Andrew Cornelio HW 6

Problem 1

a) Euler's Method

i) We will prove by induction that the explicit expression is

$$y_n = (1 + h\lambda)^n$$

Base case:

$$y_1 = y(0) + h[\lambda y(0)]$$
$$= 1 + h\lambda = (1+h\lambda)^1$$

Now we assume that the formula is true for n up till some integer k. We will prove the expression for $n = k+1$.

$$y_{k+1} = (1+h\lambda)^k + h\lambda(1+h\lambda)^k$$
$$= (1+h\lambda)^k(1+h\lambda)$$
$$= (1+h\lambda)^{k+1}$$

ii) We know that the formula is $y_n = (1+h\lambda)^{k+1}$. We want convergence for

$$\lim_{n\to\infty} (1+h\lambda)^n$$

This series will only converge for $h\lambda \in (-2, 0)$

iii) By definition of the exponential, we know

$$\lim_{n\to\infty}\left(1+\frac{\lambda t}{n}\right)^n = e^{\lambda t}$$

Now we use the hint to prove the remainder:

$$y_n - y(t) = \left(1+\frac{\lambda t}{n}\right)^n - e^{\lambda t}$$

$$= \left[e^{\frac{\lambda t}{n}}\left(1-\frac{1}{2}\left(\frac{\lambda t}{n}\right)^2 + O\left(\frac{1}{n^3}\right)\right)\right]^n - e^{\lambda t}$$

$$= e^{\lambda t} e^{n \ln\left(1-\frac{1}{2}\left(\frac{\lambda t}{n}\right)^2 + O\left(\frac{1}{n^3}\right)\right)} - e^{\lambda t}$$

We know the taylor expansion of

$$\ln(1-x) = -x - \frac{1}{2!}\left(\frac{1}{(1+\xi)^2}\right)x^2$$

Thus:

$$y_n - y(t) = e^{\lambda t} e^{n\left(-\frac{1}{2}\left(\frac{\lambda t}{n}\right)^2 + O(n^{-3})\right)} - e^{\lambda t}$$

$$= e^{\lambda t} e^{-\frac{1}{2}\frac{\lambda^2 t^2}{n} + O(n^{-2})} - e^{\lambda t}$$

$$= e^{\lambda t}\left[1 - \frac{1}{2}\frac{\lambda^2 t^2}{n} + O\left(\frac{1}{n^2}\right)\right] - e^{\lambda t}$$

$$= e^{\lambda t}\left[-\frac{1}{2}\frac{\lambda^2 t^2}{n} + O\left(\frac{1}{n^2}\right)\right]$$

$$= -\frac{1}{2}\frac{\lambda^2 t^2}{n} e^{\lambda t} + O\left(\frac{1}{n^2}\right)$$

iv) We start with the differential equation:

$$\delta'(t) = \lambda \delta(t) + \frac{\lambda^2}{2} e^{\lambda t}$$

$$\delta' - \lambda \delta = \frac{\lambda^2 e^{\lambda t}}{2}$$

The integrating factor is

$$\mu(t) = e^{\int -\lambda \, dt} = e^{-\lambda t}$$

We can then solve the diff eq using the formula:

$$(\mu(t)\delta(t))' = \mu(t) g(t)$$

$$e^{-\lambda t}\delta(t) = \int e^{\cancel{-\lambda t}} \left(\frac{\lambda^2}{2} \cancel{e^{\lambda t}}\right) dt$$

$$e^{-\lambda t}\delta(t) = \frac{\lambda^2 t}{2} + c$$

$$\delta(t) = \frac{\lambda^2 t}{2} e^{\lambda t} + c e^{\lambda t}$$

Since $\delta(0) = 0$, $c = 0$

Now, we can just plug in the solution:

$$y_n - y(t) = -\delta(t)\, h + O(n^2)$$
$$= -\frac{\lambda^2 t^2}{2n} e + O(n^2)$$

b) Heun's method

i) The update for Heun's method is:

$$\tilde{y}_{i+1} = y_i + h f(t_i, y_i) = y_i + h\lambda y_i$$

$$y_{i+1} = y_i + \frac{h}{2}\left[f(t_i, y_i) + f(t_{i+1}, \tilde{y}_{i+1})\right]$$
$$= y_i + \frac{h}{2}\left[\lambda y_i + \lambda(y_i + h\lambda y_i)\right]$$
$$= y_i + \frac{h}{2}\left[2\lambda y_i + h\lambda^2 y_i\right]$$
$$= y_i + h\lambda y_i + \frac{h^2\lambda^2}{2} y_i$$
$$= \left[1 + h\lambda + \frac{h^2\lambda^2}{2}\right] y_i$$

We can easily see now that

$$y_n = \left[1 + h\lambda + \frac{h^2\lambda^2}{2}\right] y_{n-1}$$
$$= \left(1 + h\lambda + \frac{h^2\lambda^2}{2}\right)^n$$

ii) We have the formula

$$y_n = \left(1 + h\lambda + \frac{h^2\lambda^2}{2}\right)^n$$

y_n converges when

$$-1 < 1 + h\lambda + \frac{h^2\lambda^2}{2} < 1$$

$$-4 < (h\lambda)^2 + 2h\lambda < 0$$

$$-4 < (h\lambda)(h\lambda + 2) < 0$$

We can see that the series will converge for $h\lambda \in (-2, 0)$.

iii) First we must find a suitable approximation for $1 + h\lambda + \frac{h^2\lambda^2}{2}$.

Just like we did in part a, we can use the taylor series of e^{ht}:

$$e^{\frac{\lambda t}{n}} = 1 + \frac{\lambda t}{n} + \frac{1}{2}\frac{\lambda^2 t^2}{n^2} + \frac{1}{6}\frac{\lambda^3 t^3}{n^3} + O\left(\frac{1}{n^4}\right)$$

$$1 + \frac{\lambda t}{n} + \frac{1}{2}\frac{\lambda^2 t^2}{n^2} = e^{\frac{\lambda t}{n}} - \frac{1}{6}\frac{\lambda^3 t^3}{n^3} + O\left(\frac{1}{n^4}\right)$$

$$= e^{\frac{\lambda t}{n}}\left[1 - \frac{\lambda^3 t^3}{6n^3} + O(n^{-4})\right]$$

Now we need to find

$$y_n - y(t) = \left[e^{\frac{\lambda t}{n}}\left[1 - \frac{\lambda^3 t^3}{6n^3} + O(n^{-4})\right]\right]^n - e^{\lambda t}$$

$$= e^{\lambda t} e^{n \ln\left(1 - \frac{\lambda^3 t^3}{6n^3} + O(n^{-4})\right)} - e^{\lambda t}$$

We remind ourselves of the fact that

$$\ln(1-x) = -x - \frac{1}{2!}\left(\frac{1}{(1+\xi)^2}\right) x^2$$

Thus

$$y_n - y(t) = e^{\lambda t} e^{n\left[-\frac{\lambda^3 t^3}{6n^3} + O(n^{-4})\right]} - e^{\lambda t}$$

$$= e^{\lambda t} e^{-\frac{\lambda^3 t^3}{6n^2} + O(n^{-3})} - e^{\lambda t}$$

$$= e^{\lambda t}\left[1 - \frac{\lambda^3 t^3}{6n^2} + O(n^{-3})\right] - e^{\lambda t}$$

$$= -\frac{\lambda^3 t^3}{6n^2} e^{\lambda t} + O(n^{-3})$$

iv) First we start with the diff eq:

$$\delta'(t) = \lambda \delta(t) + \frac{1}{6}\lambda^3 e^{\lambda t}$$

$$\delta'(t) - \lambda \delta(t) = \frac{1}{6}\lambda^3 e^{\lambda t}$$

Again, we have integrating factor of $\mu(t) = e^{-\lambda t}$

So

$$(\mu(t)\delta(t))' = \mu(t)g(t)$$

$$e^{-\lambda t}\delta(t) = \int \cancel{e^{-\lambda t}}\left[\frac{1}{6}\lambda^3 \cancel{e^{\lambda t}}\right]dt$$

$$e^{-\lambda t}\delta(t) = \frac{1}{6}\lambda^3 t + c$$

$$\delta(t) = \frac{1}{6}\lambda^3 t + ce^{-\lambda t}$$

If $\delta(0)=0$, then $c=0$. Now if we plug the formula into the the local error formula:

$$y_n - y(t) = -\frac{\lambda^3 t^3}{6n^2}e^{\lambda t} + O(n^{-3})$$

This agrees with the expression obtained in (iii)

Problem 2

a) First we expand the taylor series around $y(t_n)$:

$$y(t_n+h) = y(t_n) + y'(t_n)h + \frac{1}{2}y''(t_n)h^2 + \frac{1}{6}y'''(t_n)h^3 + O(h^4)$$

Rearranging, we obtain:

$$y(t_n+h) - y(t_n) = y'(t_n)h + \frac{1}{2}y''(t_n)h^2 + \frac{1}{6}y'''(t_n)h^3 + O(h^4)$$

But we also know that the local truncation error of a third order runge-kutta method is

$$\tau_n = \frac{y(t_n+h) - [y(t_n) + h\phi_n]}{h}$$

$$= \frac{y(t_n+h) - y(t_n) - h[\gamma_1 K_1 + \gamma_2 K_2 + \gamma_3 K_3]}{h}$$

$$= O(h^3)$$

If we rearrange the terms we obtain:

$$y(t_n+h) - y(t_n) \doteq h[\gamma_1 K_1 + \gamma_2 K_2 + \gamma_3 K_3] + O(h^4)$$

Now we have two equivalent expressions in terms of known quantities. The equality we must solve is:

$$(*)\quad h[\gamma_1 K_1 + \gamma_2 K_2 + \gamma_3 K_3] + O(h^4) = y'(t_n)h + \tfrac{1}{2}y''(t_n)h^2 + \tfrac{1}{6}y'''(t_n)h^3 + O(h^4)$$

Now we must find expressions for $y', y'', y''', K_1, K_2, K_3$ in terms of $\gamma_j, \alpha_j, \beta_{ij}$. Let's expand the K terms first using the taylor series. Since there is a $O(h^4)$ term, we only need to expand each series up till $O(h^2)$, since every term is multiplied by h and $hO(h^3) = O(h^4)$. We expand each series around the point (t_n, y_n).

$$K_1 = f$$

$$K_2 = f + \left[\frac{\partial f}{\partial t}(\alpha_2 h) + \frac{\partial f}{\partial y}(\beta_{21} K_1 h)\right] + \frac{1}{2}\left[\frac{\partial}{\partial t}\frac{\partial f}{\partial t}(\alpha_2 h)^2 + 2\frac{\partial}{\partial t}\frac{\partial f}{\partial y}(\alpha_2 h)(\beta_{21} K_1 h) + \frac{\partial}{\partial y}\frac{\partial f}{\partial y}(\beta_{21} K_1 h)^2\right] + O(h^3)$$

$$= f + [f_t \alpha_2 + f_y \beta_{21} K_1]h + \frac{1}{2}[f_{tt}\alpha_2^2 + 2 f_{yt}\alpha_2 \beta_{21} K_1 + f_{yy}\beta_{21}^2 K_1^2]h^2 + O(h^3)$$

$$= f + [f_t \alpha_2 + f_y f \beta_{21}]h + \frac{1}{2}[f_{tt}\alpha_2^2 + 2 f_{yt} f \alpha_2 \beta_{21} + f_{yy} f^2 \beta_{21}^2]h^2 + O(h^3)$$

$$K_3 = f + [f_t \alpha_3 + f_y(\beta_{31}K_1 + \beta_{32}K_2)]h + \frac{1}{2}[f_{tt}\alpha_3^2 + 2 f_{yt}\alpha_3(\beta_{31}K_1 + \beta_{32}K_2) + f_{yy}(\beta_{31}K_1 + \beta_{32}K_2)^2]h^2 + O(h^3)$$

$$K_3 = f + [f_t \alpha_3]h + \tfrac{1}{2}[f_{tt}\alpha_3^2]h^2 + O(h^3)$$
$$+ f_y \beta_{31} f h$$
$$+ f_y \beta_{32}\left(fh + [f_t\alpha_2 + f_y \beta_{21} f]h^2\right) + O(h^3)$$
$$+ f_{yt}\alpha_3 \beta_{31} f h^2$$
$$+ f_{yt}\alpha_3 \beta_{32} f h^2 + O(h^3)$$
$$+ \tfrac{1}{2} f_{yy} \beta_{31}^2 f^2 h^2$$
$$+ f_{yy}\beta_{31}\beta_{32} f^2 h^2$$
$$+ \tfrac{1}{2} f_{yy}\beta_{32}^2 f^2 h^2$$

$$= f + [f_t\alpha_3 + f_y f(\beta_{31}+\beta_{32})]h + [\tfrac{1}{2} f_{tt}\alpha_3^2$$
$$+ \tfrac{1}{2} f_{yy} f^2(\beta_{31}+\beta_{32})^2 + f_y f_t \alpha_2 \beta_{32} + f_y^2 f \beta_{32}\beta_{21}$$
$$+ f_{yt} f \alpha_3(\beta_{31}+\beta_{32})]h^2 + O(h^3)$$

Now we have all the K terms expanded in terms of α_i, β_{ij} and derivatives of f. We can also expand the derivatives of y in terms of derivatives of f using the chain rule:

$$D_t y = f$$
$$D_{tt} y = D_t f = \frac{\partial}{\partial t} f + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = f_t + f_y f$$
$$D_{ttt} y = D_t[f_t + f_y f] = \frac{\partial}{\partial t} f_t + \frac{\partial f_t}{\partial y}\frac{\partial y}{\partial t} + D_t[f_y]f$$
$$+ f_y D_t[f]$$
$$= f_{tt} + f_{ty} f + \left[\frac{\partial f_y}{\partial t} + \frac{\partial f_y}{\partial y}\frac{\partial y}{\partial t}\right]f$$
$$+ f_y\left[\frac{\partial}{\partial t} f + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\right]$$
$$= f_{tt} + 2 f_{ty} f + f_{yy} f^2 + f_y f_t + f_y^2 f$$

Now we just need to substitute all the expanded expressions back into equation (*). Rather than doing it naively, we can break the equality down based on powers of h.

$$[\gamma_1 f + \gamma_2 f + \gamma_3 f] h = f h$$

$$[\gamma_2(f_t \alpha_2 + f_y f \beta_{21}) + \gamma_3(f_t \alpha_3 + f_y f(\beta_{31} + \beta_{32}))] h^2 = \frac{1}{2}[f_t + f_y f] h^2$$

$$[\gamma_2(\tfrac{1}{2} f_{tt} \alpha_2^2 + f_{yt} f \alpha_2 \beta_{21} + \tfrac{1}{2} f_{yy} f^2 \beta_{21}^2) + \gamma_3(\tfrac{1}{2} f_{tt} \alpha_3^2 + \tfrac{1}{2} f_{yy} f^2 (\beta_{31} + \beta_{32})^2 + f_y f_t \alpha_2 \beta_{32} + f_y^2 f \beta_{32} \beta_{21} + f_{yt} f \alpha_3 (\beta_{31} + \beta_{32}))] h^3$$
$$= \frac{1}{6}[f_{tt} + 2 f_{ty} f + f_{yy} f^2 + f_y f_t + f_y^2 f] h^3$$

Furthermore, within each each equation, we can create more equalities by setting each derivative of f equal.

$$\gamma_1 + \gamma_2 + \gamma_3 = 1$$
$$\gamma_2 \alpha_2 + \gamma_3 \alpha_3 = \frac{1}{2}$$
$$\gamma_2 \beta_{21} + \gamma_3 \beta_{31} + \gamma_3 \beta_{32} = \frac{1}{2}$$
$$\frac{1}{2} \gamma_2 \alpha_2^2 + \frac{1}{2} \gamma_3 \alpha_3^2 = \frac{1}{6}$$
$$\gamma_2 \alpha_2 \beta_{21} + \gamma_3 \alpha_3 \beta_{31} + \gamma_3 \alpha_3 \beta_{32} = \frac{1}{3}$$
$$\frac{1}{2} \gamma_2 \beta_{21}^2 + \frac{1}{2} \gamma_3 (\beta_{31} + \beta_{32})^2 = \frac{1}{6}$$
$$\gamma_3 \alpha_2 \beta_{32} = \frac{1}{6}$$
$$\gamma_3 \beta_{32} \beta_{21} = \frac{1}{6}$$

These are the constraints

b) We just need to plug the constants into the constraints we derived. We can do this using a calculator.

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 3/4 & 0 & 3/4 & 0 \\ \hline & 2/9 & 1/3 & 4/9 \end{array}$$

✓ $\gamma_1 + \gamma_2 + \gamma_3 = 1$

✓ $\gamma_2 \alpha_2 + \gamma_3 \alpha_3 = \frac{1}{2}$

✓ $\gamma_2 \beta_{21} + \gamma_3 \beta_{31} + \gamma_3 \beta_{32} = \frac{1}{2}$

✓ $\frac{1}{2} \gamma_2 \alpha_2^2 + \frac{1}{2} \gamma_3 \alpha_3^2 = \frac{1}{6}$

✓ $\gamma_2 \alpha_2 \beta_{21} + \gamma_3 \alpha_3 \beta_{31} + \gamma_3 \alpha_3 \beta_{32} = \frac{1}{3}$

✓ $\frac{1}{2} \gamma_2 \beta_{21}^2 + \frac{1}{2} \gamma_3 (\beta_{31} + \beta_{32})^2 = \frac{1}{6}$

✓ $\gamma_3 \alpha_2 \beta_{32} = \frac{1}{6}$

✓ $\gamma_3 \beta_{32} \beta_{21} = \frac{1}{6}$

All the constraints are satisfied.

c) I modified rk4.m which was provided on the course webpage to create rk3.m. I also created main_2c.m that produces a table of values. A text version of the table is available in the file results_rk3.txt.

Problem 3

I created two main files and results files just like for prob 2c. I noticed that both have convergence order of about 1.3 rather than 2 and 4 for heun and rk4 respectively. This is due to the function we are trying to approximate.

In the derivation of heun's method, we rely on the taylor series approx. up till order 2. In order for this approximation to be valid y'' must exist and be finite over the interval. However, for this function,

$$y' = \frac{-t^{1/3}}{(1+\frac{3}{4}t^{4/3})^2} \qquad y'' = \frac{16(21t^{4/3}-4)}{3t^{2/3}(3t^{4/3}+4)^3}$$

we can see that

$$\lim_{t \to 0} y'' = -\infty$$

So the assumption is not valid. This is also true for rk4.