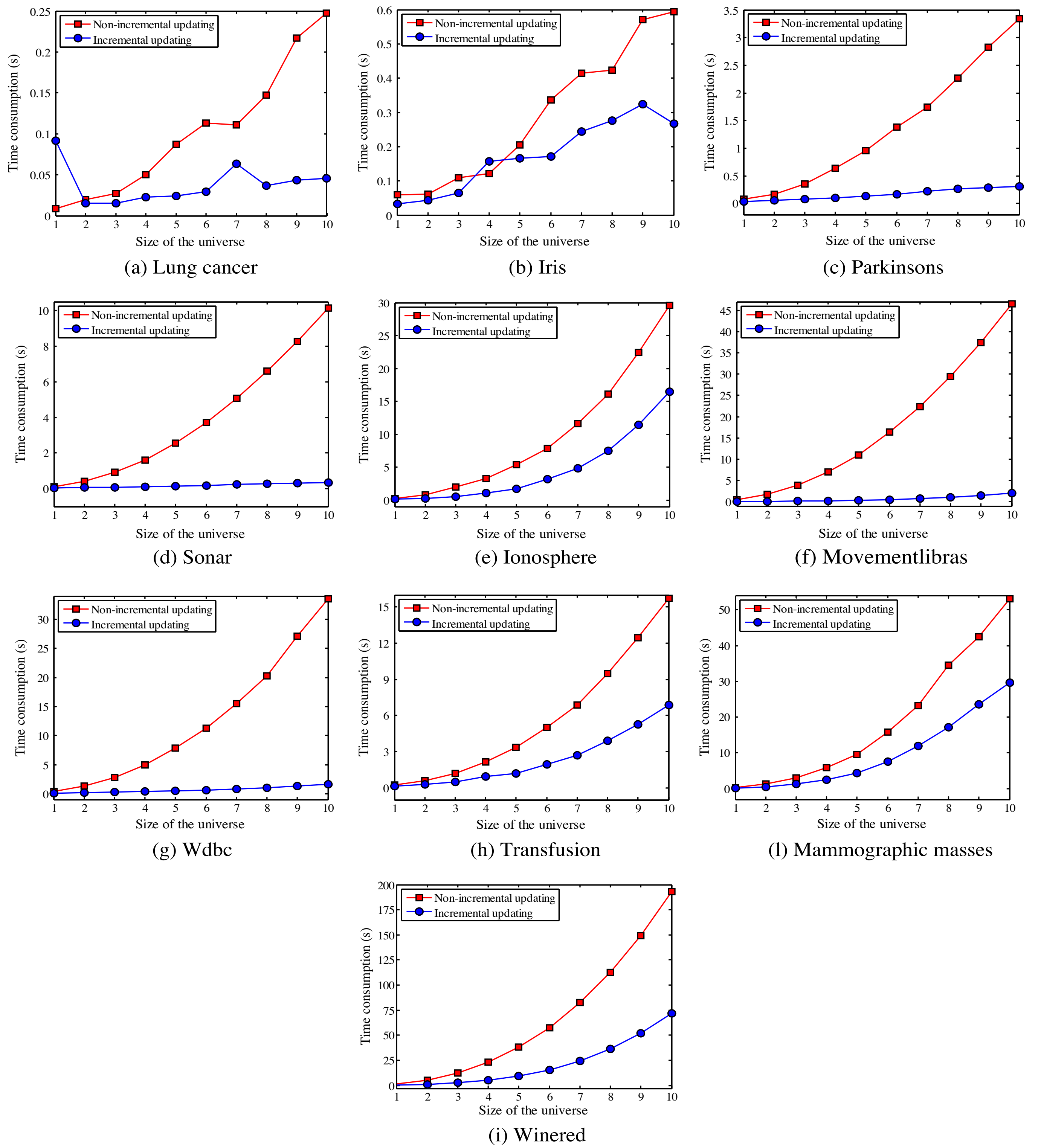


Fig. 3. A comparison of incremental updating approximations and non-incremental updating approximations when attribute values are refining in the IODS.

decrease or increase. The computation time of incremental and non-incremental updating thus decreases with the increase of the granularity measure. In Fig. 9, the granularity measure of datasets Winerred and Transfusion under different default rates λ are depicted. Comparing Figs. 7 and 9, Figs. 8 and 9, we find the alternation of computation time relate to the variation of the granularity measure.

5.4. Experiments on seven datasets with different k

We take specific values of the parameter k in this section to incrementally update approximations. The result is depicted in Fig. 10. In Fig. 10a, the computation time of datasets transfusion and mammographic decreases apparently in the case of coarsening. The computation time of datasets Wdbc, Movementlibras

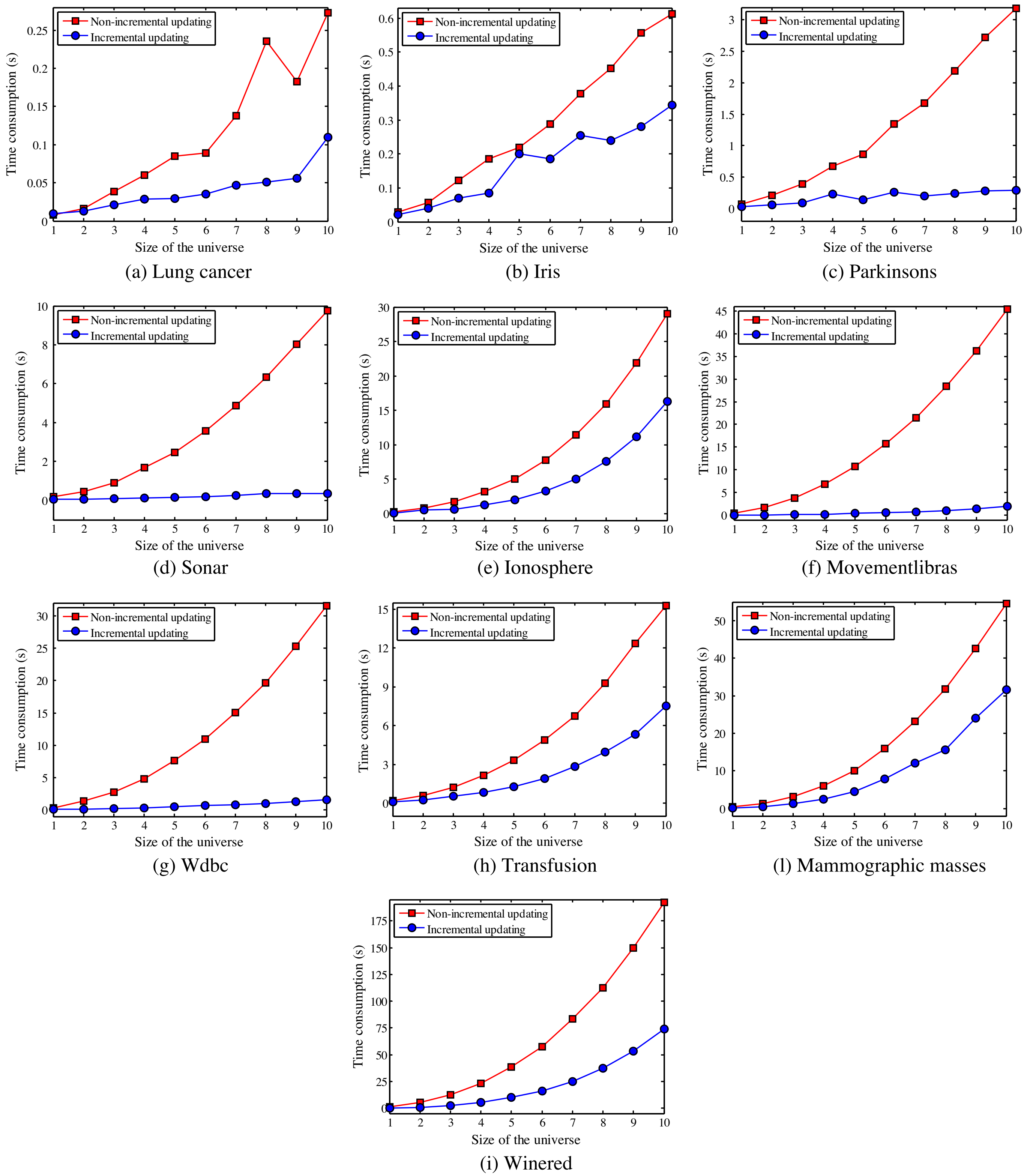


Fig. 4. A comparison of incremental updating approximations and non-incremental updating approximations when attribute values are coarsening in the IODS.

and ionosphere decreases apparently in the case of refining as shown in Fig. 10b. It follows from Property 2.3, we know the knowledge granularity decreases with increase of k. The computation time of Algorithm 3.2 for updating dominating (dominated) classes thus decreases with the increase of k.

5.5. Computational complexity analysis of the incremental algorithms in IODS

In IODS, the granules are induce by dominating classes or dominated classes. The information granulation may alter while the

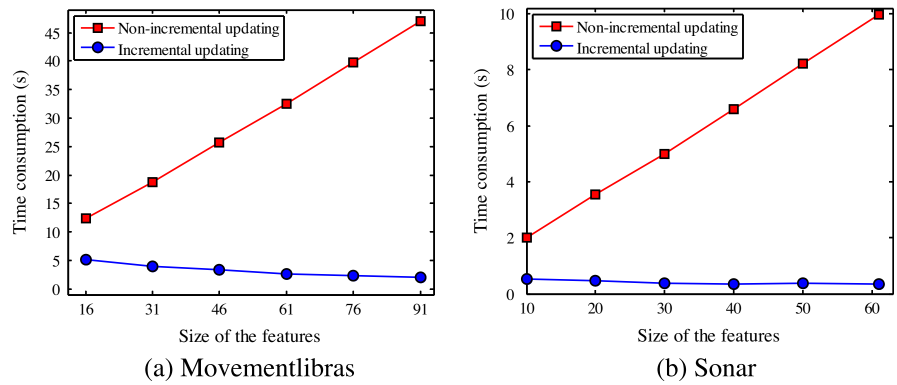


Fig. 5. A comparison of incremental updating approximations and non-incremental updating approximations when attribute values are coarsening in the IODS with the variation of attributes.

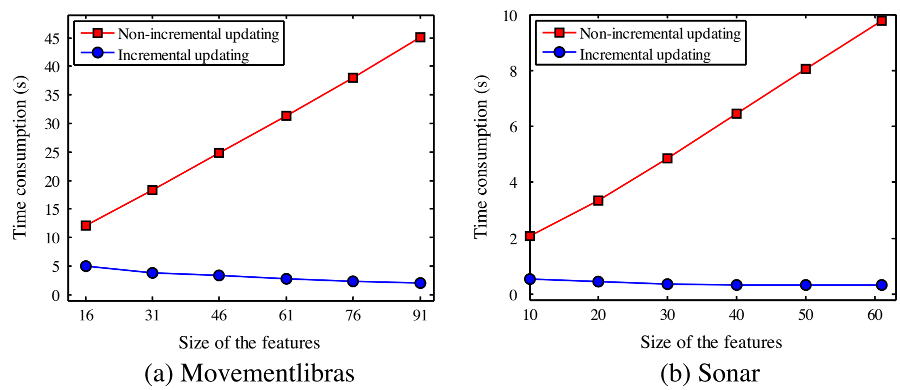


Fig. 6. A comparison of incremental updating approximations and non-incremental updating approximations when attribute values are refining in the IODS with the variation of features.

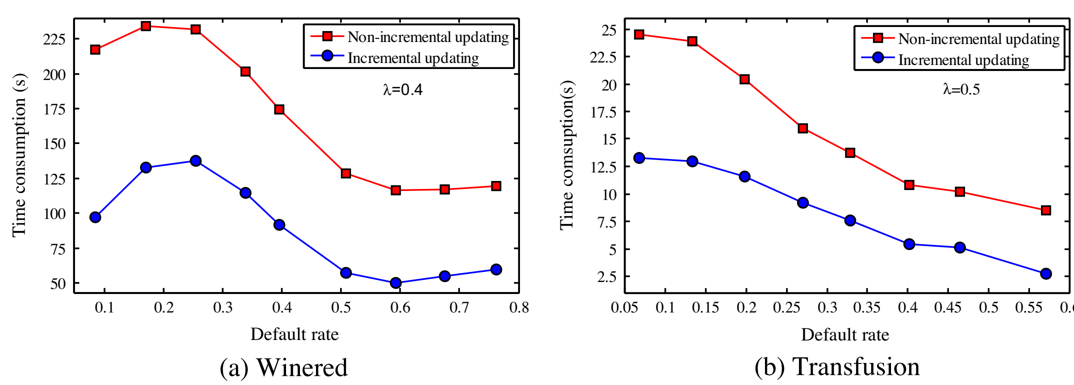
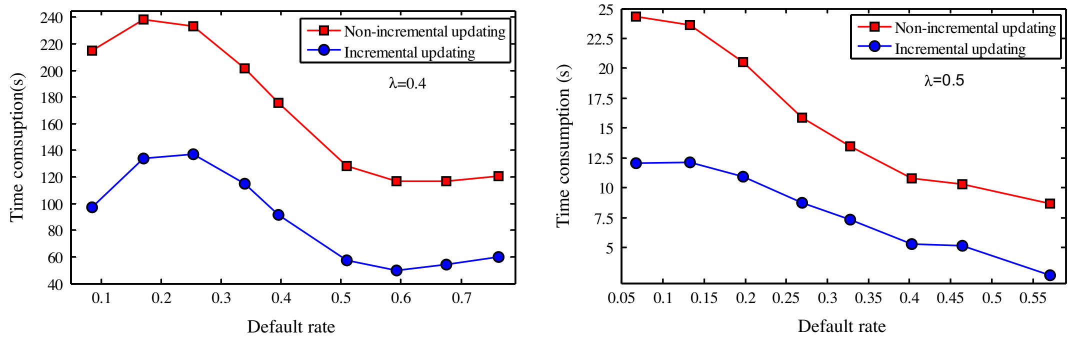


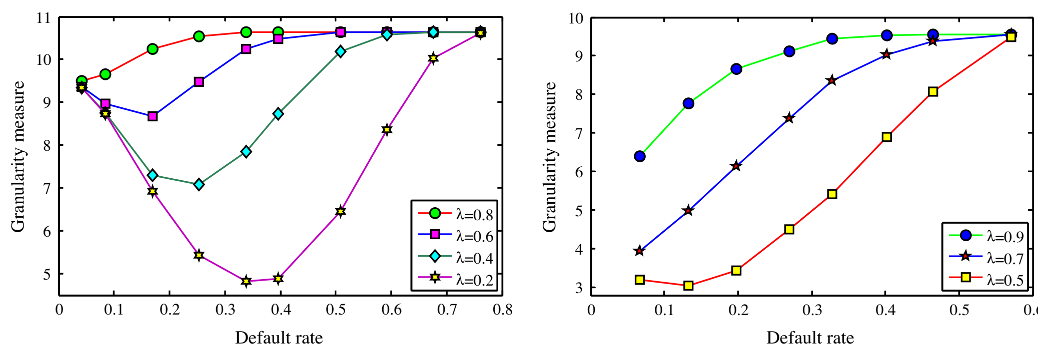
Fig. 7. A comparison of incremental updating approximations and non-incremental updating approximations when attributes coarsening in the IODS with the variation of default rates.

attribute values are coarsening or refining. Some of granules may extended (*i.e.*, the cardinality of the granules increase) and some of granules may become smaller (*i.e.*, the cardinality of the granules decrease). In IODS, upward (downward) union is approximated by dominating (dominated) classes. Therefore, the approximations of upward (downward) union may change. In our incremental updating methods, we focus on the dynamic updating of granules which vary during the attributes values coarsening or refining. The computation time include two parts, *i.e.*, incremental updating dominating (dominated) classes and incremental updating approximations. Suppose the dominating (dominated) classes extended is denoted $C_\uparrow^+ = \left\{x_{i_\uparrow}^+,\ i = 1,\ldots,m_\uparrow^+\right\}$ $\left(C_\uparrow^- = \left\{x_{i_\uparrow}^-,\ i=1,\ldots,m_\uparrow^-\right\}\right)$, the object set added to dominating (dominated) class of each object $x_{i_\uparrow}^+$ $\left(x_{i_\uparrow}^-\right)$ is $C_{i_\uparrow}^+$ $\left(C_{i_\uparrow}^-\right)$, the dominating (dominated) classes become smaller is $C_\downarrow^+ = \left\{x_{j_\downarrow}^+,\ j=1,\ldots,m_\downarrow^+\right\}$ $\left(C_\downarrow^- = \left\{x_{j_\downarrow}^-,\ j=1,\ldots,m_\downarrow^-\right\}\right)$, the object set subtracted from dominating (dominated) class of each object $x_{j_\downarrow}^+$ $\left(x_{j_\downarrow}^-\right)$ is $C_{j_\downarrow}^+$ $\left(C_{j_\downarrow}^-\right)$, the condition attribute set of the IODS is C, the upward (downward) union is $Cl_t^{\geqslant}$ $\left(Cl_t^{\leqslant}\right)$ $(1 \leqslant t \leqslant m)$, the



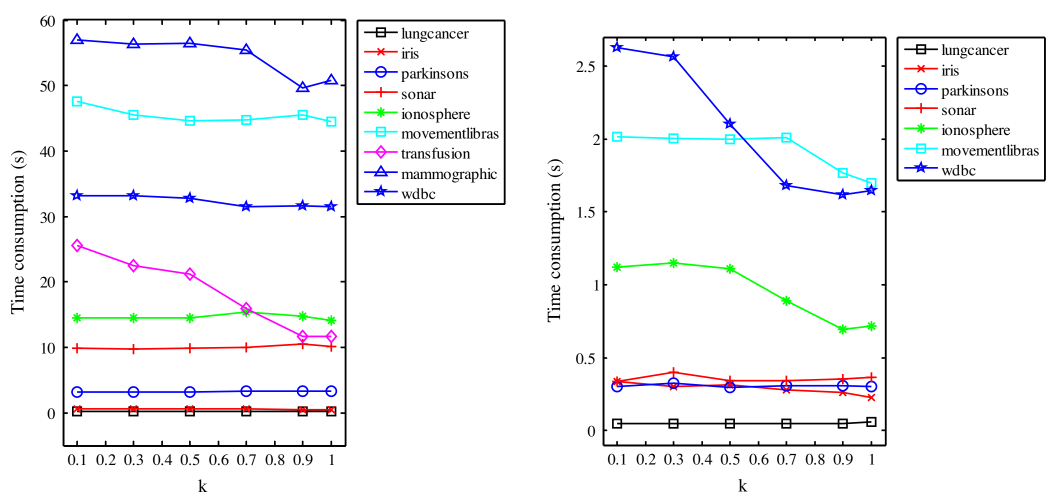
(a) Winered (b) Transfusion

Fig. 8. A comparison of incremental updating approximations and non-incremental updating approximations when attributes refining in the IODS with the variation of default rates.



(a) Winered (b) Transfusion

Fig. 9. The granularity measure with different default rates under the EDCR in the IODS.



(a) Coarsening (b) Refining

Fig. 10. The computation time of incremental updating approximations w.r.t. AVCR in the IODS with different *k*.

universe is *U*. Then, the computation complexity of incremental updating approximations for upward (downward) union is

$$O\left(\left(\sum_{i=1}^{m_{\uparrow}^{+}}\left|C_{\uparrow}^{+}\right|\left|C_{i_{\uparrow}}^{+}\right|+\sum_{i=1}^{m_{\downarrow}^{+}}\left|C_{\downarrow}^{+}\right|\left|C_{i_{\downarrow}}^{+}\right|\right)|A|+\left(\left|C_{\uparrow}^{+}\right|+\left|C_{\downarrow}^{+}\right|\right)\left|Cl_t^{\geqslant}\right|\right)$$
$$\times\left(O\left(\left(\sum_{i=1}^{m_{\uparrow}^{-}}\left|C_{\uparrow}^{-}\right|\left|C_{i_{\uparrow}}^{-}\right|+\sum_{i=1}^{m_{\downarrow}^{-}}\left|C_{\downarrow}^{-}\right|\left|C_{i_{\downarrow}}^{-}\right|\right)|A|+\left(\left|C_{\uparrow}^{-}\right|+\left|C_{\downarrow}^{-}\right|\right)\left|Cl_t^{\leqslant}\right|\right)\right)$$

The computation of approximations of non-incremental updating includes two steps, *i.e.*, compute dominating (dominated) classes and compute approximations of upward (downward) union. The computation complexity of non-incremental updating for upward (downward) union is $O\left(|U|^2|A|+|U|\left|Cl_t^{\geqslant}\right|\right)$ $\left(O\left(|U|^2|A|+|U|\left|Cl_t^{\leqslant}\right|\right)\right)$. Generally, $\sum_{i=1}^{m_{\uparrow}^{+}}\left|C_{\uparrow}^{+}\right|\left|C_{i_{\uparrow}}^{+}\right|+\sum_{i=1}^{m_{\downarrow}^{+}}\left|C_{\downarrow}^{+}\right|$

$\left|C^+_{i_\downarrow}\right| < |U|^2,\ \left|C^+_\uparrow\right| + \left|C^+_\downarrow\right| < U,\ \sum_{i=1}^{m^+_\uparrow}\left|C^+_\uparrow\right|\left|C^+_{i_\uparrow}\right| + \sum_{i=1}^{m^+_\downarrow}\left|C^+_\downarrow\right|\left|C^+_{i_\downarrow}\right| < |U|^2$, and $\left|C^-_\uparrow\right| + \left|C^-_\downarrow\right| < |U|$, then the complexity of incremental updating is less than that of non-incremental updating. Note that the cardinality of $C^+_\uparrow,\ C^-_\uparrow,\ C^+_\downarrow,\ C^-_\downarrow$ are different at different cases of attribute values coarsening and refining (*i.e.*, multi-level or single level coarsening and refining).

6. Conclusions

An information system evolves with time in the dynamic data environment. The approximations may vary according to the variation of the information system. How to dynamically maintain approximations of upward and downward unions is crucial to the efficiency of knowledge discovery. In this paper, we focused on the approaches for maintenance of approximations while attributes' values alter in the IODS. Considering the preference order exists among the attributes' values, we presented the definition of AVCR in the IODS. We then discussed the properties of dominating and dominated sets w.r.t. AVCR and further proposed the methods for incrementally updating approximations of upward and downward unions of classes. Our experimental results verified the effectiveness of the proposed methods. In the future, we will extend our algorithms to the fuzzy ordered information systems.

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