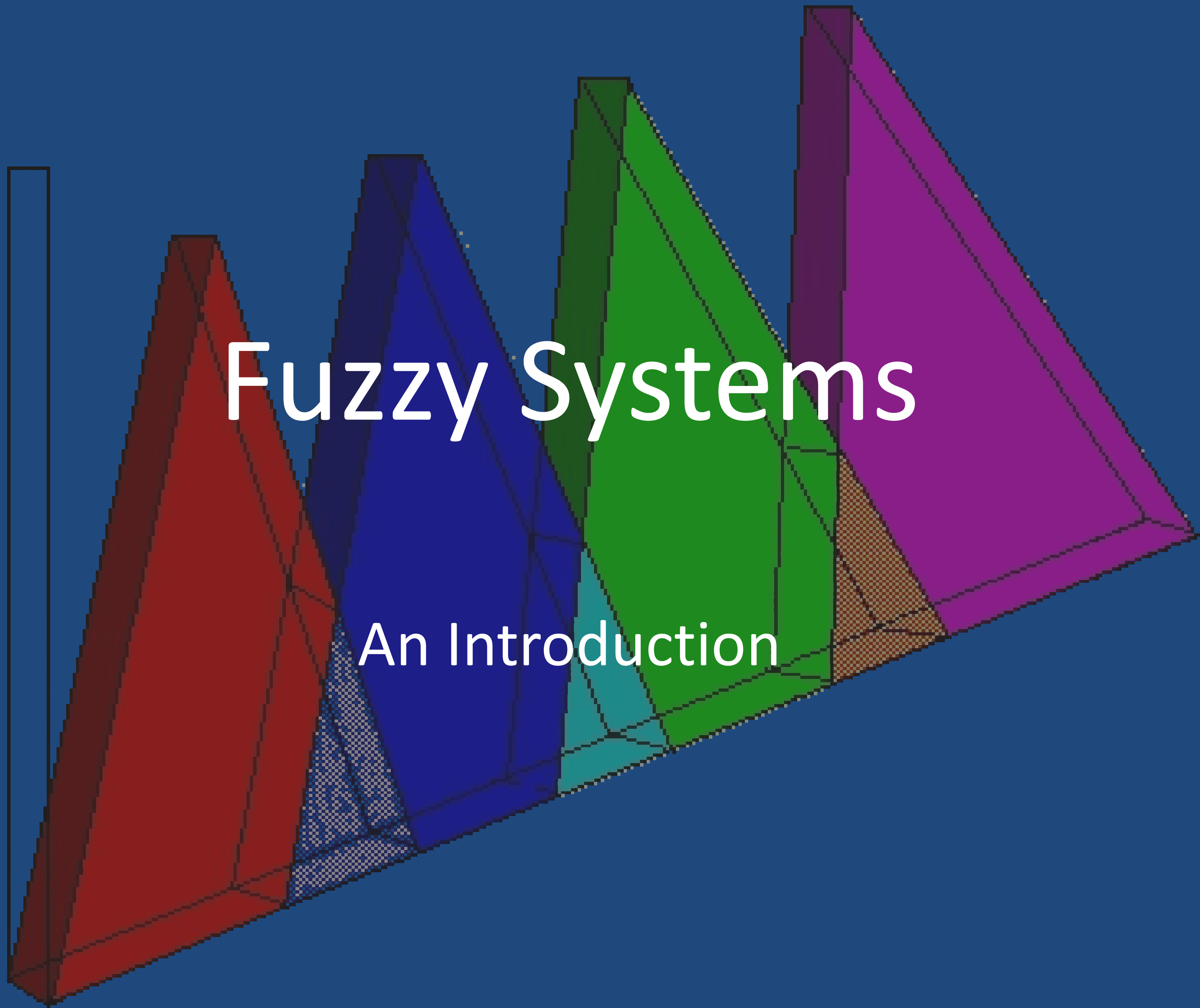
Fuzzy Systems

An Introduction



Overview

- Need for Fuzzy systems
- What are Fuzzy systems?
- Foundations of Fuzzy Sets
  - Traditional vs Fuzzy Sets
  - Fuzzy Membership
- Linguistic Variables
- Basic Fuzzy Set Operations
- Conclusion

Why do we need Fuzzy Systems?

- Real world is full of concepts which are vague/imprecise
  - Difficult to translate into precise formal/mathematical definitions
  - Could be argued that this is disadvantage – but is used successfully
  - True/False - “*Law-of-the-excluded-middle*”

FALSE	**?**	**TRUE**

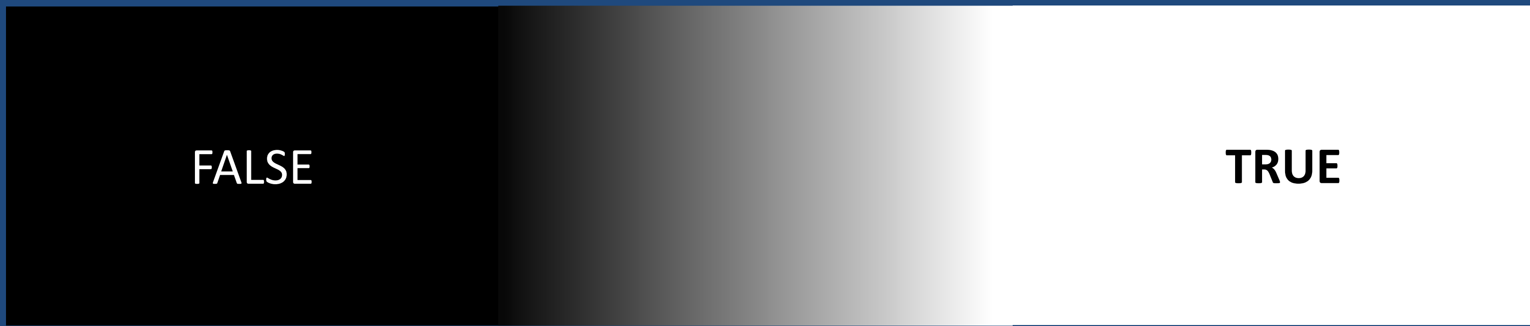
Why do we need Fuzzy Systems?

- Handling real-world vagueness
  - Early attempts (c. 1900) included *n*-valued logic – Łukasiewicz, Knuth etc.

  - 1965 Lotfi Zadeh proposed the notion of infinite-valued logic = fuzzy sets
  - '*Excluded Middle*' represented as a spectrum of possibility

Why do we need Fuzzy Systems?

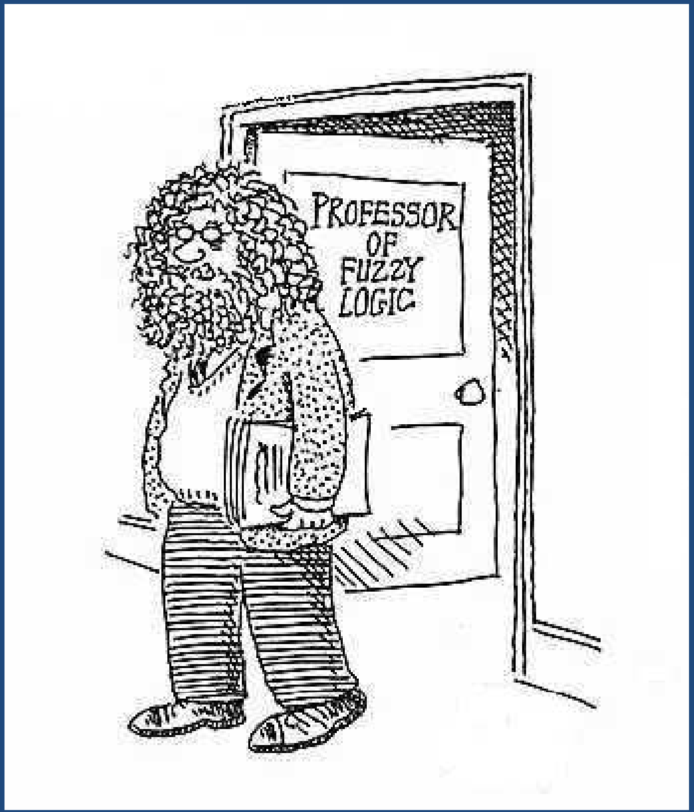
- Provides a means to reason about the ‘*excluded middle*’



- No longer an area of ‘vagueness’ or ‘don’t care’

What are Fuzzy Systems?

- Based on the foundation of Fuzzy sets



- Fuzzy Sets - to ‘make sense’ of vagueness

- Can use human concepts of fuzziness

What are Fuzzy Systems?

- Lots of applications
  - Control: Sendai Railway, Consumer Electronics, etc

  - Expert Systems: Credit card fraud, Production Process Analysis, etc.

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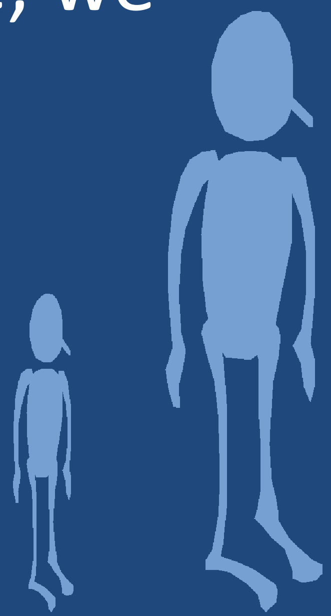
Foundations of Fuzzy Sets

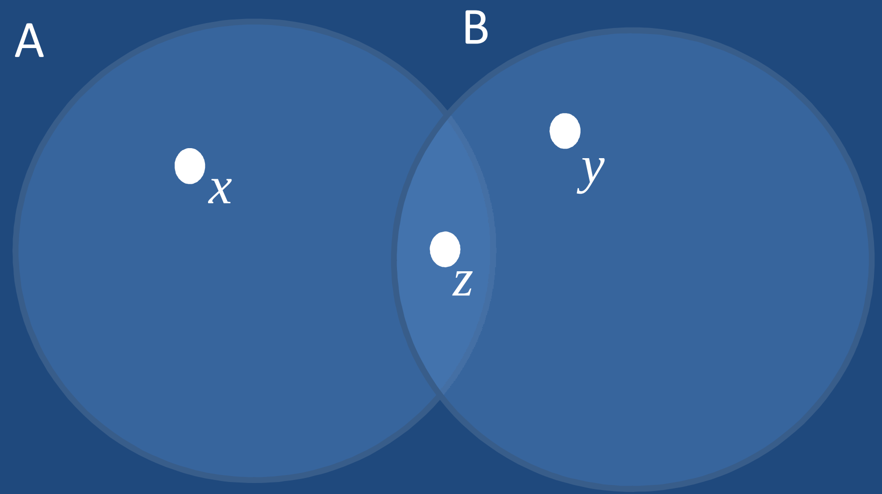
- The concept of a **set** is fundamental to mathematics.
- However, natural language is also the supreme expression of sets. For example, *house* indicates the *set of houses*. When we say a house, we mean one out of the set of houses.

- Classic example - concept “tall”. The elements of the fuzzy set “tall” are all people, but their individual membership depends on their height.



Foundations of Fuzzy Sets – Traditional sets vs Fuzzy sets

- Extension of conventional sets:



A = $\{x, z\}$
B = $\{y, z\}$

$x, z \in$ A
$y, z \in$ B

A $\cup$ B = $\{ x, y, z \}$
A $\cap$ B = $\{ z \}$

- Conventional Sets - crisp membership: {0,1}
- No ‘room’ for partial membership

Foundations of Fuzzy Sets – Traditional sets vs Fuzzy sets

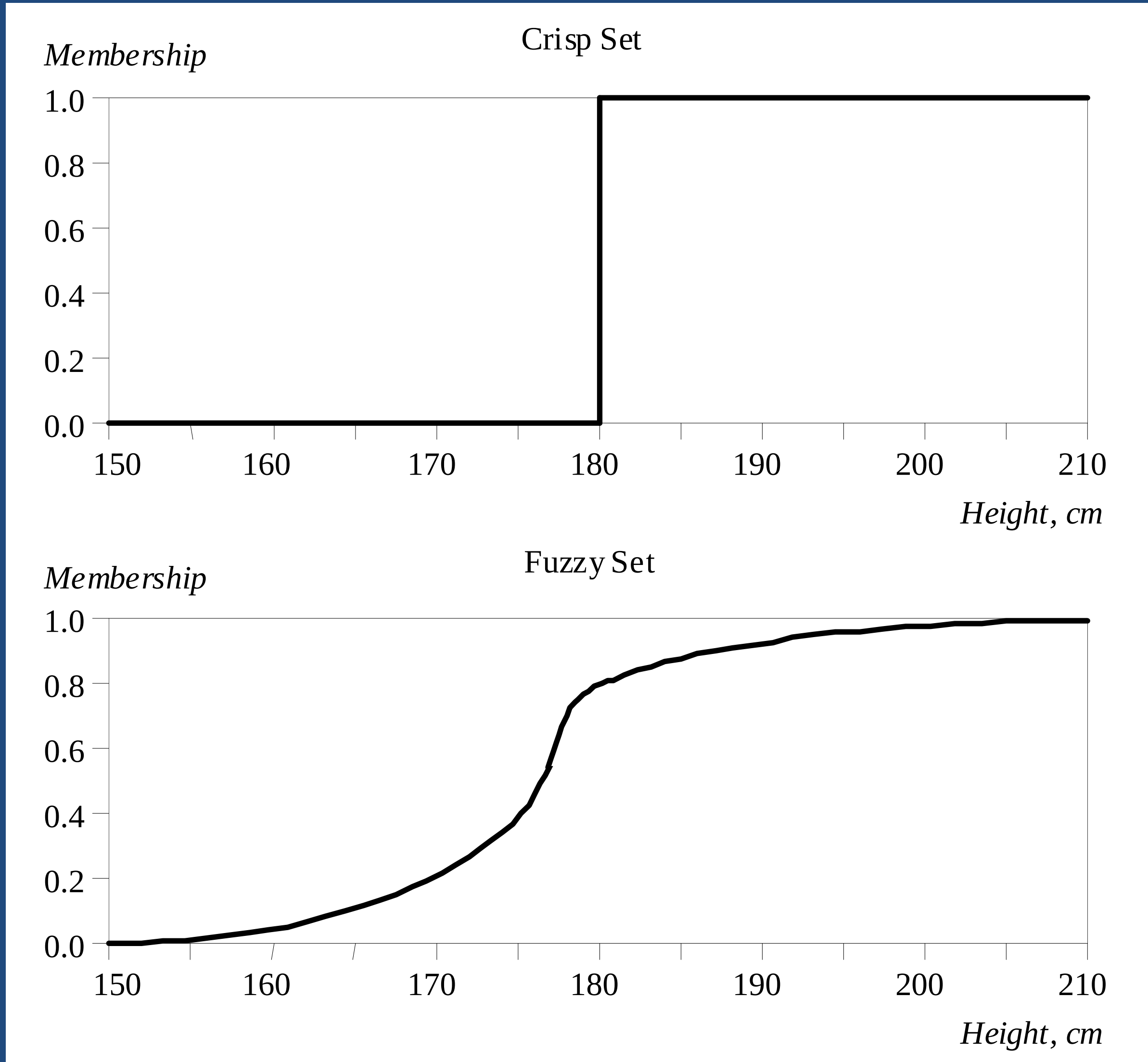
- Classical set theory, **crisp set *A* of *X* is defined as function $f_A(x)$ - characteristic function of *A***:

$f_A(x) : U \rightarrow \{0, 1\}$, where $$f_A(x) = \begin{cases} 1, \text{if } x \in A \\ 0, \text{if } x \notin A \end{cases}$$

- Maps universe *U* to a set of two elements. For any element *x* of universe *U*, characteristic function $f_A(x) = 1$ if *x* is an element of set *A*, and $f_A(x) = 0$ if it is not an element of *A*.

Foundations of Fuzzy Sets – Traditional sets vs Fuzzy sets

- The x-axis represents the **universe of discourse** – the range of all possible values.
- The universe of people's heights consists of all tall people.
- The y-axis represents the **membership value of the fuzzy set**.
- The fuzzy set "tall" maps height values into corresponding membership values.



Foundations of Fuzzy Sets – Traditional sets vs Fuzzy sets

Name	Height(cm)	Membership	
		Crisp	*Fuzzy*
Iestyn	208	1	1.00
Mark	205	1	1.00
Alex	198	1	0.98
Dewi	181	1	0.82
Mari	179	0	0.78
John	172	0	0.24
Don	167	0	0.15
Siân	158	0	0.06
Rob	155	0	0.01
Pedr	152	0	0.00

Foundations of Fuzzy Sets – Traditional sets vs Fuzzy sets

- For fuzzy sets: defined by function $\mu_A(x)$ - the membership function of set A:

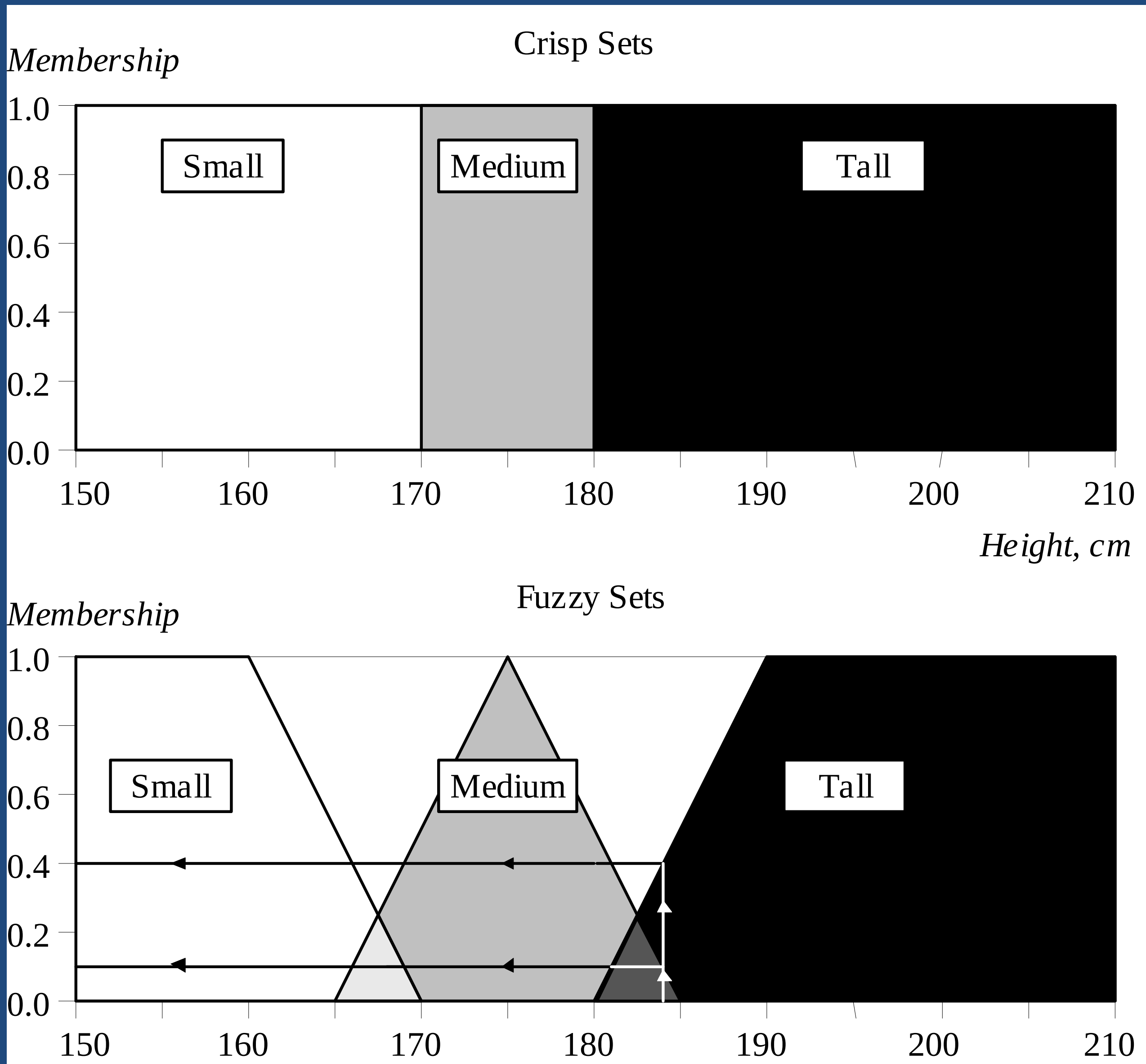
$\mu_A(x)$: U → {0, 1}, where

$\mu_A(x) = 1$ if x is completely in A;
$\mu_A(x) = 0$ if x is not in A;
$0 < \mu_A(x) < 1$ if x is partly in A.

- For any element *x* of universe *U*, membership function $\mu_A(x)$ is degree to which x is an element of set *A*

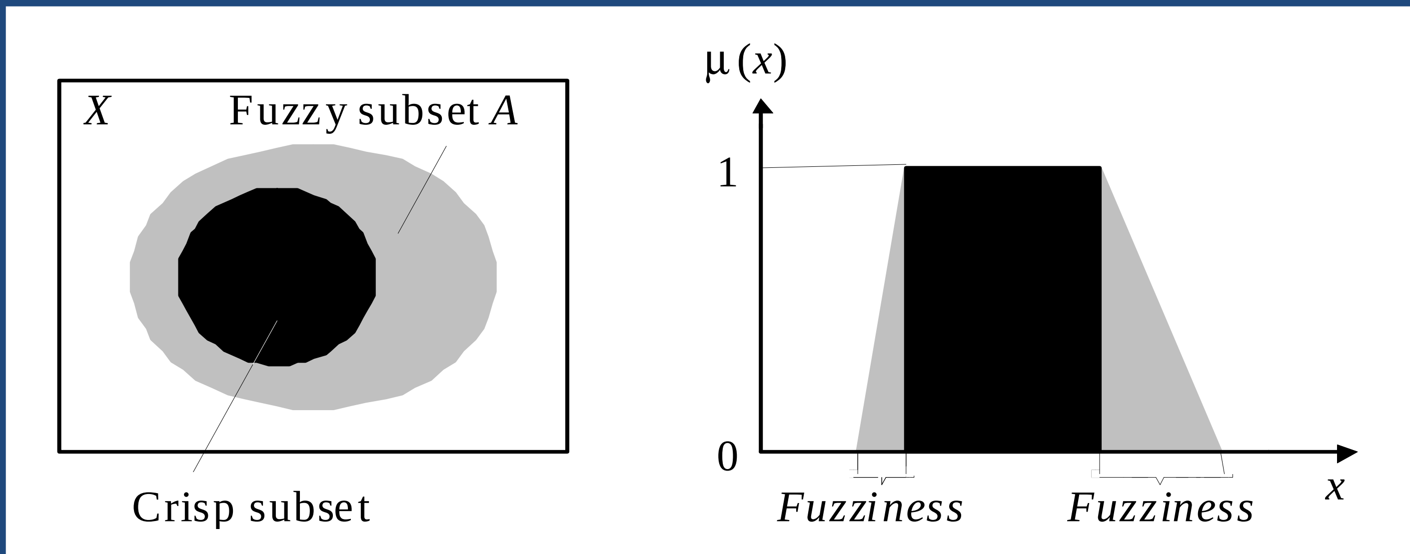
Foundations of Fuzzy Sets

- Take the concept of height further - three sets: *'small'*, *'medium'* and *'tall'*
- Person who is 184 cm - a member of the *average* with a membership of 0.1
- However, is also a member of the *'tall'* set to degree of 0.4.



Foundations of Fuzzy Sets – Fuzzy Membership Functions

- Typical functions that are to represent a fuzzy membership functions - gaussian and sigmoid.
- High computational cost - most applications employ linear fit functions (trapezoidal)



Linguistic Variables

- Central to fuzzy set theory – computing with words
- **Fuzzy variable -** "Alex is tall" implies that the linguistic label/variable *Alex* takes the linguistic value *tall*.
- Especially useful for fuzzy expert systems

e.g.

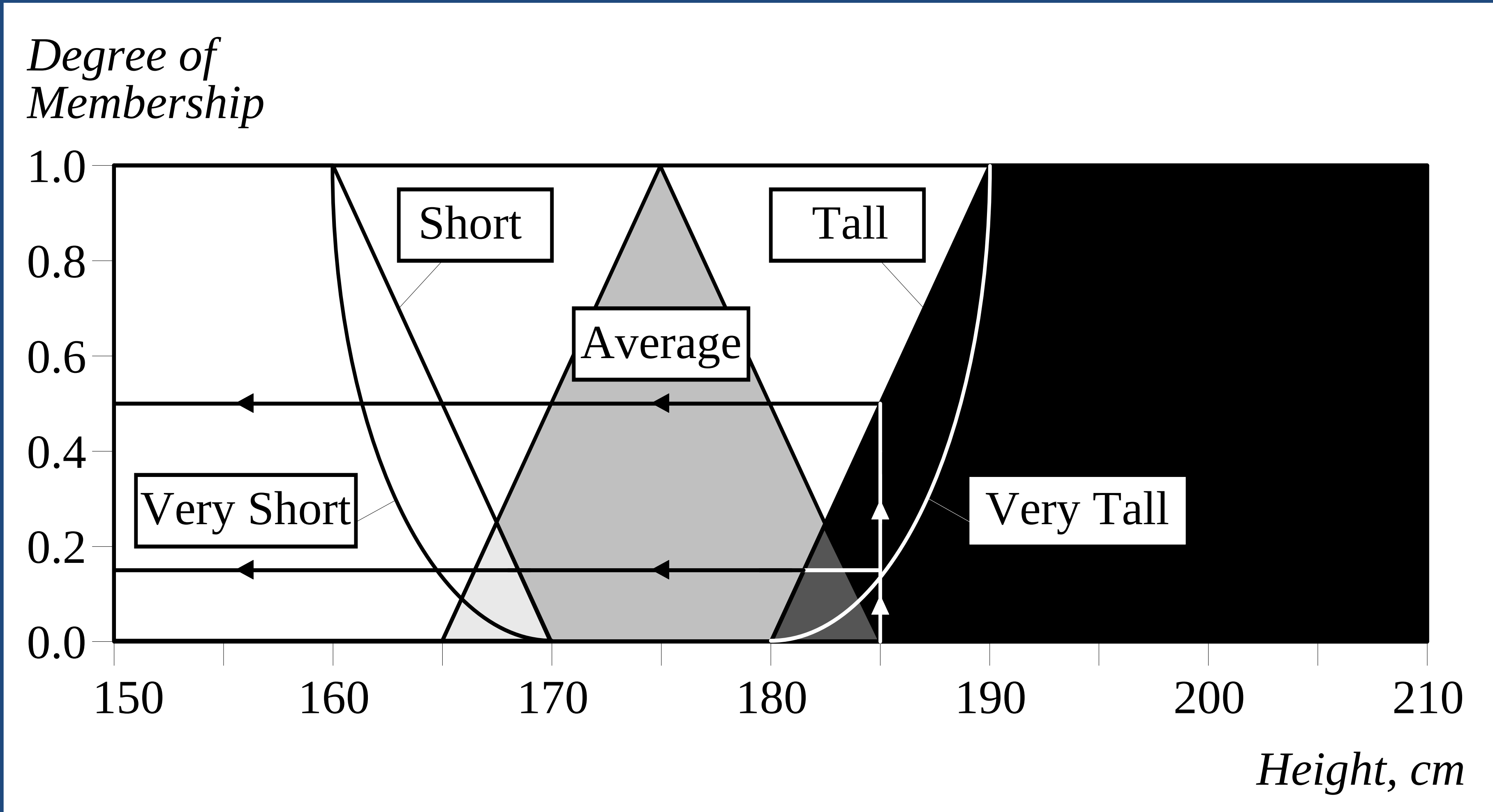
IF	*wind*	is: *strong*
THEN	*sailing*	is: *good*

Linguistic Variables - Hedges

- Range of possible values = the universe of discourse of that variable

- For example tall 150cm - 210cm, has fuzzy sets: ‘small’, ‘medium’ and ‘tall’
- Linguistic fuzzy set qualifiers, called **hedges**.
- Hedges - modify the shape of fuzzy sets.
- Natural language - *very, somewhat, quite, more or less, slightly, etc*.

Linguistic Variables - Hedges



Fuzzy Set Operations

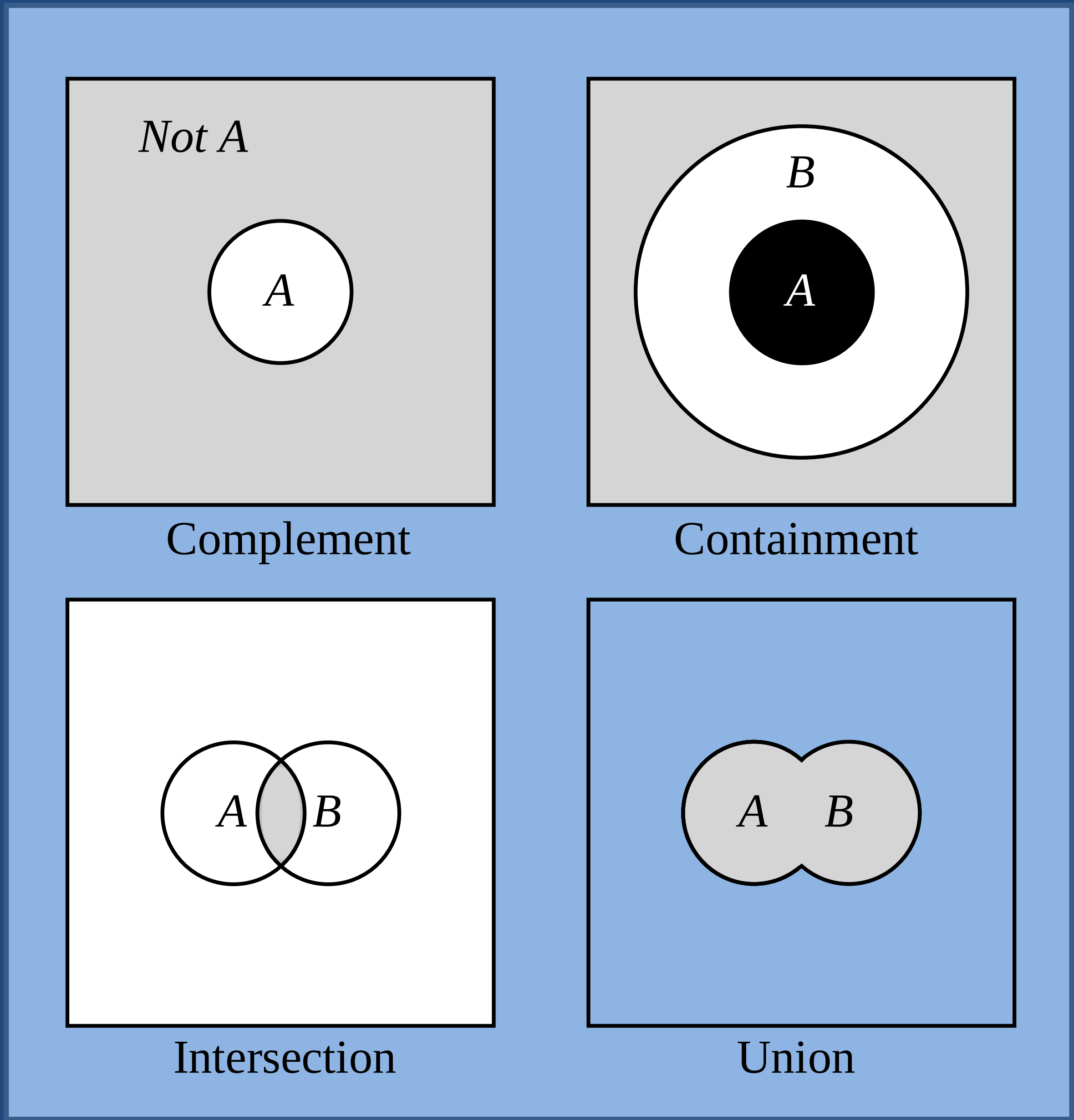
- Usually a fuzzy set is denoted as:

$$A = \mu_A(x_i)/x_i + \ldots\ldots\ldots\ldots + \mu_A(x_n)/x_n$$

where $\mu_A(x_i)/x_i$ is a pair "grade of membership" element, that belongs to a finite universe of discourse:

$$A = \{x_1, x_2, .., x_n\}$$

Fuzzy Set Operations



Fuzzy Set Operations - Complement

- Classical Sets: Which elements do not belong to the set?
- Fuzzy Sets: How much do elements not belong to the set?

- The complement of a set – sometimes called inversion.
- Earlier example: set of tall people, its complement is the set of NOT tall people.
- When we remove the tall people set from the universe = the complement.

- If A is the fuzzy set, its complement $\neg A$ can be found:

$$\mu_{\neg A}(x) = 1 - \mu_A(x)$$

Fuzzy Set Operations - Containment

- Classical Sets: Which sets belong to which other sets?
- Fuzzy Sets: Which sets belong to other sets?

- Russian dolls - a set can contain other sets.

- The smaller set is called the **subset**. For example, the set of tall people contains all tall people; very tall people is a subset of tall men.
- However, the tall people set is just a subset of the set of people.

- In crisp sets, all elements of a subset entirely belong to a larger set. In fuzzy sets, however, each element can belong less to the subset than to the larger set. Elements of the fuzzy subset have smaller memberships in it than in the larger set.

Fuzzy Set Operations - Intersection

- Classical Sets: Which element belongs to both sets?
- Fuzzy Sets: How much of the element is in both sets?

- Crisp sets - intersection between two sets contains elements shared by these sets – e.g. tall people AND thin people

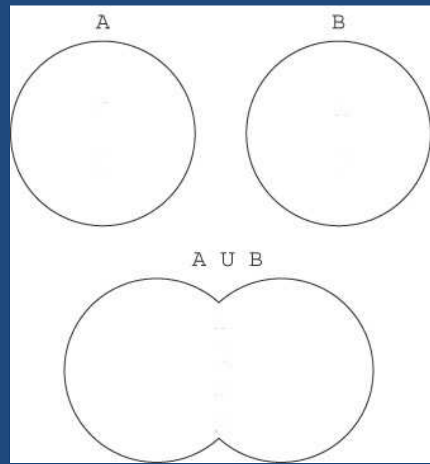
- In fuzzy sets, an element may partly belong to both sets with different memberships.

- A fuzzy intersection is the **lower membership** in both sets of each element. The fuzzy intersection of two fuzzy sets *A* and *B* on *U*:

  $$\mu_{A \cap B}(x) = \min[\mu_A(x), \mu_B(x)] = \mu_A(x) \cap \mu_B(x),$$

  where $x \in U$

Fuzzy Set Operations - Union



- Classical Sets: Which element belongs to either set?
- Fuzzy Sets: How much of the element is in either set?

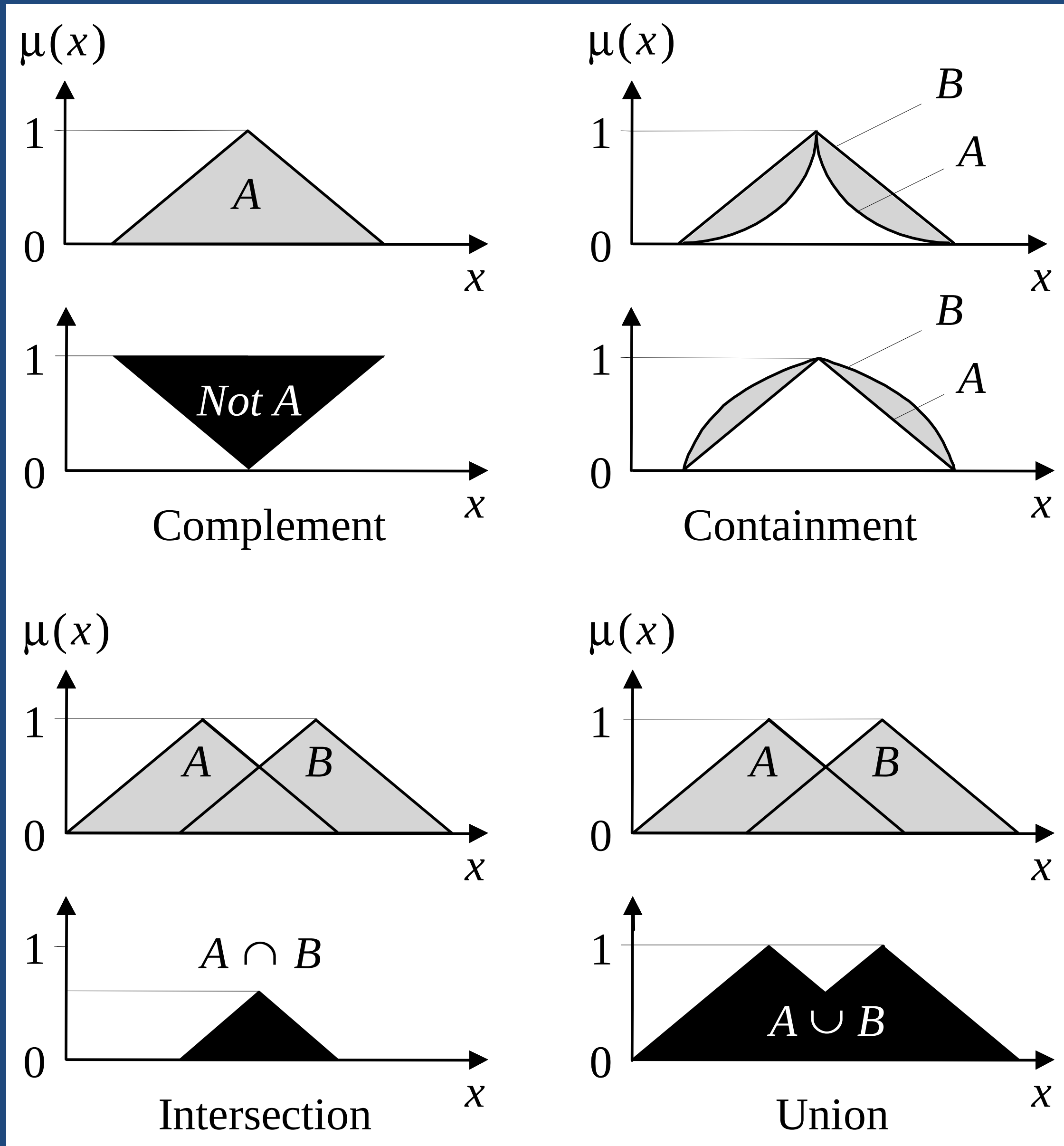
- The union of two crisp sets consists of every element that falls into either set. e.g. tall people OR thin people

- In fuzzy sets, the union is the reverse of the intersection. That is, the union is the **largest membership** value of the element in either set.
- Fuzzy operation for forming the union of two fuzzy sets A and B on universe *U* can be given as:

$$\mu A \cup B(x) = \max [\mu A(x), \mu B(x)] = \mu A(x) \cup \mu B(x),$$

where $x \in U$

Fuzzy Set Operations



Conclusion

- Provide a rich and meaningful addition – excluded middle
- FS are a powerful way of reasoning about the vagueness of the real-world
- Fuzzy sets are complementary to other forms of reasoning rather than competitive