

15.7: Triple Integrals in Cylindrical & Spherical Coordinates

Cylindrical Coordinates

- Represent point P in space by ordered triples (r, θ, z) ($r \geq 0$)
 1. r and θ are polar coordinates for the projection of P onto the xy -plane
 2. z is the rectangular vertical coordinate

Usage

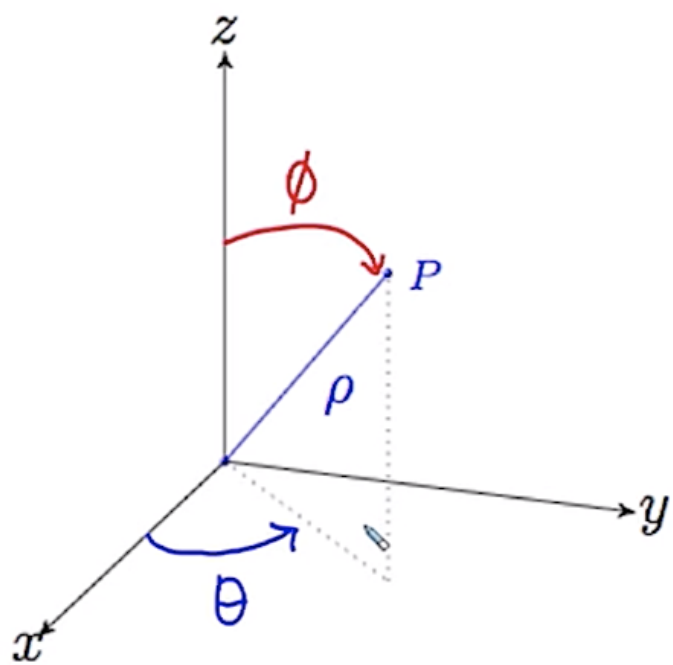
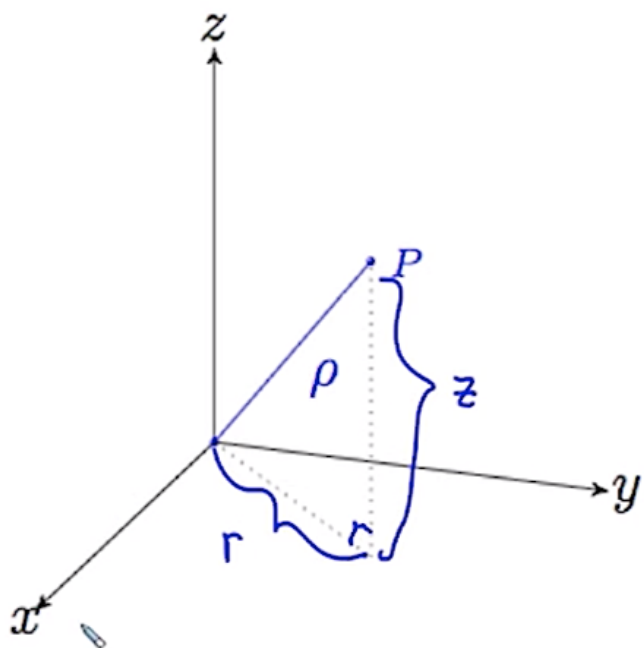
Should be used when...

- there is an axis of symmetry
- an integrand involves $x^2 + y^2$
- we're integrating over a circle (or part of) in the xy -plane

Very similar to using polar coordinates w/ double integrals, but with an added z component for triple integrals.

Spherical Coordinates

- Represent point P in space by ordered triples (ρ, ϕ, θ)
 1. ρ is distance from P to the origin ($\rho \geq 0$)
 2. ϕ is the angle \overrightarrow{OP} makes with the $+z$ -axis ($0 \leq \phi \leq \pi$)
 3. θ is the angle from cylindrical coordinates



Converting Rectangular to Spherical

$$\rho^2 = x^2 + y^2 + z^2 = r^2 + z^2$$

$$r = \rho \sin \phi$$

$$z = \rho \cos \phi$$

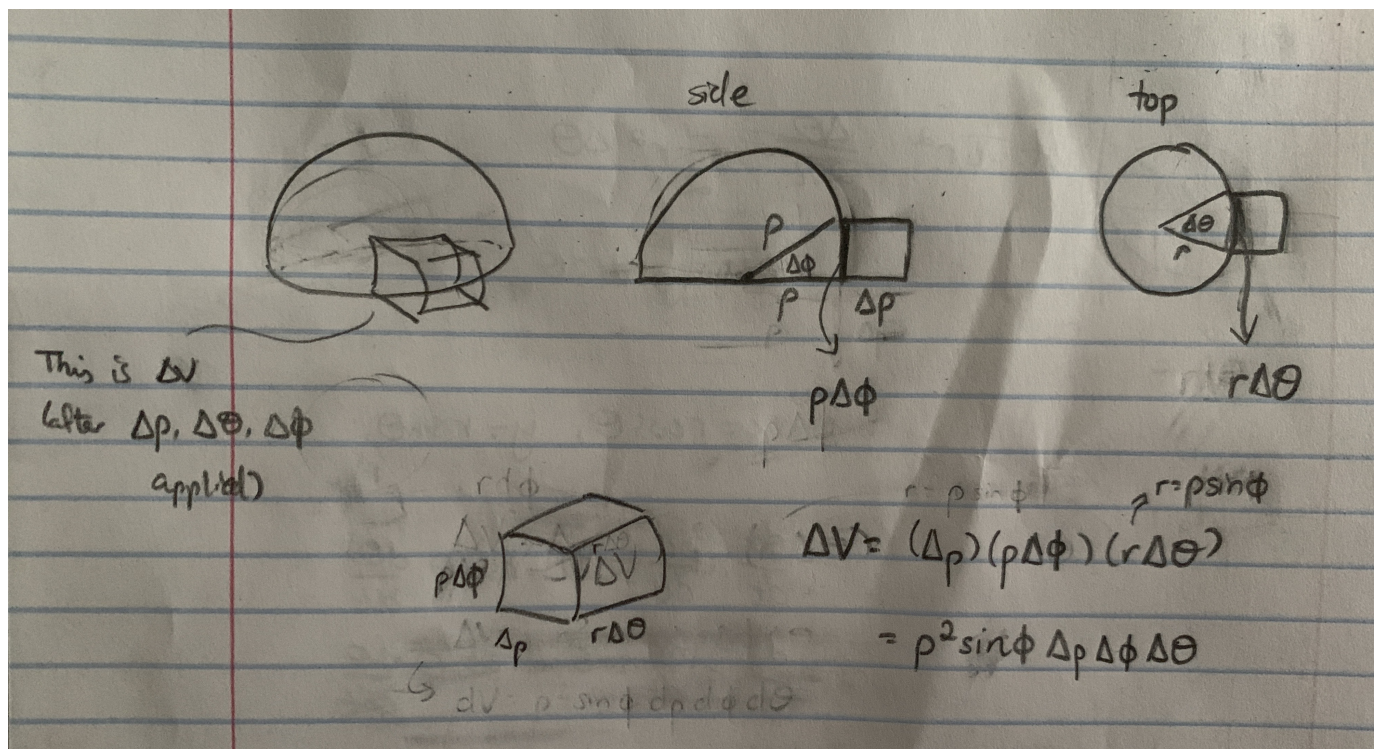
$$x = r \cos \theta = \rho \sin \phi \cos \theta$$

$$y = r \sin \theta = \rho \sin \phi \sin \theta$$

Triple Integral Definition

$$\iiint_T dV = \iiint_T \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

Why?



ΔV is the curved box above. Assuming ΔV is a rectangular prism (when ΔV is very small, it's essentially a rectangular prism),

$$\Delta V = (\Delta \rho) \overbrace{(\rho \Delta \phi)}^{\text{arclength from the side}} \underbrace{(r \Delta \theta)}_{\text{arclength from the top}}$$

$$\begin{aligned} &= \rho r \Delta \rho \Delta \phi \Delta \theta \\ &= \rho(\rho \sin \phi) \Delta \rho \Delta \phi \Delta \theta \\ &= \rho^2 \sin \phi \Delta \rho \Delta \phi \Delta \theta \end{aligned}$$

$$dV = \rho^2 \sin \phi d\rho d\phi d\theta$$

Related: [purely algebraic derivation](#)

15.8: Integration by Substitution

Jacobians

Jacobian determinate or **Jacobian** of the coordinate transformation $x = g(u, v), y = h(u, v)$:

$$J(u, v) = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}$$

The two coordinate systems are related by:

$$dx dy = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

Why?

The Jacobian transform maps the uv -coordinate system onto the xy -coordinate system.

In a mapping from uv to xy , the area $du dv$ will be multiplied by a factor of the determinant of the transform (recall from linear algebra) to get the corresponding area $dx dy$.

Extension into 3D

Jacobians can also be extended pretty trivially to 3 dimensions.

Given $x = g(u, v, w), y = h(u, v, w), z = k(u, v, w)$,

$$J(u, v, w) = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Double Integrals

Suppose $f(x, y)$ is continuous over region R . Let G be preimage of R under transform $x = g(u, v), y = h(u, v)$ (assumed to be one-to-one on interior of G). If functions g and h have continuous 1st partial derivatives within interior of G :

$$\iint_R f(x, y) \, dx \, dy = \iint_G f(g(u, v), h(u, v)) \overbrace{\left| \frac{\partial(x, y)}{\partial(u, v)} \right|}^{\text{Jacobian}} \, du \, dv$$

Triple Integrals

$$\begin{aligned} & \iiint_R f(x, y, z) \, dx \, dy \, dz \\ &= \iiint_G f(g(u, v, w), h(u, v, w), k(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \, du \, dv \, dw \end{aligned}$$

Derivation of Spherical Triple Integral by Jacobians

Spherical coordinates:

$$\begin{aligned} x &= \rho \sin \phi \cos \theta \\ y &= \rho \sin \phi \sin \theta \\ z &= \rho \cos \phi \end{aligned}$$

Derivation:

What we're trying to find

$$dx \, dy \, dz = \overbrace{\left| \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} \right|} d\rho \, d\phi \, d\theta$$

$$\begin{aligned} \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} &= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{vmatrix} \\ &= \cos \phi (\rho^2 \sin \phi \cos \phi \cos^2 \theta + \rho^2 \sin \phi \cos \phi \sin^2 \theta) \\ &\quad + \rho \sin \phi (\rho \sin^2 \phi \cos^2 \theta + \rho^2 \sin \phi^2 \sin^2 \theta) \\ &= \cos \phi (\rho^2 \sin \phi \cos \phi) \\ &\quad + \rho \sin \phi (\rho \sin^2 \phi) \\ &= \rho^2 \sin \phi \cos^2 \phi + \rho^2 \sin \phi \sin^2 \phi \\ &= \rho^2 \sin \phi \end{aligned}$$

Therefore, $dV = dx \, dy \, dz = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$