12.5: Lines & Planes in Space

Lines

The **vector equation** for line L through $P_0(x_0, y_0, z_0)$ parallel to vector $\vec{v}$:

$$\vec{r}(t) = \overbrace{\vec{r}_0}^{P_0} + t\vec{v}$$

The **standard parametrization** through $P_0(x_0, y_0, z_0)$ parallel to $\vec{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$:

$$\begin{aligned} x(t) &= x_0 + tv_1 \\ y(t) &= y_0 + tv_2 \\ z(t) &= z_0 + tv_3 \end{aligned}$$

For lines, $-\infty \leq t \leq \infty$.
For line segments, t has proper bounds.

These forms will be the most common way lines will be seen in, but there is also the **symmetric form**:

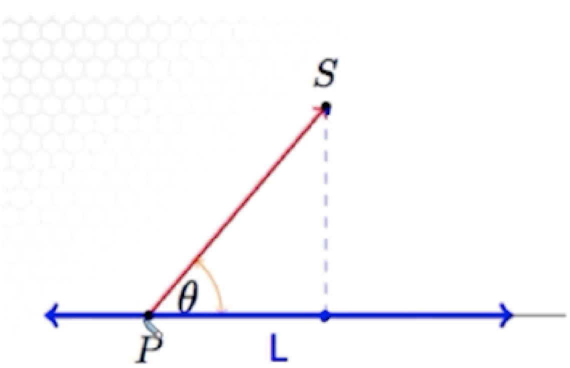
$$L : \frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_1}$$

Why?
From the standard parametrization, solve for t, set t equal. You get the symmetric form.

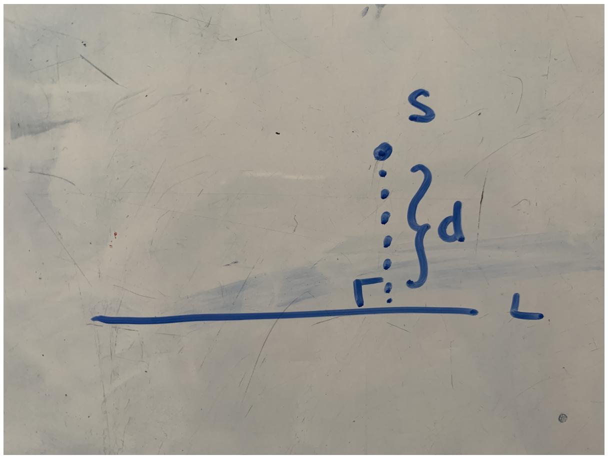
Distance from point to line

To calculate the distance from a point to a line, pick a point P on line L. Let $\vec{v}$ be the direction vector of L. Then:

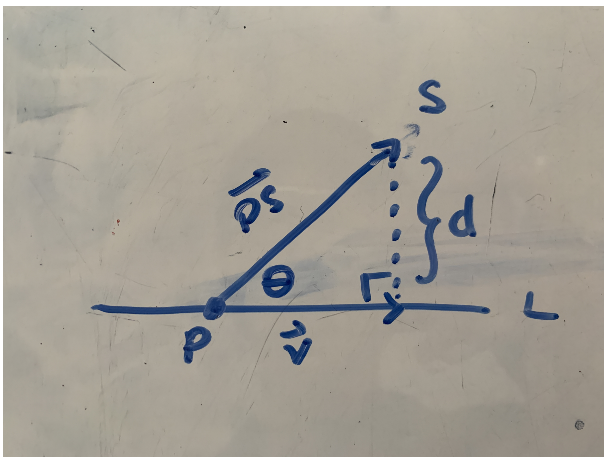
$$d = \frac{\left\|\overrightarrow{PS} \times \vec{v}\right\|}{\|\vec{v}\|}$$



Why?
The distance we're trying to find is the magnitude of the vector perpendicular to line L that reaches point S.



If we pick a point P on the line L, and create vector $\overrightarrow{PS}$, then the distance is $||\overrightarrow{PS}|| \sin\theta$ where θ is the angle between $\overrightarrow{PS}$ and L. Note that the direction of L is $\vec{v}$.



Then:

$$\begin{aligned} d &= ||\overrightarrow{PS}|| \sin\theta \\ &= ||\overrightarrow{PS}|| \sin\theta \cdot \frac{||\vec{v}||}{||\vec{v}||} \\ &= \frac{||\overrightarrow{PS}|| ||\vec{v}|| \sin\theta}{||\vec{v}||} \\ &= \frac{||\overrightarrow{PS} \times \vec{v}||}{||\vec{v}||} \end{aligned}$$

Planes

The **vector equation** for a plane through $P_0(x_0, y_0, z_0)$ with normal vector $\vec{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is given by:

$$\vec{n} \cdot (\overrightarrow{P_0P}) = 0$$

Component equation:

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$

Angle between two planes

To calculate the angle between two planes, you calculate the **acute** angle between the two planes' normals.

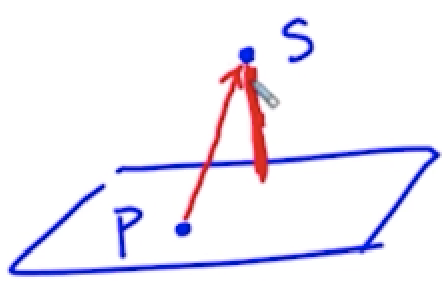
$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{||\vec{n}_1||\,||\vec{n}_2||}$$

(Note the absolute bars in the numerator.)

Distance from point to plane

To calculate the distance from a point to a line, pick a point on the plane, call it P. Let $\hat{n}$ be the **unit** vector that is normal to the plane. Then,

$$d = \left\|\operatorname{proj}_{\hat{n}} \overrightarrow{PS}\right\| = \overrightarrow{PS} \cdot \hat{n}$$



Intersecting lines & planes

Lines

Lines l_1, l_2 can be...

- parallel
- intersecting

- coincident: same line
- skew: neither parallel nor intersecting

If the direction vectors are the same, the lines must be *parallel* or *concident*.

- Pick a point on l_1. If it is on l_2, the lines are coincident.
- Otherwise, the lines are parallel.

If the direction vectors are not the same, the lines must be skew or intersecting.

- Check if there are any intersecting points. If such a point exists, then the lines are intersecting.
- Otherwise, the lines are skew.

Planes

Two planes can be: parallel, intersecting, coincident

- If the planes' normals are parallel, the planes are parallel
- If the planes' normals are not parallel, the cross product gives direction vector for line of intersection of the planes

Line & Plane

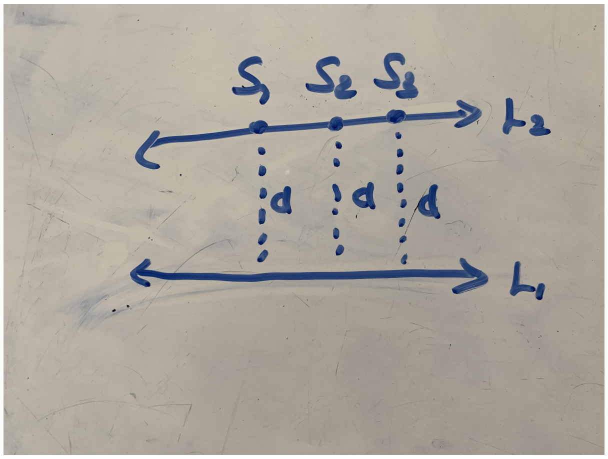
To find the intersection point between a line and a plane, substitute the line equation into the plane equation. If there's a valid point, then that is the intersecting point.

Distance between lines

Distance between parallel lines

To find the distance between two parallel lines L_1 and L_2, note that for any point on L_2, the distance to L_1 is the same.

Thus, pick any point S on L_2 and apply the distance from point to line formula.

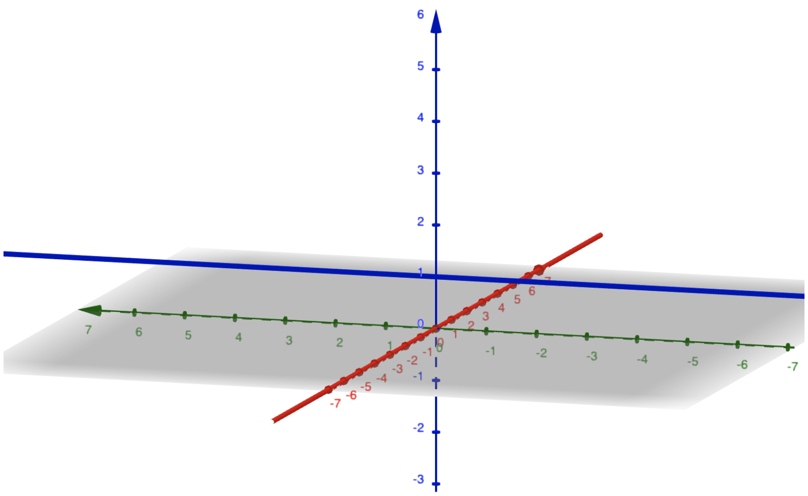


Distance between skew lines

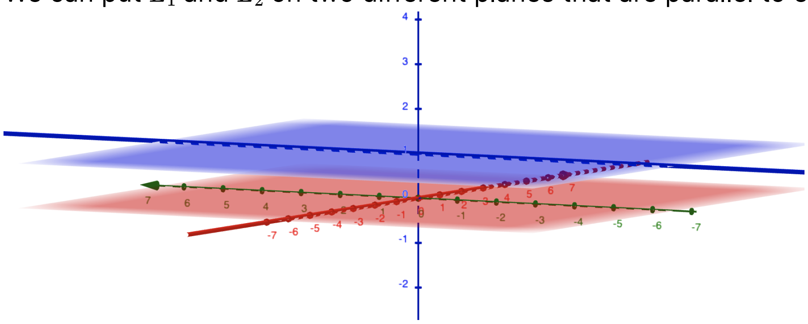
To find the distance between two skew lines L_1 and L_2,

1. Let $\vec{v}_1$ and $\vec{v}_2$ be the direction vectors of L_1 and L_2 respectively.
2. Find the unit vector normal to $\vec{v}_1$ and $\vec{v}_2$. This is $\hat{n} = \frac{\vec{v}_1 \times \vec{v}_2}{\|\vec{v}_1 \times \vec{v}_2\|}$. (Remember that the cross product gives the vector normal to v_1 and v_2.)
3. Then, apply the distance from point to plane formula. (Pick points P on L_1 and S on L_2, and calculate $d = \overrightarrow{PS} \cdot \hat{n}$.)

Why?
Imagine L_1 and L_2.



We can put L_1 and L_2 on two different planes that are parallel to each other (P_1 in red, P_2 in blue):



Since these planes are parallel, they must have the same normal.
This normal must be perpendicular to the direction of L_1 ($\vec{v}_1$) and perpendicular to the direction of L_2 ($\vec{v}_2$), so the normal must be in the direction of $\vec{v}_1 \times \vec{v}_2$.
So the **unit** normal vector is $\hat{n} = \frac{\vec{v}_1 \times \vec{v}_2}{\|\vec{v}_1 \times \vec{v}_2\|}$. (The unit normal vector isn't necessarily needed but it makes calculations down the line easier.)

Note that because these planes are parallel, the distance between them are the same no matter which point we pick on P_2.

Thus, we can find the distance by projecting any line connecting the planes onto the normal.

So, we can pick a point on P_1 and P_2 and calculate the distance by projecting.
Of course, we do not know the equation for P_1 and P_2, but we know L_1 is on P_1 and L_2 is on P_2.

So, we can pick points on L_1 and L_2.

#module1 #week2