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Solving fuzzy inequalities with concave membership functions

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Abstract

Solving systems of fuzzy inequalities could lead to the solutions of fuzzy mathematical programs. It is shown that a system of fuzzy inequalities with concave membership functions can be converted to a regular convex programming problem. A "method of centres" with "entropic regularization" techniques is proposed for solving such a problem. Some computational results are included.

Keywords: Fuzzy mathematical programming; Fuzzy membership functions; Systems of fuzzy inequalities

1. Introduction

Solving a regular mathematical programming problem essentially can be reduced to solving a system of inequalities [13–15]. Recently, Inuiguchi et al. [9] considered solving fuzzy linear programming problems in view of fuzzy linear inequalities. Here we extend the idea to study nonlinear fuzzy inequalities with concave membership functions.

Consider the following system of inequalities with $\boldsymbol{x} \in R^n$:

$$f_i(\boldsymbol{x}) \underset{\sim}{\leq} 0, \quad i=1,2,\ldots,m, \qquad g_j(\boldsymbol{x}) \leqslant 0, \quad j=1,2,\ldots,l, \tag{1}$$

where $g_j(\boldsymbol{x}) \leqslant 0$, $j=1,2,\ldots,l$, are regular inequalities, $f_i(\boldsymbol{x}) \underset{\sim}{\leq} 0$, $i=1,2,\ldots,m$, are fuzzy inequalities and "$\underset{\sim}{\leq}$" denotes the fuzzified version of "$\leqslant$" with the linguistic interpretation "approximately less than or equal to". Each fuzzy inequality $f_i(\boldsymbol{x}) \underset{\sim}{\leq} 0$ actually determines a fuzzy set $\tilde{C}_i$, whose membership function is denoted by μ_{f_i}. The membership grade $\mu_{f_i}(\boldsymbol{x})$ can be interpreted as the degree to which the regular inequality $f_i(\boldsymbol{x}) \leqslant 0$, $i=1,2,\ldots,m$, is satisfied. To specify the membership functions μ_{f_i}, it is commonly assumed that $\mu_{f_i}(\boldsymbol{x})$ should be 0 if the regular linear inequality $f_i(\boldsymbol{x}) \leqslant 0$ is strongly violated, and 1 if it is satisfied. This

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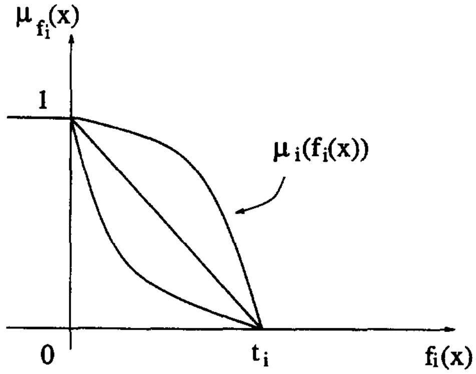

Fig. 1. The membership function $\mu_{f_i}(x)$ of the fuzzy inequality $f_i(x) \lesssim 0$.

leads to a membership function in the following form:

$$\mu_{f_i}(\boldsymbol{x}) = \begin{cases} 1 & \text{if } f_i(\boldsymbol{x}) \leqslant 0 \\ \mu_i(f_i(\boldsymbol{x})) & \text{if } 0 < f_i(\boldsymbol{x}) \leqslant t_i \quad i = 1,2,\ldots,m, \\ 0 & \text{if } f_i(\boldsymbol{x}) > t_i \end{cases} \tag{2}$$

where $t_i \geqslant 0$ is the tolerance interval which a decision maker can tolerate in the accomplishment of the fuzzy inequality $f_i(\boldsymbol{x}) \lesssim 0$. We usually assume that $\mu_i(f_i(\boldsymbol{x})) \in [0,1]$ and it is continuous and strictly decreasing over $[0, t_i]$. Fig. 1 shows different shapes of such membership functions.

Let a fuzzy decision $\tilde{D}$ of system (1) be defined as the fuzzy set resulting from the intersection of $\tilde{C}_i$, $i = 1,2,\ldots,m$, with a corresponding membership function

$$\mu_{\tilde{D}}(\boldsymbol{x}) = \min_{i=1,2,\ldots,m} \{\mu_{f_i}(\boldsymbol{x})\}. \tag{3}$$

According to Refs. [2, 16], a solution, say $\boldsymbol{x}^*$, of the system of fuzzy inequalities (1) can be taken as the solution with the highest membership in the fuzzy decision set $\tilde{D}$ and obtained by solving the following problem:

$$\begin{aligned} &\max_{\boldsymbol{x} \in R^n} \min_{i=1,2,\ldots,m} \quad \{\mu_{f_i}(\boldsymbol{x})\} \\ &\text{s.t.} \qquad\qquad g_j(\boldsymbol{x}) \leqslant 0, \quad j = 1,2,\ldots,l. \end{aligned} \tag{4}$$

Introducing one new variable α results in an equivalent problem:

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & \mu_{f_i}(\boldsymbol{x}) \geqslant \alpha, \quad i = 1,2,\ldots,m, \\ & g_j(\boldsymbol{x}) \leqslant 0, \quad j = 1,2,\ldots,l, \\ & 0 \leqslant \alpha \leqslant 1, \quad \boldsymbol{x} \in R^n. \end{aligned} \tag{5}$$

Moreover, when μ_{f_i}, $i = 1, 2, \ldots, m$, are invertible, we have

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & f_i(\boldsymbol{x}) \leqslant \mu_{f_i}^{-1}(\alpha), \quad i = 1, 2, \ldots, m, \\ & g_j(\boldsymbol{x}) \leqslant 0, \quad j = 1, 2, \ldots, l, \\ & 0 \leqslant \alpha \leqslant 1, \quad \boldsymbol{x} \in R^n. \end{aligned} \tag{6}$$

From the above procedure, we see that a system of fuzzy inequalities (1) can eventually be reduced to a nonlinear programming problem (6). Depending on the type of membership functions chosen and the properties of $f_i(\boldsymbol{x})$, $i = 1, 2, \ldots, m$, and $g_j(\boldsymbol{x})$, $j = 1, 2, \ldots, l$, the resulting optimization problem turns out to be either a linear or a nonlinear program. This paper focuses on developing solution methods for solving a system of fuzzy inequalities with concave membership functions.

2. Fuzzy inequalities with concave membership functions

Consider the case that the membership function of each fuzzy inequality $f_i(\boldsymbol{x}) \lesssim 0$ in (1) is continuous, strictly decreasing, and concave over the tolerance interval $[0, t_i]$, for $i = 1, 2, \ldots, m$. A commonly used example in fuzzy set theory is that $f(x) = 1 - ax^k$, with $k > 1$. In this case, from the theory of convex analysis [15], we have the following simple result.

Lemma 1. *If $h(\boldsymbol{x})$ is continuous, strictly decreasing and concave over a convex set Ω in R^n, then its inverse $h^{-1}(y)$ is concave.*

Lemma 1 and Problem (6) directly lead to the following result.

Theorem 1. *For the system of fuzzy inequalities (1), if $f_i(\boldsymbol{x})$, $i = 1, 2, \ldots, m$, $g_j(\boldsymbol{x})$, $j = 1, 2, \ldots, l$, are convex, and $\mu_{f_i}(\boldsymbol{x})$, $i = 1, 2, \ldots, m$, are continuous, strictly decreasing, and concave, then we can find a solution to (1) by solving the following convex programming problem:*

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & \mu_{f_i}^{-1}(\alpha) - f_i(\boldsymbol{x}) \geqslant 0, \quad i = 1, 2, \ldots, m, \\ & -g_j(\boldsymbol{x}) \geqslant 0, \quad j = 1, 2, \ldots, l, \\ & 0 \leqslant \alpha \leqslant 1, \quad \boldsymbol{x} \in R^n. \end{aligned} \tag{7}$$

Notice that problem (7) is a convex programming problem with variables $x_1, x_2, \ldots, x_n$, which are confined by the first two sets of constraints, and α, which lies in between 0 and 1. Various methods can be applied to solve general convex programming problems [1, 14]. Considering the structure of problem (7), we are interested in developing an efficient algorithm based on the framework of "method of centres". This approach can be traced back to Huard's original work [8]. The basic concepts are easy to understand and very adaptive to new developments. To describe the approach, we denote the feasible domain of problem (7) by a set D and define some terminologies. A general assumption for this approach is that D is bounded and the interior of D is non-empty.

Definition 1. Given any point $(\boldsymbol{x},\alpha)$ in the convex domain D, we define the "distance l of $(\boldsymbol{x},\alpha)$ to the boundary of D" by a continuous function

$$l((\boldsymbol{x},\alpha),D)= \min_{\substack{i=1,2,\dots,m\\ j=1,2,\dots,l}} \{\mu_{f_i}^{-1}(\alpha) - f_i(\boldsymbol{x}),\ -g_j(\boldsymbol{x}),\alpha,1-\alpha\}. \tag{8}$$

Definition 2. Given a convex domain D and a distance function $l((\boldsymbol{x},\alpha),D)$ defined on the domain, we call a point $(\bar{\boldsymbol{x}},\bar{\alpha})\in D$ the "centre of D", if it maximizes the distance function $l((\boldsymbol{x},\alpha),D)$, i.e.,

$$(\bar{\boldsymbol{x}},\bar{\alpha}):\quad l((\bar{\boldsymbol{x}},\bar{\alpha}),D)=\max\{l((\boldsymbol{x},\alpha),D)\mid(\boldsymbol{x},\alpha)\in D\}. \tag{9}$$

The basic idea of "method of centres" could be described as an iterative method in terms of the transitions from a current iterate $(\boldsymbol{x}^k,\alpha^k)$ to a new iterate $(\boldsymbol{x}^{k+1},\alpha^{k+1})$. Let $(\boldsymbol{x}^k,\alpha^k)$ be a point of D, we consider the convex domain

$$R_k \triangleq D\cap\{(\boldsymbol{x},\alpha)\mid\alpha\geqslant\alpha^k\}. \tag{10}$$

Then the new iterate $(\boldsymbol{x}^{k+1},\alpha^{k+1})$ is a solution of the centre of R_k and defined as

$$(\boldsymbol{x}^{k+1},\alpha^{k+1}):\quad l((\boldsymbol{x}^{k+1},\alpha^{k+1}),R_k)=\max\{l((\boldsymbol{x},\alpha),R_k)\mid(\boldsymbol{x},\alpha)\in R_k\}, \tag{11}$$

where

$$l((\boldsymbol{x},\alpha),R_k)= \min_{\substack{i=1,2,\dots,m\\ j=1,2,\dots,l}} \{\alpha-\alpha^k,\mu_{f_i}^{-1}(\alpha) - f_i(\boldsymbol{x}),\ -g_j(\boldsymbol{x}),\alpha,1-\alpha\} \tag{12}$$

is the distance function defined on the convex domain R_k. We start working again with $(\boldsymbol{x}^{k+1},\alpha^{k+1})$ instead of $(\boldsymbol{x}^k,\alpha^k)$. A sequence of points, $(\boldsymbol{x}^k,\alpha^k)$, are thus obtained with the following properties [8]:

(a) The sequence of domains R_k satisfies

$$R_{k'}\subset R_k\subset D,\quad \forall k'>k.$$

Since D has a non-empty interior, all the domains R_k also have a non-empty interior, except the last one in case the sequence becomes finite.

(b) The value of α is strictly increasing in every iteration.

(c) The sequence converges to an optimal solution of problem (7).

(d) The case of a finite sequence could occur only when the optimal solution belongs to the interior of the domain D.

For the above framework, the major computational work lies in the determination of the centres required, i.e., at the kth iteration, we need to resolve the following nonlinear programming problem:

$$\max_{\boldsymbol{x},\alpha}\ \min_{\substack{i=1,2,\dots,m\\ j=1,2,\dots,l}} \{\alpha-\alpha^k,\mu_{f_i}^{-1}(\alpha) - f_i(\boldsymbol{x}),\ -g_j(\boldsymbol{x}),\alpha,1-\alpha\}, \tag{13}$$

which is equivalent to the following "Min–max problem":

$$-\min_{\boldsymbol{x},\alpha} l'((\boldsymbol{x},\alpha),R_k)= \max_{\substack{i=1,2,\dots,m\\ j=1,2,\dots,l}} \{\alpha^k-\alpha, f_i(\boldsymbol{x})-\mu_{f_i}^{-1}(\alpha), g_j(\boldsymbol{x}),-\alpha,\alpha-1\}. \tag{14}$$

Again, there are many different algorithms for solving the above problem [3]. Notice that $f_i(\boldsymbol{x}) - \mu_{f_i}^{-1}(\alpha)$, $i = 1,2,\ldots,m$, and $g_j(\boldsymbol{x})$, $j = 1,2,\ldots,l$, are convex. However, they could be non-differentiable in general practice. To overcome this potential problem, we adopt the newly proposed "entropic regularization procedure" [7, 12]. This procedure guarantees that, for an arbitrarily small $\varepsilon > 0$, an ε-optimal solution of the "min–max" problem (14) can be obtained by solving the following unconstrained smooth convex program:

$$\min_{\boldsymbol{x},\alpha} l_p((\boldsymbol{x},\alpha),R_k) = \frac{1}{p}\ln\Bigg\{\exp\left[p(\alpha^k - \alpha)\right] + \sum_{i=1}^{m} \exp[p(f_i(\boldsymbol{x}) - \mu_{f_i}^{-1}(\alpha))] + \sum_{j=1}^{l} \exp[p(g_j(\boldsymbol{x}))] + \exp[p(-\alpha)] + \exp[p(\alpha - 1)]\Bigg\} \tag{15}$$

with a sufficiently large p. In other words, $\min_{\boldsymbol{x},\alpha} l_p((\boldsymbol{x},\alpha),R_k)$ provides a centre of R_k, as $p \to \infty$. It should be noted that in practice an accurate approximation can be obtained by using a moderately large p. Also because of the special "log-exponential" form of $l_p((\boldsymbol{x},\alpha),R_k)$, most overflow problems in computation can be avoided. Moreover, since it is an unconstrained, smooth, and convex optimization problem, the commonly used solution methods, such as the BFGS subroutines, can be readily applied.

3. An algorithm with a numerical example

Based on the concepts discussed in the previous section, here we propose a "method of centres with entropic regularization techniques" for finding a solution to the system of fuzzy inequalities (1). The inputs of the proposed algorithm include the initial iterate $(\boldsymbol{x}^0,\alpha^0)$ which is an interior point of D defined by (7), a vector of termination tolerances $\tau = (\tau_r, \tau_a) \in R_+^2$, and an upper bound Q which is the maximum number of unconstrained minimizations to be performed.

Algorithm

Step 0 [Initialization]: Set $k = 0$.

Step 1 [Find the centre of R_k]: Starting from $(\boldsymbol{x}^k,\alpha^k)$, apply a standard BFGS subroutine to solve the unconstrained smooth convex problem

$$\min_{\boldsymbol{x},\alpha} l_p((\boldsymbol{x},\alpha),R_k) = \frac{1}{p}\ln\Bigg\{\exp[p(\alpha^k - \alpha)] + \sum_{i=1}^{m} \exp[p(f_i(\boldsymbol{x}) - \mu_{f_i}^{-1}(\alpha))] + \sum_{j=1}^{l} \exp[p(g_j(\boldsymbol{x}))] + \exp[p(-\alpha)] + \exp[p(\alpha - 1)]\Bigg\}, \tag{16}$$

with a sufficiently large p. Denote its solution by $(\boldsymbol{x}^{k+1},\alpha^{k+1})$.

Step 2 [Update the iteration count]: Set $k \leftarrow k + 1$.

Step 3 [Check Termination Criteria]: If $\alpha^k \leqslant \tau_r\alpha^{k-1} + \tau_a$ or $k > Q$, terminate the algorithm with $(\boldsymbol{x}^k,\alpha^k)$ as a solution. Otherwise, go to step 1.

3.1. A numerical example

Consider the following system of fuzzy inequalities [11]:

$$\begin{aligned} &f_1(\boldsymbol{x}) = -4x_1 - 5x_2 - 9x_3 - 11x_4 + 111.57 \lesssim 0, \qquad f_2(\boldsymbol{x}) = x_1 + x_2 + x_3 + x_4 - 15 \lesssim 0,\\ &f_3(\boldsymbol{x}) = 7x_1 + 5x_2 + 3x_3 + 2x_4 - 80 \lesssim 0, \qquad f_4(\boldsymbol{x}) = 3x_1 + 5x_2 + 10x_3 + 15x_4 - 100 \lesssim 0,\\ &g_1(\boldsymbol{x}) = -x_1 \leqslant 0, \quad g_2(\boldsymbol{x}) = -x_2 \leqslant 0, \quad g_3(\boldsymbol{x}) = -x_3 \leqslant 0, \quad g_4(\boldsymbol{x}) = -x_4 \leqslant 0, \end{aligned} \tag{17}$$

where the membership function $\mu_{f_i}(\boldsymbol{x})$, $i = 1,2,3,4$, are specified as follows:

$$\mu_{f_1}(\boldsymbol{x}) = \begin{cases} 1 & \text{if } f_1(\boldsymbol{x}) \leqslant 0,\\ 1 - \left(\dfrac{f_1(\boldsymbol{x})}{10}\right)^2 & \text{if } 0 < f_1(\boldsymbol{x}) \leqslant 10,\\ 0 & \text{if } f_1(\boldsymbol{x}) > 10, \end{cases} \qquad \mu_{f_2}(\boldsymbol{x}) = \begin{cases} 1 & \text{if } f_2(\boldsymbol{x}) \leqslant 0,\\ 1 - \left(\dfrac{f_2(\boldsymbol{x})}{5}\right)^2 & \text{if } 0 < f_2(\boldsymbol{x}) \leqslant 5,\\ 0 & \text{if } f_2(\boldsymbol{x}) > 5, \end{cases}$$

$$\mu_{f_3}(\boldsymbol{x}) = \begin{cases} 1 & \text{if } f_3(\boldsymbol{x}) \leqslant 0,\\ 1 - \left(\dfrac{f_3(\boldsymbol{x})}{40}\right)^2 & \text{if } 0 < f_3(\boldsymbol{x}) \leqslant 40,\\ 0 & \text{if } f_3(\boldsymbol{x}) > 40, \end{cases} \qquad \mu_{f_4}(\boldsymbol{x}) = \begin{cases} 1 & \text{if } f_4(\boldsymbol{x}) \leqslant 0,\\ 1 - \left(\dfrac{f_4(\boldsymbol{x})}{30}\right)^2 & \text{if } 0 < f_4(\boldsymbol{x}) \leqslant 30,\\ 0 & \text{if } f_4(\boldsymbol{x}) > 30. \end{cases}$$

Applying Bellman and Zadeh's method of fuzzy decision making [2], the maximizing solution, $\boldsymbol{x}^*$, of this problem is given by solving the following convex programming problem:

$$\begin{aligned} \max\quad & \alpha\\ \text{s.t.}\quad & 10\sqrt{1-\alpha} + 4x_1 + 5x_2 + 9x_3 + 11x_4 - 111.57 \geqslant 0,\\ & 5\sqrt{1-\alpha} - x_1 - x_2 - x_3 - x_4 + 15 \geqslant 0, 40\sqrt{1-\alpha} - 7x_1 - 5x_2 - 3x_3 - 2x_4 + 80 \geqslant 0,\\ & 30\sqrt{1-\alpha} - 3x_1 - 5x_2 - 10x_3 - 15x_4 + 100 \geqslant 0,\\ & x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0, \quad x_4 \geqslant 0, \quad 0 \leqslant \alpha \leqslant 1. \end{aligned} \tag{18}$$

Using the proposed algorithm to solve this convex program, at the kth iteration, the following nonlinear programming problem is considered:

$$\begin{aligned} \max_{\boldsymbol{x},\alpha} \min\{&\alpha - \alpha^k, \alpha, 1-\alpha, 10\sqrt{1-\alpha} + 4x_1 + 5x_2 + 9x_3 + 11x_4 - 111.57,\\ &5\sqrt{1-\alpha} - x_1 - x_2 - x_3 - x_4 + 15, 40\sqrt{1-\alpha} - 7x_1 - 5x_2 - 3x_3 - 2x_4 + 80,\\ &30\sqrt{1-\alpha} - 3x_1 - 5x_2 - 10x_3 - 15x_4 + 100, \ x_1, \ x_2, \ x_3, \ x_4\}. \end{aligned}$$

This problem is equivalent to the min–max problem:

$$\begin{aligned} -\min_{\boldsymbol{x},\alpha} \max\{&\alpha^k - \alpha, -\alpha, \alpha - 1, 111.57 - 10\sqrt{1-\alpha} - 4x_1 - 5x_2 - 9x_3 - 11x_4,\\ &x_1 + x_2 + x_3 + x_4 - 15 - 5\sqrt{1-\alpha}, 7x_1 + 5x_2 + 3x_3 + 2x_4 - 80 - 40\sqrt{1-\alpha},\\ &3x_1 + 5x_2 + 10x_3 + 15x_4 - 100 - 30\sqrt{1-\alpha}, \ -x_1, \ -x_2, \ -x_3, \ -x_4\}. \end{aligned} \tag{19}$$

Table 1

Computational results of the "method of centres with entropic regularization techniques"

k	(x^k, α^k)	No. of iterations for minimizing $l_p((x,\alpha),R_k)$
0	((6, 2, 6, 2), 0.20)	3
1	((6, 2, 6, 2), 0.60)	37
2	((6.38, 2.18, 6.09, 1.47), 0.79)	59
3	((7.87, 0.15, 8.29, 0.14), 0.89)	35
4	((7.95, 0.05, 8.43, 0.04), 0.904)	22
5	((7.96, 0.05, 8.44, 0.04), 0.905)	13
6	((7.97, 0.05, 8.45, 0.04), 0.905)	11
7	((7.97, 0.05, 8.45, 0.04), 0.905)	

An ε-optimal solution of the "min–max" problem can be obtained by solving an unconstrained and smooth convex programming problem:

$$
\begin{aligned}
-\min_{x,\alpha} \frac{1}{p} \ln\{&\exp[p(\alpha^k - \alpha)] + \exp[p(-\alpha)] + \exp[p(\alpha - 1)] \\
&+ \exp[p(111.57 - 10\sqrt{1-\alpha} - 4x_1 - 5x_2 - 9x_3 - 11x_4)] \\
&+ \exp[p(x_1 + x_2 + x_3 + x_4 - 15 - 5\sqrt{1-\alpha})] \\
&+ \exp[p(7x_1 + 5x_2 + 3x_3 + 2x_4 - 80 - 40\sqrt{1-\alpha})] \\
&+ \exp[p(3x_1 + 5x_2 + 10x_3 + 15x_4 - 100 - 30\sqrt{1-\alpha})] \\
&+ \exp[p(-x_1)] + \exp[p(-x_2)] + \exp[p(-x_3)] + \exp[p(-x_4)]\}
\end{aligned}
$$

with p being sufficiently large.

Taking $\varepsilon = 0.025$ and fixing $p = 100$ for each iteration, the algorithm terminated after 7 iterations at the point $x^* = (7.97, 0.05, 8.45, 0.04)^{\mathrm{T}}$ with $\alpha^* = 0.905$. Computational results for this problem are listed in Table 1.

4. Concluding remarks

In this paper we study a system of fuzzy inequalities with concave membership functions. Such a system of fuzzy inequalities can be converted to a regular convex programming problem. A version of the "method of centres" with "entropic regularization" techniques has been proposed to solve the resulting convex program. Due to the entropic regularization techniques, only a commonly used BFGS subroutine is required in our implementation. An example is used to illustrate the proposed algorithm. Compared to other approaches of solving general convex programming problems, our work essentially reduces the problem to minimizing an infinitely smooth convex function without any constraints for efficient computation.

As mentioned before, using the smooth function $l_p((x,\alpha),R_k)$ as an approximation depends on the value of p. Although, for simplicity, in our implementation, a fixed p was used for every iteration to generate an ε-optimal solution. For a refined version, we may want to improve the performance of the algorithm by adjusting p values at every iteration. The adjusting mechanism and computational efficiency will be documented in a separate report. We also intend to study fuzzy inequalities with other type of membership functions and extend results to the fuzzy variational inequalities and fuzzy complementarity problems [4–6, 10].

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