

## 16.1: Line Integrals

If  $f$  is defined on a curve  $C$  given parametrically by  $\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ , the line integral of  $f$  over  $C$  is:

$$\int_C f(x, y, z) ds = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k, y_k, z_k) \Delta s_k$$

(This is the form for a **line integral of a scalar field**)

To integrate a continuous function  $f(x, y, z)$  over a curve  $C$ :

1. Find a smooth parametrization of  $C$ :

$$\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad (\text{where } a \leq t \leq b)$$

2. Evaluate the integral as:

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) |\vec{v}(t)| dt$$

(recall  $\frac{ds}{dt} = |\vec{v}|$ )

## Mass and Moment Calculations

Suppose we need to find the mass & moment for coil springs and then rods lying along a smooth curve  $C$  in space.

Recall [physics definitions](#) from 15.6.

They apply here, too.

**Mass**

$$m = \int_C \lambda ds$$

(this is a pretty straightforward extension of 15.6 so I don't think there needs to be notes here)

## 16.2: Vector Fields & Line Integrals

Let  $\vec{F}$  be a vector field with continuous components defined along smooth curve  $C$  parametrized by  $\vec{r}(t)$ ,  $a \leq t \leq b$ .

The **line integral of  $\vec{F}$  along  $C$**  is:

$$\int_C \vec{F} \cdot \vec{T} ds = \int_C \left( F \cdot \frac{d\vec{r}}{ds} \right) ds = \int_C \vec{F} \cdot d\vec{r}$$

(This is the form for a **line integral of a vector field**)

To evaluate, write  $\vec{F}$  and  $d\vec{r}$  in terms of  $t$  and apply dot product.

Line integrals may also be written as:

$$\begin{aligned} & \int_C M dx + \int_C N dy + \int_C P dz \\ &= \int_C M(x, y, z) dx + \int_C N(x, y, z) dy + \int_C P(x, y, z) dz \end{aligned}$$

(same idea, write everything in terms of  $t$ )

## Example

Evaluate  $\int_C \vec{F} \cdot d\vec{r}$ , where  $\vec{F} = \langle xy, x^2z, xyz \rangle$  along  $y = x^2$  from  $(0, 0, 0)$  to  $(1, 1, 0)$  followed by the straight-line segment from  $(1, 1, 0)$  to  $(1, 1, 1)$ .

$$\begin{aligned} C_1 : \vec{r}_1(t) &= \langle t, t^2, 0 \rangle \\ \vec{r}'_1(t) &= \langle 1, 2t, 0 \rangle \end{aligned}$$

$$\begin{aligned} C_2 : \vec{r}_2(t) &= \langle 1, 1, t \rangle \\ \vec{r}'_2(t) &= \langle 0, 0, 1 \rangle \end{aligned}$$

$$\begin{aligned} & \int_C \vec{F} \cdot d\vec{r} \\ &= \int_0^1 \langle (t)(t^2), 0, 0 \rangle \cdot \langle 1, 2t, 0 \rangle dt + \int_0^1 \langle (1)(1), (1)^2(t), (1)(1)(t) \rangle \cdot \langle 0, 0, 1 \rangle dt \\ &= \int_0^1 \langle t^3, 0, 0 \rangle \cdot \langle 1, 2t, 0 \rangle dt + \int_0^1 \langle 1, t, t \rangle \cdot \langle 0, 0, 1 \rangle dt \end{aligned}$$

## Physics

### Work

$$W = \int_C \vec{F} \cdot d\vec{r}$$

( $\vec{F}$  is force)

### Flow

$$\text{Flow} = \int_C \vec{F} \cdot \vec{T} ds$$

( $\vec{F}$  is velocity)

This integral is called a **flow integral**. If the curve starts and ends at the same point, the flow is called the *circulation* around the curve.

## Flux

$$\Phi = \int_C \vec{F} \cdot \vec{n} ds$$

( $\vec{F}$  is velocity,  $C$  is a smooth simple closed curve (starts & ends at same place and does not cross itself))

### Calculating flux across a smooth closed plane curve

Let  $\vec{F} = M\mathbf{i} + N\mathbf{j}$  and  $\vec{r} = x(t)\mathbf{i} + y(t)\mathbf{j}$ ,  
then:

$$\Phi \text{ across } C = \oint M dy - N dx$$

(Integral is evaluated at any parametrization  $\vec{r}$  that traces  $C$  counterclockwise exactly once)

$$\left\| \vec{F} \times d\vec{r} \right\| = \begin{vmatrix} M & N \\ dx & dy \end{vmatrix} = M dy - N dx$$

(Computing line integral with respect to  $ds$  gives you the above)

## 16.3: Path Independence, Conservative Fields, Potential Functions

### Definitions

Let  $\vec{F}$  be a vector field defined on open region  $D$  in space.

Suppose that for any two points  $A$  and  $B$  in  $D$ ,  $\int_C \vec{F} \cdot d\vec{r}$  along path  $C$  from  $A$  to  $B$  is the same over all paths from  $A$  to  $B$ .

The integral is **path independent** and the field is **conservative on  $D$** .

If  $\vec{F}$  is a vector field on  $D$  and  $F = \nabla f$  for some scalar function  $f$  on  $D$ ,  $f$  is called a **potential function for  $F$** .

### Example

Find a potential function  $f$  for  $\vec{F} = (y \sin z)\mathbf{i} + (x \sin z)\mathbf{j} + (xy \cos z)\mathbf{k}$ .

Let  $\vec{F} = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$ .

$$\frac{\partial f}{\partial x} = y \sin z$$

$$f = \int y \sin z \, dx = xy \sin z + \overbrace{g(y, z)}^{\text{remember } +C?}$$

$$\frac{\partial f}{\partial y} = x \sin z + \frac{\partial g}{\partial y}$$

$$\frac{\partial f}{\partial z} = xy \cos z + \frac{\partial g}{\partial z}$$

So,

$$x \sin z + \frac{\partial g}{\partial y} = x \sin z \implies \frac{\partial g}{\partial y} = 0$$

$$xy \cos z + \frac{\partial g}{\partial z} = xy \cos z \implies \frac{\partial g}{\partial z} = 0$$

Therefore,

$$f(x, y, z) = xy \sin z + C$$

## Conservative Fields & Gradient Fields

### Theorem

Let  $\vec{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$  (and  $M, N, P$  are continuous throughout open connected region  $D$ ).  
 $F$  is conservative iff  $\vec{F}$  is a gradient field  $\nabla f$  for a differentiable function  $f$ .

### Component Test for Conservative Fields

Let  $\vec{F} = X(x, y, z)\mathbf{i} + Y(x, y, z)\mathbf{j} + Z(x, y, z)\mathbf{k}$  on open simply connected domain ( $X, Y, Z$  have continuous first partial derivatives).

then  $\vec{F}$  is conservative iff:

$$\begin{aligned} \frac{\partial X}{\partial y} &= \frac{\partial Y}{\partial x} \\ \frac{\partial X}{\partial z} &= \frac{\partial Z}{\partial x} \\ \frac{\partial Y}{\partial z} &= \frac{\partial Z}{\partial y} \end{aligned}$$

If  $\vec{F}$  is a gradient field for  $f$ , then  $F = \nabla f = f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k}$ .

Then, second mixed partials need to be equivalent ( $f_{xy} = f_{yx}, f_{xz} = f_{zx}, f_{yz} = f_{zy}$ ) for this to be true.

## Fundamental Theorem of Line Integrals

Let  $C$  be a smooth curve joining points  $A$  and  $B$ , parametrized by  $\vec{r}(t)$ .

Let  $f$  be a differentiable function with continuous gradient vector  $\vec{F} = \nabla f$  on domain  $D$  containing  $C$ .

$$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

## Loop Property of Conservative Fields

Equivalent statements

1.  $\oint_C \vec{F} \cdot d\vec{r} = 0$  around every loop (every closed curve  $C$ ) in  $D$ .
2. The field  $F$  is conservative on  $D$ .

## Exactness

**Differential form:** Expression of the form  $M(x, y, z) dx + N(x, y, z) dy + P(x, y, z) dz$ .

A differential form is **exact** on domain  $D$  if:

$$M dx + N dy + P dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = df$$

for some scalar function  $f$  throughout  $D$ .

In other words, a differential form is exact iff  $\langle M, N, P \rangle = \nabla f$  (iff  $\vec{F} = \langle M, N, P \rangle$  is conservative).