16.3: Path Independence, Conservative Fields, Potential Functions

Definitions

Let $\vec{F}$ be a vector field defined on open region D in space.
Suppose that for any two points A and B in D, $\int_C \vec{F} \cdot d\vec{r}$ along path C from A to B is the same over all paths from A to B.
The integral is **path independent** and the field is **conservative on** D.

If $\vec{F}$ is a vector field on D and $F = \nabla f$ for some scalar function f on D, f is called a **potential function for** F.

Example

Find a potential function f for $\vec{F} = (y \sin z)\mathbf{i} + (x \sin z)\mathbf{j} + (xy \cos z)\mathbf{k}$.

Let $\vec{F} = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$.

$$\frac{\partial f}{\partial x} = y \sin z$$

$$f = \int y \sin z \, dx = xy \sin z + \overbrace{g(y, z)}^{\text{remember } +C?}$$

$$\frac{\partial f}{\partial y} = x \sin z + \frac{\partial g}{\partial y}$$

$$\frac{\partial f}{\partial z} = xy \cos z + \frac{\partial g}{\partial z}$$

So,

$$x \sin z + \frac{\partial g}{\partial y} = x \sin z \implies \frac{\partial g}{\partial y} = 0$$

$$xy \cos z + \frac{\partial g}{\partial z} = xy \cos z \implies \frac{\partial g}{\partial z} = 0$$

Therefore,

$$f(x, y, z) = xy \sin z + C$$

Conservative Fields & Gradient Fields

Theorem

Let $\vec{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ (and M, N, P are continuous throughout open connected region D).
F is conservative iff $\vec{F}$ is a gradient field ∇f for a differentiable function f.

Component Test for Conservative Fields

Let $\vec{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ on open simply connected domain (M, N, P have continuous first partial derivatives).
then $\vec{F}$ is conservative iff:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
$$\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$$
$$\frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$$

If $\vec{F}$ is a gradient field for f, then $F = \nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$.
Because of the mixed derivatives theorem, the second mixed partials need to be equivalent ($f_{xy} = f_{yx}, f_{xz} = f_{zx}, f_{yz} = f_{zy}$).

Fundamental Theorem of Line Integrals

Let C be a smooth curve joining points A and B, parametrized by $\vec{r}(t)$.
Let f be a differentiable function with continuous gradient vector $\vec{F} = \nabla f$ on domain D containing C.

$$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

Why?
Recall from 14.6 Tangent Planes & Differentials > More Differentials, $df = \nabla f \cdot \hat{u}\, ds$.
Then, integrating this differential form:

$$\Delta f = \int_C \nabla f \cdot \vec{T} ds$$

(which evaluates to the theorem above)

Loop Property of Conservative Fields

Equivalent statements

1. $\oint_C F \cdot d\vec{r} = 0$ around every loop (every closed curve C) in D.
2. The field F is conservative on D.

Exactness

Differential form: Expression of the form $M(x, y, z)\, dx + N(x, y, z)\, dy + P(x, y, z)\, dz$.
A differential form is **exact** on domain D if:

$$M\, dx + N\, dy + P\, dz = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz = df$$

for some scalar function f throughout D.
In other words, a differential form is exact iff $\langle M, N, P \rangle = \nabla f$ (iff $\vec{F} = \langle M, N, P \rangle$ is conservative).

#module4 #week10