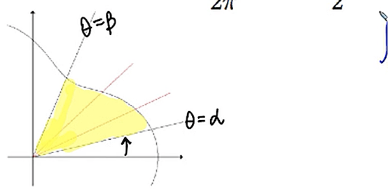
15.4: Double Integrals in Polar Form

Area of a polar region is based on the area of a sector of a circle.

- Area of a circle: πr^2
- Area of a sector: $\frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \theta$

$$\text{Area of a sector} = \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \theta$$



$$\begin{aligned} \text{Area of polar region} &= \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta \\ &= \int_{\alpha}^{\beta} \int_{a}^{b} r \, dr \, d\theta \\ &= \iint_R r \, dr \, d\theta \end{aligned}$$

(Note that $\int_a^b r\,dr = \frac{1}{2} r^2 \Big|_a^b$)

$$dA = r \, dr \, d\theta = dx \, dy$$

Given $F(r, \theta)$ is continuous on $\Gamma : a \leq r \leq b, \alpha \leq \theta \leq \beta$:

$$\iint_{\Gamma} F(r, \theta) r \, dr \, d\theta = \int_{\alpha}^{\beta} \int_{a}^{b} F(r, \theta) r \, dr \, d\theta$$

Given $F(r, \theta)$ is continuous on $\Omega : \alpha \leq \theta \leq \beta, \rho_1(\theta) \leq r \leq \rho_2(\theta)$:

$$\iint_{\Omega} F(r, \theta) r \, dr \, d\theta = \int_{\alpha}^{\beta} \int_{\rho_1(\theta)}^{\rho_2(\theta)} F(r, \theta) r \, dr \, d\theta$$

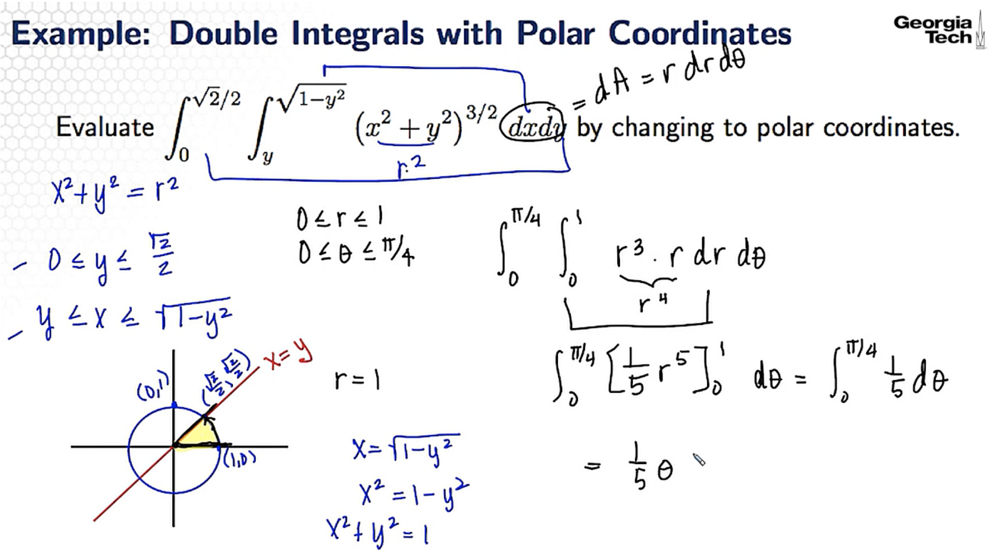
Fubini's Theorem applies here. (Bounds can be switched or rearranged as described in Fubini's.)

Volume

If $F(r, \theta) \geq 0$ over region R, then volume with R as base, bounded above by $F(r, \theta)$ is:

$$V = \iint_R F(r, \theta) r \, dr \, d\theta$$

(same as double integrals above)



15.5: Triple Integrals

Instead of working with two variables continuous over a plane, THREE variables!

Integration over a box

Given $f(x, y, z)$ continuous on box $B: a_x \le x \le b_x, a_y \le y \le b_y, a_z \le z \le b_z$

$$\iiint_B f(x,y,z)\,dV = \int_{a_z}^{b_z}\int_{a_y}^{b_y}\int_{a_x}^{b_x} f(x,y,z)\,dx\,dy\,dz$$

Fubini's Theorem applies here too.

Volume

$$V = \iiint_D dV$$

Average Value

The average value of a function F over region D in space is:

$$\text{Average Value of } F \text{ over } D = \frac{1}{\text{Volume of } D}\iiint_D F\,dV$$

15.6: Applications of Double & Triple Integrals

Physics Definitions

(mass)

$$dm = \sigma \, dA \text{ (2 dimensions)}$$
$$dm = \rho \, dV \text{ (3 dimensions)}$$

(moment)

$$dM = r \, dm$$

(moment of inertia)

$$dI = r^2 \, dm$$

Rest of these formulas can essentially be defined by these relationships.

Mass and First Moments

In three dimensions

Mass:

$$M = \iiint_D \rho \, dV$$

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x\rho \, dV$$
$$M_{xz} = \iiint_D y\rho \, dV$$
$$M_{xy} = \iiint_D z\rho \, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}$$
$$\bar{y} = \frac{M_{xz}}{M}$$
$$\bar{z} = \frac{M_{xy}}{M}$$

When density of solid object is constant ($\rho = 1$), the center of mass is called the **centroid** of the object.

In two dimensions

Mass:

$$M = \iint_D \sigma \, dA$$

First moments about the coordinate axes:

$$M_y = \iint_D x\sigma \, dA$$
$$M_x = \iint_D y\sigma \, dA$$

Center of mass:

$$\bar{x} = \frac{M_y}{M}$$
$$\bar{y} = \frac{M_x}{M}$$

Moments of Inertia

In three dimensions

$$I = \iiint r^2 \rho \, dV$$

(Around x-axis, r^2 is $(y^2 + z^2)$, etc etc)

In two dimensions

$$I = \iint r^2 \sigma \, dA$$

About origin:

$$I_O = \iint (x^2 + y^2) \sigma \, dA = I_x + I_y$$

Joint Probability Density

Joint probability density function f is a function that satisfies:

1. $f(x, y) \geq 0$
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = 1$
3. $P((X, Y) \in R) = \iint_R f(x, y) \, dx \, dy$