JHU 553.493/693, Spring 2020 – Homework #4

Fourier series and the classical Fourier transform for signals defined on $\mathbb{R}$

Due on Blackboard by the end of the day on Tuesday, April 14

Instructions: Write, on top of the first page of your assignment: Name (LAST, First), the HW# (in this case, "Homework #4"), and acknowledge others with whom you may have worked (just write "*Worked with ...*"). For the computational problems where you are asked to write computer code you may choose the programming language that you prefer, such as *Python* or *Matlab*. You do *not* have to type up your homework. If you handwrite it, please scan it, making sure it is readable. You should submit *one* PDF file on Blackboard (or at most two, if you choose to submit your code separately).

Problem 4.1 (A Fourier series)**.** The goal of this problem is to "get your hands dirty" with the computation of a Fourier series for T-periodic signals $s(t)$, $t \in \mathbb{R}$, i.e. with the property that $s(t+T) = s(t)$ for all $t \in \mathbb{R}$. Fix $T > 0$ and a number a such that $0 < a < T$. Consider the 'rectangular' signal:

$$\mathrm{rect}_a(t) = \begin{cases} 1 & \text{for } |t| \le \frac{a}{2}, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{Now let's define:} \qquad s(t) = \mathrm{rep}_T[\mathrm{rect}_a](t) = \sum_{n=-\infty}^{\infty} \mathrm{rect}_a(t - nT),$$

which is the *periodic repetition* of $\mathrm{rect}_a(t)$. This yields a signal $s(t)$ which is T-periodic.

(a) Sketch the graph of the signal $s(t)$, $t \in \mathbb{R}$.

(b) Compute the coefficients of its Fourier series (note that c_0 is just the average of s over a period):

$$c_0 = \frac{1}{T}\int_{t_0}^{t_0+T} s(t)\; dt, \quad A_k = \frac{2}{T}\int_{t_0}^{t_0+T} s(t)\cos\left(\frac{2\pi k t}{T}\right)\, dt, \quad B_k = \frac{2}{T}\int_{t_0}^{t_0+T} s(t)\sin\left(\frac{2\pi k t}{T}\right)\, dt,$$

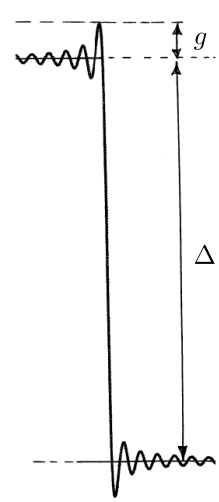
where $k \ge 1$, and you can choose t_0 (the best choice is $t_0 = -T/2$). Perform the computation by hand!

(c) We know from the theory that we can recover the signal from its Fourier coefficients.[1] To verify this for our example, we define the sequence of partial sums. The n^{th} partial sum is the function:

$$s_n(t) = c_0 + \sum_{k=1}^{n}\left\{A_k \cos\left(\frac{2\pi k t}{T}\right) + B_k \sin\left(\frac{2\pi k t}{T}\right)\right\}, \quad t \in \mathbb{R}.$$

Note that $s_n(t)$ is also T-periodic, and $s_n(t) \to s(t)$ as $n \to \infty$ at all points t where s is continuous. Fix positive T and a arbitrarily (with $a < T$), and (e.g. using *Matlab*) plot the graph of $s_n(t)$ for different values of n. Verify that, for large values of n, then $s_n(t)$ becomes a good approximation of $s(t)$.

(d) The so-called *Gibbs phonomenon*[2] is the peculiar manner in which the Fourier series of a piecewise continuously differentiable periodic function behaves at a *jump discontinuity* (also known as *discontinuity of the first kind*). The n^{th} partial sum $s_n(t)$ of the Fourier series has large oscillations near the jump, which might increase the maximum of the partial sum above that of the function $s(t)$ itself. The overshoot g does *not* die out as n increases, but approaches a finite limit (see figure on the left). It can be proven that the ratio g/Δ converges to the number 0.089489872236... as $n \to \infty$, *irrespective* of the signal $s(t)$ that we are approximating with the Fourier series! (Time permitting, we will give an explanation of this phenomenon further on in the course.) Using the code that you developed in part (c), verify numerically that for the signal $s(t)$ in the previous parts of this problem the ratio g/Δ indeed converges to a value that is reasonably close to the one predicted by the theory.



[1] Under mild conditions on the signal s, the Fourier series converges for all $t \in \mathbb{R}$ to s where the signal is continuous. If the signal has a jump discontinuity at t_0, then the Fourier series converges to the value $[s(t_0^+) + s(t_0^-)]/2$.

[2] This is named after one of its discoverers, Josiah Willard Gibbs (February 11, 1839–April 28, 1903), an American scientist who worked at Yale University and made significant theoretical contributions to physics, chemistry, and mathematics.

Problem 4.2 (numerical computation of the Fourier coefficients)**.** Sometimes integrals are difficult, or impossible, to compute analytically, and that is where one has to code up the notion of Riemann sum. Given a function $g : [a, b] \to \mathbb{C}$, fix the positive integer N and define the time step $\Delta t = \frac{b-a}{N}$. Then:

$$\int_a^b g(t)\; dt \simeq \Delta t \sum_{k=0}^{N-1} g(a + k\,\Delta t),$$

and the approximation becomes exact as $N \to \infty$. The technique can be used to compute Fourier coefficients. For example, consider the perioric function $s(t)$ of semi-circular 'bumps' defined as follows:

$$h(t) = \begin{cases} \sqrt{1-t^2} & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}, \qquad \text{and} \qquad s(t) = \text{rep}_4[h](t) = \sum_{n=-\infty}^{\infty} h(t - 4n).$$

The signal $s(t)$ is T-periodic with $T = 4$; it is *even*, so $B_k = 0$ for all k. Write code to compute c_0 and A_k, for $k \geq 1$, using the Riemann sum above, and compute some *partial* sums $s_n(t)$ of the Fourier series of $s(t)$ (e.g. for $n = 1$, 2, 5, and 20). Display the graphs of the partial sums, comparing them with $s(t)$.

Problem 4.3 (time shifting)**.** Consider the *shifted* Dirac delta function, $\delta_{t_0}(t) = \delta(t - t_0)$, $t \in \mathbb{R}$. **(a)** Show that, for any signal $s(t)$, $t \in \mathbb{R}$, it is the case that $(s * \delta_{t_0})(t) = s(t - t_0)$ (that is, the convolution with δ_{t_0} hs the effect of time-shifting by t_0). **(b)** Find the Fourier Transform $\mathcal{F}[\delta_{t_0}](f)$, $f \in \mathbb{R}$, by direct computation. **(c)** Finally, combine the above results with the convolution theorem (it states that $\mathcal{F}[g*h](f) = G(f)H(f)$) to find the Fourier Transform of the shifted signal $s(t - t_0)$ in terms of the Fourier Transform $S(f)$ of $s(t)$.

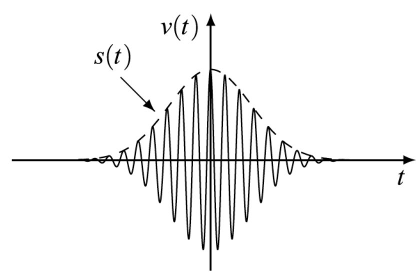
Problem 4.4. Fix a constant $\alpha > 0$, and consider the continuous-time signal $s(t) = e^{-\alpha t}\mathbb{1}(t)$, $t \in \mathbb{R}$, known as the *causal exponential*, where $\mathbb{1}(t)$ is the *Heaviside function*[3] (that is, $\mathbb{1}(t) = 1$ for $t \geq 0$, $\mathbb{1}(0) = 1/2$, and $\mathbb{1}(t) = 0$ for $t < 0$). **(a)** Compute its Fourier transform $S(f)$, $f \in \mathbb{R}$ (this is example #26 in the Fourier pairs handout on Blackboard). **(b)** Use the result from part (a) to compute the Fourier transform of $g(t) = e^{-\alpha|t|}$, $t \in \mathbb{R}$ (plotting the graph of g and comparing it with the graph of s may help). **(c)** Compute the convolution $r(t) = (s * s)(t)$, $t \in \mathbb{R}$. **(d)** Finally, using the result from part (a), what is the Fourier transform $R(f)$ of the signal $r(t)$ that you computed in part (c)?

Problem 4.5. (scaling) **(a)** Let $s(t) \xrightarrow{\mathcal{F}} S(f)$ be a given Fourier pair. Show that $s(at) \xrightarrow{\mathcal{F}} \frac{1}{a}S\big(\frac{f}{a}\big)$, for any constant a. **(b)** A known Fourier pair is $e^{-\pi t^2} \xrightarrow{\mathcal{F}} e^{-\pi f^2}$. (You do not have to prove it: this is the Fourier pair #21 in the handout. Note that since $s(t) = e^{-\pi t^2}$ Fourier-transforms to itself, i.e. $\mathcal{F}[s] = s$, we say that s is an *eigenfunction* of the linear operator $\mathcal{F}$, with eigenvalue 1.) Show that we have:

$$g(t) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2}\frac{t^2}{\sigma^2}} \quad \xrightarrow{\quad\mathcal{F}\quad} \quad G(f) = e^{-2\pi^2\sigma^2 f^2}.$$

This will be useful when we will perform the Fourier analysis of the Unmask Filter (UMF) for images.

Problem 4.6. In analog telecommunications a common operation is *modulation*, which consists of transforming an analog lowpass signal (i.e. with low frequencies), for example an audio signal, into a bandpass signal at a different frequency, to transmiti it over a limited radio frequency band. *Amplitude Modulation* (AM) consists of taking a signal s, with the property that the extension[4] of its Fourier transform S is such that $e(S) \subseteq [-B, B]$ (for some $B > 0$) and defining the new signal:



$$v(t) = s(t)\cos(2\pi f_0 t),\ t \in \mathbb{R},$$

where $f_0 > B$ is a chosen frequency (see the figure above). Find a formula for the Fourier transform $V(f) = \mathcal{F}[v](f)$ in terms of $S(f)$, $f \in \mathbb{R}$. What can you tell about the extension of V (i.e. where is V nonzero)?

[3]We use the notation $\mathbb{1}$ instead of the more conventional H, because we want to reserve the latter symbol for the Fourier stransform of a function generic function h.

[4]Remember that the extension of a function $g(x)$, $x \in \mathbb{R}$, is the set $e(g) = \{x \,|\, g(x) \neq 0\}$. A signal s whose Fourier transform S has the property that $e(S) \subseteq [-B, B]$ is called *band-limited*, and B is called the *band* of the signal.

Problem 4.7. Show that the Fourier transform $\mathcal{F}$ has "period 4", i.e. that for *any* signal s it is the case that $\mathcal{F}^4[s] = s$ (that is, $\mathcal{F}[\mathcal{F}[\mathcal{F}[\mathcal{F}[s]]]] = s$). *Hint:* For a generic signal s, what is $\mathcal{F}^2[s] = \mathcal{F}[\mathcal{F}[s]]$?

Problem 4.8. Consider a Fourier pair $s \xrightarrow{\mathcal{F}} S$. Show that: $|s'(t)| \leq 2\pi \displaystyle\int_{-\infty}^{\infty} |f|\,|S(f)|\,df$, for all $t \in \mathbb{R}$.

Remark: The right-hand side of the inequality does not depend on t (it is a constant), so we have a *uniform bound* on the derivative of s, which is the slope of its graph. Note that $|f|\,|S(f)|$ is the magnitude of the Fourier transform of s, with its high frequencies "boosted" by the function $|f|$. Therefore, if $S(f)$ does *not* have high frequencies (e.g.. $S(f) = 0$ for $|f| > B$, for some $B > 0$) or if $S(f) \to 0$ reasonably fast as $f \to \pm\infty$, then the integral will yield a small number: this bounds the absolute value of the derivative $s'(t)$ for all t, thus constraining the graph of the signal $s(t)$ to small oscillations (making it smoother).

Problem 4.9. It was proven in class that if we have the Fourier pair $s \xrightarrow{\mathcal{F}} S$, then $\mathcal{F}[s'](f) = (i2\pi f)S(f)$, $f \in \mathbb{R}$. This is illustrated by the commutative diagram[5] on the right. Consider:

$$s(t) = \text{rect}(t) = \begin{cases} 1 & \text{for } |t| \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{tikzcd} s(t) \arrow[r, "\mathcal{F}"] \arrow[d, "\frac{d}{dt}"] & S(f) \arrow[d, "(i2\pi f)\cdot"] \\ s'(t) \arrow[r, "\mathcal{F}"] & i2\pi f\, S(f) \end{tikzcd}$$

(We showed in class that $S(f) = \mathcal{F}[s](t) = \text{sinc}(f) = \frac{\sin(\pi f)}{\pi f}$, $f \in \mathbb{R}$.)
Show that the diagram indeed commutes for the function $s(t)$ above, i.e. compute the Fourier transform $U(f) = \mathcal{F}[s'](f)$ following each of the two directed paths, and show that they yield to the same result.

Problem 4.10. Consider the *sign function* (also known as *signum function*), $\text{sgn}(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t = 0 \\ -1 & \text{for } t < 0 \end{cases}$ and the *Heaviside function*,[6] i.e. $\mathbb{1}(t) = \begin{cases} 1 & \text{for } t > 0 \\ 1/2 & \text{for } t = 0 \\ 0 & \text{for } t < 0 \end{cases}$.

(a) Plot the graphs of $y = \text{sgn}(t)$ and $y = \mathbb{1}(t)$ on two separate (t, y)-planes.

(b) Show that the Fourier transform of $\text{sgn}(t)$ is $\mathcal{F}[\text{sgn}](f) = \dfrac{1}{i\pi f}$, $f \in \mathbb{R}$.
Hint: This is challenging. One way to proceed is to consider the function $X(f) = \frac{1}{i\pi f}$, $f \in \mathbb{R}$, and to show that $\mathcal{F}^{-1}[X](t) = \text{sgn}(t)$. To do so, let $x(t) = \mathcal{F}^{-1}[X](t)$, and interpret the improper integral in the inverse Fourier transform as a *Cauchy principal value*, i.e. as the limit:

$$x(t) = \int_{-\infty}^{\infty} \frac{1}{i\pi f}\, e^{i2\pi ft}\, dt = \lim_{a \to +\infty} \int_{-a}^{a} \frac{1}{i\pi f}\, e^{i2\pi ft}\, df.$$

Use Euler's formula for $e^{i2\pi ft}$ and break the integral into a real part and an imaginary part. One of the two will vanish because of the fact that its integrand is an odd function. You will then need a change of variable ($v = 2ft$, and you will get a different result depending on the sign of t).

(c) First expressing the Heaviside function in terms of the sign function, show, using the previous part of this problem, that the Fourier transform of the Heaviside function is $\mathcal{F}[\mathbb{1}](f) = \frac{1}{2}\delta(t) + \frac{1}{i2\pi f}$, $f \in \mathbb{R}$.

(d) Finally, given a generic signal $s(t)$, $t \in \mathbb{R}$, define its antiderivative:[7] $u(t) = \displaystyle\int_{-\infty}^{t} s(\tau)\,d\tau$.
Find the Fourier transform of u in terms of $S(f) = \mathcal{F}[s](f)$, $f \in \mathbb{R}$.
Hint: Can you express the integral that defines u as the *convolution* of s with another function?

[5] In mathematics, and especially in category theory, a *commutative diagram* is a diagram such that all directed paths in the diagram with the same start and endpoints lead to the same result.

[6] See footnote 3.

[7] Note that all antiderivatives of s differ by a constant. For example, $w(t) = \int_a^t s(\tau)\,d\tau$ is *the* antiderivative of s such that $w(0) = 0$. The antiderivetive u defined above is the one that vanishes at $-\infty$, i.e. the one such that $\lim_{t \to -\infty} u(t) = 0$.

Problem 4.11. Using the results from Problems 4.5, 4.9, and 4.10, find the Fourier transforms of:

(a) $s_1(t) = A_0\, e^{-t^2/T^2}$, $t \in \mathbb{R}$, where A_0 and T are real constants.

(b) $s_2(t) = te^{-t^2/T^2}$, $t \in \mathbb{R}$.

(c) $s_3(t) = \displaystyle\int_{-\infty}^{t} e^{-u^2/T^2}\, du$, $t \in \mathbb{R}$.

> ***Groups.*** A set G, with an operation $+$, is called a *group* and is indicated with $(G, +)$ if the following properties hold:
>
> **1 (Closure with respect to +).** For every $g \in G$ and $h \in G$, we have $g + h \in G$.
> **2 (Associativity).** For every $g, h, \ell \in G$, we have $(g + h) + \ell = g + (h + \ell) \in G$.
> **3 (Identity element).** There exists an element in G, indicated with 0 and called the *identity*, such that for all $g \in G$ we have $g + 0 = 0 + g = h$.
> **4 (Inverse).** For every $g \in G$, there exists an element $h \in G$, called *inverse of g* and indicated with $-g$, such that $g + h = h + g = 0$.
>
> Finally, if properties **1-4** hold, as well as property **5** below, then the group is called *Abelian* or *commutative*.
>
> **5 (Commutativity).** For every g, $h \in G$, we have $g + h = h + g$.
>
> A set H is called a *subgroup* of G if it is a subset of G (that is, $H \subseteq G$) and it is a group with the same operation in G.

Problem 4.12 (periodicity *must* be a group)**.** Consider a signal s defined on $\mathbb{R}$: that is, $y = s(t)$, with $t \in \mathbb{R}$. Consider the subset $P = \{p \,|\, s(t + p) = s(t) \text{ for all } t \in \mathbb{R}\} \subseteq \mathbb{R}$, called the *periodicity* of the signal s: it is the <u>*set*</u> of values of $p \in \mathbb{R}$ for which we have $s(t + p) = s(t)$ for all $t \in \mathbb{R}$. Show that P is an Abelian group. *Remark:* In exactly the same way, one can show that of a signal s defined on a generic Abelian group I, the set $P = \{p \,|\, s(t + p) = s(t) \text{ for all } t \in I\} \subseteq I$ is *subgroup* of I.