

14.1: Functions of Several Variables

Definition

Suppose D is a set of n -tuples of real numbers $(x_1, x_2, x_3, \dots, x_n)$.

A **real-valued function** f on D returns a real number $(f : D \rightarrow \mathbb{R})$

- domain = D
- range = set of values returned

Examples

Find domain of each:

1. $f(x, y) = \sqrt{xy}$

Answer: $\{(x, y) | xy \geq 0\}$

2. $f(x, y) = \frac{1}{\sqrt{x-y}}$

$x - y > 0$, so:

Answer: $\{(x, y) | x > y\}$

3. $f(x, y, z) = \frac{\sqrt{z}}{x^2 - y^2}$

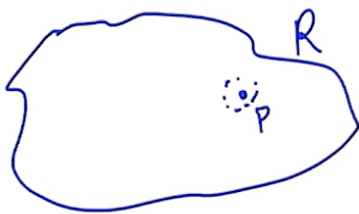
$x^2 - y^2 \neq 0$

$z \geq 0$

Answer: $\{(x, y, z) | x^2 \neq y^2, z \geq 0\}$

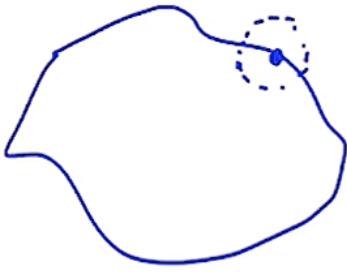
Boundary Points and Interior Points

Interior point of a set (or region) R : A point (x_0, y_0) in the center of a disk of positive radius that lies entirely in R .



Boundary point of a set (or region R): A point (x_0, y_0) where every disk of positive radius contains points that lie outside of R and points that lie in R .

(x_0, y_0) does not need to be in R .



(For the above definitions, in 3D, replace "disk" with ball)

A set is **closed** if it contains all of its boundary points.

A set is **open** if it contains none of its boundary points.

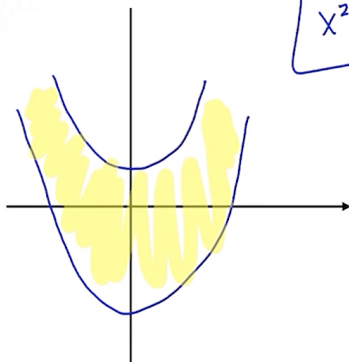
Otherwise, it is neither open nor closed.

Bounded regions: A region that lies inside a disk of finite radius

- **Unbounded:** a region that doesn't

Example: Domain and Range of a Real-Valued Function Georgia Tech

Describe the domain of $f(x, y) = \cos^{-1}(y - x^2)$



$$\begin{aligned} -1 &\leq y - x^2 \leq 1 \\ x^2 - 1 &\leq y \leq x^2 + 1 \end{aligned}$$

closed & unbounded

Level Curves and Surfaces

If c is a value in range of 2-var f , we can sketch the **level curve** $f(x, y) = c$.

If c is a value in range of 3-var f , we can sketch the **level surface** $f(x, y, z) = c$.

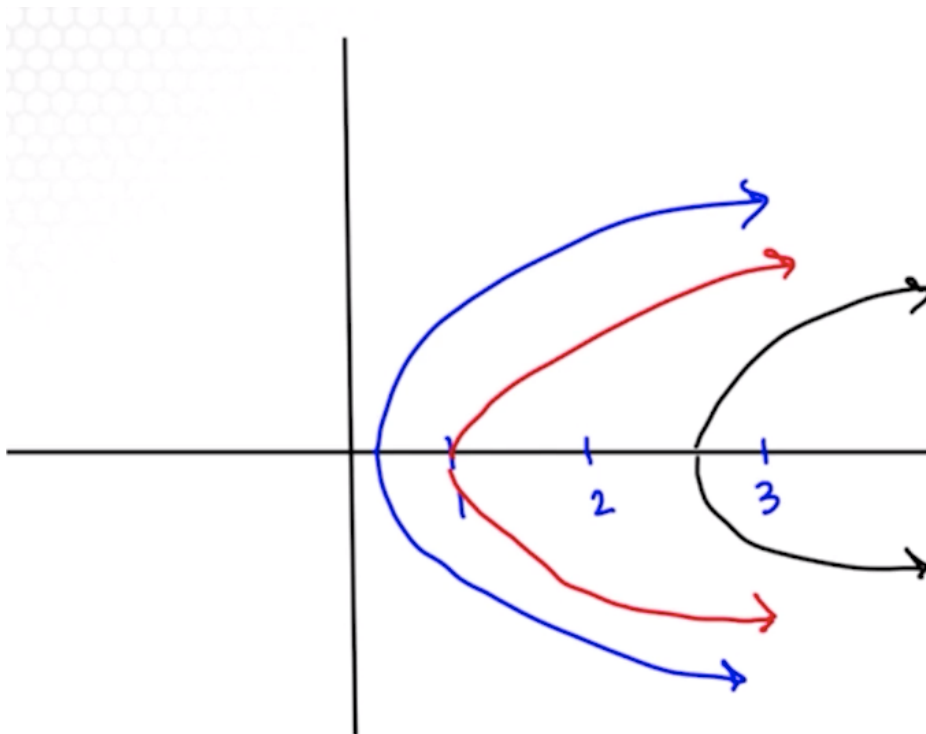
Examples

1. Graph level curves of $f(x, y) = \ln(x - y^2)$ for $c = -1, 0, 1$.

$$\begin{aligned}\ln(x - y^2) &= -1 \\ x - y^2 &= e^{-1} \\ x &= y^2 + e^{-1}\end{aligned}$$

$$\begin{aligned}\ln(x - y^2) &= 0 \\ x - y^2 &= e^0 \\ x &= y^2 + 1\end{aligned}$$

$$\begin{aligned}\ln(x - y^2) &= 1 \\ x - y^2 &= e^1 \\ x &= y^2 + e\end{aligned}$$

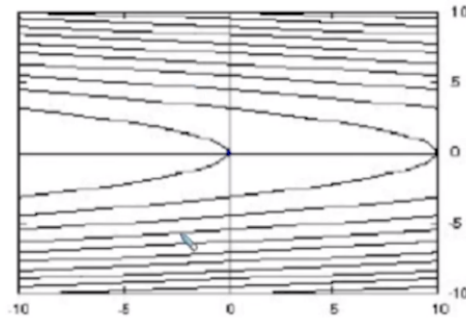


2. Describe the level curves for $f(x, y) = e^{x^2+2y^2}$.

$$\begin{aligned}e^{x^2+2y^2} &= c \\ x^2 + 2y^2 &= \underbrace{\ln c}_{\text{constant}} \quad (c > 0)\end{aligned}$$

The level curves are ellipses.

Which of the following functions is depicted in the contour plot below?



3.

- The contours are level curves.

Recognize one of the curves as $x = -y^2$, so $x + y^2 = c$, and $f(x, y) = x + y^2$.

14.2: Limits and Continuity

Limits for Functions of Several Variables

Let $f(x_1, x_2, \dots, x_n)$ be a function defined (at least) on some deleted neighborhood of \vec{x}_0 .

$$\lim_{\vec{x} \rightarrow \vec{x}_0} f(\vec{x}) = L$$

if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < \|\vec{x} - \vec{x}_0\| < \delta \implies |f(\vec{x}) - L| < \epsilon$$

(Essentially, as \vec{x} approaches \vec{x}_0 from anywhere, $f(\vec{x})$ approaches L .)

Two paths test: If the paths $\vec{x} \rightarrow \vec{x}_0$ approach different values for $f(\vec{x})$, the limit does not exist.

Example: Proving a limit using the definition

This looks like a doozy.

Find $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ for $f(x, y) = \frac{2xy^2}{x^2 + y^2}$.

If the limit is true, there must be some relationship between ϵ and δ .

On the ϵ side:

After testing, we find some paths to converge to 0

$$\begin{aligned} |f(x, y) - \widehat{L}| &< \epsilon \\ \left| \frac{2xy^2}{x^2 + y^2} - 0 \right| &< \epsilon \\ \frac{2|x|y^2}{x^2 + y^2} &< \epsilon \end{aligned}$$

On the δ side:

$$\begin{aligned} 0 &< \sqrt{(x - x_0)^2 + (y - y_0)^2} &< \delta \\ 0 &< \sqrt{x^2 + y^2} &< \delta \\ 0 &< 2\sqrt{x^2 + y^2} &< 2\delta \end{aligned}$$

Since $y^2 \leq x^2 + y^2$,

$$\frac{y^2}{x^2 + y^2} \leq 1$$

$$\text{and } \frac{2|x|y^2}{x^2 + y^2} \leq 2|x|$$

So:

$$\frac{2|x|y^2}{x^2 + y^2} \leq 2|x| = 2\sqrt{x^2} \leq 2\sqrt{x^2 + y^2} < \epsilon$$

Thus, you can let $\delta = \frac{\epsilon}{2}$ and so the limit exists.

Continuity for Functions of Several Variables

A function $f(x, y)$ is *continuous* at point (x_0, y_0) if:

1. f is defined at (x_0, y_0)
2. Limit at that point exists
3. Limit is equal to $f(x_0, y_0)$

Function is continuous if it is continuous at every point in its domain.

Define $f(1, 1)$ in a way that makes $f(x, y) = \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$ continuous at the point $(1, 1)$.

$$\lim_{(x,y) \rightarrow (1,1)} \frac{x^2 - xy}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})} = \lim_{(x,y) \rightarrow (1,1)} \frac{x(x-y)(\sqrt{x} + \sqrt{y})}{(x-y)(\sqrt{x} + \sqrt{y})} = 1(2) = 2$$

$$f(x, y) = \begin{cases} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} & (x, y) \neq (1, 1) \\ 2 & (x, y) = (1, 1) \end{cases}$$

14.3, 14.4: Partial Derivatives

Definition

If the limit exists, the **partial derivative** of $f(x, y)$ with respect to x at (x_0, y_0) is:

$$\left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

(Differentiate f with respect to x , while holding other variables constant)

(This extends trivially to more dimensions)

Differentiability

Function $z = f(x, y)$ is **differentiable** at (x_0, y_0) if f_x, f_y (the partial derivatives) and $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$ satisfies:

$$\Delta z = (f_x + \epsilon_1)\Delta x + (f_y + \epsilon_2)\Delta y$$

(where each $\epsilon_1, \epsilon_2 \rightarrow 0$ as both $\Delta x, \Delta y \rightarrow 0$)

f is **differentiable** if it is differentiable at every point in its domain.

f 's graph then is a **smooth surface**

Example

Find all first partial derivatives of $f(x, y) = 3x^2 - 2y + xy$.

$$f_x = \frac{\partial f}{\partial x} = 6x + y$$

$$f_y = \frac{\partial f}{\partial y} = -2 + x$$

Second Order Partial Derivatives

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2}$$

Mixed Derivatives Theorem

If $f(x, y)$ and all of f_x, f_y, f_{xy}, f_{yx} are defined throughout an open region containing a point (x_0, y_0) and are all continuous at (x_0, y_0) , then

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$$

Example

Find all first & second partial derivatives of $f(x, y) = 2x^2 \cos y + 3y^2 \sin x$.

$$f_x = 4x \cos y + 3y^2 \sin x$$

$$f_y = -2x^2 \sin y + 6y \sin x$$

$$f_{xx} = 4 \cos y - 3y^2 \sin x$$

$$f_{xy} = -4x \sin y + 6y \cos x$$

$$f_{yx} = -4x \sin y + 6y \cos x$$

$$f_{yy} = -2x^2 \cos y + 6 \sin x$$

Implicit Differentiation

Suppose we have a function in terms of three variables x, y, z and we cannot solve the equation for z , but we want to find $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$.

Do implicit differentiation as expected, but hold the necessary variables constant.

Example

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $xy - z^2y - 2zx = 0$.

$$y - 2zy \frac{\partial z}{\partial x} - 2z - 2x \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}(-2zy - 2x) + y - 2z = 0$$

$$\frac{\partial z}{\partial x}(-2zy - 2x) = 2z - y$$

$$\frac{\partial z}{\partial x} = \frac{y - 2z}{2zy + 2x}$$

$$x - z^2 - 2zy \frac{\partial z}{\partial y} - 2x \frac{\partial z}{\partial y} = 0$$

$$x - z^2 - (2zy + 2x) \frac{\partial z}{\partial y} = 0$$

$$(2zy + 2x) \frac{\partial z}{\partial y} = x - z^2$$

$$\frac{\partial z}{\partial y} = \frac{x - z^2}{2zy + 2x}$$

Chain Rule

Simple

If $w = f(x, y)$ is differentiable and $x(t), y(t)$ are differentiable with respect to t , then $w = f(x(t), y(t))$ is differentiable with respect to t .

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

(This is trivially extendible to more dimensions)

Example

Find $\frac{du}{dt}$: $u = x^2 - 3xy = 2y^2$, $x(t) = \cos t$, $y(t) = \sin t$

$$\begin{aligned}\frac{dx}{dt} &= -\sin t \\ \frac{dy}{dt} &= \cos t \\ \frac{\partial u}{\partial x} &= 2x - 3y \\ \frac{\partial u}{\partial y} &= -3x + 4y\end{aligned}$$

So:

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} \\ &= (2 \cos(t) - 3 \sin(t))(-\sin(t)) \\ &\quad + (-3 \cos(t) + 4 \sin(t))(\cos(t)) \\ &= -2 \sin(t) \cos(t) + 3 \sin^2(t) - 3 \cos^2(t) + 4 \sin(t) \cos(t) \\ &= 2 \sin(t) \cos(t) + 3 \sin^2(t) - 3 \cos^2(t) \\ &= \sin(2t) - 3 \cos(2t)\end{aligned}$$

UH. PANIC.

What if $u = u(x, y)$, $x = x(s, t)$, $y = y(s, t)$?

You can find partial derivatives $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$, replacing $\frac{dx}{dt}$ in the simpler chain rule with the respective partial derivative.

Example

Given $z = 4e^x \ln y$, $x = \ln(uv)$, $y = u \sin(v)$, express $\frac{\partial z}{\partial u}$ as a function of u and v .

$$\frac{\partial x}{\partial u} = \frac{1}{u}$$

$$\frac{\partial y}{\partial u} = \sin(v)$$

$$\frac{\partial z}{\partial x} = 4e^x \ln y$$

$$= 4uv \ln(u \sin(v))$$

$$\frac{\partial z}{\partial y} = \frac{4e^x}{y}$$

$$= \frac{4v}{\sin(v)}$$

So:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}$$

$$= 4uv \ln(u \sin(v)) \frac{1}{u} + \frac{4v}{\sin(v)} \sin(v)$$

$$= 4v \ln(u \sin(v)) + 4v$$

Implicit Differentiation (again)

If $F(x, y)$ is differentiable and $F(x, y) = 0$ defines y as differentiable function of x ,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

In 3D,

If $z = f(x, y)$ and $F(x, y, f(x, y)) = 0$, you can do the same thing, but to find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$

Examples

Find $\frac{dy}{dx}$ at the specified point:

$$\underbrace{xy + y^2 - 3x - 3 = 0}_{F(x,y)}, (-1, 1)$$

$$F_x = y - 3 \quad F_y = x + 2y$$

$$\frac{dy}{dx} = - \frac{(y-3)}{x+2y}$$

$$\left. \frac{dy}{dx} \right|_{(-1,1)} = 2$$

Find $\frac{\partial z}{\partial y}$ at the specified point:

$$\underbrace{xe^y + ye^z + 2\ln(x) - 2 - 3\ln(2) = 0}_{F(x,y,z)}, (1, \ln(2), \ln(3))$$

$$\frac{\partial z}{\partial y} = - F_y / F_z$$

$$F_y = xe^y + e^z$$

$$F_z = ye^z$$

$$\frac{\partial z}{\partial y} = - \frac{(xe^y + e^z)}{ye^z}$$

$$\left. \frac{\partial z}{\partial y} \right|_{(1, \ln(2), \ln(3))} = \frac{-(2+3)}{\ln(2) \cdot 3} = \frac{-5}{3\ln(2)}$$