

14.3: Partial Derivatives

Definition

If the limit exists, the **partial derivative** of $f(x, y)$ with respect to x at (x_0, y_0) is:

$$\left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

(Differentiate f with respect to x , while holding other variables constant)

(This extends trivially to other variables & more dimensions)

Differentiability

Function $z = f(x, y)$ is **differentiable** at (x_0, y_0) if f_x, f_y (the partial derivatives) and $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$ satisfies:

$$\Delta z = (f_x + \epsilon_1)\Delta x + (f_y + \epsilon_2)\Delta y$$

(where each $\epsilon_1, \epsilon_2 \rightarrow 0$ as both $\Delta x, \Delta y \rightarrow 0$)

f is **differentiable** if it is differentiable at every point in its domain.

If f is differentiable, then the graph of f is called a **smooth surface**

Why?

In single variable calculus, given a function $y = f(x)$, the change in $\Delta y = f(x_0 + \Delta x) - f(x_0)$ can be approximated by the change according to the tangent line at that point.

$$y - f(x_0) = f'(x_0)(x - x_0) \text{ (tangent line form)}$$

$$f_{approx}(x_0 + \Delta x) - f(x_0) = f'(x_0)\Delta x$$

$$\Delta y \approx f'(x_0)\Delta x$$

If we let the difference in the correct slope be ϵ , we can say $\Delta y = (f'(x_0) + \epsilon)\Delta x$.

If f is differentiable, then as $\Delta x \rightarrow 0, \epsilon \rightarrow 0$.

This idea can be extended into multivariable calculus by using multiple ϵ .

Example

Find all first partial derivatives of $f(x, y) = 3x^2 - 2y + xy$.

$$f_x = \frac{\partial f}{\partial x} = 6x + y$$
$$f_y = \frac{\partial f}{\partial y} = -2 + x$$

Second Order Partial Derivatives

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2}$$
$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x}$$
$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y}$$
$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2}$$

Mixed Derivatives Theorem

If $f(x, y)$ and all of f_x, f_y, f_{xy}, f_{yx} are defined throughout an open region containing a point (x_0, y_0) and are all continuous at (x_0, y_0) , then

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$$

Example

Find all first & second partial derivatives of $f(x, y) = 2x^2 \cos y + 3y^2 \sin x$.

$$f_x = 4x \cos y + 3y^2 \cos x$$
$$f_y = -2x^2 \sin y + 6y \sin x$$

$$\begin{aligned}
 f_{xx} &= 4 \cos y - 3y^2 \sin x \\
 f_{xy} &= -4x \sin y + 6y \cos x \\
 f_{yx} &= -4x \sin y + 6y \cos x \\
 f_{yy} &= -2x^2 \cos y + 6 \sin x
 \end{aligned}$$

Implicit Differentiation

Suppose we have a function in terms of three variables x, y, z and we cannot solve the equation for z , but we want to find $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$.

Do implicit differentiation as expected, but hold the necessary variables constant.

Example

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $xy - z^2y - 2zx = 0$.

$$y - 2zy \frac{\partial z}{\partial x} - 2z - 2x \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}(-2zy - 2x) + y - 2z = 0$$

$$\frac{\partial z}{\partial x}(-2zy - 2x) = 2z - y$$

$$\frac{\partial z}{\partial x} = \frac{y - 2z}{2zy + 2x}$$

$$x - z^2 - 2zy \frac{\partial z}{\partial y} - 2x \frac{\partial z}{\partial y} = 0$$

$$x - z^2 - (2zy + 2x) \frac{\partial z}{\partial y} = 0$$

$$(2zy + 2x) \frac{\partial z}{\partial y} = x - z^2$$

$$\frac{\partial z}{\partial y} = \frac{x - z^2}{2zy + 2x}$$

14.4: Chain Rule

x, y as functions of one variable

If $w = f(x, y)$ is differentiable and $x(t), y(t)$ are differentiable with respect to t , then $w = f(x(t), y(t))$ is differentiable with respect to t .

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

(This is trivially extendible to more dimensions)

Example

Find $\frac{du}{dt}$: $u = x^2 - 3xy = 2y^2$, $x(t) = \cos t$, $y(t) = \sin t$

$$\frac{dx}{dt} = -\sin t$$

$$\frac{dy}{dt} = \cos t$$

$$\frac{\partial u}{\partial x} = 2x - 3y$$

$$\frac{\partial u}{\partial y} = -3x + 4y$$

So:

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$$

$$= (2 \cos(t) - 3 \sin(t))(-\sin(t)) \\ + (-3 \cos(t) + 4 \sin(t))(\cos(t))$$

$$= -2 \sin(t) \cos(t) + 3 \sin^2(t) - 3 \cos^2(t) + 4 \sin(t) \cos(t) \\ = 2 \sin(t) \cos(t) + 3 \sin^2(t) - 3 \cos^2(t) \\ = \sin(2t) - 3 \cos(2t)$$

x, y as functions of multiple variables

What if $u = u(x, y)$, $x = x(s, t)$, $y = y(s, t)$?

You can find partial derivatives $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$, replacing $\frac{dx}{dt}$ in the simpler chain rule with the respective partial derivative.

$$\begin{aligned}\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} \\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t}\end{aligned}$$

Example

Given $z = 4e^x \ln y$, $x = \ln(uv)$, $y = u \sin(v)$, express $\frac{\partial z}{\partial u}$ as a function of u and v .

$$\begin{aligned}\frac{\partial x}{\partial u} &= \frac{1}{u} \\ \frac{\partial y}{\partial u} &= \sin(v)\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial x} &= 4e^x \ln y \\ &= 4uv \ln(u \sin(v))\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{4e^x}{y} \\ &= \frac{4v}{\sin(v)}\end{aligned}$$

So:

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \\ &= 4uv \ln(u \sin(v)) \frac{1}{u} + \frac{4v}{\sin(v)} \sin(v) \\ &= 4v \ln(u \sin(v)) + 4v\end{aligned}$$

Implicit Differentiation (again)

If $F(x, y)$ is differentiable and $F(x, y) = 0$ defines y as differentiable function of x ,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

In 3D,

If $z = z(x, y)$ and $F(x, y, z(x, y)) = 0$, you can do the same thing, but to find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$

Why?

Let F be written as $F(x, y(x))$.

$$\begin{aligned} F(x, y(x)) &= 0 \\ \frac{d}{dx} F(x, y(x)) &= \frac{d}{dx} 0 \\ \frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx} &= 0 \\ \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} &= 0 \\ F_x + F_y \frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{F_x}{F_y} \end{aligned}$$

Examples

Find $\frac{dy}{dx}$ at the specified point:

$$\underbrace{xy + y^2 - 3x - 3 = 0}_{F(x,y)}, (-1, 1)$$

$$F_x = y - 3 \quad F_y = x + 2y$$

$$\frac{dy}{dx} = - \frac{(y-3)}{x+2y} \quad \left. \frac{dy}{dx} \right|_{(-1,1)} = 2$$

Find $\frac{\partial z}{\partial y}$ at the specified point:

$$\underbrace{xe^y + ye^z + 2\ln(x) - 2 - 3\ln(2) = 0}_{F(x,y,z)}, (1, \ln(2), \ln(3))$$

$$\frac{\partial z}{\partial y} = - F_y / F_z \quad F_y = xe^y + e^z \quad F_z = ye^z$$

$$\frac{\partial z}{\partial y} = - \frac{(xe^y + e^z)}{ye^z} \quad \left. \frac{\partial z}{\partial y} \right|_{(1, \ln(2), \ln(3))} = \frac{-(2+3)}{\ln(2) \cdot 3} = \frac{-5}{3\ln(2)}$$