5.3 Triangular fuzzy number

5.3.1 Definition of triangular fuzzy number

Among the various shapes of fuzzy number, triangular fuzzy number(TFN) is the most popular one.

Definition(Triangular fuzzy number) It is a fuzzy number represented with three points as follows :

$A = (a_1, a_2, a_3)$

This representation is interpreted as membership functions(Fig5.6).

$$\mu_{(A)}(x) = \begin{cases} 0, & x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 \le x \le a_3 \\ 0, & x > a_3 \end{cases}$$ □

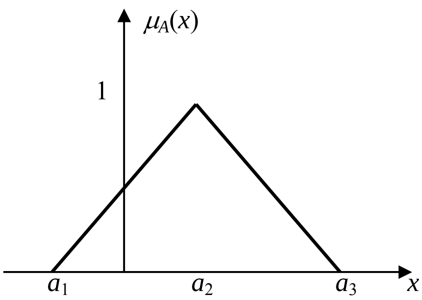


Fig 5.6 Triangular fuzzy number $A = (a_1, a_2, a_3)$

Now if you get crisp interval by α-cut operation, interval A_a shall be obtained as follows $\forall\alpha \in [0, 1]$.

From

$$\frac{a_1^{(\alpha)} - a_1}{a_2 - a_1} = \alpha, \quad \frac{a_3 - a_3^{(\alpha)}}{a_3 - a_2} = \alpha$$

we get

$a_1^{(\alpha)} = (a_2 - a_1)\alpha + a_1$

$a_3^{(\alpha)} = -(a_3 - a_2)\alpha + a_3$

Thus

$$A_\alpha = [a_1^{(\alpha)}, a_3^{(\alpha)}]$$

$$= [(a_2 - a_1)\alpha + a_1, -(a_3 - a_2)\alpha + a_3]$$

Example 5.7 In the case of the triangular fuzzy number A = (−5, −1, 1) (Fig 5.7), the membership function value will be,

$$\mu_{(A)}(x) = \begin{cases} 0, & x < -5 \\ \dfrac{x+5}{4}, & -5 \le x \le -1 \\ \dfrac{1-x}{2}, & -1 \le x \le 1 \\ 0, & x > 1 \end{cases}$$

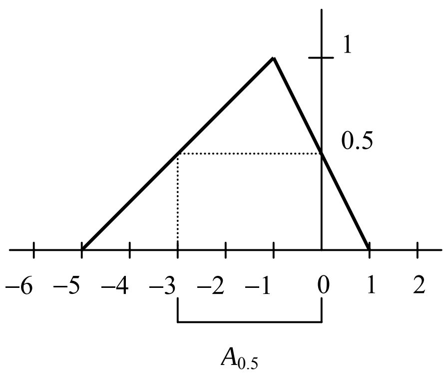


Fig 5.7 $\alpha = 0.5$ cut of triangular fuzzy number $A = (-5, -1, 1)$

α-cut interval from this fuzzy number is

$$\frac{x+5}{4} = \alpha \quad \Rightarrow \quad x = 4\alpha - 5$$

$$\frac{1-x}{2} = \alpha \quad \Rightarrow \quad x = -2\alpha + 1$$

$A_\alpha = [a_1^{(\alpha)}, a_3^{(\alpha)}] = [4\alpha - 5, -2\alpha + 1]$

If $\alpha = 0.5$, substituting 0.5 for α, we get $A_{0.5}$

$A_{0.5} = [a_1^{(0.5)}, a_3^{(0.5)}] = [-3, 0]$ □

5.3.2 Operation of Triangular Fuzzy Number

Same important properties of operations on triangular fuzzy number are summarized.

(1) The results from addition or subtraction between triangular fuzzy numbers result also triangular fuzzy numbers.

(2) The results from multiplication or division are not triangular fuzzy numbers.

(3) Max or min operation does not give triangular fuzzy number.

But we often assume that the operational results of multiplication or division to be TFNs as approximation values.

1) Operation of triangular fuzzy number

First, consider addition and subtraction. Here we need not use membership function. Suppose triangular fuzzy numbers A and B are defined as,

$A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$

i) Addition

$$\begin{aligned} A(+)B &= (a_1, a_2, a_3)(+)(b_1, b_2, b_3) \\ &= (a_1 + b_1, a_2 + b_2, a_3 + b_3) \end{aligned} \quad \text{: triangular fuzzy number}$$

ii) Subtraction

$$\begin{aligned} A(-)B &= (a_1, a_2, a_3)(-)(b_1, b_2, b_3) \\ &= (a_1 - b_3, a_2 - b_2, a_3 - b_1) \end{aligned} \quad \text{: triangular fuzzy number}$$

iii) Symmetric image

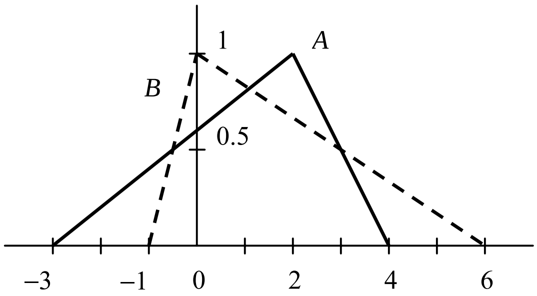
$-(A) = (-a_3, -a_2, -a_1)$: triangular fuzzy number

Example 5.8 Let's consider operation of fuzzy number A, B(Fig 5.8).

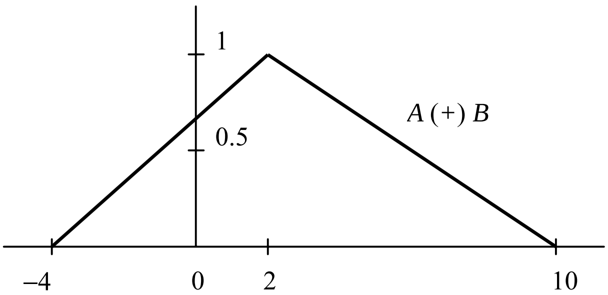
$A = (-3, 2, 4)$, $B = (-1, 0, 6)$

$A\ (+)\ B = (-4, 2, 10)$

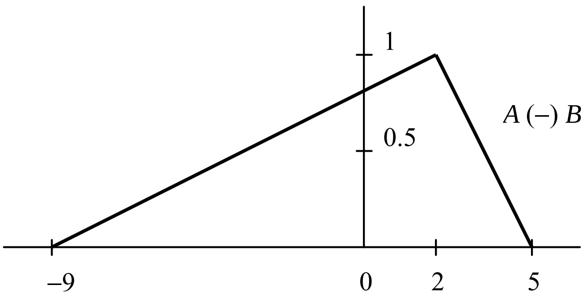
$A\ (-)\ B = (-9, 2, 5)$ □



(a) Triangular fuzzy number *A*, *B*



(b) *A* (+) *B* of triangular fuzzy numbers



(c) *A* (–) *B* triangular fuzzy numbers

Fig 5.8 *A* (+) *B* and *A* (–) *B* of triangular fuzzy numbers

2) Operations with α-cut

Example 5.9 α-level intervals from α-cut operation in the above two triangular fuzzy numbers A and B are

$$
\begin{aligned}
A_\alpha &= [a_1^{(\alpha)}, a_3^{(\alpha)}] = [(a_2 - a_1)\alpha + a_1, -(a_3 - a_2)\alpha + a_3] \\
&= [5\alpha - 3, -2\alpha + 4] \\
B_\alpha &= [b_1^{(\alpha)}, b_3^{(\alpha)}] = [(b_2 - b_1)\alpha + b_1, -(b_3 - b_2)\alpha + b_3] \\
&= [\alpha - 1, -6\alpha + 6]
\end{aligned}
$$

Performing the addition of two α-cut intervals A_α and B_α,

$A_\alpha\,(+)\,B_\alpha = [6\alpha - 4, -8\alpha + 10]$

Especially for $\alpha = 0$ and $\alpha = 1$,

A_0 (+) B_0 = [–4, 10]

A_1 (+) B_1 = [2, 2] = 2

Three points from this procedure coincide with the three points of triangular fuzzy number (-4, 2, 10) from the result A(+)B given in the previous example.

Likewise, after obtaining A_α(–)B_α, let's think of the case when $\alpha = 0$ and $\alpha = 1$.

A_α (–) B_α = $[11\alpha - 9, -3\alpha + 5]$

Substituting $\alpha = 0$ and $\alpha = 1$ for this equation,

A_0 (–) B_0 = [–9, 5]

A_1 (–) B_1 = [2, 2] = 2

These also coincide with the three points of A(–)B = (–9, 2, 5). □

Consequently, we know that we can perform operations between fuzzy number using α-cut interval.

5.3.3 Operation of general fuzzy numbers

Up to now, we have considered the simplified procedure of addition and subtraction using three points of triangular fuzzy number. However, fuzzy numbers may have general form, and thus we have to deal the operations with their membership functions.

Example 5.10 Addition A (+) B

Here we have two triangular fuzzy numbers and will calculate the addition operation using their membership functions.

A = (–3, 2, 4), B = (–1, 0, 6)

$$\mu_{(A)}(x) = \begin{cases} 0, & x < -3 \\ \dfrac{x+3}{2+3}, & -3 \le x \le 2 \\ \dfrac{4-x}{4-2}, & 2 \le x \le 4 \\ 0, & x > 4 \end{cases}$$

$$\mu_{(B)}(y) = \begin{cases} 0, & y < -1 \\ \dfrac{y+1}{0+1}, & -1 \le y \le 0 \\ \dfrac{6-y}{6-0}, & 0 \le y \le 6 \\ 0, & y > 6 \end{cases}$$

For the two fuzzy number $x \in A$ and $y \in B$, $z \in A\ (+)\ B$ shall be obtained by their membership functions.

Let's think when $z = 8$. Addition to make $z = 8$ is possible for following cases :

2 + 6, 3 + 5, 3.5 + 4.5, …

So

$$\begin{aligned} \mu_{A(+)B} &= \bigvee_{8=x+y} [\mu_A(2) \wedge \mu_B(6),\ \mu_A(3) \wedge \mu_B(5),\ \mu_A(3.5) \wedge \mu_B(4.5), \cdots] \\ &= \vee\ [1 \wedge 0,\ 0.5 \wedge 1/6,\ 0.25 \wedge 0.25, \cdots] \\ &= \vee\ [0, 1/6,\ 0.25, \cdots] \end{aligned}$$

If we go on these kinds of operations for all $z \in A\ (+)\ B$, we come to the following membership functions, and these are identical to the three point expression for triangular fuzzy number $A = (-4, 2, 10)$.

$$\mu_{A(+)B}(z) = \begin{cases} 0, & z < -4 \\ \dfrac{z+4}{6}, & -4 \le z \le 2 \\ \dfrac{10-z}{8}, & 2 \le z \le 10 \\ 0, & z > 10 \end{cases} \qquad \square$$

There in no simple method using there point expression for multiplication or division operation. So it is necessary to use membership functions.

Example 5.11 Multiplication $A\ (\bullet)\ B$

Let triangular fuzzy numbers A and B be

$A = (1, 2, 4)$, $B = (2, 4, 6)$

$$\mu_{(A)}(x) = \begin{cases} 0, & x < 1 \\ x-1, & 1 \le x < 2 \\ -\frac{1}{2}x+2, & 2 \le x < 4 \\ 0, & x \ge 4 \end{cases}$$

$$\mu_{(B)}(y) = \begin{cases} 0, & y < 2 \\ \frac{1}{2}y-1, & 2 \le y < 4 \\ -\frac{1}{2}y+3, & 4 \le y < 6 \\ 0, & y \ge 6 \end{cases}$$

Calculating multiplication A (•) B of A and B, $z = x \bullet y = 8$ is possible when $z = 2 \bullet 4$ or $z = 4 \bullet 2$

$$\begin{aligned} \mu_{A(\bullet)B} &= \vee_{x \bullet y=8} [\mu_A(2) \wedge \mu_B(4),\ \ \mu_A(4) \wedge \mu_B(2), \cdots] \\ &= \vee [1 \wedge 1,\ \ 0 \wedge 0, \cdots] \\ &= 1 \end{aligned}$$

Also when $z = x \bullet y = 12$, $3 \bullet 4$, $4 \bullet 3$, $2.5 \bullet 4.8$, … are possible.

$$\begin{aligned} \mu_{A(\bullet)B} &= \vee_{x \bullet y=12} [\mu_A(3) \wedge \mu_B(4),\quad \mu_A(4) \wedge \mu_B(3),\quad \mu_A(2.5) \wedge \mu_B(4.8), \cdots] \\ &= \vee [0.5 \wedge 1,\quad 0 \wedge 0.5,\quad 0.75 \wedge 0.6,\quad \cdots] \\ &= \vee [0.5,\quad 0,\quad 0.6,\quad \cdots] \\ &= 0.6 \end{aligned}$$

From this kind of method, if we come by membership function for all $z \in A$ (•) B, we see fuzzy number as in Fig 5.9. However, since this shape is in curve, it is not a triangular fuzzy number. For convenience, we can express it as a triangular fuzzy number by approximating A (•) B.

$$A(\bullet)B \cong (2,\quad 8,\quad 24)$$

We can wee that two end points and one peak point are used in this approximation. □

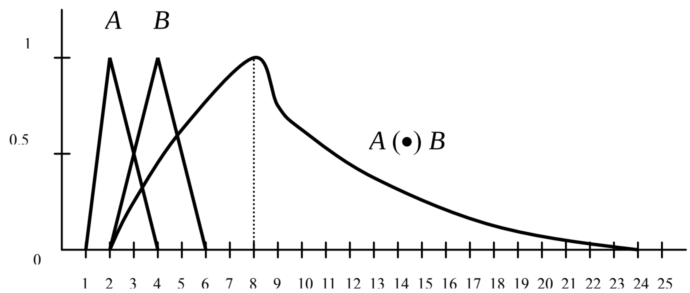


Fig 5.9 Multiplication A (•) B of triangular fuzzy number

5.3.4 Approximation of Triangular Fuzzy Number

Since it is possible to express approximated values of multiplication and division as triangular fuzzy numbers, we are now up to the fact that how to get this approximated value easily.

Example 5.12 Approximation of multiplication

First, α-cuts of two fuzzy numbers are our main concern.

$A = (1, 2, 4)$, $B=(2, 4, 6)$

$$\begin{aligned} A_\alpha &= [(2-1)\alpha+1, \quad -(4-2)\alpha+4] \\ &= [\alpha+1, \quad -2\alpha+4] \\ B_\alpha &= [(4-2)\alpha+2, \quad -(6-4)\alpha+6] \\ &= [2\alpha+2, \quad -2\alpha+6] \end{aligned}$$

For all $\alpha \in [0, 1]$, multiply A_α with B_α which are two crisp intervals. Now in $\alpha \in [0, 1]$, we see that elements of each interval are positive numbers. So multiplication operation of the two intervals is simple.

$$\begin{aligned} A_\alpha(\bullet)B_\alpha &= [\alpha+1, \quad -2\alpha+4](\bullet)[2\alpha+2, \quad -2\alpha+6] \\ &= [(\alpha+1)(2\alpha+2), \quad (-2\alpha+4)(-2\alpha+6)] \\ &= [2\alpha^2+4\alpha+2, \quad 4\alpha^2-20\alpha+24] \end{aligned}$$

When $\alpha = 0$,

$$A_0(\bullet)B_0 = [2, \quad 24]$$

When $\alpha = 1$,

$A_0(\bullet)B_1 = [2+4+2, 4-20+24] = [8, 8] = 8$

We obtain a triangular fuzzy number which is an approximation of $A (\bullet) B$ (Fig 5.9).

$A(\bullet)B \cong (2, \quad 8, \quad 24)$ □

Example 5.13 Approximation of division

In the similar way, let's express approximated value of A (/) B in a triangular fuzzy number. First, divide interval A_α by B_α. We reconsider the sets A and B in the previous example. For $\alpha \in [0, 1]$, since element in each interval has positive number, we get A_α (/) B_α as follows.

$$A_\alpha(/)B_\alpha = [(\alpha+1)/(-2\alpha+6), \quad (-2\alpha+6)/(2\alpha+2)]$$

When $\alpha = 0$,

$$
\begin{aligned}
A_0(/)B_0 &= [1/6, \quad 4/2] \\
&= [0.17, \quad 2]
\end{aligned}
$$

When $\alpha = 1$,

$$
\begin{aligned}
A_1(/)B_1 &= [(1+1)/(-2+6), \quad (-2+4)/(2+2)] \\
&= [2/4, \quad 2/4] \\
&= 0.5
\end{aligned}
$$

So the approximated value of A (/) B will be

$$A(/)B \quad = \quad (0.17, \quad 0.5, \quad 2) \qquad \square$$