Homework No.3, 553.481/681, Due March 12, 2021.
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Problem 1. Define the *Vandermonde matrix* by its elements:

$$V_{ij}[x_0, ..., x_n] = x_i^j, \quad 0 \le i, j \le n,$$

for any set of $(n+1)$ real numbers $x_0, ..., x_n$.

(a) Show that

$$\det \mathbf{V}[x_0, ..., x_n] = \prod_{i=0}^{n-1} (x_n - x_i) \cdot \det \mathbf{V}[x_0, ..., x_{n-1}].$$

Hint: Show that the determinant on the left is a polynomial of degree n in x_n and find its roots and the coefficient of its highest-order term.

Solution: We know that if we expand the determinant using the Leibniz rule on the last row, we get a polynomial in x_n whose coefficients are the matrix of minors. Note that the matrix of minors for the highest order term is

$$\det \mathbf{V}[x_0, ..., x_{n-1}]$$

We can factor this out of the polynomial. Now notice that after doing this, the last term has a coefficient of

$$\frac{A}{\det \mathbf{V}[x_0, ..., x_{n-1}]}$$

Where A is the matrix of minors for the last row and first column of the form:

$$\begin{pmatrix} x_0 & x_0^2 & \dots & x_0^{n-1} \\ x_1 & x_1^2 & \dots & x_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & x_{n-1}^2 & \dots & x_{n-1}^{n-1} \end{pmatrix}$$

But by the properties of row operations on the determinants, we can see that

$$A = \prod_{i=0}^{n-1} x_i \cdot \mathbf{V}[x_0, ..., x_{n-1}]$$

So the last term of the polynomial has coefficient $\prod_{i=0}^{n-1} x_i$. We know that the coefficient of the last term of a polynomial is the product of all the roots, so the roots of the polynomial must be x_i from $0 \le i \le n-1$. Therefore, we can write:

$$\det \mathbf{V}[x_0, ..., x_n] = \prod_{i=0}^{n-1} (x_n - x_i) \cdot \det \mathbf{V}[x_0, ..., x_{n-1}].$$

(b) Use part (a) and induction to show that

$$\det \mathbf{V}[x_0, ..., x_n] = \prod_{0 \le i < j \le n} (x_j - x_i)$$

Solution: This applies if the Vandermonde's matrix is square. If it is square, then in the case of one element, $\det \mathbf{V}[x_0] = \det [1] = 1$. If there is more than one element, we use the result from part a.

$$
\begin{aligned}
\det \mathbf{V}[x_0, ..., x_n] &= \prod_{i=0}^{n-1}(x_n - x_i) \cdot \det \mathbf{V}[x_0, ..., x_{n-1}] \\
&= \prod_{i=0}^{n-2}(x_{n-1} - x_i) \cdot \prod_{i=0}^{n-1}(x_n - x_i) \cdot \det \mathbf{V}[x_0, ..., x_{n-1}] \\
&= \dots \\
&= \prod_{0 \le i < j \le n} (x_j - x_i)
\end{aligned}
$$

Problem 2. Consider the following seven points:

$$(x_1, y_1) = (1, 16),\ (x_2, y_2) = (2, 34),\ (x_3, y_3) = (3, 58),\ (x_4, y_4) = (4, -20),$$

$$(x_5, y_5) = (5, 4),\ (x_6, y_6) = (6, 886),\ (x_7, y_7) = (7, -146)$$

For this data, find the 6th-degree interpolating polynomial (a) in the monomial basis, (b) in the barycentric Lagrange form, and (c) in the Newtonian form with divided-differences. Compare the wall clock times to compute the coefficients in the monomial basis, the barycentric weights, and the divided-differences. Which form of the interpolating polynomial is computed the fastest in this example? Plot the seven points along with the interpolating polynomial over the interval $[0, 8]$.

Polynomials:

We can write the monomial as:

$$P_M(x) = -6x^6 + 128x^5 - 1067x^4 + 4432x^3 - 9628x^2 + 10293x - 4136$$

$$P_L(x) = \Psi_6(x) \left[\frac{0.0222}{x - x_1} + \frac{-0.2833}{x - x_2} + \frac{1.2083}{x - x_3} + \frac{0.5556}{x - x_4} + \frac{0.0833}{x - x_5} + \frac{-7.3833}{x - x_6} + \frac{-0.2028}{x - x_7} \right]$$

$$P_N(x) = 16 + 18\Psi_1(x) + 3\Psi_2(x) - 18\Psi_3(x) + 13\Psi_4(x) + 2\Psi_5(x) - 6\Psi_6(x)$$

Timing: I measured 3.3060e−5 seconds for monomial coefficients, 1.3950e−5 seconds for barycentric Lagrange coefficients, and 9.7800e−6 seconds for Newton coefficients. Therefore, Newton's interpolation was the fastest in this case.

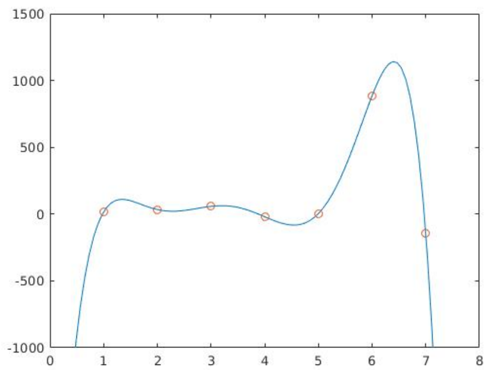


Figure 1: Interpolation

Problem 3. Use the expression

$$f[x_0, ..., x_n] = \sum_{i=0}^{n} w_i f(x_i), \quad w_i = 1/\Psi_n'(x_i)$$

for the divided-difference, with $\Psi_n(x) = \prod_{j=0}^{n}(x - x_j)$, to verify that

$$f[x_0, ..., x_n] = \frac{f[x_1, ..., x_n] - f[x_0, ..., x_{n-1}]}{x_n - x_0}.$$

Solution: When we expand out the expression, we get:

$$\frac{\sum_{i=1}^{n} w_i f(x_i) - \sum_{j=0}^{n-1} w_j f(x_j)}{x_n - x_0}$$

Where

$$w_i = \frac{1}{\prod_{k=1\neq i}^{n}(x_i - x_k)}$$

$$u_j = \frac{1}{\prod_{k=0\neq j}^{n-1}(x_j - x_k)}$$

When we combine the expressions in the numerator, there are three distinct cases. The first case is the term corresponding to $f(x_n)$. Since there no corresponding term in the second expression, there is nothing to combine it with. We can just divide by $x_n - x_0$, and get:

$$\frac{1}{x_n - x_0} \frac{f(x_n)}{\prod_{k=1\neq n}^{n}(x_n - x_k)} = \frac{f(x_n)}{\prod_{k=0}^{n-1}(x_n - x_k)}$$

The second case is the term corresponding to $f(x_0)$ in the second term in the numerator. This is similar to the last case since there is no corresponding term in the first expression. We can just divide and obtain:

$$-\frac{1}{x_n - x_0} \frac{f(x_0)}{\prod_{k=0\neq 0}^{n-1}(x_0 - x_k)} = \frac{f(x_0)}{\prod_{k=1}^{n}(x_0 - x_k)}$$

The third case is when we are combining terms corresponding to $f(x_i)$ where $0 < i < n$. In this case, there is one term in the first expression and one term in the second expression which we must combine. We get:

$$\begin{aligned}
&\frac{1}{x_n - x_0}\left(\frac{f(x_i)}{\prod_{k=1\neq i}^{n}(x_i - x_k)} - \frac{f(x_i)}{\prod_{k=0\neq i}^{n-1}(x_i - x_k)}\right) \\
&= \frac{1}{x_n - x_0}\left(\frac{(x_i - x_0)f(x_i) - (x_i - x_n)f(x_i)}{\prod_{k=0\neq i}^{n}(x_i - x_k)}\right) \\
&= \frac{f(x_i)}{\prod_{k=0\neq i}^{n}(x_i - x_k)}
\end{aligned}$$

We see in general, for any $0 < i < n$, we have the term:

$$\frac{f(x_i)}{\prod_{k=0\neq i}^{n}(x_i - x_k)}$$

Over which we sum. But this is the definition of $f[x_0, ..., x_n]$. So we have verified the result.

(b) Show that the polynomial $p_n(x)$ interpolating $f(x)$ can be written as

$$p_n(x) = \frac{\sum_{j=0}^{n} \frac{w_j f(x_j)}{x - x_j}}{\sum_{j=0}^{n} \frac{w_j}{x - x_j}}$$

provided x is not a node point.

Solution: We can start with the standard form of Lagrange interpolation. We showed in class, that the Lagrange polynomial has the form:

$$p(x) = \sum_{j=0}^{n} f(x_j) \frac{\prod_{k=0, k\neq j}^{n} (x - x_k)}{\prod_{k=0, k\neq j}^{n} (x_j - x_k)}$$

We can manipulate this to obtain the desired form

$$\begin{aligned} p(x) &= \sum_{j=0}^{n} f(x_j) \frac{\prod_{k=0, k\neq j}^{n} (x - x_k)}{\prod_{k=0, k\neq j}^{n} (x_j - x_k)} \\ &= \sum_{j=0}^{n} f(x_j) \Psi_n(x) \frac{w_j}{x - x_j} \\ &= \frac{\sum_{j=0}^{n} \frac{w_j f(x_j)}{x - x_j}}{\sum_{j=0}^{n} \frac{w_j}{x - x_j}} \end{aligned}$$

Problem 4. This problem explores the utility of *inverse polynomial interpolation* in which data (x_i, y_i), $i = 1, , , , n$ are interpolated by a polynomial $x = Q(y)$ rather than by a polynomial $y = P(x)$. Both interpolation schemes can be applied to the same data if all x_i-values are distinct for $i = 1, ..., n$ and if also all y_i-values are distinct for $i = 1, ..., n$. If both of these conditions are satisfied, then the same algorithms can be applied to evaluate both polynomial interpolants.

(a) Here we consider the following seven points:

$$(x_1, y_1) = (1, 17),\ (x_2, y_2) = (1.2, 18.0736),\ (x_3, y_3) = (1.4, 19.8416),$$
$$(x_4, y_4) = (1.6, 22.5536),\ (x_5, y_5) = (1.8, 26.4976),$$
$$(x_6, y_6) = (2, 32),\ (x_7, y_7) = (3, 33.73205)$$

Could this data have been consistently generated by $(x_i, f(x_i))$ and also by $(g(y_i), y_i)$ for a function f and its inverse $g = f^{-1}$?

Solution: Yes, the points $(x_i, f(x_i)$ are strictly monotonically increasing, so it can be modeled as a bijective function over the sets $[1, 3]$ and $[17, 33.73205]$. Thus an inverse function exists.

(b) Give both 6th-degree polynomials for the data in (a), the direct interpolant $P_6(x)$ and the inverse interpolant $Q_6(y)$, both in the Newton form.

Solution: I used the newtondif.m code to generate the divided differences. I used up to 4 decimal places of accuracy so that the expressions fit on the page, but the code has double precision. For convenience, let's denote the bases:

$$\Phi_j(x) = \prod_{j=0}^{i-1} (x - x_j)$$

and

$$\Phi_j(y) = \prod_{j=0}^{i-1} (y - y_j)$$

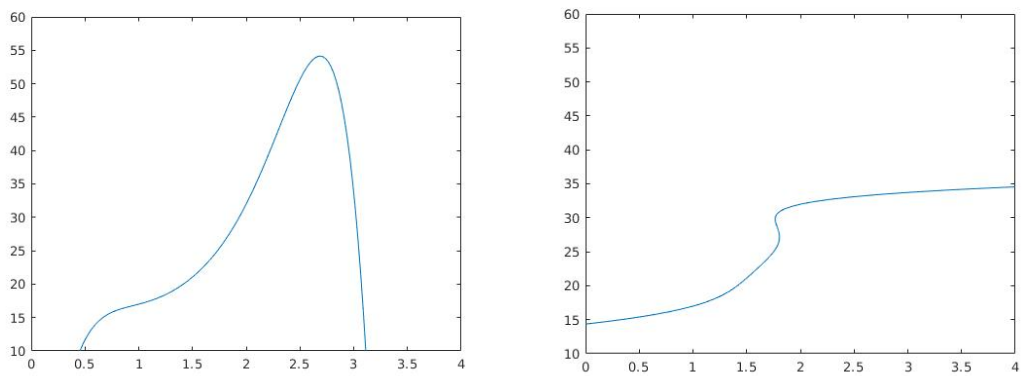
Now we can write the polynomial and inverse polynomial

$P_6(x) = 17.0000 + 5.3680\Phi_1(x) + 8.6800\Phi_2(x) + 5.2000\Phi_3(x) + 1.0000\Phi_4(x) - 0.0000\Phi_5(x) - 6.5381\Phi_6(x)$

$Q_6(y) = 1.0000 + 0.1863\Phi_1(y) - 0.0257\Phi_2(y) + 0.0031\Phi_3(y) - 0.0003\Phi_4(y) + 0.0000\Phi_5(y) + 0.0000\Phi_6(y)$

(c) Plot the data and both polynomials over the range $0 < x < 4$ and $10 < y < 60$. Is it true that $Q_6(y) = P_6^{-1}(y)$?

Solution: They are not the inverse of one another.



(a) $P_6(x)$ (b) $Q_6(y)$

Figure 2: Plots of Polynomials

Problem 5. Another example of the Runge phenomenon is provided by the function

$$f(x) = \frac{1}{(1 + 400|x|^3)^{1/3}}.$$

(a) Define the n-point Chebyshev grid in the interval $[-1, 1]$ by

$$x_k = \cos\left(\frac{(2k-1)\pi}{2n}\right), \quad k = 1, ..., n.$$

Interpolate the above function with a polynomial on the Chebyshev grid for $n = 10, 20, 30, 40, 50$ and plot the results. What would you conjecture about the limit $n \to \infty$ of the interpolating polynomial on the basis of these plots?

NOTE: TO INCREASE READABILITY, I KEPT THE PLOTS IN A FOLDER TITLED `"prob_5_images"`. EACH PLOT IS LABELED `"prob_5<a>_#".jpg`, WHERE `<a>` IS THE PROBLEM PART (a or b) AND # IS THE VALUE OF n.

Solution: The interpolation converges to the function over the set $(-1, 1)$. The Chebyshev grid interpolation does not display a Runge phenomenon, and I predict it will not display the Runge phenomenon for $n \to \infty$.

;(b) Compare the results in (a) with those obtained by polynomial interpolation on the uniform grid

$$x_k = \frac{2k - (n+1)}{2(n+1)}, \quad k = 1, ..., n$$

for $n = 10, 20, 30, 40, 50$.

Solution: The interpolation converges to the function over the set $(-1, 1)$, but displays a greater Runge effect as n increases.