# 15.7: Triple Integrals in Cylindrical & Spherical Coordinates

## Cylindrical Coordinates

- Represent point $P$ in space by ordered triples $(r, \theta, z)$ $(r \geq 0)$

1. $r$ and $\theta$ are polar coordinates for the projection of $P$ onto the $xy$-plane
2. $z$ is the rectangular vertical coordinate

### Usage

Should be used when...

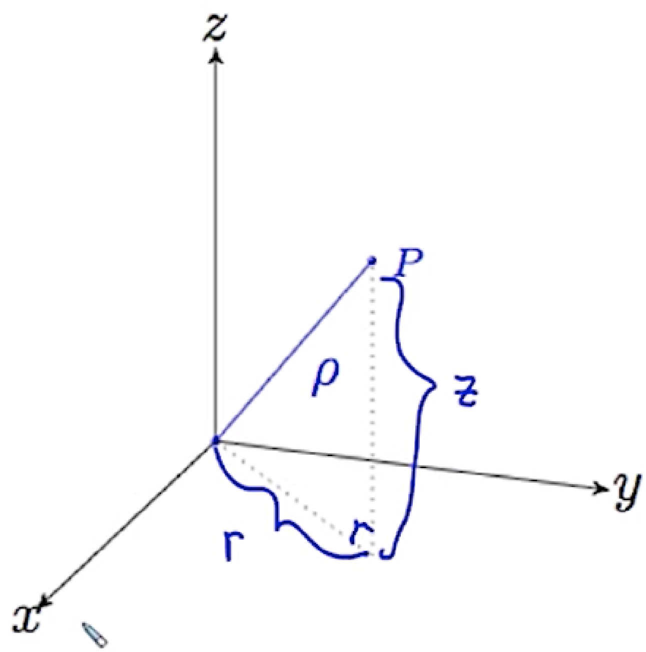
- there is an axis of symmetry
- an integrand involves $x^2 + y^2$
- we're integrating over a circle (or part of) in the $xy$-plane

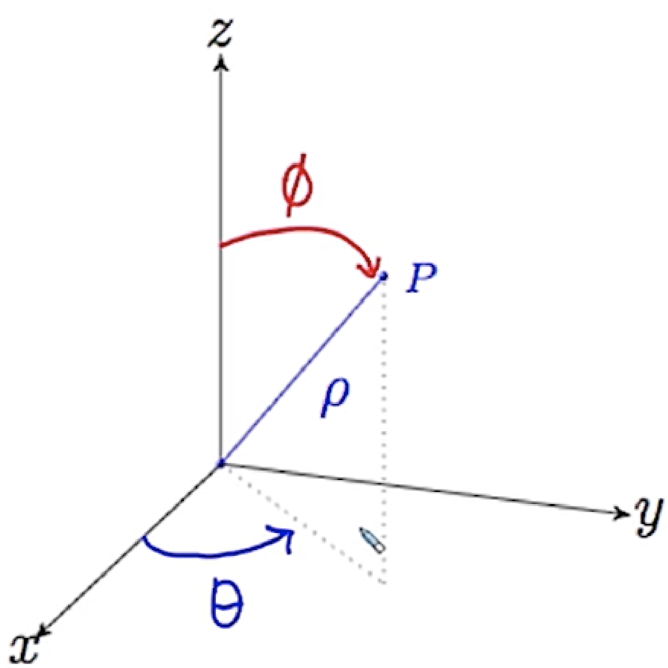
Very similar to using polar coordinates w/ double integrals, but with an added $z$ component for triple integrals.

## Spherical Coordinates

- Represent point $P$ in space by ordered triples $(\rho, \phi, \theta)$

1. $\rho$ is distance from $P$ to the origin $(\rho \geq 0)$
2. $\phi$ is the angle $\overrightarrow{OP}$ makes with the $+z$-axis $(0 \leq \phi \leq \pi)$
3. $\theta$ is the angle from cylindrical coordinates




**Converting Rectangular to Spherical**

$$\rho^2 = x^2 + y^2 + z^2 = r^2 + z^2$$

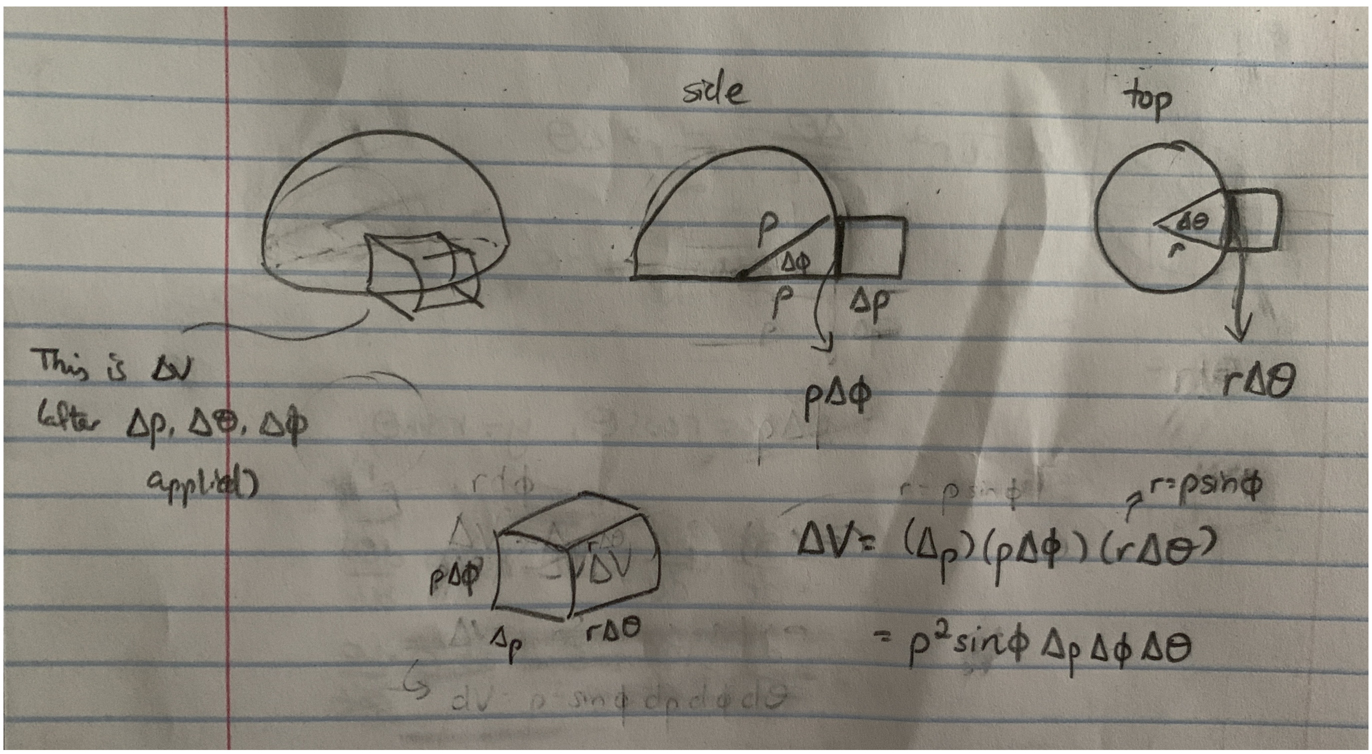
$$r = \rho \sin \phi$$
$$z = \rho \cos \phi$$

$$x = r \cos \theta = \rho \sin \phi \cos \theta$$
$$y = r \sin \theta = \rho \sin \phi \sin \theta$$

## Triple Integral Definition

$$\iiint_T dV = \iiint_T \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

**Why?**




$\Delta V$ is the curved box above. Assuming $\Delta V$ is a rectangular prism (when $\Delta V$ is very small, it's essentially a rectangular prism),

$$\Delta V = (\Delta\rho)\overbrace{(\rho\,\Delta\phi)}^{\text{arclength from the side}}\underbrace{(r\,\Delta\theta)}_{\text{arclength from the top}}$$

$$\begin{aligned} &= \rho r\,\Delta\rho\,\Delta\phi\,\Delta\theta \\ &= \rho(\rho\sin\phi)\,\Delta\rho\,\Delta\phi\,\Delta\theta \\ &= \rho^2\sin\phi\,\Delta\rho\,\Delta\phi\,\Delta\theta \end{aligned}$$

$$dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Related: purely algebraic derivation

# 15.8: Integration by Substitution

## Jacobians

**Jacobian determinate** or **Jacobian** of the coordinate transformation $x = g(u, v), y = h(u, v)$:

$$J(u, v) = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

The two coordinate systems are related by:

$$dx\,dy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv$$

**Why?**
The Jacobian transform maps the $uv$-coordinate system onto the $xy$-coordinate system.
In a mapping from $uv$ to $xy$, the area $du\,dv$ will be multiplied by a factor of the determinant of the transform (recall from linear algebra) to get the corresponding area $dx\,dy$.

**Extension into 3D**
Jacobians can also be extended pretty trivially to 3 dimensions.
Given $x = g(u, v, w), y = h(u, v, w), z = k(u, v, w)$,

$$J(u, v, w) = \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

## Double Integrals

Suppose $f(x,y)$ is continuous over region $R$. Let $G$ be preimage of $R$ under transform $x = g(u,v), y = h(u,v)$ (assumed to be one-to-one on interior of $G$). If functions $g$ and $h$ have continuous 1st partial derivatives within interior of $G$:

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(g(u,v),h(u,v)) \overbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}^{\text{Jacobian}} du\,dv$$

# Triple Integrals

$$\begin{aligned}&\iiint_R f(x,y,z)\,dx\,dy\,dz \\ &= \iiint_G f(g(u,v,w),h(u,v,w),k(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw\end{aligned}$$

## Derivation of Spherical Triple Integral by Jacobians

Spherical coordinates:

$$\begin{aligned}x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi\end{aligned}$$

Derivation:

$$dx\,dy\,dz = \overbrace{\left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right|}^{\text{What we're trying to find}} d\rho\,d\phi\,d\theta$$

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} &= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} \\
&= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \\
&= \cos\phi(\rho^2\sin\phi\cos\phi\cos^2\theta + \rho^2\sin\phi\cos\phi\sin^2\theta) \\
&+ \rho\sin\phi(\rho\sin^2\phi\cos^2\theta + \rho^2\sin\phi^2\sin^2\theta) \\
&= \cos\phi(\rho^2\sin\phi\cos\phi) \\
&+ \rho\sin\phi(\rho\sin^2\phi) \\
&= \rho^2\sin\phi\cos^2\phi + \rho^2\sin\phi\sin^2\phi \\
&= \rho^2\sin\phi
\end{aligned}$$

Therefore, $dV = dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$