15.7: Triple Integrals in Cylindrical & Spherical Coordinates

Cylindrical Coordinates

- Represent point P in space by ordered triples (r, θ, z) $(r \geq 0)$

1. r and θ are polar coordinates for the projection of P onto the xy-plane
2. z is the rectangular vertical coordinate

Usage

Should be used when...

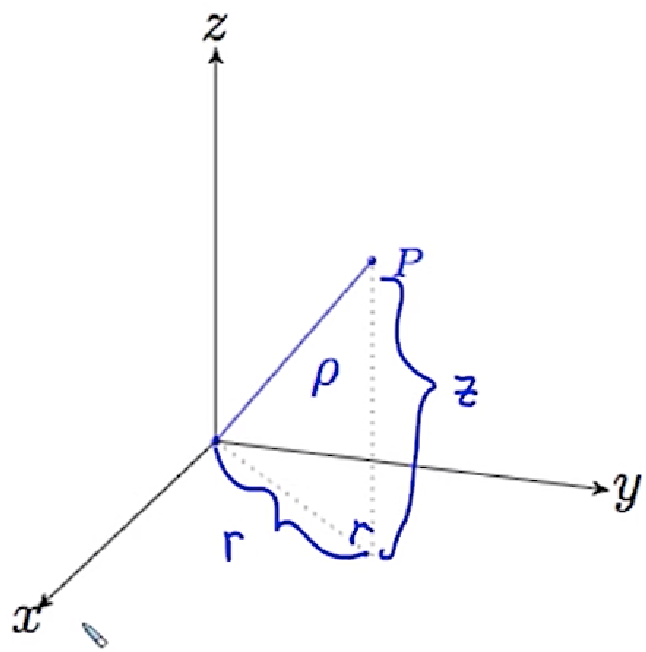
- there is an axis of symmetry
- an integrand involves $x^2 + y^2$
- we're integrating over a circle (or part of) in the xy-plane

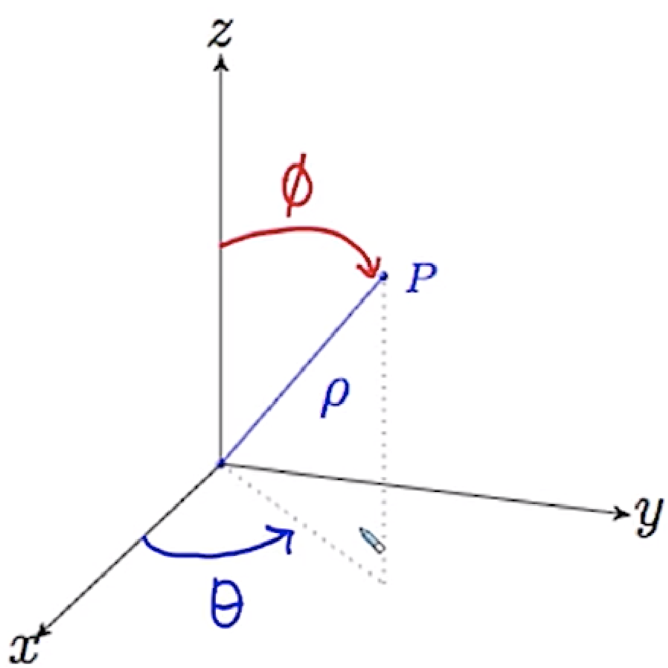
Very similar to using polar coordinates w/ double integrals, but with an added z component for triple integrals.

Spherical Coordinates

- Represent point P in space by ordered triples (ρ, ϕ, θ)

1. ρ is distance from P to the origin $(\rho \geq 0)$
2. ϕ is the angle $\overrightarrow{OP}$ makes with the $+z$-axis $(0 \leq \phi \leq \pi)$
3. θ is the angle from cylindrical coordinates





Converting Rectangular to Spherical

$$\rho^2 = x^2 + y^2 + z^2 = r^2 + z^2$$

$$\begin{aligned} r &= \rho \sin \phi \\ z &= \rho \cos \phi \end{aligned}$$

$$\begin{aligned} x &= r \cos \theta = \rho \sin \phi \cos \theta \\ y &= r \sin \theta = \rho \sin \phi \sin \theta \end{aligned}$$

Triple Integrals:

$$\iiint_T dV = \iiint_T \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

15.8: Integration by Substitution

Double Integrals

Jacobian determinate or **Jacobian** of the coordinate transformation $x = g(u, v), y = h(u, v)$:

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}$$

Substitution for Double Integrals:
Suppose $f(x, y)$ is continuous over region R. Let G be preimage of R under transform $x = g(u, v), y = h(u, v)$ (assumed to be one-to-one on interior of G). If functions g and h have continuous 1st partial derivatives within interior of G:

$$\iint_R f(x, y) \, dx \, dy = \iint_G f(g(u, v), h(u, v)) \overbrace{\left| \frac{\partial(x, y)}{\partial(u, v)} \right|}^{\text{Jacobian}} du \, dv$$

Triple Integrals

Given $x = g(u, v, w), y = h(u, v, w), z = k(u, v, w)$,

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\iiint_R f(x, y, z)\, dx\, dy\, dz$$
$$= \iiint_G f(g(u, v, w), h(u, v, w), k(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw$$