15.7: Triple Integrals in Cylindrical & Spherical Coordinates

Cylindrical Coordinates

- Represent point P in space by ordered triples (r, θ, z) $(r \geq 0)$

1. r and θ are polar coordinates for the projection of P onto the xy-plane
2. z is the rectangular vertical coordinate

Usage

Should be used when...

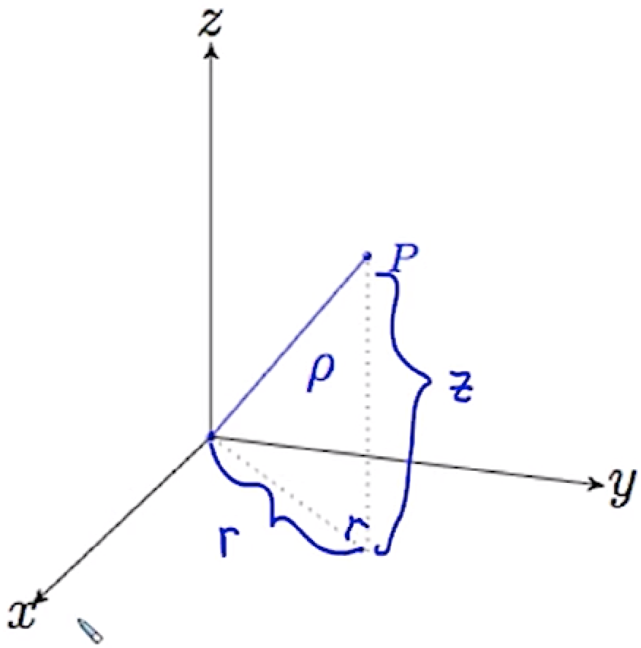
- there is an axis of symmetry
- an integrand involves $x^2 + y^2$
- we're integrating over a circle (or part of) in the xy-plane

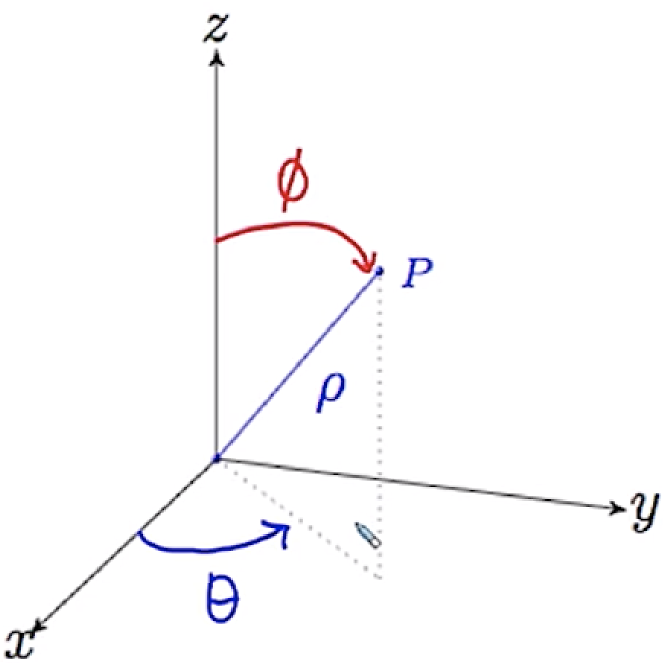
Very similar to using polar coordinates w/ double integrals, but with an added z component for triple integrals.

Spherical Coordinates

- Represent point P in space by ordered triples (ρ, ϕ, θ)

1. ρ is distance from P to the origin $(\rho \geq 0)$
2. ϕ is the angle $\overrightarrow{OP}$ makes with the $+z$-axis $(0 \leq \phi \leq \pi)$
3. θ is the angle from cylindrical coordinates





Converting Rectangular to Spherical

$$\rho^2 = x^2 + y^2 + z^2 = r^2 + z^2$$

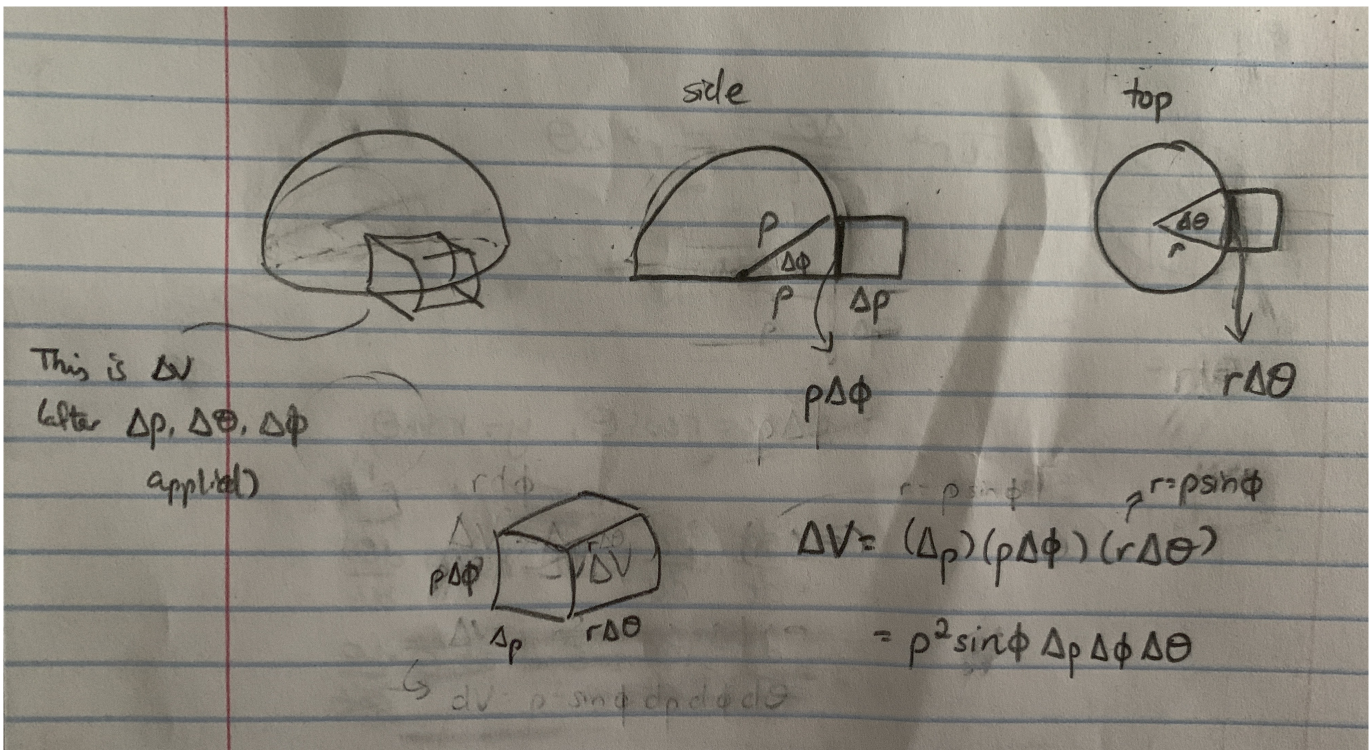
$$r = \rho \sin \phi$$
$$z = \rho \cos \phi$$

$$x = r \cos \theta = \rho \sin \phi \cos \theta$$
$$y = r \sin \theta = \rho \sin \phi \sin \theta$$

Triple Integral Definition

$$\iiint_T dV = \iiint_T \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

Why?



ΔV is the curved box above. Assuming ΔV is a rectangular prism (which is converges to at small ΔV),

$$\Delta V = (\Delta\rho)\overbrace{(\rho\,\Delta\phi)}^{\text{arclength from the side}}\underbrace{(r\,\Delta\theta)}_{\text{arclength from the top}}$$

$$\begin{aligned} &= \rho r\,\Delta\rho\,\Delta\phi\,\Delta\theta \\ &= \rho(\rho\sin\phi)\,\Delta\rho\,\Delta\phi\,\Delta\theta \\ &= \rho^2\sin\phi\,\Delta\rho\,\Delta\phi\,\Delta\theta \end{aligned}$$

$$dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

15.8: Integration by Substitution

Double Integrals

Jacobian determinate or **Jacobian** of the coordinate transformation $x = g(u,v), y = h(u,v)$:

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

Substitution for Double Integrals:
Suppose $f(x,y)$ is continuous over region R. Let G be preimage of R under transform $x = g(u,v), y = h(u,v)$ (assumed to be one-to-one on interior of G). If functions g and h have continuous 1st partial derivatives within interior of G:

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(g(u,v),h(u,v))\overbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}^{\text{Jacobian}}\,du\,dv$$

Triple Integrals

Given $x = g(u,v,w), y = h(u,v,w), z = k(u,v,w)$,

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$
\begin{aligned}
&\iiint_R f(x,y,z)\,dx\,dy\,dz \\
&= \iiint_G f(g(u,v,w),h(u,v,w),k(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw
\end{aligned}
$$