16.4: Green's Theorem

Circulation density of a vector field $\vec{F} = M\mathbf{i} + N\mathbf{j}$ at point (x, y) is the scalar expression:

$$\text{curl}\ \vec{F} \cdot \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

Divergence (flux density) of vector field $\vec{F} = M\mathbf{i} + N\mathbf{j}$ at (x, y) is:

$$\text{div}\ \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

Circulation-Curl or Tangential Form

Let C be piecewise smooth, simple closed curve enclosing region R in the plane.
Let $\vec{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N have continuous 1st partial derivatives in open region containing R.

Then:
The **counterclockwise circulation** of $\vec{F}$ around C equals the double integral of $\text{curl}\ \vec{F} \cdot \mathbf{k}$ over R.

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

Flux-Divergence or Normal Form

Let C be piecewise smooth, simple closed curve enclosing region R in the plane.
Let $\vec{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N have continuous 1st partial derivatives in open region containing R.

Then:
The **outward flux** of $\vec{F}$ around C equals the double integral of $\text{div}\ \vec{F}$ over R.

$$\oint_C \vec{F} \cdot \vec{n}\, ds = \oint_C M\, dy - N\, dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy$$

Area

Green's Theorem can be used to write area in terms of a line integral.

$$
\begin{aligned}
A_R &= \iint_R dy\,dx \\
&= \iint_R \left(\frac{1}{2} + \frac{1}{2}\right) dy\,dx \\
&= \frac{1}{2} \oint x\,dy - y\,dx
\end{aligned}
$$

16.5: Surfaces and Area

Parameterized Surfaces

A **parametrized surface** is given by: $\vec{r}(u,v) = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}$.
The domain is the set of points in the uv-plane that can be substituted into $\vec{r}$.

Sphere: $x^2 + y^2 + z^2 = a^2$
$\vec{r}(\phi, \theta) = a \sin\phi \cos\theta \mathbf{i} + a \sin\phi \sin\theta \mathbf{j} + a \cos\phi \mathbf{k}$ (pretty straightforward mapping of spherical coordinates)
$(0 \le \phi \le \pi, 0 \le \theta \le 2\pi)$

Cylinder: $x^2 + y^2 = a^2, 0 \le z \le b$
$\vec{r}(\theta, z) = a \cos\theta \mathbf{i} + a \sin\theta \mathbf{j} + z\mathbf{k}$ (pretty straightforward mapping of cylindrical coordinates)
$(0 \le \theta \le 2\pi, 0 \le z \le b)$

Cone: $z = \sqrt{x^2 + y^2}, 0 \le z \le b$
$\vec{r}(r, \theta) = r \cos\theta \mathbf{i} + r \sin\theta \mathbf{j} + r\mathbf{k}$
$(0 \le r \le b, 0 \le \theta \le 2\pi)$

Example

Find the parametrization for $z = 4 - x^2 - y^2, z \ge 0$.

$$
\begin{aligned}
z &= 4 - x^2 - y^2 \\
&= 4 - r^2 \\
\therefore r &\le 2
\end{aligned}
$$

The parametrization:

$$
\vec{r}(r, \theta) = r \cos\theta \mathbf{i} + r \sin\theta \mathbf{j} + (4 - r^2)\mathbf{k}
$$

$(0 \le r \le 2, 0 \le \theta < 2\pi)$

Surface Area

A parametrized surface $\vec{r}(u,v) = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}$ is **smooth** if $\vec{r}_u$ and $\vec{r}_v$ are continuous and $\vec{r}_u \times \vec{r}_v \neq 0$ on the interior of the parameter domain.

SA of Parametrized Surfaces

Given smooth surface $\vec{r}(u,v) = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}, a \leq u \leq b, c \leq v \leq d$:

$$\sigma = \iint_R |\vec{r}_u \times \vec{r}_v| \, dA = \int_c^d \int_a^b |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

Why?
For a small rectangular area $\Delta\sigma$ on the surface,

$$\begin{aligned} \Delta\sigma &= |(\vec{r}_u \cdot \Delta u) \times (\vec{r}_v \cdot \Delta v)| \\ &= |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v \end{aligned}$$

So, $d\sigma = |\vec{r}_u \times \vec{r}_v| \, du \, dv$.

SA of Implicit Surfaces

Area of surface $F(x,y,z) = c$ over closed & bounded region R:

$$\sigma = \iint_R \frac{\|\nabla F\|}{|\nabla F \cdot \hat{p}|} \, dA$$

(where $\hat{p} = \mathbf{i}, \mathbf{j}, \text{ or } \mathbf{k}$ is normal to R and $\nabla F \cdot \hat{p} \neq 0$)

Why?
Create $\mathfrak{shadow\ region}$ R (a projection of the surface onto a coordinate plane), and let $\hat{p}$ be the

normal vector of R.

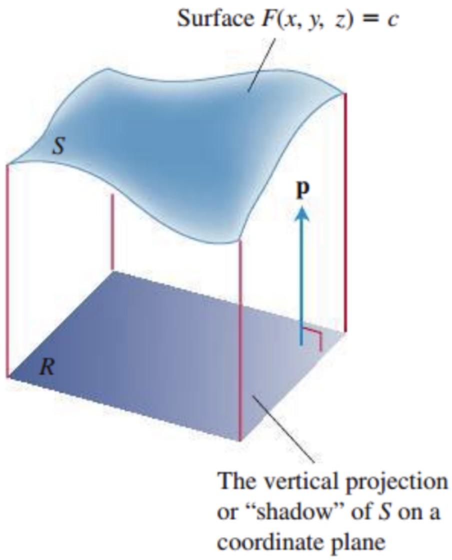

FIGURE 16.47 As we soon see, the area of a surface S in space can be calculated by evaluating a related double integral over the vertical projection or "shadow" of S on a coordinate plane. The unit vector $\mathbf{p}$ is normal to the plane.

Assume surface is smooth and require $\nabla \cdot \hat{p} \neq 0$.

Let R be the xy-plane (then $\hat{p} = \mathbf{k}$). The curve is parametrized as:

$$\vec{r}(x, y) = x\mathbf{i} + y\mathbf{j} + z(x, y)\mathbf{k}$$

(note that $z(x, y)$ is not explicitly known)

$$\begin{aligned} \vec{r}_x &= \mathbf{i} + \frac{\partial z}{\partial x}\mathbf{k} = \mathbf{i} - \frac{F_x}{F_z}\mathbf{k} \\ \vec{r}_y &= \mathbf{j} + \frac{\partial z}{\partial y}\mathbf{k} = \mathbf{j} - \frac{F_y}{F_z}\mathbf{k} \end{aligned}$$

(recall $F(x, y, z(x, y)) = 0$, so [implicit chain rule] can be applied)

Then:

$$\vec{r}_x \times \vec{r}_y = \frac{F_x}{F_z}\mathbf{i} + \frac{F_y}{F_z}\mathbf{j} + \mathbf{k}$$

(i'm too lazy to show the determinant but just scroll down like 1 scroll, there's a similar cross prod.)

$$\begin{aligned}\vec{r}_x \times \vec{r}_y &= \frac{1}{F_z}(F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}) \\ &= \frac{\nabla F}{F_z} \\ &= \frac{\nabla F}{\nabla F \cdot \hat{p}}\end{aligned}$$

Plug back in and bam equation.

SA for $z = f(x, y)$

$$\sigma = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy$$

Why?

$z = f(x, y)$ can be parametrized as $\vec{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$.

Then:

$$\begin{aligned}\vec{r}_x &= \mathbf{i} + f_x\mathbf{k} \\ \vec{r}_y &= \mathbf{j} + f_y\mathbf{k} \\ \vec{r}_x \times \vec{r}_y &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}\end{aligned}$$

So:

$$\sigma = \iint_R |\vec{r}_u \times \vec{r}_v|\, dA = \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy$$

16.6: Surface Integrals

Definition

Surface area differential:

$$d\sigma = |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

Surface integral of G over surface S

$$\iint_S G(x, y, z)\, d\sigma = \lim_{n\to\infty} \sum_{k=1}^{n} G(x_k, y_k, z_k)\Delta\sigma_k$$

Surface integrals of scalar functions

You can substitute $d\sigma$ for the surface area formulas above (Surface Area) depending on which condition you meet.

Parametrized surface
Given smooth surface $\vec{r}(u,v) = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}, (u,v) \in R$

$$\iint_S G(x,y,z)d\sigma = \iint_R G(f(u,v),g(u,v),h(u,v))|\vec{r}_u \times \vec{r}_v|\,du\,dv$$

Implicit surface

$$\iint_S G(x,y,z)d\sigma = \iint_R G(x,y,z)\frac{\|\nabla F\|}{|\nabla F \cdot \hat{p}|}\,dA$$

(S lies above its closed & bounded shadow region R in the coordinate plane beneath it)
(where $\hat{p} = \mathbf{i}, \mathbf{j}, \text{ or } \mathbf{k}$ is normal to R and $\nabla F \cdot \hat{p} \neq 0$)

For $z = f(x,y)$

$$\iint_S G(x,y,z)d\sigma = \iint_R G(x,y,f(x,y))\sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy$$

(R is the region on the xy-plane)

Surface integrals of vector fields

Let $\vec{F}$ be a vector field in 3D space with continuous components defined over a smooth surface S, with normal unit vectors $\hat{n}$ orienting S.
The **surface integral of $\vec{F}$ over** S:

$$\iint_S \vec{F} \cdot \hat{n}\,d\sigma$$

This integral is also called the **flux** of the vector field $\vec{F}$ across S.

If the surface being integrated over can be written as $g(x,y,z) = c$,

$$\hat{n} = \pm\frac{\nabla g}{\|\nabla g\|}$$

If the surface being integrated over can be parametrized as $\vec{r}(u,v)$, the surface integral can be calculated as:

$$\iint_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

Mass & Moment

Same as Week 8 > Physics Definitions, but with:

$$dm = \delta d\sigma$$