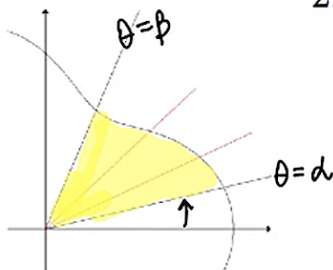


## 15.4: Double Integrals in Polar Form

The area of a polar region is based on the area of a sector of a circle.

- Area of a circle:  $\pi r^2$
- Area of a sector:  $\frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \theta$

$$\text{Area of a sector} = \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \theta$$



$$\begin{aligned} \text{Area of polar region} &= \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta \\ &= \int_{\alpha}^{\beta} \int_a^b r dr d\theta \\ &= \iint_R r dr d\theta \end{aligned}$$

(Note that  $\int_a^b r dr = \frac{1}{2} r^2 \Big|_a^b$ )

$$dA = r dr d\theta = dx dy$$

Given  $F(r, \theta)$  is continuous on  $\Gamma : a \leq r \leq b, \alpha \leq \theta \leq \beta$ :

$$\iint_{\Gamma} F(r, \theta) r dr d\theta = \int_{\alpha}^{\beta} \int_a^b F(r, \theta) r dr d\theta$$

Given  $F(r, \theta)$  is continuous on  $\Omega : \alpha \leq \theta \leq \beta, \rho_1(\theta) \leq r \leq \rho_2(\theta)$ :

$$\iint_{\Omega} F(r, \theta) r dr d\theta = \int_{\alpha}^{\beta} \int_{\rho_1(\theta)}^{\rho_2(\theta)} F(r, \theta) r dr d\theta$$

[Fubini's theorem](#) still applies, so these integration bounds can be rearranged as expected.

## Volume

If  $F(r, \theta) \geq 0$  over region  $R$ , then volume with  $R$  as base, bounded above by  $F(r, \theta)$  is:

$$V = \iint_R F(r, \theta) r \, dr \, d\theta$$

(same as double integrals above)

### Example: Double Integrals with Polar Coordinates

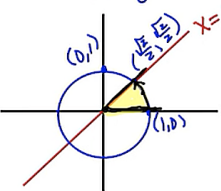
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Evaluate  $\int_0^{\sqrt{2}/2} \int_y^{\sqrt{1-y^2}} (x^2 + y^2)^{3/2} dx dy$  by changing to polar coordinates.

$x^2 + y^2 = r^2$

$- 0 \leq y \leq \frac{\sqrt{2}}{2}$

$- y \leq x \leq \sqrt{1-y^2}$



$0 \leq r \leq 1$

$0 \leq \theta \leq \pi/4$

$r = 1$

$x = \sqrt{1-y^2}$

$x^2 = 1-y^2$

$x^2 + y^2 = 1$

$dA = r \, dr \, d\theta$

$\int_0^{\pi/4} \int_0^1 \underbrace{r^3 \cdot r}_{r^4} \, dr \, d\theta$

$\int_0^{\pi/4} \left[ \frac{1}{5} r^5 \right]_0^1 d\theta = \int_0^{\pi/4} \frac{1}{5} d\theta$

$= \frac{1}{5} \theta$

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