Andrew Cornelio HW 8

Problem 1

Start with the model problem $\dot{y} = \lambda y$. Substitute this in to the formula:

$$y_{n+1} = y_n + \frac{h}{2}\left[\lambda y_n + \lambda y_{n+1}\right]$$

This is a difference equation that can be solved with $y_n = r^n$.

$$r^{n+1} = r^n + \frac{h}{2}\left(\lambda r^n + \lambda r^{n+1}\right)$$

$$r = 1 + \frac{h}{2}\left(\lambda + \lambda r\right)$$

$$2r - h\lambda r = 2 + h\lambda$$

$$r = \frac{2 + h\lambda}{2 - h\lambda}$$

We have found r_0. Now we must show $|r_0(z)| < 1$ for all $Re(z) < 0$. Let $z = h\lambda$. Let's find $|r_0(z)|^2$:

$$|r_0(z)|^2 = \frac{|2+z|^2}{|2-z|^2} = \frac{4 + 4Re(z) + |z|^2}{4 - 4Re(z) + |z|^2}$$

Since $Re(z) < 0$,

$$4 + 4Re(z) + |z|^2 < 4 - 4Re(z) + |z|^2$$

Therefore,

$$|r_0(z)|^2 < 1 \quad \text{and}$$

$$|r_0(z)| < 1$$

Problem 3

a) The classic fourth order Runge-Kutta method is

$$y_{n+1} = y_n + \frac{1}{6}h\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$k_1 = f(t_n, y_n)$$

$$k_2 = f\left(t_n + \frac{h}{2},\ y_n + h\frac{k_1}{2}\right)$$

$$k_3 = f\left(t_n + \frac{h}{2},\ y_n + h\frac{k_2}{2}\right)$$

$$k_4 = f\left(t_n + h,\ y_n + hk_3\right)$$

To find the characteristic polynomial, we subsitute $\dot{y} = \lambda y$ and $y_n = r^n$:

$$k_1 = \lambda r^n$$

$$k_2 = \lambda\left(r^n + h\frac{k_1}{2}\right) = \lambda r^n + \frac{1}{2}h\lambda^2 r^n$$

$$k_3 = \lambda\left(r^n + h\frac{k_2}{2}\right) = \lambda r^n + \frac{1}{2}h\lambda^2 r^n + \frac{1}{4}h^2\lambda^3 r^n$$

$$k_4 = \lambda\left(r^n + hk_3\right) = \lambda r^n + h\lambda^2 r^n + \frac{1}{2}h^2\lambda^3 r^n + \frac{1}{4}h^3\lambda^4 r^n$$

Substuting this back in:

$$r^{n+1} = r^n + \frac{h}{6}\Big(\lambda r^n + 2\big(\lambda r^n + \tfrac{1}{2}h\lambda^2 r^n\big) + 2\big(\lambda r^n + \tfrac{1}{2}h\lambda^2 r^n + \tfrac{1}{4}h^2\lambda^3 r^n\big) + \lambda r^n + h\lambda^2 r^n + \tfrac{1}{2}h^2\lambda^3 r^n + \tfrac{1}{4}h^3\lambda^4 r^n\Big)$$

$$= r^n + r^n\Big(\tfrac{1}{6}h\lambda + \tfrac{1}{3}h\lambda + \tfrac{1}{6}h^2\lambda^2 + \tfrac{1}{3}h\lambda + \tfrac{1}{6}h^2\lambda^2 + \tfrac{1}{12}h^3\lambda^3 + \tfrac{1}{6}h\lambda + \tfrac{1}{6}h^2\lambda^2 + \tfrac{1}{12}h^3\lambda^3 + \tfrac{1}{24}h^4\lambda^4\Big)$$

$$= r^n + r^n\Big(h\lambda + \tfrac{1}{2}h^2\lambda^2 + \tfrac{1}{6}h^3\lambda^3 + \tfrac{1}{24}h^4\lambda^4\Big)$$

So we have

$$r = 1 + h\lambda + \tfrac{1}{2}h^2\lambda^2 + \tfrac{1}{6}h^3\lambda^3 + \tfrac{1}{24}h^4\lambda^4$$

If we take $z = h\lambda$:

$$r_0(z) = 1 + z + \tfrac{1}{2}z^2 + \tfrac{1}{6}z^3 + \tfrac{1}{24}z_4$$

This is the 4th order expansion of e^x around $x=0$

b) The plot is in a file named plot_2b.png. The stability threshold is $z = -2.7853$. We can easily find the threshold of stability for the Euler method. For the Euler method, $r_0(z) = 1 + z$. If $Re(z) < 0$, then $|r_0(z)| < 1$ when $z \in (-2, 0)$. So the threshold of stability for rk4 is larger

Problem 4

a) Solution plot is titled plot_4a.png and step size plot is titled step_4a.png. There were 68 successful steps, 6 failed steps and 160 fn evals. The step sizes generally increased over time.

b) Plot named plot_4b.png

c) The same naming scheme applies as 4a. There were 141355 successful steps, 9425 failed steps, and 904681 fn evals. Step size generally decreased over time.

d) Since all the eigenvalues that we found in (b) were less than 0, the system is stiff. Since ode45 uses runge-kutta, it has difficulty handling stiff systems. The step size of ode45 is bounded by

$$\frac{Z_*}{\lambda_{min}(t)}$$

in step_4d.png, we see that this is a good bound.