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EEE-473
Homework 4

1)

a) $\mathrm{rect}(\frac{x}{4}, y) * \delta(x-1, y-2) =$

$= \left(\mathrm{rect}(\frac{x}{4}) \cdot \mathrm{rect}(y)\right) * \left(\delta(x-1) \cdot \delta(y-2)\right)$ — due to separability of functions

$= \left(\mathrm{rect}(\frac{x}{4}) * \delta(x-1)\right) \cdot \left(\mathrm{rect}(y) * \delta(y-2)\right)$ — due to convolution property (Ex. 2.11b)

$= \mathrm{rect}\left(\frac{x-1}{4}\right) \cdot \mathrm{rect}(y-2) = \mathrm{rect}\left(\frac{x-1}{4}, y-2\right)$

b) $\mathrm{rect}(\frac{x}{4}, y) * \delta(y-2) =$

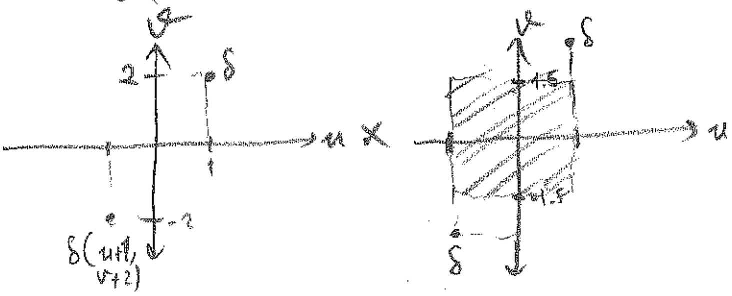
$= \left(\mathrm{rect}(\frac{x}{4}) * 1\right) \cdot \left(\mathrm{rect}(y) * \delta(y-2)\right)$

$= F^{-1}\left(4\,\mathrm{sinc}(4u) \cdot \delta(u)\right) \cdot \left(\mathrm{rect}(y-2)\right) =$

$= F^{-1}\left(4 \cdot \delta(u)\right) \cdot \mathrm{rect}(y-2) = 4 \cdot \mathrm{rect}(y-2)$

c) $\cos(2\pi x + 4\pi y) * \mathrm{sinc}(2x, 3y)$

$F^{-1}_{2D}\left[\left(\frac{1}{2}\delta(u-1, v-2) + \frac{1}{2}\delta(u+1, v+2)\right) \cdot \frac{1}{6}\mathrm{rect}\left(\frac{u}{2}, \frac{v}{3}\right)\right] = 0$



d) $\cos(2\pi(u_0 x + v_0 y)) * e^{-(x^2+y^2)}$

$$F_{2D}^{-1}\left[\left(\frac{1}{2}\,\delta(u-u_0, v-v_0) + \frac{1}{2}\,\delta(u+u_0; v+v_0)\right) \cdot \pi \cdot e^{-\pi^2 q^2}\right]$$

$$F_{2D}^{-1}\left[\frac{\pi}{2}\, e^{-\pi^2(u_0^2+v_0^2)} \cdot \left(\delta(u-u_0, v-v_0) + \delta(u+u_0, v+v_0)\right)\right]$$

$$= \pi \cdot e^{-\pi^2(u_0^2+v_0^2)} \cdot \cos(2\pi(u_0 x + v_0 y))$$

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e) $f(r) = \text{rect}\left(\frac{r-a}{b}\right) \xrightarrow{F_{2D}} b^2\, F_{2D}(\text{rect}(r-a))\Big|_{q \to bq}$

$a\cos\theta$ shift in x

$a\sin\theta$ shift in y

$$F_{2D}(\text{rect}(r-a)) = F_{2D}(\text{rect}(r)) \cdot e^{-j2\pi u \cdot a\cos\theta} \cdot e^{-j2\pi v \cdot a \cdot \sin\theta}$$

$$= \text{jinc}(q) \cdot e^{-j2\pi a \cdot (u \cdot \cos\theta + v \cdot \sin\theta)}$$

$$= \text{jinc}(q) \cdot e^{-j2\pi a \cdot (q \cdot \cos^2\theta + q\sin^2\theta)}$$

$$= \text{jinc}(q) \cdot e^{-j2\pi a \cdot q}$$

2) $L = 20\,cm$, $w_1 = 8cm \times 8cm$, $w_2 = 2cm \times 2cm$, $\mu_0 = 0.05\ cm^{-1}$

$I_d(x,0)$; $y=0$, $d=60$

M = object magnification

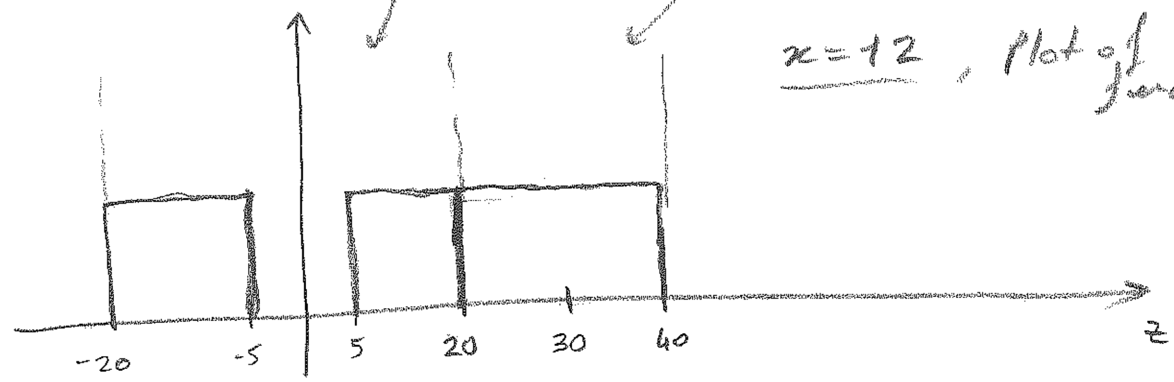
$$\mu(x,z) = \mu_0 \left[rect\left(\frac{x}{8}\right) - rect\left(\frac{x}{2}\right) \right] \cdot rect\left(\frac{z-30}{20}\right)$$

$$t_d(x,y) = \exp\left\{ -\int_z \mu\left(\frac{x}{M}, z\right) dz \right\} \qquad M = d/z$$

$$I_d(x,y) = \underbrace{\frac{I_s}{4\pi d^2}}_{I_0} \cdot \exp\left\{ -\mu_0 \underbrace{\int \left[rect\left(\frac{x}{8M}\right) - rect\left(\frac{x}{2M}\right) \right] \cdot rect\left(\frac{z-30}{20}\right) dz}_{A} \right\}$$

$$\left(rect\left(\frac{xz}{8d}\right) - rect\left(\frac{xz}{2d}\right) \right) \cdot rect\left(\frac{z-30}{20}\right)$$

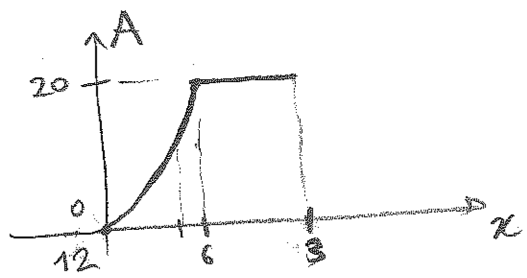
$$\left(rect\left(\frac{xz}{480}\right) - rect\left(\frac{xz}{120}\right) \right) \cdot rect\left(\frac{z-30}{20}\right)$$



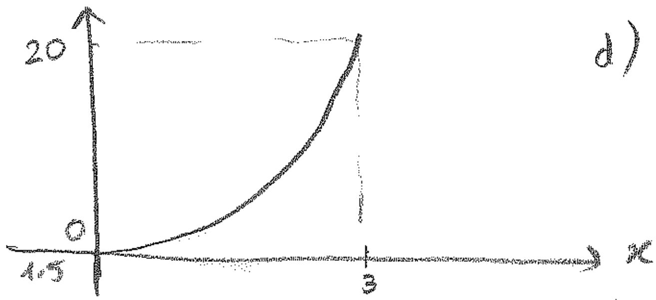
$x = 12$, Plot of two functions

a) $x > 12 \Rightarrow A = 0$ (since there is no overlap)

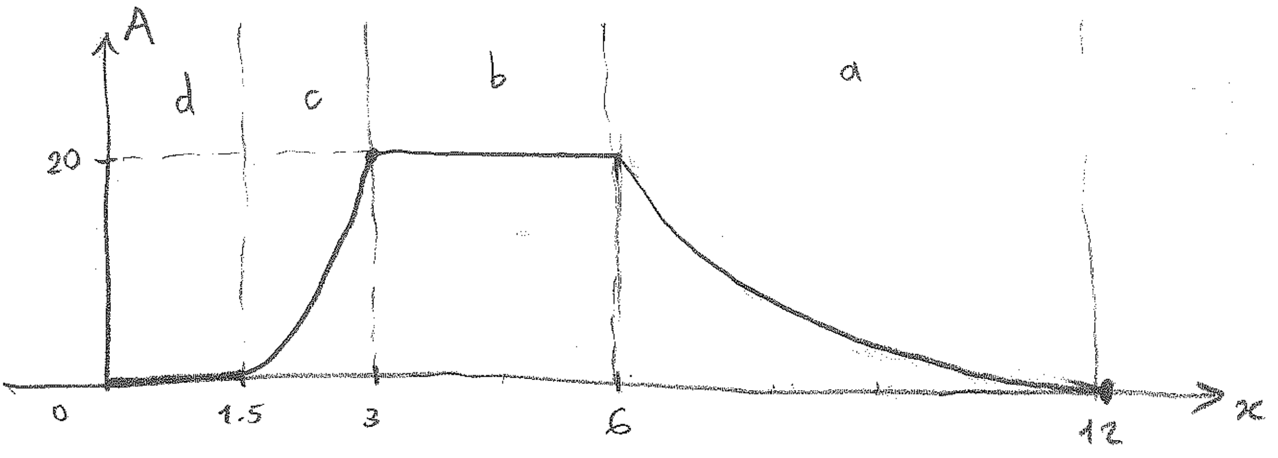
b) $3 < x < 12 \Rightarrow$



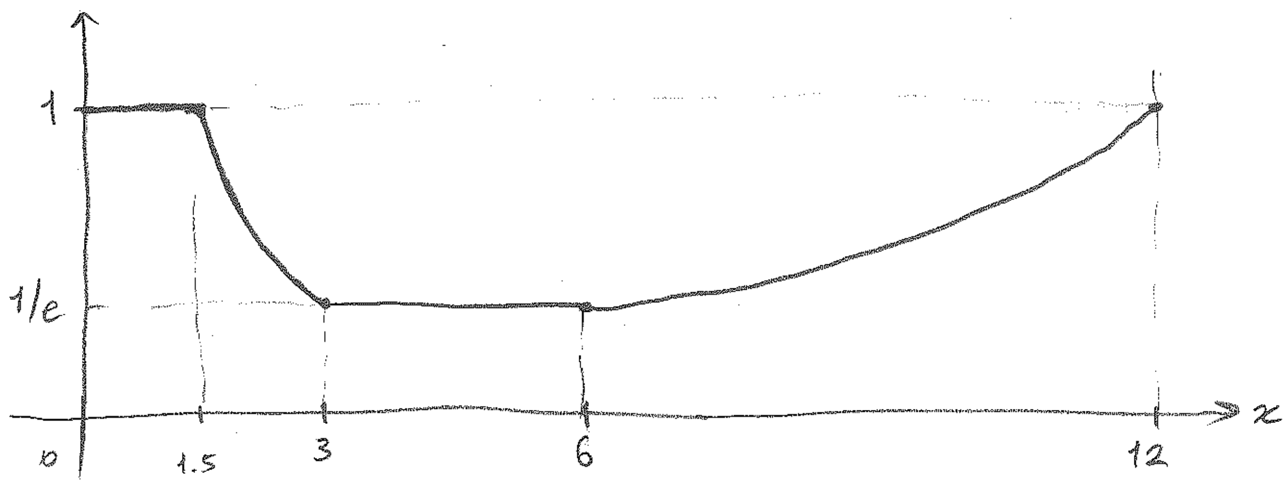
c) $1.5 < x < 3$



d) $x < 1.5 \Rightarrow A = 0$
↓
no overlap



$$\frac{I_d}{I_o} = \exp\left\{-\mu_o A\right\} \; ; \quad \mu_o = 0.05$$



3)

a) By Projection Slice Theorem,

$$f(x,y) \xrightarrow{F_{2D}} F(u,v)\Big|_{\substack{u=\rho\cos\theta \\ v=\rho\sin\theta}} \xrightarrow{F_{1D}^{-1}} g(l,\theta)$$

$$g(l,\theta) = 4\,\mathrm{sinc}(2l).\cos(4\pi l)$$

$$G(\rho,\theta) = \cancel{4}.\frac{1}{\cancel{2}}.\,\mathrm{rect}(\rho/2) * \frac{1}{\cancel{2}}\left[\delta(\rho-2)+\delta(\rho+2)\right]$$

$$= \mathrm{rect}\left(\frac{\rho-2}{2}\right) + \mathrm{rect}\left(\frac{\rho+2}{2}\right) = F(u,v) \xrightarrow{F_{2D}^{-1}} f(x,y)$$

$$f(x,y) = e^{+j2\pi.2r}.\,4.\,\mathrm{jinc}(2r) + e^{j2\pi.-2.r}.\,4.\,\mathrm{jinc}(2r)$$

$$= 8.\left(\frac{1}{2}e^{j2\pi 2r} + \frac{1}{2}e^{j2\pi -2r}\right).\,\mathrm{jinc}(2r)$$

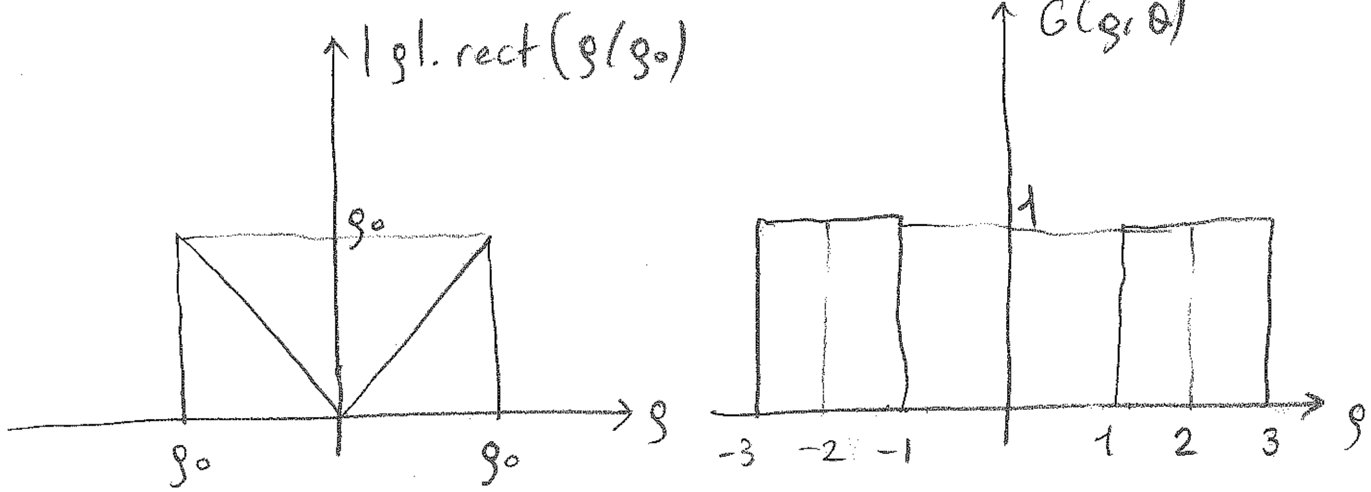
$$= 8.\cos(4\pi r).\,\mathrm{jinc}(2r)$$

3)

b) $$f(x,y) = \int_0^{\pi} \left[\int_{-\infty}^{\infty} |\rho| \cdot \mathrm{rect}\left(\frac{\rho}{\rho_0}\right) \cdot G(\rho,\theta) \cdot e^{j2\pi\rho l} \, d\rho \right] d\theta$$

$l = x\cos\theta + y\sin\theta$

$$G(\rho,\theta) = \mathrm{rect}\left(\frac{\rho-2}{2}\right) + \mathrm{rect}\left(\frac{\rho+2}{2}\right)$$



★ $\rho_0 < 1 \implies f(x,y) = 0$ since $|\rho| \cdot \mathrm{rect}(\rho/\rho_0) \cdot G(\rho,\theta) = 0$

$3 > \rho_0 > 1 \implies |\rho| \cdot \mathrm{rect}(\rho/\rho_0) \cdot G(\rho,\theta) = |\rho|$

$$f(\rho_0) = \int_1^{\rho_0} \rho \cdot e^{j2\pi\rho l} \, d\rho \underset{\text{Integration by parts}}{=} \left(\frac{\rho}{2\pi j l} + \frac{1}{4\pi^2 l^2} \right) e^{j2\pi\rho l} \Big|_1^{\rho_0}$$

$$\rho_0 > 3 \implies \int_1^3 \rho \cdot e^{j2\pi\rho l} \, d\rho = \frac{(6\pi j l + 1)}{4\pi^2 l^2} e^{j6\pi l} - \frac{(2\pi j l + 1)}{4\pi^2 l^2} e^{j2\pi l}$$

$$+ \int_{-3}^{-1} -\rho \cdot e^{j2\pi\rho l} \, d\rho = \int_{-1}^{-3} \rho \cdot e^{j2\pi\rho l} \, d\rho$$

$$= \frac{(-6\pi j l + 1)}{4\pi^2 l^2} e^{-j6\pi l} - \frac{(-2\pi j l + 1)}{4\pi^2 l^2} e^{-j2\pi l}$$

★ $1 < \varrho_0 < 3$ $\quad f(x,y) = \int_0^{\pi} \left[\int_1^3 \varrho \cdot e^{j2\pi\varrho l} d\varrho + \int_{-1}^{-3} \varrho \cdot e^{j2\pi\varrho l} d\varrho \right] d\theta$

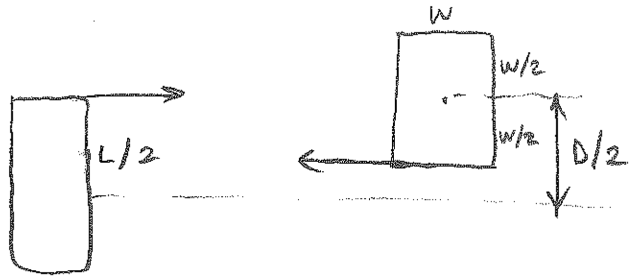
$l = x\cos\theta + y\sin\theta$

★ $\varrho_0 > 3$, Then $f_c(x,y) = f(x,y) = 8 \cdot \cos(4\pi r) \cdot \mathrm{jinc}(2r)$

Because, the $G(\varrho, \theta)$ is not affected by windowing.

4) $s(x,y) = \mathrm{rect}\left(\frac{x}{L}, \frac{y}{L}\right)$ & $t_z(x,y) = \mathrm{rect}\left(\frac{x+D/2}{W}, \frac{y}{W}\right) + \mathrm{rect}\left(\frac{x-D/2}{W}, \frac{y}{W}\right)$

a)



The lowest point of the image in the x-axis is determined by $s(x,y) = \delta\left(x - \frac{L}{2}\right)$ and $t(x,y) = \delta\left(x - \frac{D}{2} + \frac{W}{2}\right)$. Since the objects are symmetric around zy plane, in order to ensure that the images are nontouching, x axis component of the higher image must be positive.

M: object magnification

m: Source magnification

$$I_d(x,y) = s\left(\frac{x}{m}, \frac{y}{m}\right) * t\left(\frac{x}{M}, \frac{y}{M}\right)$$

$$I_d(x,y) = \delta\left(\frac{x}{m} - \frac{Lm/2}{m}\right) * \delta\left(\frac{x}{M} - \frac{(D-W).M/2}{M}\right)$$

$$I_d(x,y) = M m\, \delta\left(x - \frac{Lm}{2}\right) * \delta\left(x - \frac{(D-W).M}{2}\right)$$

$$= M.m\, \delta\left(x - \frac{Lm + (D-W).M}{2}\right)$$ must be positive

$\leftarrow \delta(x-a)$

$x=0$

$$Lm + (D-W).M \geq 0 \quad \& \quad M = \frac{d}{z}, \quad m = \frac{z-d}{z}$$

$$\frac{L.(z-d)}{z} + \frac{(D-W)\,d}{z} \geq 0 \quad \& \quad z > 0$$

$$L(z-d) + (D-W)\,d \geq 0 \implies L(z-d) \geq (W-D)\,d$$

$$L \leq (W-D)\,\frac{d}{z-d}$$

$$\boxed{L \leq (D-W)\,\frac{d}{d-z}}$$

When $z = \frac{d}{2}$, $L \leq 2(D-W)$ $\big\downarrow$ increases by z

When $z = \frac{2d}{3}$, $L \leq 3(D-W)$

To ensure all z within range satisfy the specifications.

$$\boxed{L \leq 2(D-W)}$$

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b) $L \leq 2.\left(\frac{9W}{4} - W\right) \Rightarrow L \leq \boxed{\frac{5W}{2}}$

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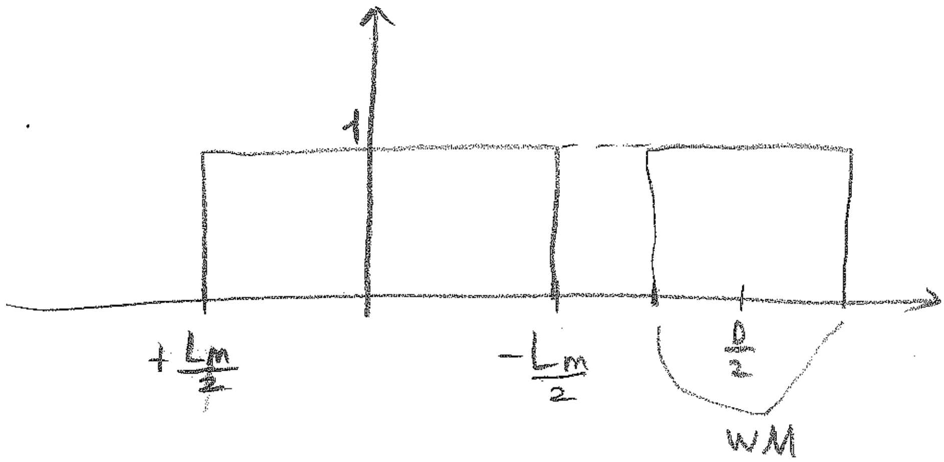
c) $s(x,y) = \text{rect}\left(\frac{x}{L}, \frac{y}{L}\right)$, $t(x,y) = \text{rect}\left(\frac{x - D/2}{W}, \frac{y}{W}\right)$

$$Id(x,y) = s\left(\frac{x}{m}, \frac{y}{m}\right) * t\left(\frac{x}{M}, \frac{y}{M}\right)$$

$$= \text{rect}\left(\frac{x}{Lm}, \frac{y}{Lm}\right) * \text{rect}\left(\frac{x - D/2}{WM}, \frac{y}{WM}\right)$$

$$= \text{rect}\left(\frac{x}{Lm}\right) * \text{rect}\left(\frac{x - D/2}{WM}\right)$$ At the center, $y = 0$



$|Lm| = \frac{5W(d-z)}{2z}$

$|Lm| = \frac{5W}{2}.(M-1)$

$=$

$\frac{5W}{4} \leq |Lm| \leq \frac{5W}{2}$

$z \in \left[\frac{d}{2}, \frac{2d}{3}\right] \Rightarrow M \in \left[\frac{3}{2}, 2\right]$

$\frac{3W}{2} \leq |WM| \leq 2W$

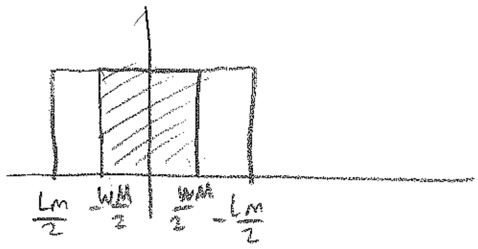


Image intensity at the center,

$Lm > WM \Rightarrow WM$

$Lm < WM \Rightarrow Lm$

$\frac{5W}{2}(M-1) > WM \qquad 5M-5 > 2M$

$M > 5/3 \implies |L_m| > \underline{WM}$

$M < 5/3 \implies |L_m|$

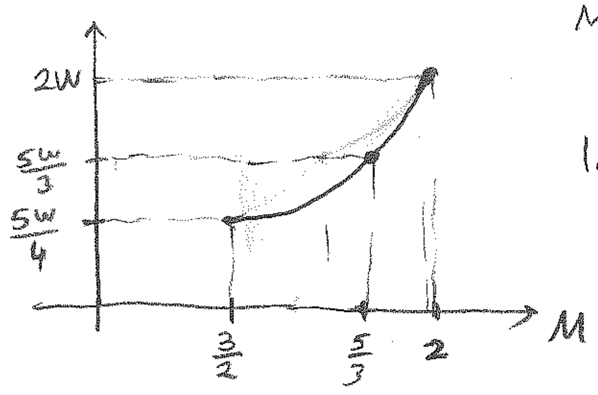


Image intensity at the center is maximum when $M = 2 \implies \boxed{z = \frac{2d}{3}}$

5) MATLAB Question:

a)

i)Sinograms

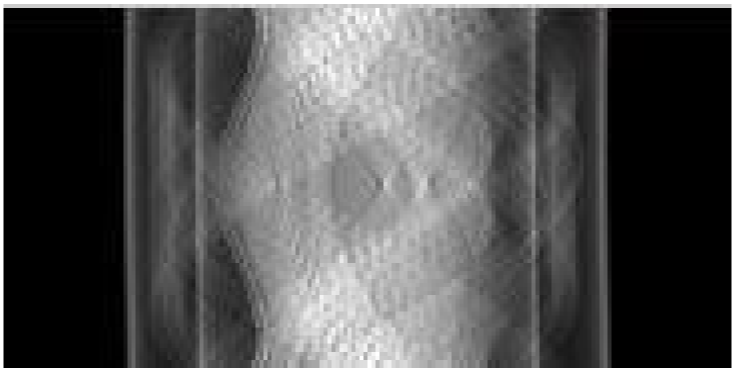


Figure 1:Sinogram for theta1

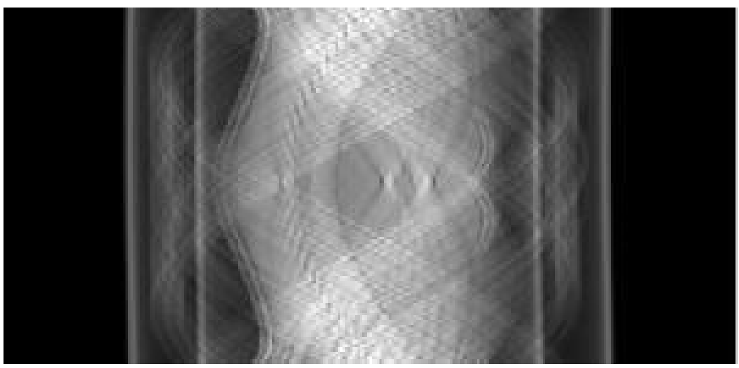


Figure 2: Sinogram for theta 2

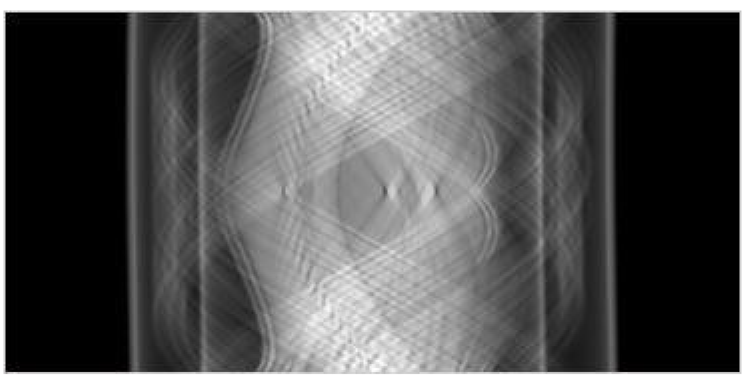


Figure 3: Sinogram for theta 3

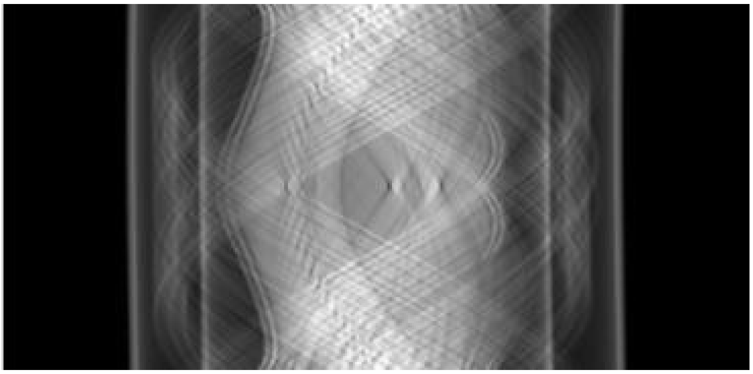


Figure 4: Sinogram for theta 4

The size of figures are adjusted so that it can be seen clearly. Thus, it can be observed that resolution of the images are different. In the last sinogram, where we use much samples of theta is the most fluent one as we expected since we get much projections.

ii) Reconstructed Images

Figure 5: Reconstructed Image for theta1

Figure 6: Reconstructed Image for theta2

Figure 7: Reconstructed Image for theta3

Figure 8: Reconstructed Image for theta4

b) In some images, there is an apparent temporal aliasing artifact due to undersampling. The artifacts are apparent in the images reconstructed for theta1, theta2 and theta3. The image for theta 4 has no apparent artifact since its sampling rate is higher. To reduce the artifacts, we can increase the sampling rate i.e. number of samples of theta or we can use aliasing filter before projections. Increasing sampling rate would be better choice, since using low pass filter cause data loss in the image.

c) Image Reconstruction for theta3

1)

Figure 9:Reconstructed Image with default filter

2)

Figure 10:Reconstructed Image with hamming windowed filter

Figure 11: Reconstructed Image using backprojection summation

Hamming windowed filter provides the best image for theta3. Since, in the reconstructed image with default filter, aliasing is more apparent. In the image, reconstructed with direct backprojection summation, there is too much blurring due to the fact that impulse response of backprojection summation is 1/r (frequency response is 1/q which acts as LPF).

d)Image Reconstruction for theta4

1)

Figure 12:Reconstructed Image with default filter

2)

Figure 13:Reconstructed Image with hamming windowed filter

3)

Figure 14: Reconstructed Image using backprojection summation

Default filter provides the best image since its contrast is higher than the other ones. The image reconstructed with Hamming filter is blurrier than the other one.

Appendix:
Matlab Code:

```
close all;
clear all;
load('projections .mat');
for i=1:360
thetan(i)=theta4(361-i);
glt4n(:,i)=glt4(:,361-i);
end
[L4,T4]=meshgrid(l4,thetan);
imshow(transpose(glt4n),[],'XData',[min(L4(:)) max(L4(:))],'YData',[min(T4(:)) max(T4(:))]);
%plotting sinogram
```