13.3: Arc Length in Space

Length of smooth curve $\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ $(a \le t \le b)$

$$L = \int_a^b \sqrt{\left(rac{dx}{dt}
ight)^2 + \left(rac{dy}{dt}
ight)^2 + \left(rac{dz}{dt}
ight)^2} \, dt$$

(pythagorean theorem between dx, dy, dz)

This is equivalent to:

$$L = \int_a^b \| ec{v} \| \, dt$$

Arc Length Parameter: Function s that finds directed distance along curve starting from $P(t_0)$ to some point P(t)

$$s(t) = \int_{t_0}^t \sqrt{\left(rac{dx}{d au}
ight)^2 + \left(rac{dy}{d au}
ight)^2 + \left(rac{dz}{d au}
ight)^2} \, d au = \int_{t_0}^t \|ec{v}\| \, d au$$

Speed:

$$speed = rac{ds}{dt} = \| ec{v}(t) \|$$

Unit Tangent Vector: Unit vector... that's tangent to the smooth curve idk what you expected Imao

$$ec{T}(t) = rac{ec{r}'(t)}{\|ec{r}'(t)\|} = rac{ec{v}(t)}{\|ec{v}(t)\|} = rac{dec{r}/dt}{ds/dt}$$

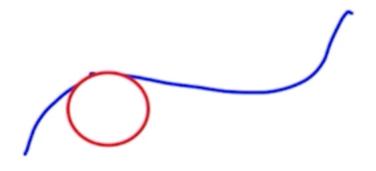
 $(\vec{v}(t) \text{ normalized})$

13.4: Curvature

If \vec{T} is a unit vector of a smooth curve, the **curvature** function of the curve is

$$\kappa = \left\| rac{dec{T}}{ds}
ight\|$$

In the blue curve, the curvature at the point is related to circle that best fit curve at that point.



(seems similar to 2nd derivative)

For smooth curve \vec{r} , curvature can be written as scalar function:

$$\kappa = \left\|rac{dec{T}/dt}{ds/dt}
ight\| = rac{\|ec{T}'(t)\|}{\|ec{v}(t)\|}$$

Circle of Curvature

The **circle of curvature** (or **osculating circle**) at point P on plane curve (2D) where $\kappa \neq 0$ is the circle of the curve that

- 1. is tangent to curve at P
- 2. has the same curvature the curve has at P
- 3. has center that lies toward the concave side of the curve

The **radius of curvature** at point P is $ho = \frac{1}{\kappa}$.

- Straight lines: curvature is constantly 0
- Circle of radius r: Curvature is constantly $\frac{1}{r}$.

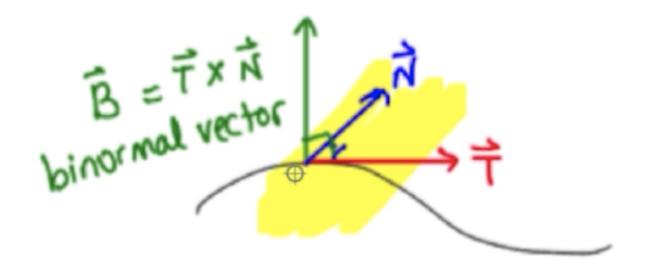
Principal Normal Vector

If $\vec{T}(t)$ is unit tangent vector and $\vec{T}'(t)=0$, then unit tangent vector d/n change direction. If $\vec{T}'(t)\neq 0$, then

Principal normal vector =
$$ec{N}(t) = rac{ec{T}'(t)}{\|ec{T}'(t)\|}$$

 $(ec{T}' ext{ normalized})$ (this vector is $oldsymbol{ar{T}}$ to $ec{T}$)

TNB Frame



Binormal vector: $ec{B}(t) = ec{T}(t) imes ec{N}(t)$

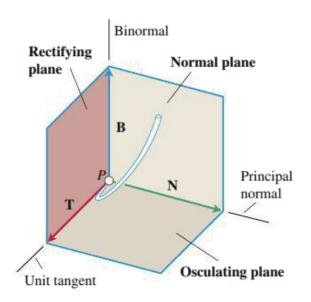
TNB frame / Frenet frame: The three vectors T, N, B

Planes

• Osculating plane: Plane between \vec{T} and \vec{N} (normal is \vec{B})

• Normal plane: Plane between \vec{N} and \vec{B} (normal is \vec{T})

• Rectifying plane: Plane between \vec{T} and \vec{B} (normal is \vec{N})



13.5: Tangential & Normal Components of Acceleration

Given position function $\vec{r}(t)$,

$$egin{align} ec{T}(t) &= rac{ec{v}(t)}{ds/dt} \ ec{v}(t) &= ec{T}(t)rac{ds}{dt} \ ec{a}(t) &= rac{dec{v}}{dt} &= ec{T}'(t)rac{ds}{dt} + ec{T}(t)rac{d^2s}{dt^2} \ \end{align}$$

Since
$$ec{N}(t)=rac{ec{T}'(t)}{\|ec{T}'(t)\|}$$
 ,

$$egin{align} ec{N}(t) \| ec{T}'(t) \| &= ec{T}'(t) \ ec{a}(t) &= ec{N}(t) \| ec{T}'(t) \| rac{ds}{dt} + ec{T}(t) rac{d^2s}{dt^2} \end{align}$$

So,

Tangential component of acceleration:

$$a_T = rac{d^2s}{dt^2} = rac{d}{dt} \|ec{v}\|$$

- Only dependent on change of speed of object
- If speed is constant, $a_T=0$ and acceleration is directed entirely towards center of curvature

Normal component of acceleration:

$$a_N = \|ec{T}'(t)\|rac{ds}{dt} = \kappaigg(rac{ds}{dt}igg)^2 = \kappa\|ec{v}\|^2$$

$$\left(\mathrm{recall} \, \kappa = rac{\|ec{T}'(t)\|}{ds/dt}
ight)$$

Curvature and Torsion

Torsion:

Let $\vec{B} = \vec{T} imes \vec{N}$.

$$au = -rac{dec{B}}{ds}\cdotec{N}$$

Measures how binormal vector changes with respect to arc length

$$au = rac{egin{array}{cccc} \cdots & ec{r}' & \cdots \ \cdots & ec{r}'' & \cdots \ \hline \cdots & ec{r}''' & \cdots \ \hline \|ec{v} imes ec{a}\|^2 \end{array}$$

Formulas for Curvature and Torsion

$$egin{aligned} ec{T}\cdotec{a} &= a_T(ec{T}\cdotec{T}) + a_N(ec{T}\cdotec{N}) = a_T \ \|ec{T} imesec{a}\| &= \|a_T(ec{T} imesec{T})\| + \|a_N(ec{T} imesec{N})\| = \|a_Nec{B}\| = a_N \end{aligned}$$

So,

$$egin{align} a_T &= rac{ec{v} \cdot ec{a}}{ds/dt} \ a_N &= rac{\|ec{v} imes ec{a}\|}{ds/dt} = \kappa igg(rac{ds}{dt}igg)^2 \ \end{align}$$

And thus,

$$\kappa = rac{\|ec{v} imesec{a}\|}{(ds/dt)^3}$$

13.6: Motion in Polar Coordinates

Given coordinates $P(r,\theta)$, position, velocity, and acceleration can be represented in terms of:

- $ec{u}_r = (\cos heta) \mathbf{i} + (\sin heta) \mathbf{j}$ (unit vector in direction of $\overrightarrow{\mathit{OP}}$)
- $ec{u}_{ heta} = -(\sin heta) \mathbf{i} + (\cos heta) \mathbf{j}$ (unit vector pointing in direction of increasing heta)

$$egin{aligned} ec{r} &= rec{u}_r = r\cos heta \mathbf{i} + r\sin heta \mathbf{j} \ ec{v} &= rac{dec{r}}{dt} = (-r heta'\sin heta + r'\cos heta)\mathbf{i} + (r heta'\cos heta + r'\sin heta)\mathbf{j} \ &= r heta'ec{u}_ heta + r'ec{u}_r \ ec{a} &= rac{dec{v}}{dt} = (-r heta'^2ec{u}_r + (r heta'' + r' heta')ec{u}_ heta) + (r''ec{u}_r + r' heta'ec{u}_ heta) \end{aligned}$$

(btw you can product rule $\vec{r}=r\vec{u}_r$ instead of expanding \vec{r} and differentiating its components)

$$egin{aligned} ec{r} &= rec{u}_r \ ec{v} &= r'ec{u}_r + r heta'ec{u}_ heta \ ec{a} &= (r'' - r heta'^2)ec{u}_r + (r heta'' + 2r' heta')ec{u}_ heta \end{aligned}$$

Cylindrical Coordinates

$$egin{aligned} ec{r} &= rec{u}_r + z\mathbf{k} \ ec{v} &= r'ec{u}_r + r heta'ec{u}_ heta + z'\mathbf{k} \ ec{a} &= (r'' - r heta'^2)ec{u}_r + (r heta'' + 2r' heta')ec{u}_ heta + z''\mathbf{k} \end{aligned}$$