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Duality Theory in Fuzzy Optimization Problems

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Abstract. A solution concept of fuzzy optimization problems, which is essentially similar to the notion of Pareto optimal solution (nondominated solution) in multiobjective programming problems, is introduced by imposing a partial ordering on the set of all fuzzy numbers. We also introduce a concept of fuzzy scalar (inner) product based on the positive and negative parts of fuzzy numbers. Then the fuzzy-valued Lagrangian function and the fuzzy-valued Lagrangian dual function for the fuzzy optimization problem are proposed via the concept of fuzzy scalar product. Under these settings, the weak and strong duality theorems for fuzzy optimization problems can be elicited. We show that there is no duality gap between the primal and dual fuzzy optimization problems under suitable assumptions for fuzzy-valued functions.

Keywords: fuzzy numbers, convex fuzzy valued functions, fuzzy-valued Lagrangian function, fuzzy-valued Lagrangian dual function, fuzzy scalar (inner) product

1. Introduction

Bellman and Zadeh (1970) inspired the development of fuzzy optimization by providing the aggregation operators, which combined the fuzzy goals and fuzzy decision space. After this motivation and inspiration, there come out a lot of articles dealing with the fuzzy optimization problems. The earliest interesting works were initiated by Zimmermann (1976, 1978, 1985) who applied fuzzy sets theory to the linear programming problems and linear multiobjective programming problems. The collection of papers on fuzzy optimization edited by Słowiński (1998) and Delgado et al. (1994) give the main stream of this topic. Lai and Hwang (1992, 1994) also give an insightful survey on this topic. On the other hand, the stochastic optimization problem (ref. Kall (1976), Vajda (1972), StancuMinasian (1984) and Prékopa (1995)) is also an interesting research topic in operations research. The book edited by Słowiński and Teghem (1990) gives the comparisons between fuzzy optimization and stochastic optimization for the multiobjective programming problems. Inuiguchi and Ramík (2000) give a brief review of fuzzy optimization and a comparison with stochastic optimization in portfolio selection problem.

The duality of fuzzy linear programming problem was first studied by Rodder and Zimmermann (1977) considering the economic interpretation of the dual variables. They used the aspiration level approach that is different from the methodology proposed in this paper, since we consider a partial ordering on the set of all fuzzy numbers (the notion of fuzzy numbers will be introduced in Section 2). This partial ordering will apply to both the fuzzy objective and fuzzy inequality constraints. Bector and Chandra (2002) discussed the duality in fuzzy linear programming problem by modifying the dual formulation of Rodder and Zimmermann.

Inuiguchi et al. (2003) and Ramik and Vlach (2002, p. 231–235) proposed the valued relations based on modality indices to formulate the primal dual pair fuzzy linear programming problems and show that the weak and strong duality theorems hold true under their formulations.

The duality theory in fuzzy linear programming problem is also discussed in Wu (2003a) by proposing a concept of fuzzy scalar (inner) product. The first attempt of discussing the duality theory in fuzzy nonlinear programming problem (fuzzy optimization problem) is formulated and discussed in Wu (2003c) by proposing the (α, β)-solution using the concept of necessity measure. In this paper, we shall discuss the duality theory in fuzzy nonlinear programming problem by introducing a partial ordering on the set of all fuzzy numbers without considering the solution in the (α, β)-level sense. Therefore, the fuzzy-valued Lagrangian function and the fuzzy valued Lagrangian dual function for the fuzzy optimization problem are proposed via the concept of fuzzy scalar product. The concept of fuzzy scalar product is also employed by Wu (2003b) to discuss the saddle point optimality condition for fuzzy optimization problem.

Verdegay (1984) defined the fuzzy dual problem with the help of parametric linear programming problem and showed that the fuzzy primal and dual problems both have the same fuzzy solution under some suitable conditions. Sakawa and Yano (1994) proposed a fuzzy dual decomposition method for large-scale multiobjective nonlinear programming problems with the block angular structure. They considered the Lagrangian function and Lagrange multipliers in the dual problem. In this paper, we propose the fuzzy-valued Lagrangian (dual) function. Liu et al. (1995) proposed the fuzzy primal and dual problems by considering the fuzzy-max and fuzzy-min in the objective functions as the traditional pattern of linear programming problems. They also discussed the fuzzy weak duality membership functions by using the "min" and "max" operators. A different notion of weak duality theorem is introduced in this paper without considering the membership functions. Let us recall that the weak duality theorem in nonlinear programming problem says that the objective value of dual problem is always less than or equal to the objective value of primal problem. In this paper, we derive the similar form of weak duality theorem for fuzzy optimization problem according to a partial ordering on the set of all fuzzy numbers. Richardt et al. (1998) also give some interesting connections between fuzzy theory, simulated annealing, and convex duality.

We attempt to develop the duality theorems in fuzzy optimization problems by providing the dual fuzzy optimization problem. We propose the fuzzy-valued Lagrangian function and fuzzy-valued Lagrangian dual function for the constrained fuzzy optimization problem via the concept of fuzzy scalar product. The concept of fuzzy scalar product adopted in this paper is the first attempt for modeling the fuzzy-valued Lagrangian (dual) function. The fuzzy-valued Lagrangian dual function is a point-to-set fuzzy-valued function. The interesting articles on the theory of point-to-set functions (set-valued functions) in optimization is edited by Huard (1979). Therefore we also provide a concept for maximizing the point-to-set-fuzzy valued function, which is also a new attempt on the topic of set-valued fuzzy optimization problem. Under these settings, we prove the weak and strong duality theorems. We show that there is no duality gap between the primal and dual fuzzy optimization problems under some suitable assumptions for fuzzy-valued functions.

In Section 2, we develop some properties of fuzzy numbers, and provide the concept of positive part and negative part of fuzzy numbers for the purpose of discussing the fuzzy scalar product in Section 3. In Sections 3 and 4, we are concerned with the concept of convexity for the fuzzy-valued function by defining a partial ordering on the set of all fuzzy numbers in a natural way, and we also provide a concept of fuzzy scalar product via the convenient consideration of positive part and negative part of fuzzy numbers proposed in Section 2. In Section 5, we provide a solution concept which is essentially similar to the notion of Pareto optimal solution in the multiobjective programming problem. In Section 6, we present the primal and dual problems of fuzzy optimization problem using the fuzzy-valued Lagrangian function and fuzzy-valued Lagrangian dual function. In Section 7, we prove the main results, the weak and strong duality theorems for the fuzzy optimization problem.

2. Analysis of Fuzzy Numbers

Let U be a universal set. The fuzzy subset $\tilde{a}$ of U is defined by its membership function $\xi_{\tilde{a}} : U \to [0, 1]$. We also assume that U is endowed with a topology τ. The α-level set of $\tilde{a}$ denoted by $\tilde{a}_\alpha$, is defined by $\tilde{a}_\alpha = \{x \in U : \xi_{\tilde{a}}(x) \geq \alpha\}$ for all $\alpha \in (0, 1]$. The 0-level set $\tilde{a}_0$ is defined as the closure of the set $\{x \in U : \xi_{\tilde{a}}(x) > 0\}$, i.e., $\tilde{a}_0 = cl(\{x \in U : \xi_{\tilde{a}}(x) > 0\})$.

Definition 2.1. *We denote by $F(U)$ the set of all fuzzy subsets $\tilde{a}$* of *U with membership function $\xi_{\tilde{a}}$ satisfying the following conditions*:

(i) *$\tilde{a}$ is normal, i.e., there exists an $x \in U$ such that $\xi_{\tilde{a}}(x) = 1$;*
(ii) $\xi_{\tilde{a}}$ is quasi concave, *i.e.*, $\xi_{\tilde{a}}(\lambda x + (1-\lambda)y) \geq \min\{\xi_{\tilde{a}}(x), \xi_{\tilde{a}}(y)\}$ *for all $\lambda \in [0, 1]$;*
(iii) *$\xi_{\tilde{a}}$ is upper semicontinuous, i.e., $\{x : \xi_{\tilde{a}}(x) \geq \alpha\}$ is a closed subset of U for all $\alpha \in [0, 1]$;*
(iv) *The 0-level set $\tilde{a}_0$ is a compact subset of U, i.e., $cl(\{x \in U : \xi_{\tilde{a}}(x) > 0\})$ is a compact subset of U.*

Throughout this paper, the universal set U is assumed as the set of all real number $\mathbb{R}$ which is endowed with the usual topology. The member $\tilde{a}$ in $\mathscr{F}(\mathbb{R})$ is called a fuzzy number. Suppose now that $\tilde{a} \in \mathscr{F}(\mathbb{R})$. From Zadeh (1965), the α-level set $\tilde{a}_\alpha$ of $\tilde{a}$ is a convex subset of $\mathbb{R}$ for all $\alpha \in [0, 1]$ from condition (ii). Combining this fact with conditions (iii) and (iv), the α-level set $\tilde{a}_\alpha$. of $\tilde{a}$ is a compact and convex subset of $\mathbb{R}$ for all $\alpha \in [0, 1]$, i.e., $\tilde{a}_\alpha$ is a closed interval in $\mathbb{R}$ for all $\alpha \in [0, 1]$. Therefore, we also write $\tilde{a}_\alpha = \left[\tilde{a}_\alpha^L, \tilde{a}_\alpha^U\right]$.

Let "$\odot$" be any binary operations $\oplus$ or $\otimes$ between two fuzzy numbers $\tilde{a}$ and $\tilde{b}$. The membership function of $\tilde{a} \odot \tilde{b}$ is defined by

$$\xi_{\tilde{a}\odot\tilde{b}}(z) = \sup_{x \circ y = z} \min\{\xi_{\tilde{a}}(x), \xi_{\tilde{b}}(y)\}$$

using the extension principle in Zadeh (1975), where the operations $\odot = \oplus$ and $\otimes$ correspond to the operations $\circ = +$ and $\times$, respectively. Then we have the following results.

Proposition 2.1 Let $\tilde{a}, \tilde{b} \in \mathscr{F}(\mathbb{R})$. Then we have

(i) $\tilde{a} \oplus \tilde{b} \in \mathscr{F}(\mathbb{R})$ and

$$(\tilde{a} \oplus \tilde{b})_\alpha = \left[\tilde{a}_\alpha^L + \tilde{b}_\alpha^L, \tilde{a}_\alpha^U + \tilde{b}_\alpha^U\right];$$

(ii) $\tilde{a} \otimes \tilde{b} \in \mathscr{F}(\mathbb{R})$ and

$$(\tilde{a} \otimes \tilde{b})_\alpha = \left[\min\{\tilde{a}_\alpha^L\tilde{b}_\alpha^L, \tilde{a}_\alpha^L\tilde{b}_\alpha^U, \tilde{a}_\alpha^U\tilde{b}_\alpha^L, \tilde{a}_\alpha^U\tilde{b}_\alpha^U\}, \max\{\tilde{a}_\alpha^L\tilde{b}_\alpha^L, \tilde{a}_\alpha^L\tilde{b}_\alpha^U, \tilde{a}_\alpha^U\tilde{b}_\alpha^L, \tilde{a}_\alpha^U\tilde{b}_\alpha^U\}\right].$$

Let $\tilde{a}$ be a fuzzy number. Then $\tilde{a}$ is called a nonnegative fuzzy number if $\xi_{\tilde{a}}(x) = 0$ for all $x < 0$ and called a nonpositive fuzzy number if $\xi_{\tilde{a}}(x) = 0$ for all $x > 0$. It is obvious that $\tilde{a}_\alpha^L$ and $\tilde{a}_\alpha^U$ are nonnegative real numbers for all $\alpha \in [0, 1]$ if $\tilde{a}$ is a nonnegative fuzzy number, and $\tilde{a}_\alpha^L$ and $\tilde{a}_\alpha^U$ are nonpositive real numbers for all $\alpha \in [0, 1]$ if $\tilde{a}$ is a nonpositive fuzzy number.

Let $\tilde{a}$ be a fuzzy number. We define the membership functions of $\tilde{a}^+$ and $\tilde{a}^-$ as

$$\xi_{\tilde{a}}+(r) = \begin{cases} \xi_{\tilde{a}}(r) & \text{if } r > 0, \\ 1 & \text{if } r = 0 \text{ and } \xi_{\tilde{a}}(r') < 1 \text{ for all } r' > 0, \\ \xi_{\tilde{a}}(0) & \text{if } r = 0 \text{ and there exists a } r' > 0 \text{ such that } \xi_{\tilde{a}}(r') = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\xi_{\tilde{a}} - (r) = \begin{cases} \xi_{\tilde{a}}(r) & \text{if } r > 0, \\ 1 & \text{if } r = 0 \text{ and } \xi_{\tilde{a}}(r') < 1 \text{ for all } r' > 0, \\ \xi_{\tilde{a}}(0) & \text{if } r = 0 \text{ and there exists a } r' < 0 \text{ such that } \xi_{\tilde{a}}(r') = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We see that $\tilde{a}^+$ is a nonnegative fuzzy number and $\tilde{a}^-$ is a nonpositive fuzzy number. If $\tilde{a}$ is a fuzzy number then $\tilde{a}^+$ and $\tilde{a}^-$ are also fuzzy numbers, since $\tilde{a}_\alpha^+$ and $\tilde{a}_\alpha^-$ are closed intervals for all $\alpha \in [0, 1]$; that is, their membership functions $\xi_{\tilde{a}+}(r)$ and $\xi_{\tilde{a}-}(r)$ are upper semicontinuous (the other conditions in Definition 2.1 are obviously true). Furthermore, we have $\tilde{a}_\alpha = \tilde{a}_\alpha^+ \oplus \tilde{a}_\alpha^- = (\tilde{a}^+ \oplus \tilde{a}^-)_\alpha$ for all $\alpha \in [0, 1]$ from

Proposition 2.1. Thus $\tilde{a} = \tilde{a}^+ \oplus \tilde{a}^-$. We call $\tilde{a}^+$ and $\tilde{a}^-$ the positive part and the negative part of $\tilde{a}$, respectively.

We say that $\tilde{a}$ is a crisp number with value m if its membership function is given by

$$\xi_{\tilde{a}}(r) = \begin{cases} 1 & \text{if } r = m, \\ 0 & \text{otherwise.} \end{cases}$$

We also use the notation $\tilde{1}_{\{m\}}$ to represent the crisp number with value m. It is easy to see that $(\tilde{1}_{\{m\}})_\alpha^L = (\tilde{1}_{\{m\}})_\alpha^U = m$ for all $\alpha \in [0,1]$. Let us remark that a real number m can be regarded as a crisp number $\tilde{1}_{\{m\}}$.

3. Convex Fuzzy Valued Functions

Let $\tilde{a}, \tilde{b} \in \mathscr{F}(\mathbb{R})$. We write $\tilde{b} \succeq \tilde{a}$ if and only if $\tilde{b}_\alpha^L \geq \tilde{a}_\alpha^L$ and $\tilde{b}_\alpha^U \geq \tilde{a}_\alpha^U$ for all $\alpha \in [0,1]$. It is not hard to show that "$\succeq$" is a partial ordering on $\mathscr{F}(\mathbb{R})$. We also write $\tilde{a} \preceq \tilde{b}$ if and only if $\tilde{b} \succeq \tilde{a}$.

Let $\tilde{f}$ be a function defined by $\tilde{f} : V \to \mathscr{F}(\mathbb{R})$, where V is a real vector space. Then $\tilde{f}$ is called a fuzzy-valued function (defined on a real vector space V). For any $x \in V$, $\tilde{f}(x)$ is a fuzzy number (since $\tilde{f}(x) \in \mathscr{F}(\mathbb{R})$). Therefore, we obtain the real numbers $(\tilde{f}(x))_\alpha^L$ and $(\tilde{f}(x))_\alpha^U$ for each $\alpha \in [0,1]$. When we write

$$\tilde{f}_\alpha^L(x) = (\tilde{f}(x))_\alpha^L \quad \text{and} \quad \tilde{f}_\alpha^U(x) = (\tilde{f}(x))_\alpha^U, \tag{1}$$

two real-valued functions $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ are then defined on V for each $\alpha \in [0,1]$. Therefore, the fuzzy-valued function $\tilde{f}$ defined on the real vector space V can induce a family of real-valued functions $\{\tilde{f}_\alpha^L, \tilde{f}_\alpha^U : \alpha \in [0,1]\}$ which are given in Eq. (1).

Definition 3.1 *Let $\tilde{f}$ be a fuzzy-valued function defined on the real vector space V and X be a convex subset of V. We say that $\tilde{f}$ is convex on X if, for all x and y in X and $0 < \lambda < 1$,*

$$\tilde{f}(\lambda x + (1-\lambda)y) \preceq \left(\tilde{1}_{\{\lambda\}} \otimes \tilde{f}(x)\right) \oplus \left(\tilde{1}_{\{1-\lambda\}} \otimes \tilde{f}(y)\right). \tag{2}$$

From Proposition 2.1 and the definition of partial ordering "$\succeq$", Eq. (2) means that

$$\begin{aligned}\tilde{f}_\alpha^L(\lambda x + (1-\lambda)y) &\leq \left(\tilde{1}_{\{\lambda\}} \otimes \tilde{f}(x)\right)_\alpha^L + \left(\tilde{1}_{\{1-\lambda\}} \otimes (y)\right)_\alpha^L \\ &= \lambda \cdot \tilde{f}_\alpha^L(x) + (1-\lambda) \cdot \tilde{f}_\alpha^L(y),\end{aligned}$$

and

$$\begin{aligned}\tilde{f}_\alpha^U(\lambda x + (1-\lambda)y) &\leq \left(\tilde{1}_{\{\lambda\}} \otimes \tilde{f}(x)\right)_\alpha^U + \left(\tilde{1}_{\{1-\lambda\}} \otimes \tilde{f}(y)\right)_\alpha^U \\ &= \lambda \cdot \tilde{f}_\alpha^U(x) + (1-\lambda) \cdot \tilde{f}_\alpha^U(y)\end{aligned}$$

for all $\alpha \in [0,1]$. Therefore, we conclude the following result.

Proposition 3.1 Let X be a convex subset of a real vector space V. $\tilde{f}$ is a convex fuzzy-valued function on X if and only if $\tilde{f}_\alpha^L$, and $\tilde{f}_\alpha^U$ are convex real-valued functions on X for all $\alpha \in [0, 1]$.

4. Fuzzy Scalar Product

We denote by $\tilde{\mathrm{x}} \in \mathscr{F}^n(\mathbb{R}) = \mathscr{F}(\mathbb{R}) \times \cdots \times \mathscr{F}(\mathbb{R})$ a fuzzy vector, i.e., $\tilde{\mathrm{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n)$, where $\tilde{x}_i \in \mathscr{F}(\mathbb{R})$ for $i = 1, \ldots, n$. We also write $\tilde{\mathrm{x}}_\alpha^L = (\tilde{x}_{1\alpha}^L, \ldots, \tilde{x}_{n\alpha}^L)$ and $\tilde{\mathrm{x}}_\alpha^U = (\tilde{x}_{1\alpha}^U, \ldots, \tilde{x}_{n\alpha}^U)$, where $\tilde{x}_{i\alpha}^L = (\tilde{x}_i)_\alpha^L$ and $\tilde{x}_{i\alpha}^U = (\tilde{x}_i)_\alpha^U$ for all $i = 1, \ldots, n$. We define the (fuzzy) addition of $\tilde{\mathrm{x}}$ and $\tilde{\mathrm{y}}$ as

$$\tilde{\mathrm{x}} \oplus \tilde{\mathrm{y}} = (\tilde{x}_1 \oplus \tilde{y}_1, \ldots, \tilde{x}_n \oplus \tilde{y}_n).$$

We also denote by $\tilde{\mathrm{x}}^+ = (\tilde{x}_1^+, \ldots, \tilde{x}_n^+)$ and $\tilde{\mathrm{x}}^- = (\tilde{x}_1^-, \ldots, \tilde{x}_n^-)$. Then, from Proposition 2.1, we have

$$(\tilde{\mathrm{x}} \oplus \tilde{\mathrm{y}})_\alpha^L = \tilde{\mathrm{x}}_\alpha^L + \tilde{\mathrm{y}}_\alpha^L, \qquad (\tilde{\mathrm{x}} \oplus \tilde{\mathrm{y}})_\alpha^U = \tilde{\mathrm{x}}_\alpha^U + \tilde{\mathrm{y}}_\alpha^U, \qquad \tilde{\mathrm{x}} = \tilde{\mathrm{x}}^+ \oplus \tilde{\mathrm{x}}^-$$

by referring to Wu (2003a).

The *fuzzy scalar (inner) product* of two fuzzy vectors $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ in $\mathscr{F}^n(\mathbb{R})$ is defined by

$$\langle\langle \tilde{x}, \tilde{y} \rangle\rangle = (\tilde{x}_1 \otimes \tilde{y}_1) \oplus \cdots \oplus (\tilde{x}_n \otimes \tilde{y}_n).$$

It is obvious that the fuzzy scalar product of two fuzzy vectors is also a fuzzy number by Proposition 2.1. Let us write $\langle \cdot, \cdot \rangle$ to denote the usual scalar product in $\mathbb{R}^n$. We say that $\tilde{\mathrm{x}}$ is nonnegative (resp. nonpositive) if each $\tilde{x}_i$ is nonnegative (resp. nonpositive) for all $i = 1, \ldots, n$. The following proposition, from Wu (2003a), is very useful in the discussion of fuzzy-valued Lagrangian function.

Proposition 4.1 (Wu (2003a)) Suppose that $\mathrm{x} \in \mathbb{R}^n$ is nonnegative or nonpositive (in this case, vector x can be regarded as a crisp fuzzy vector $\tilde{1}_{\{x\}}$ with value x).

(i) $\langle\langle \tilde{1}_{\{x\}}, \tilde{\mathrm{y}} \rangle\rangle = \langle\langle \tilde{1}_{\{x\}}, \tilde{\mathrm{y}}^+ \rangle\rangle \oplus \langle\langle \tilde{1}_{\{x\}}, \tilde{\mathrm{y}}^- \rangle\rangle$,
(ii) if x is nonnegative then $\langle\langle \tilde{1}_{\{x\}}, \tilde{\mathrm{y}} \rangle\rangle_\alpha^L = \langle \mathrm{x}, \tilde{\mathrm{y}}_\alpha^L \rangle$ and $\langle\langle \tilde{1}_{\{x\}}, \tilde{y} \rangle\rangle_\alpha^U = \langle \mathrm{x}, \tilde{\mathrm{y}}_\alpha^u \rangle$,
(iii) if x is nonpositive then $\langle\langle \tilde{1}_{\{x\}}, \tilde{y} \rangle\rangle_\alpha^L = \langle \mathrm{x}, \tilde{\mathrm{y}}_\alpha^U \rangle$ and $\langle\langle \tilde{1}_{\{x\}}, \tilde{y} \rangle\rangle_\alpha^U = \langle \mathrm{x}, \tilde{\mathrm{y}}_\alpha^L \rangle$.

For the notational convenience in Section 5, we also write $\langle\langle \tilde{1}_{\{x\}}, \tilde{\mathrm{y}} \rangle\rangle$as$\langle\langle \mathrm{x}, \tilde{\mathrm{y}} \rangle\rangle$.

5. Solution Concept

Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. We write $\tilde{a} \succ \tilde{b}$ if and only if $\tilde{a}_\alpha^L \geq \tilde{b}_\alpha^L$ and $\tilde{a}_\alpha^U \geq \tilde{b}_\alpha^U$ for each $\alpha \in [0, 1]$ and there exists an $\alpha^* \in [0, 1]$ such that $\tilde{a}_{\alpha^*}^L > \tilde{b}_{\alpha^*}^L$ or $\tilde{a}_{\alpha^*}^U > \tilde{b}_{\alpha^*}^U$ (by referring to Wu (2003b)).

Let $\tilde{f}$ and $\tilde{g}$ be two fuzzy-valued functions defined on the same real vector space V. Let X be a subset of V. Since $\tilde{f}$ and $\tilde{g}$ are going to be considered as fuzzy-valued objective functions, we call $\tilde{f}(X) = \{\tilde{f}(x) \in \mathscr{F}(\mathbb{R}) : x \in X\}$ and $\tilde{g}(X) = \{\tilde{g}(x) \in \mathscr{F}(\mathbb{R}) : x \in X\}$ as the objective subsets of $\mathscr{F}(\mathbb{R})$. Suppose that we are going to minimize $\tilde{f}$ and maximize $\tilde{g}$. Then we follow the solution concept employed by Wu (2003b) to say that $\tilde{f}(\bar{x})$ is a *nondominated objective value* in $\tilde{f}(X)$ if there exists no $x \in X$ such that $\tilde{f}(\bar{x}) \succ \tilde{f}(x)$, and $\tilde{g}(\bar{x})$ is a nondominated objective value in $\tilde{g}(X)$ if there exists no $x \in X$ such that $\tilde{g}(\bar{x}) \prec \tilde{g}(x)$. We denote by $MIN(\tilde{f}, X)$ and $MAX(\tilde{g}, X)$ the sets of all nondominated objective values of $\tilde{f}$ and $\tilde{g}$, respectively. More precisely, we write

$$MIN(\tilde{f}, X) = \{\tilde{f}(\bar{x}) : \text{there exists no } x \in X \text{ such that } \tilde{f}(\bar{x}) \succ \tilde{f}(x)\}, \tag{3}$$

and

$$MAX(\tilde{g}, X) = \{\tilde{g}(\bar{x}) : \text{there exists no } x \in X \text{ such that } \tilde{g}(\bar{x}) \prec \tilde{g}(x)\}.$$

Given any objective value $\tilde{f}(\bar{x})$, the set of all arguments of $\tilde{f}$ so that $\tilde{f}(x) \succeq \tilde{f}(\bar{x})$ for $x \in X$ is denoted by $ARG\text{ - }MIN(\tilde{f}(\bar{x}), X)$ (i.e, $\tilde{f}(x)$ dominates $\tilde{f}(\bar{x})$ in $\tilde{f}(X)$). More precisely, we write

$$ARG\text{-}MIN(\tilde{f}(\bar{x}), X) = \{x \in X : \tilde{f}(x) \succeq \tilde{f}(\bar{x})\} \tag{4}$$

Similarly, we also adopt the following notation and definition for $\tilde{g}$

$$ARG\text{-}MAX(\tilde{g}(\bar{x}), X) = \{x \in X : \tilde{g}(x) \preceq \tilde{g}(\bar{x})\}.$$

We assume that $ARG\text{-}MIN(\tilde{f}(\bar{x}), X) \neq \phi$ for all $\tilde{f}(\bar{x}) \in MIN(\tilde{f}, X)$ and $ARG\text{-}MAX(\tilde{g}(\bar{x}), X) \neq \phi$ for all $\tilde{g}(\bar{x}) \in MAX(\tilde{g}, X)$ (also referring to Wu (2003b)).

6. Primal and Dual Problems

We denote by $\tilde{0}$ a crisp number with value 0, i.e.,

$$\xi_{\tilde{0}}(r) = \begin{cases} 1 & \text{if } r = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\tilde{f}$ and $\tilde{g}_i, i = 1, \ldots, m$, be fuzzy-valued functions defined on the same real vector space V and X be a subset of V. We now consider the following primal fuzzy optimization problem (P).

$$\begin{array}{lll} \text{(P)} & \text{Minimize} & \tilde{f}(x), \\ & \text{Subject to} & \tilde{g}_i(x) \preceq \tilde{0} \quad \text{for } i = 1, \ldots, m, \\ & & x \in X. \end{array}$$

Let

$$Y = \{x \in X : \tilde{g}_i(x) \leqslant \tilde{0} \text{ for all } i = 1, \ldots, m\}$$

be the feasible set of problem (P). We say that x is a feasible solution of problem (P) if $x \in Y$ and $\tilde{f}(x)$ is the objective value of problem (P) if $x \in Y$. We use the notations

$$MIN_{\mathrm{P}}(\tilde{f}, \tilde{\mathbf{g}}, X) = MIN(\tilde{f}, Y) \tag{5}$$

and

$$ARG\text{-}MIN_{\mathrm{P}}(\tilde{f}(\tilde{x}), \tilde{\mathbf{g}}, X) = ARG\text{-}MIN(\tilde{f}(\tilde{x}), Y),$$

where $\tilde{\mathbf{g}}(x) = (\tilde{g}_1(x), \ldots, \tilde{g}_n(x))$, and $MIN(\tilde{f}, Y)$ and $ARG\text{-}MIN\ (\tilde{f}(\bar{x}), Y)$ refer to (3) and (4), respectively. It means that $MIN_{\mathrm{P}}(\tilde{f}, \tilde{\mathbf{g}}, X)$ denotes the set of all nondominated objective values of problem (P) in the set

$$\tilde{f}(Y) = \{\tilde{f}(x) : x \in X, \tilde{\boldsymbol{g}}_i(x) \preceq \tilde{0} \text{ for } i = 1, \cdots, m\}.$$

Now we say that $\bar{x}$ is a solution of primal problem (P) if $\tilde{f}(\bar{x}) \in MIN_{\mathrm{P}}(\tilde{f}, \tilde{\boldsymbol{g}}, X)$ and $\bar{x} \in Y$. Let $OPM_{\mathrm{P}}(\tilde{f}, \tilde{g}, X)$ denote the set of all solutions of primal problem (P).

Let us consider a simple example. We firstly introduce the so-called triangular fuzzy numbers. The membership function of a triangular fuzzy number $\tilde{x} = (x^L, x, x^U)$ is defined by

$$\xi_{\tilde{x}}(r) = \begin{cases} (r - x^L)/(x - x^L) & \text{if } x^L \leq r \leq x, \\ (x^U - r)/(x^U - x) & \text{if } x < r \leq x^U, \\ 0 & \text{otherwise,} \end{cases}$$

The α-level set of $\tilde{x}$ is then

$$\tilde{x}_\alpha = [(1 - \alpha)x^L + \alpha x, (1 - \alpha)x^U + \alpha x],$$

that is,

$$\tilde{x}_\alpha^L = (1 - \alpha)x^L + \alpha x \text{ and } \tilde{x}_\alpha^U = (1 - \alpha)x^U + \alpha x.$$

We also see that $-\tilde{x} = (-x^U, -x, -x^L)$.

Example 6.1 In this example, the triangular fuzzy numbers are taken as the form of $\tilde{x} = (x - 1, x, x + 1)$, where $x \in \mathbb{R}$. Then $\tilde{x}_\alpha^L = x - (1 - \alpha)$ and $\tilde{x}_\alpha^U = x + (1 - \alpha)$. Let us consider the following primal fuzzy optimization problem (Pl).

$$
\begin{array}{lll}
\text{(P1)} & \text{Minimize} & \tilde{f}(x_1, x_2) = (\tilde{x}_1 \otimes \tilde{x}_1) \oplus (\tilde{x}_2 \otimes \tilde{x}_2), \\
 & \text{Subject to} & \tilde{g}(x_1, x_2) = (-\tilde{x}_1) \oplus (-\tilde{x}_2) \oplus \tilde{4} \preceq \tilde{0}, \\
 & & x_1 \geq 1, \quad x_2 \geq 1,
\end{array}
$$

where $\tilde{x}_1 = (x_1 - 1, x_1, x_1 + 1)$, $\tilde{x}_2 = (x_2 - 1, x_2, x_2 + 1)$ and $\tilde{4} = (3, 4, 5)$. Since $x_1 \geq 1$ and $x_2 \geq 1$, we see that $\tilde{x}_1$ and $\tilde{x}_2$ are nonnegative fuzzy numbers. By definition, we have

$$
\begin{aligned}
Y &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 1, x_2 \geq 1, \tilde{g}(x_1, x_2) \preceq \tilde{0}\} \\
&= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 1, x_2 \geq 1, \tilde{g}^L_\alpha(x_1, x_2) \leq 0 \text{ and } \tilde{g}^U_\alpha(x_1, x_2) \leq 0 \text{ for all } \alpha \in [0, 1]\} \\
&= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 1, x_2 \geq 1, -x_1 - (1-\alpha) + (-x_2 - (1-\alpha)) + 3 + \alpha \leq 0, \\
&\quad -x_1 + (1-\alpha) + (-x_2 + (1-\alpha)) + 5 - \alpha \leq 0 \text{ for all } \alpha \in [0, 1]\} \\
&= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 1, x_2 \geq 1, \\
&\quad -x_1 - x_2 + 1 + 3\alpha \leq 0, -x_1 - x_2 + 7 - 3\alpha \leq 0 \text{ for all } \alpha \in [0, 1]\} \\
&= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 1, x_2 \geq 1, -x_1 - x_2 + 7 - 3\alpha \leq 0 \text{ for all } \alpha \in [0, 1]\},
\end{aligned}
$$

since $1 + 3\alpha \leq 7 - 3\alpha$ for $\alpha \in [0, 1]$. Furthermore, Y can be represented as

$$
\begin{aligned}
Y &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 1, x_2 \geq 1, 7 - 3\alpha \leq x_1 + x_2 \text{ for all } \alpha \in [0, 1]\} \\
&= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 1, x_2 \geq 1, 7 \leq x_1 + x_2\}.
\end{aligned}
$$

Since $\tilde{x}_1$ and $\tilde{x}_2$ are nonnegative fuzzy numbers, from Proposition 2.1, we have

$$\tilde{f}^L_\alpha(x_1, x_2) = (x_1 - (1 - \alpha))^2 + (x_2 - (1 - \alpha))^2,$$

and

$$\tilde{f}^U_\alpha(x_1, x_2) = (x_1 + (1 - \alpha))^2 + (x_2 + (1 - \alpha))^2.$$

That is to say, $\tilde{f}^L_\alpha$ can be regarded as the square of the distance between two points (x_1, x_2) and $(1 - \alpha, 1 - \alpha)$. Similarly, $\tilde{f}^U_\alpha$ can be regarded as the square of the distance between two points (x_1, x_2) and $(\alpha - 1, \alpha - 1)$. Therefore, it is not hard to see that $\tilde{f}^L_\alpha(\frac{7}{2}, \frac{7}{2}) \leq \tilde{f}^L_\alpha(x_1, x_2)$ and $\tilde{f}^U_\alpha(\frac{7}{2}, \frac{7}{2}) \leq \tilde{f}^U_\alpha(x_1, x_2)$ for all $(x_1, x_2) \in Y$ and all $\alpha \in [0, 1]$, *i.e.*, $\tilde{f}(\frac{7}{2}, \frac{7}{2}) \preceq \tilde{f}(x_1, x_2)$ for all $(x_1, x_2) \in Y$. It says that $(x_1, x_2) = (\frac{7}{2}, \frac{7}{2})$ is a solution of primal problem (P1). □

In the sequel, we shall propose the dual fuzzy optimization problem. First of all, we define the fuzzy-valued Lagrangian function for the primal problem (P) as follows:

$$\tilde{\phi}(x, \mathbf{u}) = \tilde{f}(x) \oplus \langle\langle \mathbf{u}, \tilde{\mathbf{g}}(\mathbf{x}) \rangle\rangle$$

for all $x \in X$ and all $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m_+$, *i.e.*, $u_i \geq 0$ for all $i = 1, \ldots, m$. Sometimes we also write $\mathbf{u} \geq \mathbf{0}$ **if** $\mathbf{u} \in \mathbb{R}^m_+$.

Example 6.2 Let us consider the same assumptions in Example 6.1 except that the constraints $x_1 \geq 1$ and $x_2 \geq 1$ are replaced by $x_1 \geq 4$ and $x_2 \geq 4$. Therefore we consider the following problem

$$\begin{array}{lll} \text{(P2)} & \text{Minimize} & \tilde{f}(x_1, x_2) = (\tilde{x}_1 \otimes \tilde{x}_1) \oplus (\tilde{x}_2 \otimes \tilde{x}_2), \\ & \text{Subject to} & \tilde{g}(x_1, x_2) = (-\tilde{x}_1) \oplus (-\tilde{x}_2) \oplus \tilde{4} \preceq \tilde{0}, \\ & & x_1 \geq 4, \quad x_2 \geq 4. \end{array}$$

Then the feasible set is

$$Y = \{(x_1, x_2) \in \mathbb{R}^2 \, : \, x_1 \geq 4, x_2 \geq 4, 7 \leq x_1 + x_2\}.$$

Using the similar arguments as in Example 6.1, it is not hard to see that $\tilde{f}(4, 4) \preceq \tilde{f}(x_1, x_2)$ for all $(x_1, x_2) \in Y$, *i.e.* $(x_1, x_2) = (4, 4)$ is a solution of the above new primal problem (P2). The fuzzy-valued Lagrangian function is then given by

$$\tilde{\phi}(x_1, x_2, u) = (\tilde{x}_1 \otimes \tilde{x}_1) \oplus (\tilde{x}_2 \otimes \tilde{x}_2) \oplus (\tilde{1}_{\{u\}} \otimes ((-\tilde{x}_1) \oplus (-\tilde{x}_2) \oplus \tilde{4}))$$

for $x_1 \geq 4, x_2 \geq 4$. Then, using Proposition 2.1, we have

$$\tilde{\phi}_\alpha^L(x_1, x_2, u) = (x_1 - (1 - \alpha))^2 + (x_2 - (1 - \alpha))^2 + u(-x_1 - x_2 + 1 + 3\alpha), \tag{6}$$

and

$$\tilde{\phi}_\alpha^U(x_1, x_2, u) = (x_1 + (1 - \alpha))^2 + (x_2 + (1 - \alpha))^2 + u(-x_1 - x_2 + 7 - 3\alpha) \tag{7}$$

for $x_1 \geq 4, x_2 \geq 4$ and all $\alpha \in [0, 1]$. □

Using the similar notation as described above, we define the fuzzy valued Lagrangian dual function as follows:

$$\begin{array}{rcl} \tilde{L}(\mathbf{u}) & = & MIN(\tilde{\phi}(\cdot, \mathbf{u}), X) \\ & = & \{\tilde{\phi}(\bar{x}, \mathbf{u}) : \text{ there exists no x } \in X \text{ such that } \tilde{\phi}(\bar{x}, \mathbf{u}) \succ \tilde{\phi}(x, \mathbf{u})\}. \end{array}$$

In other words, given $\mathbf{u} \in \mathbb{R}^m_+, \tilde{L}(\mathbf{u})$ is the set of all nondominated objective values in the set $\tilde{\phi}(X, \mathbf{u}) = \{\tilde{\phi}(x, \mathbf{u}) \, : \, x \in X\}$ when minimization is taken into account. Thus $\tilde{L}$ is a point-to-set fuzzy-valued function, i.e., $\tilde{L}(\mathbf{u})$ is a subset of $\mathscr{F}(\mathbb{R})$ for any fixed $\mathbf{u}$. The interesting literature dealing with the point-to-set functions in optimization was edited by Huard (1979).

Example 6.3 Suppose that we consider the primal problem (P2) in Example 6.2. The functions $\tilde{\phi}_\alpha^L(x_1, x_2, u)$ and $\tilde{\phi}_\alpha^U(x_1, x_2, u)$ in Eqs. (6) and (7), respectively, can be rewritten as

$$\tilde{\phi}_\alpha^L(x_1, x_2, u) = [x_1 - (1 - \alpha + u/2)]^2 + [x_2 - (1 - \alpha + u/2)]^2 + 5\alpha u - u - u^2/2, \tag{8}$$

and

$$\tilde{\phi}_\alpha^U(x_1, x_2, u) = [x_1 - (-1 + \alpha + u/2)]^2 + [x_2 - (-1 + \alpha + u/2)]^2 + 5\alpha u + 9u - u^2/2$$

after some algebraic calculations. For any fixed $u \geq 10$, since the equation of a line which passes through point $(-1 + u/2, -1 + u/2)$ and is perpendicular to line $x_1 - x_2 = 0$ is $x_1 + x_2 + 2 - u = 0$,, and the equation of a line which passes through point $(-1 + u/2, 1 + u/2)$ and is perpendicular to line $x_1 - x_2 = 0$ is $x_1 + x_2 - 2 - u = 0$, we see that $\tilde{\phi}(x_1, x_2, u)$ is dominated by

$$\begin{array}{ll} \tilde{\phi}(1 + u/2, 1 + u/2, u), & \text{if } x_1 + x_2 - 2 - u \geq 0, \\ \tilde{\phi}(-1 + u/2, -1 + u/2, u), & \text{if } x_1 + x_2 + 2 - u \leq 0, \\ \tilde{\phi}((x_1 + x_2)/2, (x_1 + x_2)/2, u), & \text{if } x_1 + x_2 - 2 - u < 0 \text{ and } x_1 + x_2 + 2 - u > 0 \end{array}$$

for $x_1 \neq x_2$. Let $(x_1 + x_2)/2 = k$. Then we see that $\tilde{L}(u) \subseteq \{\tilde{\phi}(k, k, u) : -1 + u/2 \leq k \leq 1 + u/2\}$. On the other hand, for $h \neq k$, we see that there exists $\alpha_1, \alpha_2 \in [0, 1]$ such that

$$\begin{array}{lcll} \tilde{\phi}_{\alpha_1}^L(h, h, u) < \tilde{\phi}_{\alpha_1}^L(k, k, u) & & \tilde{\phi}_{\alpha_1}^L(h, h, u) > \tilde{\phi}_{\alpha_1}^L(k, k, u), & \\ \tilde{\phi}_{\alpha_2}^L(h, h, u) > \tilde{\phi}_{\alpha_2}^L(k, k, u) & \text{or} & \tilde{\phi}_{\alpha_2}^L(h, h, u) < \tilde{\phi}_{\alpha_2}^L(k, k, u), & \text{or} \end{array}$$

$$\begin{array}{lcl} \tilde{\phi}_{\alpha_1}^L(h, h, u) < \tilde{\phi}_{\alpha_1}^L(k, k, u) & & \tilde{\phi}_{\alpha_1}^L(h, h, u) > \tilde{\phi}_{\alpha_1}^L(k, k, u), \\ \tilde{\phi}_{\alpha_2}^U(h, h, u) > \tilde{\phi}_{\alpha_2}^U(k, k, u) & \text{or} & \tilde{\phi}_{\alpha_2}^U(h, h, u) < \tilde{\phi}_{\alpha_2}^U(k, k, u). \end{array}$$

It says that $\tilde{\phi}(h, h, u)$ and $\tilde{\phi}(k, k, u)$ are not comparable. Therefore, we conclude that $\tilde{L}(u) = \{\tilde{\phi}(k, k, u) : -1 + u/2 \leq k \leq 1 + u/2\}$ for $u \geq 10$. Now, for $6 \leq u < 10$, we see that $\tilde{\phi}(x_1, x_2, u)$ is dominated by

$$\begin{array}{ll} \tilde{\phi}(1 + u/2, 1 + u/2, u), & \text{if } x_1 + x_2 - 2 - u \geq 0, \\ \tilde{\phi}((x_1 + x_2)/2, (x_1 + x_2)/2, u), & \text{if } x_1 + x_2 - 2 - u < 0 \text{ and } (x_1 + x_2)/2 \geq 4 \end{array}$$

for $x_1 \neq x_2$. Let $(x_1 + x_2)/2 = k$. Then we can similarly obtain that $\tilde{L}(u) = \{\tilde{\phi}(k, k, u) : 4 \leq k \leq 1 + u/2\}$ for $6 \leq u < 10$. This also shows that if u is a feasible solution of dual problem of (P2), then $u \geq 6$. □

We can now consider the dual fuzzy optimization problem (D) as follows:

$$\begin{array}{lll} \text{(D)} & \text{maximize} & \tilde{L}(\mathbf{u}) \\ & \text{Subject to} & \mathbf{u} \geq \mathbf{0}. \end{array}$$

We say that $\mathbf{u}$ is a feasible solution of problem (D) if $\mathbf{u} \geq \mathbf{0}$. Next we are going to introduce the solution concept of dual problem (D).

Definition 6.1 *Let A be a subset of $\mathscr{F}(\mathbb{R})$. We write $A \preceq \tilde{x}$ if $\tilde{y} \preceq \tilde{x}$ for all $\tilde{y} \in A$.*

Since $\tilde{L}$ is a point-to-set fuzzy-valued function, we say that $\bar{\mathbf{u}}$ is a solution of dual problem (D) if there exists a $\tilde{y} \in \tilde{L}(\bar{\mathbf{u}})$ such that $\tilde{y} \succeq \tilde{L}(\mathbf{u})$ for all $\mathbf{u} \neq \bar{\mathbf{u}}$ and $\mathbf{u} \geq \mathbf{0}$. Let $OPM_D(\tilde{L}, \mathbb{R}^m_+)$ denote the set of all solutions of dual problem (D). We also adopt the following notation

$$MAX_{\mathrm{D}}(\tilde{L}, \mathbb{R}^m_+) = \left\{\tilde{L}(\mathbf{u}) : \mathbf{u} \text{ is a solution of dual problem (D)}\right\}. \tag{9}$$

When $\mathbf{u}$ is fixed, following from the definition of $ARG - MIN(\cdot, \cdot)$ presented in (4), we see that

$$ARG\text{-}MIN(\tilde{\phi}(\bar{x}, \mathbf{u}), X) = \left\{x \in X : \tilde{\phi}(x, \mathbf{u}\ ) \succeq \tilde{\phi}(\bar{x}, \mathbf{u}\ )\right\}. \tag{10}$$

7. Duality Theorems

We are now in a position to prove the weak and strong duality theorems. First of all, we provide the solvability of primal problem (P) and dual problem (D).

7.1. *Solvability*

In order to present the solvability of primal problem (P) and dual problem (D), we introduce a concept of Lagrangian value.

Definition 7.1 *Let $\bar{\mathbf{u}}$ be a fixed feasible solution of dual problem* (D). *Suppose that $\bar{x}$ is a feasible solution of primal problem (P) such that $\tilde{f}(\bar{x}) \in \tilde{L}(\bar{\mathbf{u}})$. Then there exists a $\tilde{\phi}(\hat{x}, \bar{\mathbf{u}}) \in \tilde{L}(\bar{\mathbf{u}})$) (which depends on $\bar{x}$) such that $\tilde{f}(\bar{x}) = \tilde{\phi}(\hat{x}, \bar{\mathbf{u}})$. We say that $\tilde{L}(\bar{\mathbf{u}})$ is a Lagrangian value if the following conditions are satisfied*:

(i) $\bar{x} \in \bigcap_{\{\mathbf{u} \in \mathbb{R}^m_+ : \mathbf{u} \neq \bar{\mathbf{u}}\}} \bigcap_{\{y \in X : \tilde{\phi}(y, \mathbf{u}) \in \tilde{L}(\mathbf{u})\}} ARG - MIN(\tilde{\phi}(y, \mathbf{u}), X)$;

(ii) there exists no $x \in Y \backslash ARG\text{-}MIN(\tilde{\phi}(\hat{x}, \bar{\mathrm{u}}), X)$ such that $\tilde{f}(\bar{x}) \succ \tilde{f}(x)$, where Y is the feasible set of problem (P) and the set $Y \backslash ARG\text{-}MIN(\tilde{\phi}(\hat{x}, \bar{\mathrm{u}}), X)$ means that $x \in Y$ and $x \notin ARG\text{-}MIN(\tilde{\phi}(\hat{x}, \bar{\mathrm{u}}), X)$.

Condition (i) in Definition 7.1 says that, for any given $\mathbf{u} \neq \bar{\mathbf{u}}$ and $\mathbf{u} > 0, \bar{x} \in ARG - MIN(\tilde{\phi}(y, \mathbf{u}), X)$ for all $y \in X$ with $\tilde{\phi}(y, \mathbf{u}) \in \tilde{L}(\mathbf{u})$. It also means that $\tilde{\phi}(\bar{x}, \mathbf{u}) \succeq \tilde{\phi}(y, \mathbf{u})$ (see Eq. (10)) for all $y \in X$ with $\tilde{\phi}(y, \mathbf{u}) \in \tilde{L}(\mathbf{u})$; that is,

$$\tilde{\phi}(\bar{x}, \mathbf{u}) \succeq \tilde{L}(\mathbf{u}) \tag{11}$$

by Definition 6.1 for all $\mathbf{u} \neq \bar{\mathbf{u}}$ and $\mathbf{u} \geq \mathbf{0}$. Condition (ii) in Definition 7.1 says that $\tilde{f}(\bar{x})$ cannot be dominated by $\tilde{f}(x)$ for any $x \in Y \backslash ARG\text{-}MIN(\tilde{\phi}(\hat{x}, \bar{\mathbf{u}}), X)$. In other words, if we set $\bar{X} = Y \backslash ARG\text{-}MIN(\tilde{\phi}(\hat{x}, \bar{\mathbf{u}}), X)$, then $\tilde{f}(\bar{x}) \in MIN(\tilde{f}, \tilde{\mathbf{g}}, \bar{X})$.

Lemma 7.1 *If* $\mathbf{u} \geq \mathbf{0}$ *and* $\tilde{\mathbf{g}}(x) \preceq \tilde{\mathbf{0}}$, *then* $\tilde{f}(x) \oplus \langle\langle \mathbf{u}, \tilde{\mathbf{g}}(\mathbf{x}) \rangle\rangle \preceq \tilde{f}(x)$.

Proof: Let us write $\tilde{\mathbf{g}}_\alpha{}^L(x) = ((\tilde{g}_1)_\alpha{}^L(x), \ldots, (\tilde{g}_m)_\alpha{}^L(x))$. Using Propositions 2.1 and 4.1 (ii), we have, for all $\alpha \in [0, 1]$,

$$(\tilde{f}(x) \oplus \langle\langle \mathbf{u}, \tilde{\mathbf{g}}(x) \rangle\rangle)_\alpha{}^L = \tilde{f}_\alpha^L(x) + \langle \mathbf{u}, \tilde{\mathbf{g}}_\alpha{}^L(x) \rangle \leq \tilde{f}_\alpha^L(x),$$

since $\mathbf{u} \geq \mathbf{0}$ and $\tilde{\mathbf{g}}_\alpha{}^L(x) \leq 0$. Similarly, we also have $(\tilde{f}(x) \oplus \langle\langle \mathbf{u}, \tilde{\mathbf{g}}(x) \rangle\rangle)_\alpha{}^U \leq \tilde{f}_\alpha^U(x)$ for all $\alpha \in [0, 1]$. This completes the proof. □

Theorem 7.1 *(Solvability) Let* $\bar{x}$ *and* $\bar{\mathbf{u}}$ *be the feasible solutions of problems (*P*) and (D), respectively. Suppose that* $\tilde{L}(\bar{\mathbf{u}})$ *is a Lagrangian value with* $\tilde{f}(\bar{x}) \in \tilde{L}(\bar{\mathbf{u}})$. *Then* $\bar{x}$ *solves the primal problem (P) and* $\bar{\mathbf{u}}$ *solves the dual problem (D). That is,* $\tilde{f}(\bar{x}) \in MIN_P(\tilde{f}, \tilde{\mathbf{g}}, X)$ *(see Eq. (5)) and* $\tilde{L}(\bar{\mathbf{u}}) \in MAX_D(\tilde{L}, \mathbb{R}_+^m)$ *(see Eq. (*9)).

Proof: Since $\tilde{L}(\bar{\mathbf{u}})$ is a Lagrangian value, from Definition 7. 1, we have $\tilde{f}(\bar{x}) = \tilde{\phi}(\hat{x}, \bar{\mathbf{u}})$. Then

$$\tilde{f}(\bar{x}) = \tilde{\phi}(\hat{x}, \bar{\mathbf{u}}) \preceq \tilde{\phi}(x, \bar{\mathbf{u}}) \quad \text{for all } x \in ARG - MIN(\tilde{\phi}(\hat{x}, \bar{\mathbf{u}}), X) \quad \text{(see Eq.(10)).}$$

By definition and Lemma 7.1, we have $\tilde{\phi}(x, \bar{\mathbf{u}}) = \tilde{f}(x) \oplus \langle\langle \bar{\mathbf{u}}, \tilde{\mathbf{g}}(x) \rangle\rangle$ and $\tilde{f}(x) \oplus \langle\langle \bar{\mathbf{u}}, \tilde{\mathbf{g}}(x) \rangle\rangle \preceq \tilde{f}(x)$, respectively. Therefore, we obtain that

$$\tilde{f}(\bar{x}) \preceq \tilde{f}(x) \quad \text{for all } x \in ARG\text{-}MIN(\tilde{\phi}(\hat{x}, \bar{\mathbf{u}}), X).$$

Since $\tilde{L}(\bar{\mathbf{u}})$ is a Lagrangian value, we conclude that $\tilde{f}(\bar{x})$ is nondominated from condition (ii) in Definition 7.1. This shows that $\tilde{f}(\bar{x}) \in MIN_P(\tilde{f}, \tilde{\mathbf{g}}, X)$; that is, $\bar{x}$ solves the primal problem (P).

On the other hand, we see that

$$\begin{aligned} \tilde{\phi}(\hat{x}, \bar{\mathbf{u}}) &= \tilde{f}(\bar{x}) \\ &\succeq \tilde{f}(\bar{x}) \oplus \langle\langle \mathbf{u}, \tilde{\mathbf{g}}(\bar{x}) \rangle\rangle \quad \text{(by Lemma 7.1)} \\ &= \tilde{\phi}(\bar{x}, \mathbf{u}). \end{aligned}$$

Combining this fact $\tilde{\phi}(\hat{x}, \bar{\mathbf{u}}) \succeq \tilde{\phi}(\bar{x}, \mathbf{u})$ with expression (11) and using the fact of transitivity for the partial ordering $\succeq$, we then conclude that $\tilde{f}(\bar{x}) = \tilde{\phi}(\hat{x}, \bar{\mathbf{u}}) \succeq \tilde{L}(\mathbf{u})$

for all $\mathbf{u} \neq \bar{\mathbf{u}}$ and $\mathbf{u} \geq \mathbf{0}$. Thus $\tilde{L}(\bar{\mathbf{u}}) \in MAX_D(\tilde{L}, \mathbb{R}^m_+)$, since $\tilde{\phi}(\hat{x}, \bar{\mathbf{u}}) = \tilde{f}(\bar{x}) \in \tilde{L}(\bar{\mathbf{u}})$ (by the definition of solution of dual problem (D)), i.e., $\bar{\mathbf{u}}$ solves the dual problem (D). □

7.2. Weak Duality Theorem

We firstly introduce the following useful result for the sake of proving the weak duality theorem.

Proposition 7.1 Suppose that x and $\mathbf{u}$ are feasible solutions of problems (P) and (D), respectively, and

$$x \in \bigcap_{\{\bar{x} \in X : \tilde{\phi}(\bar{x}, \mathbf{u}) \in \tilde{L}(\mathbf{u})\}} ARG\text{-}MIN(\tilde{\phi}(\bar{x}, \mathbf{u}), X). \tag{12}$$

Then we have $\tilde{L}(\mathbf{u}) \preceq \tilde{f}(x)$.

Proof: Assume that $\tilde{\phi}(\bar{x}, \mathbf{u}) \in \tilde{L}(\mathbf{u})$. Then, for $x \in ARG\text{-}MIN(\tilde{\phi}(\bar{x}, \mathbf{u}), X)$, we have

$$\begin{aligned} \tilde{\phi}(\bar{x}, \mathbf{u}) &\preceq \tilde{\phi}(x, \mathbf{u}) (\text{see Eq.}(10)) \\ &= \tilde{f}(x) \oplus \langle\langle \mathbf{u}, \tilde{\mathbf{g}}(x) \rangle\rangle \\ &\preceq \tilde{f}(x) (\text{using Lemma 7.1}). \end{aligned}$$

The above inequality is satisfied for all $\tilde{\phi}(\bar{x}, \mathbf{u}) \in \tilde{L}(\mathbf{u})$ due to (12). It shows that $\tilde{L}(\mathbf{u}) \preceq \tilde{f}(x)$ by Definition 6.1. □

Next, we are going to prove the weak duality theorem. For convenience, we firstly introduce some notations.

Definition 7.2 *Let A and B be two subsets of $\mathscr{F}(\mathbb{R})$. We write $A \preceq B$ if $A \preceq \tilde{x}$ for all $\tilde{x} \in B$ (recall that $A \preceq \tilde{x}$* if *$\tilde{y} \preceq \tilde{x}$ for all $\tilde{y} \in A$. Let $A = \{A_i\}_{i \in I}$ be a family of subsets of $\mathscr{F}(\mathbb{R})$. We write $A \preceq B$ if $A_i \preceq B$ for all $A_i \in \mathscr{A}$.*

Let us recall that the weak duality theorem in nonlinear programming problem says that the objective value of dual problem is always less than or equal to the objective value of primal problem. Since $MIN_{\mathrm{P}}(\tilde{f}, \tilde{\mathbf{g}}, X)$ and $MAX_{\mathrm{D}}(\tilde{L}, \mathbb{R}^m_+)$ can be regarded as objective values of primal problem (P) and dual problem (D), respectively, we derive the similar form of weak duality theorem for fuzzy optimization problem using Definition 7.2.

Theorem 7.2 (*weak duality theorem*) *Suppose that*

$$OPM_P(\tilde{f}, \tilde{\mathbf{g}}, X) \subseteq \bigcap_{\{\mathbf{u} \in \mathbb{R}^m_+ : \mathbf{u} \in OPM_D(\tilde{L}, \mathbb{R}^m_+)\}} \bigcap_{\{\bar{x} \in X : \tilde{\phi}(\bar{x}, \mathbf{u}) \in \tilde{L}(\mathbf{u})\}} ARG\text{-}MIN(\tilde{\phi}(\bar{x}, \mathbf{u}), X). \tag{13}$$

Then

$$MAX_{\mathrm{D}}(\tilde{L}, \mathbb{R}^m_+) \preceq MIN_{\mathrm{P}}(\tilde{f}, \tilde{\mathbf{g}}, X).$$

Proof: If $x \in OPM_P(\tilde{f}, \tilde{\mathbf{g}}, X)$ then $\tilde{f}(x) \in MIN_{\mathrm{P}}(\tilde{f}, \tilde{\mathbf{g}}, X)$ by definition. From Proposition 7.1, we see that $\tilde{L}(\mathbf{u}) \preceq \tilde{f}(x)$ if x satisfies expression (12). Therefore, by Definition 7.2,

$$\tilde{L}(\mathrm{u}) \preceq MIN_{\mathrm{P}}(\tilde{f}, \tilde{\mathbf{g}}, X) \tag{14}$$

if

$$OPM_{\mathrm{P}}(\tilde{f}, \tilde{\mathbf{g}}, X) \subseteq \bigcap_{\{x \in X : \tilde{\phi}(\bar{x}, \mathbf{u}) \in \tilde{L}(\mathbf{u})\}} ARG - MIN(\tilde{\phi}(\bar{x}, \mathrm{u}), X).$$

We also see that if $\mathbf{u} \in OPM_{\mathrm{D}}(\tilde{L}, \mathbb{R}^m_+)$, then $\tilde{L}(\mathbf{u}) \in MAX_{\mathrm{D}}(\tilde{L}, \mathbb{R}^m_+)$ (see Eq. (9)). Therefore the result follows from Definition 7.2 and Eq. (14) and (13). □

7.3. *Strong Duality Theorem*

We say that there is no duality gap between the primal problem (P) and dual problem (D) if there exist $\tilde{f}(\bar{x}) \in MIN_{\mathrm{P}}(\tilde{f}, \tilde{\mathbf{g}}, X)$ and $\tilde{L}(\bar{\mathbf{u}}) \in MAX_{\mathrm{D}}(\tilde{L}, \mathbb{R}^m_+)$ such that $\tilde{f}(\bar{x}) \in \tilde{L}(\bar{\mathbf{u}})$.

In the sequel, we are going to prove the strong duality theorem; that is, there is no duality gap between the primal problem (P) and the dual problem (D).

Example 7.1 Suppose that we consider the primal problem (P2) in Example 6.2. Then $\tilde{f}(4,4) \preceq \tilde{f}(x_1, x_2)$ for all feasible solutions (x_1, x_2). It says that $MIN_{\mathrm{P}}(\tilde{f}, \tilde{\mathbf{g}}, X) = \{\tilde{f}(4,4)\}$. If the primal-dual pair problems have no duality gap, then $\tilde{f}(4,4) \in \tilde{L}(\bar{\mathbf{u}})$ for some $\tilde{L}(\bar{\mathbf{u}}) \in MAX_{\mathrm{D}}(\tilde{L}, \mathbb{R}^m_+)$. From Example 6.3, we see that $\tilde{f}(4,4) = \tilde{\phi}(k,k,u)$ for some $k \geq 4$ and $u \geq 6$, i.e., $\tilde{f}_1^L(4,4) = \tilde{\phi}_1{}^L(k,k,u)$ for some $k \geq 4$ and $u \geq 6$. We are going to get a contradiction to show that the primal-dual pair problems have a duality gap. Since $\tilde{f}_1^L(4,4) = \tilde{\phi}_1{}^L(k,k,u)$ we obtain that $k^2 - uk + 2u - 16 = 0$ (note that $\tilde{f}_1^L(4,4) = \tilde{f}_1^U(4,4)$ and $\tilde{\phi}_1{}^L(k,k,u) = \tilde{\phi}_1{}^U(k,k,u)$). Therefore, we have that

$$\begin{aligned} &-1 + \tfrac{u}{2} \leq k = \tfrac{u \pm \sqrt{u^2 - 8u + 64}}{2} \leq 1 + \tfrac{u}{2} \quad \text{if } u \geq 10 \\ &4 \leq k = \tfrac{u \pm \sqrt{u^2 - 8u + 64}}{2} \leq 1 + \tfrac{u}{2} \quad \text{if } 6 \leq u < 10. \end{aligned} \tag{15}$$

Since $\sqrt{u^2 - 8u + 64} = \sqrt{(u-4)^2 + 48} \geq \sqrt{48}$, we get a contradiction from (15). This shows that the primal-dual pair problems have a duality gap.

Lemma 7.2 (Bazarra and Shetty (1993)) *Let X be a nonempty convex set in a real vector space $V=\mathbb{R}^n$. Let $f: X\to\mathbb{R}$ and $g_i: X\to\mathbb{R}$ for $i=1,\ldots,m$ be convex functions. We consider the following systems.*

System I: $f(x)<0$ and $\mathbf{g}(x)\leq\mathbf{0}$ for some $x\in X$;

System II : $u_0 f(x)+\langle\mathbf{u},\mathbf{g}(x)\rangle\geq 0$ for all $x\in X$, $(u_0,\mathbf{u})\geq 0$ and $(u_0,\mathbf{u})\neq\mathbf{0}$.

If System I has no solution x, then System II has a solution $(u_0,\mathbf{u})$.

Let $\tilde{x}$ be a fuzzy number. We write $\tilde{x}\prec_s\tilde{0}$ if and only if $\tilde{x}_\alpha^L<0$ and $\tilde{x}_\alpha^U<0$ for all $\alpha\in[0,1]$, and we say that $\tilde{x}$ is finite if $\tilde{x}_\alpha^L$ and $\tilde{x}_\alpha^U$ are finite real numbers for all $\alpha\in[0,1]$. Then the following lemma is very useful for proving the strong duality theorem.

Lemma 7.3 *Let X be a nonempty convex subset of a real vector space $V=\mathbb{R}^n$. Let $\tilde{f}: X\to\mathscr{F}(\mathbb{R})$ and $\tilde{g}_i: X\to\mathscr{F}(\mathbb{R})$ for $i=1,\ldots,m$ be convex fuzzy-valued functions. Let $\tilde{f}(\bar{x})\in MIN_P(\tilde{f},\tilde{\mathbf{g}},X)$. We assume that $ARG\text{-}MIN_P(\tilde{f}(\bar{x}),\tilde{\mathbf{g}},X)$ is a convex set and $\tilde{f}(\bar{x})$ is finite (i.e., $\tilde{f}_\alpha^L(\bar{x})>-\infty$ and $\tilde{f}_\alpha^U(\bar{x})>-\infty$ for all $\alpha\in[0,1]$). Suppose that the Slater's constraint qualification holds true (i.e., there exists an $\breve{x}\in ARG\text{-}MIN_P(\tilde{f}(\bar{x}),\tilde{\mathbf{g}},X)$* such that $\tilde{\mathbf{g}}(\breve{x})\prec_s\tilde{\mathbf{0}}$). Then there exists an $\bar{\mathbf{u}}\geq\mathbf{0}$ such that $\tilde{\phi}(x,\bar{\mathbf{u}})\succeq\tilde{f}(\bar{x})$ for all $x\in ARG\text{-}MIN_P(\tilde{f}(\bar{x}),\tilde{\mathbf{g}},X)$.

Proof: Since $\tilde{f}(\bar{x})\in MIN_{\mathrm{P}}(\tilde{f},\tilde{\mathbf{g}},X)$, by definition, we have $\tilde{f}(\bar{x})\preceq\tilde{f}(x)$ for all $x\in ARG\text{-}MIN_{\mathrm{P}}(\tilde{f}(\bar{x}),\tilde{\mathbf{g}},X)$; that is,

$$\tilde{f}_\alpha^L(\bar{x})\leq\tilde{f}_\alpha^L(x)\quad\text{and}\quad\tilde{f}_\alpha^U(\bar{x})\leq\tilde{f}_\alpha^U(x)\tag{16}$$

for all $\alpha\in[0,1]$ and all $x\in ARG\text{-}MIN_{\mathrm{P}}(\tilde{f}(\bar{x}),\tilde{\mathbf{g}},X)$. Now, for any fixed $\alpha\in[0,1]$, since $\tilde{f}_\alpha^L(\bar{x})>-\infty$, we consider the following system

$$\tilde{f}_\alpha^L(x)-\tilde{f}_\alpha^L(\bar{x})<0\text{ and }\ \tilde{\mathbf{g}}_\alpha^L(x)\leq\mathbf{0}\quad\text{for some } x\in ARG\text{-}MIN_{\mathrm{P}}(\tilde{f}(\bar{x}),\tilde{\mathbf{g}},X).$$

Then this system has no solution on the set $ARG\text{-}MIN_{\mathrm{P}}(\tilde{f}(\bar{x}),\tilde{\mathbf{g}},X)\subseteq X$, since $\tilde{f}_\alpha^L(\bar{x})\leq\tilde{f}_\alpha^L(x)$ from (16). Hence, from Proposition 3.1 and Lemma 7.2, there exists a $(u_{0\alpha}^L,\mathbf{u}_\alpha^L)\geq\mathbf{0}$ and $(u_{0\alpha}^L,\mathbf{u}_\alpha^L)\neq\mathbf{0}$ such that

$$u_{0\alpha}^L\Big[\tilde{f}_\alpha^L(x)-\tilde{f}_\alpha^L(\bar{x})\Big]+\big\langle\mathbf{u}_\alpha^L,\tilde{\mathbf{g}}_\alpha^L(x)\big\rangle\geq 0\tag{17}$$

for all $x\in ARG\text{-}MIN_{\mathrm{P}}(\tilde{f}(\bar{x}),\tilde{\mathbf{g}},X)$. We want to show that $u_{0\alpha}^L>0$. Assume that $u_{0\alpha}^L=0$. Then we have $\big\langle\mathbf{u}_\alpha^L,\tilde{\mathbf{g}}_\alpha^L(x)\big\rangle\geq 0$ for all $x\in ARG\text{-}MIN_{\mathrm{P}}(\tilde{f}(\bar{x}),\tilde{\mathbf{g}},X)$; that is,

$$\big\langle\mathbf{u}_\alpha^L,\tilde{\mathbf{g}}_\alpha^L(\breve{x})\big\rangle\geq\mathbf{0},\tag{18}$$

since $\breve{x}\in ARG\text{-}MIN_{\mathrm{P}}(\tilde{f}(\bar{x}),\tilde{\mathbf{g}},X)$. By the assumption of Slater's constraint qualification, since $\tilde{\mathbf{g}}_\alpha^L(\breve{x})<\mathbf{0}$ and $\mathbf{u}_\alpha^L\geq\mathbf{0}$, we conclude that $\mathbf{u}_\alpha^L=\mathbf{0}$ using (18). This contradicts $(u_{0\alpha}^L, \mathbf{u}_\alpha^L) \neq \mathbf{0}$. Thus $u_{0\alpha}^L > 0$. Then, from (17) by dividing $u_{0\alpha}^L$ on both sides, we have

$$\tilde{f}_\alpha^L(x) + \langle \bar{\mathbf{u}}_\alpha^L, \tilde{\mathbf{g}}_\alpha^L(x) \rangle \geq \tilde{f}_\alpha^L(\bar{x}) \tag{19}$$

for all $x \in ARG\text{-}MIN_{\mathrm{P}}(\tilde{f}(\bar{x}), \tilde{\mathbf{g}}, X)$, where $\bar{\mathbf{u}}_\alpha^L = \mathbf{u}_\alpha^L / \mathbf{u}_{0\alpha}^L$. We write $\bar{\mathrm{u}}_\alpha^L = (\bar{u}_{1\alpha}^L, \ldots, \bar{u}_{m\alpha}^L)$. Let

$$\bar{u}_i^L = \inf_{0 \leq \alpha \leq 1} \bar{u}_{i\alpha}^L \quad \text{for } i = 1, \ldots, m. \tag{20}$$

Then we use the notation $\bar{\mathbf{u}}^L = (\bar{u}_1^L, \ldots, \bar{u}_m^L)$. We see that

$$\bar{\mathbf{u}}^L \geq \mathbf{0} \quad \text{and} \quad \bar{\mathbf{u}}^L \leq \bar{\mathbf{u}}_\alpha^L \tag{21}$$

for all $\alpha \in [0,1]$ from (20). Since $\tilde{\mathbf{g}}_\alpha^L(x) \leq \mathbf{0}$ and $\bar{\mathbf{u}}_\alpha^L \geq \mathbf{0}$, we have

$$\langle \bar{\mathbf{u}}^L, \tilde{\mathbf{g}}_\alpha^L(x) \rangle \geq \langle \bar{\mathbf{u}}_\alpha^L, \tilde{\mathbf{g}}_\alpha^L(x) \rangle \tag{22}$$

from (21). Thus, from Eqs. (22) and (19), we have

$$\tilde{f}_\alpha^L(x) + \langle \bar{\mathbf{u}}^L, \tilde{\mathbf{g}}_\alpha^L(x) \rangle \geq \tilde{f}_\alpha^L(x) + \langle \bar{\mathbf{u}}_\alpha^L, \tilde{\mathbf{g}}_\alpha^L(x) \rangle \geq \tilde{f}_\alpha^L(\bar{x})$$

for all $\alpha \in [0,1]$ and all $x \in ARG\text{-}MIN_{\mathrm{P}}(\tilde{f}(\bar{x}), \tilde{\mathbf{g}}, X)$. Similarly, we still can show that

$$\tilde{f}_\alpha^U(x) + \langle \bar{\mathbf{u}}^U, \tilde{\mathbf{g}}_\alpha^U(x) \rangle \geq \tilde{f}_\alpha^U(\bar{x})$$

for all $\alpha \in [0,1]$ and all $x \in ARG - MIN_{\mathrm{P}}(\tilde{f}(\bar{x}), \tilde{\mathbf{g}}, X)$. Let $\bar{u}_i = \min\{\bar{u}_i^L, \bar{u}_i^U\}$ and $\bar{\mathbf{u}} = (\bar{u}_1, \ldots, \bar{u}_m)$. Then $\bar{\mathbf{u}} \geq 0, \bar{\mathbf{u}} \leq \bar{\mathbf{u}}^L$ and $\bar{\mathbf{u}} \leq \bar{\mathbf{u}}^U$. Thus using the similar arguments, we also have

$$\tilde{f}_\alpha^L(x) + \langle \bar{\mathbf{u}}, \tilde{\mathbf{g}}_\alpha^L(x) \rangle \geq \tilde{f}_\alpha^L(\bar{x}) \tag{23}$$

for all $\alpha \in [0,1]$ and all $x \in ARG\text{-}MIN_{\mathrm{P}}(\tilde{f}(\bar{x}), \tilde{\mathbf{g}}, X)$, and

$$\tilde{f}_\alpha^U(x) + \langle \bar{\mathbf{u}}, \tilde{\mathbf{g}}_\alpha^U(x) \rangle \geq \tilde{f}_\alpha^U(\bar{x}) \tag{24}$$

for all $\alpha \in [0,1]$ and all $x \in ARG\text{-}MIN_{\mathrm{P}}(\tilde{f}(\bar{x}), \tilde{\mathbf{g}}, X)$. By Propositions 2.1 and 4.1, and Eqs. (23) and (24), we conclude that

$$\tilde{\phi}(x, \bar{\mathbf{u}}) = \tilde{f}(x) \oplus \langle \langle \bar{\mathbf{u}}, \tilde{\mathbf{g}}(x) \rangle \rangle \succeq \tilde{f}(\bar{x})$$

for all $x \in ARG\text{-}MIN_{\mathrm{P}}(\tilde{f}(\bar{x}), \tilde{\mathbf{g}}, X)$. This completes the proof. □

THEOREM 7.3 *(strong duality theorem) Under the assumptions and results in Lemma 7.3, we further assume that the following conditions hold true:*

(i) *there exists an* $\hat{x} \in X$ *such that* $\tilde{f}(\bar{x}) = \tilde{\phi}(\hat{x}, \bar{\mathbf{u}})$ *and* $ARG\text{-}MIN(\tilde{\phi}(\hat{x}, \bar{\mathbf{u}}), X) \subseteq ARG - MIN_{\mathrm{P}}(\tilde{f}(\bar{x}), \tilde{\mathbf{g}}, X)$;
(ii) *there exists no* $x \in X \backslash ARG\text{-}MIN(\tilde{\phi}(\hat{x}, \bar{\mathbf{u}}), X)$ *such that* $\tilde{\phi}(\hat{x}, \bar{\mathbf{u}}) \succ \tilde{\phi}(x, \bar{\mathbf{u}})$;
(iii) $\bar{x} \in \bigcap_{\{\mathbf{u} \in \mathbb{R}^m_+ : \mathbf{u} \neq \bar{\mathbf{u}}\}} \bigcap_{\{y \in X : \tilde{\phi}(y, \mathbf{u}) \in \tilde{L}(\mathbf{u})\}} ARG\text{-}MIN(\tilde{\phi}(y, \mathbf{u}), X)$.

Then there is no duality gap between the primal problem (P) and dual problem (D).

Proof: By Lemma 7.3, we have $\tilde{\phi}(x, \bar{\mathbf{u}}) \succeq \tilde{f}(\bar{x})$ for all $x \in ARG\text{-}MIN_{\mathrm{P}}(\tilde{f}(\bar{x}), \tilde{\mathbf{g}}, X)$. Then, using condition (i), we have

$$\tilde{\phi}(x, \bar{\mathbf{u}}) \succeq \tilde{f}(\bar{x}) = \tilde{\phi}(\hat{x}, \bar{\mathbf{u}}) \quad \text{for all } x \in ARG\text{-}MIN(\tilde{\phi}(\hat{x}, \bar{\mathbf{u}}), X). \tag{25}$$

Thus, from (25) and condition (ii), $\tilde{\phi}(\hat{x}, \bar{\mathbf{u}})$ is nondominated. It means that $\tilde{\phi}(\hat{x}, \bar{\mathbf{u}}) \in \tilde{L}(\bar{\mathbf{u}})$ (by the definition of $\tilde{L}(\bar{\mathbf{u}})$) , i.e., $\tilde{f}(\bar{x}) \in \tilde{L}(\bar{\mathbf{u}})$ since $\tilde{\phi}(\hat{x}, \bar{\mathbf{u}}) = \tilde{f}(\bar{x})$. It remains to prove that $\tilde{L}(\bar{\mathbf{u}}) \in MAX_{\mathrm{D}}(\tilde{L}, \mathbb{R}^m_+)$ since $\tilde{f}(\bar{x}) \in MIN_{\mathrm{P}}(\tilde{f}, \tilde{\mathbf{g}}, X)$. Or, equivalently, we need to prove that $\bar{\mathbf{u}}$ is a solution of the dual problem (D). From Proposition 7.1, we have $\tilde{f}(\bar{x}) \succeq \tilde{L}(\bar{\mathbf{u}})$ if

$$\bar{x} \in \bigcap_{\{y \in X : \tilde{\phi}(y, \mathbf{u}) \in \tilde{L}(\mathbf{u})\}} ARG\text{-}MIN(\tilde{\phi}(y, \mathbf{u}), X). \tag{26}$$

Thus, from condition (iii) and (26), we have $\tilde{f}(\bar{x}) \succeq \tilde{L}(\mathbf{u})$ for all $\mathbf{u} \neq \bar{\mathbf{u}}$ and $\mathbf{u} \geq 0$. Since $\tilde{f}(\bar{x}) \in \tilde{L}(\bar{\mathbf{u}})$, we conclude that, by definition, $\bar{\mathbf{u}}$ is a solution of the dual problem (D), i.e., $\tilde{L}(\bar{\mathbf{u}}) \in MAX_{\mathrm{D}}(\tilde{L}, \mathbb{R}^m_+)$. This shows that there is no duality gap between the primal problem (P) and dual problem (D). □

Let $\bar{x}$ be a feasible solution of primal problem (P) such that $\tilde{f}(\bar{x}) \preceq \tilde{f}(x)$ and $\tilde{g}_i(x) \preceq \tilde{0}$ for all $i = 1, \ldots, m$ and all $x \in X$. Then we see that

$$MIN_{\mathrm{P}}(\tilde{f}, \tilde{\mathbf{g}}, X) = \{\tilde{f}(\bar{x})\}. \tag{27}$$

This fact has been observed in Examples 6.1 and 6.2. We obtain the following useful lemma.

Lemma 7.4 Let $\bar{x}$ be a feasible solution of primal problem (P) such that $\tilde{f}(\bar{x}) \preceq \tilde{f}(x)$ and $\tilde{g}_i(x) \preceq \tilde{0}$ for all $i = 1, \ldots, m$ and all $x \in X$. *Let X be a nonempty convex subset of a real vector space* $V = \mathbb{R}^n$. *Let* $\tilde{f} : X \to \mathscr{F}(\mathbb{R})$ *and* $\tilde{g}_i : X \to \mathscr{F}(\mathbb{R})$ for $i = 1, \ldots, m$ *be convex fuzzy-valued functions. We assume that* $\tilde{f}(\bar{x})$ *is finite. Suppose that the Slater's constraint qualification holds true (i.e., there exists an* $\check{x} \in X$ such that $\tilde{\mathbf{g}}_i(\check{x}) \prec_s \tilde{0}$. Then there exists an $\bar{\mathbf{u}} \geq \mathbf{0}$ such that $\tilde{\phi}(x, \bar{\mathbf{u}}) \succeq \tilde{f}(\bar{x})$ for all $x \in X$.

Proof: Since $\tilde{f}(\bar{x}) \preceq \tilde{f}(x)$ and $\tilde{g}_i(x) \preceq \tilde{0}$ for all $i = 1, \ldots, m$ and all $x \in X$, we have

$$\tilde{f}_\alpha^L(\bar{x}) \leq \tilde{f}_\alpha^L(x), \qquad \tilde{f}_\alpha^U(\bar{x}) \leq \tilde{f}_\alpha^U(x), \qquad \tilde{\mathbf{g}}_\alpha{}^L(x) \leq \mathbf{0} \quad \text{and} \quad \tilde{\mathbf{g}}_\alpha{}^U(x) \leq \mathbf{0} \tag{28}$$

for all $\alpha \in [0, 1]$ and all $x \in X$. Now, for any fixed $\alpha \in [0, 1]$, we consider the following two systems

$$\tilde{f}_\alpha^L(x) - \tilde{f}_\alpha^L(\bar{x}) < 0 \quad \text{and} \quad \tilde{\mathbf{g}}_\alpha^{\mathbf{L}}(x) \leq \mathbf{0},$$

and

$$\tilde{f}_\alpha^U(x) - \tilde{f}_\alpha^U(\bar{x}) < 0 \quad \text{and} \quad \tilde{\mathbf{g}}_\alpha{}^U(x) \leq \mathbf{0}$$

for all $x \in X$. Then each of the above two systems has no solution on the set X, since $\tilde{f}_\alpha^L(\bar{x}) \leq \tilde{f}_\alpha^L(x)$ and $\tilde{f}_\alpha^U(\bar{x}) \leq \tilde{f}_\alpha^U(x)$ from (28). Using the similar arguments in the proof of Lemma 7.3, the proof is complete. □

Suppose that, for any fixed $\mathbf{u} > 0$, there exists a $y \in X$ such that $\tilde{\phi}(y, \mathbf{u}) \preceq \tilde{\phi}(x, \mathbf{u})$ for all $x \in X$. Then $\tilde{L}(\mathbf{u}) = \{\tilde{\phi}(y, \mathbf{u})\}$ which is a singleton set and *ARG-MIN* $(\tilde{\phi}(y, \mathbf{u}), X) = X$. Let us write $\tilde{\phi}(X, \bar{\mathbf{u}}) = \{\tilde{\phi}(y, \mathbf{u}) : y \in X\}$.

THEOREM 7.4 *(strong duality theorem) Under the assumptions and results in Lemma 7.4, we further assume that the following conditions hold true:*

(i) $\tilde{f}(\bar{x}) \in \tilde{\phi}(X, \bar{\mathbf{u}})$;
(ii) for any fixed $\mathbf{u} \neq \bar{\mathbf{u}}$, *then exists a* $y \in X$ *(this* y *depends on* $\mathbf{u}$*) such that* $\tilde{\phi}(y, \mathbf{u}) \preceq \tilde{\phi}(\bar{x}, \mathbf{u})$ *for all* $x \in X$.

Then there is no duality gap between the primal problem (P) and dual problem (D).

Proof: Condition (i) says that there exists an $\hat{x} \in X$ such that $\tilde{f}(\bar{x}) = \tilde{\phi}(\hat{x}, \bar{\mathbf{u}})$. Using Lemma 7.4, we have $\tilde{\phi}(x, \bar{\mathbf{u}}) \succeq \tilde{f}(\bar{x}) = \tilde{\phi}(\hat{x}, \bar{\mathbf{u}})$ for all $x \in X$. It means that $\tilde{L}(\bar{\mathbf{u}}) = \{\tilde{\phi}(\hat{x}, \bar{\mathbf{u}})\}$ and $\tilde{f}(\bar{x}) \in \tilde{L}(\bar{\mathbf{u}})$. Condition (ii) says that $\tilde{L}(\mathbf{u}) = \{\tilde{\phi}(y, \mathbf{u})\}$ and

$$\tilde{\phi}(y, \mathbf{u}) \preceq \tilde{\phi}(\bar{x}, \mathbf{u}) = \tilde{f}(\bar{x}) \oplus \langle\langle \mathbf{u}, \tilde{g}(\bar{x})\rangle\rangle \preceq \tilde{f}(\bar{x}),$$

using Lemma 7.1. In other words, $\tilde{\phi}(\hat{x}, \bar{\mathbf{u}}) = \tilde{f}(\bar{x}) \succeq \tilde{L}(\mathbf{u})$ for any $\mathbf{u} \neq \bar{\mathbf{u}}$, i.e., $\tilde{L}(\bar{\mathbf{u}}) \in MAX_{\mathrm{D}}(\tilde{L}, \mathbb{R}_+^m)$. Since $MIN_{\mathrm{P}}(\tilde{f}, \tilde{\mathbf{g}}, X) = \{\tilde{f}(\bar{x})\}$ from (27), the proof is complete. □

8. Conclusions

The weak and strong duality theorems in fuzzy optimization problems have been successfully discussed in this paper by introducing the fuzzy-valued Lagrangian (dual) function via the concept of fuzzy scalar product. Also, a solution concept, which is essentially similar to the notion of Pareto optimal solution in multiobjective programming problem, is proposed by defining a partial ordering on the set of all fuzzy numbers. We then provide some examples to clarify the solution concepts of primal and dual problems.

We have seen that the assumptions presented in Theorem 7.3 are very complex. To verify those assumptions may not be an easy task. This is one of the limitations of this paper. The main reason why those assumptions are so complex is that the ordering relation "$\preceq$" employed in this paper is just a partial ordering (rather than a total ordering) among the set of all fuzzy numbers. Therefore, the so-called "optimal" objective values $MIN_{\mathrm{P}}(\tilde{f}, \tilde{\mathbf{g}}, X)$ and $MAX_{\mathrm{D}}(\tilde{L}, \mathbb{R}_{+}^{m})$ of primal problem (P) and dual problem (D), respectively, are subsets of $\mathscr{F}(\mathbb{R})$ rather than a single value in $\mathscr{F}(\mathbb{R})$, which is another reason why the assumptions are so complex. If we can devise an example such that the ordering "$\preceq$" becomes a total ordering among the set of all fuzzy objective values, then the assumptions can be simplified as the similar assumptions presented in the conventional (crisp) optimization problem, since the duality theorems obtained in this paper follow from the similar approach of conventional optimization problem by proposing the fuzzy-valued Lagrangian dual function which is similar to the Lagrangian dual function employed in conventional optimization problem. In other words, the simple way to validate the assumptions presented in Theorem 7.3 is to take a set of fuzzy objective values such that the ordering "$\preceq$" becomes a total ordering among this set. Also, Theorem 7.4 has shown some other neat sufficient conditions to guarantee the strong duality theorem. Of course, there may exist some other ways to simplify the assumptions, which could be the future research.

The fuzzy-valued Lagrangian function and fuzzy-valued Lagrangian dual function are the first attempt adopted in this paper for modeling the dual problem of fuzzy nonlinear programming problem. We have seen that the Lagrangian dual function $\tilde{L}(u)$ is a point-to-set fuzzy-valued function. We also provide a solution concept to optimize the point-to-set fuzzy-valued function. The collection of papers edited by Huard (1979) gave some interesting techniques to optimize the point-to-set functions (set-valued functions). Therefore, in the future research, we may develop some methodology to optimize the point-to-set fuzzy-valued functions by invoking some useful ideas proposed in the book edited by Huard (1979).

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