14.1: Functions of Several Variables

Definition

Suppose D is a set of n-tuples of real numbers $(x_1, x_2, x_3, \ldots, x_n)$.

A **real-valued function** f on D returns a real number $\left(f : D \rightarrow \mathbb{R}\right)$

- domain = D
- range = set of values returned

Examples

Find domain of each:

1. $f(x, y) = \sqrt{xy}$

   Answer: $\{(x, y) | xy \geq 0\}$

2. $f(x, y) = \frac{1}{\sqrt{x-y}}$

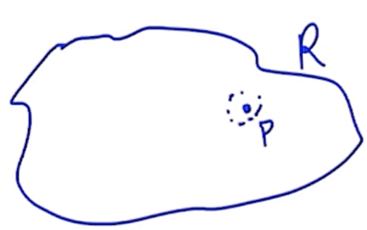
   $x - y > 0$, so:
   Answer: $\{(x, y) | x > y\}$

3. $f(x, y, z) = \frac{\sqrt{z}}{x^2 - y^2}$

   $x^2 - y^2 \neq 0$
   $z \geq 0$
   Answer: $\{(x, y, z) | x^2 \neq y^2, z \geq 0\}$

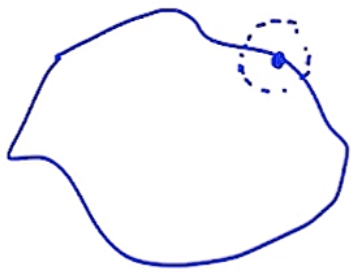
Boundary Points and Interior Points

Interior point of a set (or region) R: A point (x_0, y_0) in the center of a disk of positive radius that lies entirely in R.



Boundary point of a set (or region R): A point (x_0, y_0) where every disk of positive radius contains points that lie outside of R and points that lie in R.

(x_0, y_0) does not need to be in R.

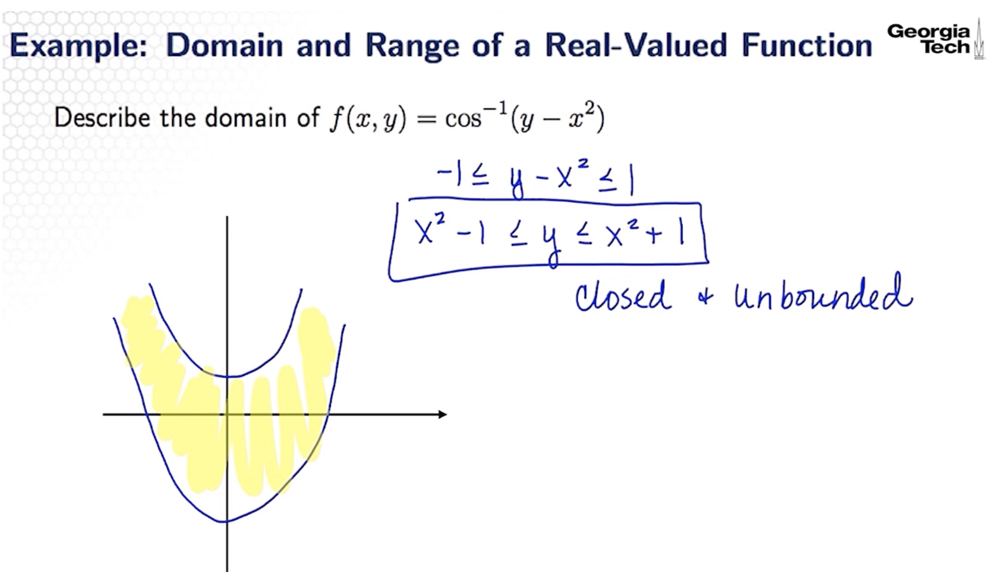


(For the above definitions, in 3D, replace "disk" with ball)

A set is **closed** if it contains <u>all</u> of its boundary points.
A set is **open** if it contains <u>none</u> of its boundary points.
Otherwise, it is neither open nor closed.

Bounded regions: A region that lies inside a disk of finite radius

- **Unbounded**: a region that doesn't



Level Curves and Surfaces

If c is a value in range of 2-var f, we can sketch the **level curve** $f(x, y) = c$.
If c is a value in range of 3-var f, we can sketch the **level surface** $f(x, y, z) = c$.

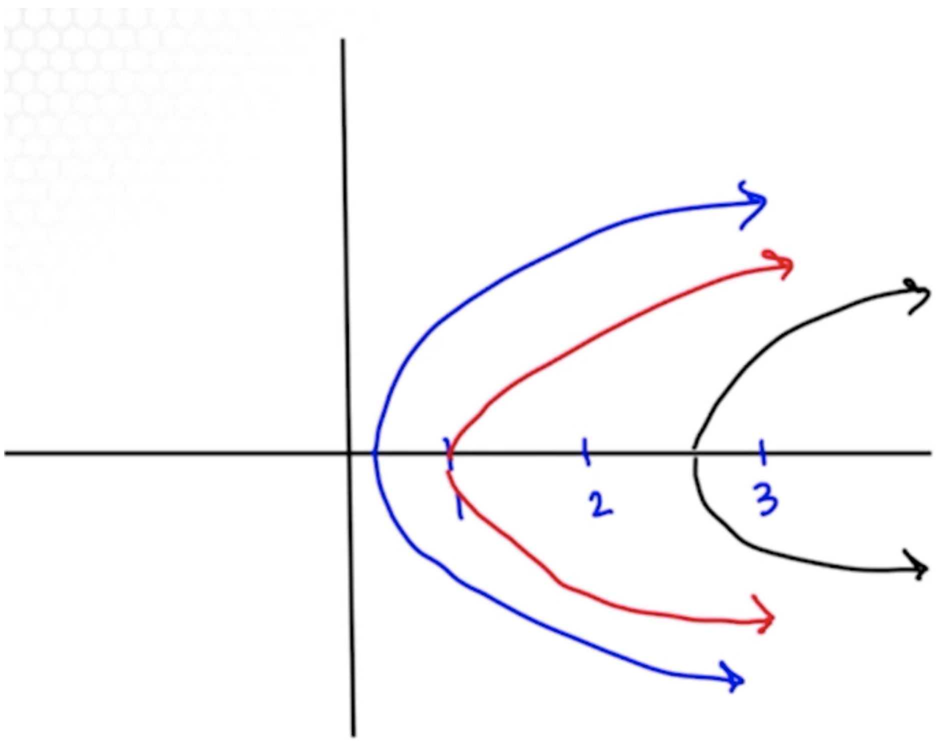
Examples

1. Graph level curves of $f(x, y) = \ln(x - y^2)$ for $c = -1, 0, 1$.

$$\begin{aligned}\ln(x-y^2)&=-1\\ x-y^2&=e^{-1}\\ x&=y^2+e^{-1}\end{aligned}$$

$$\begin{aligned}\ln(x-y^2)&=0\\ x-y^2&=e^{0}\\ x&=y^2+1\end{aligned}$$

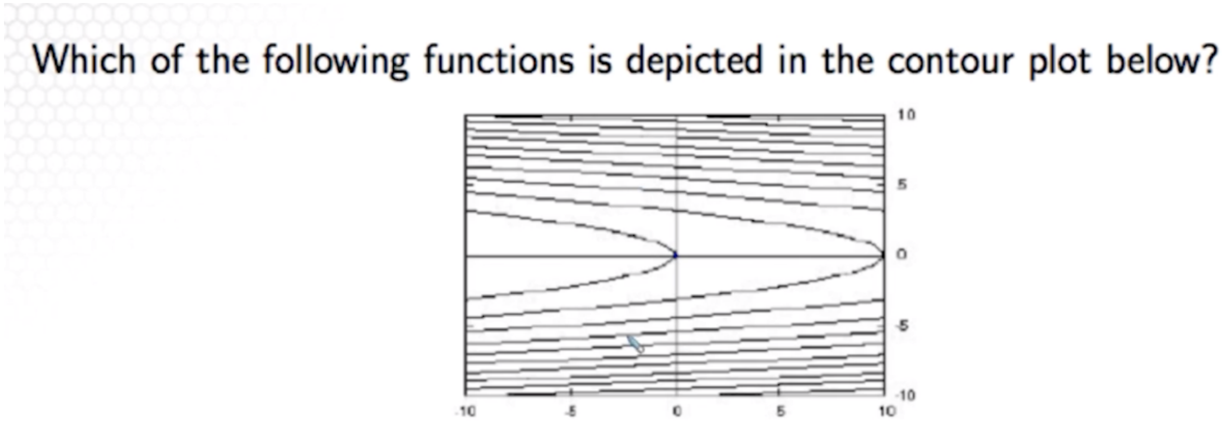
$$\begin{aligned}\ln(x-y^2)&=1\\ x-y^2&=e^{1}\\ x&=y^2+e\end{aligned}$$



2. Describe the level curves for $f(x,y)=e^{x^2+2y^2}$.

$$\begin{aligned}e^{x^2+2y^2}&=c\\ x^2+2y^2&=\underbrace{\ln c}_{\text{constant}}\ (c>0)\end{aligned}$$

The level curves are ellipses.



3.

- The contours are level curves.

Recognize one of the curves as $x = -y^2$, so $x + y^2 = c$, and $f(x, y) = x + y^2$.

14.2: Limits and Continuity

Limits for Functions of Several Variables

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined (at least) on some deleted neighborhood of $\vec{x}_0$.

$$\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = L$$

if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < \|\vec{x} - \vec{x}_0\| < \delta \implies |f(\vec{x}) - L| < \epsilon$$

(Essentially, as $\vec{x}$ approaches $\vec{x}_0$ from anywhere, $f(\vec{x})$ approaches L.)

Two paths test

If the paths $\vec{x} \to \vec{x}_0$ approach different values for $f(\vec{x})$, the limit does not exist.

Example: Proving a limit using the definition

This is a doozy.

Find $\displaystyle\lim_{(x,y) \to (0,0)} f(x, y)$ for $f(x, y) = \dfrac{2xy^2}{x^2 + y^2}$.

If the limit is true, there must be some relationship between ϵ and δ.

On the ϵ side:

$$\left|f(x,y) - \overbrace{L}^{\text{After testing, we find some paths to converge to 0}}\right| < \epsilon$$

$$\left|\frac{2xy^2}{x^2+y^2} - 0\right| < \epsilon$$

$$\frac{2|x|y^2}{x^2+y^2} < \epsilon$$

On the δ side:

$$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} \qquad < \delta$$

$$0 < \sqrt{x^2+y^2} \qquad < \delta$$

$$0 < 2\sqrt{x^2+y^2} \qquad < 2\delta$$

Since $y^2 \leq x^2 + y^2$,
$\frac{y^2}{x^2+y^2} \leq 1$
and $\frac{2|x|y^2}{x^2+y^2} \leq 2|x|$

So:

$$\frac{2|x|y^2}{x^2+y^2} \leq 2|x| = 2\sqrt{x^2} \leq 2\sqrt{x^2+y^2} < \epsilon$$

Thus, you can let $\delta = \frac{\epsilon}{2}$ and so the limit exists.

Continuity for Functions of Several Variables

A function $f(x,y)$ is *continuous* at point (x_0, y_0) if:

1. f is defined at (x_0, y_0)
2. Limit at that point exists
3. Limit is equal to $f(x_0, y_0)$

Function is continuous if it is continuous at every point in its domain.

Define $f(1,1)$ in a way that makes $f(x,y) = \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$ continuous at the point $(1,1)$.

$$\lim_{(x,y)\to(1,1)} \frac{x^2 - xy}{(\sqrt{x} - \sqrt{y})} \frac{(\sqrt{x} + \sqrt{y})}{(\sqrt{x} + \sqrt{y})} = \lim_{(x,y)\to(1,1)} \frac{x\cancel{(x-y)}(\sqrt{x} + \sqrt{y})}{\cancel{(x-y)}} = 1(2) = 2$$

$$f(x,y) = \begin{cases} \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}} & (x,y) \neq (1,1) \\ 2 & (x,y) = (1,1) \end{cases}$$

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