1 Introduction

- Visually-aided navigation (bearing), and range-aided navigation (radar) can be framed as a filtering problem. The model is non-linear, has unknown parameters, and unknown inputs (*e.g.,* accelerometer and gyrometer bias derivative), typically treated as driving noise in a random walk model.
- Observability is a necessary condition for *any* filter/observer to operate, hence a literature on observability analysis of visually-aided navigation [**?**, **?**, **?**]. Relatively little on range-aided.
- Unknown parameters are typically included in the state, thus transforming an identification problem into a filtering one, and their identifiability analysis lumped in the observability analysis of the resulting (augmented) model.
- Noise does not affect the observability of a model, so for the purpose of observability analysis, they are set to zero. This is because, by assumption, noise is "uninformative:" It is typically modeled as a realization of a white zero-mean, homoscedastic process, independent of the state of the model.
- However, the driving input to the random walk model of accelerometer and gyro bias is typically small but *not* independent of the state. In fact, far from being uninformative, it is strongly correlated with it, as it is its temporal derivative. Thus, it should be treated as an *unknown input*, rather than a "noise." As such, it should be included in the observability/identifiability analysis.
- Our first contribution is to show that while (a prototypical model of) assisted navigation and auto-calibration is *observable* in the absence of unknown input, it is *not* observable when unknown inputs are taken into account. This exposes a methodological flaw with the observability analysis of assisted navigation in the existing literature.
- Our second contribution is to reframe observability as a *sensitivity* analysis, and to show that while the set of indistinguishable trajectories is *not* a singleton (as it would be if the model was observable), but it is nevertheless bounded to a set. We explicitly characterize this set and show that, interestingly, it may not contain the "true" state trajectory. Finally, we provide bounds on the volume of this subset as a function of the characteristics of the unknown inputs.
- We do so for bearing-only augmentation, range-only augmentation, and combined augmentation.
- Rather than study observability of linearized system, or algebraically checking the rank conditions, that offers no insight on the structure of the indistinguishable states, we characterize observability directly in terms of indistinguishable sets.

1.1 Notation

A reference frame is represented by an orthogonal 3×3 positive-determinant (rotation) matrix $R \in SO(3) \doteq \{R \in R^{3\times3} \mid R^T R = RR^T = I,\ det(R) = +1\}$ and a translation vector $T \in R^3$. They are collectively indicated by $g = (R, T) \in SE(3)$. When g represents the change of coordinates from a reference frame "a" to another ("b"), it is indicated by g_{ba}. Then the columns of R_{ba} are the coordinate axes of a relative to the reference frame b, and T_{ba} is the origin of a in the reference frame b. If p_a is a point relative to the reference frame a, then its representation relative to b is $p_b = g_{ba} p_a$. In coordinates, if X_a are the coordinates of p_a, then $X_b = R_{ba} X_a + T_{ba}$ are the coordinates of p_b.

A time-varying pose is indicated with $g(t) = (R(t), T(t))$ or $g_t = (R_t, T_t)$, and the entire trajectory from an initial time t_i and a final time t_f $\{g(t)\}_{t=t_i}^{t_f}$ is indicated in short-hand notation with $g_{t_i}^{t_f}$; when the initial time is $t_0 = 0$, we omit the subscript and call g^t the trajectory "up to time t". The time-index is sometimes omitted for simplicity of notation when it is clear from the context.

We indicate with $\widehat{V} = (\widehat{\omega}, v) \in se(3)$ the (generalized) velocity or "twist", where $\widehat{\omega}$ is a skew-symmetric matrix $\widehat{\omega} \in so(3) \doteq \{S \in R^{3\times3} \mid S^T = -S\}$ corresponding to the cross product with the vector $\omega \in R^3$, so that $\widehat{\omega} v = \omega \times v$ for any vector $v \in R^3$. We indicate the generalized velocity with $V = (\omega, v)$. We indicate the group composition $g_1 \circ g_2$ simply as $g_1 g_2$. In homogeneous coordinates, $\bar{X}_b = G_{ba} \bar{X}_a$ where $\bar{X}^T = [X^T\ 1]$ and

$$G \doteq \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \in R^{4\times4} \qquad \hat{V} \doteq \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix}. \tag{1}$$

Composition of rigid motions is then represented by matrix product.

1.2 Mechanization Equations

The motion of a sensor platform is represented as the time-varying pose g_{sb} of the body relative to the spatial frame. To relate this to measurements of an inertial measurement unit (IMU) we compute the temporal derivatives of g_{sb}, which yield the (generalized) body velocity V_{sb}^b, defined by $\dot{g}_{sb}(t) = g_{sb}(t)\widehat{V}_{sb}^b(t)$, which can be broken down into the rotational and translational components $\dot{R}_{sb}(t) = R_{sb}(t)\widehat{\omega}_{sb}^b(t)$ and $\dot{T}_{sb}(t) = R_{sb}(t) v_{sb}^b(t)$. An ideal gyrometer (gyro) would measure $\omega_{\text{imu}} = \omega_{sb}^b$. The translational component of body velocity, v_{sb}^b, can be obtained from the last column of the matrix $\frac{d}{dt}\widehat{V}_{sb}^b(t)$. That is, $\dot{v}_{sb}^b = \dot{R}_{sb}^T \dot{T}_{sb} + R_{sb}^T \ddot{T}_{sb} = -\widehat{\omega}_{sb}^b v_{sb}^b + R_{sb}^T \ddot{T}_{sb} \doteq -\widehat{\omega}_{sb}^b v_{sb}^b + \alpha_{sb}^b$, which serves to define $\alpha_{sb}^b \doteq R_{sb}^T \ddot{T}_{sb}$. These equations can be simplified by defining a new linear velocity, v_{sb}, which is neither the body velocity v_{sb}^b nor the spatial velocity v_{sb}^s, but instead $v_{sb} \doteq R_{sb} v_{sb}^b$. Consequently, we have that $\dot{T}_{sb}(t) = v_{sb}(t)$ and $\dot{v}_{sb}(t) = \dot{R}_{sb} v_{sb}^b + R_{sb} \dot{v}_{sb}^b = \ddot{T}_{sb} \doteq \alpha_{sb}(t)$ where the last equation serves to define the new linear acceleration α_{sb}; as one can easily verify we have that $\alpha_{sb} = R_{sb} \alpha_{sb}^b$. An ideal accelerometer (accel) would then measure $\alpha_{\text{imu}} = R_{sb}^T(t)(\alpha_{sb}(t) - \gamma)$.

There are several reference frames to be considered in an aided navigation scenario. The *spatial frame* s, typically attached to Earth and oriented so that gravity γ takes the form $\gamma^T = [0\ 0\ 1]^T\|\gamma\|$ where $\|\gamma\|$ can be read from tabulates based on location and is typically around $9.8m/s^2$. The *body frame* b is attached to the IMU.[1] The *camera frame* c, relative to which image measurements are captured, is also unknown, although we will assume that *intrinsic calibration* has ben performed, so that measurements on the image plane are provided in metric units. Finally, the *radar frame*, or range frame r, is that of the antenna relative to which range measurements are provided.

The equations of motion (known as mechanization equations) are usually described in terms of the body frame at time t relative to the spatial frame $g_{sb}(t)$. Since the spatial frame is arbitrary (other than for being aligned to gravity), it is often chosen to be co-located with the body frame at time $t = 0$. To simplify the notation, we indicate this time-varying frame $g_{sb}(t)$ simply as g, and so for $R_{sb}, T_{sb}, \omega_{sb}, v_{sb}$, thus effectively omitting the subscript sb everywhere it appears. This yields

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$$\dot{T} = V\dot{R} = R\widehat{\omega}\dot{V} = \alpha\dot{\omega} = w\dot{\alpha}$$

(2)

where $w \in R^3$ is the rotational acceleration, and $\xi \in R^3$ the translational jerk (derivative of acceleration). Although α corresponds to neither body nor spatial acceleration, it can be easily related to accel measurements:

$$\alpha_{\text{imu}}(t) = R^T(t)(\alpha(t) - \gamma) + \underbrace{\alpha_b(t) + n_\alpha(t)} \tag{3}$$

where the measurement error in bracket includes a slowly-varying mean ("bias") $\alpha_b(t)$ and a residual term n_α that is commonly modeled as a zero-mean (its mean is captured by the bias), white, homoscedastic and Gaussian noise process. In other words, it is assumed that n_α is independent of α, hence uninformative. Here γ is the gravity vector expressed in the spatial frame.[2] Measurements from a gyro can be similarly modeled as

$$\omega_{\text{imu}}(t) = \omega(t) + \underbrace{\omega_b(t) + n_\omega(t)} \tag{4}$$

[1] In practice, the IMU has several different frames due to the fact that the gyro and accel are not co-located and aligned, and even each sensor (gyro or accel) is composed of multiple sensors, each of which can have its own reference frame. Here we will assume that the IMU has ben pre-calibrated so that accel and gyro yield measurements relative to a common reference frame, the *body frame*. In reality, it may be necessary to calibrate the alignment between the multiple-axes sensors (non-orthogonality), as well as the gains (scale factors) of each axis.

[2] The orientation of the body frame relative to gravity, R_0, is unknown, but can be approximated by keeping the IMU still (so $R^T(t) = R_0$) and averaging the accel measurements, so that $\frac{1}{T}\sum_{t=0}^{T} \alpha_{\text{imu}}(t) \simeq -R_0^T\gamma + \alpha_b$. Assuming the bias to be small (zero), this equation defines R_0 up to a rotation around gravity, which is arbitrary. Note that if $\alpha_b \neq 0$, the initial bias will affect the initial orientation estimate.

where the measurement error in bracket includes a slowly-varying bias $\omega_b(t)$ and a residual "noise" n_ω also assumed zero-mean, white, homoscedastic and Gaussian, independent of ω.

Other than the fact that the biases α_b, ω_b change *slowly*, they can change arbitrarily. One can therefore consider them an *unknown input* to the model, or a *state* in the model, in which case one has to hypothesize a dynamical model for them. For instance instance

$$\dot{\omega}_b(t) = v_b(t), \quad \dot{\alpha}_b(t) = v_\alpha(t) \tag{5}$$

for some unknown input v_b, v_α. While it is safe to assume that v_b, v_α are *small*, they certainly are not (white, zero-mean and, most importantly) uninformative. Nevertheless, it is common to consider v_b, v_α, to be realizations of a Brownian motion that is *independent* of ω_b, α_b. This is done for convenience as one can then consider all unknown inputs as "noise." Unfortunately, however, this has repercussion on the analysis of the observability and identifiability of the resulting model (Sect.).

1.3 Standard and reduced models

The mechanization equations above define a dynamical model having as output the IMU measurements. Including the initial conditions and biases, we have

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$$\dot{T} = VT(0) = 0\dot{R} = R\widehat{\omega}R(0)$$

$$\begin{aligned} \omega_{\mathrm{imu}}(t) &= \omega(t) + \omega_b(t) + n_\omega(t) \\ \alpha_{\mathrm{imu}}(t) &= \mathrm{R}^T(t)(\alpha(t) - \gamma) + \alpha_b(t) + n_\alpha(t) \end{aligned} \tag{6}$$

In this standard model, data from the IMU are considered as (output) *measurements*. However, it is customary to treat them as (known) *input* to the system, by writing ω in terms of ω_{imu} and α in terms of α_{imu}:

$$\omega = \omega_{\mathrm{imu}} - \omega_b + \underbrace{n_R}_{-n_\omega} \qquad \alpha = R(\alpha_{\mathrm{imu}} - \alpha_b) + \gamma + \underbrace{n_V}_{-Rn_\alpha} \tag{7}$$

This equality is valid for *samples* (realizations) of the stochastic processes involved, but it can be misleading as, if considered as stochastic processes, the noises above are *not* independent of the states. Such a dependency, is nevertheless typically neglected. The resulting mechanization model is

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$$\dot{T} = VT(0) = 0\dot{R} = R(\widehat{\omega}_{\mathrm{imu}} -$$

(8)

Next we will consider augmenting the models above with measurement equations coming either from *range* or *bearing* measurements for a finite set N of isolated points with coordinates $X^i \in R^3,\ i = 1, \dots, N$.

1.4 Bearing augmentation (vision)

Initially we assume there is a collection of points $X^i,\ i = 1, \dots, N$, visible from time $t = 0$ to the current time t. If $\pi : R^3 \rightarrow R^2; X \mapsto [X_1/X_3,\ X_2/X_3]$ is a canonical central (perspective) projection, assuming that the camera is *calibrated*,[3] *aligned*,[4] and that the spatial frame coincides with the body frame at time 0, we have

$$y^i(t) = black\frac{R_{1:2}^T(t)(X^i - T_{1:2}(t))}{R_3^T(t)(X^i - T_3(t))} \doteq \pi(g^{-1}(t)X^i) + n^i(t), \quad blackt \geq 0. \quad (9)$$

If the feature first appears at time $t = 0$ *and if the camera reference frame is chosen to be the origin the world reference frame* so that $T(0) = 0; R(0) = I$, then we have that $y^i(0) = \pi(X^i) + n^i(0)$, and therefore

$$X^i = \bar{y}^i(0)Z^i + \tilde{n}^i \quad (10)$$

where $\bar{y}$ is the homogeneous coordinate of y, $\bar{y} = [y^T\ 1]^T$, and $\tilde{n}^i = [{n^i}^T(0)Z^i\ \ 0]^T$. Here Z^i is the (unknown, scalar) depth of the point at time $t = 0$. With an abuse of notation, we write the map that collectively projects all points to their corresponding locations on the image plane as:

$$y(t) \doteq \begin{bmatrix} y^1 \\ y^2 \\ \vdots \\ y^N \end{bmatrix}(t) = \begin{bmatrix} \pi(R^T(X^1 - T)) \\ \pi(R^T(X^2 - T)) \\ \vdots \\ \pi(R^T(X^N - T)) \end{bmatrix} + \begin{bmatrix} n^1(t) \\ n^2(t) \\ \vdots \\ n^N(t) \end{bmatrix} \quad (11)$$

1.5 Range augmentation (radar)

If measurements of range of sparse reflectors in position $X^i \in R^3$ are given, we consider the projection function (with an abuse of notation) $\pi : R^3 \rightarrow S^2;\ X \mapsto \|X\|$, and assume that, for each point i – through pre-processing of the radar phase histories, we can measure

$$Z^i(t) = \|X^i - T(t)\| + n^i(t)\ \ \doteq \pi(g^{-1}(t)X^i) + n^i(t) \quad t \geq 0. \quad (12)$$

This equation is formally similar to (9), so the general form (23) does not change if we consider range only measurements, and even a mixture of *both* range and bearing.

[3] Intrinsic calibration parameters are known and compensated for.
[4] The pose of the camera relative to the IMU is known and compensated for.

In the case of bearing measurements, the inverse projection depends on two parameters (the unknown bearing). If we call

$$\hat{y} \doteq \frac{X}{\|X\|} \in S^2 \tag{13}$$

then we have that, noting that $Z^i(0) = \|X^i\| + n^i(0)$,

$$X^i = \hat{y}^i Z^i + \hat{n}^i \tag{14}$$

where $\hat{n}^i \doteq -\hat{y}^i n^i(0)$. Although the measurement equations for the bearing-only and range-only filters are formally identical, the observability properties and Gauge ambiguities are different.

1.6 Alignment (calibration)

Consider the model () with measurements $y^i(t)$ can representing either the range of a number of sparse reflectors or the position on the image plane of a sparse collection of point features. In the former case, the range is measured in the reference frame of the radar, and therefore we have

$$y^i(t) = \pi\left(g_{rb} g^{-1}(t) X_s^i\right) + n^i(t) \in R \tag{15}$$

where $\pi(X) = \|X\|$ and g_{rb} is the transformation from the body frame to the radar. In the latter we have

$$y^i(t) = \pi\left(g_{cb} g^{-1}(t) X_s^i\right) + n^i(t) \in R^2 \tag{16}$$

where $\pi(X) = [X_1/X_3,\ X_2/X_3]^T$, and g_{cb} is the transformation from the body frame to the camera. The *"alignment"* transformations g_{cb}, g_{rb} are typically not known and should be inferred. We can then, as done for the points X^i, add them to the state with trivial dynamics $\dot{g}_{cb} = \dot{g}_{rb} = 0$.

1.7 Groups (occlusions)

It may convenient in some cases to represent the points X_s^i in the reference frame where they first appear, say at time t_i, rather than in the spatial frame. This is because the uncertainty is highly structured in the frame where they first appear. Consider $X^i(t_i) = \bar{y}^i(t_i) Z^i(t_i)$, then $y^i(t_i)$ has the same uncertainty of the feature detector (small and isotropic on the image plane) and Z^i has a large uncertainty, but it is constrained to be positive.

However, to relate $X^i(t_i)$ to the state, we must bring it to the spatial frame, via $g(t_i)$, which is unknown. Although we may have a good approximation of it, the current estimate of the state $\hat{g}(t_i)$, the pose when the point first appears should be estimated along with the coordinates of the points. Therefore, we can represent X^i using $y^i(t_i)$, $Z^i(t_i)$ *and* $g(t_i)$:

$$X_s^i = X_s^i(g_{t_i}, y_{t_i}, Z_{t_i}) = g_{t_i} \bar{y}_{t_i} Z_{t_i} \tag{17}$$

Clearly this is an over-parametrization, since each point is now represented by $3 + 6$ parameters instead of 3. However, the pose g_{t_i} can be pooled among all points that appear at time t_i, considered therefore as a *group*. At each time, there may be a number $j = 1, \dots, K(t)$ groups, each of which has a number $i = 1, \dots, N_j(t)$ points. We indicate the group index with j and the point index with $i = i(j)$, omitting the dependency on j for simplicity. The representation of X_s^i then evolves according to

$\{$ *tabular>* r¡z< $\qquad \dot{y}^i_{t_i} = 0, \quad i = 1, \dots, N(j) \dot{Z}^i_{t_i}$

(18)

For the case of range, this is not relevant as there is no reference frame that offers a preferential treatment of uncertainty.

1.8 Compact notation

If we call the "state" $x = \{T, R, V, \alpha_b, \omega_b, X\} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ the "known input" $u = \{\omega_{\text{imu}}, \alpha_{\text{imu}}\} = \{u_1, u_2\}$, the *unknown input* $v = \{n_{\omega_b}, n_{\alpha_b}\} = \{v_1, v_2\}$, we can write the mechanization equations () as

$$\dot{x} = f(x) + c(x)u + Dv \tag{19}$$

where

$$f(x) \doteq \begin{bmatrix} x_3 \\ -x_2 x_4 \\ -x_2 x_5 + \gamma \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad c(x) \doteq \begin{bmatrix} 0 \\ R \\ R \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad D \doteq \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \tag{20}$$

and the measurement equation (11) as

$$y = h(x) + n \tag{21}$$

where

$$h(x) \doteq \begin{bmatrix} \vdots \\ \pi(x_2'(x_6^i - x_1)) \\ \vdots \end{bmatrix} \tag{22}$$

Putting together ()-(11) we have a model of the form

$$\{\dot{}\, x = f(x) + c(x)u + Dvy = h(x) + n. \tag{23}$$

1.9 Definitions

We call $y^t = \{y(\tau)\}_{\tau=0}^t$, a collection of output measurements, and $x^t = \{x(\tau)\}_{\tau=0}^t$ a state *trajectory*. Given output measurements y^t and known inputs u^t, we call

$$\mathcal{I}(y^t|u^t;\tilde{x}_0) \doteq \{\tilde{x}^t \mid y^t = h(\tilde{x}^t) \text{ s. } t. \ \dot{\tilde{x}}(t) = f(\tilde{x}) + c(\tilde{x})u(t), \ \tilde{x}(0) = \tilde{x}_0 \ \forall \ t\} \tag{24}$$

the *indistinguishable set*, or set of *indistinguishable trajectories*, for a given input u^t. If the initial condition $\tilde{x}_0 = x_0$ equals the "true" one, the indistinguishable set contains at least one element, the "true" trajectory x^t. However, if $\tilde{x}_0 \neq x_0$, the true trajectory may not even be part of this set.

If the indistinguishable set is a singleton (it contains only one element, $\tilde{x}^t$, which is a function of the initial condition $\tilde{x}_0$), we say that the model is *observable up to the initial condition*, or simply *observable*.[5] If $\{\tilde{x}^t\}$ is further independent of the initial condition, we say that the model is *strongly observable*: $\mathcal{I}(y^t|u^t;\tilde{x}_0) = \{x^t\} \ \forall \ \tilde{x}_0, \ u^t$.

If the state includes unknown parameters with a trivial dynamic, and there is no unknown input, $v = 0$, then observability of the resulting model implies that the parameters are *identifiable.* That usually requires the input u^t to be *sufficiently exciting* (SE), in order to enable disambiguating the indistinguishable states, [6] as the definition does not require that every input disambiguates states.

In the presence of *unknown inputs* $v \neq 0$, consider the following definition

$$\mathcal{I}_v(y^t|u^t;\tilde{x}_0) \doteq \{\tilde{x}^t \mid \exists \, v^t \text{ s. } t. \ y^t = h(\tilde{x}^t), \ \dot{\tilde{x}}(t) = f(\tilde{x}) + c(\tilde{x})u(t) + Dv(t) \ \forall \ t; \ \tilde{x}(0) = \tilde{x}_0\} \tag{25}$$

which is the set of *unknown-input indistinguishable states.* The model $\{f, c, D\}$ is said to be *unknown-input observable* (up to initial conditions) if the unknown-input indistinguishable set is a singleton. If such a singleton is further independent of the initial conditions, the model is strongly observable. The two definitions coincide once the only admissible unknown input is $v^t = 0$ for all t.

It is possible for a model to be observable (the indistinguishable set is a singleton), but not unknown-input observable (the unknown-input indistinguishable set is dense). In that case, the notion of *sensitivity* arises naturally, as one would want to measure the "size" of the unknown-input indistinguishable set as a function of the "size" of the unknown input. For instance, it is possible that if the set of unknown inputs is small in some sense, the resulting set of indistinguishable states is also small. If $v \in V$ and for any $\epsilon > 0$ there exists a $\delta > 0$ such that $\mathrm{v}ol(V) \leq \epsilon$ for some measure of volume implies $\mathrm{v}ol(\mathcal{I}_v(y^t|u^t;\tilde{x}_0)) < \delta$ for any $u^t, \tilde{x}_0$, then we say that the model is *bounded-unknown-input/bounded-output observable* (up to the initial condition). If the latter volume is independent of $\tilde{x}_0$ we say that model is strongly bounded-unknown-input/bounded-output observable. repetition

[5]We will assume that the solution of the differential equation $\dot{x} = f(x) + c(x)u$ is unique and continuously dependent on the initial condition, so if we impose $\tilde{x}_0 = x_0$, then $\tilde{x}^t = x^t$.

[6]Sufficient excitation means that the input is *generic*, and does not lie on a thin set. That is, even if we could find a particular input u^t that yields indistinguishable states, there will be another input that is infinitesimally close to it that will disambiguate them.

The set of indistinguishable trajectories $\mathcal{I}$ is an equivalence class, and when the model is observable *up to the initial condition*, it is parametrized by $\tilde{x}_0$. Choosing the "true" initial condition $\tilde{x}_0 = x_0$ produces an indistinguishable set consisting of the sole "true" trajectory, otherwise it is a singleton other than the true trajectory. In some cases, the initial condition corresponds to an arbitrary choice of reference frame, and therefore the equivalence class of indistinguishable trajectory are related by a *gauge transformation* (a change of coordinates). As the equivalence class can be represented by any element, enforcing a particular reference for the gauge transformation yields strong observability (although the singleton may not correspond to the true trajectory).

Related work

Unknown-input observability of linear time-invariant systems has ben addressed in [**?**, **?**], for affine systems [**?**], and non-linear systems in [**?**, **?**, **?**]. The literature on robust filtering and robust identification is relevant, if the unknown input is treated as a disturbance. However, the form of the models involved in aided navigation do not fit in the classes treated in the literature above, which motivates our analysis.

2 Analysis of Bearing-Augmented Navigation

2.1 Preliminary claims

Given $S \in SO(3)$ and $\dot{S} \in T_{SO(3)}(S)$, and $a \in R$, the matrix $(aS + \dot{S})$ is nonsingular unless $a = 0$, in which case it has rank 2 or 0. The tangent $\dot{S}$ has the form SM, where M is some skew-symmetric matrix. As such, $Mx \perp x$ for any $x \in R^3$, so

$$\|(aS + \dot{S})x\|_2^2 = \|S(aI + M)x\|_2^2 = \|ax\|_2^2 + \|Mx\|_2^2.$$

The above is zero only if $ax = 0$, so $(aS + \dot{S})$ is nonsingular. For the remaining cases, observe that a 3×3 skew-symmetric matrix has rank 2 or 0.

Let $(R(t), T(t))$ and $(\tilde{R}(t), \tilde{T}(t))$ be differentiable trajectories in $SE(3)$. For each time $t' \in [0, T]$, there exists an open, full-measure subset $\mathcal{A}_{t'} \subset R^3$ such that:

For any two static point-clouds $\{X^i\}_{i=1}^N \subset \mathcal{A}_{t'}$ and $\{\tilde{X}^i\}_{i=1}^N \subset R^3$ that satisfy

$$\pi\big(R^{-1}(t)(X^i - T(t)\big) = \pi\big(\tilde{R}^{-1}(t)(\tilde{X}^i - \tilde{T}(t))\big) \quad forall$$

iandt(26)there exist constant scalings $\sigma_{it'} > 0$ and a constant rotation $S_{t'} = \tilde{R}(t')R^{-1}(t')$ such that

$$\sigma_{it'} S_{t'}(X^i - T(t)) = (\tilde{X}^i - \tilde{T}(t)) + O((t - t')^2) \quad forall$$

i$andt.Furthermore, if$T(t$'$) $\neq 0$, then $\sigma_{it'} = \sigma_{t'}$ for all i.

Write $S(t) = \tilde{R}(t)R^{-1}(t)$. Equality under the projection π implies that there exists a scaling $\sigma_i(t)$ (possibly varying with X^i and t) such that

$$\sigma_i S\big(X^i - T\big) = \tilde{X}^i - \tilde{T}. \tag{27}$$

For a given time t', we wish to find a suitably large set $\mathcal{A}_{t'}$ such that $\dot{\sigma}_i(t') = \dot{S}(t') = 0$ and $\sigma_i(t')$ is independent of X^i, when $X^i \in \mathcal{A}_{t'}$ Taking time derivatives,

$$\big(\dot{\sigma_i}S + \sigma_i\dot{S}\big)(X^i - T) - \sigma_i S\dot{T} = -\dot{\tilde{T}}$$

or, dividing by σ_i, $\big(\dot{\sigma_i}\sigma_i S + \dot{S}\big)(X^i - T) - S\dot{T} = -1\sigma_i\dot{\tilde{T}}$. Differentiating both sides with respect to X^i,

$$\big(\dot{\sigma_i}\sigma_i S + \dot{S}\big)\delta X^i + \big(ddX^i\big(\dot{\sigma_i}\sigma_i\big)\delta X^i\big)S(X^i - T) = -\big(ddX^i\big(1\sigma_i\big)\delta X^i\big)\dot{\tilde{T}}. \tag{28}$$

Observe that $ddX^i\big(\dot{\sigma_i}\sigma_i\big)\delta X^i$ and $ddX^i\big(1\sigma_i\big)\delta X^i$ are scalars. By Lemma 1, the LHS has rank 2 or greater (as a linear map on δX^i), unless $\dot{\sigma_i}(t') = 0$. The RHS, however, has rank at most 1. Thus, () is invalid for almost all X^i, unless $\dot{\sigma_i}(t') = 0$ (two maps of different ranks can only agree on a submanifold). Plugging $\dot{\sigma_i} = 0$ into (28), we are left with

$$\dot{S}\delta X^i = -\big(ddX^i\big(1\sigma_i\big)\delta X^i\big)\dot{\tilde{T}}. \tag{29}$$

Now, the LHS has rank 2 or 0, while the RHS has rank 1 or 0. Again, () is invalid for almost all X^i, unless $\dot{S}(t') = 0$. Let $\mathcal{A}_{t'} \subset R^3$ be the open, full-measure subset (being the complement of two submanifolds) on which the latter must hold. If, in addition, $T\dot{(}t') \neq 0$, then $\dot{\tilde{T}}(t') \neq 0$ and $\frac{d\sigma_i}{dX^i}(t') = 0$, we can finally write

$$\sigma_{t'}S_{t'}(X^i - T) = \tilde{X}^i - \tilde{T} + O((t - t')^2).$$

Claim 1 (Indistinguishable Trajectories from Bearing Data Sequences) *Let $g(t)$ and $\tilde{g}(t)$ be differentiable trajectories in $SO(3)$. There exists an open, full-measure subset $\mathcal{A} \subset R^3$ such that*

> *Given two static, generic (non-coplanar) point clouds $\{X^i\}_{i=1}^N \subset \mathcal{A}$ and $\{\tilde{X}^i\}_{i=1}^N \subset R^3$, satisfying*
>
> $$\pi(g^{-1}(t)\,X^i) = \pi(\tilde{g}^{-1}(t)\,\tilde{X}^i) \quad forall$$
>
> *iandt,thereexistconstantscalings$\sigma_i > 0$ and a constant transformation $\bar{g} \in SE(3)$ such that*
>
> $\Big\{$ *tabular>*
> *r¦l<* $\qquad\qquad \tilde{X}^i = \sigma_i(\bar{g}X^i)\tilde{g}(t) = \sigma_i($

$forall iandt.$(30)Furthermore, if $g(t)$ has a non-constant translational component, then $\sigma_i = \sigma$ for all i.

Write $g(t) = (R(t), T(t))$ and $\tilde{g}(t) = (\tilde{R}(t), \tilde{T}(t))$. Let $\mathcal{A} = \{X \in R^3 : X \in \mathcal{A}_{t'} for almost all t'\}$, with $\mathcal{A}_{t'}$ defined as in Lemma 2. By Fubini's theorem, this has full measure in R^3. If $\{X^i\} \subset \mathcal{A}$, then the conditions for Lemma 2 are satisfied for almost all t, and thus there exist *constant* (being stationary for almost all t) scalings σ_i and rotation $S = \tilde{R}(t)R(t)^{-1} \in SO(3)$ such that $\tilde{X}^i = \sigma_i S(X^i - T_t) + \tilde{T}_t$.

Define $\bar{g}(t) = (\sigma_i^{-1}\tilde{g}(t))\, g(t)^{-1}$, and observe that $\tilde{X}^i = \sigma_i S(X^i - T_t) + \tilde{T}_t = \sigma_i(\tilde{R}_t(g^{-1}X^i) + \sigma_i^{-1}\tilde{T}_t) = \sigma_i\big((\sigma_i^{-1}\tilde{g}(t))\, g(t)^{-1}X^i\big) = \sigma_i(\bar{g}(t)X^i)$. If this affine relation holds for the generic set $\{X^i\}$, then $\bar{g}(t)$ must be constant. Next, $\sigma_i(\bar{g}g(t)) = \sigma_i((\sigma_i^{-1}\tilde{g}(t))\, g(t)^{-1}g(t)) = \sigma_i(\sigma_i^{-1}\tilde{g}(t)) = \tilde{g}(t)$. Finally, if $T(t') = 0$ for some t', then $\sigma_i = \sigma_i(t') = \sigma(t') = \sigma$ for all i.

In what follows, we will avoid the cumbersome discussion of sets such as $\mathcal{A} \subset R^3$, defined by a given trajectory, and will instead speak of *sufficiently exciting* trajectories, for which a given point cloud is suitable for tracking.

Definition 1 (Sufficiently Exciting Motion) *A trajectory $g(t)$ is* ***sufficiently exciting*** *relative to a point-cloud $\{X^i\}_{i=1}^N \subset R^3$ if, for all $\{\tilde{X}^i\}_{i=1}^N \subset R^3$ and $\tilde{g}(t)$ in $SE(3)$, $\pi(g(t)^{-1}(t)X^i) = \pi(\tilde{g}(t)^{-1}\tilde{X}^i) \quad for all i and t \iff$*

$$\begin{pmatrix} \tilde{X}^i & = \sigma(\bar{g}X^i) \\ \tilde{g}(t) & = \sigma(\bar{g}g(t)) \end{pmatrix} \quad for all\, i and t\, for some constant \sigma > 0 \text{ and } \bar{g} \in SE(3).$$

That is, if the projection map $\pi(g(t)X^i)$ defines $g(t)$ and $\{X^i\}$ up to a constant rotation and mapping.

Observe that the right-to-left implication is always true: if the RHS holds, then

$$\pi(\tilde{g}(t)^{-1}\tilde{X}^i) = \pi((\sigma\bar{g}g(t))^{-1}\sigma(\bar{g}X^i))\pi(g(t)^{-1}\bar{g}^{-1}\sigma^{-1}\sigma\bar{g}X^i) = \pi(g(t)^{-1}X^i).$$

We will see that the sufficient excitation condition is very easily satisfied.

Claim 2 *Given trajectories $g(t)$ and $\tilde{g}(t)$ in $SE(3)$ with non-constant translation, and a set $\{X^i\}_{i=1}^N$ of $N \geq 4$ points sampled i.i.d. from a non-singular distribution over R^3, the trajectory $g(t)$ is a.s. sufficiently exciting relative to $\{X^i\}$.*

Fix $g(t)$. By Claim 1, there exists a full-measure $\mathcal{A} \subset R^3$ such that (1) holds for any static, generic point clouds $\{X^i\}_{i=1}^N \subset \mathcal{A}$ and $\{\tilde{X}^i\}_{i=1}^N \subset R^3$. If $\{X^i\}$ is sampled i.i.d. from a non-singular distribution over R^3, then $\{X^i\} \subset \mathcal{A}$ almost surely.

Equation (1) establishes the fact that the indistinguishable trajectories are an equivalence class parameterized by a group $\sigma(\bar{g})$, called a *gauge transformation.* We now include a constant reference frame g_a. We then have the following claim.

Claim 3 (Indistinguishable Alignments) *For a point cloud $\{X^i\}_{i=1}^{N(t)}$, $N(t) > 3$, in general position (non-coplanar), and sufficiently exciting motion,*

$$\pi(g_a g^{-1}(t)X^i) = \pi(\tilde{g}_a\tilde{g}^{-1}(t)\tilde{X}^i) \tag{31}$$

if and only if there exist constants $\sigma > 0$, g_A and $g_B \in SE(3)$ such that

$\Big\{$ *tabular>*
r¡≯< $\tilde{X}^i = \sigma(g_B X^i)\tilde{g}(t) = \sigma(g_B g(t$

(32)

From Claim 1 we get constant $g_B \in SE(3)$ and $\sigma > 0$ such that $\tilde{X}^i = \sigma(g_B X^i)$ and

$$\tilde{g}(t)\tilde{g}_a^{-1} = \sigma(g_B g(t) g_a^{-1}) \tag{33}$$

Let $g_A = g_a^{-1}\sigma^{-1}(\tilde{g}_a)$. Then $\tilde{g}_a = \sigma(g_a g_A)$ and

$$\tilde{g}(t) = \sigma(g_B g(t) g_A).$$

We now include groups of points, each with its own reference frame.

Claim 4 (Indistinguishable Groups) *For a number $i = 1, \dots, K$ of groups each with a number $j = 1, \dots, N_i \geq 3$ of points in general position (non-coplanar), and sufficiently exciting motion,*

$$\pi(g_a g^{-1}(t) g_i g_a^{-1} X^j) = \pi(\tilde{g}_a \tilde{g}^{-1}(t)\tilde{g}_i \tilde{g}_a^{-1}\tilde{X}^j) \tag{34}$$

if and only if there exist constants $\sigma > 0, g_A, g_B, \bar{g}_i \in SE(3)$ such that

$\Big\{$ *tabular>*
r¡≯< $\tilde{X}^j = \sigma(g_a \bar{g}_i^{-1} g_i g_a^{-1} X^j)\tilde{g}(t) =$

(35)

From Claim 1, we get constant $g_C \in SE(3)$ and $\sigma > 0$ such that $\tilde{X}^i = \sigma(g_C X^i)$, $\tilde{g}_a \tilde{g}_i^{-1}\tilde{g}(t)\tilde{g}_a^{-1} = \sigma(g_C g_a g_i^{-1} g(t) g_a^{-1})$. Define

$$g_A := g_a^{-1}\sigma^{-1}(\tilde{g}_a), \qquad g_B := \sigma^{-1}(\tilde{g}_i g_a^{-1}) g_C g_a g_i^{-1}, \qquad \bar{g}_i := g_i g_a^{-1} g_C^{-1} g_a.$$

Then, applying the definition of $\bar{g}_i$ to (), $\tilde{X}^j = \sigma(g_C X^j) = \sigma((g_a \bar{g}_i^{-1} g_i g_a^{-1})X^j)$. Applying the definitions of g_A and g_B to (), $\tilde{g}(t) = \tilde{g}_i \tilde{g}_a^{-1}\sigma(g_C g_a g_i^{-1} g(t) g_a^{-1})\tilde{g}_a = \sigma\big(g_B \underbrace{\sigma^{-1}(g_i \tilde{g}_a^{-1}) g_C g_a g_i^{-1}}\, g(t)\, g_A \underbrace{g_a^{-1}\sigma^{-1}(\tilde{g}_a)}\big) = \sigma(g_B g(t) g_A)$. Rearranging the definitions of g_A, g_B and $\bar{g}_i$, $\tilde{g}_i = \sigma(g_B g_i g_a^{-1} g_C^{-1})\tilde{g}_a = \sigma\left(g_B g_i g_a^{-1} g_C^{-1}\sigma(\tilde{g}_a)\right) = \sigma\big(g_B \bar{g}_i \underbrace{g_i g_a^{-1} g_C^{-1} g_a}\, g_A \underbrace{g_a^{-1}\sigma(\tilde{g}_a)}\big) = \sigma(g_B \bar{g}_i g_A)$. Finally, rearrange the definition of g_A to get

$$\tilde{g}_a = \sigma(g_a g_A).$$

Eq. () describes the ambiguous state trajectories if only bearing measurement time series are given. In that case, there is no alignment to other sensor, so we can assume without loss of generality that $g_a = Id$ and so for $\tilde{g}_a$, which in turn implies $g_A = Id$. The resulting ambiguity is well-known [**?**] and shows that scale σ is constant but arbitrary, that the global reference frame is arbitrary (since g_B is), and that the reference frame of each group is also arbitrary (since $\bar{g}_i$ is). To lock these ambiguities, we can fix three directions for each group (thus fixing $\bar{g}_i$) and, in addition, for one of the groups fix the pose (thus fixing g_B); finally, we can impose that the centroid of the points in that one group (the "reference group") be one, which fixes σ. Thus, an observer designed based on the standard model, where 3 directions within each group are saturated, and where the pose of one group is fixed, and the centroid of the group is at distance one, is observable, and under the usual assumptions it should converge to a state trajectory that is related to the true one by an arbitrary unknown scaling, and global reference frame.

Now, when inertial measurements are present, of all the possible trajectories that are indistinguishable from the measurements, we are interested *only* in those that are compatible with the dynamical model driven by IMU measurements. Since the fact that X^j and g_a are constant has already ben enforced, the model will impose no constraints on $\tilde{X}^j, \tilde{g}_i$ and $\tilde{g}_a$. However, it will offer constraints on $\tilde{g}(t)$, that depends on the arbitrary constants σ, g_A, g_B.

2.2 Indistinguishable trajectories in bearing augmentation

Definition 2 *For an R^3-valued trajectory $f : R \rightarrow R^3$ and interval $\mathcal{I} \subset R^+$, define* $m(f{:}\mathcal{I}) := \inf_{\|x\|=1}\Big(\sup_{t\in\mathcal{I}} |f(t)\cdot x|\Big) = \inf_{\|x\|=1}\Big(\sup_{t\in\mathcal{I}} \|f(t)\times x\|\Big)$,
$M(f{:}\mathcal{I}) := \sup_{\|x\|=1}\Big(\sup_{t\in\mathcal{I}} |f(t)\cdot x|\Big) = \sup_{t\in\mathcal{I}} \|f(t)\|$, *and*
$\bar{m}(f{:}\mathcal{I}) := \sqrt{\max\{0,\, 2m(f{:}\mathcal{I})^2 - M(f{:}\mathcal{I})^2\}}$.

Observe that $M(f:\mathcal{I}) \geq m(f:\mathcal{I}) \geq \bar{m}(f:\mathcal{I})$, and that the inequalities are strict unless $\{\pm f(t) |\, t\in\mathcal{I}\}$ is dense on the sphere of radius $M(f{:}\mathcal{I})$. We use these "minimum-excitation" bounds in order to prove a partial converse of the Cauchy-Schwarz inequality: Let $A = c_1 I + c_2 R$, for some rotation $R \in SO(3)$ and scalars c_1 and c_2. Then, for any trajectory $f : R^+ \rightarrow R^3$ and set of times $\mathcal{I} \subset R^+$,

$$\sup_{t\in\mathcal{I}} \|Af(t)\| \;\geq\; \|A\|\,\bar{m}(f{:}\mathcal{I}).$$

First, observe that A is normal:

$$AA^T = (c_1 I + c_2 R)(c_1 I + c_2 R^T) = 2c_1c_2 I + c_1c_2(R + R^T) = A^T A.$$

Let $\{(\lambda_i, v_i)\}_{i=1}^3$ be orthonormal eigenvalue/eigenvector pairs of A, with $\lambda_1 \geq \lambda_2 \geq \lambda_3$. $\|Af(t)\|^2 = \lambda_1^2(v_1\cdot f(t))^2 + \lambda_2^2(v_2\cdot f(t)^2 + \lambda_3^2(v_3\cdot f(t))^2$
$\geq \lambda_1^2\big((v_1\cdot f(t))^2 - (v_2\cdot f(t)^2 - (v_3\cdot f(t))^2\big)$
$= \|A\|^2\big(2(v_1{\cdot}f(t))^2 - \|f(t)\|^2\big)$. Taking the supremum over $\mathcal{I}$, $\sup_{t\in\mathcal{I}} \|Af(t)\|^2 \geq \|A\|^2 \sup_{t\in\mathcal{I}}\big(2(v_1\cdot f(t))^2 - \|f(t)\|^2\big)$
$\geq \|A\|^2\big(2\sup_{t\in\mathcal{I}}(v_1\cdot f(t))^2 - \sup_{t\in\mathcal{I}}\|f(t)\|^2\big)$
$\geq \|A\|^2\big(2m(f{:}\mathcal{I})^2 - M(f{:}\mathcal{I})^2\big)$

Let $A = I - R$, for some rotation $R \in SO(3)$. Then, for trajectory $f : R^+ \to R^3$ and $\mathcal{I} \subset R^+$,

$$\sup_{t \in \mathcal{I}} \|Af(t)\| \;\geq\; \|A\|\, m(f\!:\!\mathcal{I}).$$

Let $\{(\lambda, v_1), (\bar{\lambda}, v_2), (1, 0)\}$ be the orthonormal eigenvalue/eigenvector pairs of R. Since R and I commute, $\{(\lambda - 1, v_1),\ (\bar{\lambda} - 1, v_2), (0, u)\}$ are the eigenpairs of A, and $\|A\| = |\lambda - 1| = |\bar{\lambda} - 1|$. Then,

$$\|Af(t)\|^2 = |\lambda - 1|^2 (v_1 \cdot f(t))^2 + |\bar{\lambda} - 1|^2 (v_2 \cdot f(t))^2 + 0 = \|A\|^2 (w \cdot f(t))^2,$$

where

$$w := \frac{(v_1 \cdot f(t))v_1 + (v_2 \cdot f(t))v_2}{\|(v_1 \cdot f(t))v_1 + (v_2 \cdot f(t))v_2\|} = \frac{(v_1 \cdot f(t))v_1 + (v_2 \cdot f(t))v_2}{\sqrt{(v_1 \cdot f(t))^2 + (v_2 \cdot f(t))^2}}.$$

Taking the supremum over $\mathcal{I}$,

$$\sup_{t \in \mathcal{I}} \|Af(t)\|^2 = \|A\|^2 \sup_{t \in \mathcal{I}} \|w \cdot f(t)\|^2 \geq \|A\|^2 m(f\!:\!\mathcal{I})^2.$$

Claim 5 (Indistinguishable Trajectories from IMU Data) *Let $g(t) = (R(t), T(t)) \in SE(3)$ be such that*

$$\begin{cases} tabular> \\ \quad r¡Ɨ< \end{cases} \qquad \dot{R} = R(\widehat{\omega}_{\text{imu}} - \widehat{\omega}_b)\dot{T} = V\dot{V} =$$

(36)

for some known constant γ and functions $\alpha_{\text{imu}}(t)$, $\omega_{\text{imu}}(t)$ and for some unknown functions $\alpha_b(t), \omega_b(t)$ that are constrained to have $\|\dot{\alpha}_b(t)\| \leq \epsilon$, $\|\dot{\omega}_b(t)\| \leq \epsilon$, and $\|\ddot{\omega}_b(t)\| \leq \epsilon$ at all t, for some $\epsilon < 1$.

Suppose $\tilde{g}(t) \doteq \sigma(g_B g(t) g_A)$ for some constant $g_A = (R_A, T_A)$, $g_B = (R_B, T_B)$, $\sigma > 0$, with bounds on the configuration space such that $\|T_A\| \leq M_A$ and $|\sigma| \leq M_\sigma$. Then, under sufficient excitation conditions (described in this proof), $\tilde{g}(t)$ satisfies () if and only if $\|I - R_A\| \leq \frac{2\epsilon}{m(\dot{\omega}_{\text{imu}}\!:\!R^+)}$
$|\sigma - 1| \leq \frac{k_{c_1}\epsilon + M_\sigma \|I - R_A\|}{M(\dot{\alpha}_{\text{imu}}\!:\!\mathcal{I}_{c_1})}$
$\|T_A\| \leq \frac{\epsilon(k_{c_2} + (2M_\sigma + 1)M_A)}{(1 - |\sigma - 1|)\, m(\ddot{\omega}_{\text{imu}}\!:\!\mathcal{I}_{c_2})}$
$\|(1 - R_B^T)\gamma\| \leq \frac{\epsilon(k_{c_3} + M_\sigma M_A) + (|\sigma - 1| + \epsilon) M(\omega_{\text{imu}} - \omega_b\!:\!\mathcal{I}_{c_3}) \|\gamma\|}{m(\omega_{\text{imu}} - \omega_b\!:\!\mathcal{I}_{c_3})\, (1 - |\sigma - 1|)}$ for $\mathcal{I}_i$ and k_i determined by the sufficient excitation conditions.

() The ambiguous rotation $\tilde{R}$ must satisfy $\dot{\tilde{R}} = \tilde{R}(\widehat{\omega}_{\text{imu}} - \widehat{\tilde{\omega}}_b)$ for some $\tilde{\omega}_b$:
$\dot{\tilde{R}} = R_B R(\widehat{\omega}_{\text{imu}} - \widehat{\omega}_b) R_A = \tilde{R} R_A^T (\widehat{\omega}_{\text{imu}} - \widehat{\omega}_b) R_A = \tilde{R}(\widehat{R_A^T \omega}_{\text{imu}} - \widehat{R_A^T \omega_b})$

$= \tilde{R}(\widehat{\omega}_{\text{imu}} - [\widehat{\omega}_{\text{imu}} + \widehat{R_A^T\omega}_{\text{imu}} - \widehat{R_A^T\omega_b}])$ where the quantity in brackets is $-\widehat{\tilde{\omega}}_b$, which defines

$$\tilde{\omega}_b := R_A^T\omega_b + (I - R_A^T)\,\omega_{\text{imu}}. \tag{37}$$

Taking derivatives and rearranging,

$$2\epsilon \geq \|\dot{\tilde{\omega}}_b - R_A^T\dot{\omega}_b\| = \|(I - R_A^T)\dot{\omega}_{\text{imu}}\|$$

Since this is true for all $t \in R$, we can write $2\epsilon \geq \sup_{t\in R}\|(I-R_A^T)\,\dot{\omega}_{\text{imu}}(t)\| \geq \|I - R_A^T\|\,m(\dot{\omega}_{\text{imu}}\!:\!R^+)$. This rearranges to give ().

() The ambiguous translation $\tilde{T}$ must satisfy the dynamics in (): $\ddot{\tilde{T}} = \dot{\tilde{V}} = \tilde{R}(\alpha_{\text{imu}} - \tilde{\alpha}_b) + \gamma = R_B R R_A(\alpha_{\text{imu}} - \tilde{\alpha}_b) + \gamma$. Alternatively, working with $\tilde{T} = \sigma R_B(RT_A + T)$ and applying the dynamics to T, $\ddot{\tilde{T}} = \sigma R_B(\ddot{R}T_A + \ddot{T}) = \sigma R_B(\ddot{R}T_A + R(\alpha_{\text{imu}} - \alpha_b) + \gamma)$. Taking the difference between these two expressions, $0 = \sigma R_B\ddot{R}T_A + R_B R(R_A\tilde{\alpha}_b - \sigma\alpha_b) + R_B R(\sigma\alpha_{\text{imu}} - R_A\alpha_{\text{imu}}) + (\sigma R_B - I)\gamma$, and multiplying by $R^T R_B^T$, $0 = \sigma(R^T\ddot{R})T_A + (R_A\tilde{\alpha}_b - \sigma\alpha_b) + (\sigma\alpha_{\text{imu}} - R_A\alpha_{\text{imu}}) + R^T(\sigma - R_B^T)\gamma$
$= \sigma((\widehat{\omega}_{\text{imu}} - \widehat{\omega}_b)^2 + (\dot{\widehat{\omega}}_{\text{imu}} - \dot{\widehat{\omega}}_b))T_A + (R_A\tilde{\alpha}_b - \sigma\alpha_b) + (\sigma\alpha_{\text{imu}} - R_A\alpha_{\text{imu}}) + R^T(\sigma - R_B^T)\gamma$. Differentiating again, $0 = \sigma(\dot{R}^T\ddot{R} + R^T R)T_A$
$+ ((I - R_A)\sigma + (\sigma - 1)R_A)\dot{\alpha}_{\text{imu}}$
$+ \dot{R}^T((I - R_B^T)\sigma + (\sigma - 1)R_B^T)\gamma.$
$+ (R_A\dot{\tilde{\alpha}}_b - \sigma\dot{\alpha}_b)$ As a sufficient excitation condition, assume that $\|\dot{R}(t)\| \leq \epsilon$, $\|\ddot{R}(t)\| \leq \epsilon$, and $\|R(t)\| \leq \epsilon$, for $t \in \mathcal{I}_{c_1}$. Under these constraints, () is bounded by $k_{c_1}\epsilon$, where, e.g. $k_{c_1} := 2M_\sigma M_A + (2M_\sigma + 1)(\|\gamma\| + 1)$. In that case, $\mathrm{k}_{c_1}\epsilon \geq \max_{t\in\mathcal{I}_{c_1}}\|((I - R_A)\sigma + (\sigma - 1)R_A)\dot{\alpha}_{\text{imu}}(t)\|$
$\geq |\sigma - 1|M(\dot{\alpha}_{\text{imu}}\!:\!\mathcal{I}_{c_1}) - M_\sigma\|I - R_A\|$. This rearranges to give ().

() Now, assume that $\|\dot{R}(t)\| \leq \epsilon$, $\|\ddot{R}(t)\| \leq \epsilon$, and $\|\ddot{T}(t) - \gamma\| \leq \epsilon$, for $t \in \mathcal{I}_{c_2}$. Under these constraints, $\|\dot{\alpha}_{\text{imu}}\| \leq 2\epsilon$, and () is bounded by $k_{c_2}\epsilon$, where, e.g. $k_{c_2} := (2M_\sigma + 1)(\|\gamma\| + 3)$. In that case, $\mathrm{k}_{c_2}\epsilon \geq \max_{t\in\mathcal{I}_{c_2}}\|\sigma((\widehat{\omega}_{\text{imu}} - \widehat{\omega}_b)(\dot{\widehat{\omega}}_{\text{imu}} - \dot{\widehat{\omega}}_b) + (\ddot{\widehat{\omega}}_{\text{imu}} - \ddot{\widehat{\omega}}_b))T_A\|$
$= \max_{t\in\mathcal{I}_{c_2}}\|\sigma((R^T\dot{R})(R^T\ddot{R} - (R^T\dot{R})^2) + (\ddot{\widehat{\omega}}_{\text{imu}} - \ddot{\widehat{\omega}}_b))T_A\|$
$\geq (1 - |1 - \sigma|)\max_{t\in\mathcal{I}_{c_2}}\|\ddot{\omega}_{\text{imu}}(t) \times T_A\| - (2M_\sigma + 1)M_A\epsilon$
$\geq (1 - |1 - \sigma|)\,\|T_A\|\,m(\ddot{\omega}_{\text{imu}}\!:\!\mathcal{I}_{c_2}) - (2M_\sigma + 1)M_A\epsilon$. This rearranges to give ().

() Finally, assume that $\|\ddot{R}(t)\| \leq \epsilon$, $\|R(t)\| \leq \epsilon$, and $\|\ddot{T}(t) - \gamma\| \leq \epsilon$ for $t \in \mathcal{I}_{c_3}$. As before, $\|\dot{\alpha}_{\text{imu}}\| \leq 2\epsilon$. Then, () + () is bounded by $k_{c_3}\epsilon$, where, e.g. $k_{c_3} = 2M_\sigma + 3$. In that case, $\mathrm{k}_{c_3}\epsilon \geq \|\sigma(\dot{R}^T\ddot{R} + R^T R)T_A + \dot{R}^T((I - R_B^T)\sigma + (\sigma - 1)R_B^T)\gamma\|$
$\geq \|\sigma\dot{R}^T(\ddot{R} + (I - R_B^T))\gamma\| - M_\sigma M_A\epsilon - |\sigma - 1|\,\|\dot{R}^T\|\,\|\gamma\|$
$\geq (1 - |\sigma - 1|)\,\|\dot{R}^T(I - R_B^T)\gamma\| - M_\sigma M_A\epsilon - (|\sigma - 1| + \epsilon)\,\|\dot{R}^T\|\,\|\gamma\|$
$\geq (1-|\sigma-1|)\,m(\dot{R}^T\!:\!\mathcal{I}_{c_3})\|(1-R_B^T)\gamma\| - \epsilon(k_{c_3} + M_\sigma M_A) - (|\sigma-1|+\epsilon)M(\dot{R}^T\!:\!\mathcal{I}_{c_3})\|\gamma\|$ This rearranges to give ().

2.3 Gauge transformations

repetition

The set of indistinguishable trajectories $\mathcal{I}$ is an equivalence class, and when the model is observable *up to the initial condition*, it is parametrized by $\tilde{x}_0$. Choosing the "true" initial condition $\tilde{x}_0 = x_0$ produces an indistinguishable set consisting of the sole "true" trajectory, otherwise it is a singleton other than the true trajectory. In some cases, the initial condition corresponds to an arbitrary choice of reference frame, and therefore the equivalence class of indistinguishable trajectory are related by a *gauge transformation* (a change of coordinates). As the equivalence class can be represented by any element, enforcing a particular reference for the gauge transformation yields strong observability (although the singleton may not correspond to the true trajectory).

Formally, an arbitrary choice of initial condition is sufficient to fix the gauge reference. For instance, the set of indistinguishable trajectories in the limit where $\epsilon \to 0$ is parametrized by an arbitrary $T_B \in R^3$ and $\theta \in R$,

$$\{\tilde{} T = \exp(\widehat{\gamma}\theta)T + T_B \tilde{R} = \exp(\widehat{\gamma}\theta)R\tilde{T}_{t_i} = \exp(\widehat{\gamma}\theta)\bar{T}_{t_i} + T_B \tilde{R}_{t_i} = \exp(\widehat{\gamma}\theta)\bar{R}_{t_i} \tag{38}$$

up to $\mathcal{O}\left(\frac{\|\dot{\omega}_b\|}{\|\dot{\omega}_{\text{imu}}\|}, \frac{\|\dot{\alpha}_b\|}{\|\dot{\alpha}_{\text{imu}}\|},\right.$

If we impose that $T(0) = \tilde{T}(0) = 0$, then $T_B = 0$ is determined; similarly, if we impose the initial pose to be aligned with gravity (so gravity is in the form $[0\ 0\ \|\gamma\|]^T$, then $\theta = 0$. But while we can impose this condition, we cannot *enforce* it, since the initial condition is not a part of the state of the filter, so we cannot relate the measurements at each time t directly to it.

However, if the gauge reference can be associated to *constant parameters* that are part of the state of the model, the gauge ambiguity can be enforced in a consistent manner. For instance, the ambiguous set of points is

$$\tilde{X}^j = g_a \bar{g}_i^{-1} g_i g_a^{-1} X^j. \tag{39}$$

If each group i contains at least 3 non-coplanar points, it is possible to fix $\bar{g}_i$ by parametrizing $X^j \doteq \bar{y}^j_{t_i} Z^j$ and imposing thre directions $y^j_{t_i} = \tilde{y}^j_{t_i} = y^j(t_i), j = 1, \ldots, 3$, the measurement of these directions at time t_i when they first appear. This yields $\bar{g}_i = g_i$ and $\tilde{X}^j = X^j$ for that group. Note that it is necessary to impose this constraint in *each group*.

The residual set of indistinguishable trajectories is parameterized by *constants* θ, T_B, that determine a Gauge transformation for the groups, that can be fixed by always fixing the pose of *one* of the groups. This can be done in a number of ways. For instance, if for a certain group i we impose

$$R_{t_i} = \tilde{R}_{t_i} = \hat{R}(t_i) \text{ and } T_{t_i} = \tilde{T}_{t_i} = \hat{T}(t_i) \tag{40}$$

by assigning their value to the current best estimate of pose and not including the corresponding variables in the state of the model, then we have that

$$\hat{R}(t_i) = \exp(\widehat{\gamma}\theta)\hat{R}(t_i) \tag{41}$$

and therefore $\theta = 0$; similarly,

$$T_B = (I - \exp(\widehat{\gamma}\theta))T(t_i) = 0 \tag{42}$$

Therefore, the gauge transformation is enforced explicitly at each instant of time, as each measurement provides a constraint on the states. This suggests the following modeling procedure in the design of a filter/observer for bearing-assisted navigation:

1. Set $T(0) = 0$ with zero model error covariance, and zero initial covariance.
2. Set $R(0) = R_0$ such that $[I_{2\times 2}0]R_0\alpha_{\text{imu}} = 0$, with zero model error and non-zero initial covariance.
3. Fix gravity to $[0,\ 0,\ \|\gamma\|]^T$ from tabulates
4. Initialize at rest, then perform some fast motions before groups of features are added.
5. Add K groups, each with $2N + N$ states, plus their pose for each group but one.
6. Fix 2 directions per group ([**?**] fixes all directions; this results in a non-zero mean component of the innovation, that in turn results in a small bias in all other states, that have to account for/absorb the mean)
7. Fix the pose of one group (remove its pose from the state)
8. Triage groups before adding them to the state.

After the Gauge Transformation has ben fixed, the model is observable, and therefore a properly designed observer will converge to a solution $\tilde{x}$ that is related to the true one x as follows:

$$\begin{aligned}
\tilde{X}^{ref} &= (1+\tilde{\sigma})\tilde{R}_{cb}e^{\omega_B}e^{\widehat{\gamma}\theta}e^{\omega_A}\tilde{R}_{cb}^T(X^{\text{ref}} - T_A) + (1+\tilde{\sigma})(\tilde{R}_{cb}e^{\omega_A}T_B + \tilde{R}_{cb}T_A + \tilde{T}_{cb}) && (43)\\
\tilde{X}^j &= (1+\tilde{\sigma})\tilde{R}_{cb}\bar{R}_i\tilde{R}_{t_i}\tilde{R}_{cb}^T(X^j - T_A) + (1+\tilde{\sigma})(\tilde{R}_{cb}\bar{R}_i\tilde{T}_{t_i} + \tilde{R}_{cb}\bar{T}_i + \tilde{T}_{cb}) && (44)\\
\tilde{T} &= e^{\widehat{\gamma}\theta}T + T_B(1+\tilde{\sigma}) + \omega_B e^{\widehat{\gamma}\theta}T + e^{\omega_B}e^{\widehat{\gamma}\theta}RT_A(1+\tilde{\sigma}) && (45)\\
\tilde{R} &= e^{\omega_B}e^{\widehat{\gamma}\theta}Re^{\omega_A} && (46)\\
\tilde{T}_{t_i} &= e^{\widehat{\gamma}\theta}\bar{T}_i + T_B(1+\tilde{\sigma}) + \omega_B e^{\widehat{\gamma}\theta}\bar{T}_i + e^{\omega_B}e^{\widehat{\gamma}\theta}\bar{R}_iT_A(1+\tilde{\sigma}) && (47)\\
\tilde{R}_{t_i} &= e^{\omega_B}e^{\widehat{\gamma}\theta}\bar{R}_ie^{\omega_A} && (48)\\
\tilde{T}_{cb} &= T_{cb} + \tilde{\sigma}T_{cb} + R_{cb}T_A(1+\tilde{\sigma}) && (49)\\
\tilde{R}_{cb} &= R_{cb}\exp(\omega_A) && (50)\\
\tilde{\alpha}_b &= (\mathbf{??})\\
\tilde{\omega}_b &= (\mathbf{??})
\end{aligned}$$

$where$

$$\begin{aligned}
\|T_A\| &\leq \frac{2k\min_t\|\dot{\omega}_b\|}{\max_t\|\ddot{\omega}_{\text{imu}}\|}\\
\|\omega_A\| &\leq \frac{2\min_t\|\dot{\omega}_b\|}{\max_t\|\dot{\omega}_{\text{imu}}\|}
\end{aligned}$$

$$
\begin{aligned}
\|\omega_B\| &\leq \left(\frac{3k \max(\min_t \|\dot{\omega}_b\|, \min_t \|\dot{\alpha}_b\|)}{\min(\max_t \|\dot{\omega}_{\text{imu}}\|, \max_t \|\dot{\alpha}_{\text{imu}}\|, \|\gamma\|)} \right) \\
|\tilde{\sigma}| &\leq \left(\frac{2k \min_t \|\dot{\alpha}_b\|}{\min(\max_t \|\dot{\omega}_{\text{imu}}\|, \max_t \|\dot{\alpha}_{\text{imu}}\|)} \right)
\end{aligned}
$$

and arbitrary θ, T_B and suitable constant κ. The groups will be defined up to an arbitrary reference frame $(\bar{R}_i, \bar{T}_i)$, except for the reference group where that transformation is fixed. Note that, as the reference group "switches" (when points in the reference group become occluded or otherwise disappear due to failure in the data association mechanism), a small error in pose is accumulated. This error affects the gauge transformation, not the *state* of the system, and therefore is not reflected in the innovation, nor in the covariance of the state estimate, that remains bounded. This is unlike [**?**], where the covariance of the translation state T_B and the rotation about gravity θ grows unbounded over time, possibly affecting the numerical aspects of the implementation. Notice that in the limit where $\dot{\omega}_b = \dot{\alpha}_b = 0$, we obtain back Eq. (38).

3 Analysis of Range-Augmented Navigation

3.1 Preliminary claims

We consider the reduced model with known gravity and range-only measurements, including alignment $\|y^i(t)\| = \|g_{rb} g^{-1}(t) X^i\|$

$$
\{ {}^{\cdot} T = V, \quad T(0) = T_0 \dot{R} = R(\omega - \omega_b) + n_R \quad R(0) = R_0 \dot{V} = R(\alpha - \alpha_b) + \gamma + n_V \dot{\alpha}_b = v_\alpha(t) \dot{\omega}_b = v_\omega(t) \dot{T}_{rb} = 0 \dot{R}_{rb} = 0 \tag{51}
$$

In this model, which is of the form (23), ω, α are *known inputs*, and v_α, v_ω are *unknown inputs.*

Claim 6 (Indistinguishability from range measurements) *Let $X^i \in R^3$, $i = 1, \dots, N$, then $\|\tilde{X}^i\|^2 = \|X^i\|^2 \ \forall \ i \ \Leftrightarrow \tilde{X}^i = R X^i$ for some $R \in O(3)$.*

This follows in a straightforward manner from the definitions.

Claim 7 (Indistinguishability of time-varying pose) *Let $g(t)^{-1} \in SE(3)$ for $t \in Z$, and $X^i \in R^3$ for $i \in N$, then $\tilde{g}(t) = (\tilde{R}(t), \tilde{T}(t)) \in SE(3)$ and $\tilde{X}^i \in R^3$ yields $\|\tilde{g}^{-1}(t)\tilde{X}^i\| = \|g^{-1}(t) X^i\|$ for all t and i if and only if*

$$
\{ \tilde{X}^i = \bar{R} X^i + \bar{T}, \quad \text{for constant } (\bar{R}, \bar{T}) \in SE(3) \tilde{R}(t) = \bar{R} R(t) H(t), \quad \text{for any } H(t) \in SO(3) \tilde{T}(t) = \bar{R}(t) T(t) + \tag{52}
$$

Applying Claim 6 to $g^{-1}(t) X^i$, we have that $\tilde{g}^{-1}(t)\tilde{X}^i = H(t) g^{-1}(t) X^i$ for some $H(t) \in O(3)$. Furthermore, $g^{-1}(t) X^i = g^{-1}(t) \bar{g}^{-1} \bar{g} X^i$ for any constant $\bar{g} = (\bar{R}, \bar{T}) \in SE(3)$. Isolating the time-varying component $\tilde{g}$ and the constant component $\tilde{X}$, under general-position conditions, we have that $\tilde{R} = \bar{R} R H$, where for $\tilde{R}$ to be in $SO(3)$ we must impose that $H(t) \in SO(3)$, $\tilde{T} = \bar{R} T + \bar{T}$, and $\tilde{X} = \bar{g} X$, from which the result follows.

Claim 8 (Indistinguishability of alignment) *Let $g(t), X^i$ be as in Claim 7, and $g_{rb} = (R_{rb}, T_{rb}) \in SE(3)$. Then $\tilde g(t), \tilde g_{rb}, \tilde X^i$ are such that $\|\tilde g_{rb}\tilde g^{-1}(t)\tilde X^i\| = \|g_{rb}g^{-1}(t)X^i\|$ for all t and i if and only if*

$$\{ \tilde X^i = \bar R X^i + \bar T, \quad \text{for constant } (\bar R, \bar T) \in SE(3)\tilde R(t) = \bar R R(t) R_{rb}^T H(t) \tilde R_{rb}, \quad \text{for any } H(t) \in SO(3)\tilde T(t) = \bar R(t \tag{53}$$

Applying Claim 7 to $g_{rb}g^{-1}X$, we obtain, for the rotational component

$$\tilde R \tilde R_{rb}^T = \bar R R R_{rb}^T H = \underbrace{\bar R R R_{rb}^T H \tilde R_{rb}}_{\tilde R} \tilde R_{rb}^T \tag{54}$$

and similarly for the translational component $T - RR_{rb}^T T_{rb}$, we obtain

$$\begin{aligned}
-\tilde R \tilde R_{rb}^T \tilde T_{rb} + \tilde T &= \bar R(-RR_{rb}^T T_{rb} + T) + \bar T &(55)\\
&= -\bar R R R_{rb}^T T_{rb} + \bar R T + \bar T &(56)\\
&= -\bar R R R_{rb}^T H \tilde R_{rb} \tilde R_{rb}^T H^T T_{rb} + \bar R T + \bar T &(57)\\
&= -\tilde R \tilde R_{rb}^T H^T T_{rb} + \bar R T + \bar T &(58)\\
0 &= \tilde R \tilde R_{rb}^T (H^T T_{rb} - \tilde T_{rb}) + \tilde T - \bar R T - \bar T &(59)\\
-\tilde T_{rb} + \tilde R_{rb} \tilde R^T \tilde T &= -H^T T_{rb} + \tilde R_{rb} \tilde R^T (\bar R T + \bar T) + T_0 - T_0 &(60)
\end{aligned}$$

from which we can choose $\tilde T_{rb} = T_0$ arbitrarily, and then $\tilde T = \bar R T + \bar T - \tilde R \tilde R_{rb}^T (H^T T_{rb} - \tilde T_{rb})$, from which the result follows. Of all the different trajectories $\tilde g(t)$, points $\tilde X^i$ and alignments $\tilde g_{rb}$ that yield the same range measurements, we are now interested *only* in those that satisfy the constraints imposed by the model (51).

Claim 9 (Sensitivity of range augmented navigation) *Let $g(t), g_{rb}, X^i$ as in Claim 8. The indistinguishable trajectories $\tilde g(t), \tilde g_{rb}$ and structure $\tilde X^i$ is compatible with (51), with $\|\dot\alpha_b\| \le \epsilon$ and $\|\dot\omega_b\| \le \epsilon$, if and only if*

$$\{ \| H(t) - I\| \le \frac{k\epsilon}{\max_t \|\dot\alpha\|} \|\dot H\| \le \frac{k\epsilon}{\max_t \|\alpha\|} \|T_{rb} - H\tilde T_{rb}\| \le \frac{k\epsilon}{\max_t \|\ddot\omega\|} \bar R = \exp(\widehat\gamma\theta) \tag{61}$$

for a constant k and arbitrary θ and $\tilde R_{rb}$. In particular, if biases are constant $\epsilon = 0$, then $H = I, \tilde T_{rb} = T_{rb}$, $\tilde\alpha_b = \alpha_b$, $\tilde\omega_b = \omega_b$.

Since $H(t) \in SO(3)$ in Claim 8 is arbitrary, so is $R_{rb}^T H(t) \tilde R_{rb}$. With an abuse of notation, we will refer to the latter as $H(t)$. We then have that $\tilde R(t) = \bar R R H$ is compatible with the dynamical model only if $\dot{\tilde R} = \bar R \dot R H + \bar R R \dot H = \bar R R H(\omega - \widehat{\tilde\omega}_b)$, which *defines* $\tilde\omega_b$ as

$$\tilde\omega_b = H^T\omega_b + \underbrace{(I - H^T)\omega - \omega_h} \tag{62}$$

where ω_h is defined by $\dot H = H\omega_h$. If the norm of the derivative of the left-hand side is bounded by ϵ, $\|\dot{\tilde\omega}_b\| \le \epsilon$ and so is the first term on the right-hand side, $\|\dot\omega_b\| \le \epsilon$, then by the inverse triangular inequality, the bracketed term must have a derivative that is bounded in norm by 2ϵ. Such a derivative is the driving input to the ordinary differential equation involving $H, \dot{H} = H\omega_h$ and ω_b. Since the driving input is small but otherwise arbitrary, this leaves $H(t)$ unconstrained. Note, however, that if $\epsilon = 0$ (no bias drift), then under general position conditions ($\omega(t)$ can be arbitrary), we have that $H(t) = I$ is constant, and therefore $\tilde{\omega}_b = \omega_b$. Similarly, for the translational component, we have

$$
\begin{aligned}
\tilde{T} &= \bar{R}T + \bar{T} = \bar{R}RR_{rb}^T(T_{rb} - H\tilde{T}_{rb}) && (63)\\
\dot{\tilde{T}} = \tilde{V} &= \bar{R}V - \bar{R}R(\omega - \omega_b)R_{rb}^T(T_{rb} - H\tilde{T}_{rb}) + \bar{R}RR_{rb}^T H\omega_h\tilde{T}_{rb} && (64)\\
\dot{\tilde{V}} = \tilde{R}(\alpha - \tilde{\alpha}_b) + \gamma &= \bar{R}R(\alpha - \alpha_b) + \bar{R}\gamma - \bar{R}R[(\omega - \omega_b)^2 + \dot{\omega} + \dot{\omega}_b]R_{rb}^T(T_{rb} - H\tilde{T}_{rb}) + && (65)\\
&\quad + \bar{R}R(\omega - \omega_b)R_{rb}^T H\omega_h\tilde{T}_{rb} + \bar{R}RR_{rb}^T H[\omega_h^2 + \dot{\omega}_h]\tilde{T}_{rb} && (66)\\
RH(\alpha - \tilde{\alpha}_b) + \bar{R}^T\gamma &= R(\alpha - \alpha_b) + \gamma - R[(\omega - \omega_b)^2 + \dot{\omega} + \dot{\omega}_b]R_{rb}^T(T_{rb} - H\tilde{T}_{rb}) + && (67)\\
&\quad + R(\omega - \omega_b)R_{rb}^T H\omega_h\tilde{T}_{rb} + RR_{rb}^T H[\omega_h^2 + \dot{\omega}_h]\tilde{T}_{rb}. && (68)
\end{aligned}
$$

Under general position conditions, the constant terms above must equal, which yields $\bar{R}\gamma = \gamma$ and therefore $\bar{R} = \exp(\widehat{\gamma}\theta)$ is an arbitrary rotation about the gravity vector. Removing these terms from the equation above, we obtain

$$
\begin{aligned}
H(\alpha - \tilde{\alpha}_b) &= (\alpha - \alpha_b) - [(\omega - \omega_b)^2 + \dot{\omega} + \dot{\omega}_b]R_{rb}^T(T_{rb} - H\tilde{T}_{rb}) + && (69)\\
&\quad + (\omega - \omega_b)R_{rb}^T H\omega_h\tilde{T}_{rb} + R_{rb}^T H[\omega_h^2 + \dot{\omega}_h]\tilde{T}_{rb} && (70)
\end{aligned}
$$

which defines $\tilde{\alpha}_b$, whose derivative must be bounded in norm to $\|\tilde{\alpha}_b\| \leq \epsilon$. Again, we note that if there are no bias drifts and therefore $\epsilon = 0$, from $H = I$ and the general-position conditions, from the above we obtain $\tilde{T}_{rb} = T_{rb}$ and $\tilde{\alpha}_b = \alpha_b$. More in general, however, we have that

$$
\begin{aligned}
-\dot{\alpha}_b &= -H^T\alpha_b - \omega_h H^T\alpha + (H^T - I)\dot{\alpha} + \omega_h H^T[(\omega - \omega_b)^2 + \dot{\omega} + \dot{\omega}_b]R_{rb}^T(T_{rb} - H\tilde{T}_{rb}) + && (71)\\
&\quad - H^T[2(\omega - \omega_b)(\dot{\omega} - \dot{\omega}_b) + \ddot{\omega} - \ddot{\omega}_b]R_{rb}^T(T_{rb} - H\tilde{T}_{rb}) + && (72)\\
&\quad + H^T[(\omega - \omega_b) + \dot{\omega} - \dot{\omega}_b]R_{rb}^T H\omega_h\tilde{T}_{rb} + && (73)\\
&\quad - 2\omega_h H^T(\omega - \omega_b)R_{rb}^T H\omega_h\tilde{T}_{rb} + && (74)\\
&\quad + 2H^T(\dot{\omega} - \dot{\omega}_b)R_{rb}^T H\omega_h\tilde{T}_{rb} + 2H^T(\omega - \omega_b)R_{rb}^T H\dot{\omega}_h\tilde{T}_{rb} + && (75)\\
&\quad - \omega_h H^T R_{rb}H[{\omega_h}^2 + \dot{\omega}_h]\tilde{T}_{rb} + H^T R_{rb}^T H\omega_h[{\omega_h}^2 - \dot{\omega}_h]\tilde{T}_{rb} + && (76)\\
&\quad + H^T R_{rb}^T H[2\omega_h\dot{\omega}_h + \ddot{\omega}_h]\tilde{T}_{rb} && (77)
\end{aligned}
$$

The norm of the left-hand side is bounded by ϵ, and so is the norm of the first term on the right-hand side. Because of the sufficient excitation conditions, the terms on the right-hand side that multiply the independent variables $\alpha, \dot{\alpha}, \omega, \dot{\omega}, \ddot{\omega}$ must be bounded by $k\epsilon$ where k is a constant arising from applying repeatedly the reverse triangular inequality. Isolating the term that multiples α, we have $\|\omega_h H^T\alpha\| \leq k\epsilon$, and therefore

$$
\|\omega_h\| \leq \frac{k\epsilon}{\max_t \|\alpha\|} \qquad (78)
$$

Isolating the terms multiplying $\dot{\alpha}$, $\|(H^T - I)\dot{\alpha}\|$ we obtain

$$\|H - I\| \leq \frac{k\epsilon}{\max_t \|\dot{\alpha}\|} \tag{79}$$

Isolating the terms multiplying $\ddot{\omega}$, $\|\ddot{\omega} R_{rb}^T (T_{rb} - H\tilde{T}_{rb})\|$ we obtain, after noticing that it equals $\|R_{rb}^T(\widehat{T}_{rb} - \widehat{H\tilde{T}_{rb}})R_{rb}\ddot{\omega}\|$,

$$\|T_{rb} - H\tilde{T}_{rb}\| \leq \frac{k\epsilon}{\max_t \|\ddot{\omega}\|} \tag{80}$$

which concludes the claim.

A corollary of the last claim is that range-augmented navigation is observable up to the initial conditions in the absence of unknown inputs. In the presence of unknown inputs, the above claim quantifies the sensitivity of the set of indistinguishable states as a function of the drift rate.

3.2 Initial conditions and Gauge ambiguities

So far we have not enforced consistency of the initial conditions. These are subject to a gauge transformation, since we can choose arbitrarily *either* the reference frame with respect to which the points X^i are expressed (for instance, having the origin at their centroid, and the orientation aligned with the principal axes of the point distribution), in which case $T(0), R(0)$ will be unknown; or, we can choose to impose that, say, $T(0) = 0$ and $R(0) = I$, in which case the points will be inferred relative to this reference frame up to an arbitrary rotation $\bar{R}$. Another alternative is to choose the reference frame where $\gamma = [0,\ 0,\ 9.8]^T$, in which case the initial conditions for the body frame are not known.

While a choice of initial reference associate to X can be enforced at each instant of time (*e.g.,* with a pseudo-measurement) since X is constant and part of the state, the initial condition attributed to $T(0), R(0)$ cannot, as the latter are not part of the state. Either way, however, this choice will force a particular value of $\bar{T}, \bar{R}$ – of course different than the true one – and reduce the indistinguishable set to a singleton, albeit possibly one other than the true trajectory, unless the initial conditions happen to coincide.

A typical way to choose a canonical reference frame for a collection of sparse reflector is to choose the origin of the reference frame to be the centroid, which can be accomplished by imposing

$$\sum_i X^i = \mathbf{X}1 = 0 \tag{81}$$

where $\mathbf{1}$ is a column vector of ones and $\mathbf{X}$ the $3 \times N$ matrix of the point coordinates, so that

$$\tilde{\mathbf{X}}1 = \bar{R}^T(\mathbf{X}1 - \bar{T}) = -\bar{R}^T\bar{T} = 0 \tag{82}$$

and choosing the coordinate axes to be aligned with the principal components

$$\mathbf{X}\mathbf{X}^T = \Lambda \tag{83}$$

for some diagonal matrix Λ, so that

$$\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T = \bar{R}^T \Lambda \bar{R} = \Lambda \tag{84}$$

forces $\bar{R} = I$ and $\bar{T} = 0$. Note that $\bar{R}$ is not an arbitrary rotation, but a rotation about gravity, based on the choice of reference frames we have adopted. Also, if the point cloud is symmetric, the principal directions are undefined.

This suggests a modeling procedure for designing a filter/observer as follows:

- Fix translation with pseudomeasurement $X1 = 0$.
- Fix rotational gauge by enforcing $XX^T = \Lambda$ (that is, elements (1,2), (1,3), (2,3) are zero), alternatively also (2,1), (3,1), (3,2) - but redundant). Or, write $X = \Lambda V$ and impose $VV^T = I$. This should also be equivalent.
- Let the biases float
- The estimated rotation should differ from the global one by an arbitrary constant rotational offset
- The resulting translation should differ from the global one by an arbitrary constant Euclidean offset
- Velocity should differ from true by a constant rotation
- (8/17/13): Define Gauge ambiguity when model is observable up to a constant, and that constant is in one-to-one correspondence to the initial condition. So although the IC cannot be uniquely determined, the state can be determined up to an equivalence class of trajectories each defined by a different initial condition
- Check v vs V for translational velocity (confusion with twists) from section 1 to 2 and seg.

4 Measurement model reduction

The measurement model $y = \pi(gX) + n$ involves navigation states $g \in SE(3)$ as well as constant unknown parameters $X \in R^{3\times N}$, that are also modeled as states. To distinguish them, we indicate the former with x and the latter with p. In that case, the measurement equation is of the form $y = h(x, p) + n$. In some cases,[7] one may be interested in reducing the model by *eliminating* the unknown states p. This can be done in an approximate manner via linearization, or in an exact manner using the geometry of the space of unknowns. For the case of bearing measurements, this has ben first done in [**?**] for the continuous-case, and [**?**, **?**, **?**] for the discrete-time case. The linearization version has ben done in [**?**] and successfully demonstrated on a cellphone for the case of vision measurements. The range case has bend equationn pionered by [**?**] for batch processing, and has so far never ben integrated into a filtering framework.

[7]After having augmented the model by *adding* the same states p.

4.1 Approximate model reduction via linearization (bearing)

We consider the linear approximation of the measurement equation:

$$y = h(x,p) = h(\hat{x},\hat{p}) + \underbrace{\frac{\partial h}{\partial p}(\hat{x},\hat{p})}_{H_p}\tilde{p} + \underbrace{\frac{\partial h}{\partial x}(\hat{x},\hat{p})}_{H_x}\tilde{x} + \underbrace{\mathcal{O}(\|\tilde{x}\|^2, \|\tilde{p}\|^2) + n}_{\tilde{n}} \tag{85}$$

where the matrices H_p and H_x are the Jacobians of the measurements with respect to the state and the unknown parameters p, and the new residual $\tilde{n}$ includes measurement noise as well as linearization error.

If the Jacobian H_p has full column rank, it is possible to eliminate the dependency of the measurement model from $\tilde{p}$ by multiplying both sides of the equation above by its orthogonal complement $H_p^{\perp}$, obtaining

$$r \doteq H_p^{\perp}(y - h(\hat{x},\hat{p})) = \underbrace{H_p^{\perp}(\hat{x},\hat{p})H_p\tilde{p}}_{=0} + H_p^{\perp}(\hat{x},\hat{p})H_x(\hat{x},\hat{p})\tilde{x} + \bar{n} \tag{86}$$

where $\bar{n} = H_p^{\perp}\tilde{n}$. Unfortunately, in neither the bearing-only nor the range-only case is the matrix H_p full column rank. In order to enable eliminating states $\tilde{p}$, [**?**] stack a number of temporal samples $y(t)$ on top of each other, so the stacked matrix H_p becomes full column rank, and reduction is made possible. Writing the contribution from individual points, we have

$$\begin{bmatrix} r^i(t_1) \\ r^i(t_2) \\ \vdots \\ r^i(t_i) \end{bmatrix} = \underbrace{\begin{bmatrix} H_p(\hat{x}(t_1),\hat{p}^i) \\ H_p(\hat{x}(t_2),\hat{p}^i) \\ \vdots \\ H_p(\hat{x}(t_i),\hat{p}^i) \end{bmatrix}^{\perp}}_{\mathbf{H}_p(\hat{x},\hat{p}^i)} \begin{bmatrix} \ddots & & \\ & H_x(\hat{x}(t),\hat{p}^i) & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \tilde{x}(t_1) \\ \tilde{x}(t_2) \\ \vdots \\ \tilde{x}(t_i) \end{bmatrix} + \begin{bmatrix} \bar{n}(t_1) \\ \bar{n}(t_1) \\ \vdots \\ \bar{n}(t_i) \end{bmatrix} \tag{87}$$

or in compact form as

$$red\mathbf{r}(t,t_i) = \mathbf{H}(\hat{x},\hat{p}) \begin{bmatrix} \tilde{x}(t_1) \\ \tilde{x}(t_2) \\ \vdots \\ \tilde{x}(t_i) \end{bmatrix} + \bar{\mathbf{n}}(t,t_i) \tag{88}$$

Note that the noise term $\bar{n}$ depends on the unknown structure $\hat{p}$ and the nominal motion states $\hat{x}$, in addition to the linearization error. So even if the measurement noise was reasonably uncorrelated, certainly $\bar{n}$ is heavily correlated, and even more so once the delay-line (sequence of temporally adjacent measurements) are considered as a batch. One could introduce additional states to model the correlation terms, but this would defeat the purpose of model reduction, which is to eliminate states.

In addition, even if we have eliminated the dependency on the error-state $\tilde{p}$ (the "structure correction" term), the Jacobians still depend on the (unknown) nominal structure $\hat{p}$. In [**?**], the current estimates of the motion states $\hat{x} + \tilde{x}$ are used to triangulate points in R^3, which is possible for the case of bearing measurements, to obtain a coarse approximation of the position of each point which is taken to be $\hat{p}^i$, and used to compute the Jacobians above. Specifically, given at least two time instants, for instance $\tau_i = 0$ and $t_i = t$, from (10) we get

$$\widehat{\bar{y}^i(t)}\bar{y}^i(0)Z^i = \tilde{n}^i(t) \tag{89}$$

that can be solved in a least-squares fashion to estimate $\hat{Z}^i$ and therefore $p^i = X^i$.

In the next section, we show how model reduction is possible in an *exact* fashion, in a manner that does not require linearization, and does not require an estimate of the structure states $\hat{p}$.

4.2 Model reduction via epipolar geometry (bearing): The Essential filter

Reducing the model for the case of bearing measurements can be done by eliminating the structure states via Epipolar geometry. Assuming the feature appears at $t = 0$, for any time $t > 0$ we have

$$y^{i^T}(t) \underbrace{R^T(t)\widehat{T}(t)}_{Q} y^i(0) = n^i(t) \tag{90}$$

Here, there is no linearization, the independence of structure is exact, there is no ned to triangulate the structure, and no ned to assemble multiple data in a batch. Since the equation above is linear in Q, we write it as

$$red\chi(t)Q(x) = n(t) \tag{91}$$

where $\chi(t) = (\bar{y}(t) \otimes \bar{y}(0))^T$ is the Kronecker product of $y(t)$ and $y(0)$. More details on the implementation of these equations are in the Appendix.

4.3 Approximate reduction via linearization (range)

The same procedure for eliminating structure states from the linearized model can be applied to range measurements, with the caveat that the matrix H_p is now a 1×3 row vector, and therefore at least 3 measurements are necessary for H_p to be full column rank (rather than 2 for the case of bearing measurements), and that the estimate of the nominal structure states $\hat{p}$ has to be performed by ranging triangulation, via (14): From $Z = \|RX + T\|$ we get that

$$\begin{aligned} Z(0)^2 &= X^T X \\ Z(t)^2 &= X^T R^T R X + T^T T + 2T^T R X \quad from\ which \\ 2T^T(t)R(t)X &= Z(t)^2 - Z(0)^2 - T^T(t)T(t) \end{aligned} \tag{92}$$

Given the nominal trajectory at at least thre instants, for instance $t = 0$ and $(\hat{R}(\tau), \hat{T}(\tau)), (\hat{R}(t), \hat{T}(t))$ we can solve the linear system

$$2 \begin{bmatrix} \hat{T}^T(\tau)\hat{R}(\tau) \\ \hat{T}^T(t)\hat{R}(t) \end{bmatrix} X = \begin{bmatrix} Z(\tau)^2 - Z(0)^2 - \hat{T}^T(\tau)\hat{T}(\tau) \\ Z(t)^2 - Z(0)^2 - \hat{T}^T(t)\hat{T}(t) \end{bmatrix} \tag{93}$$

subject to $X^T X = Z(0)^2$. Other than this modification, the rest proceds as for the case of bearing measurements only, yielding a measurement equation functionally identical to (88).

4.4 Model reduction via Stiefel projections (range)

The paper [**?**] presents a method to eliminate the unknown structure X from a batch of range measurements of sparse reflectors. This can be thought of as the geometrically-correct equivalent of the measurement model reduction described in the previous section. Assuming an initial time $\tau_i = 0$ for all points, and time $t_i = t$, we can write the measurement equations in the form

$$redA^\perp(x_1^t)Y^t = A^\perp(x_1^t)B(x_1^t) + n^t \tag{94}$$

where the superscript t indicates the time history up to time t, and Y^t gathers component-wise products of range measurements, given by

$$\begin{bmatrix} y_0 \\ \vdots \\ y_t \end{bmatrix} \doteq \begin{bmatrix} Z_0 \odot Z_0 \\ \vdots \\ Z_t \odot Z_t \end{bmatrix} \tag{95}$$

where $\odot$ denotes the component-wise product of the vector of range measurements and the matrices A and B are defined by the stacked measurement equation for the squared range measurements

$$Y^t \doteq \begin{bmatrix} y_1 - y_0 \\ \vdots \\ y_{t-1} - y_t \end{bmatrix} = 2 \underbrace{\begin{bmatrix} T_1^T R_1 - T_0^T R_0 \\ \vdots \\ T_{t-1}^T R_{t-1} - T_t^T R_t \end{bmatrix}}_{A(x_1^t)} X + \underbrace{\begin{bmatrix} \|T_1\|^2 - \|T_0\|^2 \\ \vdots \\ \|T_{t-1}\|^2 - \|T_t\|^2 \end{bmatrix}}_{B(x_1^t)} \mathbf{1}_N \tag{96}$$

and $\mathbf{1}_N$ is a (column) vector of N ones. Note that, unlike the case of epipolar geometry, this procedure requires the accrual of multiple measurements in the time series, but it still has the benefit of not having any dependence on the position of the points p, and not involving any linearization.

5 Delay Line State Augmentation

Representing groups of points via their pose results in augmenting the state with a collection of "key-poses," corresponding to time instants t_i when groups of features appear. Rather than picking key poses one could augment the state with a *delay-line*, or sliding window, a collection of adjacent time samples of the state, as suggested in [**?**]. For instance, if $x(t)$ is the state, the delay-line is the augmented state $x^m(t) = \{x(t), x(t-1), \dots, x(t-m)\}$.

To build a delay-line of length $m > 0$ starting from an initial time t_i and define the states $x_k(t) \doteq x(t - kdt)$ for $k = 1, \dots, m$. Assuming a constant inter-frame sampling $dt > 0$, for a linear model of the form $\dot{x} = f(x) + c(x)u = Ax + Bu$, starting at $t \geq t_i + m$ we have:

$$\begin{cases} arrayll x_m(t+dt) = x_{m-1}(t) & \mathrm{x}_m(t) = x(t_i) \qquad x_{m-1}(t+dt) \\ x_{m-1}(t) = Ax(t_i) + Bu(t_i) \qquad \vdots & \mathrm{x}_{m-k}(t+dt) = x_{m-k} \\ x_{m-k}(t) = A^k x(t_i) + \sum_{j=1}^{k-1} A^{k-j} Bu(t_i + jdt) \qquad \vdots & \mathrm{x}_2(t+dt) = x_1(t \\ x_2(t) = A^{m-2} x(t_i) + \sum_{j=1}^{m-3} A^{m-1-j} Bu(t_i + jdt) x_1(t+dt) = x(t) & \mathrm{x}_1(t) = A^{m-1} x(t_i) + \sum_{j=1}^{m-2} A^{m-1-j} Bu(\\ x(t) = A^m x(t_i) + \sum_{j=1}^{m-1} A^{m-1-j} Bu(t_i + jdt) \end{cases}$$

(97)

The initial conditions on the right can be computed recursively. In the non-linear case, this is implemented as m repeated steps of the integral $x(t+dt) = x(t) + \int_t^{t+dt} f(x)d\tau + \int_t^{t+dt} c(x)du(\tau) \doteq F(x,t) + B(x,u,t)$, each step defining the initialization of one delay block: Assuming an initial time t_i, then at time $t = t_i + m$, we have

$$\begin{aligned} x_m(t) &= x(t_i) && (98) \\ x_{m-1}(t) &= F(x_m, t) + B(x_m, u, t) && (99) \\ x_{m-2}(t) &= F(x_{m-1}, t) + B(x_{m-1}, u, t) && (100) \\ &\vdots && (101) \\ x_1(t) &= F(x, t) + B(x, u, t) && (102) \end{aligned}$$

Once initialized, the model evolves according to () for an augmented state and linearization

$$x^m(t) = \begin{bmatrix} x_m(t) \\ x_{m-1}(t) \\ \vdots \\ x_1(t) \\ x(t) \end{bmatrix}, \quad F = \begin{bmatrix} 0 & I & & & \\ & 0 & I & & \\ & & \ddots & & \\ & & & 0 & I \\ 0 & \dots & & 0 & \frac{\partial f}{\partial x}(x(t)) \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{\partial c}{\partial x}(x(t)) \end{bmatrix} \tag{103}$$

Starting from $t > t_i + m$, we have measurements available, of the form

$$\begin{aligned}
y(t-m) &= y(t_i) = h(x_m(t)) + n(t_i) &(104)\\
y(t-m+1) &= y(t_i+1) = h(x_{m-1}(t)) + n(t_i+1) &(105)\\
&\vdots &(106)\\
y(t-1) &= y(t_i+m-1) = h(x_1(t)) + n(t_i+m-1) &(107)\\
y(t) &= y(t_i+m) = h(x(t)) + n(t_i+m) &(108)
\end{aligned}$$

which we could collate into an augmented measurement equation $y^m(t)$. However, this measurement cannot be used at every time instant t, as the sliding window would result in highly (temporally) correlated noise n^m. Instead, it must be used at block intervals of km time instants, at which point it provides constraints on each of the states $x^m(t)$. Alternatively, a batch of measurements $y^m(t)$ can be used only once, at time $t = t_i + m$, to provide measurement constraints on the augmented states, as done in [**?**].

5.1 Multi-State Constraints with Epipolar Geometry

The epipolar constraint can be interpreted as an implicit measurement equation involving inter-frame pose, regardless of the position of points. If $X(t) = R^T(t)(X_0 - T(t))$, the Essential matrix at time t is of the form $Q(t) = R^T(t)\widehat{T}(t)$.[8] The inter-frame Essential matrix, $Q(t_2, t_1)$ is then the Essential matrix determined by the motion between t_1 and t_2,[9] that has a rotational component $R(t_2, t_1) \doteq R(t_2)^T R(t_1)$ and a translational component $T(t_2, t_1) = R^T(t_2)(T(t_1) - T(t_2))$. Therefore, $Q(t_2, t_1) = R^T(t_1)R(t_2)[R^T(t_2)(\widehat{T(t_1) - T(t_2)}]$, or equivalently

$$Q(t_2, t_1) = R^T(t_1)[\widehat{T}(t_1) - \widehat{T}(t_2)]R(t_2) \tag{109}$$

independent of the choice of spatial frame, since the transformation from the the camera frame at t_1 to the spatial frame is annihilated by the transformation from the spatial frame to the camera frame at t_2. In the presence of an unknown alignment transformation $g_{cb} = (R_{cb}, T_{cb})$ one can easily verify that

$$Q(t_2, t_1; R_{cb}, T_{cb}) = R^T(t_1)[\widehat{T}(t_1) - \widehat{T}(t_2) + \widehat{R(t_2)R_{cb}^T}T_{cb}]R(t_2)R_{cb}^T \tag{110}$$

or, equivalently,

$$Q(t_2, t_1; R_{cb}, T_{cb}) = Q(t_2, t_1)R_{cb}^T + R^T(t_1)R(t_2)R_{cb}^T\widehat{T}_{cb} \tag{111}$$

Now, if we grow a batch of m measurements starting from an initial t_i up to time $t = t_i + m$, we have, calling $\chi(t_1, t_2) \doteq \chi(y(t_1), y(t_2))$ and $x \doteq x(t); x_k \doteq x_k(t)$,

$$\chi(t, t-m)Q(x, x_m) \quad = \quad \chi(t_i+m, t_i)Q(x(t_i+m), x(t_i)) = 0 \tag{112}$$

[8] Note that if the motion model was instead $X(t) = R(t)X_0 + T(t)$, then the Essential matrix would have the form $Q(t) = \widehat{T}(t)R(t)$. The equivalence between the two can be arrived at by noticing that $\widehat{R^T T} = R^T\widehat{T}R$.

[9] From $X(t_1) = R^T(t_1)(X_0 - T(t_1))$ and $X(t_2) = R^T(t_2)(X_0 - T(t_2))$.

$$\begin{aligned}
\chi(t,t-m+1)Q(x,x_{m-1}) &= \chi(t_i+m,t_i+1)Q(x(t_i+m),x(t_i+1))=0 && (113)\\
&\vdots && (114)\\
\chi(t,t-m+k)Q(x,x_{m-k}) &= \chi(t_i+m,t_i+k)Q(x(t_i+m),x(t_i+k))=0 && (115)\\
&\vdots && (116)\\
\chi(t,t-1)Q(x,x_1) &= \chi(t_i+m,t_i+m-1)Q(x(t_i+m),x(t_i+m-1))=0 && (117)
\end{aligned}$$

References

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1 Description of Functions

The delay line of length $m > 0$ is defined by the following dynamical model:

$$\{ x_m(t+dt) = x_{m-1}(t) x_{m-1}(t+dt) = x_{m-2}(t) \vdots x_2(t+dt) = x_1(t) x_1(t+dt) = x(t) \dot{x} = f(x)+c(x)u \quad (118)$$

To initialize it, let $x(t+dt) = x(t) + \int_t^{t+dt} f(x)d\tau + \int_t^{t+dt} c(x)du(\tau) \doteq F(x,t) + B(x,u,t)$. Assuming an initial time t_i, then at time $t = t_i + m$, we have

$$\begin{aligned}
x_m(t) &= x(t_i) && (119)\\
x_{m-1}(t) &= F(x_m,t)+B(x_m,u,t) && (120)\\
x_{m-2}(t) &= F(x_{m-1},t)+B(x_{m-1},u,t) && (121)\\
&\vdots && (122)\\
x_1(t) &= F(x,t)+B(x,u,t) && (123)
\end{aligned}$$

The Jacobian matrices are given by

$$F = \begin{bmatrix} 0 & I & & & \\ & 0 & I & & \\ & & \ddots & & \\ & & & 0 & I \\ 0 & \dots & & 0 & \frac{\partial f}{\partial x}(x(t)) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{\partial c}{\partial x}(x(t)) \end{bmatrix} \quad (124)$$

The function `predict navistate.m` computes $\frac{\partial f}{\partial x}(x(t))$, and the function `predict augmented.m` computes the prediction and covariance propagation for the extended state above.

```
function [xip,Pp,F] = predict_augmented(xi,u,gm,P,Rv);
Tb = (length(xi)-15)/6;
F = sparse(zeros(size(P)));
for k = 1 : Tb,
      xip(6*(k-1)+1:6*k,1) = xi(6*k+1:6*(k+1));
      F(6*(k-1)+1:6*(k-1)+6,6*(k)+1:6*(k)+6) = eye(6);
end;
```

```
[xip(6*Tb+1:6*Tb+15,1),ptmp,F(6*Tb+1:6*Tb+15,6*Tb+1:6*Tb+15)] = predict_navistate(xi(6*Tb+1:6*Tb+15,1),u,gm,P(6*Tb+1:6*Tb+15,6*Tb+1:6*Tb+15),Rv);
% prediction
Rbv = sparse(zeros(size(P)));
Rbv(6*Tb+1:6*Tb+15,6*Tb+1:6*Tb+15) = Rv;
Pp = F * P * F' + Rbv;

function [xip,Pp,F] = predict_navistate(xi,u,gm,P,Rv);
T = xi(1:3,1);
Om = xi(4:6,1);
[R, dRdOm] = rodrigues(Om);
V = xi(7:9,1);
alphabias = xi(10:12,1);
ombias= xi(13:15,1);
omimu = u(1:3,1);
Rimu = rodrigues(omimu);
alphaimu = u(4:6,1);
om = omimu - ombias;
[Rt,dRtdomt] = rodrigues(om);
dRtdombias = -dRtdomt;
% prediction
Tp = T + V;
Rp = R * Rt;
[Omp,dOmpdRp] = rodrigues(Rp);
[Rp,dRpdOmp] = rodrigues(Omp);
Vp = V + R * (alphaimu - alphabias) + gm;
xip(1:3,1) = Tp;
xip(4:6,1) = Omp;
xip(7:9,1) = Vp;
xip(10:15,1) = xi(10:15,1);
F = eye(15);  % linearization of state-transition function
F(1:3,7:9) = eye(3); % dTpdV
F(4:6,4:6) = dOmpdRp*dABdA(R,Rt)*dRdOm; % dOmpdOm
F(4:6,13:15) = dOmpdRp*dABdB(R,Rt)*dRtdombias; %dOmdombias
F(7:9,4:6) = dABdA(R,alphaimu-alphabias)*dRdOm;
F(7:9,10:12) = -R; %dVdalphabias
Pp = F * P * F' + Rv;
```

The measurement equation is given, for $t > m$, by

$$\{ \chi\ (y(t), y(t{-}m))Q(x(t), x_m(t)) = n(t{-}m)\chi(y(t), y(t{-}m{+}1))Q(x(t), x_{m-1}(t)) = n(t{-}m{+}1) \ \vdots \chi(y(t), y(t{-}1))Q \tag{125}$$

which is in the form $0 = \chi Q + n$, so the innovation is $inn(t) = -\chi Q$. The linearization has the structure

$$C = \begin{bmatrix} \chi((y(t), y(t-k))\frac{\partial Q}{\partial x_{t-k}} & & & \chi((y(t), y(t-k))\frac{\partial Q}{\partial x_t} \\ & \ddots & & \\ & & \chi(y(t), y(t-1))\frac{\partial Q}{\partial x_{t-1}} & \chi(y(t), y(t-1))\frac{\partial Q}{\partial x_t} \end{bmatrix} \tag{126}$$

which is computed in the function `inessential filter augmented.m`, with the individual blocks $\chi(y(t), y(t-1))\frac{\partial Q}{\partial x_t}$ computed in the function `epipolar innovation linearization.m`. Let $y_1 = y(t{-}k)$ the first instant, and $y_2 = y(t)$ the second (most recent); similarly for x_1, x_2. Then

$$\chi(y_2, y_1)Q(x_2, x_1) = (y_2 \otimes y_1)^T Q(x_2, x_1) \tag{127}$$

where $\otimes$ is the kronecker product. In Matlab, each row of χ is given by `kron(y2,y1)'`. Note, however, that the function `makeChi` takes the arguments in reverse order, that is `makeChi(y1,y2) = kron(y2,y1)'`. The inter-frame Essential matrix is computed as follows: if x_2 has pose components (R_2, T_2) and x_1 has pose states (R_1, T_1), they are related by:

$$X_2 = R_2^T(X_0 - T_2) \Rightarrow X_2 = R_2^T(R_1 X_1 + T1 - T_2) = R_2^T R_1(X_1 + R_1^T(T_1 - T_2)) \tag{128}$$

$$X_1 = R_1^T(X_0 - T_1) \Rightarrow X_0 = R_1 X_1 + T_1 \tag{129}$$

which can be written as

$$X_2 = (\underbrace{R_1^T R_2}_{R})^T \left(X_1 - \underbrace{R_1^T (T_2 - T_1)}_{T} \right) \tag{130}$$

from which the Essential matrix

$$Q = R^T \widehat{T} = (R_1^T R_2)^T [R_1^T (\widehat{T_2 - T_1})] = R_2^T R_1 R_1^T [\widehat{T_2 - T_1}] R_1 = R_2^T [\widehat{T_2 - T_1}] R_1 \tag{131}$$

from which the innovation

$$inn = -\chi Q = -(y_2 \otimes y_1)^T R_2^T [\widehat{T_2 - T_1}] R_1 \tag{132}$$

and the linearization is computed in `epipolar innovation linearization.m`.

```
function [inn,C2,Chi,Q,C1] = epipolar_innovation_linearization(y2,y1,xi2,xi1,xihat2,xihat1);
Chi = makeChi(y1,y2,1);
[T1tildehat,dT1hatdT1] = skew3(xi1(1:3,1)); hatT1hat = skew3(xihat1(1:3,1)); T1hat = hatT1hat + T1tildehat;
[T2tildehat,dT2hatdT2] = skew3(xi2(1:3,1)); hatT2hat = skew3(xihat2(1:3,1)); T2hat = hatT2hat + T2tildehat;
[R1tilde,dR1dOm1] = rodrigues(xi1(4:6,1)); R1hat = rodrigues(xihat1(4:6,1)); R1 = R1hat*R1tilde;
[R2tilde,dR2dOm2] = rodrigues(xi2(4:6,1)); R2hat = rodrigues(xihat2(4:6,1)); R2 = R2hat*R2tilde;
T2hat_T1hat = T2hat - T1hat;
R2pT2hat_T1hat = R2'*T2hat_T1hat;
Q = R2pT2hat_T1hat*R1;
T2hat_T1hatR1 = T2hat_T1hat*R1;
dqdOm1 = dABdB(R2pT2hat_T1hat,R1)*dABdB(R1hat,R1tilde)*dR1dOm1;
dqdT1 = -dABdB(R2',T2hat_T1hatR1)*dABdA(T2hat_T1hat,R1)*dT1hatdT1;
dqdOm2 = dABdA(R2',T2hat_T1hatR1)*dAtdA(R2)*dABdB(R2hat,R2tilde)*dR2dOm2;
dqdT2 = dABdB(R2',T2hat_T1hatR1)*dABdA(T2hat_T1hat,R1)*dT2hatdT2;
dqdxi1 = [dqdT1 dqdOm1];
dqdxi2 = [dqdT2 dqdOm2];
C1 = Chi*dqdxi1;
C2 = Chi*dqdxi2;
inn = -Chi*reshape(Q',9,1);

function [xip,Pp,inn,xbp,Pbp,innb] = inessential_filter_augmented(y,y0,u,xi,P,Rn,Rv,gm,Ttrue,xb,Pb,yb);
N = size(y0,2);
Tb = (length(xb)-15)/6; % number of frames in the batch
[xbpp,Pbpp] = predict_augmented(xb,u,gm,Pb,Rv);
[xbp,Pbp,innb,Cb] = update_augmented(xbpp,Pbpp,y,yb,y0,Rn,Ttrue);
xip = xbp(6*Tb+1:6*Tb+15,1);
Pp = Pbp(6*Tb+1:6*Tb+15,6*Tb+1:6*Tb+15);
inn(1:N,1) = innb(Tb*N+1:Tb*N+N,1);

function [xbp,Pbp,innb,Cb] = update_augmented(xb,Pb,yt,yb,y0,Rn,Ttrue);
Tb = (length(xb)-15)/6; % number of frames in the batch
N = size(yt,2);
xi2 = xb(6*Tb+1:6*Tb+6,1); % current time = end of sliding window
Cb = zeros((N+1)*Tb,6*Tb+15);
for k = 1 : Tb, % compute epipolar geometry from t to t-k
    y1 = reshape(yb(2*N*(k-1)+1:2*N*k,1),2,N); % y(t-(Tb+k))
    xi1 = xb(6*(k-1)+1:6*k,1);
    [innt,C2, Chib,Qb,C1] = epipolar_innovation_linearization(yt,y1,xi2,xi1);
    innb(N*(k-1)+1:N*k,1) = innt;
    Cb(N*(k-1)+1:N*k,6*(k-1)+1:6*k) = C1;
    Cb(N*(k-1)+1:N*k,6*Tb+1:6*Tb+6) = C2;
end;
% additional measurement to time zero
[innt,C2] = epipolar_innovation_linearization(yt,y0,xb(6*Tb+1:6*Tb+15),zeros(6,1));
innb(N*Tb+1:N*Tb+N,1) = innt;
Cb(N*Tb+1:N*Tb+N,6*Tb+1:6*Tb+6) = C2;
sigma = mean(diag(Rn));
Rnn = sigma*eye(N*(Tb+1));
[xbp, Pbp,Lb] = update(xb, -innb, Pb, Rnn, Cb);
```

2 Effects of Measurement Model Reduction on Process Noise

The "signal-plus-noise" measurement model

$$y = h(x) + n \tag{133}$$

can be interpreted as a relation between *random variables* y, x, n, or between their *realizations* (samples). To make the distinction clear, we indicate random variables in capitals, Y, X, N, their realizations in lower case, and the samples with a superscript $y^{(i)}, x^{(i)}, n^{(i)}$. Then (133) can be written as $Y = h(X) + N$.

When we view (133) as a relation between realizations, given samples $n^{(i)} \sim p_N$ and $x^{(i)} \sim p_X$, assumed independent, the model tells us that a sample $y^{(i)}$ is obtained via $y^{(i)} = h(x^{(i)}) + n^{(i)}$. When we view (133) as a relation between random variables, the model tells us that the densities p_X, p_N (assuming X, N are independent) and p_Y satisfy the functional relation $p_Y(y) = \int p(y|x)dP(x) = \int p_N(y - h(x))p_X(x)dx$.

Model reduction is often performed by using the model above (133) to "solve" for some states (components of x, say x_i) as a function of y, substituting the result in the dynamical model, thus eliminating x_i and writing the remaining states $x_j,\ j \neq i$ as a function of y. Such *"pivoting"* is problematic from a statistical point of view. It consists of "fixing" a random variable to be equal to its sample value, and therefore alters the joint distribution of the other variables involved.

For instance, consider the simple case $h(x) = x$: The model (133) can be written equivalently as

$$y = x + n \quad \textit{and} \quad Y = X + N \tag{134}$$

if we "pivot" on x by writing it in terms of a *sample* of y, $y^{(i)}$, we obtain

$$x \doteq y^{(i)} - n \qquad \textit{but not} \qquad X = y - N. \tag{135}$$

To see that,

$$p_X(x) = \int p(x|y)dP(y) = \int p_N(y - x)\delta(y - y^{(i)})dy = p_N(y^{(i)} - x) \neq p_X(x) \tag{136}$$

in general, since we have made no assumptions on p_X or p_N, that can be arbitrary probability density functions. While it is possible to write $X = Y - N$, the two random variables on the right-hand side are not independent (unlike in $Y = X + N$), and therefore $Y - X|X \sim p_N$, but $Y - X|Y \neq p_N$. For p_X to be properly computed, one would have to average over all possible instances of $y^{(i)}$; that is, one would have to *marginalize* the random variable Y. In essence, pivoting consists of replacing marginalization of Y with a single sample from it.

The process is even more problematic if h is non-linear, for in that case the pivoting not only alters the distribution of X, but also that of N. For instance, assuming h to be invertible and n to be white, zero-mean, IID, homoscedastic and independent of x, it may be tempting to write

$$y = h(x) + n \Longrightarrow x = h^{-1}(y - n) \simeq h^{-1}(y) + \tilde{n} \tag{137}$$

and to consider $\tilde{n}$ to also be white, zero-mean, IID and homoscedastic. This is clearly not the case, as

$$x \doteq h^{-1}(y - n) \simeq h^{-1}(y) - \underbrace{J_h^{-1}(h(x))n}_{\tilde{n}} \tag{138}$$

where $J_h \doteq \frac{\partial h}{\partial x}$ is the Jacobian matrix of h, computed at $y = h(x)$. Therefore, not only is $\tilde{n}$ not white, but it also introduces depencies with x, thus breaking the signal-plus (independent) noise model.

In model reduction, one is often interested in reducing a mixed filtering (inferring x) and identification (inferring p):

$$\{ \dot{} x = f(x,p) + n_x y = h(x,p) + n_y \tag{139}$$

into a filtering problem, by inserting p into the state

$$\{ \dot{} x = f(x,p) + n_x \dot{p} = 0 y = h(x,p) + n_y \tag{140}$$

and then eliminating p from the measurement equation. Assume the measurements are broken into two components, in such a way that one enables pivoting to eliminate p (that is, h_2 is invertible). This can be assumed without loss of generality, assuming the model above is observable, by augmenting the state with the output delay line):

$$\{ \dot{} x = f(x,p) + n_x \dot{p} = 0 y_1 = h_1(x,p) + n_1 y_2 = h_2(x,p) + n_2 \tag{141}$$

Now pivot on y_2 to eliminate $p = h_2^{-1}(x, y_2 - n_2)$, thus obtaining the following reduced model, where part of the output is now interpreted as an input, with a known component y_2 and an unknown component n_2

$$\{ \dot{} x = f(x, h_2^{-1}(x, y_2 - n_2)) + n_x y_1 = h_1(x, h_2^{-1}(x, y_2 - n_2)) + n_1 \tag{142}$$

This can now written as a model with input, with a Taylor series expansion about $\tilde{p} = h_2^{-1}(x, y_2) \doteq h_2^{-1}(x, u)$

$$\{ \dot{} x = \tilde{f}(x,u) + \tilde{n}_x y_1 = \tilde{h}(x,u) + \tilde{n}_y \tag{143}$$

where $\tilde{f}(x,u) \doteq f(x, h_2^{-1}(x,u)), \tilde{h}(x,u) \doteq h_1(x, h_2^{-1}(x,u))$ and the (now state-dependent) noises neglect higher-order terms in the linearization:

$$\tilde{n}_x = \frac{\partial f}{\partial p}(x, h_2^{-1}(x,u)) n_2 + n_x \tag{144}$$

and

$$\tilde{n}_y = \frac{\partial h_1}{\partial p}(x, h_2^{-1}(x,u)) n_2 + n_1 \tag{145}$$

If the parameter only enters in the measurement equation,

$$\{ \dot{} x = f(x) + n_x y = h(x,p) + n_y \tag{146}$$

it is possible to eliminate it without explicit pivoting by considering an invariant statistic ϕ such that $\phi \circ h(x,p) = \phi \circ h(x, \tilde{p})$ for any $p, \tilde{p}$ and for any x. Clearly a trivial example is $\phi \circ h = 0 \ \forall \ x, p$, but this is not viable, as ϕ (as a function of x, p) must satisfy the conditions of the implicit function theorem in order to uniquely constrain p as a function of x, so the Jacobian of $\phi \circ h$ with respect to x has to be non-singular (transversality conditions). In general, however, we cannot compute $\phi \circ h$, because of the noise n_y:

$$\{ \dot{x} = f(x) + n_x \phi(y) = \phi(h(x,p) + n_y) \tag{147}$$

In general, ϕ is non-linear, and therefore we do not have $\phi(y) = \phi \circ h(x,p) + \phi(n_y)$. But even if this was the case, note that ϕ does, in general, depend on x (because of the transversality conditions). Therefore, we would again have a state-dependent residual $\tilde{n}_y \doteq \phi(n_y)$.

A particularly simple case can be had when $h(x,p)$ is linear in p, $h(x,p) = H(x)p$, in which case, assuming H has full column rank (which again can always be assumed without loss of generality at the cost of augmenting the state with the output delay line), we have that

$$\phi(y) = H^{\perp}(x)y = H^{\perp}(x)n_y \tag{148}$$

and we have eliminated the dependency on p, at the expense of having a state-dependent noise $\tilde{n}_y = H^{\perp}(x)n_y$.

As an approximation of the above approach, one can consider a linearization of the measurement equation, assuming *some* estimate of the parameter p being available, based on the state x, $\hat{p} = \hat{p}(x)$,

$$y = h(x,p) + n \simeq h(x, \hat{p}(x)) + \underbrace{\frac{\partial h}{\partial p}(x, \hat{p}(x))}_{J_h} \delta p + \tilde{n}(x) \tag{149}$$

where $\tilde{n}(x)$ includes the linearization error, and from this, assuming that the Jacobian J_h is full column rank,

$$J_h^{\perp}(x, \hat{p}(x))y \simeq h(x, \hat{p}(x)) + \underbrace{J_h^{\perp}(x, \hat{p}(x))\tilde{n}(x)}_{\tilde{n}_y(x)}. \tag{150}$$

Note that in this case the dependency on x is not only through the Jacobian, but also on the function $\hat{p}(x)$.