15.8: Integration by Substitution

Jacobians

Jacobian determinate or **Jacobian** of the coordinate transformation $x = g(u, v), y = h(u, v)$:

$$J(u, v) = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}$$

The two coordinate systems are related by:

$$dx \, dy = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du \, dv$$

Why?
The Jacobian transform maps the uv-coordinate system onto the xy-coordinate system.
In a mapping from uv to xy, the area $du \, dv$ will be multiplied by a factor of the determinant of the transform (recall from linear algebra) to get the corresponding area $dx \, dy$.

Extension into 3D
Jacobians can also be extended pretty trivially to 3 dimensions.
Given $x = g(u, v, w), y = h(u, v, w), z = k(u, v, w)$,

$$J(u, v, w) = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Double Integrals

Suppose $f(x, y)$ is continuous over region R. Let G be preimage of R under transform $x = g(u, v), y = h(u, v)$ (assumed to be one-to-one on interior of G). If functions g and h have continuous 1st partial derivatives within interior of G:

$$\iint_R f(x, y) \, dx \, dy = \iint_G f(g(u, v), h(u, v)) \overbrace{\left| \frac{\partial(x, y)}{\partial(u, v)} \right|}^{\text{Jacobian}} du \, dv$$

Triple Integrals

$$\iiint_R f(x, y, z)\, dx\, dy\, dz$$

$$= \iiint_G f(g(u, v, w), h(u, v, w), k(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw$$

Derivation of Spherical Triple Integral by Jacobians

Spherical coordinates:

$$x = \rho \sin \phi \cos \theta$$
$$y = \rho \sin \phi \sin \theta$$
$$z = \rho \cos \phi$$

Derivation:

$$dx\, dy\, dz = \overbrace{\left| \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} \right|}^{\text{What we're trying to find}} d\rho\, d\phi\, d\theta$$

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{vmatrix}$$

$$= \cos \phi(\rho^2 \sin \phi \cos \phi \cos^2 \theta + \rho^2 \sin \phi \cos \phi \sin^2 \theta)$$
$$+ \rho \sin \phi(\rho \sin^2 \phi \cos^2 \theta + \rho^2 \sin \phi^2 \sin^2 \theta)$$

$$= \cos \phi(\rho^2 \sin \phi \cos \phi)$$
$$+ \rho \sin \phi(\rho \sin^2 \phi)$$

$$= \rho^2 \sin \phi \cos^2 \phi + \rho^2 \sin \phi \sin^2 \phi$$

$$= \rho^2 \sin \phi$$

Therefore, $dV = dx\, dy\, dz = \rho^2 \sin \phi\, d\rho\, d\phi\, d\theta$