12.5: Lines & Planes in Space

Lines

The **vector equation** for line L through $P_0(x_0, y_0, z_0)$ parallel to vector $\vec{v}$:

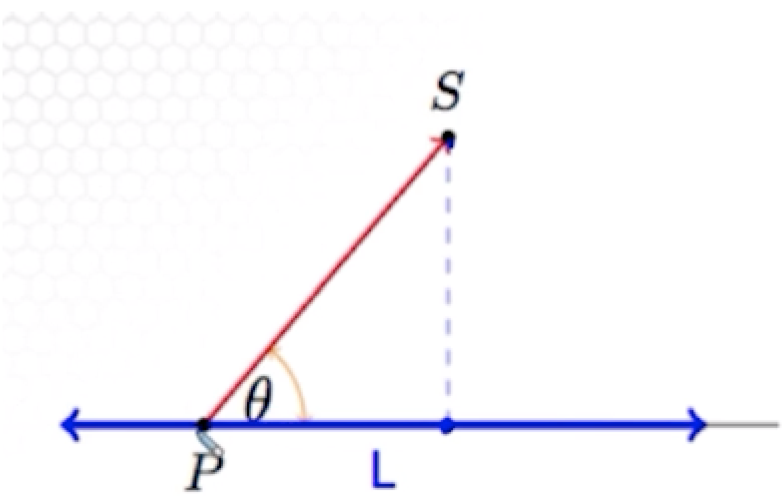
$$\vec{r}(t) = \overbrace{\vec{r}_0}^{P_0} + t\vec{v} \ \ (-\infty < t < \infty)$$

The **standard parametrization** through $P_0(x_0, y_0, z_0)$ parallel to $\vec{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$:

$$\begin{aligned} x(t) &= x_0 + tv_1 \\ y(t) &= y_0 + tv_2 \\ z(t) &= z_0 + tv_3 \end{aligned}$$

If is line, $-\infty < t < \infty$. If t is bounded, is line segment.

Distance from point to line



$$d = \frac{\left\|\overrightarrow{PS} \times \vec{v}\right\|}{\|\vec{v}\|}$$

Planes

The **vector equation** for a plane through $P_0(x_0, y_0, z_0)$ with normal vector $\vec{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is given by:

$$\vec{n} \cdot (\overrightarrow{P_0P}) = 0$$

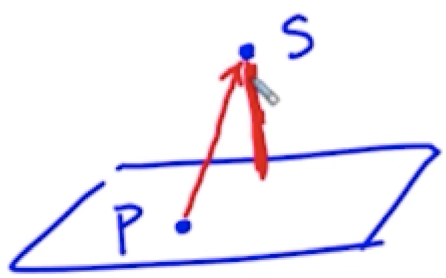
Component equation:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Angle between two planes

- Parallel planes have the same normal.
- Angle between two intersecting planes = acute angle between normals $\left(\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\|\|\vec{n}_2\|}\right)$

Distance from point to plane



$$d = \left\| \text{proj}_{\hat{n}} \overrightarrow{PS} \right\| = \overrightarrow{PS} \cdot \overbrace{\hat{n}}^{\text{normalized } \vec{n}}$$

Use this to find **distance between skew lines**

- Given lines l_1, l_2, find unit normal vector of the $l_1 l_2$ plane, $\hat{n} = \frac{\vec{l_1} \times \vec{l_2}}{\|\vec{l_1} \times \vec{l_2}\|}$, and project $\overrightarrow{PS}$ onto $\hat{n}$

Intersecting lines & planes

Lines

Lines l_1, l_2 can be...

- parallel
- intersecting

- coincident: same line
- skew: neither parallel nor intersecting

If the direction vectors are the same, the lines must be *parallel* or *concident*.

- Pick a point on l_1. If it is on l_2, the lines are coincident.
- Otherwise, the lines are parallel.

If the direction vectors are not the same, the lines must be skew or intersecting.

- Check if there are any intersecting points. If such a point exists, then the lines are intersecting.
- Otherwise, the lines are skew.

Planes

Two planes can be: parallel, intersecting, coincident

- If the planes' normals are parallel, the planes are parallel
- If the planes' normals are not parallel, the cross product gives direction vector for line of intersection of the planes

Line & Plane

To find the intersection point between a line and a plane, substitute the line equation into the plane equation.
If there's a valid point, then that is the intersecting point.

#week2