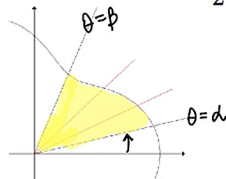


## 15.4: Double Integrals in Polar Form

Area of a polar region is based on the area of a sector of a circle.

- Area of a circle:  $\pi r^2$
- Area of a sector:  $\frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \theta$

$$\text{Area of a sector} = \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \theta$$



$$\begin{aligned} \text{Area of polar region} &= \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta \\ &= \int_{\alpha}^{\beta} \int_a^b r dr d\theta \\ &= \iint_R r dr d\theta \end{aligned}$$

(Note that  $\int_a^b r dr = \frac{1}{2} r^2 \Big|_a^b$ )

$$dA = r dr d\theta = dx dy$$

Given  $F(r, \theta)$  is continuous on  $\Gamma : a \leq r \leq b, \alpha \leq \theta \leq \beta$ :

$$\iint_{\Gamma} F(r, \theta) r dr d\theta = \int_{\alpha}^{\beta} \int_a^b F(r, \theta) r dr d\theta$$

Given  $F(r, \theta)$  is continuous on  $\Omega : \alpha \leq \theta \leq \beta, \rho_1(\theta) \leq r \leq \rho_2(\theta)$ :

$$\iint_{\Omega} F(r, \theta) r dr d\theta = \int_{\alpha}^{\beta} \int_{\rho_1(\theta)}^{\rho_2(\theta)} F(r, \theta) r dr d\theta$$

(Same applies for other orientations of this)

If  $F(r, \theta) \geq 0$  over region  $R$ , then volume with  $R$  as base, bounded above by  $F(r, \theta)$  is:

$$V = \iint_R F(r, \theta) r dr d\theta$$

(same as double integrals above)

### Example: Double Integrals with Polar Coordinates

Evaluate  $\int_0^{\sqrt{2}/2} \int_y^{\sqrt{1-y^2}} (x^2 + y^2)^{3/2} dx dy$  by changing to polar coordinates.

$x^2 + y^2 = r^2$

$- 0 \leq y \leq \frac{\sqrt{2}}{2}$

$- y \leq x \leq \sqrt{1-y^2}$

$0 \leq r \leq 1$

$0 \leq \theta \leq \pi/4$

$dA = r dr d\theta$

$\int_0^{\pi/4} \int_0^1 r^3 \cdot r dr d\theta$

$\int_0^{\pi/4} \left[ \frac{1}{5} r^5 \right]_0^1 d\theta = \int_0^{\pi/4} \frac{1}{5} d\theta$

$= \frac{1}{5} \theta$

$x = \sqrt{1-y^2}$

$x^2 = 1-y^2$

$x^2 + y^2 = 1$

## 15.5: Triple Integrals

Instead of working with two variables continuous over a plane, THREE variables!

### Integration over a box

Given  $f(x, y, z)$  continuous on box  $B : a_x \leq x \leq b_x, a_y \leq y \leq b_y, a_z \leq z \leq b_z$

$$\iiint_B f(x, y, z) dV = \int_{a_z}^{b_z} \int_{a_y}^{b_y} \int_{a_x}^{b_x} f(x, y, z) dx dy dz$$

Bounds are the same idea for general regions, too

### Volume

$$V = \iiint_D dV$$

### Average Value

The average value of a function  $F$  over region  $D$  in space is:

$$\text{Average Value of } F \text{ over } D = \frac{1}{\text{Volume of } D} \iiint_D F dV$$

## 15.6: Applications of Double & Triple Integrals

### Mass and First Moments

## In three dimensions

Mass:

$$M = \iiint_D \rho \, dV$$

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x \rho \, dV$$

$$M_{xz} = \iiint_D y \rho \, dV$$

$$M_{xy} = \iiint_D z \rho \, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}$$

$$\bar{y} = \frac{M_{xz}}{M}$$

$$\bar{z} = \frac{M_{xy}}{M}$$

When density of solid object is constant ( $\rho = 1$ ), the center of mass is called the **centroid** of the object.

## In two dimensions

Mass:

$$M = \iint_D \sigma \, dA$$

First moments about the coordinate planes:

$$M_x = \iint_D x \sigma \, dA$$

$$M_y = \iint_D y \sigma \, dA$$

Center of mass:

$$\bar{x} = \frac{M_y}{M}$$

$$\bar{y} = \frac{M_x}{M}$$

## Moments of Inertia

### In three dimensions

$$I = \iiint r^2 \rho \, dV$$

(Around x-axis,  $r^2$  is  $(y^2 + z^2)$ , etc etc)

### In two dimensions

$$I = \iint r^2 \sigma \, dA$$

About origin:

$$I_O = \iint (x^2 + y^2) \sigma \, dA = I_x + I_y$$

## Joint Probability Density

**Joint probability density function**  $f$  is a function that satisfies:

1.  $f(x, y) \geq 0$
2.  $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = 1$
3.  $P((X, Y) \in R) = \iint_R f(x, y) \, dx \, dy$