

16.1, 16.2: Line Integrals

in Scalar Fields

If f is defined on a curve C given parametrically by $\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, the line integral of f over C is:

$$\int_C f(x, y, z) ds = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k, y_k, z_k) \Delta s_k$$

To integrate a continuous function $f(x, y, z)$ over a curve C :

1. Find a smooth parametrization of C :

$$\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad (\text{where } a \leq t \leq b)$$

2. Evaluate the integral as:

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) |\vec{v}(t)| dt$$

(recall $\frac{ds}{dt} = |\vec{v}|$)

Mass and Moment Calculations

Suppose we need to find the mass & moment for coil springs and then rods lying along a smooth curve C in space.

Recall [physics definitions](#) from 15.6.

They apply here, too.

Mass

$$m = \int_C \lambda ds$$

(this is a pretty straightforward extension of 15.6 so I don't think there needs to be notes here)

in Vector Fields

Let \vec{F} be a vector field with continuous components defined along smooth curve C parametrized by $\vec{r}(t)$, $a \leq t \leq b$.

The **line integral of \vec{F} along C** is:

$$\int_C \vec{F} \cdot \vec{T} \, ds = \int_C \left(F \cdot \frac{d\vec{r}}{ds} \right) ds = \int_C \vec{F} \cdot d\vec{r}$$

To evaluate, write \vec{F} and $d\vec{r}$ in terms of t and apply dot product.

Line integrals may also be written as:

$$\begin{aligned} & \int_C M \, dx + \int_C N \, dy + \int_C P \, dz \\ &= \int_C M(x, y, z) \, dx + \int_C N(x, y, z) \, dy + \int_C P(x, y, z) \, dz \end{aligned}$$

(same idea, write everything in terms of t)

Example

Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = \langle xy, x^2z, xyz \rangle$ along $y = x^2$ from $(0, 0, 0)$ to $(1, 1, 0)$ followed by the straight-line segment from $(1, 1, 0)$ to $(1, 1, 1)$.

$$\begin{aligned} C_1 : \vec{r}_1(t) &= \langle t, t^2, 0 \rangle \\ \vec{r}'_1(t) &= \langle 1, 2t, 0 \rangle \end{aligned}$$

$$\begin{aligned} C_2 : \vec{r}_2(t) &= \langle 1, 1, t \rangle \\ \vec{r}'_2(t) &= \langle 0, 0, 1 \rangle \end{aligned}$$

$$\begin{aligned} & \int_C \vec{F} \cdot d\vec{r} \\ &= \int_0^1 \langle (t)(t^2), 0, 0 \rangle \cdot \langle 1, 2t, 0 \rangle \, dt + \int_0^1 \langle (1)(1), (1)^2(t), (1)(1)(t) \rangle \cdot \langle 0, 0, 1 \rangle \, dt \\ &= \int_0^1 \langle t^3, 0, 0 \rangle \cdot \langle 1, 2t, 0 \rangle \, dt + \int_0^1 \langle 1, t, t \rangle \cdot \langle 0, 0, 1 \rangle \, dt \end{aligned}$$

Applications to Physics

Work

$$W = \int_C \vec{F} \cdot d\vec{r}$$

- \vec{F} is force

Flow

$$\text{Flow} = \int_C \vec{F} \cdot \vec{T} \, ds$$

- \vec{F} is velocity

This integral is called a **flow integral**. If the curve starts and ends at the same point, the flow is called the *circulation* around the curve.

Flux (across a smooth simple closed plane curve)

$$\Phi = \int_C \vec{F} \cdot \hat{n} \, ds$$

- \vec{F} is a vector field in the plane, $M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$
- C is a smooth simple closed curve (starts & ends at same place and does not cross itself)
- \hat{n} is the outward-pointing unit vector normal to C

Alternative form:

$$\Phi \text{ across } C = \oint M \, dy - N \, dx$$

(Integral is evaluated at any parametrization \vec{r} that traces C counterclockwise exactly once)

Why?

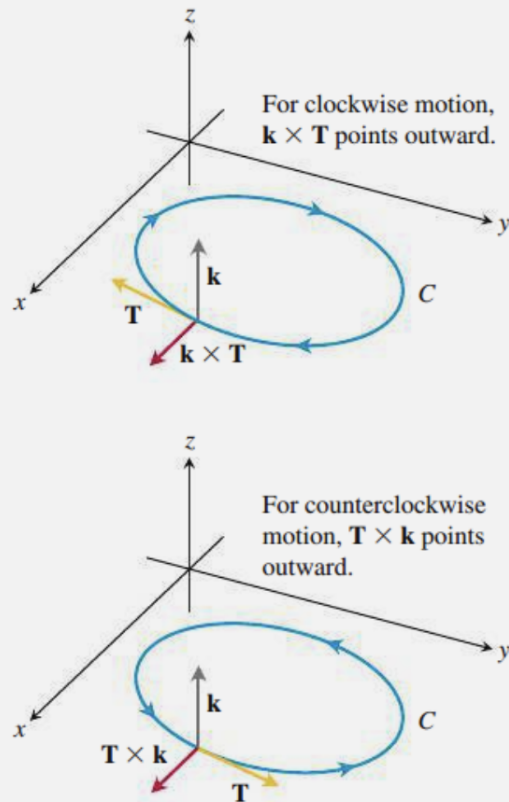


FIGURE 16.24 To find an outward unit normal vector for a smooth simple curve C in the xy -plane that is traversed counterclockwise as t increases, we take $\mathbf{n} = \mathbf{T} \times \mathbf{k}$. For clockwise motion, we take $\mathbf{n} = \mathbf{k} \times \mathbf{T}$.

Assuming counterclockwise,

$$\hat{\mathbf{n}} = \vec{T} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{dy}{ds} \mathbf{i} - \frac{dx}{ds} \mathbf{j}$$

Then:

$$\begin{aligned} \Phi &= \int_C \vec{F} \cdot \hat{\mathbf{n}} \, ds \\ &= \int_C \langle M, N \rangle \cdot \left\langle \frac{dy}{ds}, \frac{-dx}{ds} \right\rangle ds \\ &= \oint M \, dy - N \, dx \end{aligned}$$

[#week10](#)