14.3: Partial Derivatives

Definition

If the limit exists, the **partial derivative** of $f(x, y)$ with respect to x at (x_0, y_0) is:

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

(Differentiate f with respect to x, while holding other variables constant)
(This extends trivially to other variables & more dimensions)

Differentiability

Function $z = f(x, y)$ is **differentiable** at (x_0, y_0) if f_x, f_y (the partial derivatives) and $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$ satisfies:

$$\Delta z = (f_x + \epsilon_1)\Delta x + (f_y + \epsilon_2)\Delta y$$

(where each $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$)

f is **differentiable** if it is differentiable at every point in its domain.
If f is differentiable, then the graph of f is called a **smooth surface**

Why?
In single variable calculus, given a function $y = f(x)$, the change in $\Delta y = f(x_0 + \Delta x) - f(x_0)$ can be approximated by the change according to the tangent line at that point.

$$\begin{aligned} y - f(x_0) &= f'(x_0)(x - x_0) \text{ (tangent line form)} \\ f_{approx}(x_0 + \Delta x) - f(x_0) &= f'(x_0)\Delta x \\ \Delta y &\approx f'(x_0)\Delta x \end{aligned}$$

If we let the difference in the correct slope be ϵ, we can say $\Delta y = (f'(x_0) + \epsilon)\Delta x$.
If f is differentiable, then as $\Delta x \to 0, \epsilon \to 0$.

This idea can be extended into multivariable calculus by using multiple ϵ.

Example

Find all first partial derivatives of $f(x, y) = 3x^2 - 2y + xy$.

$$
\begin{aligned}
f_x &= \frac{\partial f}{\partial x} = 6x + y \\
f_y &= \frac{\partial f}{\partial y} = -2 + x
\end{aligned}
$$

Second Order Partial Derivatives

$$
\begin{aligned}
(f_x)_x = f_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \\
(f_x)_y = f_{xy} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \\
(f_y)_x = f_{yx} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \\
(f_y)_y = f_{yy} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}
\end{aligned}
$$

Mixed Derivatives Theorem

If $f(x, y)$ and all of f_x, f_y, f_{xy}, f_{yx} are defined throughout an open region containing a point (x_0, y_0) and are all continuous at (x_0, y_0), then

$$
f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)
$$

Example

Find all first & second partial derivatives of $f(x, y) = 2x^2 \cos y + 3y^2 \sin x$.

$$
\begin{aligned}
f_x &= 4x \cos y + 3y^2 \cos x \\
f_y &= -2x^2 \sin y + 6y \sin x
\end{aligned}
$$

$$
\begin{aligned}
f_{xx} &= 4 \cos y - 3y^2 \sin x \\
f_{xy} &= -4x \sin y + 6y \cos x \\
f_{yx} &= -4x \sin y + 6y \cos x \\
f_{yy} &= -2x^2 \cos y + 6 \sin x
\end{aligned}
$$

Implicit Differentiation

Suppose we have a function in terms of three variables x, y, z and we cannot solve the equation for z, but we want to find $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$.

Do implicit differentiation as expected, but hold the necessary variables constant.

Example

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $xy - z^2y - 2zx = 0$.

$$\begin{aligned} y - 2zy\frac{\partial z}{\partial x} - 2z - 2x\frac{\partial z}{\partial x} &= 0 \\ \frac{\partial z}{\partial x}(-2zy - 2x) + y - 2z &= 0 \\ \frac{\partial z}{\partial x}(-2zy - 2x) &= 2z - y \\ \frac{\partial z}{\partial x} &= \frac{y - 2z}{2zy + 2x} \end{aligned}$$

$$\begin{aligned} x - z^2 - 2zy\frac{\partial z}{\partial y} - 2x\frac{\partial z}{\partial y} &= 0 \\ x - z^2 - (2zy + 2x)\frac{\partial z}{\partial y} &= 0 \\ (2zy + 2x)\frac{\partial z}{\partial y} &= x - z^2 \\ \frac{\partial z}{\partial y} &= \frac{x - z^2}{2zy + 2x} \end{aligned}$$

14.4: Chain Rule

x, y as functions of one variable

If $w = f(x, y)$ is differentiable and $x(t), y(t)$ are differentiable with respect to t, then $w = f(x(t), y(t))$ is differentiable with respect to t.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

(This is trivially extendible to more dimensions)

Example

Find $\frac{du}{dt}$: $u = x^2 - 3xy = 2y^2,\ x(t) = \cos t,\ y(t) = \sin t$

$$\frac{dx}{dt} = -\sin t$$
$$\frac{dy}{dt} = \cos t$$
$$\frac{\partial u}{\partial x} = 2x - 3y$$
$$\frac{\partial u}{\partial y} = -3x + 4y$$

So:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$\begin{aligned} &= (2\cos(t) - 3\sin(t))(-\sin(t)) \\ &\quad + (-3\cos(t) + 4\sin(t))(\cos(t)) \end{aligned}$$

$$\begin{aligned} &= -2\sin(t)\cos(t) + 3\sin^2(t) - 3\cos^2(t) + 4\sin(t)\cos(t) \\ &= 2\sin(t)\cos(t) + 3\sin^2(t) - 3\cos^2(t) \\ &= \sin(2t) - 3\cos(2t) \end{aligned}$$

x, y as functions of multiple variables

What if $u = u(x, y)$, $x = x(s, t)$, $y = y(s, t)$?

You can find partial derivatives $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$, replacing $\frac{dx}{dt}$ in the simpler chain rule with the respective partial derivative.

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

Example

Given $z = 4e^x \ln y, x = \ln(uv), y = u\sin(v)$, express $\frac{\partial z}{\partial u}$ as a function of u and v.

$$
\begin{aligned}
\frac{\partial x}{\partial u} &= \frac{1}{u} \\
\frac{\partial y}{\partial u} &= \sin(v)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= 4e^x \ln y \\
&= 4uv \ln(u \sin(v))
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial z}{\partial y} &= \frac{4e^x}{y} \\
&= \frac{4v}{\sin(v)}
\end{aligned}
$$

So:

$$
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}
$$

$$
\begin{aligned}
&= 4uv \ln(u \sin(v))\frac{1}{u} + \frac{4v}{\sin(v)}\sin(v) \\
&= 4v \ln(u \sin(v)) + 4v
\end{aligned}
$$

Implicit Differentiation (again)

If $F(x, y)$ is differentiable and $F(x, y) = 0$ defines y as differentiable function of x,

$$
\frac{dy}{dx} = -\frac{F_x}{F_y}
$$

In 3D,
If $z = z(x, y)$ and $F(x, y, z(x, y)) = 0$, you can do the same thing, but to find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

Why?
Let F be written as $F(x, y(x))$.

$$
\begin{aligned}
F(x, y(x)) &= 0 \\
\frac{d}{dx} F(x, y(x)) &= \frac{d}{dx} 0 \\
\frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx} &= 0 \\
\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} &= 0 \\
F_x + F_y \frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= -\frac{F_x}{F_y}
\end{aligned}
$$

Examples

Find $\frac{dy}{dx}$ at the specified point:

$$\underbrace{xy + y^2 - 3x - 3}_{F(x,y)} = 0, \; (-1, 1)$$

$$F_x = y - 3 \qquad F_y = x + 2y$$

$$\frac{dy}{dx} = -\frac{(y-3)}{x+2y} \qquad \left.\frac{dy}{dx}\right|_{(-1,1)} = 2$$

Find $\frac{\partial z}{\partial y}$ at the specified point:

$$\underbrace{xe^y + ye^z + 2\ln(x) - 2 - 3\ln(2)}_{F(x,y,z)} = 0, \; (1, \ln(2), \ln(3))$$

$$\frac{\partial z}{\partial y} = -F_y/F_z \qquad F_y = xe^y + e^z \qquad F_z = ye^z$$

$$\frac{\partial z}{\partial y} = -\frac{(xe^y + e^z)}{ye^z} \qquad \left.\frac{\partial z}{\partial y}\right|_{(1,\ln(2),\ln(3))} = \frac{-(2+3)}{\ln(2)\cdot 3} = \frac{-5}{3\ln(2)}$$

#module2 #week4