16.4: Green's Theorem

Circulation density of a vector field $\vec{F} = M\mathbf{i} + N\mathbf{j}$ at point (x, y) is the scalar expression:

$$\text{curl}\,\vec{F}\cdot\mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

Divergence (flux density) of vector field $\vec{F} = M\mathbf{i} + N\mathbf{j}$ at (x, y) is:

$$\text{div}\,\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

Circulation-Curl or Tangential Form

Let C be piecewise smooth, simple closed curve enclosing region R in the plane.
Let $\vec{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N have continuous 1st partial derivatives in open region containing R.

Then:
The **counterclockwise circulation** of $\vec{F}$ around C equals the double integral of $\text{curl}\,\vec{F}\cdot\mathbf{k}$ over R.

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$$

Flux-Divergence or Normal Form

Let C be piecewise smooth, simple closed curve enclosing region R in the plane.
Let $\vec{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N have continuous 1st partial derivatives in open region containing R.

Then:
The **outward flux** of $\vec{F}$ around C equals the double integral of $\text{div}\,\vec{F}$ over R.

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dx\,dy$$

Area

Green's Theorem can be used to write area in terms of a line integral.

$$
\begin{aligned}
A_R &= \iint_R dy\,dx \\
&= \iint_R \left(\frac{1}{2} + \frac{1}{2}\right) dy\,dx \\
&= \frac{1}{2} \oint x\,dy - y\,dx
\end{aligned}
$$

16.5: Surfaces and Area

Parameterized Surfaces

A **parametrized surface** is given by: $\vec{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$.
The domain is the set of points in the uv-plane that can be substituted into $\vec{r}$.

Sphere: $x^2 + y^2 + z^2 = a^2$
$\vec{r}(\phi, \theta) = a \sin\phi \cos\theta \mathbf{i} + a \sin\phi \sin\theta \mathbf{j} + a \cos\phi \mathbf{k}$ (pretty straightforward mapping of spherical coordinates)
$(0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi)$

Cylinder: $x^2 + y^2 = a^2, 0 \leq z \leq b$
$\vec{r}(\theta, z) = a \cos\theta \mathbf{i} + a \sin\theta \mathbf{j} + z\mathbf{k}$ (pretty straightforward mapping of cylindrical coordinates)
$(0 \leq \theta \leq 2\pi, 0 \leq z \leq b)$

Cone: $z = \sqrt{x^2 + y^2}, 0 \leq z \leq b$
$\vec{r}(r, \theta) = r \cos\theta \mathbf{i} + r \sin\theta \mathbf{j} + r\mathbf{k}$
$(0 \leq r \leq b, 0 \leq \theta \leq 2\pi)$

Example

Find the parametrization for $z = 4 - x^2 - y^2, z \geq 0$.

$$
\begin{aligned}
z &= 4 - x^2 - y^2 \\
&= 4 - r^2 \\
\therefore r &\leq 2
\end{aligned}
$$

The parametrization:

$$
\vec{r}(r, \theta) = r \cos\theta \mathbf{i} + r \sin\theta \mathbf{j} + (4 - r^2)\mathbf{k}
$$

$(0 \leq r \leq 2, 0 \leq \theta < 2\pi)$

Surface Area

A parametrized surface $\vec{r}(u,v) = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}$ is **smooth** if $\vec{r}_u$ and $\vec{r}_v$ are continuous and $\vec{r}_u \times \vec{r}_v \neq 0$ on the interior of the parameter domain.

Given smooth surface $\vec{r}(u,v) = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}, a \leq u \leq b, c \leq v \leq d$:

$$A = \iint_R |\vec{r}_u \times \vec{r}_v| \, dA = \int_c^d \int_a^b |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

Surface area of an implicit surface:
Area of surface $F(x,y,z) = c$ over closed & bounded region R:

$$SA = \iint_R \frac{\|\nabla F\|}{|\nabla F \cdot \hat{p}|} \, dA$$

(where $\hat{p} = \mathbf{i}, \mathbf{j}, \text{ or } \mathbf{k}$ is normal to R and $\nabla F \cdot \hat{p} \neq 0$)

Surface area for $z = f(x,y)$:

$$A = \iint_R \sqrt{f_x^2 + f_y^2 + 1} \, dx \, dy$$

(This can be derived by creating parametrization $\vec{r}(x,y) = x\mathbf{i} + y\mathbf{j} + f(x,y)\mathbf{k}$ and applying parametrized surface integral formula)

16.6: Surface Integrals

Definition

Surface area differential:

$$d\sigma = |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

Surface integral of G over surface S

$$\iint_S G(x,y,z) \, d\sigma = \lim_{n \to \infty} \sum_{k=1}^{n} G(x_k, y_k, z_k) \Delta\sigma_k$$

Surface integrals of scalar functions

You can substitute $d\sigma$ for the surface area formulas above (Surface Area) depending on which condition you meet.

Parametrized surface
Given smooth surface $\vec{r}(u,v) = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}, (u,v) \in R$

$$\iint_S G(x,y,z)d\sigma = \iint_R G(f(u,v),g(u,v),h(u,v))|\vec{r}_u \times \vec{r}_v|\,du\,dv$$

Implicit surface

$$\iint_S G(x,y,z)d\sigma = \iint_R G(x,y,z)\frac{\|\nabla F\|}{|\nabla F \cdot \hat{p}|}\,dA$$

(S lies above its closed & bounded shadow region R in the coordinate plane beneath it)
(where $\hat{p} = \mathbf{i}, \mathbf{j}, \text{ or } \mathbf{k}$ is normal to R and $\nabla F \cdot \hat{p} \neq 0$)

For $z = f(x,y)$

$$\iint_S G(x,y,z)d\sigma = \iint_R G(x,y,f(x,y))\sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy$$

(R is the region on the xy-plane)

Surface integrals of vector fields

Let $\vec{F}$ be a vector field in 3D space with continuous components defined over a smooth surface S, with normal unit vectors $\hat{n}$ orienting S.
The **surface integral of** $\vec{F}$ **over** S:

$$\iint_S \vec{F} \cdot \hat{n}\,d\sigma$$

This integral is also called the **flux** of the vector field $\vec{F}$ across S.

If the surface being integrated over can be written as $g(x,y,z) = c$,

$$\hat{n} = \pm\frac{\nabla g}{\|\nabla g\|}$$

If the surface being integrated over can be parametrized as $\vec{r}(u,v)$, the surface integral can be calculated as:

$$\iint_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\,du\,dv$$

Mass & Moment

Same as [Week 8 > Physics Definitions], but with:

$$dm = \delta d\sigma$$