1 Introduction

Visually-aided navigation (bearing), and range-aided navigation (radar) can be framed as a filtering problem. The model is non-linear, has unknown parameters, and unknown inputs (*e.g.,* accelerometer and gyrometer bias derivative), typically treated as driving noise in a random walk model. Observability is a necessary condition for *any* filter/observer to operate, hence a literature on observability analysis of visually-aided navigation [8, 10, 7]. Relatively little on range-aided. Unknown parameters are typically included in the state, thus transforming an identification problem into a filtering one, and their identifiability analysis lumped in the observability analysis of the resulting (augmented) model. Noise does not affect the observability of a model, so for the purpose of observability analysis, they are set to zero. This is because, by assumption, noise is "uninformative:" It is typically modeled as a realization of a white zero-mean, homoscedastic process, independent of the state of the model. However, the driving input to the random walk model of accelerometer and gyro bias is typically small but *not* independent of the state. In fact, far from being uninformative, it is strongly correlated with it, as it is its temporal derivative. Thus, it should be treated as an *unknown input*, rather than a "noise." As such, it should be included in the observability/identifiability analysis. Our first contribution is to show that while (a prototypical model of) assisted navigation and auto-calibration is *observable* in the absence of unknown input, it is *not* observable when unknown inputs are taken into account. This exposes a methodological flaw with the observability analysis of assisted navigation in the existing literature. Our second contribution is to reframe observability as a *sensitivity* analysis, and to show that while the set of indistinguishable trajectories is *not* a singleton (as it would be if the model was observable), but it is nevertheless bounded to a set. We explicitly characterize this set and show that, interestingly, it may not contain the "true" state trajectory. Finally, we provide bounds on the volume of this subset as a function of the characteristics of the unknown inputs. We do so for bearing-only augmentation, range-only augmentation, and combined augmentation. Rather than study observability of linearized system, or algebraically checking the rank conditions, that offers no insight on the structure of the indistinguishable states, we characterize observability directly in terms of indistinguishable sets.

1.1 Notation

A reference frame is represented by an orthogonal 3×3 positive-determinant (rotation) matrix $R \in \mathrm{SO}(3) \doteq \{R \in \mathbb{R}^{3\times 3} \mid R^T R = RR^T = I,\ \det(R) = +1\}$ and a translation vector $T \in \mathbb{R}^3$. They are collectively indicated by $g = (R, T) \in \mathrm{SE}(3)$. When g represents the change of coordinates from a reference frame "a" to another ("b"), it is indicated by g_{ba}. Then the columns of R_{ba} are the coordinate axes of a relative to the reference frame b, and T_{ba} is the origin of a in the reference frame b. If p_a is a point relative to the reference frame a, then its representation relative to b is $p_b = g_{ba}p_a$. In coordinates, if X_a are the coordinates of p_a, then $X_b = R_{ba}X_a + T_{ba}$ are the coordinates of p_b.

A time-varying pose is indicated with $g(t) = (R(t), T(t))$ or $g_t = (R_t, T_t)$, and the entire trajectory from an initial time t_i and a final time t_f $\{g(t)\}_{t=t_i}^{t_f}$ is indicated in short-hand notation with $g_{t_i}^{t_f}$; when the initial time is $t_0 = 0$, we omit the subscript and call g^t the trajectory "up to time t". The time-index is sometimes omitted for simplicity of notation when it is clear from the context.

We indicate with $\widehat{V} = (\widehat{\omega}, v) \in \mathfrak{se}(3)$ the (generalized) velocity or "twist", where $\widehat{\omega}$ is a skew-symmetric matrix $\widehat{\omega} \in \mathfrak{so}(3) \doteq \{S \in \mathbb{R}^{3\times 3} \mid S^T = -S\}$ corresponding to the cross product with the vector $\omega \in \mathbb{R}^3$, so that $\widehat{\omega}v = \omega \times v$ for any vector $v \in \mathbb{R}^3$. We indicate the generalized velocity with $V = (\omega, v)$. We indicate the group composition $g_1 \circ g_2$ simply as $g_1 g_2$. In homogeneous coordinates, $\bar{X}_b = G_{ba}\bar{X}_a$ where $\bar{X}^T = [X^T\ 1]$ and

$$G \doteq \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \qquad \hat{V} \doteq \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix}. \tag{1}$$

Composition of rigid motions is then represented by matrix product.

1.2 Mechanization Equations

The motion of a sensor platform is represented as the time-varying pose g_{sb} of the body relative to the spatial frame. To relate this to measurements of an inertial measurement unit (IMU) we compute the temporal derivatives of g_{sb}, which yield the (generalized) body velocity V_{sb}^b, defined by $\dot{g}_{sb}(t) = g_{sb}(t)\widehat{V}_{sb}^b(t)$, which can be broken down into the rotational and translational components $\dot{R}_{sb}(t) = R_{sb}(t)\widehat{\omega}_{sb}^b(t)$ and $\dot{T}_{sb}(t) = R_{sb}(t)v_{sb}^b(t)$. An ideal gyrometer (gyro) would measure $\omega_{\text{imu}} = \omega_{sb}^b$. The translational component of body velocity, v_{sb}^b, can be obtained from the last column of the matrix $\frac{d}{dt}\widehat{V}_{sb}^b(t)$. That is, $\dot{v}_{sb}^b = \dot{R}_{sb}^T\dot{T}_{sb} + R_{sb}^T\ddot{T}_{sb} = -\widehat{\omega}_{sb}^b v_{sb}^b + R_{sb}^T\ddot{T}_{sb} \doteq -\widehat{\omega}_{sb}^b v_{sb}^b + \alpha_{sb}^b$, which serves to define $\alpha_{sb}^b \doteq R_{sb}^T\ddot{T}_{sb}$. These equations can be simplified by defining a new linear velocity, v_{sb}, which is neither the body velocity v_{sb}^b nor the spatial velocity v_{sb}^s, but instead $v_{sb} \doteq R_{sb}v_{sb}^b$. Consequently, we have that $\dot{T}_{sb}(t) = v_{sb}(t)$ and $\dot{v}_{sb}(t) = \dot{R}_{sb}v_{sb}^b + R_{sb}\dot{v}_{sb}^b = \ddot{T}_{sb} \doteq \alpha_{sb}(t)$ where the last equation serves to define the new linear acceleration α_{sb}; as one can easily verify we have that $\alpha_{sb} = R_{sb}\alpha_{sb}^b$. An ideal accelerometer (accel) would then measure $\alpha_{\text{imu}} = R_{sb}^T(t)(\alpha_{sb}(t) - \gamma)$.

There are several reference frames to be considered in an aided navigation scenario. The *spatial frame* s, typically attached to Earth and oriented so that gravity γ takes the form $\gamma^T = [0\ 0\ 1]^T\|\gamma\|$ where $\|\gamma\|$ can be read from tabulates based on location and is typically around $9.8m/s^2$. The *body frame* b is attached to the IMU.[1] The *camera frame* c, relative to which image measurements are captured, is also unknown, although we will assume that *intrinsic calibration* has ben performed, so that measurements on the image plane are provided in metric units. Finally, the *radar frame*, or range frame r, is that of the antenna relative to which range measurements are provided.

The equations of motion (known as mechanization equations) are usually described in terms of the body frame at time t relative to the spatial frame $g_{sb}(t)$. Since the spatial frame is arbitrary (other than for being aligned to gravity), it is often chosen to be co-located with the body frame at time $t = 0$.

[1] In practice, the IMU has several different frames due to the fact that the gyro and accel are not co-located and aligned, and even each sensor (gyro or accel) is composed of multiple sensors, each of which can have its own reference frame. Here we will assume that the IMU has ben pre-calibrated so that accel and gyro yield measurements relative to a common reference frame, the *body frame*. In reality, it may be necessary to calibrate the alignment between the multiple-axes sensors (non-orthogonality), as well as the gains (scale factors) of each axis.

To simplify the notation, we indicate this time-varying frame $g_{sb}(t)$ simply as g, and so for $R_{sb}, T_{sb}, \omega_{sb}, v_{sb}$, thus effectively omitting the subscript sb everywhere it appears. This yields

$$\begin{cases} \dot{T} &= V \\ \dot{R} &= R\widehat{\omega} \\ \dot{V} &= \alpha \\ \dot{\omega} &= w \\ \dot{\alpha} &= \xi \end{cases} \tag{2}$$

where $w \in \mathbb{R}^3$ is the rotational acceleration, and $\xi \in \mathbb{R}^3$ the translational jerk (derivative of acceleration). Although α corresponds to neither body nor spatial acceleration, it can be easily related to accel measurements:

$$\boxed{\alpha_{\text{imu}}(t) = R^T(t)(\alpha(t) - \gamma) + \underbrace{\alpha_b(t) + n_\alpha(t)}} \tag{3}$$

where the measurement error in bracket includes a slowly-varying mean ("bias") $\alpha_b(t)$ and a residual term n_α that is commonly modeled as a zero-mean (its mean is captured by the bias), white, homoscedastic and Gaussian noise process. In other words, it is assumed that n_α is independent of α, hence uninformative. Here γ is the gravity vector expressed in the spatial frame.[2] Measurements from a gyro can be similarly modeled as

$$\boxed{\omega_{\text{imu}}(t) = \omega(t) + \underbrace{\omega_b(t) + n_\omega(t)}} \tag{4}$$

where the measurement error in bracket includes a slowly-varying bias $\omega_b(t)$ and a residual "noise" n_ω also assumed zero-mean, white, homoscedastic and Gaussian, independent of ω.

Other than the fact that the biases α_b, ω_b change *slowly*, they can change arbitrarily. One can therefore consider them an *unknown input* to the model, or a *state* in the model, in which case one has to hypothesize a dynamical model for them. For instance instance

$$\dot{\omega}_b(t) = v_b(t), \quad \dot{\alpha}_b(t) = v_\alpha(t) \tag{5}$$

for some unknown input v_b, v_α. While it is safe to assume that v_b, v_α are *small*, they certainly are not (white, zero-mean and, most importantly) uninformative. Nevertheless, it is common to consider v_b, v_α, to be realizations of a Brownian motion that is *independent* of ω_b, α_b. This is done for convenience as one can then consider all unknown inputs as "noise." Unfortunately, however, this has repercussion on the analysis of the observability and identifiability of the resulting model (Sect. 2).

[2] The orientation of the body frame relative to gravity, R_0, is unknown, but can be approximated by keeping the IMU still (so $R^T(t) = R_0$) and averaging the accel measurements, so that $\frac{1}{T}\sum_{t=0}^{T} \alpha_{\text{imu}}(t) \simeq -R_0^T\gamma + \alpha_b$. Assuming the bias to be small (zero), this equation defines R_0 up to a rotation around gravity, which is arbitrary. Note that if $\alpha_b \neq 0$, the initial bias will affect the initial orientation estimate.

1.3 Standard and reduced models

The mechanization equations above define a dynamical model having as output the IMU measurements. Including the initial conditions and biases, we have

$$\begin{cases} \dot{T} = V & T(0) = 0 \\ \dot{R} = R\widehat{\omega} & R(0) = R_0 \\ \dot{V} = \alpha \\ \dot{\omega} = w \\ \dot{\alpha} = \xi \\ \dot{\omega}_b = n_{\omega_b} \\ \dot{\alpha}_b = n_{\alpha_b} \\ \dot{\gamma} = 0 \\ \omega_{\text{imu}}(t) = \omega(t) + \omega_b(t) + n_\omega(t) \\ \alpha_{\text{imu}}(t) = R^T(t)(\alpha(t) - \gamma) + \alpha_b(t) + n_\alpha(t) \end{cases} \tag{6}$$

In this standard model, data from the IMU are considered as (output) *measurements*. However, it is customary to treat them as (known) *input* to the system, by writing ω in terms of ω_{imu} and α in terms of α_{imu}:

$$\boxed{\omega = \omega_{\text{imu}} - \omega_b + \underbrace{n_R}_{-n_\omega}} \qquad \boxed{\alpha = R(\alpha_{\text{imu}} - \alpha_b) + \gamma + \underbrace{n_V}_{-Rn_\alpha}} \tag{7}$$

This equality is valid for *samples* (realizations) of the stochastic processes involved, but it can be misleading as, if considered as stochastic processes, the noises above are *not* independent of the states. Such a dependency, is nevertheless typically neglected. The resulting mechanization model is

$$\boxed{\begin{cases} \dot{T} = V & T(0) = 0 \\ \dot{R} = R(\widehat{\omega}_{\text{imu}} - \widehat{\omega}_b) + n_R & R(0) = R_0 \\ \dot{V} = R(\alpha_{\text{imu}} - \alpha_b) + \gamma + n_V \\ \dot{\omega}_b = n_{\omega_b} \\ \dot{\alpha}_b = n_{\alpha_b}. \end{cases}} \tag{8}$$

Next we will consider augmenting the models above with measurement equations coming either from *range* or *bearing* measurements for a finite set N of isolated points with coordinates $X^i \in \mathbb{R}^3,\ i = 1, \dots, N$.

1.4 Bearing augmentation (vision)

Initially we assume there is a collection of points $X^i,\ i = 1, \dots, N$, visible from time $t = 0$ to the current time t. If $\pi : \mathbb{R}^3 \to \mathbb{R}^2; X \mapsto [X_1/X_3,\ X_2/X_3]$ is a canonical central (perspective) projection, assuming that the camera is *calibrated*,[3] *aligned*,[4] and that the spatial frame coincides with the body frame

[3]Intrinsic calibration parameters are known and compensated for.
[4]The pose of the camera relative to the IMU is known and compensated for.

at time 0, we have

$$\boxed{y^i(t) = \frac{R_{1:2}^T(t)(X^i - T_{1:2}(t))}{R_3^T(t)(X^i - T_3(t))} \doteq \pi(g^{-1}(t)X^i) + n^i(t), \quad t \geq 0.} \tag{9}$$

If the feature first appears at time $t = 0$ *and if the camera reference frame is chosen to be the origin the world reference frame* so that $T(0) = 0; R(0) = I$, then we have that $y^i(0) = \pi(X^i) + n^i(0)$, and therefore

$$\boxed{X^i = \bar{y}^i(0)Z^i + \tilde{n}^i} \tag{10}$$

where $\bar{y}$ is the homogeneous coordinate of y, $\bar{y} = [y^T \; 1]^T$, and $\tilde{n}^i = [{n^i}^T(0)Z^i \;\; 0]^T$. Here Z^i is the (unknown, scalar) depth of the point at time $t = 0$. With an abuse of notation, we write the map that collectively projects all points to their corresponding locations on the image plane as:

$$y(t) \doteq \begin{bmatrix} y^1 \\ y^2 \\ \vdots \\ y^N \end{bmatrix}(t) = \begin{bmatrix} \pi(R^T(X^1 - T)) \\ \pi(R^T(X^2 - T)) \\ \vdots \\ \pi(R^T(X^N - T)) \end{bmatrix} + \begin{bmatrix} n^1(t) \\ n^2(t) \\ \vdots \\ n^N(t) \end{bmatrix} \tag{11}$$

1.5 Range augmentation (radar)

If measurements of range of sparse reflectors in position $X^i \in \mathbb{R}^3$ are given, we consider the projection function (with an abuse of notation) $\pi : \mathbb{R}^3 \to \mathbb{S}^2;\; X \mapsto \|X\|$, and assume that, for each point i – through pre-processing of the radar phase histories, we can measure

$$\boxed{Z^i(t) = \|X^i - T(t)\| + n^i(t) \;\; \doteq \pi(g^{-1}(t)X^i) + n^i(t) \quad t \geq 0.} \tag{12}$$

This equation is formally similar to (9), so the general form (23) does not change if we consider range only measurements, and even a mixture of *both* range and bearing.

In the case of bearing measurements, the inverse projection depends on two parameters (the unknown bearing). If we call

$$\hat{y} \doteq \frac{X}{\|X\|} \in \mathbb{S}^2 \tag{13}$$

then we have that, noting that $Z^i(0) = \|X^i\| + n^i(0)$,

$$X^i = \hat{y}^i Z^i + \hat{n}^i \tag{14}$$

where $\hat{n}^i \doteq -\hat{y}^i n^i(0)$. Although the measurement equations for the bearing-only and range-only filters are formally identical, the observability properties and Gauge ambiguities are different.

1.6 Alignment (calibration)

Consider the model (8) with measurements $y^i(t)$ can representing either the range of a number of sparse reflectors or the position on the image plane of a sparse collection of point features. In the former case, the range is measured in the reference frame of the radar, and therefore we have

$$y^i(t) = \pi\left(g_{rb} g^{-1}(t) X_s^i\right) + n^i(t) \in \mathbb{R} \tag{15}$$

where $\pi(X) = \|X\|$ and g_{rb} is the transformation from the body frame to the radar. In the latter we have

$$\boxed{y^i(t) = \pi\left(g_{cb} g^{-1}(t) X_s^i\right) + n^i(t) \in \mathbb{R}^2} \tag{16}$$

where $\pi(X) = [X_1/X_3,\ X_2/X_3]^T$, and g_{cb} is the transformation from the body frame to the camera. The *"alignment"* transformations g_{cb}, g_{rb} are typically not known and should be inferred. We can then, as done for the points X^i, add them to the state with trivial dynamics $\dot{g}_{cb} = \dot{g}_{rb} = 0$.

1.7 Groups (occlusions)

It may convenient in some cases to represent the points X_s^i in the reference frame where they first appear, say at time t_i, rather than in the spatial frame. This is because the uncertainty is highly structured in the frame where they first appear. Consider $X^i(t_i) = \bar{y}^i(t_i) Z^i(t_i)$, then $y^i(t_i)$ has the same uncertainty of the feature detector (small and isotropic on the image plane) and Z^i has a large uncertainty, but it is constrained to be positive.

However, to relate $X^i(t_i)$ to the state, we must bring it to the spatial frame, via $g(t_i)$, which is unknown. Although we may have a good approximation of it, the current estimate of the state $\hat{g}(t_i)$, the pose when the point first appears should be estimated along with the coordinates of the points. Therefore, we can represent X^i using $y^i(t_i)$, $Z^i(t_i)$ *and* $g(t_i)$:

$$X_s^i = X_s^i(g_{t_i}, y_{t_i}, Z_{t_i}) = g_{t_i} \bar{y}_{t_i} Z_{t_i} \tag{17}$$

Clearly this is an over-parametrization, since each point is now represented by $3 + 6$ parameters instead of 3. However, the pose g_{t_i} can be pooled among all points that appear at time t_i, considered therefore as a *group*. At each time, there may be a number $j = 1, \dots, K(t)$ groups, each of which has a number $i = 1, \dots, N_j(t)$ points. We indicate the group index with j and the point index with $i = i(j)$, omitting the dependency on j for simplicity. The representation of X_s^i then evolves according to

$$\begin{cases} \dot{y}_{t_i}^i = 0, & i = 1, \dots, N(j) \\ \dot{Z}_{t_i}^i = 0 & \\ \dot{g}_j = 0, & j = 1, \dots, K(t). \end{cases} \tag{18}$$

For the case of range, this is not relevant as there is no reference frame that offers a preferential treatment of uncertainty.

1.8 Compact notation

If we call the "state" $x = \{T, R, V, \alpha_b, \omega_b, X\} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ the "known input" $u = \{\omega_{\text{imu}}, \alpha_{\text{imu}}\} = \{u_1, u_2\}$, the *unknown input* $v = \{n_{\omega_b}, n_{\alpha_b}\} = \{v_1, v_2\}$, we can write the mechanization equations (8) as

$$\dot{x} = f(x) + c(x)u + Dv \tag{19}$$

where

$$f(x) \doteq \begin{bmatrix} x_3 \\ -x_2 x_4 \\ -x_2 x_5 + \gamma \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad c(x) \doteq \begin{bmatrix} 0 \\ R \\ R \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad D \doteq \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \tag{20}$$

and the measurement equation (11) as

$$y = h(x) + n \tag{21}$$

where

$$h(x) \doteq \begin{bmatrix} \vdots \\ \pi(x_2'(x_6^i - x_1)) \\ \vdots \end{bmatrix} \tag{22}$$

Putting together (8)-(11) we have a model of the form

$$\boxed{\begin{cases} \dot{x} = f(x) + c(x)u + Dv \\ y = h(x) + n. \end{cases}} \tag{23}$$

1.9 Definitions

We call $y^t = \{y(\tau)\}_{\tau=0}^t$, a collection of output measurements, and $x^t = \{x(\tau)\}_{\tau=0}^t$ a state *trajectory*. Given output measurements y^t and known inputs u^t, we call

$$\mathcal{I}(y^t|u^t;\tilde{x}_0) \doteq \{\tilde{x}^t \mid y^t = h(\tilde{x}^t) \text{ s. } t. \ \dot{\tilde{x}}(t) = f(\tilde{x}) + c(\tilde{x})u(t), \ \tilde{x}(0) = \tilde{x}_0 \ \forall \ t\} \tag{24}$$

the *indistinguishable set*, or set of *indistinguishable trajectories*, for a given input u^t. If the initial condition $\tilde{x}_0 = x_0$ equals the "true" one, the indistinguishable set contains at least one element, the "true" trajectory x^t. However, if $\tilde{x}_0 \neq x_0$, the true trajectory may not even be part of this set.

If the indistinguishable set is a singleton (it contains only one element, $\tilde{x}^t$, which is a function of the initial condition $\tilde{x}_0$), we say that the model is *observable up to the initial condition*, or simply *observable*.[5] If $\{\tilde{x}^t\}$ is further independent of the initial condition, we say that the model is *strongly observable*: $\mathcal{I}(y^t|u^t;\tilde{x}_0) = \{x^t\} \ \forall \ \tilde{x}_0, \ u^t$.

If the state includes unknown parameters with a trivial dynamic, and there is no unknown input, $v = 0$, then observability of the resulting model implies that the parameters are *identifiable*. That usually requires the input u^t to be *sufficiently exciting* (SE), in order to enable disambiguating the indistinguishable states, [6] as the definition does not require that every input disambiguates states.

[5]We will assume that the solution of the differential equation $\dot{x} = f(x) + c(x)u$ is unique and continuously dependent on the initial condition, so if we impose $\tilde{x}_0 = x_0$, then $\tilde{x}^t = x^t$.

[6]Sufficient excitation means that the input is *generic*, and does not lie on a thin set. That is, even if we could find a particular input u^t that yields indistinguishable states, there will be another input that is infinitesimally close to it that will disambiguate them.

In the presence of *unknown inputs* $v \neq 0$, consider the following definition

$$\mathcal{I}_v(y^t|u^t;\tilde{x}_0) \doteq \{\tilde{x}^t \mid \exists\, v^t \text{ s. } t.\, y^t = h(\tilde{x}^t),\ \dot{\tilde{x}}(t) = f(\tilde{x}) + c(\tilde{x})u(t) + Dv(t)\ \forall\, t;\ \tilde{x}(0) = \tilde{x}_0\} \tag{25}$$

which is the set of *unknown-input indistinguishable states*. The model $\{f, c, D\}$ is said to be *unknown-input observable* (up to initial conditions) if the unknown-input indistinguishable set is a singleton. If such a singleton is further independent of the initial conditions, the model is strongly observable. The two definitions coincide once the only admissible unknown input is $v^t = 0$ for all t.

It is possible for a model to be observable (the indistinguishable set is a singleton), but not unknown-input observable (the unknown-input indistinguishable set is dense). In that case, the notion of *sensitivity* arises naturally, as one would want to measure the "size" of the unknown-input indistinguishable set as a function of the "size" of the unknown input. For instance, it is possible that if the set of unknown inputs is small in some sense, the resulting set of indistinguishable states is also small. If $v \in V$ and for any $\epsilon > 0$ there exists a $\delta > 0$ such that $\mathrm{v}ol(V) \leq \epsilon$ for some measure of volume implies $\mathrm{v}ol(\mathcal{I}_v(y^t|u^t;\tilde{x}_0)) < \delta$ for any $u^t, \tilde{x}_0$, then we say that the model is *bounded-unknown-input/bounded-output observable* (up to the initial condition). If the latter volume is independent of $\tilde{x}_0$ we say that model is strongly bounded-unknown-input/bounded-output observable.

The set of indistinguishable trajectories $\mathcal{I}$ is an equivalence class, and when the model is observable *up to the initial condition*, it is parametrized by $\tilde{x}_0$. Choosing the "true" initial condition $\tilde{x}_0 = x_0$ produces an indistinguishable set consisting of the sole "true" trajectory, otherwise it is a singleton other than the true trajectory. In some cases, the initial condition corresponds to an arbitrary choice of reference frame, and therefore the equivalence class of indistinguishable trajectory are related by a *gauge transformation* (a change of coordinates). As the equivalence class can be represented by any element, enforcing a particular reference for the gauge transformation yields strong observability (although the singleton may not correspond to the true trajectory).

Related work

Unknown-input observability of linear time-invariant systems has ben addressed in [1, 4], for affine systems [5], and non-linear systems in [3, 9, 2]. The literature on robust filtering and robust identification is relevant, if the unknown input is treated as a disturbance. However, the form of the models involved in aided navigation do not fit in the classes treated in the literature above, which motivates our analysis.

2 Analysis of Bearing-Augmented Navigation

2.1 Preliminary claims

Lemma 2.1. Given $S \in \mathrm{SO}(3)$ and $\dot{S} \in T_{\mathrm{SO}(3)}(S)$, and $a \in \mathbb{R}$, the matrix $(aS + \dot{S})$ is nonsingular unless $a = 0$, in which case it has rank 2 or 0.

Proof. The tangent $\dot{S}$ has the form SM, where M is some skew-symmetric matrix. As such, $Mx \perp x$ for any $x \in \mathbb{R}^3$, so

$$\|(aS + \dot{S})x\|_2^2 = \|S(aI + M)x\|_2^2 = \|ax\|_2^2 + \|Mx\|_2^2.$$

The above is zero only if $ax = 0$, so $(aS + \dot{S})$ is nonsingular. For the remaining cases, observe that a 3×3 skew-symmetric matrix has rank 2 or 0. □

Lemma 2.2. Let $(R(t), T(t))$ and $(\tilde{R}(t), \tilde{T}(t))$ be differentiable trajectories in SE(3). For each time $t' \in [0, T]$, there exists an open, full-measure subset $\mathcal{A}_{t'} \subset \mathbb{R}^3$ such that:

For any two static point-clouds $\{X^i\}_{i=1}^N \subset \mathcal{A}_{t'}$ and $\{\tilde{X}^i\}_{i=1}^N \subset \mathbb{R}^3$ that satisfy

$$\pi\big(R^{-1}(t)(X^i - T(t)\big) = \pi\big(\tilde{R}^{-1}(t)(\tilde{X}^i - \tilde{T}(t))\big) \quad \text{for all } i \text{ and } t \tag{26}$$

there exist constant scalings $\sigma_{it'} > 0$ and a constant rotation $S_{t'} = \tilde{R}(t')R^{-1}(t')$ such that

$$\sigma_{it'} S_{t'}(X^i - T(t)) = (\tilde{X}^i - \tilde{T}(t)) + O((t - t')^2) \quad \text{for all } i \text{ and } t.$$

Furthermore, if $T(t') \neq 0$, then $\sigma_{it'} = \sigma_{t'}$ for all i.

Proof. Write $S(t) = \tilde{R}(t)R^{-1}(t)$. Equality under the projection π implies that there exists a scaling $\sigma_i(t)$ (possibly varying with X^i and t) such that

$$\sigma_i S\big(X^i - T\big) = \tilde{X}^i - \tilde{T}. \tag{27}$$

For a given time t', we wish to find a suitably large set $\mathcal{A}_{t'}$ such that $\dot{\sigma_i}(t') = \dot{S}(t') = 0$ and $\sigma_i(t')$ is independent of X^i, when $X^i \in \mathcal{A}_{t'}$ Taking time derivatives,

$$\big(\dot{\sigma_i} S + \sigma_i \dot{S}\big)(X^i - T) - \sigma_i S\dot{T} = -\dot{\tilde{T}}$$

or, dividing by σ_i,

$$\big(\tfrac{\dot{\sigma_i}}{\sigma_i} S + \dot{S}\big)(X^i - T) - S\dot{T} = -\tfrac{1}{\sigma_i}\dot{\tilde{T}}. \tag{28}$$

Differentiating both sides with respect to X^i,

$$\big(\tfrac{\dot{\sigma_i}}{\sigma_i} S + \dot{S}\big)\delta X^i + \big(\tfrac{d}{dX^i}\big(\tfrac{\dot{\sigma_i}}{\sigma_i}\big)\delta X^i\big)S(X^i - T) = -\big(\tfrac{d}{dX^i}\big(\tfrac{1}{\sigma_i}\big)\delta X^i\big)\dot{\tilde{T}}. \tag{29}$$

Observe that $\frac{d}{dX^i}\big(\frac{\dot{\sigma_i}}{\sigma_i}\big)\delta X^i$ and $\frac{d}{dX^i}\big(\frac{1}{\sigma_i}\big)\delta X^i$ are scalars. By Lemma 2.1, the LHS has rank 2 or greater (as a linear map on δX^i), unless $\dot{\sigma_i}(t') = 0$. The RHS, however, has rank at most 1. Thus, (28) is invalid for almost all X^i, unless $\dot{\sigma_i}(t') = 0$ (two maps of different ranks can only agree on a submanifold). Plugging $\dot{\sigma_i} = 0$ into (29), we are left with

$$\dot{S}\delta X^i = -\big(\tfrac{d}{dX^i}\big(\tfrac{1}{\sigma_i}\big)\delta X^i\big)\dot{\tilde{T}}. \tag{30}$$

Now, the LHS has rank 2 or 0, while the RHS has rank 1 or 0. Again, (28) is invalid for almost all X^i, unless $\dot{S}(t') = 0$. Let $\mathcal{A}_{t'} \subset \mathbb{R}^3$ be the open, full-measure subset (being the complement of two submanifolds) on which the latter must hold. If, in addition, $\dot{T}(t') \neq 0$, then $\dot{\tilde{T}}(t') \neq 0$ and $\frac{d\sigma_i}{dX^i}(t') = 0$, we can finally write

$$\sigma_{t'} S_{t'}(X^i - T) = \tilde{X}^i - \tilde{T} + O((t - t')^2).$$

□

Claim 1 (Indistinguishable Trajectories from Bearing Data Sequences)**.** Let $g(t)$ and $\tilde{g}(t)$ be differentiable trajectories in SO(3). There exists an open, full-measure subset $\mathcal{A} \subset \mathbb{R}^3$ such that

> Given two static, generic (non-coplanar) point clouds $\{X^i\}_{i=1}^N \subset \mathcal{A}$ and $\{\tilde{X}^i\}_{i=1}^N \subset \mathbb{R}^3$, satisfying
>
> $$\pi(g^{-1}(t)\, X^i) = \pi(\tilde{g}^{-1}(t)\, \tilde{X}^i) \quad \text{for all } i \text{ and } t,$$
>
> there exist constant scalings $\sigma_i > 0$ and a constant transformation $\bar{g} \in \mathrm{SE}(3)$ such that
>
> $$\begin{cases} \tilde{X}^i &= \sigma_i(\bar{g}X^i) \\ \tilde{g}(t) &= \sigma_i(\bar{g}g(t)) \end{cases} \qquad \text{for all } i \text{ and } t. \tag{31}$$
>
> Furthermore, if $g(t)$ has a non-constant translational component, then $\sigma_i = \sigma$ for all i.

Proof. Write $g(t) = (R(t), T(t))$ and $\tilde{g}(t) = (\tilde{R}(t), \tilde{T}(t))$. Let $\mathcal{A} = \{X \in \mathbb{R}^3 : X \in \mathcal{A}_{t'} \text{ for almost all } t'\}$, with $\mathcal{A}_{t'}$ defined as in Lemma 2.2. By Fubini's theorem, this has full measure in $\mathbb{R}^3$. If $\{X^i\} \subset \mathcal{A}$, then the conditions for Lemma 2.2 are satisfied for almost all t, and thus there exist *constant* (being stationary for almost all t) scalings σ_i and rotation $S = \tilde{R}(t)R(t)^{-1} \in \mathrm{SO}(3)$ such that $\tilde{X}^i = \sigma_i S(X^i - T_t) + \tilde{T}_t$.

Define $\bar{g}(t) = (\sigma_i^{-1}\tilde{g}(t))\, g(t)^{-1}$, and observe that

$$\tilde{X}^i = \sigma_i S(X^i - T_t) + \tilde{T}_t = \sigma_i(\tilde{R}_t(g^{-1}X^i) + \sigma_i^{-1}\tilde{T}_t) = \sigma_i\big((\sigma_i^{-1}\tilde{g}(t))\, g(t)^{-1}X^i\big) = \sigma_i(\bar{g}(t)X^i).$$

If this affine relation holds for the generic set $\{X^i\}$, then $\bar{g}(t)$ must be constant. Next,

$$\sigma_i(\bar{g}g(t)) = \sigma_i((\sigma_i^{-1}\tilde{g}(t))\, g(t)^{-1}g(t)) = \sigma_i(\sigma_i^{-1}\tilde{g}(t)) = \tilde{g}(t).$$

Finally, if $T(t') = 0$ for some t', then $\sigma_i = \sigma_i(t') = \sigma(t') = \sigma$ for all i. □

In what follows, we will avoid the cumbersome discussion of sets such as $\mathcal{A} \subset \mathbb{R}^3$, defined by a given trajectory, and will instead speak of *sufficiently exciting* trajectories, for which a given point cloud is suitable for tracking.

Definition 1 (Sufficiently Exciting Motion)**.** A trajectory $g(t)$ is **sufficiently exciting** relative to a point-cloud $\{X^i\}_{i=1}^N \subset \mathbb{R}^3$ if, for all $\{\tilde{X}^i\}_{i=1}^N \subset \mathbb{R}^3$ and $\tilde{g}(t)$ in SE(3),

$$\pi(g(t)^{-1}(t)X^i) = \pi(\tilde{g}(t)^{-1}\tilde{X}^i) \quad \text{for all } i \text{ and } t \iff \tag{32}$$
$$\left(\begin{array}{rl} \tilde{X}^i &= \sigma(\bar{g}X^i) \\ \tilde{g}(t) &= \sigma(\bar{g}g(t)) \end{array} \quad \text{for all } i \text{ and } t\right) \text{ for some constant } \sigma > 0 \text{ and } \bar{g} \in \mathrm{SE}(3).$$

That is, if the projection map $\pi(g(t)X^i)$ defines $g(t)$ and $\{X^i\}$ up to a constant rotation and mapping.

Observe that the right-to-left implication is always true: if the RHS holds, then

$$\pi(\tilde{g}(t)^{-1}\tilde{X}^i) = \pi((\sigma\bar{g}g(t))^{-1}\sigma(\bar{g}X^i))\pi(g(t)^{-1}\bar{g}^{-1}\sigma^{-1}\sigma\bar{g}X^i) = \pi(g(t)^{-1}X^i).$$

We will see that the sufficient excitation condition is very easily satisfied.

Claim 2. Given trajectories $g(t)$ and $\tilde{g}(t)$ in SE(3) with non-constant translation, and a set $\{X^i\}_{i=1}^N$ of $N \geq 4$ points sampled i.i.d. from a non-singular distribution over $\mathbb{R}^3$, the trajectory $g(t)$ is a.s. sufficiently exciting relative to $\{X^i\}$.

Proof. Fix $g(t)$. By Claim 1, there exists a full-measure $\mathcal{A} \subset \mathbb{R}^3$ such that (32) holds for any static, generic point clouds $\{X^i\}_{i=1}^N \subset \mathcal{A}$ and $\{\tilde{X}^i\}_{i=1}^N \subset \mathbb{R}^3$. If $\{X^i\}$ is sampled i.i.d. from a non-singular distribution over $\mathbb{R}^3$, then $\{X^i\} \subset \mathcal{A}$ almost surely. □

Equation (31) establishes the fact that the indistinguishable trajectories are an equivalence class parameterized by a group $\sigma(\bar{g})$, called a *gauge transformation.* We now include a constant reference frame g_a. We then have the following claim.

Claim 3 (Indistinguishable Alignments)**.** For a point cloud $\{X^i\}_{i=1}^{N(t)}$, $N(t) > 3$, in general position (non-coplanar), and sufficiently exciting motion,

$$\pi(g_a g^{-1}(t) X^i) = \pi(\tilde{g}_a \tilde{g}^{-1}(t) \tilde{X}^i) \tag{33}$$

if and only if there exist constants $\sigma > 0$, g_A and $g_B \in$ SE(3) such that

$$\begin{cases} \tilde{X}^i &= \sigma(g_B X^i) \\ \tilde{g}(t) &= \sigma(g_B g(t) g_A) \\ \tilde{g}_a &= \sigma(g_a g_A). \end{cases} \tag{34}$$

Proof. From Claim 1 we get constant $g_B \in$ SE(3) and $\sigma > 0$ such that $\tilde{X}^i = \sigma(g_B X^i)$ and

$$\tilde{g}(t)\tilde{g}_a^{-1} = \sigma(g_B g(t) g_a^{-1}) \tag{35}$$

Let $g_A = g_a^{-1}\sigma^{-1}(\tilde{g}_a)$. Then $\tilde{g}_a = \sigma(g_a g_A)$ and

$$\tilde{g}(t) = \sigma(g_B g(t) g_A).$$

□

We now include groups of points, each with its own reference frame.

Claim 4 (Indistinguishable Groups)**.** For a number $i = 1, \ldots, K$ of groups each with a number $j = 1, \ldots, N_i \geq 3$ of points in general position (non-coplanar), and sufficiently exciting motion,

$$\pi(g_a g^{-1}(t) g_i g_a^{-1} X^j) = \pi(\tilde{g}_a \tilde{g}^{-1}(t) \tilde{g}_i \tilde{g}_a^{-1} \tilde{X}^j) \tag{36}$$

if and only if there exist constants $\sigma > 0, g_A, g_B, \bar{g}_i \in$ SE(3) such that

$$\begin{cases} \tilde{X}^j &= \sigma(g_a \bar{g}_i^{-1} g_i g_a^{-1} X^j) \\ \tilde{g}(t) &= \sigma(g_B g(t) g_A) \\ \tilde{g}_i &= \sigma(g_B \bar{g}_i g_A) \\ \tilde{g}_a &= \sigma(g_a g_A) \end{cases} \tag{37}$$

Proof. From Claim 1, we get constant $g_C \in SE(3)$ and $\sigma > 0$ such that

$$\tilde{X}^i = \sigma(g_C X^i), \tag{38}$$

$$\tilde{g}_a \tilde{g}_i^{-1} \tilde{g}(t) \tilde{g}_a^{-1} = \sigma(g_C g_a g_i^{-1} g(t) g_a^{-1}). \tag{39}$$

Define

$$g_A := g_a^{-1} \sigma^{-1}(\tilde{g}_a), \qquad g_B := \sigma^{-1}(\tilde{g}_i g_a^{-1}) g_C g_a g_i^{-1}, \qquad \bar{g}_i := g_i g_a^{-1} g_C^{-1} g_a.$$

Then, applying the definition of $\bar{g}_i$ to (38),

$$\tilde{X}^j = \sigma(g_C X^j) = \sigma((g_a \bar{g}_i^{-1} g_i g_a^{-1}) X^j).$$

Applying the definitions of g_A and g_B to (39),

$$\tilde{g}(t) = \tilde{g}_i \tilde{g}_a^{-1} \sigma(g_C g_a g_i^{-1} g(t) g_a^{-1}) \tilde{g}_a = \sigma\big(\underbrace{\sigma^{-1}(g_i \tilde{g}_a^{-1}) g_C g_a g_i^{-1}}_{g_B} g(t) \underbrace{g_a^{-1} \sigma^{-1}(\tilde{g}_a)}_{g_A}\big) = \sigma(g_B g(t) g_A).$$

Rearranging the definitions of g_A, g_B and $\bar{g}_i$,

$$\tilde{g}_i = \sigma(g_B g_i g_a^{-1} g_C^{-1}) \tilde{g}_a = \sigma\left(g_B g_i g_a^{-1} g_C^{-1} \sigma(\tilde{g}_a)\right) = \sigma\big(g_B \underbrace{g_i g_a^{-1} g_C^{-1} g_a}_{\bar{g}_i} \underbrace{g_a^{-1} \sigma(\tilde{g}_a)}_{g_A}\big) = \sigma(g_B \bar{g}_i g_A).$$

Finally, rearrange the definition of g_A to get

$$\tilde{g}_a = \sigma(g_a g_A).$$

□

Eq. (37) describes the ambiguous state trajectories if only bearing measurement time series are given. In that case, there is no alignment to other sensor, so we can assume without loss of generality that $g_a = Id$ and so for $\tilde{g}_a$, which in turn implies $g_A = Id$. The resulting ambiguity is well-known [11] and shows that scale σ is constant but arbitrary, that the global reference frame is arbitrary (since g_B is), and that the reference frame of each group is also arbitrary (since $\bar{g}_i$ is). To lock these ambiguities, we can fix three directions for each group (thus fixing $\bar{g}_i$) and, in addition, for one of the groups fix the pose (thus fixing g_B); finally, we can impose that the centroid of the points in that one group (the "reference group") be one, which fixes σ. Thus, an observer designed based on the standard model, where 3 directions within each group are saturated, and where the pose of one group is fixed, and the centroid of the group is at distance one, is observable, and under the usual assumptions it should converge to a state trajectory that is related to the true one by an arbitrary unknown scaling, and global reference frame.

Now, when inertial measurements are present, of all the possible trajectories that are indistinguishable from the measurements, we are interested *only* in those that are compatible with the dynamical model driven by IMU measurements. Since the fact that X^j and g_a are constant has already ben enforced, the model will impose no constraints on $\tilde{X}^j, \tilde{g}_i$ and $\tilde{g}_a$. However, it will offer constraints on $\tilde{g}(t)$, that depends on the arbitrary constants σ, g_A, g_B.

2.2 Indistinguishable trajectories in bearing augmentation

Definition 2. For an $\mathbb{R}^3$-valued trajectory $f:\mathbb{R}\to\mathbb{R}^3$ and interval $\mathcal{I}\subset\mathbb{R}^+$, define

$$\begin{aligned} m(f:\mathcal{I}) &:= \inf_{\|x\|=1}\Big(\sup_{t\in\mathcal{I}} |f(t)\cdot x|\Big) = \inf_{\|x\|=1}\Big(\sup_{t\in\mathcal{I}} \|f(t)\times x\|\Big), \\ M(f:\mathcal{I}) &:= \sup_{\|x\|=1}\Big(\sup_{t\in\mathcal{I}} |f(t)\cdot x|\Big) = \sup_{t\in\mathcal{I}}\|f(t)\|, \quad \text{and} \\ \bar{m}(f:\mathcal{I}) &:= \sqrt{\max\{0,\, 2m(f:\mathcal{I})^2 - M(f:\mathcal{I})^2\}}. \end{aligned}$$

Observe that $M(f:\mathcal{I}) \geq m(f:\mathcal{I}) \geq \bar{m}(f:\mathcal{I})$, and that the inequalities are strict unless $\{\pm f(t)|\, t\in\mathcal{I}\}$ is dense on the sphere of radius $M(f:\mathcal{I})$. We use these "minimum-excitation" bounds in order to prove a partial converse of the Cauchy-Schwarz inequality:

Lemma 2.3. Let $A = c_1 I + c_2 R$, for some rotation $R\in \mathrm{SO}(3)$ and scalars c_1 and c_2. Then, for any trajectory $f:\mathbb{R}^+\to\mathbb{R}^3$ and set of times $\mathcal{I}\subset\mathbb{R}^+$,

$$\sup_{t\in\mathcal{I}}\|Af(t)\| \;\geq\; \|A\|\,\bar{m}(f:\mathcal{I}).$$

Proof. First, observe that A is normal:

$$AA^T = (c_1 I + c_2 R)(c_1 I + c_2 R^T) = 2c_1c_2 I + c_1c_2(R+R^T) = A^T A.$$

Let $\{(\lambda_i, v_i)\}_{i=1}^3$ be orthonormal eigenvalue/eigenvector pairs of A, with $\lambda_1 \geq \lambda_2 \geq \lambda_3$.

$$\begin{aligned} \|Af(t)\|^2 &= \lambda_1^2(v_1\cdot f(t))^2 + \lambda_2^2(v_2\cdot f(t)^2 + \lambda_3^2(v_3\cdot f(t))^2 \\ &\geq \lambda_1^2\big((v_1\cdot f(t))^2 - (v_2\cdot f(t)^2 - (v_3\cdot f(t))^2\big) \\ &= \|A\|^2\big(2(v_1\cdot f(t))^2 - \|f(t)\|^2\big). \end{aligned}$$

Taking the supremum over $\mathcal{I}$,

$$\begin{aligned} \sup_{t\in\mathcal{I}}\|Af(t)\|^2 &\geq \|A\|^2\sup_{t\in\mathcal{I}}\big(2(v_1\cdot f(t))^2 - \|f(t)\|^2\big) \\ &\geq \|A\|^2\big(2\sup_{t\in\mathcal{I}}(v_1\cdot f(t))^2 - \sup_{t\in\mathcal{I}}\|f(t)\|^2\big) \\ &\geq \|A\|^2\big(2m(f:\mathcal{I})^2 - M(f:\mathcal{I})^2\big) \end{aligned}$$

□

Lemma 2.4. Let $A = I - R$, for some rotation $R\in\mathrm{SO}(3)$. Then, for trajectory $f:\mathbb{R}^+\to\mathbb{R}^3$ and $\mathcal{I}\subset\mathbb{R}^+$,

$$\sup_{t\in\mathcal{I}}\|Af(t)\| \;\geq\; \|A\|\,m(f:\mathcal{I}).$$

Proof. Let $\{(\lambda, v_1), (\bar{\lambda}, v_2), (1,0)\}$ be the orthonormal eigenvalue/eigenvector pairs of R. Since R and I commute, $\{(\lambda-1, v_1),\, (\bar{\lambda}-1, v_2), (0,u)\}$ are the eigenpairs of A, and $\|A\| = |\lambda-1| = |\bar{\lambda}-1|$. Then,

$$\|Af(t)\|^2 = |\lambda-1|^2(v_1\cdot f(t))^2 + |\bar{\lambda}-1|^2(v_2\cdot f(t))^2 + 0 = \|A\|^2(w\cdot f(t))^2,$$

where

$$w := \frac{(v_1 \cdot f(t))v_1 + (v_2 \cdot f(t))v_2}{\|(v_1 \cdot f(t))v_1 + (v_2 \cdot f(t))v_2\|} = \frac{(v_1 \cdot f(t))v_1 + (v_2 \cdot f(t))v_2}{\sqrt{(v_1 \cdot f(t))^2 + (v_2 \cdot f(t))^2}}.$$

Taking the supremum over $\mathcal{I}$,

$$\sup_{t\in\mathcal{I}} \|Af(t)\|^2 = \|A\|^2 \sup_{t\in\mathcal{I}} \|w \cdot f(t)\|^2 \geq \|A\|^2 m(f\colon \mathcal{I})^2.$$

□

Claim 5 (Indistinguishable Trajectories from IMU Data)**.** Let $g(t) = (R(t), T(t)) \in \mathrm{SE}(3)$ be such that

$$\begin{cases} \dot{R} = R(\widehat{\omega}_{\mathrm{imu}} - \widehat{\omega}_b) \\ \dot{T} = V \\ \dot{V} = R(\alpha_{\mathrm{imu}} - \alpha_b) + \gamma \end{cases} \tag{40}$$

for some known constant γ and functions $\alpha_{\mathrm{imu}}(t)$, $\omega_{\mathrm{imu}}(t)$ and for some unknown functions $\alpha_b(t), \omega_b(t)$ that are constrained to have $\|\dot{\alpha}_b(t)\| \leq \epsilon$, $\|\dot{\omega}_b(t)\| \leq \epsilon$, and $\|\ddot{\omega}_b(t)\| \leq \epsilon$ at all t, for some $\epsilon < 1$.

Suppose $\tilde{g}(t) \doteq \sigma(g_B g(t) g_A)$ for some constant $g_A = (R_A, T_A)$, $g_B = (R_B, T_B)$, $\sigma > 0$, with bounds on the configuration space such that $\|T_A\| \leq M_A$ and $|\sigma| \leq M_\sigma$. Then, under sufficient excitation conditions (described in this proof), $\tilde{g}(t)$ satisfies (40) if and only if

$$\|I - R_A\| \leq \frac{2\epsilon}{m(\dot{\omega}_{\mathrm{imu}} \colon \mathbb{R}^+)} \tag{41}$$

$$|\sigma - 1| \leq \frac{k_{c_1}\epsilon + M_\sigma \|I - R_A\|}{M(\dot{\alpha}_{\mathrm{imu}} \colon \mathcal{I}_{c_1})} \tag{42}$$

$$\|T_A\| \leq \frac{\epsilon(k_{c_2} + (2M_\sigma + 1)M_A)}{(1 - |\sigma - 1|)\, m(\ddot{\omega}_{\mathrm{imu}} \colon \mathcal{I}_{c_2})} \tag{43}$$

$$\|(1 - R_B^T)\gamma\| \leq \frac{\epsilon(k_{c_3} + M_\sigma M_A) + (|\sigma - 1| + \epsilon) M(\omega_{\mathrm{imu}} - \omega_b \colon \mathcal{I}_{c_3}) \|\gamma\|}{m(\omega_{\mathrm{imu}} - \omega_b \colon \mathcal{I}_{c_3})\,(1 - |\sigma - 1|)} \tag{44}$$

for $\mathcal{I}_i$ and k_i determined by the sufficient excitation conditions.

Proof.

(41) The ambiguous rotation $\tilde{R}$ must satisfy $\dot{\tilde{R}} = \tilde{R}(\widehat{\omega}_{\mathrm{imu}} - \widehat{\tilde{\omega}}_b)$ for some $\tilde{\omega}_b$:

$$\begin{aligned} \dot{\tilde{R}} &= R_B R(\widehat{\omega}_{\mathrm{imu}} - \widehat{\omega}_b) R_A = \tilde{R} R_A^T (\widehat{\omega}_{\mathrm{imu}} - \widehat{\omega}_b) R_A = \tilde{R}(\widehat{R_A^T \omega_{\mathrm{imu}}} - \widehat{R_A^T \omega_b}) \\ &= \tilde{R}(\widehat{\omega}_{\mathrm{imu}} - \ [\widehat{\omega}_{\mathrm{imu}} + \widehat{R_A^T \omega_{\mathrm{imu}}} - \widehat{R_A^T \omega_b}]) \end{aligned}$$

where the quantity in brackets is $-\widehat{\tilde{\omega}}_b$, which defines

$$\tilde{\omega}_b := R_A^T \omega_b + (I - R_A^T)\, \omega_{\mathrm{imu}}. \tag{45}$$

Taking derivatives and rearranging,

$$2\epsilon \geq \|\dot{\tilde{\omega}}_b - R_A^T \dot{\omega}_b\| = \|(I - R_A^T)\dot{\omega}_{\mathrm{imu}}\|$$

Since this is true for all $t \in \mathbb{R}$, we can write

$$2\epsilon \geq \sup_{t\in\mathbb{R}} \|(I - R_A^T)\,\dot{\omega}_{\text{imu}}(t)\| \geq \|I - R_A^T\|\, m(\dot{\omega}_{\text{imu}}\!:\!\mathbb{R}^+).$$

This rearranges to give (41).

(42) The ambiguous translation $\tilde{T}$ must satisfy the dynamics in (40):

$$\ddot{\tilde{T}} = \dot{\tilde{V}} = \tilde{R}(\alpha_{\text{imu}} - \tilde{\alpha}_b) + \gamma = R_B R R_A(\alpha_{\text{imu}} - \tilde{\alpha}_b) + \gamma.$$

Alternatively, working with $\tilde{T} = \sigma R_B(RT_A + T)$ and applying the dynamics to T,

$$\ddot{\tilde{T}} = \sigma R_B(\ddot{R}T_A + \ddot{T}) = \sigma R_B(\ddot{R}T_A + R(\alpha_{\text{imu}} - \alpha_b) + \gamma).$$

Taking the difference between these two expressions,

$$0 = \sigma R_B \ddot{R} T_A + R_B R(R_A\tilde{\alpha}_b - \sigma\alpha_b) + R_B R(\sigma\alpha_{\text{imu}} - R_A\alpha_{\text{imu}}) + (\sigma R_B - I)\gamma,$$

and multiplying by $R^T R_B^T$,

$$\begin{aligned}
0 &= \sigma(R^T\ddot{R})T_A + (R_A\tilde{\alpha}_b - \sigma\alpha_b) + (\sigma\alpha_{\text{imu}} - R_A\alpha_{\text{imu}}) + R^T(\sigma - R_B^T)\gamma \\
&= \sigma((\widehat{\omega}_{\text{imu}} - \widehat{\omega}_b)^2 + (\dot{\widehat{\omega}}_{\text{imu}} - \dot{\widehat{\omega}}_b))T_A + (R_A\tilde{\alpha}_b - \sigma\alpha_b) + (\sigma\alpha_{\text{imu}} - R_A\alpha_{\text{imu}}) + R^T(\sigma - R_B^T)\gamma.
\end{aligned}$$

Differentiating again,

$$\begin{aligned}
0 &= \sigma(\dot{R}^T\ddot{R} + R^T\dddot{R})T_A && (46)\\
&\quad + ((I - R_A)\sigma + (\sigma - 1)R_A)\dot{\alpha}_{\text{imu}} && (47)\\
&\quad + \dot{R}^T((I - R_B^T)\sigma + (\sigma - 1)R_B^T)\gamma. && (48)\\
&\quad + (R_A\dot{\tilde{\alpha}}_b - \sigma\dot{\alpha}_b) && (49)
\end{aligned}$$

As a sufficient excitation condition, assume that $\|\dot{R}(t)\| \leq \epsilon$, $\|\ddot{R}(t)\| \leq \epsilon$, and $\|\dddot{R}(t)\| \leq \epsilon$, for $t \in \mathcal{I}_{c_1}$. Under these constraints, (47) is bounded by $k_{c_1}\epsilon$, where, e.g. $k_{c_1} := 2M_\sigma M_A + (2M_\sigma + 1)(\|\gamma\| + 1)$. In that case,

$$\begin{aligned}
k_{c_1}\epsilon &\geq \max_{t\in\mathcal{I}_{c_1}} \|((I - R_A)\sigma + (\sigma - 1)R_A)\dot{\alpha}_{\text{imu}}(t)\| \\
&\geq |\sigma - 1| M(\dot{\alpha}_{\text{imu}}\!:\!\mathcal{I}_{c_1}) - M_\sigma\|I - R_A\|.
\end{aligned}$$

This rearranges to give (42).

(43) Now, assume that $\|\dot{R}(t)\| \leq \epsilon$, $\|\ddot{R}(t)\| \leq \epsilon$, and $\|\ddot{T}(t) - \gamma\| \leq \epsilon$, for $t \in \mathcal{I}_{c_2}$. Under these constraints, $\|\dot{\alpha}_{\text{imu}}\| \leq 2\epsilon$, and (46) is bounded by $k_{c_2}\epsilon$, where, e.g. $k_{c_2} := (2M_\sigma + 1)(\|\gamma\| + 3)$. In that case,

$$\begin{aligned}
k_{c_2}\epsilon &\geq \max_{t\in\mathcal{I}_{c_2}} \|\sigma((\widehat{\omega}_{\text{imu}} - \widehat{\omega}_b)(\dot{\widehat{\omega}}_{\text{imu}} - \dot{\widehat{\omega}}_b) + (\ddot{\widehat{\omega}}_{\text{imu}} - \ddot{\widehat{\omega}}_b))T_A\| \\
&= \max_{t\in\mathcal{I}_{c_2}} \|\sigma((R^T\dot{R})(R^T\ddot{R} - (R^T\dot{R})^2) + (\ddot{\widehat{\omega}}_{\text{imu}} - \ddot{\widehat{\omega}}_b))T_A\| \\
&\geq (1 - |1 - \sigma|)\max_{t\in\mathcal{I}_{c_2}} \|\ddot{\omega}_{\text{imu}}(t) \times T_A\| - (2M_\sigma + 1)M_A\epsilon \\
&\geq (1 - |1 - \sigma|)\,\|T_A\|\, m(\ddot{\omega}_{\text{imu}}\!:\!\mathcal{I}_{c_2}) - (2M_\sigma + 1)M_A\epsilon.
\end{aligned}$$

This rearranges to give (43).

(44) Finally, assume that $\|\ddot{R}(t)\| \leq \epsilon$, $\|\dddot{R}(t)\| \leq \epsilon$, and $\|\ddot{T}(t) - \gamma\| \leq \epsilon$ for $t \in \mathcal{I}_{c_3}$. As before, $\|\dot{\alpha}_{\text{imu}}\| \leq 2\epsilon$. Then, (46) + (47) is bounded by $k_{c_3}\epsilon$, where, e.g. $k_{c_3} = 2M_\sigma + 3$. In that case,

$$\begin{aligned}
k_{c_3}\epsilon &\geq \|\sigma(\dot{R}^T\ddot{R} + R^T\dddot{R})T_A + \dot{R}^T((I - R_B^T)\sigma + (\sigma-1)R_B^T)\gamma\| \\
&\geq \|\sigma\dot{R}^T(\ddot{R} + (I - R_B^T))\gamma\| - M_\sigma M_A\epsilon - |\sigma - 1|\,\|\dot{R}^T\|\,\|\gamma\| \\
&\geq (1 - |\sigma - 1|)\,\|\dot{R}^T(I - R_B^T)\gamma\| - M_\sigma M_A\epsilon - (|\sigma - 1| + \epsilon)\,\|\dot{R}^T\|\,\|\gamma\| \\
&\geq (1 - |\sigma - 1|)\,m(\dot{R}^T : \mathcal{I}_{c_3})\|(1 - R_B^T)\gamma\| - \epsilon(k_{c_3} + M_\sigma M_A) - (|\sigma - 1| + \epsilon)M(\dot{R}^T : \mathcal{I}_{c_3})\|\gamma\|
\end{aligned}$$

This rearranges to give (44).

□

2.3 Gauge transformations

The set of indistinguishable trajectories $\mathcal{I}$ is an equivalence class, and when the model is observable *up to the initial condition*, it is parametrized by $\tilde{x}_0$. Choosing the "true" initial condition $\tilde{x}_0 = x_0$ produces an indistinguishable set consisting of the sole "true" trajectory, otherwise it is a singleton other than the true trajectory. In some cases, the initial condition corresponds to an arbitrary choice of reference frame, and therefore the equivalence class of indistinguishable trajectory are related by a *gauge transformation* (a change of coordinates). As the equivalence class can be represented by any element, enforcing a particular reference for the gauge transformation yields strong observability (although the singleton may not correspond to the true trajectory).

Formally, an arbitrary choice of initial condition is sufficient to fix the gauge reference. For instance, the set of indistinguishable trajectories in the limit where $\epsilon \to 0$ is parametrized by an arbitrary $T_B \in \mathbb{R}^3$ and $\theta \in \mathbb{R}$,

$$\begin{cases}
\tilde{T} = \exp(\widehat{\gamma}\theta)T + T_B \\
\tilde{R} = \exp(\widehat{\gamma}\theta)R \\
\tilde{T}_{t_i} = \exp(\widehat{\gamma}\theta)\bar{T}_{t_i} + T_B \\
\tilde{R}_{t_i} = \exp(\widehat{\gamma}\theta)\bar{R}_{t_i} \qquad\qquad \text{up to } \mathcal{O}\left(\frac{\|\dot{\omega}_b\|}{\|\dot{\omega}_{\text{imu}}\|}, \frac{\|\dot{\alpha}_b\|}{\|\dot{\alpha}_{\text{imu}}\|}, \frac{1}{\|\gamma\|}\right) \\
\tilde{T}_{cb} = T_{cb} \\
\tilde{R}_{cb} = R_{cb}
\end{cases} \tag{50}$$

If we impose that $T(0) = \tilde{T}(0) = 0$, then $T_B = 0$ is determined; similarly, if we impose the initial pose to be aligned with gravity (so gravity is in the form $[0\ 0\ \|\gamma\|]^T$, then $\theta = 0$. But while we can impose this condition, we cannot *enforce* it, since the initial condition is not a part of the state of the filter, so we cannot relate the measurements at each time t directly to it.

However, if the gauge reference can be associated to *constant parameters* that are part of the state of the model, the gauge ambiguity can be enforced in a consistent manner. For instance, the ambiguous set of points is

$$\tilde{X}^j = g_a\bar{g}_i^{-1}g_i g_a^{-1}X^j. \tag{51}$$

If each group i contains at least 3 non-coplanar points, it is possible to fix $\bar{g}_i$ by parametrizing $X^j \doteq \bar{y}_{t_i}^j Z^j$ and imposing thre directions $y_{t_i}^j = \tilde{y}_{t_i}^j = y^j(t_i), j =$

$1, \ldots, 3$, the measurement of these directions at time t_i when they first appear. This yields $\bar{g}_i = g_i$ and $\tilde{X}^j = X^j$ for that group. Note that it is necessary to impose this constraint in *each group.*

The residual set of indistinguishable trajectories is parameterized by *constants* θ, T_B, that determine a Gauge transformation for the groups, that can be fixed by always fixing the pose of *one* of the groups. This can be done in a number of ways. For instance, if for a certain group i we impose

$$R_{t_i} = \tilde{R}_{t_i} = \hat{R}(t_i) \text{ and } T_{t_i} = \tilde{T}_{t_i} = \hat{T}(t_i) \tag{52}$$

by assigning their value to the current best estimate of pose and not including the corresponding variables in the state of the model, then we have that

$$\hat{R}(t_i) = \exp(\widehat{\gamma}\theta)\hat{R}(t_i) \tag{53}$$

and therefore $\theta = 0$; similarly,

$$T_B = (I - \exp(\widehat{\gamma}\theta))T(t_i) = 0 \tag{54}$$

Therefore, the gauge transformation is enforced explicitly at each instant of time, as each measurement provides a constraint on the states. This suggests the following modeling procedure in the design of a filter/observer for bearing-assisted navigation:

1. Set $T(0) = 0$ with zero model error covariance, and zero initial covariance.
2. Set $R(0) = R_0$ such that $[I_{2\times 2}0]R_0\alpha_{\text{imu}} = 0$, with zero model error and non-zero initial covariance.
3. Fix gravity to $[0,\ 0,\ \|\gamma\|]^T$ from tabulates
4. Initialize at rest, then perform some fast motions before groups of features are added.
5. Add K groups, each with $2N + N$ states, plus their pose for each group but one.
6. Fix 2 directions per group ([6] fixes all directions; this results in a non-zero mean component of the innovation, that in turn results in a small bias in all other states, that have to account for/absorb the mean)
7. Fix the pose of one group (remove its pose from the state)
8. Triage groups before adding them to the state.

After the Gauge Transformation has ben fixed, the model is observable, and therefore a properly designed observer will converge to a solution $\tilde{x}$ that is related to the true one x as follows:

$$
\begin{aligned}
\tilde{X}^{ref} &= (1+\tilde{\sigma})\tilde{R}_{cb}e^{\omega_B}e^{\widehat{\gamma}\theta}e^{\omega_A}\tilde{R}_{cb}^T(X^{\text{ref}}-T_A)+(1+\tilde{\sigma})(\tilde{R}_{cb}e^{\omega_A}T_B+\tilde{R}_{cb}T_A+\tilde{T}_{cb}) && (55)\\
\tilde{X}^j &= (1+\tilde{\sigma})\tilde{R}_{cb}\bar{R}_i\tilde{R}_{t_i}\tilde{R}_{cb}^T(X^j-T_A)+(1+\tilde{\sigma})(\tilde{R}_{cb}\bar{R}_i\tilde{T}_{t_i}+\tilde{R}_{cb}\bar{T}_i+\tilde{T}_{cb}) && (56)\\
\tilde{T} &= e^{\widehat{\gamma}\theta}T+T_B(1+\tilde{\sigma})+\omega_B e^{\widehat{\gamma}\theta}T+e^{\omega_B}e^{\widehat{\gamma}\theta}RT_A(1+\tilde{\sigma}) && (57)\\
\tilde{R} &= e^{\omega_B}e^{\widehat{\gamma}\theta}Re^{\omega_A} && (58)\\
\tilde{T}_{t_i} &= e^{\widehat{\gamma}\theta}\bar{T}_i+T_B(1+\tilde{\sigma})+\omega_B e^{\widehat{\gamma}\theta}\bar{T}_i+e^{\omega_B}e^{\widehat{\gamma}\theta}\bar{R}_iT_A(1+\tilde{\sigma}) && (59)\\
\tilde{R}_{t_i} &= e^{\omega_B}e^{\widehat{\gamma}\theta}\bar{R}_ie^{\omega_A} && (60)\\
\tilde{T}_{cb} &= T_{cb}+\tilde{\sigma}T_{cb}+R_{cb}T_A(1+\tilde{\sigma}) && (61)\\
\tilde{R}_{cb} &= R_{cb}\exp(\omega_A) && (62)\\
\tilde{\alpha}_b &= (\mathbf{??})\\
\tilde{\omega}_b &= (\mathbf{??})
\end{aligned}
$$

w$here$

$$
\begin{aligned}
\|T_A\| &\leq \frac{2k\min_t\|\dot{\omega}_b\|}{\max_t\|\ddot{\omega}_{\text{imu}}\|}\\
\|\omega_A\| &\leq \frac{2\min_t\|\dot{\omega}_b\|}{\max_t\|\dot{\omega}_{\text{imu}}\|}\\
\|\omega_B\| &\leq \left(\frac{3k\max(\min_t\|\dot{\omega}_b\|,\min_t\|\dot{\alpha}_b\|)}{\min(\max_t\|\dot{\omega}_{\text{imu}}\|,\max_t\|\dot{\alpha}_{\text{imu}}\|,\|\gamma\|)}\right)\\
|\tilde{\sigma}| &\leq \left(\frac{2k\min_t\|\dot{\alpha}_b\|}{\min(\max_t\|\dot{\omega}_{\text{imu}}\|,\max_t\|\dot{\alpha}_{\text{imu}}\|)}\right)
\end{aligned}
$$

and arbitrary θ, T_B and suitable constant κ. The groups will be defined up to an arbitrary reference frame $(\bar{R}_i, \bar{T}_i)$, except for the reference group where that transformation is fixed. Note that, as the reference group "switches" (when points in the reference group become occluded or otherwise disappear due to failure in the data association mechanism), a small error in pose is accumulated. This error affects the gauge transformation, not the *state* of the system, and therefore is not reflected in the innovation, nor in the covariance of the state estimate, that remains bounded. This is unlike [**?**], where the covariance of the translation state T_B and the rotation about gravity θ grows unbounded over time, possibly affecting the numerical aspects of the implementation. Notice that in the limit where $\dot{\omega}_b = \dot{\alpha}_b = 0$, we obtain back Eq. (50).