

16.5: Surfaces and Area

Parameterized Surfaces

A **parametrized surface** is given by: $\vec{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$.

The domain is the set of points in the uv -plane that can be substituted into \vec{r} .

Sphere: $x^2 + y^2 + z^2 = a^2$

$\vec{r}(\phi, \theta) = a \sin \phi \cos \theta \mathbf{i} + a \sin \phi \sin \theta \mathbf{j} + a \cos \phi \mathbf{k}$ (pretty straightforward mapping of spherical coordinates)

$$(0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi)$$

Cylinder: $x^2 + y^2 = a^2, 0 \leq z \leq b$

$\vec{r}(\theta, z) = a \cos \theta \mathbf{i} + a \sin \theta \mathbf{j} + z \mathbf{k}$ (pretty straightforward mapping of cylindrical coordinates)

$$(0 \leq \theta \leq 2\pi, 0 \leq z \leq b)$$

Cone: $z = \sqrt{x^2 + y^2}, 0 \leq z \leq b$

$\vec{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + r \mathbf{k}$

$$(0 \leq r \leq b, 0 \leq \theta \leq 2\pi)$$

Example

Find the parametrization for $z = 4 - x^2 - y^2, z \geq 0$.

$$\begin{aligned} z &= 4 - x^2 - y^2 \\ &= 4 - r^2 \\ \therefore r &\leq 2 \end{aligned}$$

The parametrization:

$$\vec{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + (4 - r^2) \mathbf{k}$$

$$(0 \leq r \leq 2, 0 \leq \theta < 2\pi)$$

Surface Area

A parametrized surface $\vec{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$ is **smooth** if \vec{r}_u and \vec{r}_v are continuous and $\vec{r}_u \times \vec{r}_v \neq 0$ on the interior of the parameter domain.

SA of Parametrized Surfaces

Given smooth surface $\vec{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, a \leq u \leq b, c \leq v \leq d$:

$$\sigma = \iint_R |\vec{r}_u \times \vec{r}_v| dA = \int_c^d \int_a^b |\vec{r}_u \times \vec{r}_v| du dv$$

Why?

For a small rectangular area $\Delta\sigma$ on the surface,

$$\begin{aligned}\Delta\sigma &= |(\vec{r}_u \cdot \Delta u) \times (\vec{r}_v \cdot \Delta v)| \\ &= |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v\end{aligned}$$

So, $d\sigma = |\vec{r}_u \times \vec{r}_v| du dv$.

SA of Implicit Surfaces

Area of surface $F(x, y, z) = c$ over closed & bounded region R :

$$\sigma = \iint_R \frac{\|\nabla F\|}{|\nabla F \cdot \hat{p}|} dA$$

(where $\hat{p} = \mathbf{i}, \mathbf{j}$, or \mathbf{k} is normal to R and $\nabla F \cdot \hat{p} \neq 0$)

Why?

Create shadow region R (a projection of the surface onto a coordinate plane), and let \hat{p} be the normal vector of R .

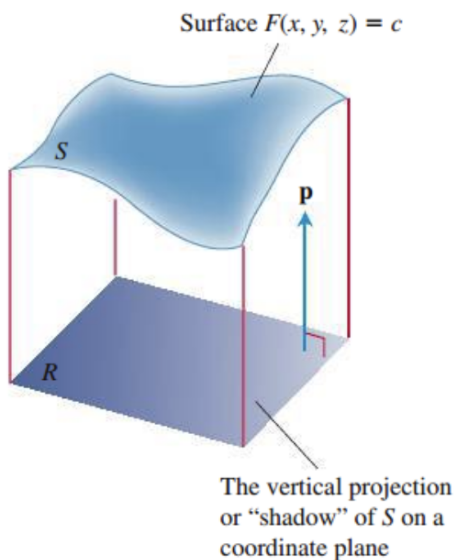


FIGURE 16.47 As we soon see, the area of a surface S in space can be calculated by evaluating a related double integral over the vertical projection or “shadow” of S on a coordinate plane. The unit vector \mathbf{p} is normal to the plane.

Assume surface is smooth and require $\nabla \cdot \hat{p} \neq 0$.

Let R be the xy -plane (then $\hat{p} = \mathbf{k}$). The curve is parametrized as:

$$\vec{r}(x, y) = x\mathbf{i} + y\mathbf{j} + z(x, y)\mathbf{k}$$

(note that $z(x, y)$ is not explicitly known)

$$\begin{aligned}\vec{r}_x &= \mathbf{i} + \frac{\partial z}{\partial x}\mathbf{k} = \mathbf{i} - \frac{F_x}{F_z}\mathbf{k} \\ \vec{r}_y &= \mathbf{j} + \frac{\partial z}{\partial y}\mathbf{k} = \mathbf{j} - \frac{F_y}{F_z}\mathbf{k}\end{aligned}$$

(recall $F(x, y, z(x, y)) = 0$, so [implicit chain rule](#) can be applied)

Then:

$$\begin{aligned}\vec{r}_x \times \vec{r}_y &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -\frac{F_x}{F_z} \\ 0 & 1 & -\frac{F_y}{F_z} \end{vmatrix} \\ &= \frac{F_x}{F_z}\mathbf{i} + \frac{F_y}{F_z}\mathbf{j} + \mathbf{k} \\ &= \frac{1}{F_z}(F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}) \\ &= \frac{\nabla F}{F_z} \\ &= \frac{\nabla F}{\nabla F \cdot \hat{p}}\end{aligned}$$

SA for $z = f(x, y)$

$$\sigma = \iint_R \sqrt{f_x^2 + f_y^2 + 1} \, dx \, dy$$

Why?

$z = f(x, y)$ can be parametrized as $\vec{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$.

Then:

$$\begin{aligned}\vec{r}_x &= \mathbf{i} + f_x \mathbf{k} \\ \vec{r}_y &= \mathbf{j} + f_y \mathbf{k} \\ \vec{r}_x \times \vec{r}_y &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x \mathbf{i} - f_y \mathbf{j} + \mathbf{k}\end{aligned}$$

So:

$$\sigma = \iint_R |\vec{r}_u \times \vec{r}_v| dA = \sqrt{f_x^2 + f_y^2 + 1} dx dy$$

16.6: Surface Integrals

Definition

Surface area differential:

$$d\sigma = |\vec{r}_u \times \vec{r}_v| du dv$$

Surface integral of G over surface S

$$\iint_S G(x, y, z) d\sigma = \lim_{n \rightarrow \infty} \sum_{k=1}^n G(x_k, y_k, z_k) \Delta\sigma_k$$

Surface integrals of scalar functions

To evaluate a surface integral, substitute $d\sigma$ for [the surface area formulas above](#) depending on which type of surface is being evaluated against.

Parametrized surface

Given smooth surface $\vec{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, (u, v) \in R$

$$\iint_S G(x, y, z) d\sigma = \iint_R G(f(u, v), g(u, v), h(u, v)) |\vec{r}_u \times \vec{r}_v| du dv$$

Implicit surface

$$\iint_S G(x, y, z) d\sigma = \iint_R G(x, y, z) \frac{\|\nabla F\|}{|\nabla F \cdot \hat{p}|} dA$$

(S lies above its closed & bounded shadow region R in the coordinate plane beneath it)
(where $\hat{p} = \mathbf{i}, \mathbf{j}$, or \mathbf{k} is normal to R and $\nabla F \cdot \hat{p} \neq 0$)

For $z = f(x, y)$

$$\iint_S G(x, y, z) d\sigma = \iint_R G(x, y, f(x, y)) \sqrt{f_x^2 + f_y^2 + 1} dx dy$$

(R is the region on the xy -plane)

Surface integrals of vector fields

Let \vec{F} be a vector field in 3D space with continuous components defined over a smooth surface S , with normal unit vectors \hat{n} orienting S .

The **surface integral of \vec{F} over S** :

$$\iint_S \vec{F} \cdot \hat{n} d\sigma$$

This integral is also called the **flux** of the vector field \vec{F} across S .

Evaluation across level surface

If the surface being integrated over can be written as $g(x, y, z) = c$,

$$\hat{n} = \pm \frac{\nabla g}{\|\nabla g\|}$$

(since [the gradient is normal to level surfaces](#))

Evaluation across parametrized surface

If the surface being integrated over can be parametrized as $\vec{r}(u, v)$,

$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$$

(since the axes of the [area on the surface](#) are in the direction of \vec{r}_u and \vec{r}_v).

Because of this, the integral can be simplified as:

$$\iint_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) du dv$$

Mass & Moment

Same as [15.6 Apps of Double & Triple Integrals > Physics Definitions](#), but with:

$$dm = \delta d\sigma$$

[#module4 #week12](#)

