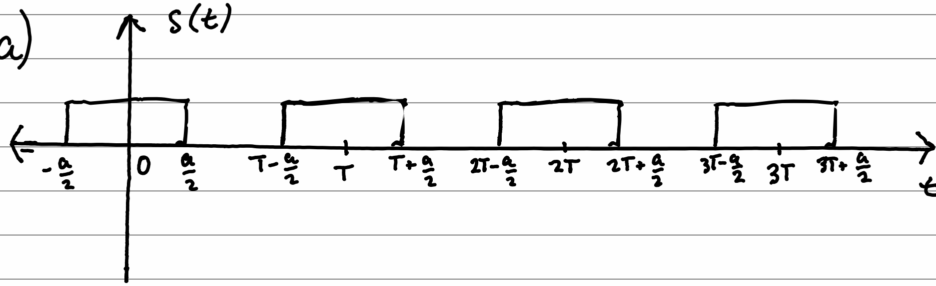
Cornelio, Andrew HW #4

1) a)



b) $$C_0 = \frac{1}{T}\int_{-\frac{a}{2}}^{a/2} dt = \frac{a}{T}$$

$$A_k = \frac{2}{T}\int_{-\frac{a}{2}}^{\frac{a}{2}} \cos\left(\frac{2\pi k t}{T}\right) dt$$

$$= \frac{\cancel{2}}{\cancel{T}}\left(\frac{\cancel{T}}{\cancel{2}\pi k}\right)\sin\left(\frac{2\pi k t}{T}\right)\Big|_{-\frac{a}{2}}^{\frac{a}{2}}$$

$$= \frac{2}{\pi k}\sin\left(\frac{\pi k a}{T}\right)$$

$$B_k = \frac{2}{T}\int_{-\frac{a}{2}}^{\frac{a}{2}} \sin\left(\frac{2\pi k t}{T}\right) dt$$

$$= -\frac{\cancel{2}}{\cancel{T}}\left(\frac{\cancel{T}}{\cancel{2}\pi k}\right)\cos\left(\frac{2\pi k t}{T}\right)\Big|_{-\frac{a}{2}}^{\frac{a}{2}}$$

$$= 0$$

c+d) See insert

2) See insert

3) a) We compute:

$$(s * \delta_{t_0})(t) = \int_{-\infty}^{\infty} s(\tau)\,\delta_{t_0}(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} s(\tau)\,\delta_{-t_0}(\tau - t)\,d\tau$$

$$= \int_{-\infty}^{\infty} s(\tau)\,\delta_{t-t_0}(\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} s(t-t_0)\,\delta_{t-t_0}(\tau)\,d\tau$$

$$= s(t-t_0)\int_{-\infty}^{\infty} \delta_{t-t_0}(\tau)\,d\tau$$

$$= s(t-t_0)$$

b) We compute the fourier transform

$$\mathcal{F}[\delta_{t_0}](f) = \int_{-\infty}^{\infty} \delta(t-t_0)\,e^{-2\pi i f t}\,dt$$

$$= \int_{-\infty}^{\infty} \delta(t-t_0)\,e^{-2\pi i f t_0}\,dt$$

$$= e^{-2\pi i f t_0}\int_{-\infty}^{\infty} \delta(t-t_0)\,dt$$

$$= e^{-2\pi i f t_0}$$

c) $\mathcal{F}[s * \delta_{t_0}](f) = \mathcal{F}[s](f)\,\mathcal{F}[\delta_{t_0}](f)$

$$= S(t)e^{-2\pi i f t_0}$$

$$= \mathcal{F}[s(t-t_0)](f)$$

4) a) $\mathcal{F}[e^{-\alpha t}\mathbb{1}(t)](f) = \int_{-\infty}^{\infty} e^{-\alpha t - 2\pi i f t}\mathbb{1}(t)\,dt$

$$= \int_{0}^{\infty} e^{-(\alpha+2\pi i f)t}\,dt$$

$$= -\frac{1}{\alpha+2\pi i f}\, e^{-(\alpha+2\pi i f)t}\Big|_{0}^{\infty}$$

$$= \frac{1}{\alpha+2\pi i f}$$

b) $\mathcal{F}[e^{-\alpha|t|}](f) = \int_{-\infty}^{\infty} e^{-\alpha|t| - 2\pi i f t}\,dt$

$$= \int_{0}^{\infty} e^{-(\alpha+2\pi i f)t}\,dt + \int_{-\infty}^{0} e^{(\alpha-2\pi i f)t}\,dt$$

$$= \frac{-e^{-(\alpha+2\pi i f)t}}{\alpha+2\pi i f}\Big|_{0}^{\infty} + \frac{e^{(\alpha-2\pi i f)t}}{2\pi i f-\alpha}\Big|_{-\infty}^{0}$$

(where $|\alpha| > |2\pi i f|$ $\quad = \frac{1}{\alpha+2\pi i f} + \frac{1}{\alpha-2\pi i f}$

$$= \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$$

c) $(s*s)(t) = (e^{-\alpha t}\mathbb{1} * e^{-\alpha t}\mathbb{1})(t)$

$$= \int_{-\infty}^{\infty} e^{-\alpha\tau-\alpha(t-\tau)}\mathbb{1}(\tau)\mathbb{1}(t-\tau)\,d\tau$$

$$= \int_{0}^{t} e^{-\alpha t}\,d\tau$$

$$= e^{-\alpha t}t \quad (\text{where } t>0)$$

$$= e^{-\alpha t}t\,\mathbb{1}(t)$$

d) $\mathcal{F}[s*s](f) = \mathcal{F}[s](f)\,\mathcal{F}[s](f)$

$$= \frac{1}{(\alpha+2\pi i f)^2}$$

$$= \mathcal{F}[e^{-\alpha t}t\,\mathbb{1}](f)$$

5 a) $\mathcal{F}[s(at)](f) = \int_{-\infty}^{\infty} s(at)e^{-2\pi i f t}\,dt$

let $v = at$
$dv = a\,dt$

$$= \frac{1}{a}\int_{-\infty}^{\infty} s(v)e^{-2\pi i \frac{f}{a} v}\,dv$$

$$= \frac{1}{a}S\left(\frac{f}{a}\right)$$

b) We use part a. Let $s(t) = e^{-\pi t^2}$.
Note that $h(t) = s((2\pi\sigma^2)^{-1/2})$. From (a) we know:

$$\mathcal{F}[h(t)] = \sqrt{2\pi\sigma^2}\,S\left(\frac{f}{\sqrt{2\pi\sigma^2}}\right) = \sqrt{2\pi\sigma^2}\,e^{-2\pi^2\sigma^2 f^2}$$

By linearity of the fourier transform, we can say

$$g(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{t^2}{\sigma^2}} \xrightarrow{\mathcal{F}} e^{-2\pi^2\sigma^2 f^2}$$

6) $\mathcal{F}[v](f) = \int_{-\infty}^{\infty} v(t) e^{-2\pi i f t} dt$

$$= \int_{-\infty}^{\infty} s(t)\cos(2\pi f_0 t) e^{-2\pi i f t} dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} s(t)\left(e^{2\pi i f_0 t} + e^{-2\pi i f_0 t}\right) e^{-2\pi i f t} dt$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} s(t) e^{-2\pi i (f-f_0)t} dt + \int_{-\infty}^{\infty} s(t) e^{-2\pi i (f_0+f)t} dt\right]$$

$$= \frac{1}{2}\left[S(f-f_0) + S(f+f_0)\right]$$

The extension of V is broken in two parts: $[-f_0-B, f_0+B] \cup [f_0-B, f_0+B]$. We might be able to select one of those extensions & approximate the signal at a higher frequency.

7 $\mathcal{F}[\mathcal{F}[s](f)](t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(t) e^{-2\pi i f t} dt \, e^{-2\pi i f t} df$

let $u = -f$, $du = -df$

$$= \int_{-\infty}^{\infty} S(f) e^{-2\pi i f t} df$$

$$= -\int_{-\infty}^{\infty} S(-u) e^{2\pi i u t} du = -s(-t)$$

Let $g(t) = -s(-t)$. Then,

$$g(t) \xrightarrow{\mathcal{F}^2} -g(-t)$$

or

$$s(t) \xrightarrow{\mathcal{F}^4} s(t)$$

8) $$|s'(t)| = \left|\frac{d}{dt}\int_{-\infty}^{\infty} S(f)e^{2\pi i f t}\,df\right|$$
$$= \left|\int_{-\infty}^{\infty} \frac{\partial}{\partial t} S(f)e^{2\pi i f t}\,df\right|$$
$$= \left|\int_{-\infty}^{\infty} S(f)(2\pi i f)e^{2\pi i f t}\,df\right|$$
$$= 2\pi\left|\int_{-\infty}^{\infty} S(f) f e^{2\pi i f t}\,df\right|$$
$$\le 2\pi\int_{-\infty}^{\infty} \left|S(f) f e^{2\pi i f t}\right|df$$
$$= 2\pi\int_{-\infty}^{\infty} |S(f)|\,|f|\,df$$

9) First we do the easier path. We know:

$$s \xrightarrow{\mathcal{F}} \text{sinc}(f)$$

So, $$\mathcal{F}[s'](f) = (2\pi i f)\,\text{sinc}(f)$$
$$= (2\cancel{\pi} i \cancel{f})\frac{\sin(\pi f)}{\cancel{\pi f}}$$
$$= 2i\sin(\pi f)$$

Now we take the other path

$$s'(t) = \delta(t - \tfrac{1}{2}) - \delta(t + \tfrac{1}{2})$$

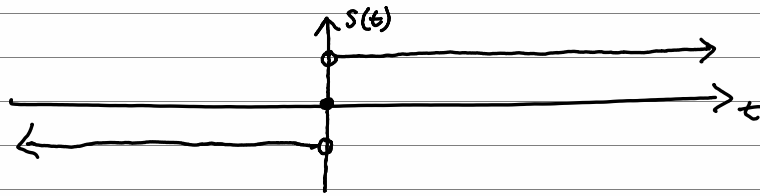
So, $$\mathcal{F}[s'](f) = \int_{-\infty}^{\infty} \left(\delta(t - \tfrac{1}{2}) - \delta(t + \tfrac{1}{2})\right) e^{-2\pi i f t}\, dt$$

$$= e^{-2\pi i f(-\frac{1}{2})} - e^{-2\pi i f(\frac{1}{2})}$$

$$= e^{\pi i f} - e^{-\pi i f}$$

$$= 2i \sin(\pi f)$$

10) a)



sgn(t)

$\mathbb{1}(t)$

b) Note that $\frac{d}{dt}\operatorname{sgn}(t) = 2\delta(t)$. We can use the result from 9:

$$\mathcal{F}[2\delta(t)](f) = 2e^{-2\pi i f(\frac{0}{2})} = 2$$

We know from 9:

$$\mathcal{F}\left[\tfrac{d}{dt}\operatorname{sgn}(t)\right] = (2\pi i f)\,\mathcal{F}[\operatorname{sgn}(t)](f)$$

Combining these we get:

$$2 = (2\pi i f)\,\mathcal{F}[sgn](f)$$

Or:

$$\mathcal{F}[sgn](f) = \frac{1}{\pi i f}$$

c) We can let: $\mathbb{1}(t) = \frac{1}{2}(sgn(t)+1)$. By linearity, we can say:

$$\mathcal{F}[\mathbb{1}(t)](f) = \frac{1}{2}\mathcal{F}[sgn](f) + \frac{1}{2}\mathcal{F}[1](f)$$

$$= \frac{1}{2}\frac{1}{\pi i f} + \frac{1}{2}\delta(f)$$

d) Note that $\frac{d}{dt}v(t) = s(t)$. From a, we know:

$$\mathcal{F}[\tfrac{d}{dt}v](f) = (2\pi i f)\,\mathcal{F}[v](f)$$

So,

$$\mathcal{F}[v](f) = \frac{1}{2\pi i f}\mathcal{F}[\tfrac{d}{dt}v](f)$$

$$= \frac{S(f)}{2\pi i f}$$

11) a) Let $s(t) = e^{-\pi t^2}$. Then $s_1(t) = A_0\, s\left(\frac{t}{\sqrt{\pi}T}\right)$

So,

$$\mathcal{F}[s_1](f) = A_0(\sqrt{\pi}\,T)\,S(\sqrt{\pi}\,Tf)$$

$$= A_0(\sqrt{\pi}\,T)\,e^{-\pi^2 T^2 f^2}$$

b) Let $g(t) = e^{-\pi t^2/T^2}$, Then,

$$\frac{d}{dt} g(t) = \frac{-2\pi t}{T^2} e^{-\pi t^2/T^2}$$

So, $$s_2(t) = -\frac{T^2}{2\pi} \frac{d}{dt} g(t)$$

From linearity & derivative properties:

$$\mathcal{F}[s_2](f) = -\frac{T^2}{2\pi} \mathcal{F}[\tfrac{d}{dt} g](f)$$

$$= -\frac{T^2}{\cancel{2\pi}} (\cancel{2\pi} i f) \mathcal{F}[g](f)$$

$$= -T^2 (if)(\sqrt{\pi}\, T) e^{-\pi^2 T^2 f^2}$$

In the last step we used our result from (a), where $A_0 = 1$.

c) Note that $\frac{d}{dt} s_3(t) = s(t)$. So:

$$\mathcal{F}[\tfrac{d}{dt} s_3](f) = (2\pi i f) \mathcal{F}[s_3](f)$$

$$\mathcal{F}[s_3](f) = \frac{S(t)}{2\pi i f} = \frac{\sqrt{\pi}\, T e^{-\pi^2 T^2 f^2}}{2\pi i f}$$

12) Closurer: Let $p, q \in P$. Then

$$s(t+p+q) = s(t+q) = s(t)$$

So $p+q \in P$.

Associativity: Let $p, q, r \in P$. Then

$$s(t+(p+q)+r) = s(t+p+(q+r))$$

By additive associativity. So,

$$(p+q)+r = p+(q+r)$$

Identity: Let $p \in P$. We know $0 \in P$ since:

$$s(t+0) = s(t)$$

We see 0 is the identity of G, since

$$s(t+p+0) = s(t+0+p) = s(t+p)$$

So $\quad 0+p = p+0 = p$

Inverse: Let $v = t+p$. Then

$$s(v) = s(t+p) = s(t)$$

We can see that $-p \in P$ since

$$s(v-p) = s(t+p-p) = s(t+0) = s(t)$$

Since t is arbitrary, so is v. Hence $-p \in P$. And by additive inverse

$$p+(-p) = 0.$$

So G contains the inverse.

Commutativity: Let $p, q \in P$. Then, by additive commutativity

$$s(t+p+q) = s(t+q+p)$$

So $p+q, q+p \in P$. Also by additive commutativity: $p+q = q+p$.

Hence $(P, +)$ is abelian.