13.3: Arc Length in Space

Length of smooth curve $\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ $(a \leq t \leq b)$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

(pythagorean theorem between dx, dy, dz)

This is equivalent to:

$$L = \int_a^b \|\vec{v}\|\, dt$$

Arc Length Parameter: Function *s* that finds directed distance along curve starting from $P(t_0)$ to some point $P(t)$

$$s(t) = \int_{t_0}^t \sqrt{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2}\, d\tau = \int_{t_0}^t \|\vec{v}\|\, d\tau$$

Speed:

$$speed = \frac{ds}{dt} = \|\vec{v}(t)\|$$

Unit Tangent Vector: Unit vector... that's tangent to the smooth curve idk what you expected lmao

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\vec{v}(t)}{\|\vec{v}(t)\|} = \frac{d\vec{r}/dt}{ds/dt}$$

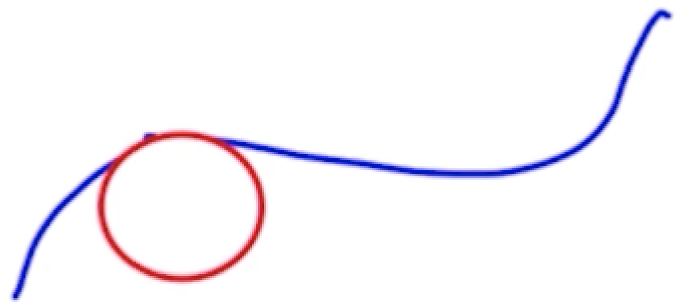
($\vec{v}(t)$ normalized)

13.4: Curvature

If $\vec{T}$ is a unit vector of a smooth curve, the **curvature** function of the curve is

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\|$$

In the blue curve, the curvature at the point is related to circle that best fit curve at that point.



(seems similar to 2nd derivative)

For smooth curve $\vec{r}$, curvature can be written as scalar function:

$$\kappa = \left\| \frac{d\vec{T}/dt}{ds/dt} \right\| = \frac{\|\vec{T}'(t)\|}{\|\vec{v}(t)\|}$$

Circle of Curvature

The **circle of curvature** (or **osculating circle**) at point P on plane curve (2D) where $\kappa \neq 0$ is the circle of the curve that

1. is tangent to curve at P
2. has the same curvature the curve has at P
3. has center that lies toward the concave side of the curve

The **radius of curvature** at point P is $\rho = \frac{1}{\kappa}$.

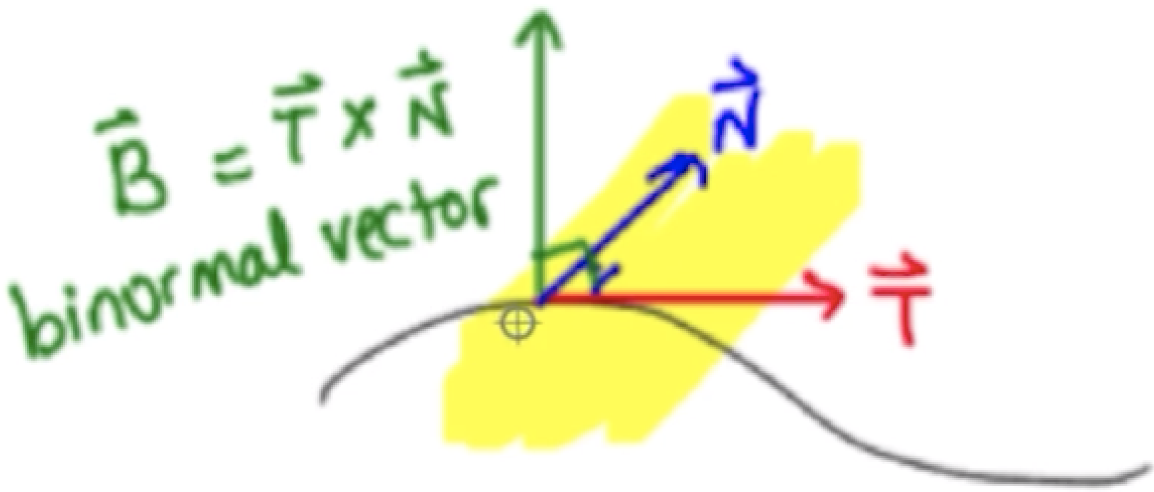
- Straight lines: curvature is constantly 0
- Circle of radius r: Curvature is constantly $\frac{1}{r}$.

Principal Normal Vector

If $\vec{T}(t)$ is unit tangent vector and $\vec{T}'(t) = 0$, then unit tangent vector d/n change direction.
If $\vec{T}'(t) \neq 0$, then

Principal normal vector = $\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$
($\vec{T}'$ normalized)
(this vector is $\perp$ to $\vec{T}$)

TNB Frame

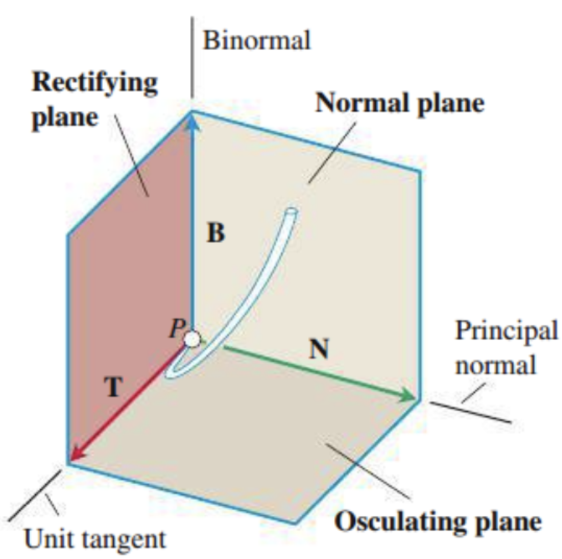


Binormal vector: $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

TNB frame / Frenet frame: The three vectors T, N, B

Planes

- **Osculating plane**: Plane between $\vec{T}$ and $\vec{N}$ (normal is $\vec{B}$)
- **Normal plane**: Plane between $\vec{N}$ and $\vec{B}$ (normal is $\vec{T}$)
- **Rectifying plane**: Plane between $\vec{T}$ and $\vec{B}$ (normal is $\vec{N}$)



13.5: Tangential & Normal Components of Acceleration

Given position function $\vec{r}(t)$,

$$\vec{T}(t) = \frac{\vec{v}(t)}{ds/dt}$$
$$\vec{v}(t) = \vec{T}(t)\frac{ds}{dt}$$
$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \vec{T}'(t)\frac{ds}{dt} + \vec{T}(t)\frac{d^2s}{dt^2}$$

Since $\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$,

$$\vec{N}(t)\|\vec{T}'(t)\| = \vec{T}'(t)$$
$$\vec{a}(t) = \vec{N}(t)\|\vec{T}'(t)\|\frac{ds}{dt} + \vec{T}(t)\frac{d^2s}{dt^2}$$

So,
Tangential component of acceleration:

$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}\|\vec{v}\|$$

- Only dependent on change of speed of object
- If speed is constant, $a_T = 0$ and acceleration is directed entirely towards center of curvature

Normal component of acceleration:

$$a_N = \|\vec{T}'(t)\|\frac{ds}{dt} = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa\|\vec{v}\|^2$$

$\left(\text{recall } \kappa = \frac{\|\vec{T}'(t)\|}{ds/dt}\right)$

Curvature and Torsion

Torsion:

Let $\vec{B} = \vec{T} \times \vec{N}$.

$$\tau = -\frac{d\vec{B}}{ds} \cdot \vec{N}$$

- Measures how binormal vector changes with respect to arc length

$$\tau = \frac{\begin{vmatrix} \cdots & \vec{r}' & \cdots \\ \cdots & \vec{r}'' & \cdots \\ \cdots & \vec{r}''' & \cdots \end{vmatrix}}{\|\vec{v} \times \vec{a}\|^2}$$

Formulas for Curvature and Torsion

$$\vec{T} \cdot \vec{a} = a_T(\vec{T} \cdot \vec{T}) + a_N(\vec{T} \cdot \vec{N}) = a_T$$
$$\|\vec{T} \times \vec{a}\| = \|a_T(\vec{T} \times \vec{T})\| + \|a_N(\vec{T} \times \vec{N})\| = \|a_N \vec{B}\| = a_N$$

So,

$$a_T = \frac{\vec{v} \cdot \vec{a}}{ds/dt}$$
$$a_N = \frac{\|\vec{v} \times \vec{a}\|}{ds/dt} = \kappa \left(\frac{ds}{dt}\right)^2$$

And thus,

$$\kappa = \frac{\|\vec{v} \times \vec{a}\|}{(ds/dt)^3}$$

13.6: Motion in Polar Coordinates

Given coordinates $P(r, \theta)$,
position, velocity, and acceleration can be represented in terms of:

- $\vec{u}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$ (unit vector in direction of $\overrightarrow{OP}$)
- $\vec{u}_\theta = -(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$ (unit vector pointing in direction of increasing θ)

$$\vec{r} = r\vec{u}_r = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}$$
$$\vec{v} = \frac{d\vec{r}}{dt} = (-r\theta'\sin\theta + r'\cos\theta)\mathbf{i} + (r\theta'\cos\theta + r'\sin\theta)\mathbf{j}$$
$$= r\theta'\vec{u}_\theta + r'\vec{u}_r$$
$$\vec{a} = \frac{d\vec{v}}{dt} = (-r\theta'^2\vec{u}_r + (r\theta'' + r'\theta')\vec{u}_\theta) + (r''\vec{u}_r + r'\theta'\vec{u}_\theta)$$

(btw you can product rule $\vec{r} = r\vec{u}_r$ instead of expanding $\vec{r}$ and differentiating its components)

$$\vec{r} = r\vec{u}_r$$
$$\vec{v} = r'\vec{u}_r + r\theta'\vec{u}_\theta$$
$$\vec{a} = (r'' - r\theta'^2)\vec{u}_r + (r\theta'' + 2r'\theta')\vec{u}_\theta$$

Cylindrical Coordinates

$$\vec{r} = r\vec{u}_r + z\mathbf{k}$$
$$\vec{v} = r'\vec{u}_r + r\theta'\vec{u}_\theta + z'\mathbf{k}$$
$$\vec{a} = (r'' - r\theta'^2)\vec{u}_r + (r\theta'' + 2r'\theta')\vec{u}_\theta + z''\mathbf{k}$$