16.1: Line Integrals

If f is defined on a curve C given parametrically by $\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, the line integral of f over C is:

$$\int_C f(x, y, z)\, ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k$$

(This is the form for a **line integral of a scalar field**)

To integrate a continuous function $f(x, y, z)$ over a curve C:

1. Find a smooth parametrization of C:

$$\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}\$\$(where\$a \leq t \leq b\$)$$

2. Evaluate the integral as:

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))|\vec{v}(t)|\, dt$$

(recall $\frac{ds}{dt} = |\vec{v}|$)

Mass and Moment Calculations

Suppose we need to find the mass & moment for coil springs and then rods lying along a smooth curve C in space.

Recall physics definitions from 15.6.
They apply here, too.
Mass

$$m = \int_C \lambda\, ds$$

(this is a pretty straightforward extension of 15.6 so I don't think there needs to be notes here)

16.2: Vector Fields & Line Integrals

Let $\vec{F}$ be a vector field with continuous components defined along smooth curve C parametrized by $\vec{r}(t), a \leq t \leq b$.

The **line integral of $\vec{F}$ along** C is:

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \left(F \cdot \frac{d\vec{r}}{ds} \right) ds = \int_C \vec{F} \cdot d\vec{r}$$

(This is the form for a **line integral of a vector field**)

To evaluate, write $\vec{F}$ and $d\vec{r}$ in terms of t and apply dot product.

Line integrals may also be written as:

$$\begin{aligned} &\int_C M\, dx + \int_C N\, dy + \int_C P\, dz \\ &= \int_C M(x,y,z)\, dx + \int_C N(x,y,z)\, dy + \int_C P(x,y,z)\, dz \end{aligned}$$

(same idea, write everything in terms of t)

Example

Evaluate $\int_C \vec{F} \cdot dr$, where $\vec{F} = \langle xy, x^2z, xyz \rangle$ along $y = x^2$ from $(0,0,0)$ to $(1,1,0)$ followed by the straight-line segment from $(1,1,0)$ to $(1,1,1)$.

$$\begin{aligned} C_1 : \vec{r}_1(t) &= \langle t, t^2, 0 \rangle \\ \vec{r}_1'(t) &= \langle 1, 2t, 0 \rangle \end{aligned}$$

$$\begin{aligned} C_2 : \vec{r}_2(t) &= \langle 1, 1, t \rangle \\ \vec{r}_2'(t) &= \langle 0, 0, 1 \rangle \end{aligned}$$

$$\begin{aligned} &\int_C \vec{F} \cdot d\vec{r} \\ &= \int_0^1 \langle (t)(t^2), 0, 0 \rangle \cdot \langle 1, 2t, 0 \rangle\, dt + \int_0^1 \langle (1)(1), (1)^2(t), (1)(1)(t) \rangle \cdot \langle 0, 0, 1 \rangle\, dt \\ &= \int_0^1 \langle t^3, 0, 0 \rangle \cdot \langle 1, 2t, 0 \rangle\, dt + \int_0^1 \langle 1, t, t \rangle \cdot \langle 0, 0, 1 \rangle\, dt \end{aligned}$$

Physics

Work

$$W = \int_C \vec{F} \cdot d\vec{r}$$

- $\vec{F}$ is force

Flow

$$\text{Flow} = \int_C \vec{F} \cdot \vec{T}\, ds$$

- $\vec{F}$ is velocity

This integral is called a **flow integral**. If the curve starts and ends at the same point, the flow is called the *circulation* around the curve.

Flux (across a smooth simple closed plane curve)

$$\Phi = \int_C \vec{F} \cdot \hat{n}\, ds$$

- $\vec{F}$ is a vector field in the plane, $M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$
- C is a smooth simple closed curve (starts & ends at same place and does not cross itself)
- $\hat{n}$ is the outward-pointing unit vector normal to C

Alternative form:

$$\Phi \text{ across } C = \oint M\, dy - N\, dx$$

(Integral is evaluated at any parametrization $\vec{r}$ that traces C counterclockwise exactly once)

Why?

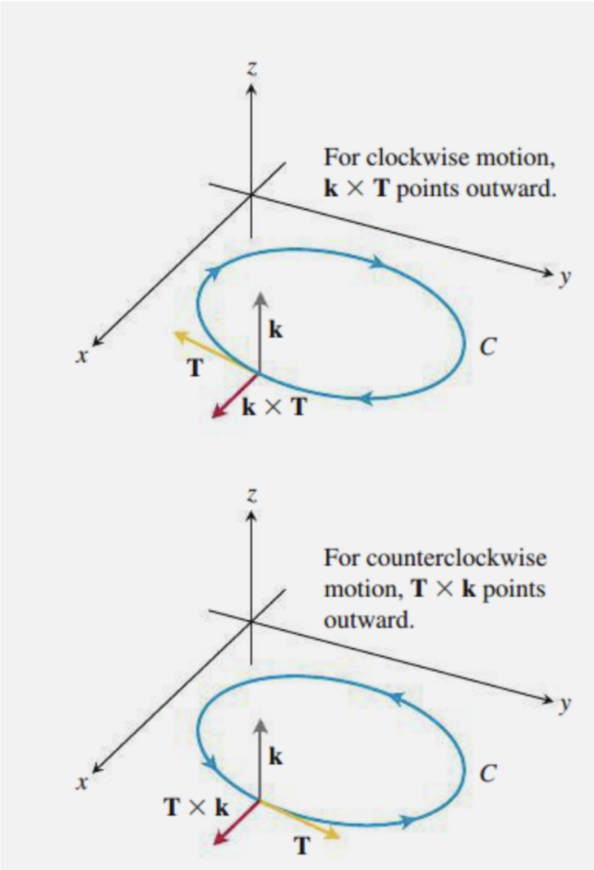

FIGURE 16.24 To find an outward unit normal vector for a smooth simple curve C in the xy-plane that is traversed counterclockwise as t increases, we take $\mathbf{n} = \mathbf{T} \times \mathbf{k}$. For clockwise motion, we take $\mathbf{n} = \mathbf{k} \times \mathbf{T}$.

Assuming counterclockwise,

$$\hat{n} = \vec{T} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}$$

Then:

$$\begin{aligned} \Phi &= \int_C \vec{F} \cdot \hat{n}\, ds \\ &= \int_C \langle M, N \rangle \cdot \left\langle \frac{dy}{ds}, \frac{-dx}{ds} \right\rangle ds \\ &= \oint M\, dy - N\, dx \end{aligned}$$

16.3: Path Independence, Conservative Fields, Potential Functions

Definitions

Let $\vec{F}$ be a vector field defined on open region D in space.
Suppose that for any two points A and B in D, $\int_C \vec{F} \cdot d\vec{r}$ along path C from A to B is the same over all paths from A to B.
The integral is **path independent** and the field is **conservative on** D.

If $\vec{F}$ is a vector field on D and $F = \nabla f$ for some scalar function f on D, f is called a **potential function for** F.

Example

Find a potential function f for $\vec{F} = (y \sin z)\mathbf{i} + (x \sin z)\mathbf{j} + (xy \cos z)\mathbf{k}$.

Let $\vec{F} = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$.

$$\frac{\partial f}{\partial x} = y \sin z$$

$$f = \int y \sin z \, dx = xy \sin z + \overbrace{g(y, z)}^{\text{remember } +C?}$$

$$\frac{\partial f}{\partial y} = x \sin z + \frac{\partial g}{\partial y}$$

$$\frac{\partial f}{\partial z} = xy \cos z + \frac{\partial g}{\partial z}$$

So,

$$x \sin z + \frac{\partial g}{\partial y} = x \sin z \implies \frac{\partial g}{\partial y} = 0$$

$$xy \cos z + \frac{\partial g}{\partial z} = xy \cos z \implies \frac{\partial g}{\partial z} = 0$$

Therefore,

$$f(x, y, z) = xy \sin z + C$$

Conservative Fields & Gradient Fields

Theorem

Let $\vec{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ (and M, N, P are continuous throughout open connected region D).
F is conservative iff $\vec{F}$ is a gradient field ∇f for a differentiable function f.

Component Test for Conservative Fields

Let $\vec{F} = X(x,y,z)\mathbf{i} + Y(x,y,z)\mathbf{j} + Z(x,y,z)\mathbf{k}$ on open simply connected domain (X, Y, Z have continuous first partial derivatives).
then $\vec{F}$ is conservative iff:

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}$$
$$\frac{\partial X}{\partial z} = \frac{\partial Z}{\partial x}$$
$$\frac{\partial Y}{\partial z} = \frac{\partial Z}{\partial y}$$

If $\vec{F}$ is a gradient field for f, then $F = \nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$.
Then, second mixed partials need to be equivalent ($f_{xy} = f_{yx}, f_{xz} = f_{zx}, f_{yz} = f_{zy}$) for this to be true.

Fundamental Theorem of Line Integrals

Let C be a smooth curve joining points A and B, parametrized by $\vec{r}(t)$.
Let f be a differentiable function with continuous gradient vector $\vec{F} = \nabla f$ on domain D containing C.

$$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

Loop Property of Conservative Fields

Equivalent statements

1. $\oint_C F \cdot d\vec{r} = 0$ around every loop (every closed curve C) in D.
2. The field F is conservative on D.

Exactness

Differential form: Expression of the form $M(x,y,z)\,dx + N(x,y,z)\,dy + P(x,y,z)\,dz$.
A differential form is **exact** on domain D if:

$$M\,dx + N\,dy + P\,dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz = df$$

for some scalar function f throughout D.
In other words, a differential form is exact iff $\langle M, N, P \rangle = \nabla f$ (iff $\vec{F} = \langle M, N, P \rangle$ is conservative).