15.1, 15.2, 15.3: Double Integrals

Over Rectangles

Let $f(x, y)$ be continuous over rectangle $\mathcal{R} : a \leq x \leq b, c \leq y \leq d$.
Let $\mathcal{P}$ be a partition of $\mathcal{R}$ and let m_{ij} and M_{ij} be maximum values of f on the i, j sub-rectangle $\mathcal{R}_{ij}$. Then:

1. Lower sum

$$L_f(\mathcal{P}) = \sum_{i=1}^{m} \sum_{j=1}^{n} \overbrace{\Delta x_i \Delta y_j}^{\text{area}} m_{ij}$$

2. Upper sum

$$U_f(\mathcal{P}) = \sum_{i=1}^{m} \sum_{j=1}^{n} \overbrace{\Delta x_i \Delta y_j}^{\text{area}} M_{ij}$$

The **double integral** of f over $\mathcal{P}$ is the unique number I satisfying $L_f(\mathcal{P}) \leq I \leq U_f(\mathcal{P})$ for all partitions $\mathcal{P}$.

$$\begin{gathered} I = \iint_{\mathcal{R}} f(x, y)\, dx\, dy \\ \text{or} \\ I = \iint_{\mathcal{R}} f(x, y)\, dA \end{gathered}$$

Fubini's Theorem (first form)

If $f(x, y)$ is continuous throughout rectangular region $R : a \leq x \leq b, c \leq y \leq d$:

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

Volume

If $f(x, y)$ is a positive function over a rectangular region R in xy-plane, the volume of the solid region over the xy-plane bounded below by R and above by $f(x, y)$ is:

$$V = \iint_R f(x, y)\, dA$$

Example

Evaluate $\int_0^3 \int_{-2}^0 (x^2y - 2xy)\, dy\, dx$.

(From bounds: $-2 \le y \le 0, 0 \le x \le 3$)

Like partials, hold one constant while integrating other.

$$\begin{aligned}
&\int_{-2}^{0} (x^2y - 2xy)\, dy \\
&= x^2\frac{y^2}{2} - xy^2 \bigg|_{-2}^{0} \\
&= -2x^2 + 4x \\
&\int_0^3 (-2x^2 + 4x)\, dx \\
&= -\frac{2}{3}x^3 + 2x^2 \bigg|_0^3 \\
&= 0
\end{aligned}$$

Over General Regions

Fubini's Theorem (stronger form)

Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b, g_1(x) \le y \le g_2(x)$ (such that g_1 and g_2 are continuous on $[a, b]$)

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

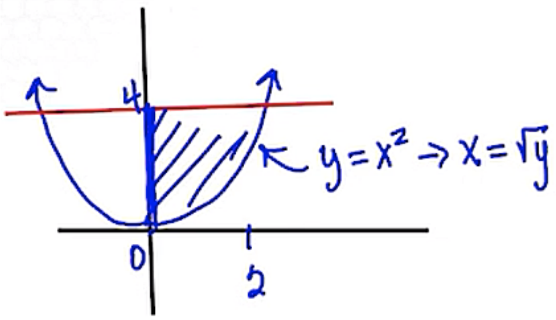
2. If R is defined by $c \le y \le d, h_1(y) \le x \le h_2(y)$ (such that h_1 and h_2 are continuous on $[c, d]$)

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x, y)\, dx\, dy$$

Changing Order of Integration

Vertical cross sections (1) and horizontal cross sections (2) are used.
You can switch between them.

$x^2 \le y \le 4$
$0 \le x \le 2$



$$\left.\begin{array}{l} 0 \le y \le 4 \\ 0 \le x \le \sqrt{y} \end{array}\right\}$$

Properties

Given continuous functions $f(x,y), g(x,y)$ on bounded region R:

1. Constant mutiple

$$\iint_R cf(x,y)\,dA = c\iint_R f(x,y)dA$$

2. Sum and difference

$$\iint_R (f(x,y) \pm g(x,y))\,dA = \iint_R f(x,y)dA \pm \iint_R g(x,y)dA$$

3. Domination

$$\iint_R f(x,y)\,dA \ge 0 \text{ if } f(x,y) \ge 0 \text{ on } R$$

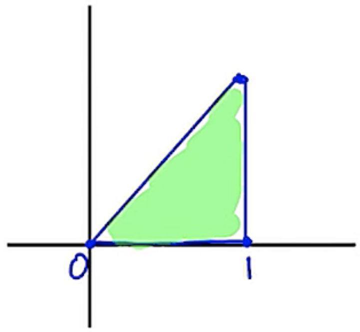
$$\iint_R f(x,y)\,dA \ge \iint_R g(x,y)\,dA \text{ if } f(x,y) \ge g(x,y) \text{ on } R$$

4. Additivity: If R is the union of two non-overlapping regions R_1 and R_2:

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

Example

Evaluate $\iint_R x^3 y \, dA$ where $R : 0 \leq x \leq 1, 0 \leq y \leq x$.
R:



$$\int_0^1 \underbrace{\int_0^x x^3 y \, dy}_{\begin{gathered} \frac{1}{2}x^3y^2\Big|_0^x \\ = \frac{1}{2}x^5 \end{gathered}} \, dx$$

$$\int_0^1 \frac{1}{2}x^5 \, dx$$
$$= \frac{1}{12}x^6\Big|_0^1$$
$$= \frac{1}{12}$$

Application: Area

The **area** of a closed, bounded plane region R is:

$$A = \iint_R dA$$

Application: Average Value

$$\text{Average value of } f \text{ over } R = \frac{1}{\text{area of } R} \iint_R f \, dA$$