15.8: Integration by Substitution

Jacobians

Jacobian determinate or **Jacobian** of the coordinate transformation $x = g(u, v), y = h(u, v)$:

$$J(u, v) = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}$$

The two coordinate systems are related by:

$$dx\,dy = \left| \det \left(\frac{\partial(x, y)}{\partial(u, v)} \right) \right| du\,dv$$

$dx\,dy$ = *the **absolute value of the determinant of the Jacobian matrix*** $\cdot\, du\,dv$

Why?
The Jacobian transform maps the uv-coordinate system onto the xy-coordinate system.
In a mapping from uv to xy, the area $du\,dv$ will be multiplied by a factor of the determinant of the transform (recall from linear algebra) to get the corresponding area $dx\,dy$.

Extension into 3D
Jacobians can also be extended pretty trivially to 3 dimensions.
Given $x = g(u, v, w), y = h(u, v, w), z = k(u, v, w)$,

$$J(u, v, w) = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Double Integrals

Suppose $f(x, y)$ is continuous over region R. Let G be preimage of R under transform $x = g(u, v), y = h(u, v)$ (assumed to be one-to-one on interior of G). If functions g and h have continuous 1st partial derivatives within interior of G:

$$\iint_R f(x, y)\,dx\,dy = \iint_G f(g(u, v), h(u, v)) \overbrace{\left| \frac{\partial(x, y)}{\partial(u, v)} \right|}^{\text{Jacobian}} du\,dv$$

Triple Integrals

$$\iiint_R f(x,y,z)\,dx\,dy\,dz$$
$$= \iiint_G f(g(u,v,w),h(u,v,w),k(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

Derivation of Spherical Triple Integral by Jacobians

Spherical coordinates:

$$\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}$$

Derivation:

$$dx\,dy\,dz = \overbrace{\left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right|}^{\text{What we're trying to find}} d\rho\,d\phi\,d\theta$$

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} &= \begin{vmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial\theta} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial\theta} \end{vmatrix} \\
&= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \\
\\
&= \cos\phi(\rho^2\sin\phi\cos\phi\cos^2\theta + \rho^2\sin\phi\cos\phi\sin^2\theta) \\
&+ \rho\sin\phi(\rho\sin^2\phi\cos^2\theta + \rho^2\sin\phi^2\sin^2\theta) \\
\\
&= \cos\phi(\rho^2\sin\phi\cos\phi) \\
&+ \rho\sin\phi(\rho\sin^2\phi) \\
\\
&= \rho^2\sin\phi\cos^2\phi + \rho^2\sin\phi\sin^2\phi \\
\\
&= \rho^2\sin\phi
\end{aligned}$$

Therefore, $dV = dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

#module3 #week9