16.7: Stokes' Theorem

∇, div, and curl

The del operator:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

Two formulas use the ∇ operator:

$$\begin{aligned} \text{del } \vec{F} &= \nabla \cdot \vec{F} \\ \text{curl } \vec{F} &= \nabla \times \vec{F} \end{aligned}$$

Curl Identity

$$\begin{aligned} \text{curl grad } f &= 0 \\ \nabla \times \nabla f &= 0 \end{aligned}$$

Why?

$$\nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

Mixed partials have to be equal if continuous, so everything here evals to 0.

Example

Find the div and curl for $\vec{F} = (x^2 - yz)\mathbf{i} + ye^x\mathbf{j} + (xy+z)\mathbf{k}$.

$$\begin{aligned} \text{div } \vec{F} = \nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(ye^x) + \frac{\partial}{\partial x}(xy+z) \\ &= 2x + e^x + 1 \end{aligned}$$

$$
\begin{aligned}
\text{curl } \vec{F} = \nabla \times \vec{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & ye^x & xy + z \end{vmatrix} \\
&= \mathbf{i}\left(\frac{\partial}{\partial y}(xy+z) - \frac{\partial}{\partial z}(ye^x)\right) \\
&- \mathbf{j}\left(\frac{\partial}{\partial x}(xy+z) - \frac{\partial}{\partial z}(x^2 - yz)\right) \\
&+ \mathbf{k}\left(\frac{\partial}{\partial x}(ye^x) - \frac{\partial}{\partial y}(x^2 - yz)\right) \\
&= x\mathbf{i} - 2y\mathbf{j} + (ye^x + z)\mathbf{k}
\end{aligned}
$$

Stokes' Theorem

Let S be a piecewise smooth oriented surface with piecewise smooth boundary curve C.
Let $\vec{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field with continuous 1st partial derivatives on open region containing S.

Then the circulation of $\vec{F}$ around C in the CCW dir:

$$
\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, d\sigma = \iint_S (\text{curl } \vec{F}) \cdot \hat{n}\, d\sigma
$$

($\hat{n}$ is the unit normal vector with respect to the surface)

Closed Loop Property:
If $\text{curl F} = 0$ at every point of a simply connected open region D in space, then on any piecewise-smooth closed path C in D:

$$
\oint_C F \cdot d\vec{r} = 0
$$

(pretty straightforward extension of the loop property of conservative fields)

16.8: The Divergence Theorem and a Unified Theory

Divergence Theorem

Let S be a piecewise smooth oriented surface.
Let F be a vector field whose components have continuous 1st partial derivatives.

Then, the flux of $\vec{F}$ across S in the direction of the surface's outward unit normal field $\hat{n}$:

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iiint_D \nabla \cdot \vec{F}\, dV$$

Corollary:
The outward flux across a piecewise smooth oriented closed surface is 0 for any vector field F with 0 divergence at every point of the region.

Divergence & Curl

$$\text{div }(\text{curl }\vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = 0$$

Unifying Fundamental Theorem of Vector Integral Calculus

The integral of a differential operator (∇) acting on a field over a region = the sum of the field components (appropriate to the operator) over the boundary of the region

Example with Stokes' theorem:

$$\overbrace{\oint_C \vec{F} \cdot d\vec{r}}^{\Sigma \text{ of field components over boundary } C} = \overbrace{\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, d\sigma}^{\int \text{ of } \nabla \text{ acting on } \vec{F} \text{ over } S}$$

Generalizations of Green's Theorem

Tangential Form

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \iint_R (\nabla \times \vec{F}) \cdot \mathbf{k}\, dA \text{ (tangential form)}$$
$$\oint_C \vec{F} \cdot \vec{T}\, ds = \iint_R (\nabla \times \vec{F}) \cdot \hat{n}\, dA \text{ (Stokes' theorem)}$$

Normal Form

$$\oint_C \vec{F} \cdot \hat{n}\, ds = \iint_R \nabla \cdot \vec{F}\, dA \text{ (normal form)}$$
$$\iint_S \vec{F} \cdot \hat{n}\, d\sigma = \iiint_D \nabla \cdot \vec{F}\, dV \text{ (Divergence theorem)}$$