Andrew Cornelio HW 7

Problem 1

a) We use the constraints presented in lecture:

Consistency:

$$\sum_{j=0}^{p} a_j = 1$$

$$-\sum_{j=0}^{p} j a_j + \sum_{j=-1}^{p} b_j = 1$$

formal m-th order convergence

$$\sum_{j=0}^{p} (-j)^i a_j + i \sum_{j=-1}^{p} (-j)^{i-1} b_j = 1$$
for $i = 2, \ldots, m$

Writing these constraints explicitly we have the linear system:

$$a_0 + a_1 = 1$$
$$-\cancel{0 a_0} - 1 a_1 + b_0 + b_1 = 1$$
$$\cancel{(-0)^2 a_0} + (-1)^2 a_1 + 2[\cancel{(-0)^1 b_0} + (-1)^1 b_1] = 1$$
$$\cancel{(-0)^3 a_0} + (-1)^3 a_1 + 3[\cancel{(-0)^2 b_0} + (-1)^2 b_1] = 1$$

If we clean this up, we get:

$$\begin{aligned} a_0 + a_1 \quad\quad\quad\quad &= 1 \\ -a_1 + b_0 + b_1 &= 1 \\ a_1 \quad -2b_1 &= 1 \\ -a_1 \quad +3b_1 &= 1 \end{aligned}$$

We can just use a numerical solver to find:

$$a_0 = -4 \qquad a_1 = 5$$
$$b_0 = 4 \qquad b_1 = 2$$

b) We know that the method has third order convergence. This means that the truncation error must be of the form

$$y_{n+1} - y_n = \frac{c}{4!} y^{IV}(t_n) h^4 + O(h^5)$$

For some constant c. We use the formula presented in lecture to find the value of c:

$$c = 1 - \left[\cancel{(-0)^4 a_0} + (-1)^4 a_1 + 4\left[\cancel{(-0)^3} b_0 + (-1)^3 b_1\right]\right]$$
$$= 1 - (-3) = 4$$

Thus the leading term has form

$$\frac{1}{3} y^{IV}(t_n) h^4$$

c) Let's use the model problem

$$\dot{y} = \lambda y \qquad y(0) = 1$$

We plug this into the method:

$$y_{n+1} = -4y_n + 5y_{n-1} + h[4\lambda y_n + 2\lambda y_{n-1}]$$

This is a linear difference eqn that has the solution $y_n = r^n$. Plugging this in, we get

$$r^{n+1} = -4r^n + 5r^{n-1} + h[4\lambda r^n + 2\lambda r^{n-1}]$$

$$r^2 = -4r + 5 + h\lambda[4r + 2]$$

$$r^2 + (4 - 4h\lambda)r - (5 + 2h\lambda) = 0$$

$$r = 2h\lambda - 2 \pm \sqrt{(2-2h\lambda)^2 + (5+2h\lambda)}$$

$$= 2h\lambda - 2 \pm \sqrt{4 - 8h\lambda + 4h^2\lambda^2 + 5 + 2h\lambda}$$

$$= 2h\lambda - 2 \pm \sqrt{4h^2\lambda^2 - 6h\lambda + 9}$$

If we set $h\lambda = 0$, we get

$$r = -2 \pm \sqrt{9} = 1, -5$$

$r_0 = 1$ is the principle solution and $r_1 = -5$ is the parasitic solution

Since $|r_1| > |r_0|$, the parasitic solution grows faster, and the method is not relatively stable.

Problem 2

We showed in lecture that for the midpoint method,

$$r_0 = h\lambda + \sqrt{1 + h^2\lambda^2}$$

$$= \frac{\lambda t}{n} + \sqrt{1 + \frac{t^2\lambda^2}{n^2}}$$

Now we want to find r_0^n. If we use the hint, we can do

$$e^{n \ln r_0} = e^{n \ln\left(\frac{\lambda t}{n} + \sqrt{1 + \frac{t^2\lambda^2}{n^2}}\right)}$$

$$= e^{n\left(\frac{\lambda t}{n} - \frac{1}{6}\left(\frac{\lambda t}{n}\right)^3 + O\left(\left(\frac{\lambda t}{n}\right)^5\right)\right)}$$

$$= e^{\left(\lambda t - \frac{1}{6}\frac{(\lambda t)^3}{n^2} + O(n^{-5})\right)}$$

$$= \exp\left[\lambda t + O\left(\frac{\lambda^3 t^3}{n^2}\right)\right]$$

Problem 3

I modified trapezoid.m to create trapezoid2.m. The main function is called main_3.m. The two methods produce a solution that are within the desired tolerance, but the Newton iteration uses 2 iterations per step on average where as fixed point iteration uses an average of 14.8 iterations.

Problem 4

I created a program main_4.m to solve the problem. The graphs produced by the methods are saved as png files with the name Solution_N.png, where N is the number of iterations. Errors are saved in files of name Error_N.m.

All three methods appear to converge to the solution as $N \to \infty$. The trapezoid appears to do the best. Unlike the midpoint method it is relatively so there are no oscillations around the solution.

Problem 5

a) We use the model problem

$$\dot{y} = \lambda y$$

Substituting this in, we have

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left[\lambda y_{n-1} + 4\lambda y_n + \lambda y_{n+1}\right]$$

This is a difference equation with solution $y_n = r^n$. Again, substituting it in, we get:

$$r^{n+1} = r^{n-1} + \frac{h}{3}\left[\lambda r^{n-1} + 4\lambda r^n + \lambda r^{n+1}\right]$$

$$r^2 = 1 + \frac{h}{3}\left[\lambda + 4\lambda r + \lambda r^2\right]$$

$$\left(1 - \frac{h\lambda}{3}\right) r^2 - \frac{4h\lambda}{3} r - \left(1 + \frac{h\lambda}{3}\right) = 0$$

$$(3 - h\lambda) r^2 - 4h\lambda r - (3 + h\lambda) = 0$$

$$r = \frac{2h\lambda \pm \sqrt{(2h\lambda)^2 + (3-h\lambda)(3+h\lambda)}}{3 - h\lambda}$$

$$r_0 = \frac{2h\lambda \pm \sqrt{9 - 3h^2\lambda^2}}{3 - h\lambda}$$

So we have

$$r_0 = \frac{2h\lambda + \sqrt{9 - 3h^2\lambda^2}}{3 - h\lambda}$$

$$r_1 = \frac{2h\lambda - \sqrt{9 - 3h^2\lambda^2}}{3 - h\lambda}$$

b) If we plot the roots by substituting $x = h\lambda$, we can see that

$$|r_1| > |r_0|$$

when $x \in (-1.732, 0)$. So if λ is negative, the parasitic solution dominates.