Uniform Estimators

1. MSE(ˆ θ) = bias(ˆ θ)2 + var(ˆ θ)

$$
\begin{aligned}
MSE(\hat\theta) &= E(\hat\theta^2 - 2\hat\theta\theta + \theta^2) \\
&= E(\hat\theta^2) - E(\hat\theta)^2 + E(\hat\theta)^2 - 2\theta E(\hat\theta) + \theta^2 \\
&= \theta^2 - 2\theta E(\hat\theta) + E(\hat\theta)^2 + E(\hat\theta^2) - 2E(\hat\theta)^2 + E(\hat\theta)^2 \\
&= bias(\hat\theta)^2 + E(\hat\theta^2) + E(2\hat\theta(E(\hat\theta))) + E(E(\hat\theta)^2) \\
&= bias(\hat\theta)^2 + E((\hat\theta - E(\hat\theta))^2) \\
&= bias(\hat\theta)^2 + var(\hat\theta)
\end{aligned}
$$

2. ˆ LMOM is unbiased, but that ˆ LMLE has bias

A. $\hat{L}_{MOM}$

$$
\begin{aligned}
bias(\hat L_{MOM}) &= L - E(2\bar X_n) \\
&= L - \frac{2}{n}\sum_{i=1}^{n} E(X_i) \\
&= L - \frac{2}{n} n \frac{L}{2} = L - L = 0
\end{aligned}
$$

B. $\hat{L}_{MLE}$

Notice that the pdf of $\hat L_{MLE} = \max\limits_i X_i$ is:

> $f(x) = n\frac{x^{n-1}}{L^n}$. [1]

Therefore, $E(\hat L_{MLE}) = \int\limits_0^L xn\frac{x^{n-1}}{L^n}dx = \frac{n}{n+1}L$ (, which is less than L.)

$bias(\hat L_{MLE}) = \frac{1}{n+1}L$

3. the variance

A. $\hat{L}_{MOM}$

$var(\hat L_{MOM}) = \frac{4}{n^2}\sum\limits_{i=1}^{n} var(X_i) = \frac{4}{n}\frac{L^2}{12} = \frac{L^2}{3n}$

B. $\hat{L}_{MLE}$

$var(\hat L_{MLE}) = E(\hat L_{MLE}^2) - E(\hat L_{MLE})^2 = \int\limits_0^L x^2 n\frac{x^{n-1}}{L^n}dx - (\frac{n}{n+1}L)^2 = (\frac{n}{n+2} - \frac{n^2}{(n+1)^2})L^2 = \frac{n}{(n+2)(n+1)^2}L^2$

4. the better estimator

$MSE(\hat{L}_{MOM}) = 0^2 + \frac{L^2}{3n} = \frac{L^2}{3n}$
$MSE(\hat{L}_{MLE}) = (\frac{1}{n+1}L)^2 + \frac{n}{(n+2)(n+1)^2}L^2 = \frac{2}{(n+2)(n+1)}L^2$

Let $MSE(\hat{L}_{MLE}) \leq MSE(\hat{L}_{MOM})$:
$\frac{2}{(n+2)(n+1)}L^2 \leq \frac{1}{3n}L^2$
$0 \leq n^2 - 3n + 2$
$n \leq 1,\ or\, n \geq 2$

Notice that $n \in Z^+$:
$\forall n \in Z^+, MSE(\hat{L}_{MLE}) \leq MSE(\hat{L}_{MOM})$
Therefore, $\hat{L}_{MLE}$ is the better estimator.

5. Experimentally verify

```
D:\Users\endlesstory\Desktop\536\hw1>python UE.py
Estimated:
MSE_{MOM} = 0.3307
MSE_{MLE} = 0.0204
Theoretical:
MSE_{MOM} = 0.3333
MSE_{MLE} = 0.0194
```

6. an explanation

Here is the idea:

Assuming $k = \max_i X_i, \hat{L} = 10^6 k$ is obviously better than $\hat{L} = 10^9 k$. If L was that large, it would be much likely to get a larger k. Therefore, $\hat{L} = 10k$ is also better than $\hat{L} = 10^6 k$ for the same reason, and $\hat{L} = k$ is even better.

Notice that here are 2 cases: if $\hat{L}_{MLE} \leq \hat{L}_{MOM}$, by previous argument, $\hat{L}_{MLE}$ is the better one. If $\hat{L}_{MLE} > \hat{L}_{MOM}$, we have seen a k, and $\hat{L}_{MOM}$ estimates a number less than k. Obviously, in this case, $\hat{L}_{MOM}$ is out.

7. P(^ LMLE < L−ε)

$P(\hat{L}_{MLE} < L - \epsilon) = \int_{L-\epsilon}^{L} n\frac{x^{n-1}}{L^n}dx = 1 - (\frac{L-\epsilon}{L})^n$

Let $P(\hat{L}_{MLE} < L - \epsilon) \leq \delta$:
$1 - (\frac{L-\epsilon}{L})^n \leq \delta$
$n \geq log_{(1-\frac{\epsilon}{L})}(1 - \delta) = \frac{log(1-\delta)}{log(1-\frac{\epsilon}{L})}$

8.L =(n n−1) max i=1,…,nXi

$E(\hat{L}) = \frac{n+1}{n} \int_0^L xn\frac{x^{n-1}}{L^n}dx = L$

$bias(\hat{L}) = 0$

$var(\hat{L}) = (\frac{n+1}{n})^2 var(\hat{L}_{MLE}) = \frac{(n+1)^2}{n^2} \frac{n}{(n+2)(n+1)^2} L^2 = \frac{1}{n(n+2)} L^2$

$MSE(\hat{L}) = 0^2 + \frac{L^2}{n(n+2)} = \frac{L^2}{n(n+2)}$

Compared to $MSE(\hat{L}_{MOM}) = \frac{L^2}{3n}$:

Let $MSE(\hat{L}) \leq MSE(\hat{L}_{MOM})$:

$n^2 - n \leq 0$

$n \leq 0, \ or\, n \geq 1$

Therefore, $\forall n \in Z^+, MSE(\hat{L}) \leq MSE(\hat{L}_{MOM})$

---

1. HW0-Answers https://content.sakai.rutgers.edu/access/content/group/13eb977e-0fb8-4ee0-9b17-4edccb798315/HW0-Answers.pdf ↩