ELEC 530

Detection and Estimation

Homework #6

1. Exponentials in Noise ... Kay 4.1

$$x[n] = \sum_{i=1}^{p} A_i r_i^n + w[n] \quad , n = 0, 1, \dots, N-1$$

$$\underline{x} = \underline{\underline{R}} . \underline{A} + \underline{w} \longrightarrow \sim N(0, \sigma^2 I)$$

where

$$\underline{\underline{R}} = \begin{bmatrix} 1 & \cdots & 1 \\ r_1 & \cdots & r_p \\ \vdots & & \vdots \\ r_1^{N-1} & \cdots & r_p^{N-1} \end{bmatrix}_{N\times p} \quad , \quad \underline{A} = \begin{bmatrix} A_1 \\ \vdots \\ A_p \end{bmatrix}_{p\times 1}$$

This is lineas model, since noise is independently normally distributed.

$$p(x; A) = p_{\underline{w}}(x - \underline{\underline{R}}\underline{A}; A) = (2\pi\sigma^2)^{-N/2} \cdot \exp\left\{ -\frac{1}{2\sigma^2} (x - RA)^T (x - RA) \right\}$$

$$\ln p(x; A) = -\frac{N}{2} \ln(\sigma^2 2\pi) - \frac{1}{2\sigma^2} (x - RA)^T (x - RA)$$

$$\hat{A} = \underset{A}{\operatorname{argmin}} \| \underline{x} - \underline{\underline{R}}\underline{A} \|_2^2$$ solution of Least square problem.

$$* \quad \hat{A}_{MVU} = \hat{A}_{LS} = \hat{A}_{BLUE} = (R^T \underline{\underline{R}})^{-1} \underline{\underline{R}}.\underline{x}$$

$$E\{\hat{A}_{MVU}\} = E\{(R^T R)^{-1} R^T . x\} = (R^T R)^{-1} . R . \underbrace{E\{x\}} = (R^T R)^{-1} R^T . \underline{R}\,\underline{A}$$

$$= \underline{A} \checkmark$$

$$\mathrm{Cov}\{\hat{A}_{MVU}\} = E\{\hat{A} . \hat{A}^T\} - \mu_{\hat{A}} \mu_{\hat{A}}^T$$

$$= E\{(R^T R)^{-1} R^T \underbrace{x . x^T} R . (R^T R)^{-1}\} - AA^T$$

$$= (R^T R)^{-1} R^T . E\{\underline{x} x^T\} R (R^T R)^{-1} \} - AA^T$$

$$* \ Cov(\hat{A}_{MVU}) = (R^T R)^{-1} R^T (\underset{C_w}{\underline{\underline{C_x}}} + \underbrace{R A A^T R^T}_{\mu_x \cdot \mu_x^T}) R (R^T R)^{-1} - A A^T$$

$$= (R^T R)^{-1} R^T . \sigma^2 . I . R (R^T R)^{-1}$$

$$+ (R^T R)^{-1} . R^T . R A A^T R^T R (R^T R)^{-1}$$

$$- A A^T$$

$$= \underline{\sigma^2 . (R^T R)^{-1}}$$

Case: $p = 2$, $r_1 = 1$, $r_2 = -1$, N even

$$R = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \\ & \vdots \\ 1 & -1 \end{bmatrix}_{N \times 2}, \quad R^T R = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix}_{2 \times 2} = N . I$$

$$\hat{A}_{MVU} = \frac{1}{N} I . \underline{\underline{R}} . \underline{x} = \frac{1}{N} . \underline{R} \, \underline{x} = \frac{1}{N} \left[1^T x \, , \, 1_{+-}^T x \right]$$

$$* \ \hat{A}_{1MVU} = \bar{x} = \frac{1}{N} \sum_{i=0}^{N-1} x[i] \quad \text{(sample mean)}$$

$$* \ \hat{A}_{2MVU} = x[0] - x[1] + x[2] - x[3] + \dots + x[N-2] - x[N-1]$$

$$= \sum_{n=0}^{N-1} (-1)^n . x[n]$$

$$* \ Cov(\hat{A}_{MVU}) = \sigma^2 . \frac{1}{N} . I = \frac{\sigma^2}{N}$$

2. A Relative of Normal Distribution... (Kay 5.2)

Neyman-Fischer Factorization Theorem

$T(\underline{x})$ is a sufficient statistic for $\theta \Leftrightarrow p(x;\theta) = g(T(x),\theta).h(x)$

$\{x[n], n=0,1,\dots,N-1\}$ iid observations

$$p(x[n];\sigma^2) = \begin{cases} \frac{x[n]}{\sigma^2} \exp\left\{-\frac{1}{2\sigma^2} x^2[n]\right\}, & x[n] \geq 0 \\ 0, & x[n] < 0 \end{cases}$$

$$p(x[n];\sigma^2) = \frac{x[n]}{\sigma^2} \cdot \exp\left\{-\frac{1}{2}\frac{x^2[n]}{\sigma^2}\right\} \cdot u[x[n]]$$

$$p(\underline{x},\sigma^2) = \frac{1}{\sigma^{2N}} \prod_{n=0}^{N-1} x[n] \cdot \exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1} x^2[n]\right\} \cdot u\left[\min\{\underline{x}\}\right]$$

$$(\text{ie } \min\{x[0],\dots x[N-1]\} > 0 \Leftrightarrow x[0] \geq 0, \dots x[N-1] \geq 0)$$

$$p(\underline{x},\sigma^2) = \prod_{n=0}^{N-1} x[n] \cdot u\left[\min\{x[0],\dots x[N-1]\}\right] \cdot \frac{1}{\sigma^{2N}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1} x^2[n]\right\}$$

$$p(x,\sigma^2) = h(\underline{x}) \cdot g(T(\underline{x}),\sigma^2)$$

where $h(\underline{x}) = u\left[\min\{x[0],\dots x[N-1]\}\right] \cdot \prod_{n=0}^{N-1} x[n]$

$$g(T(x),\sigma^2) = \frac{1}{\sigma^{2N}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1} x^2[n]\right\}$$

Then,
$T(x) = \sum_{n=0}^{N-1} x^2[n]$ is a sufficient statistic for σ^2.

3. DC in Uniform Noise vs Gaussian Noise

$x[n] = A + w[n]$, $n = 0, \ldots\ldots, N-1$

$\underline{x} = \underline{1}A + \underline{w}$ $\quad w[n] \sim$ independent $(0, \sigma^2)$

(a) $\underline{w} \sim N(0, \sigma^2 I)$

$\underline{x} = \underline{1} A + \underline{w}$ (Linear Model)

$\hat{Q}_G = \underset{A}{\operatorname{argmin}} \|\underline{x} - \underline{1}A\|_2^2 = (\underline{1}^T\underline{1})^{-1} \underline{1}^T \underline{x}$

$= \frac{1}{N} 1^T x = \bar{x}$ (sample mean)

(b) $w_n \sim U[a, b]$ independent

& $E\{w_n\} = 0$ & $Var\{w_n\} = \sigma^2$ known

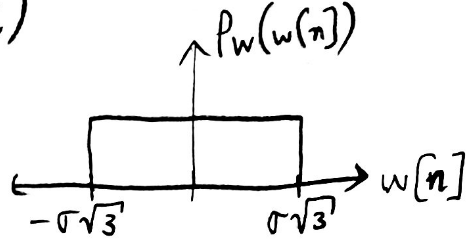
Then, $E\{w_n\} = \frac{b+a}{2} = 0 \Rightarrow b + a = 0$

$Var\{w_n\} = \frac{(b-a)^2}{12} = \sigma^2 \Rightarrow b - a = \sigma\sqrt{12}$

$b - a = 2\sigma\sqrt{3}$

Then, $b = -a = \sigma\sqrt{3}$

(i)



* distribution of $w(n)$ is known

$p(w[n]) = u[w[n] + \sigma\sqrt{3}] - u[w[n] - \sigma\sqrt{3}]$

$* \; p(\underline{w}) = \prod_{n=0}^{N-1} w[n] = \prod_{n=0}^{N-1} (u[w[n] + \sigma\sqrt{3}] - u[w[n] - \sigma\sqrt{3}])$

$= 1. \; \underbrace{u[\min\{\underline{w}\} + \sigma\sqrt{3}] \,.\, u[\sigma\sqrt{3} - \max\{\underline{w}\}]}_{\delta(\forall w_n \in [-\sigma\sqrt{3}, \sigma\sqrt{3}))}$

$* \; p(\underline{x}; A) = p_{\underline{w}}(\underline{x} - \underline{1}A)$

$= u[\min\{\underline{x} - 1A\} + \sigma\sqrt{3}] \,.\, u[\sigma\sqrt{3} - \max\{\underline{x} - 1A\}]$

$$p(\underline{x};A) = u\left[\min\{x[0]-A, x[1]-A, \dots x[N-1]-A\} + \sigma\sqrt{3}\right] \cdot$$
$$u\left[\sigma\sqrt{3} - \max\{x[0]-A, x[1]-A, \dots, x[N-1]-A\}\right]$$

$$= u\left[\min\{x[0], \dots x[N-1]\} - A + \sigma\sqrt{3}\right] \cdot$$
$$\cdot\, u\left[A + \sigma\sqrt{3} - \max\{x[0], \dots x[N-1]\}\right]$$

$$= u\left[m(\underline{x}) - (A - \sigma\sqrt{3})\right] \cdot u\left[A + \sigma\sqrt{3} - M(\underline{x})\right]$$

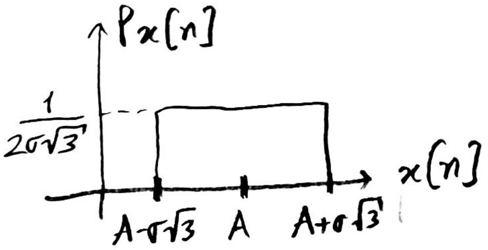
According to Neyman Fischer Factorization Theorem

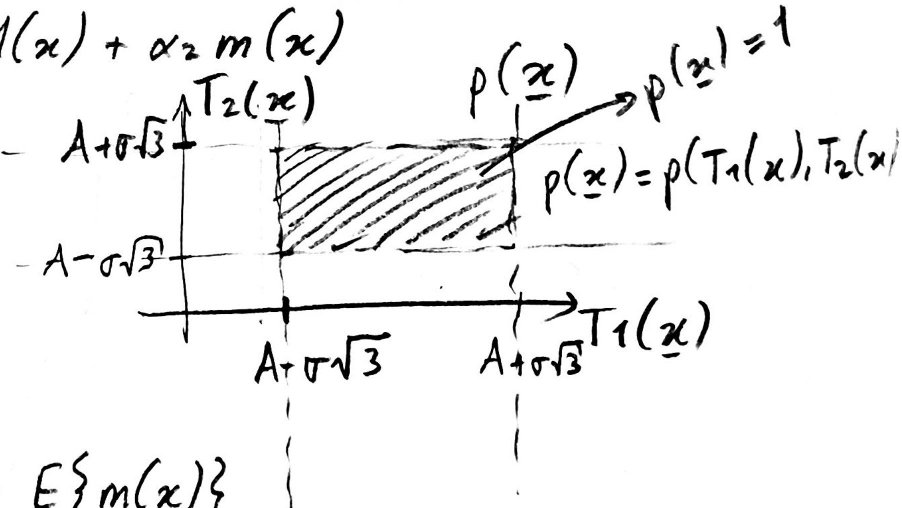
$$p(x, A) = h(x) \cdot g(T_1(x), T_2(x), A)$$

where $h(\underline{x}) = 1$ & $g(T_1(x), T_2(x), A) = u\left[T_1(x) - (A - \sigma\sqrt{3})\right] \cdot$
$$\cdot\, u\left(A + \sigma\sqrt{3} - T_2(x)\right]$$

$T_1(x) = m(x)$ & $T_2(x) = M(x)$ are sufficient statistics for A. $\underline{T}(x) = \left[M(x) \;\; m(x)\right]^T$ is a sufficient statistic for A.

(ii) $\hat{\theta}_u = \alpha^T \cdot \underline{T}(x) = \alpha_1 M(x) + \alpha_2 m(x)$





We need to find $E\{M(\underline{x})\}$ & $E\{m(\underline{x})\}$

$*$ $F_{M(x)}(t) = P(M(x) \le t) = P(\forall x[n] \le t) = \left(t - (A - \sigma\sqrt{3})\right)^N \cdot \left(\frac{1}{2\sigma\sqrt{3}}\right)^N$

$\left(\underline{\text{for } A - \sigma\sqrt{3} \le t \le A + \sigma\sqrt{3}}\right)$

$$\frac{\partial F_{M(x)}(t)}{\partial t} = (2\sigma\sqrt{3})^{-N} \cdot N \cdot \left(t - (A - \sigma\sqrt{3})\right)^{N-1}$$

$$f_{M(x)}(z) = \left(\frac{1}{2\sigma\sqrt{3}}\right)^N \cdot N \cdot \left(z - (A - \sigma\sqrt{3})\right)^{N-1}, \quad t \in \left[A - \sigma\sqrt{3}, A + \sigma\sqrt{3}\right]$$

$$E\{M(x)\} = \left\{ \int_{A-\sigma\sqrt{3}}^{A+\sigma\sqrt{3}} z \cdot \left(z-(A-\sigma\sqrt{3})\right)^{N-1} dz \right\} \cdot N \cdot \left(\frac{1}{2\sigma\sqrt{3}}\right)^{N}$$

$z \longrightarrow z + A - \sigma\sqrt{3}$ transformation of variables

$$= \left(\int_{0}^{2\sigma\sqrt{3}} (z + A - \sigma\sqrt{3}) \cdot z^{N-1} \cdot N \, dz \right) \cdot (2\sigma\sqrt{3})^{-N}$$

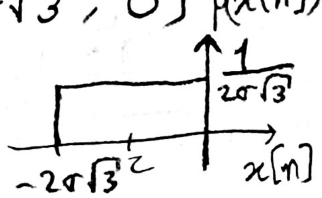
$$= \left(\int_{0}^{2\sigma\sqrt{3}} z^{N} \cdot N \, dz + (A-\sigma\sqrt{3}) \int_{0}^{2\sigma\sqrt{3}} z^{N-1} \cdot N \, dz \right) \cdot (2\sigma\sqrt{3})^{-N}$$

$$= \left(\frac{z^{N+1}}{N+1} \cdot N \Bigg|_{0}^{2\sigma\sqrt{3}} + (A-\sigma\sqrt{3}) \cdot z^{N} \Bigg|_{0}^{2\sigma\sqrt{3}} \right) \cdot (2\sigma\sqrt{3})^{-N}$$

$$= \left((2\sigma\sqrt{3})^{N+1} \cdot \frac{N}{N+1} + (A-\sigma\sqrt{3}) \cdot (2\sigma\sqrt{3})^{N} \right) \cdot (2\sigma\sqrt{3})^{-N}$$

$$* = \underbrace{2\sigma\sqrt{3} \cdot \frac{N}{N+1}}_{E\{M(x)\} \text{ where } x \sim U(0,\, 2\sigma\sqrt{3}] \text{ as we expected}} + \underbrace{(A-\sigma\sqrt{3})}_{\text{shift from } U(0,\, 2\sigma\sqrt{3}]} = \boxed{A + \frac{N-1}{N+1}\sigma\sqrt{3}}$$

$$E\{m(\underline{x})\} = \underbrace{E\{m(\underline{x}_1)\}}_{\min\{\underline{x}_1\} \text{ where } x_{1n} \sim U[-2\sigma\sqrt{3},\, 0]} + (A+\sigma\sqrt{3}) \qquad \underline{x}_1 = \underline{x} - \underline{1}(A+\sigma\sqrt{3})$$



$$F_{m(x_1)}(z) = P(m(x_1) \le z) = 1 - P(m(x_1) \ge z)$$

$$= 1 - P(\forall x_{1n} \ge z) = 1 - \prod_{n=0}^{N-1} \left(1 - \underbrace{P(x_{1n} \le z)}_{z}\right)$$

$$F_{m(x_1)}(z) = 1 - (-z)^{N} (2\sigma\sqrt{3})^{-N} \quad \text{for } z \in [-2\sigma\sqrt{3}, 0]$$

I used transformation of rv $\underline{x}_1 = \underline{x} - \underline{1}(A + \sigma\sqrt{3})$ just for simplicity of integral evaluation to find cdf.

$$f_{m(x_1)}(\tau) = \frac{\partial F_{m(x_1)}(\tau)}{\partial \tau} = \frac{\partial}{\partial \tau}\left(1-(-\tau)^N (2\sigma\sqrt{3})^{-N}\right)$$

$$= N.(-\tau)^{N-1}.(2\sigma\sqrt{3})^{-N}$$

$$E\{m(x_1)\} = \left\{\int_{-2\sigma\sqrt{3}}^{0} \tau.N.(-\tau)^{N-1} d\tau\right\}(2\sigma\sqrt{3})^{-N}$$

$$= \int_{0}^{-2\sigma\sqrt{3}} (-\tau)^N .N.d\tau = \frac{N}{N+1}(-\tau)^{N+1}\Big|_{-2\sigma\sqrt{3}}^{0} .(2\sigma\sqrt{3})^{-N}$$

$$= -\frac{N}{N+1}(2\sigma\sqrt{3})^{N+1}.(2\sigma\sqrt{3})^{-N}$$

$$= -\frac{N}{N+1} 2\sigma\sqrt{3}$$

$$* \; \mathbb{E}\{m(x)\} = \mathbb{E}\{m(x_1)\} + A + \sigma\sqrt{3}$$

$$= -\frac{N}{N+1} 2\sigma\sqrt{3} + A + \sigma\sqrt{3}$$

$$= \boxed{A - \frac{(N-1)}{N+1}\sigma\sqrt{3}}$$

$$\mathbb{E}\{\alpha_1 M(x) + \alpha_2 m(x)\} = \alpha_1\left(A + \frac{N-1}{N+1}\sigma\sqrt{3}\right) +$$

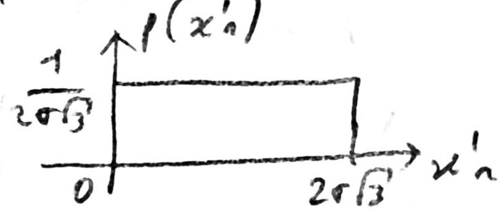
$$+ \alpha_2\left(A - \frac{N-1}{N+1}\sigma\sqrt{3}\right)$$

For $\alpha_1 = \alpha_2 = \frac{1}{2}$, $\mathbb{E}\{\hat{\theta}_u(x)\} = A$

$$\boxed{\hat{\theta}_u(x) = \frac{1}{2}\left(M(\underline{x}) + m(\underline{x})\right)}$$

iii) $Var(\hat{Q}u) = \frac{1}{4} Var(M(\underline{x})) + \frac{1}{4} Var(m(\underline{x}))$

We denote $x' = \underline{x} - \underline{1}(A - \sigma\sqrt{3})$



$Var\{M(\underline{x})\} = Var\{M(x')\}$

$$Var\{M(x')\} = E\{M(x')^2\} - \underbrace{E\{M(x')\}}_{2\sigma\sqrt{3}\left(\frac{N}{N+1}\right) \text{ found in (ii)}}^2$$

$$E\{M(x')^2\} = \left(\int_0^{2\sigma\sqrt{3}} \tau^2 \cdot \tau^{N-1} \cdot N \, d\tau\right) \cdot (2\sigma\sqrt{3})^{-N}$$

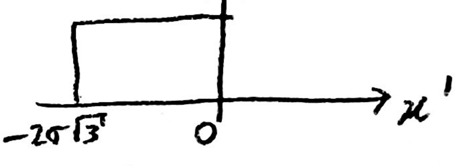
$$= (2\sigma\sqrt{3})^{-N} \cdot \frac{N}{N+2} \cdot \tau^{N+2}\Big|_0^{2\sigma\sqrt{3}} = (2\sigma\sqrt{3})^2 \cdot \frac{N}{N+2}$$

$$Var\{M(x)\} = Var\{M(x')\} = 12\sigma^2 \cdot \frac{N}{N+2} - 12\sigma^2 \frac{N^2}{(N+1)^2}$$

$$= 12\sigma^2 \left(\frac{N}{N+2} - \frac{N^2}{(N+1)^2}\right)$$

$$= 12\sigma^2 \cdot \frac{N}{(N+1)^2 (N+2)}$$

$*\ Var\{m(x)\} = Var\{m(x')\}$ where $\underline{x}' = \underline{x} - \underline{1}(A + \sigma\sqrt{3})$



$$E\{m(x')^2\} = (2\sigma\sqrt{3})^{-N} \int_{-2\sigma\sqrt{3}}^{0} \tau^2 \cdot N(-\tau)^{N-1} d\tau$$

$$= (2\sigma\sqrt{3})^{-N} \cdot N \cdot \int (-\tau)^{N+1} d\tau = (2\sigma\sqrt{3})^{-N} \cdot \frac{N}{N+2} \cdot (-\tau)^{N+2}\Big|_0^{-2\sigma\sqrt{3}}$$

$$= \frac{N}{N+2} (2\sigma\sqrt{3})^2 = \frac{N}{N+2} 12\sigma^2$$

$$\mathrm{Var}\{m(x)\} = \mathrm{Var}\{m(x')\} = \mathbb{E}\{m(x')^2\} - E\{m(x')\}^2$$

$$= \frac{N}{N+2}\, 12\sigma^2 - \left(-\frac{N}{N+1}\, 2\sigma\sqrt{3}\right)^2$$

$$= \frac{N}{N+2}\, 12\sigma^2 - \frac{N^2}{(N+1)^2}\, 12\sigma^2 = 12\sigma^2 \frac{N}{(N+1)^2(N+2)}$$

$= \mathrm{Var}\{M(x)\}$ as intuitively expected.

$$\mathrm{Var}\{\hat{Q}_u\} = \frac{1}{4}\left(\mathrm{Var}\{M(x)\} + \mathrm{Var}\{m(x)\}\right)$$

$$= \frac{1}{4} \cdot 2 \cdot 12\sigma^2 \cdot \frac{N}{(N+1)^2(N+2)} = 6\sigma^2 \frac{N}{(N+1)^2(N+2)}$$

(iv) $\mathrm{Var}\{\hat{Q}_g\} = \mathrm{Var}\{\bar{x}\} = \dfrac{\sigma_x^2}{N} = \dfrac{\sigma^2}{N}$

ELEC-530 HW6 Question 3

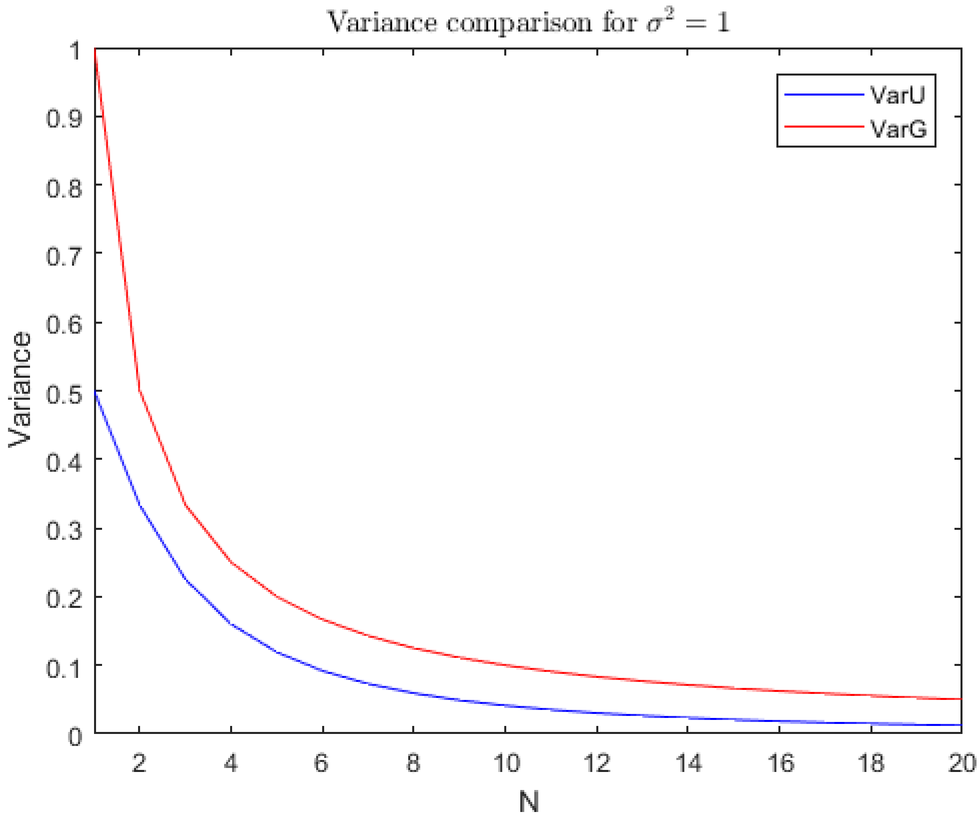
Table of Contents

written by Ender Erkaya

Part-1

Variance Comparison

```
N=1:20;
sigma=1;
VarU=6*sigma*N./(((N+1).^2).*(N+2));
VarG=sigma./N;
figure
plot(N,VarU,'b');
hold on
plot(N,VarG,'r');
xlim([1 20]);
legend('VarU','VarG');
xlabel('N');
ylabel('Variance');
title('Variance comparison for $\sigma^2=1$','Interpreter','Latex');
```



Part-2

Uniform Noise

```
No_of_Sim=10000;
N=100;
sigma=1/12;%sigma^2
A=5;

X = A-sqrt(3*sigma)+2*sqrt(3*sigma)*rand(N,No_of_Sim);% noise is
 uniform
% applying estimators
Est1=@(x) mean(x); % Estimator 1 is sample mean estimator
Est2=@(x) 1/2*(min(x)+max(x)); % Estimator 2 is 0.5*(m(x)+M(x))

x=X(:,1);
A_est1=Est1(x);
A_est2=Est2(x);

A_hat1=zeros(No_of_Sim,1);
A_hat2=zeros(No_of_Sim,1);

for k=1:No_of_Sim
    x=X(:,k);
    A_hat1(k)=Est1(x);
    A_hat2(k)=Est2(x);
```

```
end

MSE1_U=mean((A_hat1-A).^2);
MSE2_U=mean((A_hat2-A).^2);

fprintf('For X is Uniform, MSE1 is %.6f and MSE2 is %.6f
 \n',MSE1_U,MSE2_U);
disp(['Since we use uniform noise MVU for uniform noise is Estimator
 2']);
disp([' hence Average Square Error1 > Average Square Error 2, ie
 variance increases','since we use A^g for uniform distribution']);
```

```
For X is Uniform, MSE1 is 0.000840 and MSE2 is 0.000048
Since we use uniform noise MVU for uniform noise is Estimator 2
 hence Average Square Error1 > Average Square Error 2, ie variance
 increasessince we use A^g for uniform distribution
```

Part-3

Gaussian Distribution X ~ N(A,sigma^2)

```
X = A+sqrt(sigma)*randn(N,No_of_Sim);

% applying estimators
Est1=@(x) mean(x); % Estimator 1 is sample mean estimator
Est2=@(x) 1/2*(min(x)+max(x)); % Estimator 2 is 0.5*(m(x)+M(x))

x=X(:,1);
A_est1=Est1(x);
A_est2=Est2(x);

A_hat1=zeros(No_of_Sim,1);
A_hat2=zeros(No_of_Sim,1);

for k=1:No_of_Sim
    x=X(:,k);
    A_hat1(k)=Est1(x);
    A_hat2(k)=Est2(x);
end

MSE1_G=mean((A_hat1-A).^2);
MSE2_G=mean((A_hat2-A).^2);

fprintf('For X is Gaussian, MSE1 is %.5f and MSE2 is %.5f
 \n',MSE1_G,MSE2_G);
disp(['Since we use uniform noise MVU for gaussian noise in Estimator
 2']);
disp([' hence Average Square Error2 > Average Square Error 1, ie
 variance increases','since we use A^u for Gaussian distribution']);
```

```
For X is Gaussian, MSE1 is 0.00083 and MSE2 is 0.00772
Since we use uniform noise MVU for gaussian noise in Estimator 2
 hence Average Square Error2 > Average Square Error 1, ie variance
 increasessince we use A^u for Gaussian distribution
```

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