1 Introduction

The work described here is the result of a collaboration of Andrea Censi, who

1.1 Previous Work

2 Formalization

Definition 2.1 (Policy)**.** A decision policy is a tuple $\langle \mathcal{C}, \mathcal{U}, T, \mathcal{Y} \rangle$, where $\mathcal{U}$ is a set of actions, $\mathcal{Y}$ is a set of observations, $\mathcal{C}$ is the set of finite sequences in $\mathcal{Y}$, and and $T : \mathcal{C} \to \mathcal{U} \cup \{\epsilon\}$.

A given

Definition 2.2 (Finite State Machine)**.** A finite state machine is a tuple

$$\langle \Sigma, \Gamma, S, S_0, \delta, \omega \rangle ,$$

where Σ and Γ are finite input and ouput alphabets, S is a finite set of states, $s_0 \in S$ is an initial state, $\delta : S \times \Sigma \to S$ is a state-transition function, and $\omega : S \times \Sigma \to \Gamma$ is an output function.

this generalizes to

Definition 2.3 (Incompletely-Specified Finite State Machine (ISFSM))**.** An incompletely-specified finite state machine is a tuple

$$\langle \Sigma, \Gamma, S, s_0, \delta, \omega \rangle ,$$

where Σ and Γ are finite input and ouput alphabets, S is a finite set of states, $s_0 \in S$ is an initial state, $\delta : S \times \Sigma \to S \cup \{\phi\}$ is a state-transition function, and $\omega : S \times \Sigma \to \Gamma \cup \{\epsilon\}$ is an output function. The extra symbols $\phi \notin S$ and $\epsilon \notin \Gamma$ denote "unspecified" outputs, whose associated input and state are not expected to occur.

Example 2.1 (Equivalent Decision Tables)**.** Suppose $\mathcal{Y} = \{1, 2\}$, $\mathcal{U} = \{A, B\}$, $\mathcal{C} = \bigcup_{i=1}^{2} \mathcal{Y}^i$ and

$$\mathcal{T}(c) = \begin{cases} A & c \in \{(1), (1, 2)\} \\ B & c \in \{(2), (2, 1)\} \\ \mathcal{T}'(c) & \text{otherwise.} \end{cases} ,$$

is an optimal policy, for arbitrary $\mathcal{T}' : \mathcal{C} \to \mathcal{U}$. However, depending on the choice of $\mathcal{T}'$ (highlighted in the tables below), the completed policy can have differently-sized minimal representations:

Instead, we propose an "incompletely-determined" formalization:

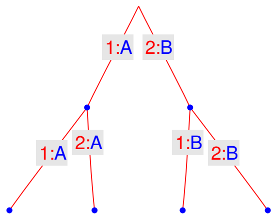
Definition 2.4 (Policies)**.** Given a set $\mathcal{Y}$ of observations, recursively construct

$$\mathcal{C}_0 = \{\emptyset\} \quad \text{and} \quad \mathcal{C}_{i+1} = \{(c, y_{i+1}) : c \in \mathcal{C}_i,\, y_{i+1} \in \mathcal{Y}_c\}, \tag{1}$$

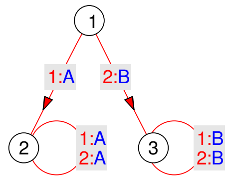
where $\mathcal{Y}_c \subseteq \mathcal{Y}$ are the observations that may be seen in context c. Let $\mathcal{C} = \bigcup_{i=0}^{\infty} \mathcal{C}_i$. A **policy** P is then a tuple $\langle \mathcal{C}, \mathcal{U}, \mathcal{T}, \mathcal{Y} \rangle$, where $\mathcal{U}$ is some decision set and $\mathcal{T} : \mathcal{C} \setminus \{\emptyset\} \to \mathcal{U}$.

$\mathcal{T} : \mathcal{C} \to \mathcal{U}$	
(1)	$\mapsto A$
(2)	$\mapsto B$
$(1,1)$	$\mapsto A$
$(1,2)$	$\mapsto A$
$(2,1)$	$\mapsto B$
$(2,2)$	$\mapsto B$

(a) Decision Table



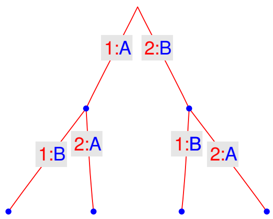
(b) Decision Tree



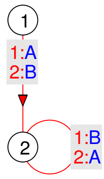
(c) Minimal Policy

$\mathcal{T} : \mathcal{C} \to \mathcal{U}$	
(1)	$\mapsto A$
(2)	$\mapsto B$
$(1,1)$	$\mapsto B$
$(1,2)$	$\mapsto A$
$(2,1)$	$\mapsto B$
$(2,2)$	$\mapsto A$

(d) Decision Table



(e) Decision Tree



(f) Minimal Policy

Definition 2.5 (Completely-Determined Policies)**.** If $\mathcal{Y} = \bigcup_{\mathcal{C}} \mathcal{Y}_c$ and $\mathcal{C} = \mathcal{Y}^{\leq n}$ for some $n \in \mathbb{N}$, then $P = \langle \mathcal{C}, \mathcal{U}, \mathcal{T}, \mathcal{Y} \rangle$ is **completely determined**. A **completion** of P is a policy $P' = \langle \mathcal{C}', \mathcal{U}', \mathcal{T}', \mathcal{Y} \rangle$ such that

$$\mathcal{Y} \subseteq \mathcal{Y}', \quad \mathcal{C}' = \bigcup_{i=0}^{\infty} (\mathcal{Y}')^i, \quad \mathcal{U} \subseteq \mathcal{U}', \quad \text{and} \quad \mathcal{T}'|_{\mathcal{C}} = \mathcal{T}. \tag{2}$$

Let $\mathrm{Comp}(P)$ be the set of completions of the policy P.

Definition 2.6 (FSM Representations)**.** An **FSM representation** (or just **representation**) is a tuple $\langle \mathcal{C}, \mathcal{R}, \mathcal{U}, \mathcal{S}, \mathcal{T}, \mathcal{Y} \rangle$ (abbreviated to $\langle \mathcal{R}, \mathcal{S} \rangle$ when $P = \langle \mathcal{C}, \mathcal{U}, \mathcal{T}, \mathcal{Y} \rangle$ is given), with "states" $\mathcal{S} \subseteq \mathbb{N}$ and state assignments $\mathcal{R} : \mathcal{C} \to \mathcal{S}$, such that

$$\mathcal{R}(c) = \mathcal{R}(c') \quad \text{and} \quad y \in \mathcal{Y}_c \cap \mathcal{Y}_{c'} \implies \mathcal{T}(c, y) = \mathcal{T}(c', y). \tag{3}$$

Let $\mathrm{Rep}(P)$ be the set of representations of the policy P.

Definition 2.7 (Minimal Representations)**.** The **size** of an FSM representation is the cardinality of its state set. A representation $\langle \mathcal{R}, \mathcal{S} \rangle$ of P is **minimal** if $|\mathcal{S}| = \min\{|\mathcal{S}'| : \langle \mathcal{R}', \mathcal{S}' \rangle \in \mathrm{Rep}(P)\}$. A representation $\langle \mathcal{R}', \mathcal{S}' \rangle$ is a **reduction** of the representation $\langle \mathcal{R}, \mathcal{S} \rangle$ if there is a surjection $\phi : \mathcal{S} \to \mathcal{S}'$ such that $\mathcal{R}' = \phi(\mathcal{R})$.

Example 2.2. If $\mathcal{C} = \{c_1, c_2, \ldots\}$, then we have a canonical representation $\langle \mathcal{R}, \mathcal{S} \rangle$, where

$$\mathcal{S} = \{1, \ldots, |\mathcal{C}|\} \quad \text{and} \quad \mathcal{R} : c_k \mapsto k. \tag{4}$$

It can be shown that the size of a minimal representation of a policy P is equal to the minimum size of the minimal representations of its completions, i.e.

$$\min\{|\mathcal{S}'| : \langle \mathcal{R}', \mathcal{S}' \rangle \in \mathrm{Rep}(P)\} = \min\{|\mathcal{S}'| : \langle \mathcal{R}', \mathcal{S}' \rangle \in \mathrm{Rep}(P'),\ P' \in \mathrm{Comp}(P)\} \tag{5}$$

Incompletely-determined policies allow more freedom in representation reduction, as shown in the next example.

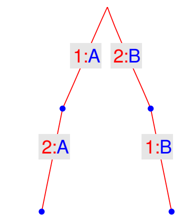
Example 2.3 (Incompletely-Determined Policies)**.** Let $\mathcal{C} = \{\emptyset, (1), (2), (1,2), (2,1)\}$, $\mathcal{U} = \{A, B\}$, and

$$\mathcal{T}(c, y) = \begin{cases} A & c \in \{(1), (1,2)\} \\ B & c \in \{(2), (2,1)\} \end{cases}.$$

Observe that the minimal policy is the same as that of the completely-determined policy in Example 1(f).

$\mathcal{T} : \mathcal{C} \setminus \{\emptyset\} \to \mathcal{U}$	
(1)	$\mapsto A$
(2)	$\mapsto B$
$(1,2)$	$\mapsto A$
$(2,1)$	$\mapsto B$

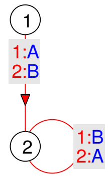
(a) Decision Table



(b) Decision Tree

$\mathcal{R} : \mathcal{C} \to \mathcal{S}$	
$\emptyset$	$\mapsto 1$
(1)	$\mapsto 2$
(2)	$\mapsto 2$
$(1,2)$	$\mapsto 2$
$(2,1)$	$\mapsto 2$

(c) Representation



(d) Minimal Policy

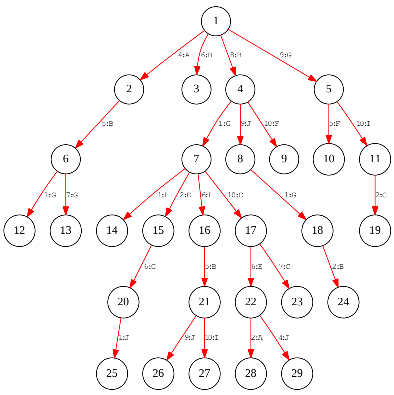
2.1 FSM Reduction

Given a decision policy (or an ISFSM) how do we find an obedient (or equivalent) ISFSM with the smallest possible state set?

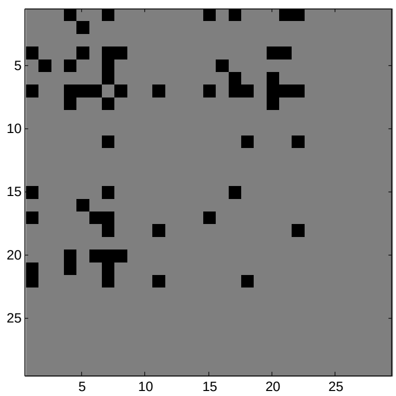
for completely-specified FSM, this is

3 Representation Reduction Strategies

To find a minimum representation of a given policy, we first compute a graph of reducibility relations, then compute a minimal clique-covering.

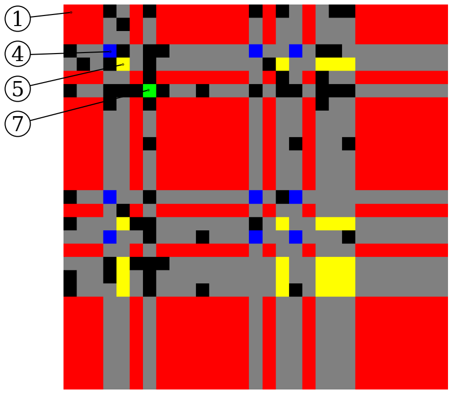


(a) Canonical Representation

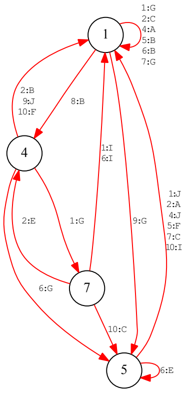


(b) Compatibility Matrix

For practical computation of reducibility, we'll start with the weaker condition of compatibility.



(c) Greedy Clique Covering of 4(b)



(d) Reduced Representation

Figure 4: *Greedy Reduction Algorithm* A "running clique" is kept, to which new states are added until all remaining states are incompatible with at least one state in the clique. Then, a new clique is begun. Here, the cliques are $\{1, 2, 3, 6, 8, 9, 10, 11, 12, 13, 14, 16, 19, 22, 26, 27, 28, 29, 30, 31, 32\}$ (red), $\{4, 15, 18\}$ (blue), $\{5, 17, 21, 22, 23\}$ (yellow), and $\{7\}$ (green). (d) shows the resulting policy graph once

3.1 Reducibility Relations

Definition 3.1 (Reducibility)**.** For a given policy $P = \langle \mathcal{C}, \mathcal{U}, \mathcal{T}, \mathcal{Y} \rangle$, two contexts $c_1, c_2 \in \mathcal{C}$ are **reducible** (write $c_1 \sim c_2$) if there exists a representation $\langle \mathcal{R}, \mathcal{S} \rangle$ of P such that $\mathcal{R}(c_1) = \mathcal{R}(c_2)$. Likewise, for a given representation $R = \langle \mathcal{R}, \mathcal{S} \rangle$, two states $s_1, s_2 \in \mathcal{S}$ are **reducible** if there exists a reduction $(\phi, \langle \mathcal{R}', \mathcal{S}' \rangle)$ of R such that $\phi(s_1) = \phi(s_2)$.

Observe that for any representation $\langle \mathcal{C}, \mathcal{R}, \mathcal{U}, \mathcal{S}, \mathcal{T}, \mathcal{Y} \rangle$, the contexts $c_1, c_2 \in \mathcal{C}$ are reducible if and only if the states $\mathcal{R}(c_1)$ and $\mathcal{R}(c_2)$ are reducible. Observe also that for incompletely-determined policies, reducibility is a symmetric but not-necessarily-transitive relation

Example 3.1 (Non-Transitive Reducibility)**.** Suppose $\mathcal{Y} = \{1, 2, 3\}$, $\mathcal{C} = \{\emptyset, (1), (2), (1, 3), (2, 3)\}$, $\mathcal{U} = \{A, B\}$, and

$$\mathcal{T}(c) = \begin{cases} A & c \in \{(1), (1, 3)\} \\ B & c \in \{(2), (2, 3)\} \end{cases}. \tag{6}$$

Observe that, under this policy, $\emptyset \sim (1)$ and $\emptyset \sim (2)$, but $(1) \not\sim (2)$, since $\mathcal{T}(1, 3) \neq \mathcal{T}(2, 3)$.

However, it can be shown that, under a completely-determined policy, reducibility induces an equivalence relation. In either case, we compute reducibility using the following criterion:

Lemma 3.1. Two contexts $c_1, c_2 \in \mathcal{C}$ are reducible iff

$$\mathcal{T}(c_1, s) = \mathcal{T}(c_2, s) \quad \text{for all} \quad s \in \mathcal{Y}^* \quad \text{such that} \quad (c_1, s), (c_2, s) \in \mathcal{C} \tag{7}$$

This informs the following algorithm

Algorithm 1: Compute Reducibility Relations

Input: A representation $\langle \mathcal{C}, \mathcal{R}, \mathcal{U}, \mathcal{S}, \mathcal{T}, \mathcal{Y} \rangle$
Output: A reducibility matrix $A : \mathcal{S} \times \mathcal{S} \to \{\textbf{true}, \textbf{false}\}$.

$A(s_1, s_2) \Leftarrow$ **true** for all $s_1, s_2 \in \mathcal{S}$.
repeat
  $isChanged \Leftarrow$ **false**
  for $s_1 < s_2 \in \mathcal{S}$ **do**
    if $A(s_1, s_2) =$ **true then**
      for $c_1 \in \mathcal{R}^{-1}(s_1)$, $c_2 \in \mathcal{R}^{-1}(s_2)$ **do**
        for $y \in \mathcal{Y}_{c_1} \cap \mathcal{Y}_{c_2}$ **do**
          if $\mathcal{T}(c_1, y) \neq \mathcal{T}(c_2, y)$ or $\sim A(\mathcal{R}(c_1, y), \mathcal{R}(c_2, y))$ **then**
            $A(s_1, s_2) \Leftarrow$ **false**.
            $isChanged \Leftarrow$ **true**.
          end if
        end for
      end for
    end if
  end for
until $\sim isChanged$

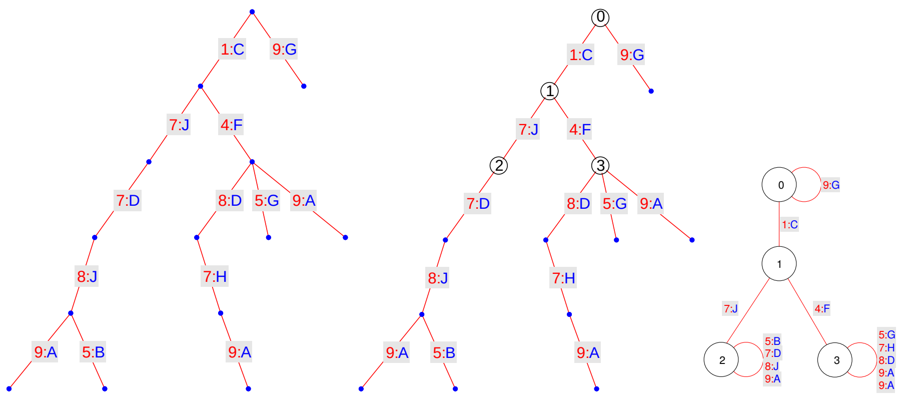
3.2 Bit-at-a-Time

Proposed by Andrea Censi, MIT-LIDS: Greedily separate ambiguous contexts along decision tree.

3.3 Greedy Covering

Now, although reducibility is not an equivalence relation, any reduction $\phi : \mathcal{S} \to \mathcal{S}'$ induces an equivalence relation, partitioning $\mathcal{S}$ into cliques of mutually-reducible states, i.e.

$$\mathcal{S} = \bigsqcup_{s' \in \mathcal{S}'} \phi^{-1}(s'), \quad \text{where} \quad \phi(s_1) = \phi(s_2) \implies A(s_1, s_2) \tag{8}$$



(a) Decision Tree (b) Partitioned Tree (c) Reduced FSM

Thus, a minimum [1] reduction induces a minimum clique partition of the reducible states of a representation.

3.4 Assembling Cliques

Notation 1 (Arrow notation)**.** For a policy $P = \langle \mathcal{C}, \mathcal{U}, \mathcal{T}, \mathcal{Y} \rangle$, write $c \to c'$ if $c = (c_1, \ldots, c_i) \in \mathcal{C}_i \subseteq \mathcal{C}$ and $c' = (c_1, \ldots, c_i, y) \in \mathcal{C}_{i+1} \subseteq \mathcal{C}$, for some i. For a representation $\langle \mathcal{R}, \mathcal{S} \rangle$ of P, write $s_1 \to s_2$ if there are $c_1 \in f^{-1}(s_1)$ and $c_2 \in f^{-1}(s_2)$ such that $c_1 \to c_2$.

We propose the following, greedy, approximate algorithm Although we have

Algorithm 2: Greedy Clique Covering

Input: A representation $\langle \mathcal{C}, \mathcal{R}, \mathcal{U}, \mathcal{S}, \mathcal{T}, \mathcal{Y} \rangle$ with $s_1 < s_2$ only if $s_2 \not\to s_1$.
Input: A reducibility matrix $A : \mathcal{S} \times \mathcal{S} \to \{\mathbf{true}, \mathbf{false}\}$ as computed by Algorithm 1.
Output: A partition function $\phi : \mathcal{S} \to \mathcal{S}'$ with $\phi(s_1) = \phi(s_2)$ only if $A(s_1, s_2)$.

$\mathcal{S}' \Leftarrow \mathcal{S}$
$\phi \Leftarrow id_{\mathcal{S}}$
$unused \Leftarrow \mathcal{S}$
while $|unused| > 0$ **do**
  $s_1 \Leftarrow \min(unused)$
  $unused \Leftarrow unused \setminus \{s_1\}$
  for $s_2 \in unused$ **do**
    if $A(s_1, s_2)$ **then**
      $\phi(s_2) \Leftarrow s_1$
      $unused \Leftarrow unused \setminus \{s_2\}$
    end if
  end for
end while

no proof that this algorithm produces minimal representations of a given policy, it is not inconceivable that this or another greedy policy could work. In general, the Minimal Clique Covering problem is NP-Complete, but the tree structure of the decision policy is an a constraint that may simplify the problem.

3.5 Alberto and Simão

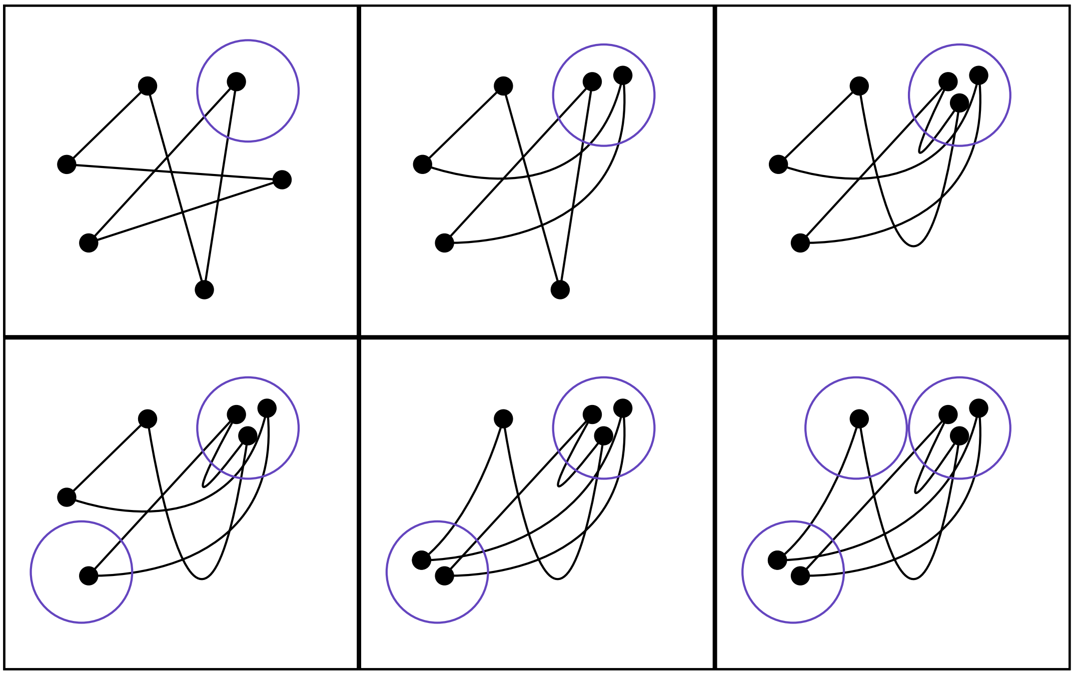
3.5.1 Maximal Anticlique

3.5.2 Limitations

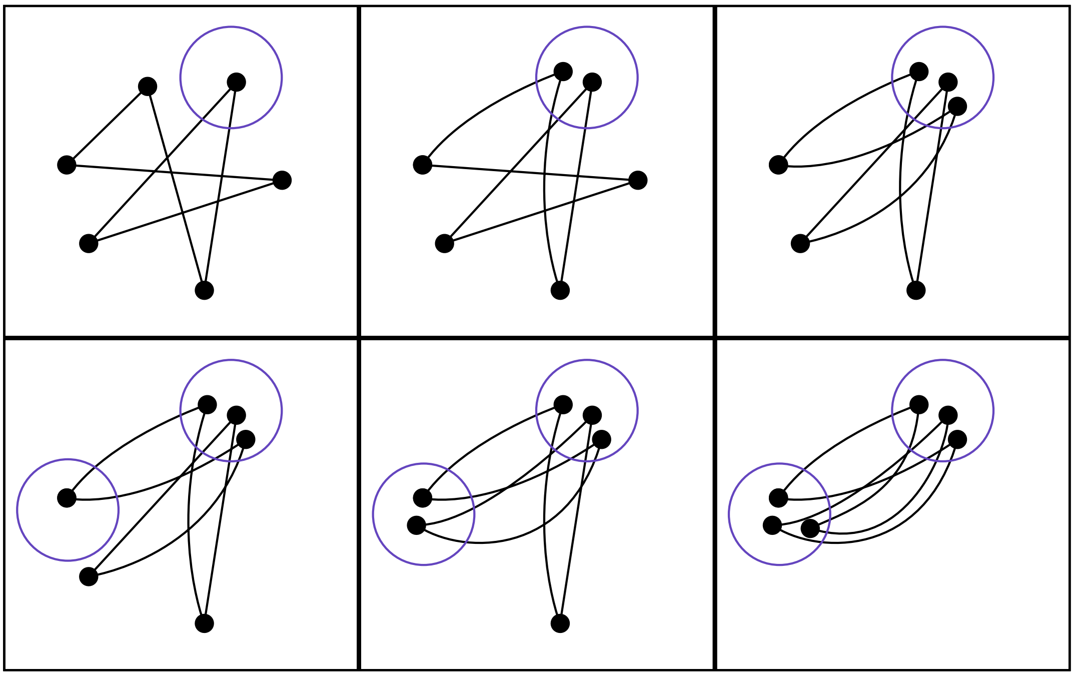
3.6 Exhaustive Search

In order find the absolute minimum representation of a given policy, it suffices to run the greedy algorithm on all possible orderings of its states: Given a minimum clique covering $S = \{s_{c_{11}}, s_{c_{12}}, \ldots, s_{c_{1N_1}}\} \cup \{s_{c_{21}}, s_{c_{22}}, \ldots, s_{c_{2N_2}}\} \cup$

[1] Here, we distinguish "minimal" from "minimum" reductions. A minimal reduction is one that cannot be further reduced by combining any of its states

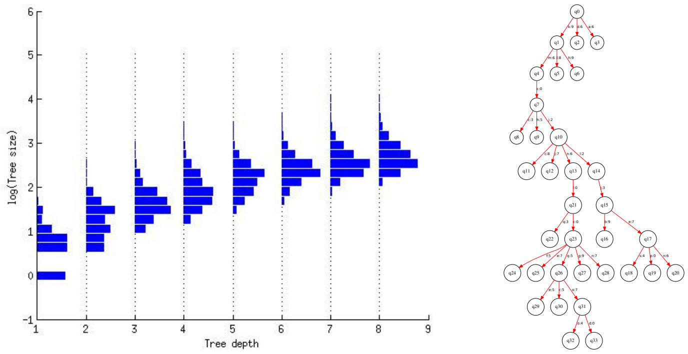


(d) Bad Ordering



(e) Good Ordering

Figure 5: In



(a) Poisson Tree Growth Trend (b) Tree, Depth 10

Figure 6: *Poisson Random Tree* Generated by recursively adding $X \sim \text{Poisson}(\lambda)$ children to each new node. Result is conditioned on process not terminating before depth H. Models a birth/death process where individuals continuously produce offspring at a rate of λ per lifetime.

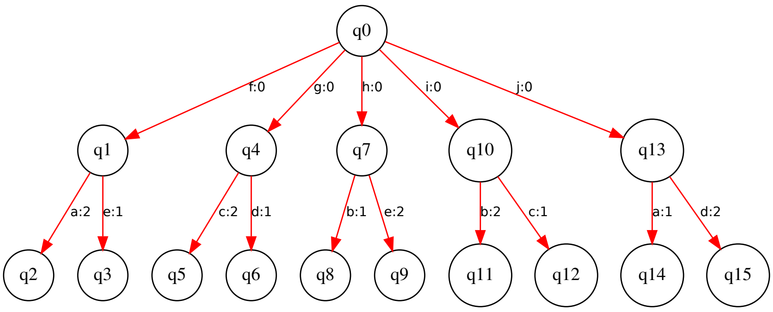


Figure 7: *Pathological Tree* This tree was designed to frustrate the algorithm of Alberto and Simão. Each of its states at depth 1 is incompatible with exactly two others. The resulting distinction graph consists of disjoint rings.

$\cdots \cup \{s_{c_{K1}}, s_{c_{K2}}, \ldots, s_{c_{KN_K}}\}$, feed states to the greedy algorithm in the order in which they are written. Failure to add states to a running clique will occur only once per clique in the minimal covering (exactly K times) [2], so the greedy algorithm will produce a minimum covering.

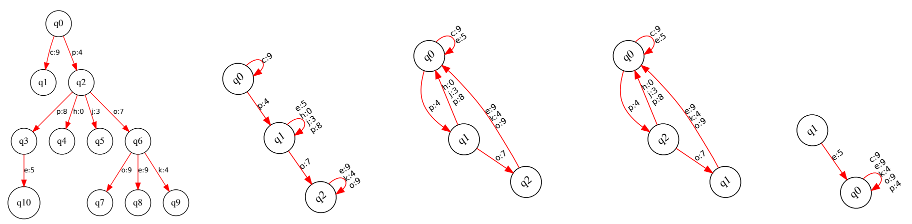
3.7 Comparisons

3.7.1 Poisson Random Tree

3.7.2 Pathological Tree

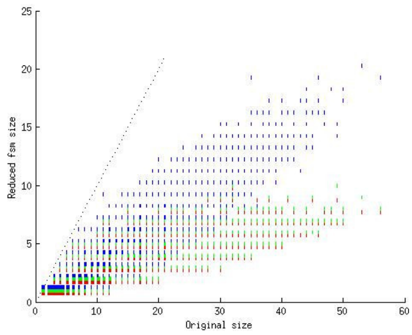
we can

[2]If more than K times, then some running clique

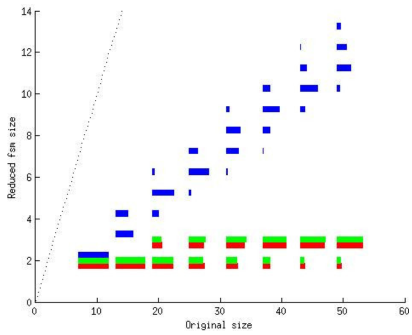


(a) Original (b) Bit-at-a-Time (c) Greedy (d) Alberto-Simão (e) Exact

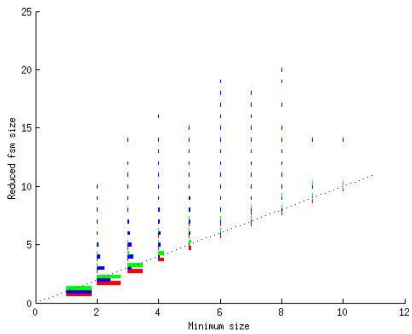
Figure 8: *Typical Reductions* Here we contrast the results of the various reduction algorithms introduced above. The Bit-at-a-Time method (b) produces a minimal sub-tree of the canonical policy. The last three methods are equivalent up to a reordering of states, so reductions (c) and (d) are practically identical.



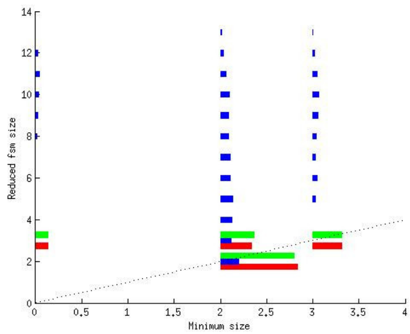
(a) Poisson Trees Reduced vs. Orig. Size



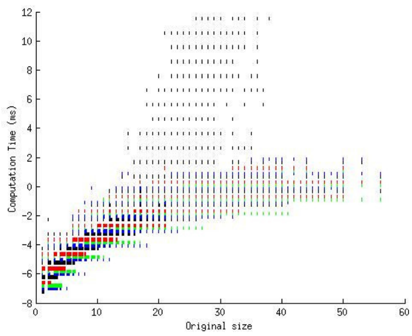
(b) Patho. Trees Reduced vs. Orig. Size



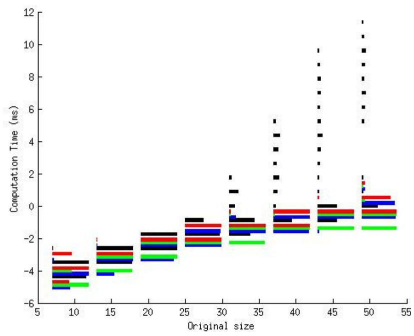
(c) Poisson Trees Reduced vs. Min. Size



(d) Patho. Trees Reduced vs. Min. Size



(e) Algo. Runtimes on Poison Trees



(f) Algo. Runtimes on Patho. Trees

Figure 9: Red bars pertain to Alberto and Simão's method, blue bars pertain to the bit-at-a time method, green bars pertain to the greedy clique completion method, and black bars pertain to an exhaustive search for a minimal reduction.