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A comparative study of fuzzy rough sets

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Abstract

The notion of a rough set was originally proposed by Pawlak (1982). Later on, Dubois and Prade (1990) introduced *fuzzy rough sets* as a fuzzy generalization of rough sets. In this paper, we present a more general approach to the fuzzification of rough sets. Specifically, we define a broad family of fuzzy rough sets, each one of which, called an $(\mathscr{I},\mathscr{T})$-fuzzy rough set, is determined by an implicator $\mathscr{I}$ and a triangular norm $\mathscr{T}$. Basic properties of fuzzy rough sets are investigated. In particular, we define three classes of fuzzy rough sets, relatively to three main classes of implicators well known in the literature, and analyse their properties in the context of basic rough equalities. Finally, we refer to an operator-oriented characterization of rough sets as proposed by Lin and Liu (1994) and show soundness of this axiomatization for the Łukasiewicz fuzzy rough sets.

Keywords: Fuzzy set theory; Rough set theory; Fuzzy implicator; Fuzzy rough set

1. Introduction

Modelling imprecise and incomplete information is one of the main research topics in the area of knowledge representation. Many of the existing approaches are based on some extensions of classical set theory: fuzzy set theory and rough set theory, among others.

The concept of a rough set was originally proposed by Pawlak [12–14] as a formal tool for modelling and processing incomplete information in information systems. This theory evolved into a far-reaching methodology centering on analysis of incomplete information [5,16,17,23]. It soon invoked a natural question concerning possible connections between rough sets and fuzzy sets. Basically, both theories address the problem of information granulation: the theory of fuzzy sets is centred upon fuzzy information granulation, whereas rough set theory is focused on crisp information granulation. Originally, the basic notion in rough set theory

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was indistinguishability (i.e. indistinguishability between objects in information systems induced by different values of attributes characterizing these objects), yet in recent extensions [9] the focus moves to the notion of similarity, which is in fact a fuzzy concept. It is therefore apparent that these two theories have become much closer to each other.

Dubois and Prade (see [1,2]) were one of the first who investigated the problem of fuzzification of a rough set. [2] In this paper, we propose a more general approach to this issue. Specifically, by applying the extension principle, we define fuzzy rough sets dependently on fuzzy conjunction and fuzzy implication operators. As a consequence, we propose a broad class of fuzzy rough sets, each one of which, called $(\mathscr{I},\mathscr{T})$-*fuzzy rough sets*, is represented by an implicator $\mathscr{I}$ and a triangular norm $\mathscr{T}$. We define three classes of fuzzy rough sets taking into account three well-known classes of implicators, namely S-, R- and QL-implicators. Our aim is to investigate properties of these classes. In perspective, this approach is intended to allow for considering a variety of methods for representing and analysing of both incomplete and imprecise information.

The paper is organized as follows. In Section 2 we recall basic notions of rough set theory. In Section 3 the definition of fuzzy rough approximation and fuzzy rough set are introduced. Next, in Section 4, we investigate several properties of fuzzy rough sets. The paper is completed by concluding remarks and some options for further work.

2. Rough approximations and rough sets

Let $\mathfrak{X}$ be a nonempty universe and let R be an equivalence relation on $\mathfrak{X}$. The main question addressed by rough sets is: *how to represent subsets $A \subseteq \mathfrak{X}$ by means of elements of the quotient set $\mathfrak{X}/R$?*

The following notation will be used. Given a nonempty universe $\mathfrak{X}$, by $\wp(\mathfrak{X})$ we will denote a power-set on $\mathfrak{X}$. If R is an equivalence relation on $\mathfrak{X}$ then for every $x \in \mathfrak{X}$, $[x]_R$ stands for the equivalence class of R with the representant x, i.e. $[x]_R = \{y \in \mathfrak{X}\colon (x,y) \in R\}$. Finally, for any $A \subseteq \mathfrak{X}$, we write $\mathrm{co}\,A$ to denote the complementation of A in $\mathfrak{X}$, that is the set $\mathfrak{X} \backslash A$.

Definition 1. A pair $\mathrm{As} = (\mathfrak{X}, R)$, where $\mathfrak{X} \neq \emptyset$ and R is an equivalence relation on $\mathfrak{X}$, is called an *approximation space*.

Definition 2. For an approximation space $\mathrm{As}=(\mathfrak{X},R)$, by a *rough approximation in* As we mean a mapping $\mathtt{Apr}_{\mathrm{As}}\colon \wp(\mathfrak{X}) \to \wp(\mathfrak{X}) \times \wp(\mathfrak{X})$ defined by for every $A \in \wp(\mathfrak{X})$, $\mathtt{Apr}_{\mathrm{As}}(A) = (\underline{\mathrm{As}}(A), \overline{\mathrm{As}}(A))$, where

$$\underline{\mathrm{As}}(A) = \{x \in \mathfrak{X}\colon [x]_R \subseteq A\}, \tag{1}$$

$$\overline{\mathrm{As}}(A) = \{x \in \mathfrak{X}\colon [x]_R \cap A \neq \emptyset\}. \tag{2}$$

$\underline{\mathrm{As}}(A)$ is called a *lower rough approximation of A in* As, whereas $\overline{\mathrm{As}}(A)$ is called an *upper rough approximation of A in* As.

Definition 3. Given an approximation space $\mathrm{As} = (\mathfrak{X}, R)$, a pair $(L, U) \in \wp(\mathfrak{X}) \times \wp(\mathfrak{X})$ is called a *rough set in* As iff $(L, U) = \mathtt{Apr}_{\mathrm{As}}(A)$ for some $A \in \wp(\mathfrak{X})$.

For any approximation space $\mathrm{As} = (\mathfrak{X}, R)$, a subset $A \subseteq \mathfrak{X}$ is called *definable in* As iff $\underline{\mathrm{As}}(A) = \overline{\mathrm{As}}(A)$.

Let us recall several basic properties of rough sets (see [12]).

Theorem 1. *For every approximation space* $\mathrm{As} = (\mathfrak{X}, R)$ *and every subsets* $A, B \subseteq \mathfrak{X}$, *we have*:
- (P1) $\underline{\mathrm{As}}(A) \subseteq A \subseteq \overline{\mathrm{As}}(A)$,
- (P2) $\underline{\mathrm{As}}(\emptyset) = \emptyset = \overline{\mathrm{As}}(\emptyset)$,
- (P3) $\underline{\mathrm{As}}(\mathfrak{X}) = \mathfrak{X} = \overline{\mathrm{As}}(\mathfrak{X})$,
- (P4) *if* $A \subseteq B$, *then* $\underline{\mathrm{As}}(A) \subseteq \underline{\mathrm{As}}(B)$ *and* $\overline{\mathrm{As}}(A) \subseteq \overline{\mathrm{As}}(B)$,
- (P5) $\underline{\mathrm{As}}(\underline{\mathrm{As}}(A)) = \underline{\mathrm{As}}(A)$,
- (P6) $\overline{\mathrm{As}}(\overline{\mathrm{As}}(A)) = \overline{\mathrm{As}}(A)$,
- (P7) $\overline{\mathrm{As}}(\underline{\mathrm{As}}(A)) = \underline{\mathrm{As}}(A)$,
- (P8) $\underline{\mathrm{As}}(\overline{\mathrm{As}}(A)) = \overline{\mathrm{As}}(A)$,
- (P9) $\underline{\mathrm{As}}(A) = \mathrm{co}\,\overline{\mathrm{As}}(\mathrm{co}\,A)$,
- (P10) $\overline{\mathrm{As}}(A) = \mathrm{co}\,\underline{\mathrm{As}}(\mathrm{co}\,A)$,
- (P11) $\underline{\mathrm{As}}(A \cap B) = \underline{\mathrm{As}}(A) \cap \underline{\mathrm{As}}(B)$,
- (P12) $\overline{\mathrm{As}}(A \cap B) \subseteq \underline{\mathrm{As}}(A) \cap \underline{\mathrm{As}}(B)$,
- (P13) $\underline{\mathrm{As}}(A \cup B) \supseteq \overline{\mathrm{As}}(A) \cup \overline{\mathrm{As}}(B)$,

[2] Other approaches were proposed by Nakamura [8] and Thiele [19–21].

(P14) $\overline{\mathrm{As}}(A \cup B) = \overline{\mathrm{As}}(A) \cup \overline{\mathrm{As}}(B)$,
(P15) $\underline{\mathrm{As}}([x]_R) = \overline{\mathrm{As}}([x]_R)$ *for all* $x \in \mathfrak{X}$.

By Theorem 1, we immediately get:

Corollary 1. *For every approximation space* $\mathrm{As} = (\mathfrak{X}, R)$,
(i) *for every* $A \subseteq \mathfrak{X}$, $\underline{\mathrm{As}}(A)$ *and* $\overline{\mathrm{As}}(A)$ *are definable in* As,
(ii) *for every* $x \in \mathfrak{X}$, $[x]_R$ *is definable in* As.

In [6] an operator-oriented approach to rough sets was proposed. The authors investigated a problem of axiomatic characterization of Pawlak's rough sets. More specifically, given a universe $\mathfrak{X}$, two operators $L, H : \wp(\mathfrak{X}) \to \wp(\mathfrak{X})$ were considered such that the following conditions are satisfied for every $A, B \subseteq \mathfrak{X}$:
(A.1) $L(A) = \mathrm{co}\, H(\mathrm{co}\, A)$,
(A.2) $H(\emptyset) = \emptyset$,
(A.3) $H(A \cup B) = H(A) \cup H(B)$,
(A.4) $L(A) \subseteq A$,
(A.5) $L(A) = L(L(A))$,
(A.6) $L(A) = H(L(A))$.

Theorem 2 (Lin and Liu [5]). *For two operators* $L, H : \wp(\mathfrak{X}) \to \wp(\mathfrak{X})$ *satisfying conditions* (A.1)–(A.6), *there is an equivalence relation* R *on* $\mathfrak{X}$ *such that for* $\mathrm{As} = (\mathfrak{X}, R)$ *and every* $A \subseteq \mathfrak{X}$,

$$L(A) = \underline{\mathrm{As}}(A) \quad \textit{and} \quad H(A) = \overline{\mathrm{As}}(A).$$

It is easy to check that the binary relation R on $\mathfrak{X}$ given by $(x, y) \in R$ iff $x \in H(\{y\})$, is such a relation.

3. Fuzzy rough sets

In this section we introduce definitions of fuzzy rough approximations and fuzzy rough sets. To begin with, let us recall some basic terminology and notation.

3.1. Fuzzy logical operators

A *triangular norm*, or shortly t-norm, is an increasing, associative and commutative mapping $\mathscr{T} : [0,1]^2 \to [0,1]$ that satisfies the boundary condition $(\forall x \in [0,1])(\mathscr{T}(x,1) = x)$. The most popular continuous t-norms are:

- the standard *min* operator $\mathscr{T}_M(x,y) = \min\{x,y\}$ (the largest t-norm [4]),
- the algebraic product $\mathscr{T}_P(x,y) = x * y$,
- the bold intersection (also called the Łukasiewicz t-norm) $\mathscr{T}_L(x,y) = \max\{0, x+y-1\}$.

A *triangular conorm* (shortly t-conorm) is an increasing, associative and commutative mapping $\mathscr{S} : [0,1]^2 \to [0,1]$ that satisfies the boundary condition $(\forall x \in [0,1])(\mathscr{S}(x,0) = x)$. Three well-known continuous t-conorms are:

- the standard *max* operator $\mathscr{S}_M(x,y) = \max\{x,y\}$ (the smallest t-conorm),
- the probabilistic sum $\mathscr{S}_P(x,y) = x + y - x * y$,
- the bounded sum $\mathscr{S}_L(x,y) = \min\{1, x+y\}$.

A *negator* $\mathscr{N}$ is a decreasing $[0,1]$–$[0,1]$ mapping satisfying $\mathscr{N}(0) = 1$ and $\mathscr{N}(1) = 0$. The negator $\mathscr{N}_s(x) = 1 - x$ is usually referred to as the standard negator. A negator $\mathscr{N}$ is called *involutive* iff $\mathscr{N}(\mathscr{N}(x)) = x$ for all $x \in [0,1]$; it is called *weakly involutive* iff $\mathscr{N}(\mathscr{N}(x)) \geqslant x$ for all $x \in [0,1]$. Every involutive negator is continuous [4].

Given a negator $\mathscr{N}$, a t-norm $\mathscr{T}$ and a t-conorm $\mathscr{S}$ are called *dual with respect to* $\mathscr{N}$ iff de Morgan laws are satisfied, i.e.

$$\mathscr{S}(\mathscr{N}(x), \mathscr{N}(y)) = \mathscr{N}(\mathscr{T}(x,y)),$$

$$\mathscr{T}(\mathscr{N}(x), \mathscr{N}(y)) = \mathscr{N}(\mathscr{S}(x,y)).$$

It is well known [4] that for an involutive negator $\mathscr{N}$ and a t-conorm $\mathscr{S}$, the function $\mathscr{T}_{\mathscr{S}}(x,y) = \mathscr{N}(\mathscr{S}(\mathscr{N}(x), \mathscr{N}(y)))$, $x, y \in [0,1]$, is a t-norm such that $\mathscr{T}$ and $\mathscr{S}$ are dual with respect to $\mathscr{N}$. In the following it will be referred to as a *t-norm dual to* $\mathscr{S}$ *wrt* $\mathscr{N}$.

For a t-norm $\mathscr{T}$ (resp. a t-conorm $\mathscr{S}$), we will write $\bigcap_{\mathscr{T}}$ (resp. $\bigcup_{\mathscr{S}}$) to denote fuzzy intersection (resp. fuzzy union) determined by $\mathscr{T}$ (resp. $\mathscr{S}$). Zadeh's fuzzy union and fuzzy intersection will be denoted by $\cup$ and $\cap$, respectively. The symbol $\mathrm{co}_{\mathscr{N}}$ will be used to denote fuzzy complement determined by a negator $\mathscr{N}$, i.e. for every $A \in \mathscr{F}(\mathfrak{X})$ and every $x \in \mathfrak{X}$, $(\mathrm{co}_{\mathscr{N}} A)(x) = \mathscr{N}(A(x))$.

By an *implicator* we mean a function $\mathscr{I} : [0,1]^2 \to [0,1]$ satisfying $\mathscr{I}(1,0) = 0$ and $\mathscr{I}(1,1) = \mathscr{I}(0,1) = \mathscr{I}(0,0) = 1$. An implicator $\mathscr{I}$ is called *left monotonic* (resp. *right monotonic*) iff for every $x \in [0,1], \mathscr{I}(\cdot, x)$ is decreasing (resp. $\mathscr{I}(x, \cdot)$ is increasing). If $\mathscr{I}$ is both left monotonic and right monotonic, then it is called *hybrid monotonic.* [3]

For a left monotonic implicator $\mathscr{I}$, the function $\mathscr{N}(x)=\mathscr{I}(x,0)$, $x\in[0,1]$, is a negator, called a negator *induced by* $\mathscr{I}$. For example, the Łukasiewicz implicator $\mathscr{I}_{\mathrm{L}}(x,y)=\min\{1,1-x+y\}$ induces the standard negator $\mathscr{N}_{\mathrm{s}}$.

Several classes of implicators have been studied in the literature. Recall the definitions of three main classes of these operators [4].

Let $\mathscr{T}$, $\mathscr{S}$ and $\mathscr{N}$ be a t-norm, a t-conorm and a negator, respectively. An implicator $\mathscr{I}$ is called

- an *S-implicator based on* $\mathscr{S}$ *and* $\mathscr{N}$ iff

  $$\mathscr{I}(x,y)=\mathscr{S}(\mathscr{N}(x),y)\quad \text{for all } x,y\in[0,1];$$

- an *R-implicator* (*residual implicator*) *based on* $\mathscr{T}$ iff for every $x,y\in[0,1]$,

  $$\mathscr{I}(x,y)=\sup\{\lambda\in[0,1]\colon\ \mathscr{T}(x,\lambda)\leqslant y\}$$

  provided that $\mathscr{T}$ is continuous;
- a QL-*implicator based on* $\mathscr{S}$, $\mathscr{T}$ *and* $\mathscr{N}$ iff for every $x,y\in[0,1]$,

  $$\mathscr{I}(x,y)=\mathscr{S}(\mathscr{N}(x),\mathscr{T}(x,y))$$

  provided that $\mathscr{T}$ and $\mathscr{S}$ are dual wrt $\mathscr{N}$.

Three most popular S-implicators are:

- the Łukasiewicz implicator $\mathscr{I}_{\mathrm{L}}(x,y)=\min\{1,1-x+y\}$, based on $\mathscr{S}_{\mathrm{L}}$ and $\mathscr{N}_{\mathrm{s}}$,
- the Kleene–Dienes implicator $\mathscr{I}_{\mathrm{KD}}(x,y)=\max\{1-x,y\}$, based on $\mathscr{S}_{\mathrm{M}}$ and $\mathscr{N}_{\mathrm{s}}$,
- the Kleene–Dienes–Łukasiewicz implicator $\mathscr{I}_{\bigstar}(x,y)=1-x+x*y$, based on $\mathscr{S}_{\mathrm{P}}$ and $\mathscr{N}_{\mathrm{s}}$.

The most popular R-implicators are:

- the Łukasiewicz implicator $\mathscr{I}_{\mathrm{L}}$, based on $\mathscr{T}_{\mathrm{L}}$,
- the Gödel implicator $\mathscr{I}_{\mathrm{G}}(x,y)=1$ for $x\leqslant y$ and $\mathscr{I}_{\mathrm{G}}(x,y)=y$ elsewhere, based on $\mathscr{T}_{\mathrm{M}}$,
- the Gaines implicator $\mathscr{I}_{\triangle}(x,y)=1$ for $x\leqslant y$ and $\mathscr{I}_{\triangle}(x,y)=y/x$ elsewhere, based on $\mathscr{T}_{\mathrm{P}}$.

Two well-known QL-implicators are:

- the Kleene–Dienes implicator $\mathscr{I}_{\mathrm{KD}}$, based on $\mathscr{T}_{\mathrm{L}}$, $\mathscr{S}_{\mathrm{L}}$ and $\mathscr{N}_{\mathrm{s}}$,
- the Early Zadeh implicator $\mathscr{I}_{\mathrm{Z}}(x,y)=\max\{1-x,\min\{x,y\}\}$, based on $\mathscr{T}_{\mathrm{M}}$, $\mathscr{S}_{\mathrm{M}}$ and $\mathscr{N}_{\mathrm{s}}$.

Observe the following simple property.

Proposition 1. *Every S-implicator and every R-implicator is hybrid monotonic. Every* QL-*implicator is right monotonic.*

Proof. Let $y_1,y_2\in[0,1]$ be such that $y_1\leqslant y_2$.

- Assume that $\mathscr{I}$ is an S-implicator. Then for every $x\in[0,1]$,

  $$\mathscr{I}(x,y_1)=\mathscr{S}(\mathscr{N}(x),y_1)\leqslant\mathscr{S}(\mathscr{N}(x),y_2)$$
  $$=\mathscr{I}(x,y_2),$$
  $$\mathscr{I}(y_1,x)=\mathscr{S}(\mathscr{N}(y_1),x)\geqslant\mathscr{S}(\mathscr{N}(y_2),x)$$
  $$=\mathscr{I}(y_2,x).$$

  Hence $\mathscr{I}$ is hybrid monotonic.
- Let $\mathscr{I}$ be an R-implicator based on a continuous t-norm $\mathscr{T}$. Then for every $x\in[0,1]$,

  $$\mathscr{I}(x,y_1)=\sup\{\lambda\in[0,1]\colon\ \mathscr{T}(x,\lambda)\leqslant y_1\}$$
  $$\leqslant\sup\{\lambda\in[0,1]\colon\ \mathscr{T}(x,\lambda)\leqslant y_2\}$$
  $$=\mathscr{I}(x,y_2).$$

  Also, since $\{\lambda\in[0,1]\colon\ \mathscr{T}(y_2,\lambda)\leqslant x\}\subseteq\{\lambda\in[0,1]\colon \mathscr{T}(y_1,\lambda)\leqslant x\}$,

  $$\mathscr{I}(y_2,x)=\sup\{\lambda\in[0,1]\colon\ \mathscr{T}(y_2,\lambda)\leqslant x\}$$
  $$\leqslant\sup\{\lambda\in[0,1]\colon\ \mathscr{T}(y_1,\lambda)\leqslant x\}$$
  $$=\mathscr{I}(y_1,x).$$

  Hence $\mathscr{I}$ is also hybrid monotonic.
- Finally, assume that $\mathscr{I}$ is a QL-implicator. Then for every $x\in[0,1]$,

  $$\mathscr{I}(x,y_1)=\mathscr{S}(\mathscr{N}(x),\mathscr{T}(x,y_1))$$
  $$\leqslant\mathscr{S}(\mathscr{N}(x),\mathscr{T}(x,y_2))=\mathscr{T}(x,y_2),$$

  so $\mathscr{I}$ is right monotonic. $\square$

[3] Hybrid monotonicity is one of the axioms for implicators as proposed by Smets and Magrez [18]. For a complete study of 18 widely used implicators with respect to the Smets and Magrez axioms we refer to [15].

In general, QL-implicators are not left monotonic. For instance, for the Early Zadeh implicator $\mathscr{I}_Z$, $\mathscr{I}_Z(0.5, 0.6) = 0.5 < 0.6 = \mathscr{I}_Z(0.8, 0.6)$.

An implicator $\mathscr{I}$ is said to be a *border implicator* [4] iff for every $x \in [0,1]$, $\mathscr{I}(1, x) = x$. By straightforward verification it is easy to show the following fact.

Proposition 2. *Every S-implicator, R-implicator and* QL-*implicator is a border implicator.*

3.2. Similarity relations

Let $\mathfrak{X}$ be a nonempty universe. A fuzzy binary relation R on $\mathfrak{X}$ is called a *similarity relation* iff R is

- reflexive $R(x,x) = 1$ for all $x \in \mathfrak{X}$,
- symmmetric: $R(x,y) = R(y,x)$ for all $x, y \in \mathfrak{X}$,
- sup-min transitive: $R(x,y) \geqslant \sup\min_{z\in\mathfrak{X}}\{R(x,z), R(z,y)\}$ for all $x, y \in \mathfrak{X}$.

Recall that for a similarity relation R on $\mathfrak{X}$, the similarity class $[x]_R$ with $x \in \mathfrak{X}$ as a representant, is a fuzzy set on $\mathfrak{X}$ defined by

$$[x]_R(y) = R(x,y) \quad \text{for all } y \in \mathfrak{X}.$$

The following two auxiliary lemmas will be useful later.

Lemma 1. *If R is a similarity relation on $\mathfrak{X}$, then for every t-norm $\mathscr{T}$ and every $x, y \in \mathfrak{X}$,*

$$R(x,z) = \sup_{z\in\mathfrak{X}} \mathscr{T}(R(x,z), R(z,y)).$$

Proof. Since *min* is the largest t-norm, for all t-norms $\mathscr{T}$ and all $x, y \in \mathfrak{X}$, we have

$$\begin{aligned} R(x,y) &\geqslant \sup_{z\in\mathfrak{X}} \min\{R(x,z), R(z,y)\} \\ &\geqslant \sup_{z\in\mathfrak{X}} \mathscr{T}(R(x,z), R(z,y)). \end{aligned} \tag{3}$$

Moreover, for every t-norm $\mathscr{T}$ and every $x, y \in \mathfrak{X}$,

$$\begin{aligned} &\sup_{z\in\mathfrak{X}} \mathscr{T}(R(x,z), R(z,y)) \\ &\quad \geqslant \mathscr{T}(R(x,x), R(x,y)) \\ &\quad = \mathscr{T}(1, R(x,y)) = R(x,y), \end{aligned} \tag{4}$$

by reflexivity of R and the definition of a t-norm. Eqs. (3) and (4) imply the result. □

Lemma 2. *Let $\mathscr{T}$ be a continuous t-norm and let $\mathscr{I}$ be an R-implicator based on $\mathscr{T}$. Then for every similarity relation R on $\mathfrak{X}$,*

$$(\forall x, y \in \mathfrak{X}) \quad \left(\inf_{z\in\mathfrak{X}} \mathscr{I}(R(x,z), R(z,y)) = R(x,y) \right). \tag{5}$$

Proof. For every $x, y \in \mathfrak{X}$,

$$\begin{aligned} &\inf_{z\in\mathfrak{X}} \mathscr{I}(R(x,z), R(z,y)) \\ &\quad \leqslant \mathscr{I}(R(x,x), R(x,y)) = \mathscr{I}(1, R(x,y)) = R(x,y). \end{aligned} \tag{6}$$

On the other hand, for every $x, y \in \mathfrak{X}$,

$$\begin{aligned} &\inf_{z\in\mathfrak{X}} \mathscr{I}(R(x,z), R(z,y)) \\ &\quad = \inf_{z\in\mathfrak{X}} \sup\{\lambda \in [0,1]\colon \mathscr{T}(R(x,z), \lambda) \leqslant R(z,y)\}. \end{aligned}$$

By symmetry of R and Lemma 1,

$$\begin{aligned} \mathscr{T}(R(x,z), R(x,y)) &= \mathscr{T}(R(z,x), R(x,y)) \\ &\leqslant \sup_{x\in\mathfrak{X}} \mathscr{T}(R(z,x), R(x,y)) \\ &= R(z,y), \end{aligned}$$

so

$$\sup\{\lambda \in [0,1]\colon \mathscr{T}(R(x,z), \lambda) \leqslant R(z,y)\} \geqslant R(x,y).$$

Therefore,

$$\begin{aligned} &\inf_{z\in\mathfrak{X}} \sup\{\lambda \in [0,1]\colon \mathscr{T}(R(x,z), \lambda) \leqslant R(z,y)\} \\ &\quad \geqslant \inf_{z\in\mathfrak{X}} R(x,y) = R(x,y). \end{aligned}$$

Thus, we have

$$\inf_{z\in\mathfrak{X}} \mathscr{I}(R(x,z), R(z,y)) \geqslant R(x,y).$$

Hence, by (6), we get the result. □

In general, however, neither S-implicators nor QL-implicators satisfy (5).

[4] In [7] one can get an idea about the huge number of border implicators defined on a finite chain.

Table 1
Similarity relation R

	a	b	c	d
a	1	0.25	0.25	0.50
b	0.25	1	0.75	0.25
c	0.25	0.75	1	0.25
d	0.50	0.25	0.25	1

Example 1. Let $\mathfrak{X}=\{a,b,c\}$ and R be a similarity relation on $\mathfrak{X}$ given in Table 1. Note that

$$\inf_{x\in\mathfrak{X}} \mathscr{I}_{\mathrm{Z}}(R(a,x),R(x,a)) = 0.5 \neq 1 = R(a,a),$$

$$\inf_{x\in\mathfrak{X}} \mathscr{I}_{\mathrm{KD}}(R(a,x),R(x,a)) = 0.5 \neq 1 = R(a,a).$$

3.3. Fuzzy rough approximations and fuzzy rough sets

In this section we define fuzzy rough approximations and fuzzy rough sets. Let us start with introducing the following basic notion.

Definition 4 (*Fuzzy approximation space*). For a nonempty universe $\mathfrak{X}$ and a similarity relation R on $\mathfrak{X}$, a pair $\textsc{Fas}=(\mathfrak{X},R)$ is called a *fuzzy approximation space*.

Consider again definitions (1) and (2). They mean that for every subset $A\subseteq\mathfrak{X}$ and every $x\in\mathfrak{X}$,

$$\chi_{\underline{\mathrm{As}}(A)}(x) = 1 \quad \text{iff}$$
$$(\forall y\in\mathfrak{X})\ (\chi_R(x,y)=1 \Rightarrow \chi_A(y)=1),$$

$$\chi_{\overline{\mathrm{As}}(A)}(x) = 1 \quad \text{iff}$$
$$(\exists y\in\mathfrak{X})\ (\chi_R(x,y)=1 \wedge \chi_A(y)=1),$$

where, as usual, χ_A stands for the characteristic function of the set A. While interpreting $\Rightarrow$ and $\wedge$ as an implicator $\mathscr{I}$ and a triangular norm $\mathscr{T}$, respectively, we get by extension the following definition.

Definition 5 (*Fuzzy rough approximation*). Let $\textsc{Fas}=(\mathfrak{X},R)$ be a fuzzy approximation space and $\mathscr{I}$ and $\mathscr{T}$ be a border implicator and a t-norm, respectively. An $(\mathscr{I},\mathscr{T})$-*fuzzy rough approximation in* $\textsc{Fas}$ is a mapping $\mathtt{Apr}_{\textsc{Fas}}^{\mathscr{I},\mathscr{T}} : \mathscr{F}(\mathfrak{X})\to\mathscr{F}(\mathfrak{X})\times\mathscr{F}(\mathfrak{X})$ defined by for every $A\in\mathscr{F}(\mathfrak{X})$,

$$\mathtt{Apr}_{\textsc{Fas}}^{\mathscr{I},\mathscr{T}}(A) = (\underline{\textsc{Fas}}_{\mathscr{I}}(A), \overline{\textsc{Fas}}^{\mathscr{T}}(A)),$$

where for every $x\in\mathfrak{X}$,

$$\underline{\textsc{Fas}}_{\mathscr{I}}(A)(x) = \inf_{y\in\mathfrak{X}} \mathscr{I}(R(x,y),A(y)),$$

$$\overline{\textsc{Fas}}^{\mathscr{T}}(A)(x) = \sup_{y\in\mathfrak{X}} \mathscr{T}(R(x,y),A(y)).$$

The fuzzy set $\underline{\textsc{Fas}}_{\mathscr{I}}(A)$ (resp. $\overline{\textsc{Fas}}^{\mathscr{T}}(A)$) is called an $\mathscr{I}$*-lower* (resp. $\mathscr{T}$*-upper*) *fuzzy rough approximation of A in* $\textsc{Fas}$.

Given $\textsc{Fas}=(\mathfrak{X},R)$, we say that $F\in\mathscr{F}(\mathfrak{X})$ is a *lower* (resp. *upper*) *fuzzy rough approximation of* $A\in\mathscr{F}(\mathfrak{X})$ *in* $\textsc{Fas}$ iff $F=\underline{\textsc{Fas}}_{\mathscr{I}}(A)$ for some border implicator $\mathscr{I}$ (resp. $F=\overline{\textsc{Fas}}^{\mathscr{T}}(A)$ for some t-norm $\mathscr{T}$).

Remark 1. Recall that for any fuzzy set $A\in\mathscr{F}(\mathfrak{X})$, its height and plinth are defined as $\mathrm{height}(A)=\sup_{y\in\mathfrak{X}}A(y)$ and $\mathrm{plinth}(A)=\inf_{y\in\mathfrak{X}}A(y)$. By Definition 5, for every $x\in\mathfrak{X}$,

$$\underline{\textsc{Fas}}_{\mathscr{I}}(A)(x) = \mathrm{plinth}([x]_R \underset{\mathscr{I}}{\rightarrowtail} A),$$

$$\overline{\textsc{Fas}}^{\mathscr{T}}(A)(x) = \mathrm{height}([x]_R \cap_{\mathscr{T}} A),$$

where $\underset{\mathscr{I}}{\rightarrowtail}$ is a pointwise extension of $\mathscr{I}$, i.e. $(A \underset{\mathscr{I}}{\rightarrowtail} B)(x)=\mathscr{I}(A(x),B(x))$ for all $A,B\in\mathscr{F}(\mathfrak{X})$ and all $x\in\mathfrak{X}$.

Let us define three classes of fuzzy rough approximations:

- S-FRA determined by $(\mathscr{I}_{\mathscr{S}},\mathscr{T}_{\mathscr{S}})$, where $\mathscr{I}_{\mathscr{S}}$ is a continuous S-implicator based on a continuous t-conorm $\mathscr{S}$ and an involutive negator $\mathscr{N}$, and $\mathscr{T}_{\mathscr{S}}$ is dual to $\mathscr{S}$ wrt $\mathscr{N}$,
- R-FRA (residual fuzzy rough approximations) determined by $(\mathscr{I},\mathscr{T})$, where $\mathscr{I}$ is a continuous R-implicator based on a continuous t-norm $\mathscr{T}$,
- Q-FRA determined by $(\mathscr{I},\mathscr{T})$, where $\mathscr{I}$ is a QL-implicator based on a continuous t-norm $\mathscr{T}$ and an involutive negator $\mathscr{N}$.

Definition 6 (*Fuzzy rough set*). Let $\mathrm{FAS} = (\mathfrak{X}, R)$ be a fuzzy approximation space and let $\mathscr{I}$ and $\mathscr{T}$ be a border implicator and a t-norm, respectively. A pair $(L, U) \in \mathscr{F}(\mathfrak{X}) \times \mathscr{F}(\mathfrak{X})$ is called an $(\mathscr{I}, \mathscr{T})$-*fuzzy rough set in* FAS iff $(L, U) = \mathtt{Apr}^{\mathscr{I},\mathscr{T}}_{\mathrm{FAS}}(A)$ for some $A \in \mathscr{F}(\mathfrak{X})$.

Clearly, for every class of fuzzy rough approximations, there is a corresponding class of fuzzy rough sets. Accordingly, we have the class S-FRS, R-FRS and Q-FRS of fuzzy rough sets.

Remark 2. The concept of fuzzy rough set was proposed by Dubois and Prade [1,2]. Their idea was as follows. Let $\mathfrak{X}$ be a nonempty universe, R be a similar ity relation on $\mathfrak{X}$ and let $F \in \mathscr{F}(\mathfrak{X})$. A *fuzzy rough set* is a pair $(R_*(F), R^*(F))$ of fuzzy sets on $\mathfrak{X}$ such that for every $x \in \mathfrak{X}$,

$$R_*(F)(x) = \inf_{y\in\mathfrak{X}} \max\{1 - R(x, y), F(y)\},$$

$$R^*(F)(x) = \sup_{y\in\mathfrak{X}} \min\{R(x, y), F(y)\}.$$

Note that this notion is exactly the $(\mathscr{I}_{\mathrm{KD}}, \mathscr{T}_{\mathrm{M}})$-fuzzy rough set.

The following notion is a fuzzy analogon of that of *definability*.

Definition 7. Let $\mathrm{FAS} = (\mathfrak{X}, R)$ be a fuzzy approximation space and let $\mathscr{I}$ and $\mathscr{T}$ be a border implicator and a t-norm, respectively. A fuzzy set $A \in \mathscr{F}(\mathfrak{X})$ is $(\mathscr{I}, \mathscr{T})$-*definable* in FAS iff $\underline{\mathrm{FAS}}_{\mathscr{I}}(A) = A = \overline{\mathrm{FAS}}^{\mathscr{T}}(A)$.

4. Properties of fuzzy rough sets

Let us start with some simple observations.

Proposition 3. *Let $\mathscr{I}$ and $\mathscr{T}$ be a border implicator and a t-norm, respectively. Then for every fuzzy approximation space* $\mathrm{FAS} = (\mathfrak{X}, R)$,

(F1) $\underline{\mathrm{FAS}}_{\mathscr{I}}(A) \subseteq A \subseteq \overline{\mathrm{FAS}}^{\mathscr{T}}(A)$ *for all* $A \in \mathfrak{X}$.

(F2) $\underline{\mathrm{FAS}}_{\mathscr{I}}(\emptyset) = \emptyset = \overline{\mathrm{FAS}}^{\mathscr{T}}(\emptyset)$.

(F3) (a) $\overline{\mathrm{FAS}}^{\mathscr{T}}(\mathfrak{X}) = \mathfrak{X}$,
(b) $\underline{\mathrm{FAS}}_{\mathscr{I}}(\mathfrak{X}) = \mathfrak{X}$, *provided that $\mathscr{I}$ is left monotonic.*

(F4) *for every $A, B \in \mathscr{F}(\mathfrak{X})$, if $A \subseteq B$ then*
(a) $\overline{\mathrm{FAS}}^{\mathscr{T}}(A) \subseteq \overline{\mathrm{FAS}}^{\mathscr{T}}(B)$,
(b) $\underline{\mathrm{FAS}}_{\mathscr{I}}(A) \subseteq \underline{\mathrm{FAS}}_{\mathscr{I}}(B)$, *provided that $\mathscr{I}$ is right monotonic.*

Proof. (F1) For every $x \in \mathfrak{X}$,

$$\begin{aligned}\underline{\mathrm{FAS}}_{\mathscr{I}}(A)(x) &= \inf_{y\in\mathfrak{X}} \mathscr{I}(R(x, y), A(y)) \\ &\leqslant \mathscr{I}(R(x,x), A(x)) = \mathscr{I}(1, A(x)) \\ &= A(x),\end{aligned}$$

$$\begin{aligned}\overline{\mathrm{FAS}}^{\mathscr{T}}(A)(x) &= \sup_{y\in\mathfrak{X}} \mathscr{T}(R(x, y), A(y)) \\ &\geqslant \mathscr{T}(R(x,x), A(x)) = \mathscr{T}(1, A(x)) \\ &= A(x).\end{aligned}$$

Hence $\underline{\mathrm{FAS}}_{\mathscr{I}}(A) \subseteq A \subseteq \overline{\mathrm{FAS}}^{\mathscr{T}}(A)$.

(F2) For every $x \in \mathfrak{X}$, $\overline{\mathrm{FAS}}^{\mathscr{T}}(\emptyset)(x) = \sup_{y\in\mathfrak{X}} \mathscr{T}(R(x, y), 0) = 0$. By (F1), for every border implicator $\mathscr{I}$, $\underline{\mathrm{FAS}}_{\mathscr{I}}(\emptyset) \subseteq \emptyset$, so $\underline{\mathrm{FAS}}_{\mathscr{I}}(\emptyset) = \emptyset$. Hence $\underline{\mathrm{FAS}}_{\mathscr{I}}(\emptyset) = \emptyset = \overline{\mathrm{FAS}}^{\mathscr{T}}(\emptyset)$.

(F3) For every $x \in \mathfrak{X}$,
(a) $\overline{\mathrm{FAS}}^{\mathscr{T}}(\mathfrak{X})(x) = \sup_{y\in\mathfrak{X}} \mathscr{T}(R(x, y), 1) = \sup_{y\in\mathfrak{X}} R(x, y) = 1$ by reflexivity of R.
(b) If $\mathscr{I}$ is left monotonic, then for every $x \in \mathfrak{X}$,

$$\begin{aligned}\underline{\mathrm{FAS}}_{\mathscr{I}}(\mathfrak{X})(x) &= \inf_{y\in\mathfrak{X}} \mathscr{I}(R(x, y), 1) \\ &\geqslant \inf_{y\in\mathfrak{X}} \mathscr{I}(1, 1) = 1.\end{aligned}$$

(F4) Let $A, B \in \mathscr{F}(\mathfrak{X})$ be such that $A \subseteq B$. By monotonicity of $\mathscr{T}$, we have for every $x \in \mathfrak{X}$,
(a) $\overline{\mathrm{FAS}}^{\mathscr{T}}(A)(x) = \sup_{y\in\mathfrak{X}} \mathscr{T}(R(x, y), A(y)) \leqslant \sup_{y\in\mathfrak{X}} \mathscr{T}(R(x, y), B(y)) = \overline{\mathrm{FAS}}^{\mathscr{T}}(B)(x)$, so $\overline{\mathrm{FAS}}^{\mathscr{T}}(A) \subseteq \overline{\mathrm{FAS}}^{\mathscr{T}}(B)$.
(b) Let $\mathscr{I}$ be right monotonic. Then for every $x \in \mathfrak{X}$,

$$\begin{aligned}\underline{\mathrm{FAS}}_{\mathscr{I}}(A)(x) &= \inf_{y\in\mathfrak{X}} \mathscr{I}(R(x, y), A(y)) \\ &\leqslant \inf_{y\in\mathfrak{X}} \mathscr{I}(R(x, y), B(y)) \\ &= \underline{\mathrm{FAS}}_{\mathscr{I}}(B)(x).\end{aligned}$$

Hence $\underline{\mathrm{FAS}}_{\mathscr{I}}(A) \subseteq \underline{\mathrm{FAS}}_{\mathscr{I}}(B)$. $\square$

Remark 3. Let $\mathscr{I}$ be a QL-implicator based on a t-norm $\mathscr{T}$, a t-conorm $\mathscr{S}$ and a negator $\mathscr{N}$. Clearly, for every $x, y \in [0,1]$,

$$\mathscr{I}(x,y) = \mathscr{S}(\mathscr{N}(x), \mathscr{T}(x,y)) \leqslant \mathscr{S}(\mathscr{N}(x), y) = \mathscr{I}_{\mathscr{S}}(x,y),$$

where $\mathscr{I}_{\mathscr{S}}$ is an S-implicator. It implies, in turn, that for every fuzzy approximation space $\mathrm{FAS} = (\mathfrak{X}, R)$ and every fuzzy set $A \in \mathscr{F}(\mathfrak{X})$, $\underline{\mathrm{FAS}}_{\mathscr{I}}(A) \subseteq \underline{\mathrm{FAS}}_{\mathscr{I}_{\mathscr{S}}}(A)$. Therefore, for every QL-implicator $\mathscr{I}$, we can get the better (in the sense of fuzzy inclusion) lower fuzzy rough approximation (of any fuzzy set A) by using the corresponding S-implicator $\mathscr{I}_{\mathscr{S}}$.

Remark 4. If a border implicator is not left monotonic (resp. right monotonic), then (F3)(b) (resp. (F4)(b)) need not hold. Consider, for example, a fuzzy approximation space $\mathrm{FAS} = (\mathfrak{X}, R)$ as defined in Example 1 and the Willmott implicator defined by

$$\mathscr{I}_{\sharp}(x,y) = \min\{\max\{1-x, y\}, \max\{1-x, x\}, \max\{1-y, y\}\} \quad \text{for all } x, y \in [0,1].$$

It is easy to verify that $\mathscr{I}_{\sharp}$ is a border implicator. Since $\mathscr{I}_{\sharp}(0.4, 0.2) = 0.6 > 0.5 = \mathscr{I}_{\sharp}(0.4, 0.5)$, $\mathscr{I}_{\sharp}$ is not right monotonic. For fuzzy sets A and B:

$$A = \begin{pmatrix} a & b & c & d \\ 0 & 1 & 0.75 & 0 \end{pmatrix} \subseteq \begin{pmatrix} a & b & c & d \\ 0.5 & 1 & 0.75 & 0.25 \end{pmatrix} = B,$$

by simple calculations we get

$$\underline{\mathrm{FAS}}_{\mathscr{I}_{\sharp}}(A) = \begin{pmatrix} a & b & c & d \\ 0 & 0.75 & 0.75 & 0 \end{pmatrix} \not\subseteq \begin{pmatrix} a & b & c & d \\ 0.5 & 0.5 & 0.5 & 0.25 \end{pmatrix} = \underline{\mathrm{FAS}}_{\mathscr{I}_{\sharp}}(B).$$

Also, for the Early Zadeh implicator $\mathscr{I}_{\mathrm{Z}}$,

$$\underline{\mathrm{FAS}}_{\mathscr{I}_{\mathrm{Z}}}(\mathfrak{X}) = \begin{pmatrix} a & b & c & d \\ 0.5 & 0.75 & 0.75 & 0.5 \end{pmatrix} \neq \mathfrak{X}. \quad \square$$

By Propositions 1 and 3 we immediately get the following corollary.

Corollary 2. *For every fuzzy approximation space* $\mathrm{FAS} = (\mathfrak{X}, R)$,

(i) *if $\mathscr{I}$ is an S-implicator, an R-implicator or a* QL-*implicator and $A, B \in \mathscr{F}(\mathfrak{X})$, $A \subseteq B$, then* $\underline{\mathrm{FAS}}_{\mathscr{I}}(A) \subseteq \underline{\mathrm{FAS}}_{\mathscr{I}}(B)$;

(ii) *if $\mathscr{I}$ is an S-implicator or an R-implicator, then* $\underline{\mathrm{FAS}}_{\mathscr{I}}(\mathfrak{X}) = \mathfrak{X}$.

Let us consider now fuzzy rough approximations of fuzzy sets $\mathbb{I}_{\xi}$, $\xi \in [0,1]$, defined by: for every $x \in \mathfrak{X}$, $\mathbb{I}_{\xi}(x) = \xi$.

Proposition 4. *For every t-norm $\mathscr{T}$, every fuzzy approximation space* $\mathrm{FAS} = (\mathfrak{X}, R)$ *and every* $\xi \in [0,1]$, $\overline{\mathrm{FAS}}^{\mathscr{T}}(\mathbb{I}_{\xi}) = \mathbb{I}_{\xi}$.

Proof. Let $\xi \in [0,1]$. Then for every $x \in \mathfrak{X}$,

$$\overline{\mathrm{FAS}}^{\mathscr{T}}(\mathbb{I}_{\xi})(x) = \sup_{y \in \mathfrak{X}} \mathscr{T}(R(x,y), \xi) \leqslant \mathscr{T}\left(\sup_{y \in \mathfrak{X}} R(x,y), \xi\right) = \mathscr{T}(1, \xi) = \xi = \mathbb{I}_{\xi}(x),$$

so $\overline{\mathrm{FAS}}^{\mathscr{T}}(\mathbb{I}_{\xi}) \subseteq \mathbb{I}_{\xi}$. Hence, by (F1), we get the result. $\square$

Proposition 5. *For every fuzzy approximation space* $\mathrm{FAS} = (\mathfrak{X}, R)$ *and every* $\xi \in [0,1]$,

(i) *for every t-norm $\mathscr{T}$ and every S-implicator $\mathscr{I}$, $\mathbb{I}_{\xi}$ is $(\mathscr{I}, \mathscr{T})$-definable,*

(ii) *for every R-implicator $\mathscr{I}$ based on a continuous t-norm $\mathscr{T}$, $\mathbb{I}_{\xi}$ is $(\mathscr{I}, \mathscr{T})$-definable.*

Proof. Let $\xi \in [0,1]$.

(i) Let $\mathscr{I}$ be an S-implicator based on a t-conorm $\mathscr{S}$. Then for every $x \in \mathfrak{X}$,

$$\underline{\mathrm{FAS}}_{\mathscr{I}}(\mathbb{I}_{\xi})(x) = \inf_{y \in \mathfrak{X}} \mathscr{I}(R(x,y), \xi) = \inf_{y \in \mathfrak{X}} \mathscr{S}(\mathscr{N}(R(x,y)), \xi) \geqslant \inf_{y \in \mathfrak{X}} \mathscr{S}(0, \xi) = \xi = \mathbb{I}_{\xi}(x).$$

By (F1), $\underline{\mathrm{FAS}}_{\mathscr{I}}(\mathbb{l}_\xi) \subseteq \mathbb{l}_\xi$. Hence $\underline{\mathrm{FAS}}_{\mathscr{I}}(\mathbb{l}_\xi) = \mathbb{l}_\xi$. Taking into account Proposition 4, we get

$$\underline{\mathrm{FAS}}_{\mathscr{I}}(\mathbb{l}_\xi) = \overline{\mathrm{FAS}}^{\mathscr{T}}(\mathbb{l}_\xi).$$

(ii) Let $\mathscr{I}$ be an R-implicator based on a continuous t-norm $\mathscr{T}$. Then for every $x \in \mathfrak{X}$,

$$\begin{aligned}&\underline{\mathrm{FAS}}_{\mathscr{I}}(\mathbb{l}_\xi)(x)\\ &= \inf_{y\in\mathfrak{X}} \mathscr{I}(R(x,y),\xi)\\ &= \inf_{y\in\mathfrak{X}} \sup\{\alpha \in [0,1]\colon\ \mathscr{T}(R(x,y),\alpha) \leqslant \xi\} \geqslant \xi,\end{aligned}$$

since

$$\sup\{\alpha \in [0,1]\colon\ \mathscr{T}(R(x,y),\alpha) \leqslant \xi\} \geqslant \xi \quad \text{for all } x, y \in \mathfrak{X}.$$

Therefore,

$$\underline{\mathrm{FAS}}_{\mathscr{I}}(\mathbb{l}_\xi) \supseteq \mathbb{l}_\xi.$$

By (F1), $\underline{\mathrm{FAS}}_{\mathscr{I}}(\mathbb{l}_\xi) \subseteq \mathbb{l}_\xi$, so

$$\underline{\mathrm{FAS}}_{\mathscr{I}}(\mathbb{l}_\xi) = \mathbb{l}_\xi.$$

Whence, in view of Proposition 4, we immediately obtain the result. □

4.1. Compositions of fuzzy rough approximations

Observe first that by (F1), for every fuzzy rough approximation $\mathrm{FAS} = (\mathfrak{X}, R)$ and every $A \in \mathscr{F}(\mathfrak{X})$, we have

$$\underline{\mathrm{FAS}}_{\mathscr{I}}(\underline{\mathrm{FAS}}_{\mathscr{I}}(A)) \subseteq \underline{\mathrm{FAS}}_{\mathscr{I}}(A), \tag{7}$$

$$\overline{\mathrm{FAS}}^{\mathscr{T}}(\overline{\mathrm{FAS}}^{\mathscr{T}}(A)) \supseteq \overline{\mathrm{FAS}}^{\mathscr{T}}(A). \tag{8}$$

We aim to establish classes of fuzzy rough approximations for which (7) and (8) are equalities. To this end, let us consider the following auxiliary condition: for an implicator $\mathscr{I}$ and a t-norm $\mathscr{T}$, we say that $\mathscr{I}$ *satisfies* C1 *for* $\mathscr{T}$ iff

C1. $(\forall x, y, z \in [0,1])$
$(\mathscr{I}(x, \mathscr{I}(y,z)) \geqslant \mathscr{I}(\mathscr{T}(x,y),z))$.

We say that $\mathscr{I}$ *satisfies* C1 iff it satisfies C1 for some t-norm $\mathscr{T}$.

Remark 5. Condition C1 may be interpreted as follows. Recall the well-known classical tautology of the form $(\alpha \to (\beta \to \gamma)) \leftrightarrow ((\alpha \wedge \beta) \to \gamma)$. While interpreting $\wedge$ and $\to$ as a t-norm $\mathscr{T}$ and an implicator $\mathscr{I}$, respectively, we obtain $\mathscr{I}(\alpha, \mathscr{I}(\beta,\gamma)) = \mathscr{I}(\mathscr{T}(\alpha,\beta),\gamma)$. Condition C1 is just a weakened form of this condition.

Proposition 6. *Every R-implicator and every S-implicator based on an involutive negator $\mathscr{N}$, satisfies* C1.

Proof. (1) Let $\mathscr{I}$ be an S-implicator based on an involutive negator $\mathscr{N}$. By the definition of S-implicators, for every $x, y, z \in [0,1]$, we have

$$\begin{aligned}\mathscr{I}(x,\mathscr{I}(y,z)) &= \mathscr{S}(\mathscr{N}(x), \mathscr{S}(\mathscr{N}(y),z))\\ &= \mathscr{S}(\mathscr{S}(\mathscr{N}(x),\mathscr{N}(y)),z)\\ &= \mathscr{S}(\mathscr{N}(\mathscr{T}_{\mathscr{S}}(x,y)),z)\\ &= \mathscr{I}(\mathscr{T}_{\mathscr{S}}(x,y),z).\end{aligned}$$

(2) Assume on the contrary that there exists an R-implicator $\mathscr{I}$ based on some t-norm $\mathscr{T}$, such that for every t-norm $\mathscr{T}_1$ it holds that

$$\mathscr{I}(x,\mathscr{I}(y,z)) < \mathscr{I}(\mathscr{T}_1(x,y),z) \quad \text{for some } x, y, z \in [0,1].$$

By the definition of R-implicators, it means that

$$\begin{aligned}&\sup\{\lambda \in [0,1]\colon\ \mathscr{T}(x,\lambda) \leqslant \mathscr{I}(y,z)\}\\ &\quad < \sup\{\lambda \in [0,1]\colon\ \mathscr{T}(\mathscr{T}_1(x,y),\lambda) \leqslant z\}\end{aligned} \tag{9}$$

for some $x, y, z \in [0,1]$. In particular, (9) holds for $\mathscr{T}_1 \equiv \mathscr{T}$. Therefore, there is $\lambda_0 \in [0,1]$ such that

$$\mathscr{T}(\mathscr{T}(x,y),\lambda_0) \leqslant z, \tag{10}$$

$$\mathscr{T}(x,\lambda_0) > \mathscr{I}(y,z). \tag{11}$$

Again, from the definition of R-implicators, (11) means that

$$\mathscr{T}(x,\lambda_0) > \sup\{\lambda \in [0,1]\colon\ \mathscr{T}(y,\lambda) \leqslant z\}.$$

Hence, by associativity of t-norms, we have

$$\mathscr{T}(y, \mathscr{T}(x,\lambda_0)) = \mathscr{T}(\mathscr{T}(x,y),\lambda_0) > z,$$

which contradicts (10).

Proposition 7. *Let $\mathscr{I}$ be a continuous and hybrid monotonic border implicator satisfying* C1. *Then for every fuzzy approximation space* $\text{FAS} = (\mathfrak{X}, R)$ *and every* $A \in \mathscr{F}(\mathfrak{X})$,
(F5) $\underline{\text{FAS}}_{\mathscr{I}}(\underline{\text{FAS}}_{\mathscr{I}}(A)) = \underline{\text{FAS}}_{\mathscr{I}}(A)$.

Proof. For every $x \in \mathfrak{X}$,

$$\begin{aligned}
&\underline{\text{FAS}}_{\mathscr{I}}(\underline{\text{FAS}}_{\mathscr{I}}(A))(x)\\
&\quad = \inf_{y\in\mathfrak{X}} \mathscr{I}\left(R(x,y), \inf_{z\in\mathfrak{X}} \mathscr{I}(R(y,z),A(z))\right)\\
&\quad = \inf_{y\in\mathfrak{X}} \inf_{z\in\mathfrak{X}} \mathscr{I}(R(x,y),\mathscr{I}(R(y,z),A(z)))
\end{aligned}$$

by continuity of $\mathscr{I}$. Furthermore, applying C1, we get

$$\begin{aligned}
&\inf_{y\in\mathfrak{X}} \inf_{z\in\mathfrak{X}} \mathscr{I}(R(x,y),\mathscr{I}(R(y,z),A(z)))\\
&\quad \geqslant \inf_{z\in\mathfrak{X}} \inf_{y\in\mathfrak{X}} \mathscr{I}(\mathscr{T}(R(x,y),R(y,z)),A(z))\\
&\quad = \inf_{z\in\mathfrak{X}} \mathscr{I}\left(\sup_{y\in\mathfrak{X}} \mathscr{T}(R(x,y),R(y,z)),A(z)\right).
\end{aligned}$$

By Lemma 1, $R(x,z) = \sup_{y\in\mathfrak{X}} \mathscr{T}(R(x,y),R(y,z))$ for all $x,z \in \mathfrak{X}$, so

$$\begin{aligned}
\underline{\text{FAS}}_{\mathscr{I}}(\underline{\text{FAS}}_{\mathscr{I}}(A))(x) &\geqslant \inf_{z\in\mathfrak{X}} \mathscr{I}(R(x,z),A(z))\\
&= \underline{\text{FAS}}_{\mathscr{I}}(A)(x) \quad \text{for all } x \in \mathfrak{X}.
\end{aligned}$$

Therefore, we obtain

$$\underline{\text{FAS}}_{\mathscr{I}}(\underline{\text{FAS}}_{\mathscr{I}}(A)) \supseteq \underline{\text{FAS}}_{\mathscr{I}}(A).$$

Hence, by (7), we get the result. □

By Propositions 1, 6 and 7 we get the following corollary.

Corollary 3. *For every continuous R-implicator and every continuous S-implicator based on an involutive negator*, (F5) *holds for all* $\text{FAS} = (\mathfrak{X}, R)$ *and all* $A \in \mathscr{F}(\mathfrak{X})$.

Also we have:

Proposition 8. *Let $\mathscr{T}$ be a continuous t-norm. Then for every fuzzy approximation space* $\text{FAS} = (\mathfrak{X}, R)$ *and every* $A \in \mathscr{F}(\mathfrak{X})$,

(F6) $\overline{\text{FAS}}^{\mathscr{T}}(\overline{\text{FAS}}^{\mathscr{T}}(A)) = \overline{\text{FAS}}^{\mathscr{T}}(A)$.

Proof. For every $x \in \mathfrak{X}$,

$$\begin{aligned}
&\overline{\text{FAS}}^{\mathscr{T}}(\overline{\text{FAS}}^{\mathscr{T}}(A))(x)\\
&\quad = \sup_{y\in\mathfrak{X}} \mathscr{T}\left(R(x,y), \sup_{z\in\mathfrak{X}} \mathscr{T}(R(y,z),A(z))\right)\\
&\quad = \sup_{y\in\mathfrak{X}} \sup_{z\in\mathfrak{X}} \mathscr{T}(R(x,y),\mathscr{T}(R(y,z),A(z)))\\
&\quad = \sup_{z\in\mathfrak{X}} \sup_{y\in\mathfrak{X}} \mathscr{T}(\mathscr{T}(R(x,y),R(y,z)),A(z))\\
&\quad = \sup_{z\in\mathfrak{X}} \mathscr{T}\left(\sup_{y\in\mathfrak{X}} \mathscr{T}(R(x,y),R(y,z)),A(z)\right)
\end{aligned}$$

by continuity and associativity of $\mathscr{T}$. By Lemma 1, $\sup_{y\in\mathfrak{X}} \mathscr{T}(R(x,y),R(y,z)) = R(x,z)$, so

$$\begin{aligned}
\overline{\text{FAS}}^{\mathscr{T}}(\overline{\text{FAS}}^{\mathscr{T}}(A))(x) &= \sup_{z\in\mathfrak{X}} \mathscr{T}(R(x,z),A(z))\\
&= \overline{\text{FAS}}^{\mathscr{T}}(A)(x) \quad \text{for all } x \in \mathfrak{X}.
\end{aligned}$$

Hence, $\overline{\text{FAS}}^{\mathscr{T}}(\overline{\text{FAS}}^{\mathscr{T}}(A)) = \overline{\text{FAS}}^{\mathscr{T}}(A)$. □

Let us examine now mixed compositions of lower and upper fuzzy rough approximations, i.e. the fuzzy analogous of (P7) and (P8). Again, by (F1), we have in general

$$\overline{\text{FAS}}^{\mathscr{T}}(\underline{\text{FAS}}_{\mathscr{I}}(A)) \supseteq \underline{\text{FAS}}_{\mathscr{I}}(A), \tag{12}$$

$$\underline{\text{FAS}}_{\mathscr{I}}(\overline{\text{FAS}}^{\mathscr{T}}(A)) \subseteq \overline{\text{FAS}}^{\mathscr{T}}(A) \tag{13}$$

for all fuzzy approximation spaces $\text{FAS} = (\mathfrak{X}, R)$ and all $A \in \mathscr{F}(\mathfrak{X})$. We shall show that for every R-implicator $\mathscr{I}$ based on a continuous t-norm $\mathscr{I}$, (12) and (13) are equalities for $\mathscr{I}$ and $\mathscr{T}$, and all $\text{FAS} = (\mathfrak{X}, R)$ and $A \in \mathscr{F}(\mathfrak{X})$.

First, consider the following auxiliary lemma.

Lemma 3. *For every continuous t-norm $\mathscr{T}$ and an R-implicator $\mathscr{I}$ based on $\mathscr{T}$, it holds that for every* $x,y,z \in [0,1]$,

$$\mathscr{T}(x,\mathscr{I}(y,z)) \leqslant \mathscr{I}(\mathscr{I}(x,y),z).$$

Proof. Observe first that for every R-implicator $\mathscr{I}$ based on a continuous t-norm $\mathscr{T}$, it holds that

$$\mathscr{T}(x,\mathscr{I}(x,y)) = \mathscr{T}(x,\sup\{\lambda\in[0,1]\colon\ \mathscr{T}(x,\lambda)\leqslant y\}) \leqslant y \quad \text{for all } x,y\in[0,1] \tag{14}$$

by continuity of $\mathscr{T}$. Consider the following two cases.

(1) Let $x\leqslant y$. Then, by the definition of R-implicators, $\mathscr{I}(x,y)=1$, which implies

$$\mathscr{I}(\mathscr{I}(x,y),z) = \mathscr{I}(1,z) = z. \tag{15}$$

- Let $y\leqslant z$. Then $\mathscr{I}(y,z)=1$, so $\mathscr{T}(x,\mathscr{I}(y,z)) = \mathscr{T}(x,1)=x$. Since $x\leqslant z$, we get

  $$\mathscr{T}(x,\mathscr{I}(y,z))\leqslant\mathscr{I}(\mathscr{I}(x,y),z).$$

- Let $y>z$. By assumption and (14),

  $$\mathscr{T}(x,\mathscr{I}(y,z))\leqslant\mathscr{T}(y,\mathscr{I}(y,z))\leqslant z.$$

  Hence, by (15), we get

  $$\mathscr{T}(x,\mathscr{I}(y,z))\leqslant\mathscr{I}(\mathscr{I}(x,y),z).$$

(2) Let $x>y$ and, as before, consider the following two cases.

- Assume that $y\leqslant z$. Then $\mathscr{I}(y,z)=1$, so

  $$\mathscr{T}(x,\mathscr{I}(y,z)) = x. \tag{16}$$

  By assumption and (14), $\mathscr{T}(x,\mathscr{I}(x,y))\leqslant y\leqslant z$. Therefore,

  $$\mathscr{I}(\mathscr{I}(x,y),z) = \sup\{\lambda\in[0,1]:\mathscr{T}(\mathscr{I}(x,y),\lambda)\leqslant z\}\geqslant x.$$

  Whence, by (16), we get $\mathscr{T}(x,\mathscr{I}(y,z))\leqslant\mathscr{I}(\mathscr{I}(x,y),z)$.

- Let $y>z$ and assume on the contrary that

  $$\mathscr{T}(x,\mathscr{I}(y,z)) > \mathscr{I}(\mathscr{I}(x,y),z) = \sup\{\lambda\in[0,1]:\mathscr{T}(\mathscr{I}(x,y),\lambda)\leqslant z\}.$$

  It means that

  $$\mathscr{T}(\mathscr{I}(x,y),\mathscr{T}(x,\mathscr{I}(y,z)))>z.$$

  However,

  $$\mathscr{T}(\mathscr{I}(x,y),\ \mathscr{T}(x,\mathscr{I}(y,z))) = \mathscr{T}(\mathscr{T}(x,\mathscr{I}(x,y)),\mathscr{I}(y,z)).$$

  Therefore, by (14), we obtain the following contradiction:

  $$z < \mathscr{T}(\mathscr{T}(x,\mathscr{I}(x,y)),\mathscr{I}(y,z)) \leqslant \mathscr{T}(y,\mathscr{I}(y,z))\leqslant z. \qquad \square$$

Proposition 9. *Let $\mathscr{T}$ be a continuous t-norm and let $\mathscr{I}$ be on R-implicator based on $\mathscr{T}$. Then for every fuzzy approximation space* $\text{FAS}=(\mathfrak{X},R)$ *and every* $A\in\mathscr{F}(\mathfrak{X})$,

(F7) $\overline{\text{FAS}}^{\mathscr{T}}(\underline{\text{FAS}}_{\mathscr{I}}(A))=\underline{\text{FAS}}_{\mathscr{I}}(A)$

(F8) $\underline{\text{FAS}}_{\mathscr{I}}(\overline{\text{FAS}}^{\mathscr{T}}(A))=\overline{\text{FAS}}^{\mathscr{T}}(A)$.

Proof. (F7) For every $x\in\mathfrak{X}$,

$$\begin{aligned}\overline{\text{FAS}}^{\mathscr{T}}(\underline{\text{FAS}}_{\mathscr{I}}(A))(x) &= \sup_{y\in\mathfrak{X}}\mathscr{T}\left(R(x,z),\inf_{z\in\mathfrak{X}}\mathscr{I}(R(y,z),A(z))\right)\\ &= \inf_{z\in\mathfrak{X}}\sup_{y\in\mathfrak{X}}\mathscr{T}(R(x,y),\mathscr{I}(R(y,z),A(z))).\end{aligned}$$

By Lemma 3,

$$\begin{aligned}\overline{\text{FAS}}^{\mathscr{T}}(\underline{\text{FAS}}_{\mathscr{I}}(A))(x) &\leqslant \inf_{z\in\mathfrak{X}}\sup_{y\in\mathfrak{X}}\mathscr{I}(\mathscr{I}(R(x,y),R(y,z)),A(z))\\ &\leqslant \inf_{z\in\mathfrak{X}}\mathscr{I}\left(\inf_{y\in\mathfrak{X}}\mathscr{I}(R(x,y),R(y,z)),A(z)\right).\end{aligned}$$

Next, by Lemma 2,

$$\inf_{y\in\mathfrak{X}}\mathscr{I}(R(x,y),R(y,z)) = R(x,z) \quad \text{for all } x,z\in\mathfrak{X}.$$

Hence we get

$$\begin{aligned}\overline{\text{FAS}}^{\mathscr{T}}(\underline{\text{FAS}}_{\mathscr{I}}(A))(x) &\leqslant \inf_{z\in\mathfrak{X}}\mathscr{I}(R(x,z),A(z))\\ &= \underline{\text{FAS}}_{\mathscr{I}}(A)(x) \quad \text{for all } x\in\mathfrak{X}\end{aligned}$$

or equivalently,

$$\overline{\mathrm{FAS}}^{\mathscr{T}}(\underline{\mathrm{FAS}}_{\mathscr{I}}(A)) \subseteq \underline{\mathrm{FAS}}_{\mathscr{I}}(A).$$

Applying (F1) we get

$$\overline{\mathrm{FAS}}^{\mathscr{T}}(\underline{\mathrm{FAS}}_{\mathscr{I}}(A)) \supseteq \underline{\mathrm{FAS}}_{\mathscr{I}}(A),$$

whence $\overline{\mathrm{FAS}}^{\mathscr{T}}(\underline{\mathrm{FAS}}_{\mathscr{I}}(A)) = \underline{\mathrm{FAS}}_{\mathscr{I}}(A)$.
(F8) For every $x \in \mathfrak{X}$,

$$\begin{aligned}
&\underline{\mathrm{FAS}}_{\mathscr{I}}(\overline{\mathrm{FAS}}^{\mathscr{T}}(A))(x) \\
&\quad = \inf_{y\in\mathfrak{X}} \mathscr{I}(R(x,y), \overline{\mathrm{FAS}}^{\mathscr{T}}(A)(y)) \\
&\quad = \inf_{y\in\mathfrak{X}} \sup\{\alpha \in [0,1]\colon\ \mathscr{T}(R(x,y),\alpha) \\
&\qquad\qquad \leqslant \overline{\mathrm{FAS}}^{\mathscr{T}}(A)(y)\}.
\end{aligned}$$

By continuity of $\mathscr{T}$, for every $x, y \in \mathfrak{X}$,

$$\begin{aligned}
&\mathscr{T}(R(x,y), \overline{\mathrm{FAS}}^{\mathscr{T}}(A)(x)) \\
&\quad = \mathscr{T}\left(R(x,y), \sup_{z\in\mathfrak{X}} \mathscr{T}(R(x,z),A(z))\right) \\
&\quad = \sup_{z\in\mathfrak{X}} \mathscr{T}(R(x,y), \mathscr{T}(R(x,z),A(z))) \\
&\quad = \sup_{z\in\mathfrak{X}} \mathscr{T}(\mathscr{T}(R(x,y),R(x,z)),A(z)).
\end{aligned}$$

Next, by symmetry of R and Lemma 1,

$$\begin{aligned}
\mathscr{T}(R(x,y),R(x,z)) &= \mathscr{T}(R(y,x),R(x,z)) \\
&\leqslant R(y,z) \quad \text{for all } x,y,z \in \mathfrak{X},
\end{aligned}$$

so

$$\begin{aligned}
\mathscr{T}(R(x,y), \overline{\mathrm{FAS}}^{\mathscr{T}}(A)(x)) &\leqslant \sup_{z\in\mathfrak{X}} \mathscr{T}(R(y,z),A(z)) \\
&= \overline{\mathrm{FAS}}^{\mathscr{T}}(A)(y).
\end{aligned}$$

Then

$$\begin{aligned}
&\sup\{\alpha \in [0,1]\colon\ \mathscr{T}(R(x,y),\alpha) \leqslant \overline{\mathrm{FAS}}^{\mathscr{T}}(A)(y)\} \\
&\quad \geqslant \overline{\mathrm{FAS}}^{\mathscr{T}}(A)(x),
\end{aligned}$$

so

$$\begin{aligned}
&\underline{\mathrm{FAS}}_{\mathscr{I}}(\overline{\mathrm{FAS}}^{\mathscr{T}}(A))(x) \\
&\quad \geqslant \inf_{y\in\mathfrak{X}} \overline{\mathrm{FAS}}^{\mathscr{T}}(A)(x) \\
&\quad = \overline{\mathrm{FAS}}^{\mathscr{T}}(A)(x) \quad \text{for all } x \in \mathfrak{X}. \qquad (17)
\end{aligned}$$

By (F1),

$$\underline{\mathrm{FAS}}_{\mathscr{I}}(\overline{\mathrm{FAS}}^{\mathscr{T}}(A))(x) \leqslant \overline{\mathrm{FAS}}^{\mathscr{T}}(A)(x) \quad \text{for all } x \in \mathfrak{X}. \qquad (18)$$

Eqs. (17) and (18) imply the result. □

In general, however, neither (F7) nor (F8) hold for QL-implicators based on $\mathscr{T}$. Similarly, if $\mathscr{I}$ is an S-implicator based on a t-conorm $\mathscr{S}$ and an involutive negator $\mathscr{N}$, neither (F7) nor (F8) hold for $\mathscr{T}_{\mathscr{S}}(x,y) = \mathscr{N}(\mathscr{S}(\mathscr{N}(x), \mathscr{N}(y)))$. Below we give the corresponding counterexamples.

Example 2. Let $\mathrm{FAS} = (\mathfrak{X}, R)$ be as in Example 1 and let the Kleene–Dienes–Lukasiewicz implicator $\mathscr{I}_\bigstar$ be a representant of the class of S-implicators. $\mathscr{I}_\bigstar$ is based on the probabilistic sum $\mathscr{S}_\mathrm{P}$ and the standard negator $\mathscr{N}_\mathrm{s}$. Then $\mathscr{T}_{\mathscr{S}} \equiv \mathscr{T}_\mathrm{P}$. For the fuzzy set A given by

$$A = \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we can easily calculate

$$\begin{aligned}
\underline{\mathrm{FAS}}_{\mathscr{I}_\bigstar}(A) &= \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 0.5 \end{pmatrix} \\
&\subset \begin{pmatrix} a & b & c & d \\ 0.25 & 0.125 & 0.125 & 0.25 \end{pmatrix} \\
&= \overline{\mathrm{FAS}}^{\mathscr{T}_\mathrm{P}}(\underline{\mathrm{FAS}}_{\mathscr{I}_\bigstar}(A)),
\end{aligned}$$

$$\begin{aligned}
\overline{\mathrm{FAS}}^{\mathscr{T}_\mathrm{P}}(A) &= \begin{pmatrix} a & b & c & d \\ 0.5 & 0.25 & 0.25 & 1 \end{pmatrix} \\
&\supset \begin{pmatrix} a & b & c & d \\ 0.5 & 0.25 & 0.25 & 0.75 \end{pmatrix} \\
&= \underline{\mathrm{FAS}}_{\mathscr{I}_\bigstar}(\overline{\mathrm{FAS}}^{\mathscr{T}_\mathrm{P}}(A)).
\end{aligned}$$

Moreover, for the QL-implicator $\mathscr{I}_Z$ (based on $\mathscr{T}_M$), we get

$$\underline{\mathrm{FAS}}_{\mathscr{I}_Z}(A) = \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 0.5 \end{pmatrix}$$

$$\subset \begin{pmatrix} a & b & c & d \\ 0.5 & 0.25 & 0.25 & 0.5 \end{pmatrix}$$

$$= \overline{\mathrm{FAS}}^{\mathscr{T}_M}(\underline{\mathrm{FAS}}_{\mathscr{I}_Z}(A)),$$

$$\overline{\mathrm{FAS}}^{\mathscr{T}_M}(A) = \begin{pmatrix} a & b & c & d \\ 0.5 & 0.25 & 0.25 & 1 \end{pmatrix}$$

$$\supset \begin{pmatrix} a & b & c & d \\ 0.5 & 0.25 & 0.25 & 0.5 \end{pmatrix}$$

$$= \underline{\mathrm{FAS}}_{\mathscr{I}_Z}(\overline{\mathrm{FAS}}^{\mathscr{T}_M}(A)). \qquad \square$$

Propositions 7–9 immediately imply:

Corollary 4. *For every R-implicator $\mathscr{I}$ based on a continuous t-norm $\mathscr{T}$, every fuzzy approximation space $\mathrm{FAS} = (\mathfrak{X}, R)$ and every $A \in \mathscr{F}(\mathfrak{X})$, both $\underline{\mathrm{FAS}}_{\mathscr{I}}(A)$ and $\overline{\mathrm{FAS}}^{\mathscr{T}}(A)$ are $(\mathscr{I}, \mathscr{T})$-definable.*

4.2. Duality

Let us consider now dualities of fuzzy lower and fuzzy upper rough approximations, corresponding to crisp properties (P9) and (P10). Note first the following fact.

Lemma 4. *Let $\mathscr{I}$ and $\mathscr{T}$ be a border implicator and a t-norm, respectively, and let $\mathscr{N}$ be a negator. Then for every fuzzy approximation space $\mathrm{FAS} = (\mathfrak{X}, R)$ and every $A \in \mathscr{F}(\mathfrak{X})$,*

(a) *if $\mathscr{N}$ is weakly involutive and $\mathscr{N}(\mathscr{I}(x, y)) \geqslant \mathscr{T}(x, \mathscr{N}(y))$ for all $x, y \in [0, 1]$, then*

$$\underline{\mathrm{FAS}}_{\mathscr{I}}(A) \subseteq \mathrm{co}_{\mathscr{N}}\, \overline{\mathrm{FAS}}^{\mathscr{T}}(\mathrm{co}_{\mathscr{N}} A),$$

$$\overline{\mathrm{FAS}}^{\mathscr{T}}(A) \subseteq \mathrm{co}_{\mathscr{N}}\, \underline{\mathrm{FAS}}_{\mathscr{I}}(\mathrm{co}_{\mathscr{N}} A),$$

(b) *if $\mathscr{N}$ is involutive and $\mathscr{N}(\mathscr{I}(x, y)) \leqslant \mathscr{T}(x, \mathscr{N}(y))$ for all $x, y \in [0, 1]$, then*

$$\underline{\mathrm{FAS}}_{\mathscr{I}}(A) \supseteq \mathrm{co}_{\mathscr{N}}\, \overline{\mathrm{FAS}}^{\mathscr{T}}(\mathrm{co}_{\mathscr{N}} A),$$

$$\overline{\mathrm{FAS}}^{\mathscr{T}}(A) \supseteq \mathrm{co}_{\mathscr{N}}\, \underline{\mathrm{FAS}}_{\mathscr{I}}(\mathrm{co}_{\mathscr{N}} A).$$

Proof. (a) For every $x \in \mathfrak{X}$,

$$(\mathrm{co}_{\mathscr{N}}\, \overline{\mathrm{FAS}}^{\mathscr{T}}(\mathrm{co}_{\mathscr{N}} A))(x) = \mathscr{N}\left(\sup_{y \in \mathfrak{X}} \mathscr{T}(R(x, y), \mathscr{N}(A(y)))\right).$$

By assumption,

$$\mathscr{N}\left(\sup_{y \in \mathfrak{X}} \mathscr{T}(R(x, y), \mathscr{N}(A(y)))\right) \geqslant \mathscr{N}\left(\sup_{y \in \mathfrak{X}} \mathscr{N}(\mathscr{I}(R(x, y), A(y)))\right).$$

Furthermore,

$$\mathscr{N}\left(\sup_{y \in \mathfrak{X}} \mathscr{N}(\mathscr{I}(R(x, y), A(y)))\right) \geqslant \mathscr{N}\left(\mathscr{N}\left(\inf_{y \in \mathfrak{X}} \mathscr{I}(R(x, y), A(y))\right)\right).$$

Since $\mathscr{N}$ is weakly involutive, we get

$$\mathscr{N}\left(\mathscr{N}\left(\inf_{y \in \mathfrak{X}} \mathscr{I}(R(x, y), A(y))\right)\right) \geqslant \inf_{y \in \mathfrak{X}} \mathscr{I}(R(x, y), A(y)) = \overline{\mathrm{FAS}}^{\mathscr{T}}(A)(x).$$

Therefore, $\overline{\mathrm{FAS}}^{\mathscr{T}}(A) \subseteq \mathrm{co}_{\mathscr{N}}\, \overline{\mathrm{FAS}}^{\mathscr{T}}(\mathrm{co}_{\mathscr{N}} A)$.

Similarly, for every $x \in \mathfrak{X}$,

$$(\mathrm{co}_{\mathscr{N}}\, \underline{\mathrm{FAS}}_{\mathscr{I}}(\mathrm{co}_{\mathscr{N}} A))(x) = \mathscr{N}\left(\inf_{y \in \mathfrak{X}} \mathscr{I}(R(x, y), \mathscr{N}(A(y)))\right)$$

$$\geqslant \sup_{y \in \mathfrak{X}} \mathscr{N}(\mathscr{I}(R(x, y), \mathscr{N}(A(y))))$$

$$\geqslant \sup_{y \in \mathfrak{X}} \mathscr{T}(R(x, y), \mathscr{N}(\mathscr{N}(A(y))))$$

$$\geqslant \sup_{y\in\mathfrak{X}} \mathscr{T}(R(x,y),A(y))$$

$$= \overline{\mathrm{FAS}}^{\mathscr{T}}(A)(x).$$

Hence $\underline{\mathrm{FAS}}_{\mathscr{I}}(A) \subseteq \mathrm{co}_{\mathscr{N}}\, \overline{\mathrm{FAS}}^{\mathscr{T}}(\mathrm{co}_{\mathscr{N}} A)$.

In the analogous way (b) can be proved. □

We have the following fuzzy counterparts of properties (P9) and (P10).

Proposition 10. *For every fuzzy approximation space* $\mathrm{FAS}=(\mathfrak{X},R)$ *and every* $A\in\mathscr{F}(\mathfrak{X})$,

(a) *if* $\mathscr{I}$ *is an S-implicator based on a t-conorm* $\mathscr{S}$ *and an involutive negator* $\mathscr{N}$, *then*
(F9)(a) $\underline{\mathrm{FAS}}_{\mathscr{I}}(A) = \mathrm{co}_{\mathscr{N}}\, \overline{\mathrm{FAS}}^{\mathscr{T}}(\mathrm{co}_{\mathscr{N}} A)$,
(F10)(a) $\overline{\mathrm{FAS}}^{\mathscr{T}}(A) = \mathrm{co}_{\mathscr{N}}\, \underline{\mathrm{FAS}}_{\mathscr{I}}(\mathrm{co}_{\mathscr{N}} A)$,
where $\mathscr{T}$ *is dual to* $\mathscr{S}$ *wrt* $\mathscr{N}$;

(b) *if* $\mathscr{I}$ *is an R-implicator based on a continuous t-norm* $\mathscr{T}$ *and* $\mathscr{N}$ *is a negator induced by* $\mathscr{I}$, *or* $\mathscr{I}$ *is a* QL-*implicator based on a t-norm* $\mathscr{T}$ *and an involutive negator* $\mathscr{N}$, *then*
(F9)(b) $\underline{\mathrm{FAS}}_{\mathscr{I}}(A) \subseteq \mathrm{co}_{\mathscr{N}}\, \overline{\mathrm{FAS}}^{\mathscr{T}}(\mathrm{co}_{\mathscr{N}} A)$,
(F10)(b) $\overline{\mathrm{FAS}}^{\mathscr{T}}(A) \subseteq \mathrm{co}_{\mathscr{N}}\, \underline{\mathrm{FAS}}_{\mathscr{I}}(\mathrm{co}_{\mathscr{N}} A)$.

Proof. Notice that for every S-implicator $\mathscr{I}$ based on a t-conorm $\mathscr{S}$ and an involutive negator $\mathscr{N}$, we have for every $x,y\in[0,1]$

$$\mathscr{N}(\mathscr{I}(x,y)) = \mathscr{N}(\mathscr{S}(\mathscr{N}(x),y)) = \mathscr{T}_{\mathscr{S}}(x,\mathscr{N}(y)).$$

Then conditions (a) and (b) in Lemma 4 are satisfied. Hence we get the result.

Let $\mathscr{I}$ be an R-implicator based on a continuous t-norm $\mathscr{T}$ and let $\mathscr{N}$ be a negator induced by $\mathscr{I}$. By Lemma 3, for every $x,y\in[0,1]$,

$$\mathscr{N}(\mathscr{I}(x,y)) = \mathscr{I}(\mathscr{I}(x,y),0) \geqslant \mathscr{T}(x,\mathscr{I}(y,0))$$
$$= \mathscr{T}(x,\mathscr{N}(y)).$$

Moreover, $\mathscr{N}$ is weakly involutive, since for every $x\in[0,1]$,

$$\mathscr{N}(\mathscr{N}(x)) = \mathscr{I}(\mathscr{I}(x,0),0) \geqslant \mathscr{T}(x,\mathscr{I}(0,0))$$
$$= \mathscr{T}(x,1) = x.$$

Then, by Lemma 4(a), we get the result.

Finally, if $\mathscr{I}$ is a QL-implicator based on a t-norm $\mathscr{T}$, a t-conorm $\mathscr{S}$ and an involutive negator $\mathscr{N}$, then for every $x,y\in[0,1]$,

$$\mathscr{N}(\mathscr{I}(x,y)) = \mathscr{N}(\mathscr{S}(\mathscr{N}(x),\mathscr{T}(x,y)))$$
$$= \mathscr{T}(x,\mathscr{N}(\mathscr{T}(x,y))) \geqslant \mathscr{T}(x,\mathscr{N}(y)),$$

which, in view of Lemma 4(a), implies the result. □

Example 3. The Early Zadeh implicator $\mathscr{I}_{\mathrm{Z}}$ is a QL-implicator based on $\mathscr{T}_{\mathrm{M}}$, $\mathscr{S}_{\mathrm{M}}$ and $\mathscr{N}_{\mathrm{s}}$. The Łukasiewicz implicator $\mathscr{I}_{\mathrm{L}}$ is the S-implicator based on $\mathscr{S}_{\mathrm{L}}$ and $\mathscr{N}_{\mathrm{s}}$. By Proposition 10, for all $\mathrm{FAS}=(\mathfrak{X},R)$ and $A\in\mathscr{F}(\mathfrak{X})$,

$$\underline{\mathrm{FAS}}_{\mathscr{I}_{\mathrm{Z}}}(\mathrm{co}_{\mathscr{N}_{\mathrm{s}}} A) \subseteq \mathrm{co}_{\mathscr{N}_{\mathrm{s}}}\, \overline{\mathrm{FAS}}^{\mathscr{T}_{\mathrm{M}}}(A)$$

and

$$\underline{\mathrm{FAS}}_{\mathscr{I}_{\mathrm{L}}}(\mathrm{co}_{\mathscr{N}_{\mathrm{s}}} A) = \mathrm{co}_{\mathscr{N}_{\mathrm{s}}}\, \overline{\mathrm{FAS}}^{\mathscr{T}_{\mathrm{L}}}(A).$$

4.3. *Interactions with union and intersections*

In this point we will examine lower and upper fuzzy rough approximations of union and intersection of fuzzy sets, i.e. the fuzzy counterparts of (P11)–(P14) in Theorem 1. To begin with, consider (P11) and (P12). We will use the following auxiliary conditions.

Let $\mathscr{I}$ be an implicator and let $\mathscr{T}$ and $\mathscr{T}'$ be t-norms:

C2. $(\forall x,y,z\in[0,1])$
$(\mathscr{I}(x,\mathscr{T}(y,z)) \geqslant \mathscr{T}(\mathscr{I}(x,y),\mathscr{I}(x,z)))$,

C3. $(\forall x,y,z\in[0,1])$
$(\mathscr{T}(x,\mathscr{T}'(y,z)) \leqslant \mathscr{T}'(\mathscr{T}(x,y),\mathscr{T}(x,z)))$.

If C3 holds for $\mathscr{T}$ and $\mathscr{T}'$, then we say that the pair $(\mathscr{T},\mathscr{T}')$ satisfies C3.

Remark 6. It is easily noted that C2 results from fuzzifying the classical tautology of the form $\alpha\to(\beta\wedge\gamma)\leftrightarrow(\alpha\to\beta)\wedge(\alpha\to\gamma)$, where $\to$ and $\wedge$ are interpreted as an implicator $\mathscr{I}$ and a t-norm $\mathscr{T}$, respectively. Analogously, C3 is a weakened fuzzy counterpart of the tautology $\alpha\wedge(\beta\wedge\gamma)\leftrightarrow(\alpha\wedge\beta)\wedge(\alpha\wedge\gamma)$.

Observe also that C2 has the form of equality for $\mathscr{T}_{\mathrm{M}}$ and every right monotonic fuzzy implicator $\mathscr{I}$. Similarly, if $\mathscr{T}'\equiv\mathscr{T}_{\mathrm{M}}$, then C3 is equality, so $(\mathscr{T},\mathscr{T}_{\mathrm{M}})$ satisfies C3 for every t-norm $\mathscr{T}$.

Taking into account conditions C2 and C3, we have the following results on upper (resp. lower) fuzzy rough approximations of a $\mathscr{T}$-intersection of fuzzy sets.

Proposition 11. *For every fuzzy approximation space* $\mathrm{FAS} = (\mathfrak{X}, R)$ *and every* $A, B \in \mathscr{F}(\mathfrak{X})$,

(F11) $\underline{\mathrm{FAS}}_{\mathscr{I}}(A \cap_{\mathscr{T}} B) \supseteq \underline{\mathrm{FAS}}_{\mathscr{I}}(A) \cap_{\mathscr{T}} \underline{\mathrm{FAS}}_{\mathscr{I}}(B)$, *provided that* $\mathscr{I}$ *and* $\mathscr{T}$ *satisfy* C2. *If* $\mathscr{I}$ *is right monotonic, then*

$$\underline{\mathrm{FAS}}_{\mathscr{I}}(A \cap B) = \underline{\mathrm{FAS}}_{\mathscr{I}}(A) \cap \underline{\mathrm{FAS}}_{\mathscr{I}}(B).$$

(F12) $\overline{\mathrm{FAS}}^{\mathscr{T}}(A \cap_{\mathscr{T}'} B) \subseteq \overline{\mathrm{FAS}}^{\mathscr{T}}(A) \cap_{\mathscr{T}'} \overline{\mathrm{FAS}}^{\mathscr{T}}(B)$, *provided that* $(\mathscr{T}, \mathscr{T}')$ *satisfies* C3.

Proof. (F11) For every $x \in \mathfrak{X}$,

$$\begin{aligned}
&\underline{\mathrm{FAS}}_{\mathscr{I}}(A \cap_{\mathscr{T}} B)(x)\\
&\quad = \inf_{y\in\mathfrak{X}} \mathscr{I}(R(x,y), \mathscr{T}(A(y), B(y)))\\
&\quad \geqslant \inf_{y\in\mathfrak{X}} \mathscr{T}(\mathscr{I}(R(x,y), A(y)), \mathscr{I}(R(x,y), B(y)))\\
&\quad \geqslant \mathscr{T}\left(\inf_{y\in\mathfrak{X}} \mathscr{I}(R(x,y), A(y)),\right.\\
&\qquad\qquad \left.\inf_{y\in\mathfrak{X}} \mathscr{I}(R(x,y), B(y))\right)\\
&\quad = \mathscr{T}(\underline{\mathrm{FAS}}_{\mathscr{I}}(A)(x), \underline{\mathrm{FAS}}_{\mathscr{I}}(B)(x))\\
&\quad = (\underline{\mathrm{FAS}}_{\mathscr{I}}(A) \cap_{\mathscr{T}} \underline{\mathrm{FAS}}_{\mathscr{I}}(B))(x).
\end{aligned}$$

Hence $\underline{\mathrm{FAS}}_{\mathscr{I}}(A \cap_{\mathscr{T}} B) \supseteq \underline{\mathrm{FAS}}_{\mathscr{I}}(A) \cap_{\mathscr{T}} \underline{\mathrm{FAS}}_{\mathscr{I}}(B)$.

Also, for a right monotonic border implicator $\mathscr{I}$ and $x \in \mathfrak{X}$,

$$\begin{aligned}
&\underline{\mathrm{FAS}}_{\mathscr{I}}(A \cap B)(x)\\
&\quad = \inf_{y\in\mathfrak{X}} \mathscr{I}(R(x,y), \min\{A(y), B(y)\})\\
&\quad = \inf_{y\in\mathfrak{X}} \min\{\mathscr{I}(R(x,y), A(y)), \mathscr{I}(R(x,y), B(y))\}\\
&\quad = \min\left\{\inf_{y\in\mathfrak{X}} \mathscr{I}(R(x,y), A(y)),\right.\\
&\qquad\qquad \left.\inf_{y\in\mathfrak{X}} \mathscr{I}(R(x,y), B(y))\right\}\\
&\quad = (\underline{\mathrm{FAS}}_{\mathscr{I}}(A) \cap \underline{\mathrm{FAS}}_{\mathscr{I}}(B))(x),
\end{aligned}$$

so $\underline{\mathrm{FAS}}_{\mathscr{I}}(A \cap B) = \underline{\mathrm{FAS}}_{\mathscr{I}}(A) \cap \underline{\mathrm{FAS}}_{\mathscr{I}}(B)$.

Proceeding in the similar way we can show (F12). □

In view of Remark 6, from Proposition 11 we get:

Corollary 5. *Let* $\mathscr{T}$ *be a t-norm and let* $\mathscr{I}$ *be an S-implicator, an R-implicator or a* QL*-implicator. Then for every fuzzy approximation space* $\mathrm{FAS} = (\mathfrak{X}, R)$ *and every* $A, B \in \mathscr{F}(\mathfrak{X})$,

$$\overline{\mathrm{FAS}}^{\mathscr{T}}(A \cap B) \subseteq \overline{\mathrm{FAS}}^{\mathscr{T}}(A) \cap \overline{\mathrm{FAS}}^{\mathscr{T}}(B),$$

$$\underline{\mathrm{FAS}}_{\mathscr{I}}(A \cap B) = \underline{\mathrm{FAS}}_{\mathscr{I}}(A) \cap \underline{\mathrm{FAS}}_{\mathscr{I}}(B).$$

Example 4. Observe that the Kleene–Dienes implicator $\mathscr{I}_{\mathrm{KD}}$ and the t-norm $\mathscr{T}_{\mathrm{P}}$, satisfy C2. Indeed, for every $x, y, z \in [0, 1]$, we have

$$\begin{aligned}
&\mathscr{T}_{\mathrm{P}}(\mathscr{I}_{\mathrm{KD}}(x,y), \mathscr{I}_{\mathrm{KD}}(x,z))\\
&\quad = \max\{1-x, y\} * \max\{1-x, z\}\\
&\quad = \max\{(1-x)^2, y*(1-x), z*(1-x), y*z\}\\
&\quad \leqslant \max\{1-x, y*z\} = \mathscr{I}_{\mathrm{KD}}(x, \mathscr{T}_{\mathrm{P}}(y,z)).
\end{aligned}$$

Therefore,

$$\underline{\mathrm{FAS}}_{\mathscr{I}_{\mathrm{KD}}}(A \cap_{\mathscr{T}_{\mathrm{P}}} B) \supseteq \underline{\mathrm{FAS}}_{\mathscr{I}_{\mathrm{KD}}}(A) \cap_{\mathscr{T}_{\mathrm{P}}} \underline{\mathrm{FAS}}_{\mathscr{I}_{\mathrm{KD}}}(B).$$

Also, in view of Remark 6,

$$\overline{\mathrm{FAS}}^{\mathscr{T}_{\mathrm{P}}}(A \cap B) \subseteq \overline{\mathrm{FAS}}^{\mathscr{T}_{\mathrm{P}}}(A) \cap \overline{\mathrm{FAS}}^{\mathscr{T}_{\mathrm{P}}}(B). \qquad \square$$

Let us examine lower (resp. upper) fuzzy rough approximations of union of fuzzy sets, i.e. the fuzzy counterparts of (P13) and (P14). Consider first the following condition: for an implicator $\mathscr{I}$ and a t-conorm $\mathscr{S}$,

C4. $(\forall x, y, z \in [0,1])$ $(\mathscr{I}(x, \mathscr{S}(y,z)) \geqslant \mathscr{S}(\mathscr{I}(x,y), \mathscr{I}(x,z)))$.

Remark 7. It is easily noted that C4 is the equality for $\mathscr{S}_{\mathrm{M}}$, provided that $\mathscr{I}$ is right monotonic. Then, in view of Proposition 1, C4 holds for pairs $(\mathscr{I}, \mathscr{S}_{\mathrm{M}})$, where $\mathscr{I}$ is an S-implicator, R-implicator or QL-implicator.

Recall [3] that a t-norm $\mathscr{T}$ and a t-conorm $\mathscr{S}$ satisfy *weakened distributivity laws* iff

$$\mathscr{T}(x,\mathscr{S}(y,z)) \leqslant \mathscr{S}(\mathscr{T}(x,y),\mathscr{T}(x,z)),$$
$$\mathscr{T}(\mathscr{S}(x,y),\mathscr{S}(x,z)) \leqslant \mathscr{S}(x,\mathscr{T}(y,z)).$$

Proposition 12. *Let $\mathscr{I},\mathscr{T},\mathscr{S}$ be a border implicator, a t-norm and a t-conorm, respectively. Then for every fuzzy approximation space* $\mathrm{FAS}=(\mathfrak{X},R)$ *and every $A,B\in\mathscr{F}(\mathfrak{X})$,*

(F13) $\underline{\mathrm{FAS}}_{\mathscr{I}}(A\cup_{\mathscr{S}}B) \supseteq \underline{\mathrm{FAS}}_{\mathscr{I}}(A)\cup_{\mathscr{S}}\underline{\mathrm{FAS}}_{\mathscr{I}}(B)$, *provided that $\mathscr{I}$ and $\mathscr{S}$ satisfy* C4,

(F14) $\overline{\mathrm{FAS}}^{\mathscr{T}}(A\cup_{\mathscr{S}}B) \subseteq \overline{\mathrm{FAS}}^{\mathscr{T}}(A)\cup_{\mathscr{S}}\overline{\mathrm{FAS}}^{\mathscr{T}}(B)$, *provided that $\mathscr{T}$ and $\mathscr{S}$ satisfy the weakened distributivity laws*
$\overline{\mathrm{FAS}}^{\mathscr{T}}(A\cup B)=\overline{\mathrm{FAS}}^{\mathscr{T}}(A)\cup\overline{\mathrm{FAS}}^{\mathscr{T}}(B)$.

Proof. For every $x\in\mathfrak{X}$,

$$\begin{aligned}
&\underline{\mathrm{FAS}}_{\mathscr{I}}(A\cup_{\mathscr{S}}B)(x)\\
&\quad=\inf_{y\in\mathfrak{X}}\mathscr{I}(R(x,y),\mathscr{S}(A(y),B(y)))\\
&\quad\geqslant\inf_{y\in\mathfrak{X}}\mathscr{S}(\mathscr{I}(R(x,y),A(y)),\mathscr{I}(R(x,y),B(y)))\\
&\quad\geqslant\mathscr{S}\left(\inf_{y\in\mathfrak{X}}\mathscr{I}(R(x,y),A(y)),\right.\\
&\qquad\qquad\left.\inf_{y\in\mathfrak{X}}\mathscr{I}(R(x,y),B(y))\right)\\
&\quad=(\underline{\mathrm{FAS}}_{\mathscr{I}}(A)\cup_{\mathscr{S}}\underline{\mathrm{FAS}}_{\mathscr{I}}(B))(x).
\end{aligned}$$

Hence $\underline{\mathrm{FAS}}_{\mathscr{I}}(A\cup_{\mathscr{S}}B)\supseteq\underline{\mathrm{FAS}}_{\mathscr{I}}(A)\cup_{\mathscr{S}}\underline{\mathrm{FAS}}_{\mathscr{I}}(B)$.
(F14) For every $x\in\mathfrak{X}$,

$$\begin{aligned}
&\overline{\mathrm{FAS}}^{\mathscr{T}}(A\cup_{\mathscr{S}}B)(x)\\
&\quad=\sup_{y\in\mathfrak{X}}\mathscr{T}(R(x,y),\mathscr{S}(A(y),B(y)))\\
&\quad\leqslant\sup_{y\in\mathfrak{X}}\mathscr{S}(\mathscr{T}(R(x,y),A(y)),\mathscr{T}(R(x,y),B(y)))\\
&\quad\leqslant\mathscr{S}\left(\sup_{y\in\mathfrak{X}}\mathscr{T}(R(x,y),A(y)),\right.\\
&\qquad\qquad\left.\sup_{y\in\mathfrak{X}}\mathscr{T}(R(x,y),B(y))\right)\\
&\quad=(\overline{\mathrm{FAS}}^{\mathscr{T}}(A)\cup_{\mathscr{S}}\overline{\mathrm{FAS}}^{\mathscr{T}}(B))(x).
\end{aligned}$$

Hence $\overline{\mathrm{FAS}}^{\mathscr{T}}(A\cup_{\mathscr{S}}B)\subseteq\overline{\mathrm{FAS}}^{\mathscr{T}}(A)\cup_{\mathscr{S}}\overline{\mathrm{FAS}}^{\mathscr{T}}(B)$.
Also, for every $x\in\mathfrak{X}$,

$$\begin{aligned}
&\overline{\mathrm{FAS}}^{\mathscr{T}}(A\cup B)(x)\\
&\quad=\sup_{y\in\mathfrak{X}}\mathscr{T}(R(x,y),\max\{A(y),B(y)\})\\
&\quad=\sup_{y\in\mathfrak{X}}\max\{\mathscr{T}(R(x,y),A(y)),\mathscr{T}(R(x,y),B(y))\}\\
&\quad=\max\left\{\sup_{y\in\mathfrak{X}}\mathscr{T}(R(x,y),A(y)),\right.\\
&\qquad\qquad\left.\sup_{y\in\mathfrak{X}}\mathscr{T}(R(x,y),B(y))\right\}\\
&\quad=(\overline{\mathrm{FAS}}^{\mathscr{T}}(A)\cup\overline{\mathrm{FAS}}^{\mathscr{T}}(B))(x),
\end{aligned}$$

so $\overline{\mathrm{FAS}}^{\mathscr{T}}(A\cup B)=\overline{\mathrm{FAS}}^{\mathscr{T}}(A)\cup\overline{\mathrm{FAS}}^{\mathscr{T}}(B)$. □

By Proposition 12 and Remark 7, we get the following corollary.

Corollary 6. *Let $\mathscr{I}$ be an S-implicator, R-implicator or* QL-*implicator. Then for every approximation space* $\mathrm{FAS}=(\mathfrak{X},R)$ *and every $A,B\in\mathscr{F}(\mathfrak{X})$,*

$$\underline{\mathrm{FAS}}_{\mathscr{I}}(A\cup B)\supseteq\underline{\mathrm{FAS}}_{\mathscr{I}}(A)\cup\underline{\mathrm{FAS}}_{\mathscr{I}}(B).$$

It is worth noting that (F13) holds neither for S-implicators nor QL-implicators (based on $\mathscr{S}$). Consider the following example.

Example 5. Let $\mathrm{FAS}=(\mathfrak{X},R)$ be as in Example 1 and A,B and C be fuzzy sets on $\mathfrak{X}$ given by

$$A=\begin{pmatrix} a & b & c & d\\ 0 & 0 & 0 & 0.25\end{pmatrix},$$

$$B=\begin{pmatrix} a & b & c & d\\ 0 & 0 & 0 & 0.5\end{pmatrix},$$

$$C=\begin{pmatrix} a & b & c & d\\ 0.25 & 0.75 & 1 & 0\end{pmatrix}.$$

For the S-implicator $\mathscr{I}_L$ (based on $\mathscr{S}_L$), one can easily calculate

$$\underline{\text{FAS}}_{\mathscr{I}_L}(A \cup_{\mathscr{S}_B} B) = \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 0.5 \end{pmatrix},$$

$$\underline{\text{FAS}}_{\mathscr{I}_L}(A) \cup_{\mathscr{S}_B} \underline{\text{FAS}}_{\mathscr{I}_L}(B) = \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 0.75 \end{pmatrix},$$

$$\underline{\text{FAS}}_{\mathscr{I}_L}(A \cup_{\mathscr{S}_B} C) = \begin{pmatrix} a & b & c & d \\ 0.25 & 0.75 & 1 & 0.25 \end{pmatrix},$$

$$\underline{\text{FAS}}_{\mathscr{I}_L}(A) \cup_{\mathscr{S}_B} \underline{\text{FAS}}_{\mathscr{I}_L}(C) = \begin{pmatrix} a & b & c & d \\ 0.25 & 0.75 & 0.75 & 0.25 \end{pmatrix}.$$

Obviously,

$$\underline{\text{FAS}}_{\mathscr{I}_L}(A \cup_{\mathscr{S}_B} B) \subset \underline{\text{FAS}}_{\mathscr{I}_L}(A) \cup_{\mathscr{S}_B} \underline{\text{FAS}}_{\mathscr{I}_L}(B),$$

$$\underline{\text{FAS}}_{\mathscr{I}_L}(A \cup_{\mathscr{S}_B} C) \supset \underline{\text{FAS}}_{\mathscr{I}_L}(A) \cup_{\mathscr{S}_B} \underline{\text{FAS}}_{\mathscr{I}_L}(C).$$

The analogous relations hold for the QL-implicator $\mathscr{I}_k(x, y) = 1 - x + x^2 * y$ (based on $\mathscr{S}_P$) and $\mathscr{S}_P$.

4.4. Fuzzy rough approximations of similarity classes

Proposition 13. *For every t-norm $\mathscr{T}$ and every fuzzy approximation space* $\text{FAS} = (\mathfrak{X}, R)$,

$$\overline{\text{FAS}}^{\mathscr{T}}([x]_R) = [x]_R \quad \textit{for all } x \in \mathfrak{X}.$$

Proof. By Lemma 1, for every $x, y \in \mathfrak{X}$,

$$\begin{aligned} &\overline{\text{FAS}}^{\mathscr{T}}([x]_R)(y) \\ &\quad = \sup_{z\in\mathfrak{X}} \mathscr{T}(R(y,z), R(x,z)) \\ &\quad = \sup_{z\in\mathfrak{X}} \mathscr{T}(R(x,z), R(z,y)) \\ &\quad = R(x,y) = [x]_R(y). \qquad \square \end{aligned}$$

Proposition 14. *For every R-implicator $\mathscr{I}$ and every fuzzy approximation space* $\text{FAS} = (\mathfrak{X}, R)$,

$$\underline{\text{FAS}}_{\mathscr{I}}([x]_R) = [x]_R \quad \textit{for all } x \in \mathfrak{X}. \tag{19}$$

Proof. Follows directly from Lemma 2. □

By Propositions 13 and 14 we immediately get:

Corollary 7. *For every continuous t-norm $\mathscr{T}$, every R-implicator $\mathscr{I}$ based on $\mathscr{T}$, every fuzzy approximation space* $\text{FAS} = (\mathfrak{X}, R)$ *and every* $x \in \mathfrak{X}$, $[x]_R$ *is* $(\mathscr{I}, \mathscr{T})$*-definable.*

In general, property (19) holds neither for S-implicators nor QL-implicators.

Example 6. Let $\text{FAS} = (\mathfrak{X}, R)$ be defined as in Example 1. For the QL-implicator $\mathscr{I}_Z$ and the S-implicator $\mathscr{I}_\star$, we can easily get

$$\underline{\text{FAS}}_{\mathscr{I}_Z}([a]_R) = \begin{pmatrix} a & b & c & d \\ 0.5 & 0.25 & 0.25 & 0.5 \end{pmatrix} \subset \begin{pmatrix} a & b & c & d \\ 1 & 0.25 & 0.25 & 0.5 \end{pmatrix} = [a]_R,$$

$$\underline{\text{FAS}}_{\mathscr{I}_*}([a]_R) = \begin{pmatrix} a & b & c & d \\ 0.75 & 0.25 & 0.25 & 0.5 \end{pmatrix} \subset [a]_R.$$

5. Concluding remarks

In this paper, we have proposed a general definition of fuzzy rough sets and have investigated several properties of them. Three classes of fuzzy rough approximations have been defined, namely the classes S-FRA, R-FRA and Q-FRA, and properties of these classes were essentially investigated.

In view of the established results the following conclusions are worth emphasizing:

1. For residual fuzzy rough approximations all properties (from among 15 under consideration) are satisfied, except (F9) and (F10) (in (F11) and (F13) we take Zadeh's fuzzy intersection and fuzzy union).
2. For the class S-FRA only (F6), (F7) and (F15) do not hold.
3. For Q-FRA, most considered properties do not hold. Specifically, only (F1), (F2), (F11), (F13) are actually satisfied (together with (F9)(b) and (F10)(b) representing the weakened forms of duality).

It suggests therefore that the classes S-FRA and R-FRA exhibit essential similarities to crisp rough approximations and might be viewed as the most natural fuzzy generalizations of Pawlak's original concept. Furthermore, since the Łukasiewicz implicator $\mathscr{I}_{\mathrm{L}}$ is both an S-implicator and a residual implicator, fuzzy rough approximations determined by $(\mathscr{I}_{\mathrm{L}}, \mathscr{T}_{\mathrm{L}})$, which might be called *Łukesiewicz fuzzy rough approximations*, satisfy all properties considered here.

Finally, let us refer our results to the operator-oriented characteristics of rough sets, as it was given in Section 2. First, let us introduce the following notion.

Definition 8. Let $L, H : \mathscr{F}(\mathfrak{X}) \to \mathscr{F}(\mathfrak{X})$ be two mappings and let $\Lambda(L,H)$ be a collection of conditions characterizing these mappings. Furthermore, let $\mathscr{I}$ and $\mathscr{T}$ be a border implicator and a t-norm, respectively. We say that $\Lambda(L,H)$ is *sound* for $(\mathscr{I},\mathscr{T})$-fuzzy rough approximations iff every condition in $\Lambda(\underline{\mathrm{FAS}}_{\mathscr{I}}, \overline{\mathrm{FAS}}^{\mathscr{T}})$ holds. It is called *complete* for $(\mathscr{I},\mathscr{T})$-fuzzy rough approximations iff every two mappings $L', H' : \mathscr{F}(\mathfrak{X}) \to \mathscr{F}(\mathfrak{X})$ satisfying all conditions in $\Lambda(L', H')$ determine a similarity relation R on $\mathfrak{X}$ such that $L \equiv \underline{\mathrm{FAS}}_{\mathscr{I}}$ and $H \equiv \overline{\mathrm{FAS}}^{\mathscr{T}}$.

The results obtained in Section 4 show that (A.1)–(A.6) is not sound for any of the three classes of fuzzy rough approximations considered in this paper. In particular, (A.6) need not hold for S-FRA (Example 2), duality does not hold in general for R-FRA (Proposition 10(b)), whereas for Q-FRA, both duality and (A.5), (A.6) are not generally satisfied (Example 2, Proposition 10(b)). However, we have:

Proposition 15. *The characterization* (A.1)–(A.6) *is sound for the Łukasiewicz fuzzy rough approximations.*

Proof. Follows from Propositions 3(F1), (F2), 9(F6), 10(F9)(a), (F10)(a), 12(F14) and Corollary 3. □

As one may expect, these axioms do not give sufficient characterization of Łukasiewicz fuzzy rough sets, i.e. (A.1)–(A.6) is not *complete* for this class. Consider, for example, two mappings $L, H : \mathscr{F}(\mathfrak{X}) \to \mathscr{F}(\mathfrak{X})$ defined by

$$L(A) = \begin{cases} \emptyset & \text{if } A \neq \mathfrak{X}, \\ \mathfrak{X} & \text{if } A = \mathfrak{X}, \end{cases} \qquad H(A) = \begin{cases} \emptyset & \text{if } A = \emptyset, \\ \mathfrak{X} & \text{if } A \neq \emptyset. \end{cases}$$

It is easy to see that L and H satisfy (A.1)–(A.6), where the fuzzy complementation $\mathrm{co}_{\mathscr{N}}$ in (A.1) is defined with respect to the standard negator $\mathscr{N}_{\mathrm{s}}$. In the crisp case they induce the equivalence relation $R = \mathfrak{X} \times \mathfrak{X}$, yet they do not determine any fuzzy binary relation. To see that, let us take a fuzzy set $\mathbb{I}_\alpha$ for an arbitrary $\alpha \in]0,1[$. Then $H(\mathbb{I}_\alpha) = \mathfrak{X}$, i.e. for all $x \in \mathfrak{X}$, $H(\mathbb{I}_\alpha)(x) = 1$, but for every fuzzy binary relation R on $\mathfrak{X}$,

$$\overline{\mathrm{FAS}}^{\mathscr{T}_{\mathrm{L}}}(\mathbb{I}_\alpha) = \sup_{y \in \mathfrak{X}} \mathscr{T}_{\mathrm{L}}(R(x,y), \mathbb{I}_\alpha(y)) = \sup_{y \in \mathfrak{X}} \mathscr{T}_{\mathrm{L}}(R(x,y), \alpha) < 1.$$

It is still an open problem concerning a complete operator-oriented characterization of Łukasiewicz fuzzy rough sets. Moreover, it would be certainly worthwhile to consider similar characteristics for other classes of fuzzy rough sets. In order to formulate such characterization, one needs to have underlying knowledge about properties of particular classes of these structures. In this respect, the present contribution is the first step in this direction.

A natural extension of the presented approach is considering fuzzy rough sets defined relatively to arbitrary fuzzy binary relations. In the crisp case, this problem was broadly discussed in the literature (see, for example, [13,17,19,20]). Our aim is to construct a general classification of fuzzy rough sets.

Another step towards extension of the present approach is to consider fuzzification of modal-like *information logics*. These logical systems, extensively investigated by Orłowska [9,10] and Vakarelov [22] in the framework of rough set theory, allow for representing and reasoning about properties of objects, as well as relations among these objects (referred to as *information relations*). Recently [11], many-valued information logics have been proposed. It is interesting to consider a fuzzy generalization of this approach, in particular with the aim to formulate inference patterns allowing for the analysis of mereological relations between objects in fuzzy information systems.

These topics will be investigated in our ongoing work.

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