Interpreting Fuzzy Membership Functions in the Theory of Rough Sets

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Abstract. A fundamental difficulty with fuzzy set theory is the semantical interpretations of membership functions. We address this issue in the theory of rough sets. Rough membership functions are viewed as a special type of fuzzy membership functions interpretable using conditional probabilities. Rough set approximations are related to the core and support of a fuzzy set. A salient feature of the interpretation is that membership values of equivalent or similar elements are related to each other. Two types of similarities are considered, one is defined by a partition of the universe, and the other is defined by a covering.

1 Introduction

The theory of fuzzy sets is a generalization of the classical sets by allowing partial membership. It provides a more realistic framework for modeling the ill-definition of the boundary of a class [2, 8]. However, a fundamental difficulty with fuzzy set theory is the semantical interpretations of the degrees of membership. The objectives of this paper are to investigate interpretations of fuzzy membership functions in the theory of rough sets, and to establish the connections between core and support of fuzzy sets and rough set approximations.

There are at least two views for interpreting rough set theory [4, 6]. The operator-oriented view treats rough set theory as an *extension* of the classical set theory. Two additional unary set-theoretic operators are introduced, and the meanings of sets and standard set-theoretic operators are unchanged. This view is closely related to modal logics. The set-oriented view treats rough set theory as a *deviation* of the classical set theory. Sets and set-theoretic operators are associated with non-standard interpretations, and no additional set-theoretic operator is introduced. This view is related to many-valued logics and fuzzy sets. A particular set-oriented view is characterized by rough membership functions [3]. The formulation and interpretation of rough membership functions are based on partitions of a universe. By viewing rough membership functions as a special type of fuzzy membership functions, one may be able to provide a sound semantical interpretation of fuzzy membership functions.

The rest of the paper is organized as follows. In Section 2, we examine the interpretation of fuzzy membership functions and show the connections between

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rough set approximations and the core and support of fuzzy sets, based on rough membership functions defined by a partition of a universe. A unified framework is used for studying both fuzzy sets and rough sets. In Section 3, within the established framework, we apply the arguments to a more general case by extending partitions to coverings of the universe. Three different rough membership functions are introduced. They lead to commonly used rough set approximations.

2 Review and Comparison of Fuzzy Sets and Rough Sets

In this section, some basic issues of fuzzy sets and rough sets are reviewed, examined, and compared by using a unified framework. Fuzzy membership functions are interpreted in terms of rough membership functions. The concepts of core and support of a fuzzy set are related to rough set approximations.

2.1 Fuzzy Sets

The notion of fuzzy sets provides a convenient tool for representing vague concepts by allowing partial memberships. Let U be a finite and non-empty set called universe. A fuzzy subset $\mathcal{A}$ of U is defined by a membership function:

$$\mu_{\mathcal{A}} : U \longrightarrow [0,1]. \tag{1}$$

There are many definitions for fuzzy set complement, intersection, and union. The standard min-max system proposed by Zadeh is defined component-wise by [8]:

$$\begin{aligned}
\mu_{\neg\mathcal{A}}(x) &= 1 - \mu_{\mathcal{A}}(x),\\
\mu_{\mathcal{A}\cap\mathcal{B}}(x) &= \min(\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)),\\
\mu_{\mathcal{A}\cup\mathcal{B}}(x) &= \max(\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)).
\end{aligned} \tag{2}$$

In general, one may interpret fuzzy set operators using triangular norms (t-norms) and conorms (t-conorms) [2]. Let t and s be a pair of t-norm and t-conorms, we have:

$$\begin{aligned}
\mu_{\mathcal{A}\cap\mathcal{B}}(x) &= t(\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)),\\
\mu_{\mathcal{A}\cup\mathcal{B}}(x) &= s(\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)).
\end{aligned} \tag{3}$$

A crisp set may be viewed as a degenerated fuzzy set. A pair of t-norm and t-conorm reduce to standard set intersection and union when applied to crisp subsets of U. The min is an example of t-norms and the max is an example of t-conorms. An important feature of fuzzy set operators as defined by t-norms and t-conorms is that they are truth-functional operators. In other words, membership functions of complement, intersection, and union of fuzzy sets are defined based solely on membership functions of the fuzzy sets involved [6]. Although they have some desired properties, such as $\mu_{\mathcal{A}\cap\mathcal{B}}(x) \leq \min[\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)] \leq$

$\max[\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)] \leq \mu_{\mathcal{A}\cup\mathcal{B}}(x)$, there is a lack of a well accepted semantical interpretation of fuzzy set operators.

The concepts of core and support have been introduced and used as approximations of a fuzzy set [2]. The core of a fuzzy set $\mathcal{A}$ is a crisp subset of U consisting of elements with full membership:

$$core(\mathcal{A}) = \{x \in U \mid \mu_{\mathcal{A}}(x) = 1\}. \tag{4}$$

The support is a crisp subset of U consisting of elements with non-zero membership:

$$support(\mathcal{A}) = \{x \in U \mid \mu_{\mathcal{A}}(x) > 0\}. \tag{5}$$

With $1 - (\cdot)$ as fuzzy set complement, and t-norms and t-conorms as fuzzy set intersection and union, the following properties hold:

$$\begin{array}{ll}
\text{(F1)} & core(\mathcal{A}) = \neg(support(\neg\mathcal{A})), \\
& support(\mathcal{A}) = \neg(core(\neg\mathcal{A})), \\
\text{(F2)} & core(\mathcal{A}\cap\mathcal{B}) = core(\mathcal{A}) \cap core(\mathcal{B}), \\
& support(\mathcal{A}\cap\mathcal{B}) \subseteq support(\mathcal{A}) \cap support(\mathcal{B}), \\
\text{(F3)} & core(\mathcal{A}\cup\mathcal{B}) \supseteq core(\mathcal{A}) \cup core(\mathcal{B}), \\
& support(\mathcal{A}\cup\mathcal{B}) = support(\mathcal{A}) \cup support(\mathcal{B}), \\
\text{(F4)} & core(\mathcal{A}) \subseteq \mathcal{A} \subseteq support(\mathcal{A}).
\end{array}$$

According to (F1), one may interpret *core* and *support* as a pair of dual operators on the set of all fuzzy sets. They map a fuzzy set to a pair of crisp sets. By properties (F2) and (F3), one may say that *core* is distributive over $\cap$ and *support* is distributive over $\cup$. However, *core* is not necessarily distributive over $\cup$ and *support* is not necessarily distributive over $\cap$. These two properties follow from the properties of t-norm and t-conorm. When the min-max system is used, we have equality in (F2) and (F3). Property (F4) suggests that a fuzzy set lies within its core and support.

2.2 Rough Sets

A fundamental concept of rough set theory is indiscernibility. Let $R \subseteq U \times U$ be an equivalence relation on a finite and non-empty universe U. That is, the relation R is reflexive, symmetric, and transitive. The pair $apr = (U, R)$ is called an approximation space. The equivalence relation R partitions the universe U into disjoint subsets called equivalence classes. Elements in the same equivalence class are said to be indistinguishable. The partition of the universe is referred to as the quotient set and is denoted by $U/R = \{E_1, \ldots, E_n\}$.

An element $x \in U$ belongs to one and only one equivalence class. Let

$$[x]_R = \{y \mid xRy\}, \tag{6}$$

denote the equivalence class containing x. For a subset $A \subseteq U$, we have the following well defined rough membership function [3]:

$$\mu_A(x) = \frac{|[x]_R \cap A|}{|[x]_R|}, \tag{7}$$

where $|\cdot|$ denotes the cardinality of a set. One can easily see the similarity between rough membership functions and conditional probabilities. As a matter of fact, the rough membership value $\mu_A(x)$ may be interpreted as the conditional probability that an arbitrary element belongs to A given that the element belongs to $[x]_R$.

Rough membership functions may be interpreted as a special type of fuzzy membership functions interpretable in terms of probabilities defined simply by cardinalities of sets. In general, one may use a probability function on U to define rough membership functions [7]. One may view the fuzzy set theory as an uninterpreted mathematical theory of abstract membership functions. The theory of rough set thus provides a more specific and more concrete interpretation of fuzzy membership functions. The source of the fuzziness in describing a concept is the indiscernibility of elements. In the theoretical development of fuzzy set theory, fuzzy membership functions are treated as abstract mathematical functions without any constraint imposed [2]. When we interpret fuzzy membership functions in the theory of rough sets, we have the following constraints:

$$\begin{array}{ll}
\text{(rm1)} & \mu_U(x) = 1, \\
\text{(rm2)} & \mu_\emptyset(x) = 0, \\
\text{(rm3)} & y \in [x]_R \Longrightarrow \mu_A(x) = \mu_A(y), \\
\text{(rm4)} & x \in A \Longrightarrow \mu_A(x) \neq 0 \\
\text{(rm5)} & \mu_A(x) = 1 \Longrightarrow x \in A, \\
\text{(rm6)} & A \subseteq B \Longrightarrow \mu_A(x) \leq \mu_B(x).
\end{array}$$

Property (rm3) is particularly important. It shows that elements in the same equivalence class must have the same degree of membership. That is, indiscernible elements should have the same membership value. Such a constraint, which ties the membership values of individual elements according to their connections, is intuitively appealing. Although this topic has been investigated by some authors, there is still a lack of systematic study [1]. Property (rm4) can be equivalently expressed as $\mu_A(x) = 0 \Longrightarrow x \notin A$, and property (rm5) expressed as $x \notin A \Longrightarrow \mu_A(x) \neq 1$.

The constraints on rough membership functions have significant implications on rough set operators. There does not exist a one-to-one relationship between rough membership functions and subsets of U. Two distinct subsets of U may define the same rough membership function. Rough membership functions corresponding to $\neg A$, $A \cap B$, and $A \cup B$ must be defined using set operators and equation (7). By laws of probability, we have:

$$\mu_{\neg A}(x) = 1 - \mu_A(x),$$

$$\mu_{A\cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_{A\cap B}(x),$$
$$\max(0, \mu_A(x) + \mu_B(x) - 1) \leq \mu_{A\cap B}(x) \leq \min(\mu_A(x), \mu_B(x)),$$
$$\max(\mu_A(x), \mu_B(x)) \leq \mu_{A\cup B}(x) \leq \min(1, \mu_A(x) + \mu_B(x)). \tag{8}$$

Unlike the commonly used fuzzy set operators, the new intersection and union operators are non-truth-functional. That is, it is impossible to obtain rough membership functions of $A \cap B$ and $A \cup B$ based solely on the rough membership functions of A and B. One must also consider their relationships to the equivalence class $[x]_R$.

In an approximation space, a subset $A \subseteq U$ is approximated by a pair of sets called lower and upper approximations as follows [3]:

$$\begin{aligned}\underline{apr}(A) &= \{x \in U \mid \mu_A(x) = 1\}\\ &= core(\mu_A),\\ \overline{apr}(A) &= \{x \in U \mid \mu_A(x) > 0\}\\ &= support(\mu_A).\end{aligned} \tag{9}$$

That is, the lower and upper approximation are indeed the core and support of the fuzzy set μ_A. For any subsets $A, B \subseteq U$, we have:

$$\begin{array}{ll}\text{(R1)} & \underline{apr}(A) = \neg(\overline{apr}(\neg A)),\\ & \overline{apr}(A) = \neg(\underline{apr}(\neg A)),\\ \text{(R2)} & \underline{apr}(A \cap B) = \underline{apr}(A) \cap \underline{apr}(B),\\ & \overline{apr}(A \cap B) \subseteq \overline{apr}(A) \cap \overline{apr}(B),\\ \text{(R3)} & \underline{apr}(A \cup B) \supseteq \underline{apr}(A) \cup \underline{apr}(B),\\ & \overline{apr}(A \cup B) = \overline{apr}(A) \cup \overline{apr}(B),\\ \text{(R4)} & \underline{apr}(A) \subseteq A \subseteq \overline{apr}(A),\end{array}$$

By comparing with (F1)-(F4), we can see that rough set approximation operators satisfy the same properties of core and support of fuzzy sets.

Using the equivalence classes $U/R = \{E_1, \ldots, E_n\}$, lower and upper approximations can be equivalently defined as follows:

$$\begin{aligned}\underline{apr}(A) &= \bigcup\{E_i \in U/R \mid E_i \subseteq A\},\\ \overline{apr}(A) &= \bigcup\{E_i \in U/R \mid E_i \cap A \neq \emptyset\}.\end{aligned} \tag{10}$$

The lower approximation $\underline{apr}(A)$ is the union of all the equivalence classes which are subsets of A. The upper approximation $\overline{apr}(A)$ is the union of all the equivalence classes which have a non-empty intersection with A.

3 Generalized Rough Membership Functions based on Coverings of the Universe

In a partition, an element belongs to one equivalence class and two distinct equivalence classes have no overlap. The rough set theory built on a partition, although easy to analyze, may not provide a realistic view of relationships between elements of the universe. One may consider a more realistic model by extending partitions to coverings of the universe [6,9].

A covering of the universe, $C = \{C_1, \ldots, C_n\}$, is a family of subsets of U such that $U = \bigcup\{C_i \mid i = 1, \ldots, n\}$. Two distinct sets in C may have a non-empty overlap. An arbitrary element x of U may belong to more than one set in C. The family $C(x) = \{C_i \in C \mid x \in C_i\}$ consists of sets in C containing x. The sets in $C(x)$ may describe different types or various degrees of similarity between elements of U. For a set $C_i \in C(x)$, we may compute a value $|C_i \cap A|/|C_i|$ by extending equation (7). It may be interpreted as the membership value of x from the view point of C_i. From the set $C(x)$, we have a family of values $\{|C_i \cap A|/|C_i| \mid x \in C_i\}$. Generalized rough membership functions may be defined by using this family of values. We consider the following three definitions:

$$\text{(minimum)} \qquad \mu_A^m(x) = \min\left\{\frac{|C_i \cap A|}{|C_i|} \mid x \in C_i\right\}, \tag{11}$$

$$\text{(maximum)} \qquad \mu_A^M(x) = \max\left\{\frac{|C_i \cap A|}{|C_i|} \mid x \in C_i\right\}, \tag{12}$$

$$\text{(average)} \qquad \mu_A^*(x) = \text{avg}\left\{\frac{|C_i \cap A|}{|C_i|} \mid x \in C_i\right\}. \tag{13}$$

The minimum, maximum, and average definitions may be regarded as the most permissive, the most optimistic view, and the balanced view in defining rough membership functions. The minimum rough membership function is determined by a set in $C(x)$ which has the smallest overlap with A, and the maximum rough membership function by a set in $C(x)$ which has the largest overlap with A. The average rough membership function depends on every set in $C(x)$. The three rough membership functions are related by:

$$\mu_A^m(x) \leq \mu_A^*(x) \leq \mu_A^M(x). \tag{14}$$

A partition is a special type of coverings. In this case, three rough membership functions reduce to the same rough membership function.

The generalized rough membership functions have the following properties:

(grm1) $\quad \mu_U^m(x) = \mu_U^*(x) = \mu_U^M(x) = 1,$

(grm2) $\quad \mu_\emptyset^m(x) = \mu_\emptyset^*(x) = \mu_\emptyset^M(x) = 0,$

(grm3) $\quad [\forall C_i \in C(x \in C_i \Longleftrightarrow y \in C_i)] \Longrightarrow$
$\quad [\mu_A^m(x) = \mu_A^m(y), \mu_A^*(x) = \mu_A^*(y), \mu_A^M(x) = \mu_A^M(y)],$

(grm4) $\quad x, y \in C_i \Longrightarrow [\mu_A^m(x) \neq 1 \Longrightarrow \mu_A^m(y) \neq 1, \mu_A^M(x) \neq 0 \Longrightarrow \mu_A^M(y) \neq 0],$

(grm5) $\quad x \in A \Longrightarrow \mu_A^m(x) \neq 0,$

(grm6) $\quad \mu_A^M(x) = 1 \Longrightarrow x \in A,$

(grm7) $\quad A \subseteq B \Longrightarrow [\mu_A^m(x) \leq \mu_B^m(x), \mu_A^*(x) \leq \mu_B^*(x), \mu_A^M(x) \leq \mu_B^M(x)].$

Both (grm3) and (grm4) show the constraints on rough membership functions imposed by the similarity of objects. From the relation $\mu_A^m(x) \leq \mu_A^*(x) \leq \mu_A^M(x)$, we can obtain additional properties. For example, (grm5) implies that $x \in A \Longrightarrow \mu^*_A(x) \neq 0$ and $x \in A \Longrightarrow \mu^M_A(x) \neq 0$. Similarly, (grm6) implies that $\mu^m_A(x) = 1 \Longrightarrow x \in A$ and $\mu^*_A(x) = 1 \Longrightarrow x \in A$.

For set-theoretic operators, one can verify that the following properties:

$$
\begin{aligned}
&\mu^m_{\neg A}(x) = 1 - \mu^M_A(x),\\
&\mu^M_{\neg A}(x) = 1 - \mu^m_A(x),\\
&\mu^*_{\neg A}(x) = 1 - \mu^*_A(x),\\
&\max(0, \mu^m_A(x) + \mu^m_B(x) - \mu^M_{A\cup B}(x)) \leq \mu^m_{A\cap B}(x) \leq \min(\mu^m_A(x), \mu^m_B(x)),\\
&\max(\mu^M_A(x), \mu^M_B(x)) \leq \mu^M_{A\cup B}(x) \leq \min(1, \mu^M_A(x) + \mu^M_B(x) - \mu^m_{A\cap B}(x)),\\
&\mu^*_{A\cap B}(x) = \mu^*_A(x) + \mu^*_B(x) - \mu^*_{A\cup B}(x).
\end{aligned} \tag{15}
$$

We again obtain non-truth-functional rough set operators.

The minimum rough membership function may be viewed as the lower bound on all possible rough membership functions definable using a covering, while the maximum rough membership as the upper bound. The pair $(\mu^m_A(x), \mu^M_A(x))$ may also be used to define an interval-valued fuzzy set [2]. The interval $[\mu^m_A(x), \mu^M_A(x)]$ is the membership value of x with respect to A.

From the three rough membership functions, we define three pairs of lower and upper approximations. For the minimum definition, we have:

$$
\begin{aligned}
\underline{apr}^m(A) &= core(\mu^m_A)\\
&= \{x \in U \mid \mu^m_A(x) = 1\}\\
&= \{x \in U \mid \forall C_i \in C(x \in C_i \Longrightarrow C_i \subseteq A)\},\\
\overline{apr}^m(A) &= spport(\mu^m_A)\\
&= \{x \in U \mid \mu^m_A(x) > 0\}\\
&= \{x \in U \mid \forall C_i \in C(x \in C_i \Longrightarrow C_i \cap A \neq \emptyset)\}.
\end{aligned} \tag{16}
$$

For the maximum definition, we have:

$$
\begin{aligned}
\underline{apr}^M(A) &= core(\mu^M_A)\\
&= \{x \in U \mid \mu^M_A(x) = 1\}\\
&= \{x \in U \mid \exists C_i \in C(x \in C_i, C_i \subseteq A)\},\\
&= \bigcup\{C_i \in C \mid C_i \subseteq A\},\\
\overline{apr}^M(A) &= spport(\mu^M_A)\\
&= \{x \in U \mid \mu^M_A(x) > 0\}\\
&= \{x \in U \mid \exists C_i \in C(x \in C_i, C_i \cap A \neq \emptyset)\}\\
&= \bigcup\{C_i \in C \mid C_i \cap A \neq \emptyset\}.
\end{aligned} \tag{17}
$$

The lower and upper approximations in each pair are no longer dual operators. However, $(\underline{apr}^m, \overline{apr}^M)$ and $(\underline{apr}^M, \overline{apr}^m)$ are two pairs of dual operators. The first pair can be derived from the average definition, namely:

$$
\underline{apr}^*(A) = \underline{apr}^m(A), \qquad \overline{apr}^*(A) = \overline{apr}^M(A). \tag{18}
$$

These approximation operator have been studied extensively in rough set theory. Their connections and properties can be found in a recent paper by Yao [5].

4 Conclusion

Rough membership functions can be viewed as a special type of fuzzy membership functions, and rough set approximations as the core and support of fuzzy sets. This provides a starting point for the interpretation of fuzzy membership functions in the theory of rough sets. We study rough membership functions defined based on partitions and coverings of a universe.

The formulation and interpretation of rough membership functions are inseparable parts of the theory of rough sets. Each rough membership function has a well defined semantical interpretation. The source of uncertainty modeled by rough membership functions is the indiscernibility or similarity of objects. Constraints are imposed on rough membership functions by the relationships between objects. More specifically, equivalent objects must have the same membership value, and similar objects must have similar membership values. These observations may have significant implications for the understanding of fuzzy set theory. The interpretation of fuzzy membership functions in the theory of rough sets provides a more restrictive, but more concrete, view of fuzzy sets. Such semantically sound models may provide possible solutions to the fundamental difficulty with fuzzy set theory regarding semantical interpretations of fuzzy membership functions.

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