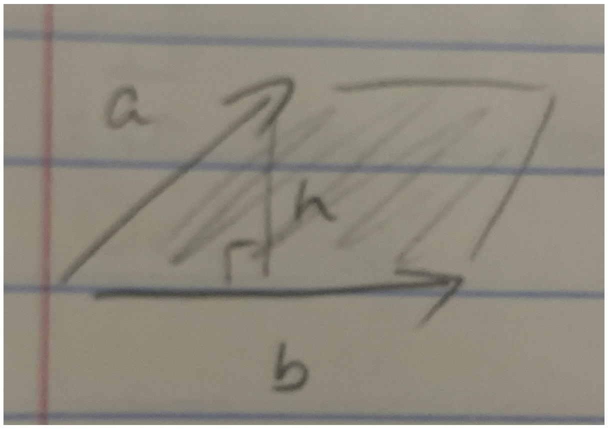
12.4: Cross Products

The **cross product** of $\vec{a}$ and $\vec{b}$ is the vector normal to $\vec{a}$ and $\vec{b}$ whose length is the area of the parallelogram between $\vec{a}$ and $\vec{b}$.



$$\text{area} = \left\| \vec{A} \times \vec{B} \right\| = \|\vec{A}\| \|\vec{B}\| \sin\theta$$

In full:

$$\vec{A} \times \vec{B} = \left(\|\vec{A}\| \|\vec{B}\| \sin\theta \right) \underbrace{\hat{n}}_{\text{Unit vector } \perp \text{ to plane AB}}$$

Note that if $\vec{a} \times \vec{b} = 0$, $\vec{a}$ and $\vec{b}$ are parallel.

Properties

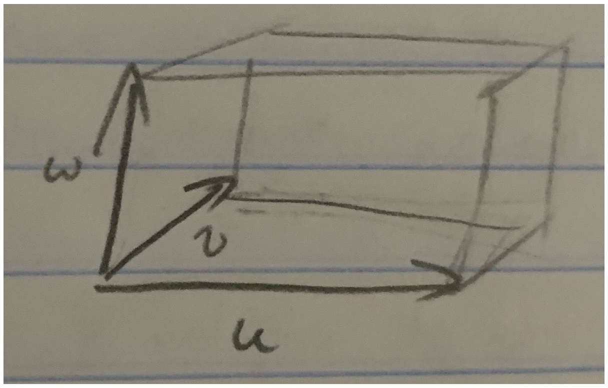
1. $r\vec{u} \times s\vec{v} = (rs)(\vec{u} \times \vec{v})$
2. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$ (*cross product is <u>anticommutative</u> not commutative*)
3. $\vec{0} \times \vec{u} = \vec{0}$
4. $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$ (left distributive)
5. $(\vec{v} + \vec{w}) \times \vec{u} = (\vec{v} \times \vec{u}) + (\vec{w} \times \vec{u})$ (right distributive)
6. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$ (**not** associative)

Cross product as determinant

$$\vec{u} \times \vec{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

Triple scalar product

Used to find the volume of a parallelopiped (3D parallelogram)



$$(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} \vdots & \vdots & \vdots \\ \vec{u} & \vec{v} & \vec{w} \\ \vdots & \vdots & \vdots \end{vmatrix}$$

Why?
The cross product creates a vector in a direction perpendicular to $\vec{u}$ and $\vec{v}$, with a magnitude of the area between $\vec{u}$ and $\vec{v}$.

The volume of a parallelopiped is $V = lwh$, where h is the height perpendicular to the area lw.

h would be $\vec{w} \cos \theta$ (where θ is the angle between $\vec{u} \times \vec{v}$ and $\vec{w}$).
Hence, $V = (\vec{u} \times \vec{v}) \cdot \vec{w}$.

#module1 #week1