14.1: Functions of Several Variables

Definition

Suppose D is a set of n-tuples of real numbers $(x_1, x_2, x_3, \ldots, x_n)$.

A **real-valued function** f on D returns a real number $(f : D \to \mathbb{R})$

- domain = D
- range = set of values returned

Examples

Find domain of each:

1. $f(x,y) = \sqrt{xy}$

   Answer: $\{(x,y) | xy \geq 0\}$

2. $f(x,y) = \frac{1}{\sqrt{x-y}}$

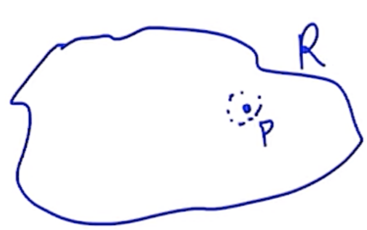
   $x - y > 0$, so:
   Answer: $\{(x,y) | x > y\}$

3. $f(x,y,z) = \frac{\sqrt{z}}{x^2-y^2}$

   $x^2 - y^2 \neq 0$
   $z \geq 0$
   Answer: $\{(x,y,z) | x^2 \neq y^2, z \geq 0\}$

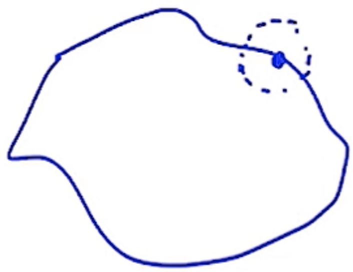
Boundary Points and Interior Points

Interior point of a set (or region) R: A point (x_0, y_0) in the center of a disk of positive radius that lies entirely in R.



Boundary point of a set (or region R): A point (x_0, y_0) where every disk of positive radius contains points that lie outside of R and points that lie in R.

(x_0, y_0) does not need to be in R.

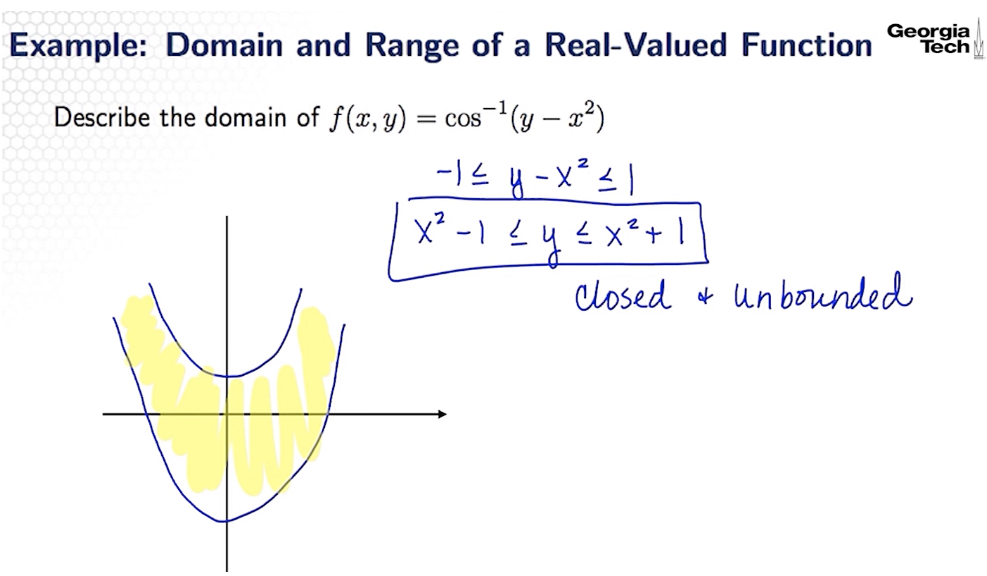


(For the above definitions, in 3D, replace "disk" with ball)

A set is **closed** if it contains all of its boundary points.
A set is **open** if it contains none of its boundary points.
Otherwise, it is neither open nor closed.

Bounded regions: A region that lies inside a disk of finite radius

- **Unbounded**: a region that doesn't



Level Curves and Surfaces

If c is a value in range of 2-var f, we can sketch the **level curve** $f(x, y) = c$.
If c is a value in range of 3-var f, we can sketch the **level surface** $f(x, y, z) = c$.

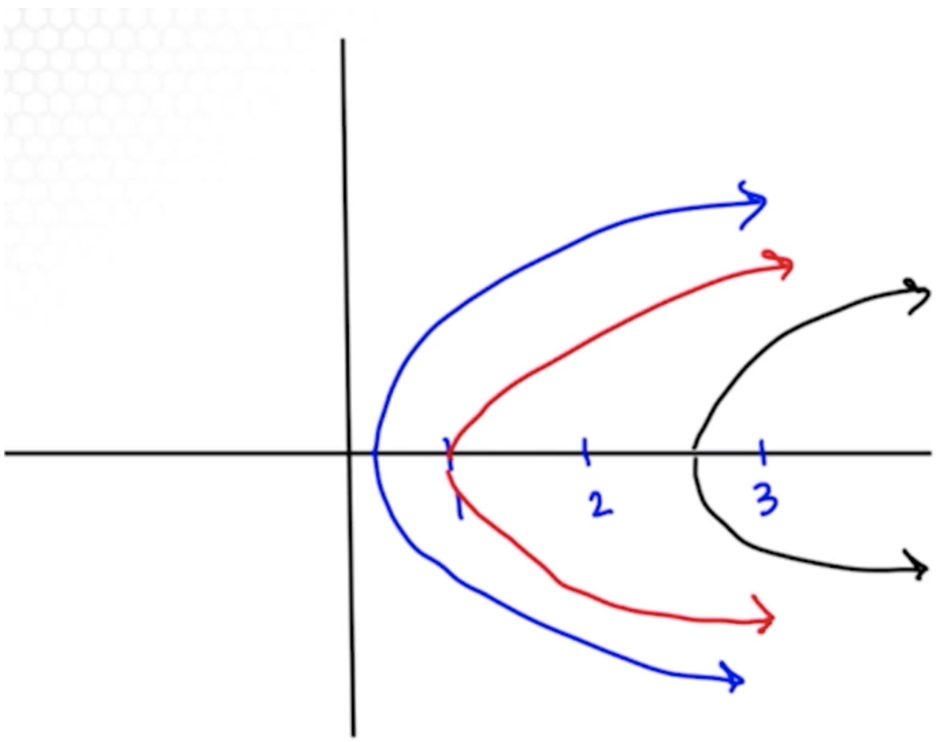
Examples

1. Graph level curves of $f(x, y) = \ln(x - y^2)$ for $c = -1, 0, 1$.

$$\ln(x - y^2) = -1$$
$$x - y^2 = e^{-1}$$
$$x = y^2 + e^{-1}$$

$$\ln(x - y^2) = 0$$
$$x - y^2 = e^{0}$$
$$x = y^2 + 1$$

$$\ln(x - y^2) = 1$$
$$x - y^2 = e^{1}$$
$$x = y^2 + e$$

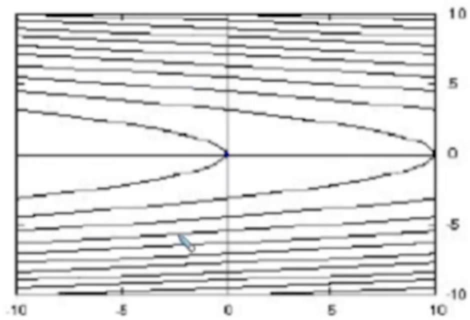


2. Describe the level curves for $f(x, y) = e^{x^2+2y^2}$.

$$e^{x^2+2y^2} = c$$
$$x^2 + 2y^2 = \underbrace{\ln c}_{\text{constant}} \ (c > 0)$$

The level curves are ellipses.

3. Which of the following functions is depicted in the contour plot below?



- The contours are level curves.

Recognize one of the curves as $x = -y^2$, so $x + y^2 = c$, and $f(x, y) = x + y^2$.

14.2: Limits and Continuity

Limits for Functions of Several Variables

Let $f(x_1, x_2, \ldots, x_n)$ be a function defined (at least) on some deleted neighborhood of $\vec{x}_0$.

$$\lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = L$$

if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < \|\vec{x} - \vec{x}_0\| < \delta \implies |f(\vec{x}) - L| < \epsilon$$

(Essentially, as $\vec{x}$ approaches $\vec{x}_0$ from anywhere, $f(\vec{x})$ approaches L.)

Two paths test: If the paths $\vec{x} \to \vec{x}_0$ approach different values for $f(\vec{x})$, the limit does not exist.

Example: Proving a limit using the definition

This looks like a doozy.

Find $\displaystyle\lim_{(x,y) \to (0,0)} f(x, y)$ for $f(x, y) = \dfrac{2xy^2}{x^2 + y^2}$.

If the limit is true, there must be some relationship between ϵ and δ.

On the ϵ side:

$$\begin{aligned}
|f(x,y) - \overbrace{L}^{\text{After testing, we find some paths to converge to 0}}| &< \epsilon \\
\left|\frac{2xy^2}{x^2+y^2} - 0\right| &< \epsilon \\
\frac{2|x|y^2}{x^2+y^2} &< \epsilon
\end{aligned}$$

On the δ side:

$$\begin{aligned}
0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} &\qquad < \delta \\
0 < \sqrt{x^2+y^2} &\qquad < \delta \\
0 < 2\sqrt{x^2+y^2} &\qquad < 2\delta
\end{aligned}$$

Since $y^2 \leq x^2 + y^2$,
$\frac{y^2}{x^2+y^2} \leq 1$
and $\frac{2|x|y^2}{x^2+y^2} \leq 2|x|$

So:

$$\frac{2|x|y^2}{x^2+y^2} \leq 2|x| = 2\sqrt{x^2} \leq 2\sqrt{x^2+y^2} < \epsilon$$

Thus, you can let $\delta = \frac{\epsilon}{2}$ and so the limit exists.

Continuity for Functions of Several Variables

A function $f(x,y)$ is *continuous* at point (x_0, y_0) if:

1. f is defined at (x_0, y_0)
2. Limit at that point exists
3. Limit is equal to $f(x_0, y_0)$

Function is continuous if it is continuous at every point in its domain.

Define $f(1,1)$ in a way that makes $f(x,y) = \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$ continuous at the point $(1,1)$.

$$\lim_{(x,y)\to(1,1)} \frac{x^2 - xy}{(\sqrt{x} - \sqrt{y})} \frac{(\sqrt{x} + \sqrt{y})}{(\sqrt{x} + \sqrt{y})} = \lim_{(x,y)\to(1,1)} \frac{x\cancel{(x-y)}(\sqrt{x} + \sqrt{y})}{\cancel{(x-y)}} = 1(2) = 2$$

$$f(x,y) = \begin{cases} \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}} & (x,y) \neq (1,1) \\ 2 & (x,y) = (1,1) \end{cases}$$

14.3, 14.4: Partial Derivatives

Definition

If the limit exists, the **partial derivative** of $f(x,y)$ with respect to x at (x_0, y_0) is:

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x,y)}{\Delta x}$$

(Differentiate f with respect to x, while holding other variables constant)
(This extends trivially to more dimensions)

Differentiability

Function $z = f(x,y)$ is **differentiable** at (x_0, y_0) if f_x, f_y (the partial derivatives) and $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$ satisfies:

$$\Delta z = (f_x + \epsilon_1)\Delta x + (f_y + \epsilon_2)\Delta y$$

(where each $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$)

f is **differentiable** if it is differentiable at every point in its domain.
f's graph then is a **smooth surface**

Example

Find all first partial derivatives of $f(x,y) = 3x^2 - 2y + xy$.

$$f_x = \frac{\partial f}{\partial x} = 6x + y$$
$$f_y = \frac{\partial f}{\partial y} = -2 + x$$

Second Order Partial Derivatives

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$
$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$
$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$
$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Mixed Derivatives Theorem

If $f(x, y)$ and all of f_x, f_y, f_{xy}, f_{yx} are defined throughout an open region containing a point (x_0, y_0) and are all continuous at (x_0, y_0), then

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$$

Example

Find all first & second partial derivatives of $f(x, y) = 2x^2 \cos y + 3y^2 \sin x$.

$$f_x = 4x \cos y + 3y^2 \cos x$$
$$f_y = -2x^2 \sin y + 6y \sin x$$

$$f_{xx} = 4 \cos y - 3y^2 \sin x$$
$$f_{xy} = -4x \sin y + 6y \cos x$$
$$f_{yx} = -4x \sin y + 6y \cos x$$
$$f_{yy} = -2x^2 \cos y + 6 \sin x$$

Implicit Differentiation

Suppose we have a function in terms of three variables x, y, z and we cannot solve the equation for z, but we want to find $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$.

Do implicit differentiation as expected, but hold the necessary variables constant.

Example

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $xy - z^2y - 2zx = 0$.

$$\begin{aligned}
y - 2zy\frac{\partial z}{\partial x} - 2z - 2x\frac{\partial z}{\partial x} &= 0 \\
\frac{\partial z}{\partial x}(-2zy - 2x) + y - 2z &= 0 \\
\frac{\partial z}{\partial x}(-2zy - 2x) &= 2z - y \\
\frac{\partial z}{\partial x} &= \frac{y - 2z}{2zy + 2x}
\end{aligned}$$

$$\begin{aligned}
x - z^2 - 2zy\frac{\partial z}{\partial y} - 2x\frac{\partial z}{\partial y} &= 0 \\
x - z^2 - (2zy + 2x)\frac{\partial z}{\partial y} &= 0 \\
(2zy + 2x)\frac{\partial z}{\partial y} &= x - z^2 \\
\frac{\partial z}{\partial y} &= \frac{x - z^2}{2zy + 2x}
\end{aligned}$$

Chain Rule

x, y as functions of one variable

If $w = f(x, y)$ is differentiable and $x(t), y(t)$ are differentiable with respect to t, then $w = f(x(t), y(t))$ is differentiable with respect to t.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

(This is trivially extendible to more dimensions)

Example

Find $\frac{du}{dt}$: $u = x^2 - 3xy = 2y^2,\ x(t) = \cos t,\ y(t) = \sin t$

$$
\begin{aligned}
\frac{dx}{dt} &= -\sin t \\
\frac{dy}{dt} &= \cos t \\
\frac{\partial u}{\partial x} &= 2x - 3y \\
\frac{\partial u}{\partial y} &= -3x + 4y
\end{aligned}
$$

So:

$$
\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}
$$

$$
\begin{aligned}
&= (2\cos(t) - 3\sin(t))(-\sin(t)) \\
&\quad + (-3\cos(t) + 4\sin(t))(\cos(t))
\end{aligned}
$$

$$
\begin{aligned}
&= -2\sin(t)\cos(t) + 3\sin^2(t) - 3\cos^2(t) + 4\sin(t)\cos(t) \\
&= 2\sin(t)\cos(t) + 3\sin^2(t) - 3\cos^2(t) \\
&= \sin(2t) - 3\cos(2t)
\end{aligned}
$$

x, y as functions of multiple variables

What if $u = u(x, y)$, $x = x(s, t)$, $y = y(s, t)$?

You can find partial derivatives $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$, replacing $\frac{dx}{dt}$ in the simpler chain rule with the respective partial derivative.

$$
\begin{aligned}
\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \\
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}
\end{aligned}
$$

Example

Given $z = 4e^x \ln y, x = \ln(uv), y = u\sin(v)$, express $\frac{\partial z}{\partial u}$ as a function of u and v.

$$
\frac{\partial x}{\partial u} = \frac{1}{u}
$$
$$
\frac{\partial y}{\partial u} = \sin(v)
$$

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= 4e^x \ln y \\
&= 4uv \ln(u \sin(v))
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial z}{\partial y} &= \frac{4e^x}{y} \\
&= \frac{4v}{\sin(v)}
\end{aligned}
$$

So:

$$
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}
$$

$$
\begin{aligned}
&= 4uv \ln(u \sin(v))\frac{1}{u} + \frac{4v}{\sin(v)}\sin(v) \\
&= 4v \ln(u \sin(v)) + 4v
\end{aligned}
$$

Implicit Differentiation (again)

If $F(x, y)$ is differentiable and $F(x, y) = 0$ defines y as differentiable function of x,

$$
\frac{dy}{dx} = -\frac{F_x}{F_y}
$$

In 3D,
If $z = z(x, y)$ and $F(x, y, z(x, y)) = 0$, you can do the same thing, but to find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

Why?
Let F be written as $F(x, y(x))$.

$$
\begin{aligned}
F(x, y(x)) &= 0 \\
\frac{d}{dx} F(x, y(x)) &= \frac{d}{dx} 0 \\
\frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx} &= 0 \\
\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} &= 0 \\
F_x + F_y \frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= -\frac{F_x}{F_y}
\end{aligned}
$$

Examples

Find $\dfrac{dy}{dx}$ at the specified point:

$$\underbrace{xy + y^2 - 3x - 3}_{F(x,y)} = 0, \ (-1, 1)$$

$$F_x = y - 3 \qquad F_y = x + 2y$$

$$\frac{dy}{dx} = -\frac{(y-3)}{x+2y} \qquad \left.\frac{dy}{dx}\right|_{(-1,1)} = 2$$

Find $\dfrac{\partial z}{\partial y}$ at the specified point:

$$\underbrace{xe^y + ye^z + 2\ln(x) - 2 - 3\ln(2)}_{F(x,y,z)} = 0, \ (1, \ln(2), \ln(3))$$

$$\frac{\partial z}{\partial y} = -F_y/F_z \qquad F_y = xe^y + e^z \qquad F_z = ye^z$$

$$\frac{\partial z}{\partial y} = -\frac{(xe^y + e^z)}{ye^z} \qquad \left.\frac{\partial z}{\partial y}\right|_{(1,\ln(2),\ln(3))} = \frac{-(2+3)}{\ln(2)\cdot 3} = \frac{-5}{3\ln(2)}$$