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A characterization of a general class of ranking functions on triangular fuzzy numbers

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Abstract

This paper aims to the discussion of properties for ranking fuzzy numbers. An approach of axiomatic type is followed providing the definition of two groups of requirements which characterize a ranking function through its behaviour on triangular fuzzy numbers. Particularly, the problem of making the evaluations on triangular fuzzy numbers sensitive to their spread has been deeply analyzed.

Keywords: Triangular fuzzy numbers; Ranking functions; Ambiguity

1. Introduction

The problem of ordering fuzzy numbers has been studied by many authors who have followed quite different methods [2]. One straightforward idea is to transform fuzzy numbers into real numbers, with the purpose of inducing on the set of the involved fuzzy numbers the order of the real line. The instruments of the transformation are called ranking functions and generally they arise from intuitions of different nature (geometrical, possibilistic, etc.). For instance, in [12], given an arbitrary triangular fuzzy number (TFN), we define both its pessimistic and optimistic alternative as basic notions for generating an order.

The aim of this paper is to formulate some axiomatic requirements in order to qualify an arbitrary ranking function through its behaviour on the remarkable subdomain of the TFNs. A first attempt in this direction is present in [14], but here we generalize the class of functions considered and propose a different approach in determining some of the axioms.

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We will present four axioms divided into two groups: the first is conceived as essential for any ranking function to be considered acceptable at least in the above-cited subdomain; the second attempts to characterize, in a more specific way than the first, a reasonable behaviour of a ranking function through the idea to weight, in some way, the spread of a TFN.

In Section 2, besides an in-depth discussion on such requirements, we formalize an idea on which many of the ranking functions present in literature are based, when applied to TFN: at first, it consists in two independent evaluations of the right (RTFN) and left side (LTFN) of a TFN, then in the connection of the partial results through a real parameter, indicator of the pessimism–optimism of the decision maker.

In Section 3, we present a large review of the indexes found in current literature, showing that almost all fulfill our axioms.

Finally, in Section 4, we illustrate a general method of building ranking functions which extend a classical case and satisfy all our axioms.

2. A set of essential axioms

In this work, we say that a fuzzy subset Ω of $\mathbb{R}_0^+ = [0,\infty[$, with membership function μ_Ω, is a fuzzy number iff

(i) $\Omega_\alpha = \{x \in \mathbb{R}_0^+ : \mu_\Omega(x) \geqslant \alpha\}$ is a convex set, denoted by $[a_\alpha, b_\alpha]$;
(ii) $\mu_\Omega(\cdot)$ is an upper semicontinuous function;
(iii) Ω is normal, i.e. it exists $m \in \mathbb{R}_0^+$ such that $\mu_\Omega(m) = 1$;
(iv) $supp(\Omega) = \{x \in \mathbb{R}_0^+ : \mu_\Omega(x) > 0\}$ is a bounded set of $\mathbb{R}_0^+$.

Particularly, A is a TFN, if its membership function μ_A is of the following type:

$$\mu_A(z) = \begin{cases} 0, & 0 \leqslant z \leqslant a_1, \\ \dfrac{z-a_1}{a_2-a_1}, & a_1 \leqslant z \leqslant a_2, \\ \dfrac{z-a_3}{a_2-a_3}, & a_2 \leqslant z \leqslant a_3, \\ 0, & z \geqslant a_3 \end{cases}$$

with $0 \leqslant a_1 \leqslant a_2 \leqslant a_3$, but not at the same time $a_1 = a_2 = a_3$, except for the *crisp* numbers. More simply, we will identify a TFN A with the triplet (a_1, a_2, a_3). When A only has a RTFN, that is when $a_1 = a_2 < a_3$, we will replace the symbol A with A_r; symmetrically, we will denote by A_l a TFN with only a LTFN, that is when $a_2 = a_3 > a_1$.

Let $\mathscr{H}$ be an assigned set of fuzzy numbers such that $\mathscr{H} \supseteq \mathscr{A}$, where $\mathscr{A}$ is the class of all the TFNs on $\mathbb{R}_0^+$. Let F be a ranking function on $\mathscr{H}$, briefly $F \in RF(\mathscr{H})$: it is intended that $F : \mathscr{H} \to \mathbb{R}$ induces on its domain an ordering formulated by

$$\Omega \succ \Omega' \quad \text{iff } F(\Omega) > F(\Omega'),$$

$$\Omega \sim \Omega' \quad \text{iff } F(\Omega) = F(\Omega')$$

for every $\Omega, \Omega' \in \mathscr{H}$. Clearly, $\Omega \succcurlyeq \Omega'$ by F iff $F(\Omega) \geqslant F(\Omega')$. We remark the fact that when F is applied only to TFNs, in brief $F/\mathscr{A}$, it depends only on the parameters a_1, a_2, a_3. In the following, we will always suppose that $F/\mathscr{A} = F/\mathscr{A}\ (a_1, a_2, a_3)$ is at least of class C^1. Our conviction is that TFNs play a key role, because we consider the behaviour of an arbitrary $F \in RF(\mathscr{H})$ as reasonable, if it verifies some minimal requirements just in the subdomain $\mathscr{A}$. The axiomatic conditions which every $F \in RF(\mathscr{H})$ must satisfy are

(A_1) $F(A) \in supp(A)$ for every $A \in \mathscr{A}$,
(A_2) $F/\mathscr{A}$ is strictly increasing in the variables a_1, a_2, a_3.

Remark 1. The first axiom is actually a very weak condition, satisfied by almost all the RFs presented in literature. The implicit meaning of such a RF is actually an instrument of defuzzification: given any TFN A, whatever is represented on the x-axis, has a degree of reliability different from zero only within the interval $[a_1, a_3]$ and must be necessarily "summarized" by a value belonging to the $supp(A)$.

Remark 2. Given an arbitrary $F \in RF(\mathscr{H})$, suppose that $F/\mathscr{A}$ is at least continuous in the variables a_1, a_2, a_3 and satisfying (A_1). Then, when F is applied to a crisp number, it has to restore the same value. It is interesting to note that this obvious consequence of (A_1), which could be considered as an accepted fact, is not verified by some ranking approaches as the Chang index [5], which assigns to any (normal) TFN A the value $(a_1 + a_2 + a_3)(a_3 - a_1)/6$ and associates zero to any crisp number, or the one proposed in [6]. Namely, its RF, when applied to a (normal) TFN A, is exactly $F(A) = \sqrt{x_c^2 + y_c^2}$, where x_c and y_c are, respectively, the x and y coordinates of the centroid of A, i.e.

$$x_c = \frac{a_1 + a_2 + a_3}{3}, \qquad y_c = \frac{1}{3}\left(1 + \frac{2a_2}{a_1 + 2a_2 + a_3}\right).$$

When a TFN A tends to the crisp number given by its central value, that is when $a_1, a_3 \to a_2$, then $F(A) \to \sqrt{a_2^2 + \frac{1}{4}}$, which is clearly greater than a_2.

Remark 3. The second axiom is based upon the crucial assumption that what is represented on the x-axis (for instance, monetary profits) is more and more preferable as it moves rightwards (for this reason, we have chosen to treat fuzzy numbers on $\mathbb{R}_0^+$). In this case, it is evident that a TFN $A_h = (a_1 + h, a_2, a_3)$ (or $A_h = (a_1, a_2, a_3 + h)$) must have an higher evaluation than $A = (a_1, a_2, a_3)$ for any arbitrarily small $h > 0$. It could be only a little more controversial that $A_h = (a_1, a_2 + h, a_3)$ has to be preferred to A, but, as a matter of fact, also in this situation, all the ranking methods found in literature are in accordance with our axiom (see [20, Table 3, Example (a)]). On the other hand, we are in agreement with Tran–Duckstein (TD) (see [20]), who state that, in cases really controversial such as different TFNs symmetrical to the same central value, a "good" ranking method should allow to obtain different orderings, depending on the attitude of the decision maker (e.g. risk-prone or risk-averse). Therefore, we consider empty of meaning rules as "higher mean value and at the same time lower spread" (see [18]), which would always assign, in the just mentioned case, the higher evaluation to the smallest TFN. Among the very few examples of RFs which do not satisfy (A_2) (and really not so easy to accept in all their consequences) we recall the examples seen in the previous remark: in fact, concerning Chang index, we have that $\partial F/\partial a_1 < 0$ for all a_i, $i = 1, 2, 3$, when A is a TFN with $a_2 < 2a_1$, while the second case yields $\partial F/\partial a_3 < 0$ for all a_i, $i = 1, 2, 3$, when A is a TFN symmetrical to its central value with $a_2 < \frac{1}{4}$.

Suppose now to consider a generic $G \in RF(\mathscr{H})$ and to focus on its behaviour in the subdomain $\mathscr{A}$. Our purpose is to give a representation of $G/\mathscr{A}$ through a real parameter $\lambda \in [0, 1]$ and an opportune $F \in RF(\mathscr{H})$ which verifies (A_1) but not necessarily (A_2). In particular, we think of a representation of the type

$$G(A) = \lambda F(A_r) + (1 - \lambda) F(A_l), \quad A \in \mathscr{A}. \tag{1}$$

Note that, if we apply any $F \in RF(\mathscr{H})$, satisfying at least (A_1), to an arbitrary $A_r = (a_2, a_3)$, it is easy to see that it is always writable in the form

$$F(a_2, a_3) = a_2 + \rho(a_2, a_3)(a_3 - a_2), \tag{2}$$

where $\rho = \rho(a_2, a_3)$ is a C^1 function from $D = \{(v, \xi) : v, \xi \in \mathbb{R},\ \xi > v \geqslant 0\}$ to $]0, 1[$. Symmetrically, applying F to any $A_l = (a_1, a_2)$, it can be reduced to the form

$$F(a_1, a_2) = a_2 - \sigma(a_1, a_2)(a_2 - a_1), \tag{3}$$

where $\sigma = \sigma(a_1, a_2)$ is a C^1 function from D to $]0, 1[$.

Remark 4. We point out that F is required to satisfy at least (A_1), but not necessarily (A_2): this depends on the fact that only the first axiom is necessary to ensure the absolutely general form illustrated in (2), (3). Further, a representation of type (1) and the assumption of (A_1) on F easily imply that also G fulfills (A_1). Conversely, G could verify (A_2) even if such requirement is not valid for F, as we shall see in the next section.

Now, we are able to write (1) as follows:

$$G(a_1, a_2, a_3) = a_2 + \lambda \rho(a_2, a_3)(a_3 - a_2) - (1 - \lambda)\sigma(a_1, a_2)(a_2 - a_1), \quad A \in \mathscr{A}. \tag{4}$$

In this case, we say that the pair (F, λ) is a *generator* of $G/\mathscr{A}$. Note that, given any $\lambda \in [0, 1]$, (G, λ) cannot be the generator of $G/\mathscr{A}$: in fact, pair $(G, 1)$ would verify (1) only for RTFNs, while $(G, 0)$ only for LTFNs. Form (1) is based on the crucial idea that it is always possible to divide the evaluation process of a "complete" TFN A, where complete means $a_1 < a_2 < a_3$, in two separate and independent subcases, i.e. the evaluations of the right and left side, or optimistic and pessimistic side. Once we have calculated the results due to the right and left side, we derive the final evaluation through a convex combination of such partial results by means of a "weight" λ, which represents an optimism–pessimism indicator selected by the decision maker. We emphasize the fact that most RFs proposed in literature, when applied to (normal) TFNs, actually admit the form shown in (4). Evidently, this means that the philosophy of this kind of representation is quite natural and it will be followed also in the example of the last section.

A generic $F \in RF(\mathscr{H})$, satisfying at least (A_1), applied to any TFN A, is always writable in the form

$$F(a_1, a_2, a_3) = a_1 + \varphi(a_1, a_2, a_3)(a_3 - a_1),$$

where $\varphi = \varphi(a_1, a_2, a_3)$, the *core* of $F/\mathscr{A}$, is a C^1 function to be valued on $]0, 1[$.

Remark 5. From (2) and (3), it is easy to obtain that $\varphi(a_2, a_2, a_3) = \rho(a_2, a_3)$ and $\varphi(a_1, a_2, a_2) = 1 - \sigma(a_1, a_2)$. Hence, we could call ρ and $1 - \sigma$, respectively, the right and left core of $F/\mathscr{A}$.

Remark 6. Given any $G \in RF(\mathscr{H})$, assume that $G/\mathscr{A}$ is generated by the pair (F, λ). Then, the core of $G/\mathscr{A}$, denoted by $\varphi_\lambda = \varphi_\lambda(a_1, a_2, a_3)$, is

$$\varphi_\lambda(a_1, a_2, a_3) = \rho_\lambda(a_2, a_3)\frac{a_3 - a_2}{a_3 - a_1} + (1 - \sigma_\lambda(a_1, a_2))\frac{a_2 - a_1}{a_3 - a_1}, \tag{5}$$

where $\sigma_\lambda \equiv (1 - \lambda)\sigma$ and $\rho_\lambda \equiv \lambda\rho$.

It is our conviction that $\varphi(A)$ must be sensitive to the spread of A, but the point is: is it possible to request a priori a specific behaviour of φ as "reasonable"? Let us try to explain better this point: assume that

(A_3) $\dfrac{\partial\varphi(a_1, .)}{\partial a_1} \leqslant 0$, for all a_1, a_2, a_3,

(A_4) $\dfrac{\partial\varphi(., a_3)}{\partial a_3} \leqslant 0$, for all a_1, a_2, a_3,

where $\varphi(., a_i)$ conventionally means that the mapping depends only on the explicitly written variable. The meaning of (A_3) and (A_4) has to be searched in their strong influence on the rapidity of growth of F with regard to the variables a_1 and a_3. In fact, (A_3) implies that

$$F(a_1, .) \geqslant Ka_1$$

for some $K > 0$ and a_1 sufficiently great, while from (A_4) we can easily derive that

$$F(., a_3) \leqslant Ka_3$$

for some $K > 0$ and $a_3 \to \infty$.

Let us try to relate the spread of a TFN A to its associated "uncertainty": on the same line as Delgado–Vila–Voxman [8,9], we could interpret the spread of A as its *ambiguity*, i.e. the imprecision in determining the exact value of the magnitude represented by A (see also [17], where the concept of ambiguity is subdivided into three particular terms, each one with a more stringent meaning. Relatively to our case, the most appropriate is certainly "non-specificity").

With this in mind, we note that, as a_3 increases, the ambiguity of A also grows, so the request to slow down the rapidity of growth of the evaluation of A appears reasonable, while an opposite request seems acceptable as a_1 increases and consequently the ambiguity of A decreases. In the same way, we could argue for (strict) increasing monotonicity of φ in a_2, but this request is already contained in (A_2). As we said before, (A_3) and (A_4) could appear reasonable, but these two properties have not to be taken for granted in every case: the next step, in the following section, is to examine the fulfillment of RFs found in current literature with respect to (A_3) and (A_4) to figure out how many authors, more or less consciously, have shared with us the "rationality" of these two requirements. Finally, from now on, we will distinguish the two groups of axioms so far defined as the *basic* (A_1) –(A_2) and the *reasonable* (A_3)–(A_4), precising that the term "reasonable" is not absolutely used as unquestionable.

3. A review of ranking functions on TFNs

In the first part of this section, we will show that many of the RFs found in literature satisfy the basic axioms and, applied to TFNs, are generated by a pair (F, λ) in which right and left core are constant, that is not depending on the variables a_1, a_2, a_3. We assert that such RFs fulfill the reasonable axioms. In fact, for general ρ_λ and σ_λ, from (5) we easily get

$$\frac{\partial \varphi_\lambda}{\partial a_1} = \frac{a_3 - a_2}{(a_3 - a_1)^2}(\rho_\lambda + \sigma_\lambda - 1) - \frac{a_2 - a_1}{a_3 - a_1}(1 - \lambda)\frac{\partial \sigma}{\partial a_1}$$

and

$$\frac{\partial \varphi_\lambda}{\partial a_3} = \frac{a_2 - a_1}{(a_3 - a_1)^2}(\rho_\lambda + \sigma_\lambda - 1) + \frac{a_3 - a_2}{a_3 - a_1}\lambda\frac{\partial \rho}{\partial a_3}.$$

Under the assumptions

$$\frac{\partial \sigma(a_1, a_2)}{\partial a_1} = \frac{\partial \rho(a_2, a_3)}{\partial a_3} = 0,$$

that certainly include the special case in which ρ and σ are constant, the previous derivatives reduce to

$$\frac{\partial \varphi_\lambda}{\partial a_1} = \frac{a_3 - a_2}{(a_3 - a_1)^2}(\rho_\lambda + \sigma_\lambda - 1)$$

and

$$\frac{\partial \varphi_\lambda}{\partial a_3} = \frac{a_2 - a_1}{(a_3 - a_1)^2}(\rho_\lambda + \sigma_\lambda - 1)$$

so that, according to the clear negativity of the term $\rho_\lambda + \sigma_\lambda - 1$, our assertion is proved.

The most complete and explicit example of RF which, restricted to TFNs, is generated by a pair as in (1), is given by Campos–Gonzalez (CG) in [3]. Their index, applied to any fuzzy number Ω, is

$$V_P^\lambda(\Omega) = \int_0^1 (\lambda b_\alpha + (1 - \lambda)a_\alpha)\,\mathrm{d}P(\alpha),$$

where P is a probability measure on $[0, 1]$, for some $\lambda \in [0, 1]$. When we apply V_P^λ to a TFN A and P is a normalized Stieltjes measure, generated by a strictly increasing, continuous function $s : [0, 1] \to [0, 1]$ such that $s(0) = 0$ and $s(1) = 1$, in symbols $s \in \mathcal{N}$, we find that

$$V_s^\lambda(A) \equiv V_P^\lambda(A) = a_2 + \lambda k(s)(a_3 - a_2) - (1 - \lambda)k(s)(a_2 - a_1), \tag{6}$$

where $k(s) = \int_0^1 s(t)\,dt$. By the evident fact that $k(s) \in]0,1[$, it is immediate to see that V_s^λ verifies the basic axioms. Then, after some calculations, it is possible to show that a possible pair generator of $V_s^\lambda/\mathscr{A}$ is given by (F_s, λ), where

$$F_s(\Omega) = k(s)(a_0 + b_0) + (\tfrac{1}{2} - k(s))(a_1 + b_1), \quad \Omega \in \mathscr{H}. \tag{7}$$

We emphasize that, in general, such F_s fulfills (A_1), but not (A_2): namely, as

$$F_s(A) = k(s)(a_1 + a_3) + (1 - 2k(s))a_2, \quad A \in \mathscr{A}$$

we find that $\partial F_s/\partial a_2 < 0$ when $k(s) > \frac{1}{2}$. Note that the CG model could appear restrictive, because, comparing (6) to (4), we can immediately deduce that ρ and σ are always coincident, exactly equal to $k(s)$. Further, when we use the Lebesgue measure, corresponding to $s(t) = t$, we get $\rho = \sigma = \frac{1}{2}$. Actually, the list of different RFs which reduce to a special case (or generate the same ordering) of CG, not only when applied to (normal) TFNs, but also in more general cases, is very long: see, for instance, the indexes proposed by Adamo [1], Carlsson–Fuller [4], Choobineh–Li [7], Delgado–Vila–Voxman [8], Dubois–Prade [11], Facchinetti–Ghiselli Ricci–Muzzioli [12], Fortemps–Roubens [13], Heilpern [15,16], Liou–Wang [19], Yager [23,24], Yao–Wu [26] and a review of Wang–Kerre [21,22] for a careful comparison. Here, we recall explicitly only two different approaches due to Yager: the first, called centroid method, is given by

$$Y_1(\Omega) = \frac{\int x\mu_\Omega(x)\,dx}{\int \mu_\Omega(x)\,dx}, \quad \Omega \in \mathscr{H}.$$

When we apply Y_1 to the subdomain $\mathscr{A}$, we find

$$Y_1(A) = \frac{a_1 + a_2 + a_3}{3},$$

therefore, given any $s \in \mathscr{N}$ such that $k(s) = \frac{2}{3}$ (for instance, $s(t) = \sqrt{t}$), it is immediate to see that $Y_1/\mathscr{A} = V_s^{1/2}/\mathscr{A}$. Hence, from (7), we deduce that $Y_1/\mathscr{A}$ is generated by $(F_s, \frac{1}{2})$, where

$$F_s(A) = \tfrac{2}{3}(a_1 + a_3) - \tfrac{1}{3}a_2, \quad A \in \mathscr{A}$$

for every $s \in \mathscr{N}$ such that $k(s) = \frac{2}{3}$. The second approach, proposed by Yager and Filev (YF) (see [10,25]), is given by

$$Y_2(\Omega) = \frac{1}{2}\,\frac{\int_0^1 (a_\alpha + b_\alpha) f(\alpha)\,d\alpha}{\int_0^1 f(\alpha)\,d\alpha}, \quad \Omega \in \mathscr{H},$$

where f is an assigned "weight" mapping from $[0,1]$ to $[0,1]$, generally continuous and positive. In this case, we have, for every $A \in \mathscr{A}$,

$$Y_2(A) = a_2 + \frac{1}{2}\,\frac{\int_0^1 (1-\alpha) f(\alpha)\,d\alpha}{\int_0^1 f(\alpha)\,d\alpha}(a_3 - a_2) - \frac{1}{2}\,\frac{\int_0^1 (1-\alpha) f(\alpha)\,d\alpha}{\int_0^1 f(\alpha)\,d\alpha}(a_2 - a_1).$$

Hence, given any $s \in \mathcal{N}$ such that

$$k(s) = \frac{\int_0^1 (1-\alpha) f(\alpha)\,\mathrm{d}\alpha}{\int_0^1 f(\alpha)\,\mathrm{d}\alpha}$$

it is not difficult to verify that $Y_2/\mathcal{A} = V_s^{1/2}/\mathcal{A}$ and consequently there exists at least a pair $(F_s, \frac{1}{2})$ generator of $Y_2/\mathcal{A}$ with F_s as in (7). It is very interesting that the role of the weight mapping f in YF is the same played by the introduction of other integrals than the Lebesgue one in CG, i.e. fundamentally to generalize the standard case, obtained choosing $f \equiv 1$ in YF and the Lebesgue measure in CG, in which right and left core of $Y_2/\mathcal{A}, V_s^{\lambda}/\mathcal{A}$ are equal to $\frac{1}{2}$.

In the second part of this section, we will analyse in detail the approach proposed by TD (see [20]), which appears deeply different from CG. The method of TD consists in a comparison of distance D_f, where f is a weight mapping, defined on $[0,1]$, continuous and positive, from fuzzy numbers to predetermined crisp ideals of the best (the crisp maximum, called Max), and the worst (the crisp minimum, called Min). The idea is that a fuzzy number is ranked first if its distance to the Max is the smallest but its distance to the Min is the greatest. This approach presumes that the union of the supports of all the fuzzy numbers to rank, in symbols $\mathcal{S}$, is bounded in $\mathbb{R}$, because the values $M = \sup \mathcal{S}$ and $m = \inf \mathcal{S}$, which represent, respectively, Max and Min, are deeply involved in the calculation of the distances. However, as in our model we have chosen to treat fuzzy numbers on $\mathbb{R}_0^+$ and, in particular, the whole class of TFNs defined on $[0,\infty[$, we can put $m = 0$, corresponding to the crisp minimum given by $0 = (0,0,0)$, and not consider the distance to the Max, because $M = \infty$. Practically, not taking into account the index due to the Max, we define a RF based only on the index referred, in the same way suggested by TD, to the Min and described as follows:

$$F_{\mathrm{TD}}(\Omega) = D(\Omega, 0), \quad \Omega \in \mathcal{H},$$

where $D \equiv D_{f=1}$, since we have chosen $f = 1$ for sake of simplicity. Let us refer to the paper of TD for the complete expression of the distance D between two generic fuzzy numbers, in order to give directly the explicit form of F_{TD} on $\mathcal{A}$

$$F_{\mathrm{TD}}(A) = \sqrt{\int_0^1 g(t)\,\mathrm{d}t}, \quad g(t) = (c_1 + tc_2)^2 + c_3(1-t)^2, \quad t \in [0,1], \tag{8}$$

where $c_1 = c_1(a_1, a_3)$, $c_2 = c_2(a_1, a_2, a_3)$, $c_3 = c_3(a_1, a_3)$ are given by

$$c_1 = \frac{a_1 + a_3}{2}, \qquad c_2 = a_2 - \frac{a_1 + a_3}{2}, \qquad c_3 = \frac{(a_3 - a_1)^2}{12}.$$

We divide our examination into three steps.

S1: F_{TD} verifies the basic axioms.

Relatively to the first, let us start with the assertion that $F_{\mathrm{TD}}(A) < a_3$: in fact, being g strictly convex on its domain $[0,1]$ (an easy calculation yields $g''(t) = 2c_2^2 + 2c_3 > 0$), the inequality

$$\int_0^1 g(t)\,\mathrm{d}t < \frac{g(1) + g(0)}{2} \tag{9}$$

necessarily holds. Hence, developing the right term of (9), we find that

$$F_{\mathrm{TD}}^2(A) < \frac{a_1^2+3a_2^2+a_3^2+a_1a_3}{6},$$

so the easy inequality, direct consequence of $0\leqslant a_1\leqslant a_2\leqslant a_3$, given by

$$\frac{a_1^2+3a_2^2+a_3^2+a_1a_3}{6} \leqslant a_3^2$$

proves our assertion. Now we claim that $F_{\mathrm{TD}}(A)>a_1$: since $g'(t)=t\,(2c_2^2+2c_3)+2(c_1c_2-c_3)$, we analyze separately two subcases, that is $c_1c_2\geqslant c_3$ and $c_1c_2<c_3$. In the first one, we deduce that $g'(t)>0$ for every $t\in[0,1]$, hence $g_{\min}=g(t_{\min})=g(0)$ and consequently

$$\int_0^1 g(t)\,\mathrm{d}t > g(0)=c_1^2+c_3=\frac{a_1^2+a_3^2+a_1a_3}{3},$$

so the easy inequality

$$\frac{a_1^2+a_3^2+a_1a_3}{3} \geqslant a_1^2$$

covers the first part of the claim. Relatively to the second part, we can immediately find that $t_{\min}=(c_3-c_1c_2)/(c_2^2+c_3)$ and $g_{\min}=c_3(c_1+c_2)^2/(c_2^2+c_3)$. The strict increasing monotonicity of $g_{\min}$ with regard to c_3, seen as positive variable, being c_3 bounded below from c_1c_2, implies

$$g_{\min} > c_1c_2\,\frac{(c_1+c_2)^2}{c_2^2+c_1c_2}=c_1(c_1+c_2)=\frac{a_1+a_3}{2}\,a_2 \geqslant a_1^2$$

and the fact that $\int_0^1 g(t)\,\mathrm{d}t>g_{\min}$ concludes definitively the proof of (A_1). Finally, with a simple calculation directly from (8), we find that

$$F_{\mathrm{TD}}(A)=\frac{1}{6}\sqrt{4a_1^2+12a_2^2+4a_3^2+4a_1a_3+6a_1a_2+6a_2a_3}$$

and consequently, for every (a_1,a_2,a_3),

$$\frac{\partial F_{\mathrm{TD}}^2}{\partial a_1}=\frac{1}{36}\,(8a_1+6a_2+4a_3) > 0$$

$$\frac{\partial F_{\mathrm{TD}}^2}{\partial a_2}=\frac{1}{36}\,(6a_1+24a_2+6a_3) > 0$$

$$\frac{\partial F_{\mathrm{TD}}^2}{\partial a_3}=\frac{1}{36}\,(4a_1+6a_2+8a_3) > 0.$$

These conditions are sufficient to assure (A_2), since $\partial F_{\mathrm{TD}}^2/\partial a_i=2F_{\mathrm{TD}}\partial F_{\mathrm{TD}}/\partial a_i,\ i=1,2,3$.

S2: $F_{\mathrm{TD}}/\mathscr{A}$ is not generated by any pair (F,λ).

Suppose for contradiction there exists a pair (F^*,λ^*) which generates $F_{\mathrm{TD}}/\mathscr{A}$. Hence, we would have

$$F_{\mathrm{TD}}(A)=\lambda^*F^*(A_{\mathrm{r}})+(1-\lambda^*)F^*(A_{\mathrm{l}}),\quad A\in\mathscr{A}. \tag{10}$$

Applying for instance (10) to the TFN $(0,1,2)$ and then to the RTFN $(1,2)$ and the LTFN $(0,1)$, we would find that

$$F_{\mathrm{TD}}(1,2)+F_{\mathrm{TD}}(0,1)=1+F_{\mathrm{TD}}(0,1,2) \tag{11}$$

or, more explicitly, replacing the real values of F_{TD} in (11),

$$\tfrac{1}{6}\sqrt{58}+\tfrac{1}{6}\sqrt{22}=1+\tfrac{1}{6}\sqrt{40},$$

which is evidently false, concluding the proof of $S2$.

S3: $F_{\mathrm{TD}}/\mathscr{A}$ does not fulfill the reasonable axioms.
We choose, for example, to show that (A_4) does not hold for the right core of $F_{\mathrm{TD}}/\mathscr{A}$. Let us apply F_{TD} to an arbitrary RTFN $A_{\mathrm{r}}=(a_2,a_3)$: then, we get

$$F_{\mathrm{TD}}(A_{\mathrm{r}})=\tfrac{1}{6}\sqrt{22a_2^2+4a_3^2+10a_2a_3}$$

and consequently, by (2)

$$\rho(a_2,a_3)=\frac{\sqrt{22a_2^2+4a_3^2+10a_2a_3}-6a_2}{6(a_3-a_2)}. \tag{12}$$

From (12), after long calculations, we obtain that $\partial\rho/\partial a_3>0$ iff $(a_3-a_2)^2>0$, which is evidently true, hence we have shown that

$$\frac{\partial\rho(a_2,a_3)}{\partial a_3}=\frac{\partial\varphi(a_2,a_2,a_3)}{\partial a_3}>0$$

for every $a_2<a_3$, so contradicting (A_4).

The final comment to S1–S3 is that F_{TD}, directly obtained from an index of TD, is another concrete example of RF which verifies the basic axioms, but it is not decomposable into two separate processes of evaluation, related to right and left side of a complete TFN. On the other hand, S3 indicates that what we have judged as reasonable is not always to be expected in any case. This difference actually arises from a different point of view in the interpretation of the core of a RF. Indeed, we have seen that $\partial\rho(a_2,a_3)/\partial a_3>0$, but it is possible also to show that $\partial\rho(a_2,a_3)/\partial a_2<0$ for all $a_2<a_3$: it is very interesting to note that such properties let ρ satisfy two axiomatic requirements given in [14], where the right core of a RF is interpreted as *degree of risk* associated to the evaluation of a TFN A, instead of being related to the ambiguity of A.

4. The generalized centroid method

In this section, we will present an interesting class of ranking functions for TFNs, built according to the idea illustrated in (1), with the purpose of finding out the conditions under which the basic and the reasonable axioms hold. The RF generator is a generalization of the centroid method, cited in the previous section as Y_1, and it has the form

$$F_f(\Omega)=\frac{\int x f(x)\mu_\Omega(x)\,\mathrm{d}x}{\int f(x)\mu_\Omega(x)\,\mathrm{d}x},\quad \Omega\in\mathscr{H},$$

where f is any real mapping defined on $\mathbb{R}_0^+$, positive (possibly zero in $x=0$) and continuous, in symbols $f\in\mathscr{C}_0^+[0,\infty[$. When $f=1$, $F_{f=1}$ reduces to Y_1. Given any $\lambda\in[0,1]$, the pair generator is therefore (F_f,λ), which determines a RF G_f on $\mathscr{A}$ given by

$$G_f(A)=\lambda F_f(a_2,a_3)+(1-\lambda)F_f(a_1,a_2),$$

where

$$F_f(a_2,a_3)=\frac{\int_{a_2}^{a_3} xf(x)\mu_{A_\mathrm{r}}(x)\,\mathrm{d}x}{\int_{a_2}^{a_3} f(x)\mu_{A_\mathrm{r}}(x)\,\mathrm{d}x} \tag{13}$$

and

$$F_f(a_1,a_2)=\frac{\int_{a_1}^{a_2} xf(x)\mu_{A_\mathrm{l}}(x)\,\mathrm{d}x}{\int_{a_1}^{a_2} f(x)\mu_{A_\mathrm{l}}(x)\,\mathrm{d}x}. \tag{14}$$

The RF generator, as requested in the definition given in Section 2, has to satisfy at least (A_1): this is quite easy to check, since, for every $A\in\mathscr{A}$, we have

$$F_f(A)=\frac{\int_{a_1}^{a_3} xf(x)\mu_A(x)\,\mathrm{d}x}{\int_{a_1}^{a_3} f(x)\mu_A(x)\,\mathrm{d}x} \tag{15}$$

and applying the mean value theorem for integrals to the numerator of (15), being $f\mu_A\geqslant 0$ on $\mathbb{R}_0^+$, we get the searched result. This obviously implies that also G_f satisfies (A_1), as noted in Remark 4 of Section 2. The next step is to analyze the validity of the second basic axiom.

Proposition 4.1. *G_f fulfills (A_2) for every $f\in\mathscr{C}_0^+[0,\infty[$.*

Proof. Given any $f\in\mathscr{C}_0^+[0,\infty[$, let us define the function

$$T_f(\alpha,\beta)=\frac{\int_\alpha^\beta xf(x)(x-\alpha)\,\mathrm{d}x}{\int_\alpha^\beta f(x)(x-\alpha)\,\mathrm{d}x},$$

where $\alpha,\beta\geqslant 0$ and $\alpha\neq\beta$. Suppose that, for every α and β, we find

$$\partial T_f/\partial\alpha>0;\ \ \partial T_f/\partial\beta>0. \tag{16}$$

Then, since (13)–(14) are also writable, respectively, as

$$F_f(a_2,a_3)=\frac{\int_{a_3}^{a_2} xf(x)(x-a_3)\,\mathrm{d}x}{\int_{a_3}^{a_2} f(x)(x-a_3)\,\mathrm{d}x}$$

and

$$F_f(a_1,a_2)=\frac{\int_{a_1}^{a_2} xf(x)(x-a_1)\,\mathrm{d}x}{\int_{a_1}^{a_2} f(x)(x-a_1)\,\mathrm{d}x}$$

it is clear that $T_f(a_3,a_2)=F_f(a_2,a_3)$ and $T_f(a_1,a_2)=F_f(a_1,a_2)$. Hence, we would obtain the strict increasing of $F_f(A_\mathrm{r})$, $F_f(A_\mathrm{l})$ in all their variables and immediately, by the structure of G_f, also the final result. Therefore, this proposition is based on the validity of (16).

Let us begin with the variable β: it is not difficult to see that $\partial T_f/\partial\beta > 0$ iff

$$\int_\alpha^\beta f(x)(\beta-\alpha)(x-\alpha)(\beta-x)\,\mathrm{d}x > 0,$$

which is easy to check, because $f(x)(x-\alpha)(\beta-x)$ is positive in both of the cases $|\alpha-\beta|>0$. Instead, if we consider the variable α, we find that $\partial T_f/\partial\alpha > 0$ iff

$$\int_\alpha^\beta xf(x)(x-\alpha)\,\mathrm{d}x \int_\alpha^\beta f(x)\,\mathrm{d}x > \int_\alpha^\beta xf(x)\,\mathrm{d}x \int_\alpha^\beta f(x)(x-\alpha)\,\mathrm{d}x. \tag{17}$$

Fixed an arbitrary α, let us define in $[\alpha,\infty[$, when $\alpha<\beta$, or in $[0,\alpha]$, when $\alpha>\beta$, the following functions:

$$\mathscr{F}_1(t) = \int_\alpha^t xf(x)(x-\alpha)\,\mathrm{d}x \int_\alpha^t f(x)\,\mathrm{d}x; \quad \mathscr{F}_2(t) = \int_\alpha^t xf(x)\,\mathrm{d}x \int_\alpha^t f(x)(x-\alpha)\,\mathrm{d}x.$$

Writing (17) in terms of such functions, we have to prove that $\mathscr{F}_1(t) > \mathscr{F}_2(t)$. With regard to the derivatives, we get $\mathscr{F}_1'(t) > \mathscr{F}_2'(t)$ on the respective domains iff

$$\int_\alpha^t f(x)(t-x)^2 > 0 \quad \text{if } \alpha < \beta; \qquad \int_\alpha^t f(x)(t-x)^2 < 0 \quad \text{if } \alpha > \beta$$

which are both evidently true. From $\mathscr{F}_1'(t) > \mathscr{F}_2'(t)$ and the clear condition $\mathscr{F}_1(\alpha) = \mathscr{F}_2(\alpha)$, we can immediately deduce that $\mathscr{F}_1(t) > \mathscr{F}_2(t)$, concluding completely the proof. □

Remark. The generality of Proposition 4.1 is confirmed by the complexity of the expression of G_f, even with a rather simple f. See, for instance, the form of $G^* \equiv G_{f^*}$, generated by

$$f^*(x) = \begin{cases} 1, & x \in [0,1], \\ 1/x, & x \in [1,\infty[\end{cases}$$

for every TFN A such that $a_1 \geqslant 1$:

$$G^*(A) = \frac{\lambda}{2}\,\frac{(a_3-a_2)^2}{a_2-a_3-a_3\,\ln(a_2/a_3)} + \frac{1-\lambda}{2}\,\frac{(a_2-a_1)^2}{a_2-a_1-a_1\,\ln(a_2/a_1)}.$$

As noted in the previous section, in any RF reducible on TFNs to the CG form (6), right and left core are constant. Conversely, the generalized centroid method gives the opportunity to produce RFs with right and left core really variable with respect to a_1, a_2, a_3. For example, the right core of the just mentioned G^*, called ρ^*, in case $a_2 \geqslant 1$, is given by

$$\rho^*(a_2,a_3) = \frac{(a_3-a_2)^2 + 2a_2(a_3-a_2) + 2a_2a_3\,\ln(a_2/a_3)}{2(a_3-a_2)(a_2-a_3-a_3\,\ln(a_2/a_3))}.$$

The expression of ρ^* makes clear the complexity of an investigation on the validity of the reasonable axioms in G_f, for any $f \in \mathscr{C}_0^+[0,\infty[$. Thus, we will present such an analysis for the subclass $\{f_r\}_{r\geqslant 1}$

of $\mathscr{C}_0^+[0,\infty[$, where $f_r(x)=x^r$. Let us put $G_r \equiv G_{f_r}$ and ρ_r, $1-\sigma_r$, respectively, right and left core of G_r: we are now able to prove the following statement.

Proposition 4.2. *Given any real number $r \geqslant 1$, we have that, for every (a_1,a_2,a_3)*

$$\partial\rho_r/\partial a_3 \leqslant 0; \qquad \partial\sigma_r/\partial a_1 \geqslant 0.$$

Proof. Let us define the function

$$T_r(\alpha,\beta) = \frac{\int_\alpha^\beta f_r(x)(x-\alpha)(\beta-x)\,\mathrm{d}x}{(\beta-\alpha)\int_\alpha^\beta f_r(x)(x-\alpha)\,\mathrm{d}x},$$

where $\alpha,\beta \geqslant 0$ and $\alpha \neq \beta$. Suppose that, for every α and β, we find

$$\partial T_r/\partial\alpha \geqslant 0. \tag{18}$$

Then, since $T_r(a_3,a_2) = -\rho_r(a_2,a_3)$ and $T_r(a_1,a_2)=\sigma_r(a_1,a_2)$, we would immediately obtain our thesis. Therefore, this proposition is based on the validity of (18). We can easily find that (18) holds iff

$$\mathscr{F}_r^1(\alpha,\beta) \geqslant \mathscr{F}_r^2(\alpha,\beta), \tag{19}$$

where $\mathscr{F}_r^1(\alpha,\beta) = (\beta-\alpha)L_r(\alpha,\beta)\int_\alpha^\beta f_r(x)\,\mathrm{d}x$ and

$$L_r(\alpha,\beta) = \int_\alpha^\beta f_r(x)(x-\alpha)(\beta-x)\,\mathrm{d}x,$$

while $\mathscr{F}_r^2(\alpha,\beta) = M_r(\alpha,\beta)N_r(\alpha,\beta)$, where

$$M_r(\alpha,\beta) = \int_\alpha^\beta f_r(x)(x-\alpha)\,\mathrm{d}x$$

and

$$N_r(\alpha,\beta) = \int_\alpha^\beta f_r(x)(\beta-x)^2\,\mathrm{d}x.$$

Let us define the function $\mathscr{F}_r(\alpha,\beta) = \mathscr{F}_r^1(\alpha,\beta) - \mathscr{F}_r^2(\alpha,\beta)$ in the domain $[0,\beta]$ when $0 \leqslant \alpha < \beta$, or in $[\beta,\infty[$ when $0 \leqslant \beta < \alpha$. Obviously, we have $\mathscr{F}_r(\beta,\beta)=0$ for every β, while it takes only some calculation to see that $\mathscr{F}_r(\alpha,0)=0$ for every α, so (19) is certainly assured by

$$\frac{\partial\mathscr{F}_r(\alpha,\beta)}{\partial\alpha}\operatorname{sign}(\alpha-\beta) > 0, \quad |\alpha-\beta| > 0, \;\; \beta \neq 0. \tag{20}$$

It is not difficult to show that, after some algebraic manipulation, (20) is equivalent to

$$(\mathscr{G}_r^1(\alpha,\beta) - \mathscr{G}_r^2(\alpha,\beta))\operatorname{sign}(\alpha-\beta) > 0, \quad |\alpha-\beta| > 0, \;\; \beta \neq 0, \tag{21}$$

where

$$\mathscr{G}_r^1(\alpha,\beta) = f_r(\alpha)(\beta-\alpha)\int_\alpha^\beta f_r(x)(x-\alpha)^2\,\mathrm{d}x,$$

while

$$\mathscr{G}_r^2(\alpha,\beta) = 2\int_\alpha^\beta f_r(x)\,\mathrm{d}x\int_\alpha^\beta f_r(x)(x-\alpha)(\beta-x)\,\mathrm{d}x.$$

Developing the integrals which characterize $\mathscr{G}_r^1$ and $\mathscr{G}_r^2$, after long calculations, we find that (21) is equivalent to

$$\psi_r(t)(t-1) > 0 \quad \text{for all } t \geqslant 0,\ t \neq 1 \tag{22}$$

in the new variable $t=\alpha/\beta$, where

$$\psi_r(t) = 4t^{2r+3} - C_{r+3}t^{r+3} + C_{r+2}t^{r+2} - C_{r+1}t^{r+1} + C_r t^r + C_1 t - C_0$$

while

$$C_{r+3} = r^3+6r^2+9r+4, \qquad C_{r+2} = 3r^3+16r^2+21r, \qquad C_{r+1} = 3r^3+14r^2+17r+6,$$

$$C_r = r^3+4r^2+5r+2, \qquad C_1 = 2r+6, \qquad C_0 = 2r+2.$$

For instance, $\psi_1(t)$ is equal to $4(t-1)^5$ and clearly satisfies (22). We have to show that (22) is valid also for $r>1$: since $\psi_r(0) = -C_0 < 0$ and $\psi_r(1)=0$, the searched condition is certainly assured by

$$\psi_r'(t) > 0 \quad \text{for all } t > 0,\ t \neq 1. \tag{23}$$

With regard to the derivative of ψ_r', we put $\eta_r(t) = \psi_r''(t)/t^{r-2}$ in the domain $]0,\infty[$: it is possible to see that

$$\eta_r(t) = D_{r+3}t^{r+3} - D_3t^3 + D_2t^2 - D_1t + D_0,$$

where

$$D_{r+3} = 16r^2+40r+24, \qquad D_3 = r^5+11r^4+45r^3+85r^2+74r+24,$$

$$D_2 = 3r^5+25r^4+75r^3+95r^2+42r, \qquad D_1 = 3r^5+17r^4+31r^3+23r^2+6r,$$

$$D_0 = r^5+3r^4+r^3-3r^2-2r.$$

Calculating the derivatives of η_r until the third order and checking all the boundary conditions, after long calculations, we obtain

$$\eta_r(0), \eta_r''(0) > 0, \quad \eta_r'(0) < 0, \quad \eta_r(1) = \eta_r'(1) = \eta_r''(1) = 0 \tag{24}$$

and η_r''' strictly increasing on $[0,\infty[$, with $\eta_r'''(0)\eta_r'''(1) < 0$. The combination of the properties of η_r''' with (24) implies the existence of a unique value $\xi \in]0,1[$ such that

$$\eta_r(t)(t-\xi)(t-1) > 0 \quad \text{for all } t > 0,\ t \neq 1, \xi \tag{25}$$

and the same for ψ_r''. Consequently, the boundary conditions $\psi_r'(0)>0$ and $\psi_r'(1)=0$, together with (25), easily imply (23), concluding definitively the proof. □

Proposition 4.2 is essential for our final purpose, illustrated in the next statement.

Proposition 4.3. *G_r fulfills the reasonable axioms for every $r\geqslant 1$.*

Proof. If we denote by $\varphi_{\lambda,r}$ the core of G_r (and by analogy $\rho_{\lambda,r}$, $1-\sigma_{\lambda,r}$, respectively, for right and left core), from (5) we obtain

$$\frac{\partial\varphi_{\lambda,r}}{\partial a_1}=\frac{a_3-a_2}{(a_3-a_1)^2}(\rho_{\lambda,r}+\sigma_{\lambda,r}-1)-\frac{a_2-a_1}{a_3-a_1}\frac{\partial\sigma_{\lambda,r}}{\partial a_1} \tag{26}$$

and

$$\frac{\partial\varphi_{\lambda,r}}{\partial a_3}=\frac{a_2-a_1}{(a_3-a_1)^2}(\rho_{\lambda,r}+\sigma_{\lambda,r}-1)+\frac{a_3-a_2}{a_3-a_1}\frac{\partial\rho_{\lambda,r}}{\partial a_3}. \tag{27}$$

The form of (26) and (27), the negativity of the term $\rho_{\lambda,r}+\sigma_{\lambda,r}-1$ and Proposition 4.2 easily imply our thesis. □

5. Conclusions

We have defined two groups of axioms to test the rationality of RFs for fuzzy numbers through their behaviour on TFNs. A discussion on the nature of such axioms, combined with a large review of RFs present in current literature, has given us the idea to call them, respectively, basic and reasonable, even if we have precised, with a detailed analysis, that there are robust methods that satisfy the first group, but not the second. Next step should be the check of reliability of RFs through domains which include TFNs as particular cases, such as trapezoidal or Left–Right fuzzy numbers, at least for the "basic" axioms. This topic will be dealt with in a forthcoming paper.

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