12.5: Lines & Planes in Space

Lines

The **vector equation** for line L through $P_0(x_0, y_0, z_0)$ parallel to vector $\vec{v}$:

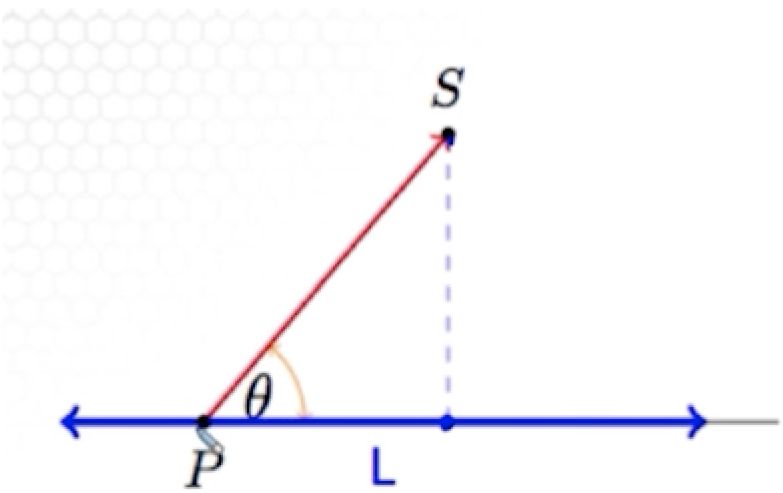
$$\vec{r}(t) = \overbrace{\vec{r}_0}^{P_0} + t\vec{v} \ (-\infty < t < \infty)$$

The **standard parametrization** through $P_0(x_0, y_0, z_0)$ parallel to $\vec{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$:

$$\begin{aligned} x(t) &= x_0 + tv_1 \\ y(t) &= y_0 + tv_2 \\ z(t) &= z_0 + tv_3 \end{aligned}$$

If is line, $-\infty < t < \infty$. If t is bounded, is line segment.

Distance from point to line



$$d = \frac{\left\| \overrightarrow{PS} \times \vec{v} \right\|}{\|\vec{v}\|}$$

Planes

The **vector equation** for a plane through $P_0(x_0, y_0, z_0)$ with normal vector $\vec{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is given by:

$$\vec{n} \cdot (\overrightarrow{P_0 P}) = 0$$

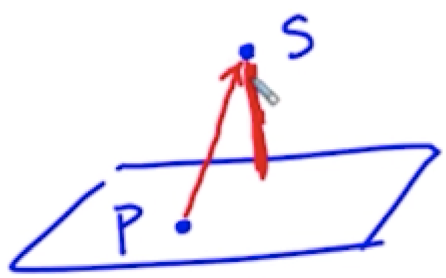
Component equation:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Angle between two planes

- Parallel planes have the same normal.
- Angle between two intersecting planes = acute angle between normals $\left(\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\| \|\vec{n}_2\|}\right)$

Distance from point to plane



$$d = \left\| \text{proj}_{\hat{n}} \overrightarrow{PS} \right\| = \overrightarrow{PS} \cdot \overbrace{\hat{n}}^{\text{normalized } \vec{n}}$$

Use this to find **distance between skew lines**

- Given lines l_1, l_2, find unit normal vector of the $l_1 l_2$ plane, $\hat{n} = \frac{\vec{l_1} \times \vec{l_2}}{\|\vec{l_1} \times \vec{l_2}\|}$, and project $\overrightarrow{PS}$ onto $\hat{n}$

Intersecting lines & planes

Lines

Lines: l_1, l_2 can be: parallel, coincident, skew, intersecting

- coincident: same line
- skew: neither parallel nor intersecting

If direction vectors are same, parallel or coincident

- Pick point on l_1. If on l_2, coincident

- Else parallel

If not, skew or intersecting

- Check for intersecting point. If exists, intersecting
- Else skew

Planes

Parallel, intersecting, coincident

- If normals are parallel, planes are parallel
- If normals are not parallel, cross product gives direction vector for line of intersection of the planes

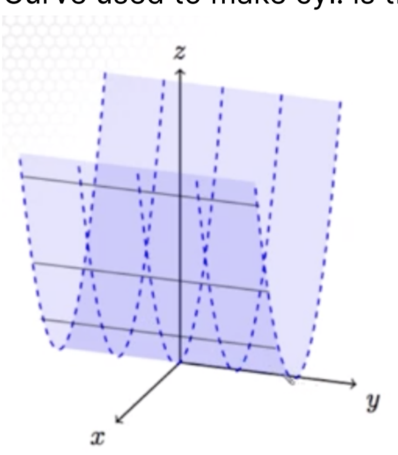
Line & Plane

- Substitute line into plane and find point where line & plane intersect

12.6: Cylinders and Quadric Surfaces

Cylinder: surface generated by moving straight line along given planar curve, holding line parallel to given fixed line

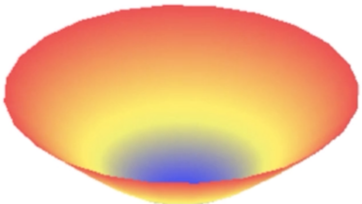
- Curve used to make cyl. is the **generating curve**



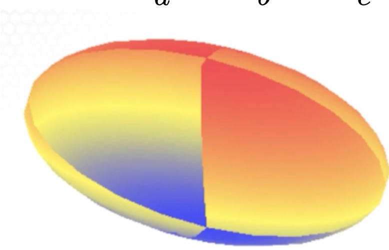
Quadric Surface: 2nd degree equation in x, y, z

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Jz + K = 0$$

Elliptical Paraboloid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$

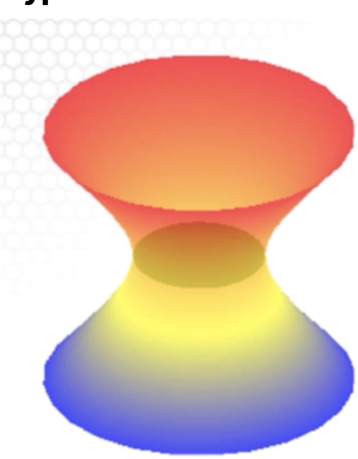


Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

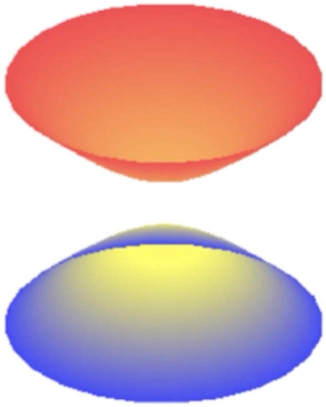


Elliptical Cone: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$

Hyperboloid of One Sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



Hyperboloid of Two Sheets: $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



Hyperbolic Paraboloid: $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}$

Saddle from "how to theoretically turn a sphere inside out" lookin ass

13.1, 13.2: Vector Functions

Vector function $\vec{f}\colon \mathbb{R} \mapsto \mathbb{R}^n$

- Given real valued functions f_1, f_2, f_3,
  - $\vec{f}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$
- f_1, f_2, f_3 are the **components** of $\vec{f}$
- Parametrization should be obvious

Limits of Vector Functions

Let $\vec{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$,

$$\lim_{t \to t_0} \vec{r}(t) = \vec{L}$$

if for every $\epsilon > 0$, there exists corresponding $\delta > 0$ such that for all $t \in D$,

$$\|\vec{r}(t) - \vec{L}\| < \epsilon \text{ whenever } 0 < |t - t_0| < \delta$$

idk why i wrote that cause basically the idea is: to find lim, find lim of each component

$\vec{f}$ is **continuous** at point $t = t_0$ if $\lim_{t \to t_0} \vec{f}(t) = \vec{f}(t_0)$
($\vec{f}$ is continuous if all points are continuous)

Limit rules

Given $\vec{f}(t) \to \vec{L}, \vec{g}(t) \to \vec{M}, u(t) \to U$, then:

1. $\vec{f}(t) + \vec{g}(t) \to \vec{L} + \vec{M}$
2. $\alpha \vec{f}(t) \to \alpha \vec{L}$
3. $u(t)\vec{f}(t) \to U\vec{L}$
4. $\vec{f}(t) \cdot \vec{g}(t) \to \vec{L} \cdot \vec{M}$
5. $\vec{f}(t) \times \vec{g}(t) \to \vec{L} \times \vec{M}$

Derivatives of Vector Functions

$$\vec{r}'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}$$

If limit exists, $\vec{r}$ is **differentiable** at t.
($\vec{r}$ is differentiable if all points are differentiable)

$\vec{r}'$ is the same as the sum of the derivatives of each component

Derivative rules

1. $(\vec{f} + \vec{g})'(t) = \vec{f}'(t) + \vec{g}'(t)$
2. $(\alpha \vec{f})'(t) = \alpha \vec{f}'(t)$
3. $(\vec{f} \cdot \vec{g})'(t) = \vec{f}'(t) \cdot \vec{g}(t) + \vec{f}(t) \cdot \vec{g}'(t)$ (product rule, applies for *dot product*, *cross product*, *scalar multiplication*)
4. $(\vec{f} \circ u)'(t) = \vec{f}'(u(t))u'(t)$ (chain rule)

Tangent Lines, Velocity, Acceleration

$\vec{r}(t)$ is **smooth** if $\frac{dr}{dt}$ is continuous & never zero

Tangent line to smooth curve $\vec{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \text{ at } t = t_0$
is the line that passes $\vec{r}(t_0)$ & is parallel to $\vec{r}'(t_0)$.

If $\vec{r}$ is position,

- $\vec{v} = \frac{d\vec{r}}{dt}$ (velocity)
- $\vec{a} = \frac{d\vec{v}}{dt}$ (acceleration)
- direction of motion = direction of $\vec{v}$
- speed = $\|\vec{v}\|$

Integrals of Vector Functions

$\vec{R}(t)$ is antiderivative of $\vec{r}(t)$ on interval I if $\frac{d\vec{R}}{dt} = \vec{r}$ at each point in I.

$$\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{C}$$

(componentize to find indef/def integral)

Integral rules

1. $\int_a^b (\vec{f} + \vec{g})(t)\,dt = \int_a^b \vec{f}(t)\,dt + \int_a^b \vec{g}(t)\,dt$
2. $\int_a^b (\alpha\vec{f})(t)\,dt = \alpha\int_a^b \vec{f}(t)\,dt$
3. $\int_a^b (\vec{c}\vec{f})(t)\,dt = \vec{c}\int_a^b \vec{f}(t)\,dt$
4. $\left\|\int_a^b (\vec{f})(t)\,dt\right\| \leq \int_a^b \left\|\vec{f}(t)\right\|dt$

Projectile Motion

Ideal projectile motion: bro use kinematics