16.4: Green's Theorem

Circulation density of a vector field $\vec{F} = M\mathbf{i} + N\mathbf{j}$ at point (x, y) is the scalar expression:

$$\text{curl } \vec{F} \cdot \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

Divergence (flux density) of vector field $\vec{F} = M\mathbf{i} + N\mathbf{j}$ at (x, y) is:

$$\text{div } \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

Circulation-Curl or Tangential Form

Let C be piecewise smooth, simple closed curve enclosing region R in the plane.
Let $\vec{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N have continuous 1st partial derivatives in open region containing R.

Then:
The **counterclockwise circulation** of $\vec{F}$ around C equals the double integral of $\text{curl } \vec{F} \cdot \mathbf{k}$ over R.

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

Flux-Divergence or Normal Form

Let C be piecewise smooth, simple closed curve enclosing region R in the plane.
Let $\vec{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N have continuous 1st partial derivatives in open region containing R.

Then:
The **outward flux** of $\vec{F}$ around C equals the double integral of $\text{div } \vec{F}$ over R.

$$\oint_C \vec{F} \cdot \vec{n}\, ds = \oint_C M\, dy - N\, dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy$$

Area

Green's Theorem can be used to write area in terms of a line integral.

$$
\begin{aligned}
A_R &= \iint_R dy\,dx \\
&= \iint_R \left(\frac{1}{2} + \frac{1}{2}\right) dy\,dx \\
&= \frac{1}{2} \oint x\,dy - y\,dx
\end{aligned}
$$

#week12