16.7, 16.8: Stokes' Theorem and Divergence Theorem

∇, div, and curl

The del operator (∇)

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

Two formulas use the ∇ operator:

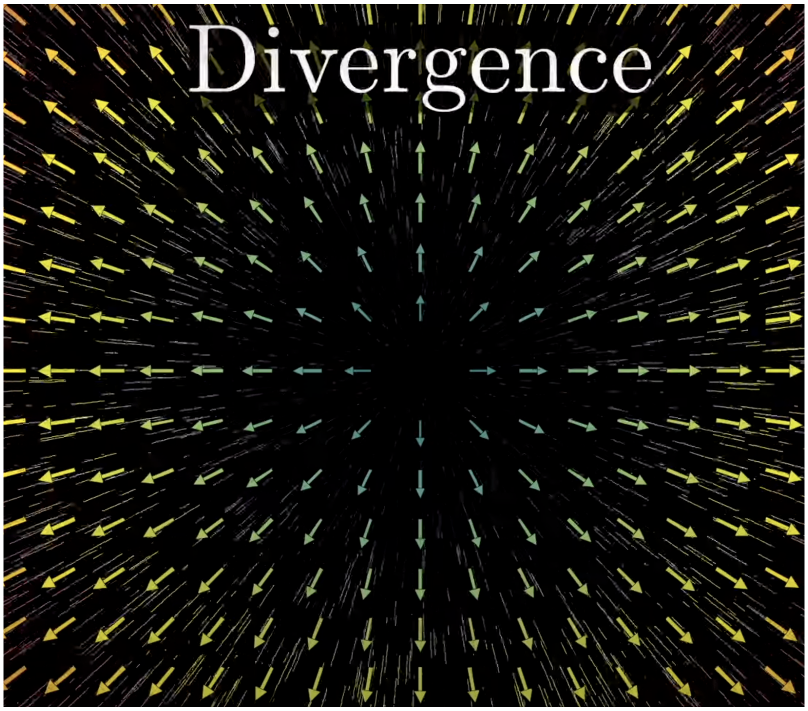
$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$
$$\operatorname{curl} \vec{F} = \nabla \times \vec{F}$$

See 3b1b video on divergence and curl (https://www.youtube.com/watch?v=rB83DpBJQsE)

Divergence

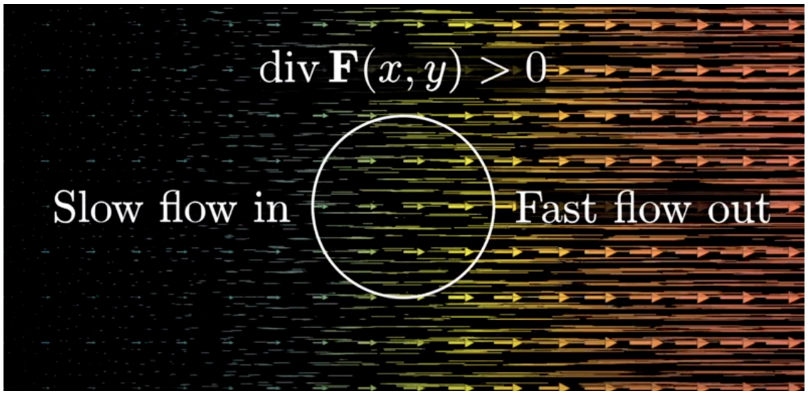
Given vector field $\vec{F}$ (where $\vec{F}$ represents velocity of a flowing fluid), the **divergence** of $\vec{F}$ represents the rate at which that fluid compresses or expands.

It's the flux density at that point (measured in flux/volume).



At a point:

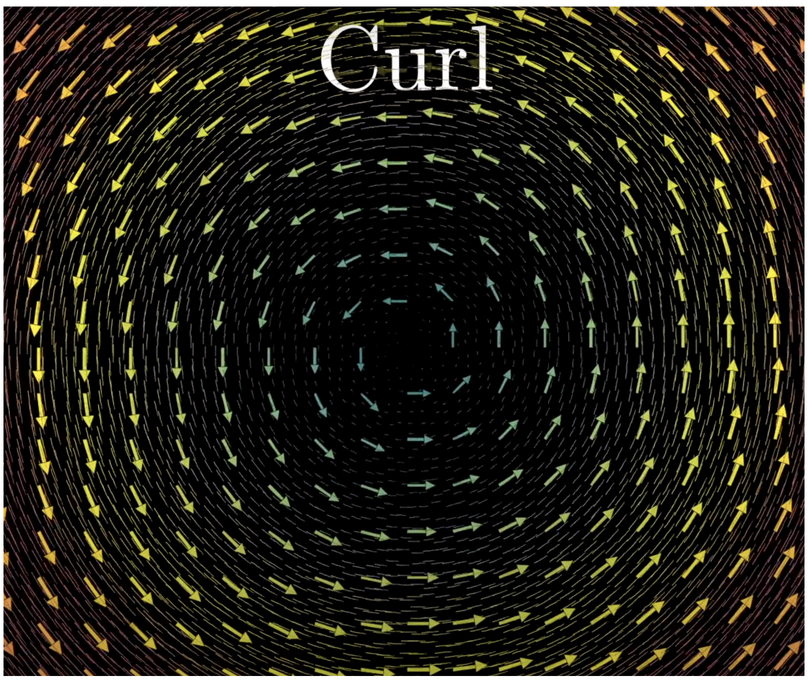
- If $\operatorname{div} \vec{F}$ is positive, then in a small circle region near that point, more fluid is exporting than importing (fluid moves outwards).
- If $\operatorname{div} \vec{F}$ is negative, then in a small circle region near that point, more fluid is importing than exporting (fluid moves inwards).



Curl

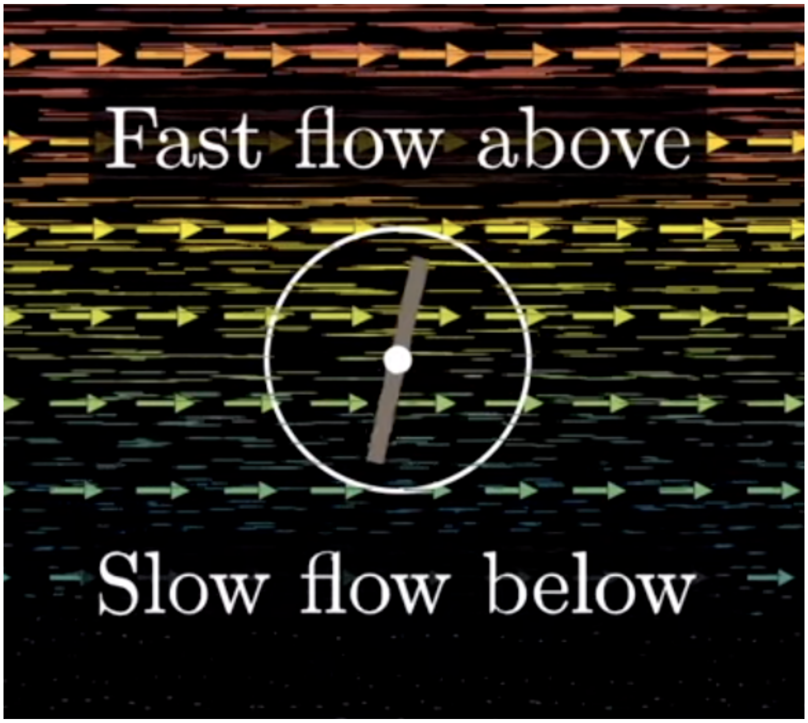
The **curl** of vector field $\vec{F}$ is a vector that describes the rate of rotation of that field $\vec{F}$.

Given normal vector $\hat{n}$, the rate of rotation around axis $\hat{n}$ is $\text{curl}\,\vec{F} \cdot \hat{n}$. This is the circulation density around that point.



At a point:

- If $\text{curl}\,\vec{F} \cdot \hat{n}$ is positive, then in a small circle region near that point, the vector field tends to rotate counterclockwise.
- If $\text{curl}\,\vec{F} \cdot \hat{n}$ is negative, then in a small circle region near that point, the vector field tends to rotate clockwise.



(Note how a spinny wheel would rotate clockwise in this vector field.)

Curl Identity

$$\text{curl grad } f = \nabla \times \nabla f = 0$$

Why?

$$\nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

Mixed partials have to be equal if continuous, so all the determinants evaluate to 0.

Example

Find the div and curl for $\vec{F} = (x^2 - yz)\mathbf{i} + ye^x\mathbf{j} + (xy + z)\mathbf{k}$.

$$\begin{aligned} \text{div } \vec{F} = \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(ye^x) + \frac{\partial}{\partial x}(xy + z) \\ &= 2x + e^x + 1 \end{aligned}$$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & ye^x & xy + z \end{vmatrix}$$

$$\begin{aligned} &= \mathbf{i}\left(\frac{\partial}{\partial y}(xy+z) - \frac{\partial}{\partial z}(ye^x)\right) \\ &- \mathbf{j}\left(\frac{\partial}{\partial x}(xy+z) - \frac{\partial}{\partial z}(x^2 - yz)\right) \\ &+ \mathbf{k}\left(\frac{\partial}{\partial x}(ye^x) - \frac{\partial}{\partial y}(x^2 - yz)\right) \\ &= x\mathbf{i} - 2y\mathbf{j} + (ye^x + z)\mathbf{k} \end{aligned}$$

Stokes' Theorem

Let S be a piecewise smooth oriented surface with piecewise smooth boundary curve C.
Let $\vec{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field with continuous 1st partial derivatives on open region containing S.

Then the circulation of $\vec{F}$ around C in the CCW dir:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \overbrace{(\nabla \times \vec{F})}^{\text{curl } \vec{F}} \cdot \hat{n}\, d\sigma$$

($\hat{n}$ is the unit normal vector with respect to the surface)

Closed Loop Property:
If $\text{curl F} = 0$ at every point of a simply connected open region D in space, then on any piecewise-smooth closed path C in D:

$$\oint_C F \cdot d\vec{r} = 0$$

(pretty straightforward extension of the loop property of conservative fields)

Divergence Theorem

Let S be a piecewise smooth oriented surface.
Let F be a vector field whose components have continuous 1st partial derivatives.

Then, the flux of $\vec{F}$ across S in the direction of the surface's outward unit normal field $\hat{n}$:

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iiint_D \overbrace{\nabla \cdot \vec{F}}^{\text{div } \vec{F}}\, dV$$

Corollary:
The outward flux across a piecewise smooth oriented closed surface is 0 for any vector field F with 0 divergence at every point of the region.

Divergence & Curl

$$\text{div}\ (\text{curl}\ \vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = 0$$

Unifying Fundamental Theorem of Vector Integral Calculus

Generalizations of Green's Theorem

Recall Green's Theorem. Stokes' theorem and the divergence theorem are both extensions of Green's.

Tangential Form

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \iint_R (\nabla \times \vec{F}) \cdot \mathbf{k}\, dA \text{ (tangential form)}$$
$$\oint_C \vec{F} \cdot \vec{T}\, ds = \iint_R (\nabla \times \vec{F}) \cdot \hat{n}\, dA \text{ (Stokes' theorem)}$$

Normal Form

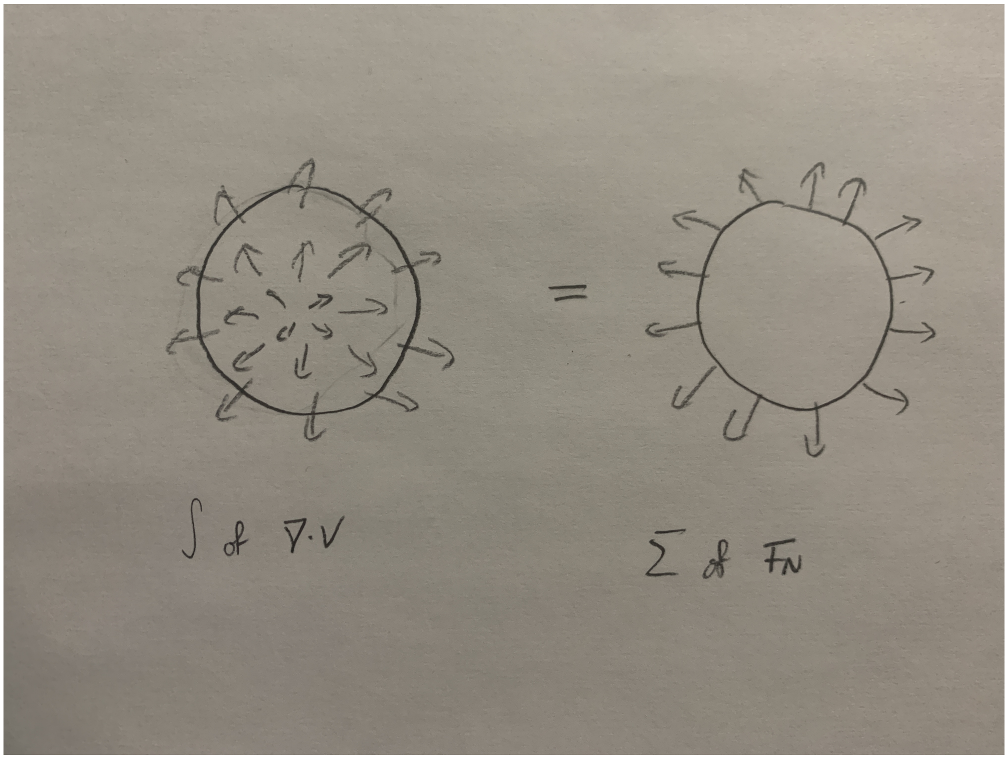
$$\oint_C \vec{F} \cdot \hat{n}\, ds = \iint_R \nabla \cdot \vec{F}\, dA \text{ (normal form)}$$
$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iiint_D \nabla \cdot \vec{F}\, dV \text{ (Divergence theorem)}$$

The Theory

The integral of a differential operator (div or curl) acting on a field over a region = the sum of the field components (appropriate to the operator) over the boundary of the region

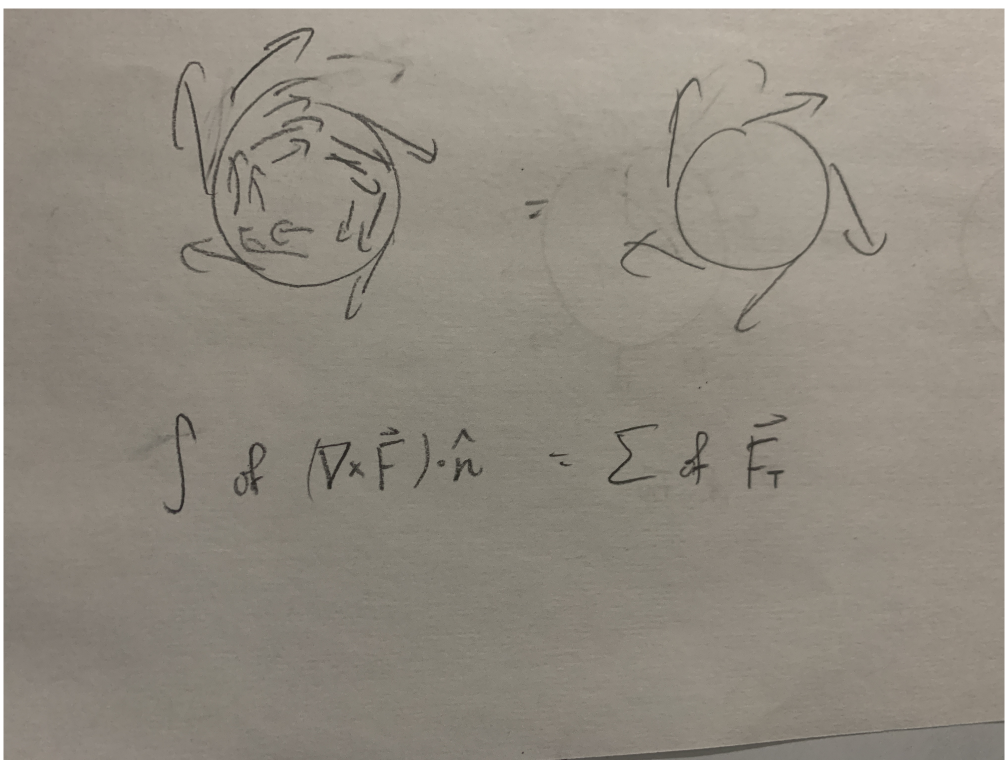
huh what what huh what what huh?
For the Divergence Theorem (and the normal form of Green's), integrating $\text{div}\ \vec{F}$ over a region is equal to the normal components of the field at the region's boundary.

$$\overbrace{\iiint_D \overset{\text{div}\ \vec{F}}{\nabla \cdot \vec{F}}\, dV}^{\text{integral over region}} = \overbrace{\iint_S \overset{F_N}{\vec{F} \cdot \hat{n}}\, d\sigma}^{\text{sum over boundary}}$$



For Stokes' Theorem (and the tangential form of Green's), integrating $(\text{curl}\,\vec{F}) \cdot \hat{n}$ over a region is equal to the tangential components of the field at the region's boundary.

$$\overbrace{\iint_S \overset{\text{curl}\,\vec{F}}{(\nabla \times \vec{F})} \cdot \hat{n}\, dA}^{\text{integral over region}} = \overbrace{\oint_C \overset{F_T}{\vec{F} \cdot \vec{T}\, ds}}^{\text{sum over boundary}}$$



#module4 #week13