16.1: Line Integrals over Scalar Fields

If f is defined on a curve C given parametrically by $\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, the line integral of f over C is:

$$\int_C f(x, y, z)\, ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k$$

To integrate a continuous function $f(x, y, z)$ over a curve C:

1. Find a smooth parametrization of C:

$$\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}\$\$(where\$a \leq t \leq b\$)$$

2. Evaluate the integral as:

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))|\vec{v}(t)|\, dt$$

(recall $\frac{ds}{dt} = |\vec{v}|$)

Mass and Moment Calculations

Suppose we need to find the mass & moment for coil springs and then rods lying along a smooth curve C in space.

Recall physics definitions from 15.6.
They apply here, too.
Mass

$$m = \int_C \lambda\, ds$$

(this is a pretty straightforward extension of 15.6 so I don't think there needs to be notes here)

16.2: Line Integrals over Vector Fields

Let $\vec{F}$ be a vector field with continuous components defined along smooth curve C parametrized by $\vec{r}(t), a \leq t \leq b$.

The **line integral of $\vec{F}$ along C** is:

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \left(F \cdot \frac{d\vec{r}}{ds}\right) ds = \int_C \vec{F} \cdot d\vec{r}$$

To evaluate, write $\vec{F}$ and $d\vec{r}$ in terms of t and apply dot product.

Line integrals may also be written as:

$$\begin{aligned} &\int_C M\, dx + \int_C N\, dy + \int_C P\, dz \\ &= \int_C M(x, y, z)\, dx + \int_C N(x, y, z)\, dy + \int_C P(x, y, z)\, dz \end{aligned}$$

(same idea, write everything in terms of t)

Example

Evaluate $\int_C \vec{F} \cdot dr$, where $\vec{F} = \langle xy, x^2 z, xyz \rangle$ along $y = x^2$ from $(0, 0, 0)$ to $(1, 1, 0)$ followed by the straight-line segment from $(1, 1, 0)$ to $(1, 1, 1)$.

$$\begin{aligned} C_1 : \vec{r}_1(t) &= \langle t, t^2, 0 \rangle \\ \vec{r}_1'(t) &= \langle 1, 2t, 0 \rangle \end{aligned}$$

$$\begin{aligned} C_2 : \vec{r}_2(t) &= \langle 1, 1, t \rangle \\ \vec{r}_2'(t) &= \langle 0, 0, 1 \rangle \end{aligned}$$

$$\begin{aligned} &\int_C \vec{F} \cdot d\vec{r} \\ &= \int_0^1 \langle (t)(t^2), 0, 0 \rangle \cdot \langle 1, 2t, 0 \rangle\, dt + \int_0^1 \langle (1)(1), (1)^2(t), (1)(1)(t) \rangle \cdot \langle 0, 0, 1 \rangle\, dt \\ &= \int_0^1 \langle t^3, 0, 0 \rangle \cdot \langle 1, 2t, 0 \rangle\, dt + \int_0^1 \langle 1, t, t \rangle \cdot \langle 0, 0, 1 \rangle\, dt \end{aligned}$$

Applications to Physics

Work

$$W = \int_C \vec{F} \cdot d\vec{r}$$

- $\vec{F}$ is force

Flow

$$\text{Flow} = \int_C \vec{F} \cdot \vec{T}\, ds$$

- $\vec{F}$ is velocity

This integral is called a **flow integral**. If the curve starts and ends at the same point, the flow is called the *circulation* around the curve.

Flux

$$\Phi = \int_C \vec{F} \cdot \hat{n}\, ds$$

- $\vec{F}$ is a vector field in the plane, $M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$
- C is a smooth simple closed curve (starts & ends at same place and does not cross itself)
- $\hat{n}$ is the outward-pointing unit vector normal to C

Flux across a planar curve

$$\Phi \text{ across } C = \oint M\, dy - N\, dx$$

(Integral is evaluated at any parametrization $\vec{r}$ that traces C counterclockwise exactly once)

Why?

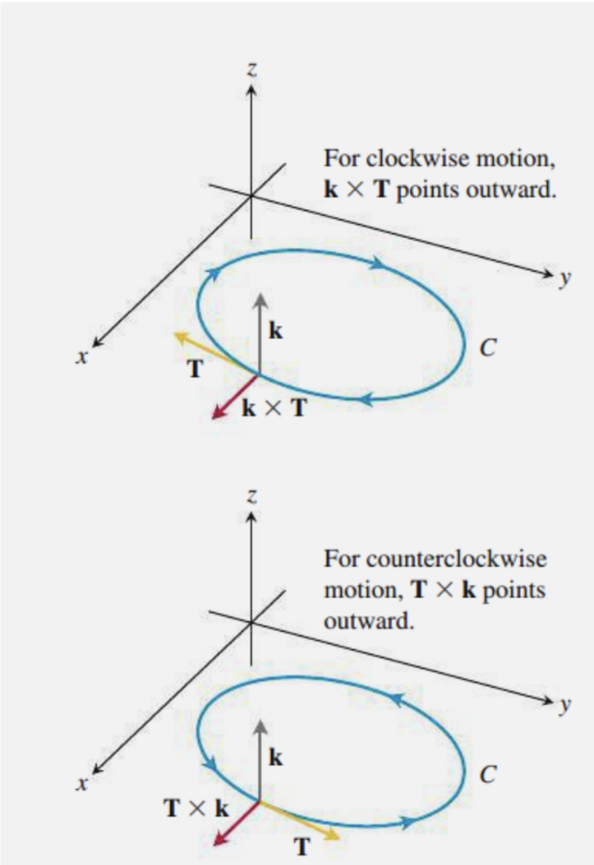

FIGURE 16.24 To find an outward unit normal vector for a smooth simple curve C in the xy-plane that is traversed counterclockwise as t increases, we take $\mathbf{n} = \mathbf{T} \times \mathbf{k}$. For clockwise motion, we take $\mathbf{n} = \mathbf{k} \times \mathbf{T}$.

Assuming counterclockwise,

$$\hat{n} = \vec{T} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}$$

Then:

$$
\begin{aligned}
\Phi &= \int_C \vec{F} \cdot \hat{n}\, ds \\
&= \int_C \langle M, N \rangle \cdot \left\langle \frac{dy}{ds}, \frac{-dx}{ds} \right\rangle ds \\
&= \oint M\, dy - N\, dx
\end{aligned}
$$

#module4 #week10