13.3: Arc Length in Space

Length of smooth curve $\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ $(a \le t \le b)$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_a^b \|\vec{v}\|\, dt$$

(Derives from $ds^2 = dx^2 + dy^2 + dz^2$)

Arc Length Parameter: Function *s* that finds directed distance along curve starting from $P(t_0)$ to some point $P(t)$

$$s(t) = \int_{t_0}^t \sqrt{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2}\, d\tau = \int_{t_0}^t \|\vec{v}\|\, d\tau$$

Speed:

$$\text{speed} = \frac{ds}{dt} = \|\vec{v}(t)\|$$

Unit Tangent Vector: The unit vector that is tangent to the curve

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\vec{v}(t)}{\|\vec{v}(t)\|} = \frac{d\vec{r}/dt}{ds/dt}$$

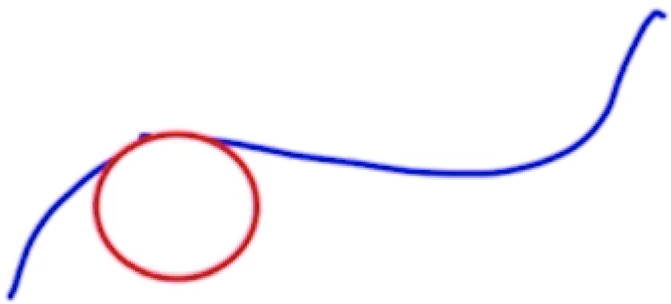
(This is the same as normalizing $\vec{v}$)

13.4: Curvature

If $\vec{T}$ is a unit vector of a smooth curve, the **curvature** function of the curve is

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\|$$

In the blue curve, the curvature at the point is related to circle that best fit curve at that point.



(seems similar to 2nd derivative)

For smooth curve $\vec{r}$, curvature can be written as scalar function:

$$\kappa = \left\| \frac{d\vec{T}/dt}{ds/dt} \right\| = \frac{\|\vec{T}'(t)\|}{\|\vec{v}(t)\|}$$

Circle of Curvature

The **circle of curvature** (or **osculating circle**) at point P on plane curve (2D) where $\kappa \neq 0$ is the circle of the curve that

1. is tangent to curve at P
2. has the same curvature the curve has at P
3. has center that lies toward the concave side of the curve

The **radius of curvature** at point P is $\rho = \frac{1}{\kappa}$.

- Straight lines: curvature is constantly 0
- Circle of radius r: Curvature is constantly $\frac{1}{r}$.

Principal Normal Vector

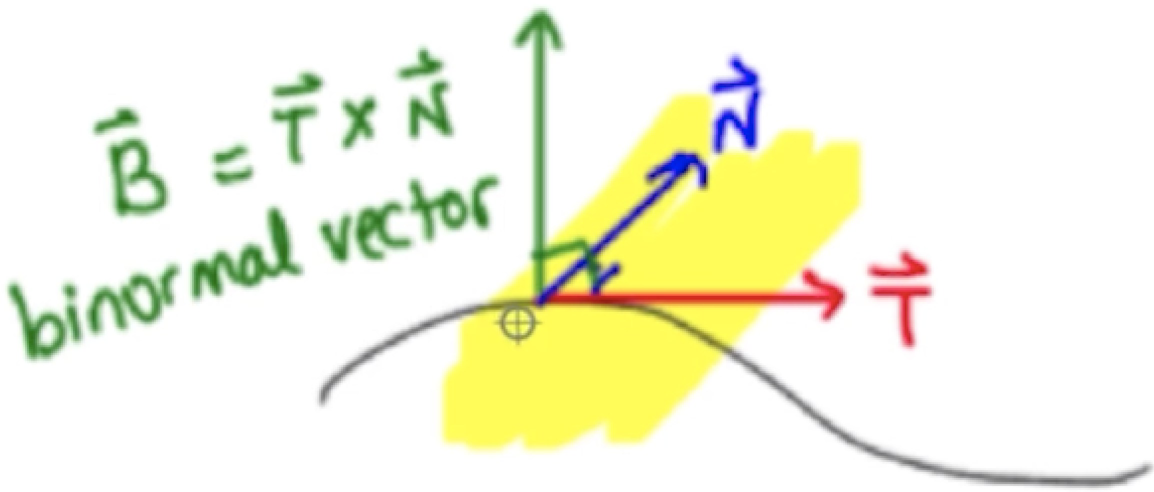
If $\vec{T}(t)$ is unit tangent vector and $\vec{T}'(t) = 0$, then unit tangent vector d/n change direction.

If $\vec{T}'(t) \neq 0$, then

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

(This is the same as normalizing $T'(t)$, and is $\perp$ to $\vec{T}$)

TNB Frame

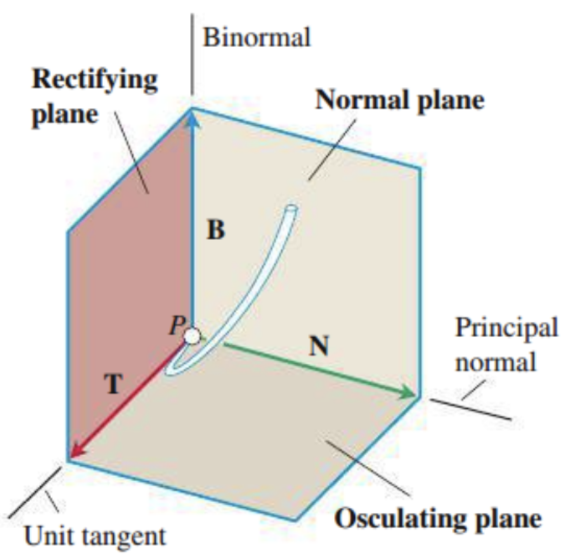


Binormal vector: $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

TNB frame / Frenet frame: The three vectors $\vec{T}$, $\vec{N}$, $\vec{B}$

Planes

- **Osculating plane**: Plane between $\vec{T}$ and $\vec{N}$ (normal is $\vec{B}$)
- **Normal plane**: Plane between $\vec{N}$ and $\vec{B}$ (normal is $\vec{T}$)
- **Rectifying plane**: Plane between $\vec{T}$ and $\vec{B}$ (normal is $\vec{N}$)



13.5: Tangential & Normal Components of Acceleration

How do we write $\vec{a}$ as components of the tangential and normal components?
In other words, how do we find a_T and a_N in:

$$\vec{a}(t) = \vec{N}(t)a_N + \vec{T}(t)a_T$$

Definitions

Tangential component of acceleration:

$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}\|\vec{v}\|$$

- Only dependent on change of speed of object
- If speed is constant, $a_T = 0$ and acceleration is directed entirely towards center of curvature

Normal component of acceleration:

$$a_N = \|\vec{T}'(t)\|\frac{ds}{dt} = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa\|\vec{v}\|^2$$

It may also be simpler to use $a_N = \sqrt{\|a\|^2 - a_T^2}$ to avoid having to calculate curvature.

Why?
Given position function $\vec{r}(t)$,

$$\vec{T}(t) = \frac{\vec{v}(t)}{ds/dt}$$
$$\vec{v}(t) = \vec{T}(t)\frac{ds}{dt}$$
$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \vec{T}'(t)\frac{ds}{dt} + \vec{T}(t)\frac{d^2s}{dt^2}$$

Since $\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$,

$$\vec{N}(t)\|\vec{T}'(t)\| = \vec{T}'(t)$$
$$\vec{a}(t) = \vec{N}(t)\underbrace{\|\vec{T}'(t)\|\frac{ds}{dt}}_{a_N} + \vec{T}(t)\underbrace{\frac{d^2s}{dt^2}}_{a_T}$$

Curvature and Torsion

Torsion

Let $\vec{B} = \vec{T} \times \vec{N}$.

$$\tau = -\frac{d\vec{B}}{ds} \cdot \vec{N}$$

- Measures how binormal vector changes with respect to arc length

$$\tau = \frac{\begin{vmatrix} \dots & \vec{r}' & \dots \\ \dots & \vec{r}'' & \dots \\ \dots & \vec{r}''' & \dots \end{vmatrix}}{\|\vec{v} \times \vec{a}\|^2}$$

This formula is derived in more advanced texts, so no explanation here.

Additional Formulas for Curvature

$$\vec{T} \cdot \vec{a} = a_T(\vec{T} \cdot \vec{T}) + a_N(\vec{T} \cdot \vec{N}) = a_T$$
$$\|\vec{T} \times \vec{a}\| = \|a_T(\vec{T} \times \vec{T})\| + \|a_N(\vec{T} \times \vec{N})\| = \|a_N\vec{B}\| = a_N$$

(An alternative way of thinking about it is $\vec{T} \cdot \vec{a}$ is the projection of $\vec{a}$ onto $\vec{v}$, and cross-prod is the projection onto the perpendicular.)

Therefore:

$$a_T = \frac{\vec{v} \cdot \vec{a}}{ds/dt}$$
$$a_N = \frac{\|\vec{v} \times \vec{a}\|}{ds/dt} = \kappa\left(\frac{ds}{dt}\right)^2$$

$$\kappa = \frac{\|\vec{v} \times \vec{a}\|}{(ds/dt)^3}$$

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