# 16.5: Surfaces and Area

## Parameterized Surfaces

A **parametrized surface** is given by: $\vec{r}(u,v) = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}$.
The domain is the set of points in the $uv$-plane that can be substituted into $\vec{r}$.

**Sphere**: $x^2 + y^2 + z^2 = a^2$
$\vec{r}(\phi,\theta) = a\sin\phi\cos\theta\mathbf{i} + a\sin\phi\sin\theta\mathbf{j} + a\cos\phi\mathbf{k}$ (pretty straightforward mapping of spherical coordinates)
$(0 \le \phi \le \pi, 0 \le \theta \le 2\pi)$

**Cylinder**: $x^2 + y^2 = a^2, 0 \le z \le b$
$\vec{r}(\theta,z) = a\cos\theta\mathbf{i} + a\sin\theta\mathbf{j} + z\mathbf{k}$ (pretty straightforward mapping of cylindrical coordinates)
$(0 \le \theta \le 2\pi, 0 \le z \le b)$

**Cone**: $z = \sqrt{x^2 + y^2}, 0 \le z \le b$
$\vec{r}(r,\theta) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + r\mathbf{k}$
$(0 \le r \le b, 0 \le \theta \le 2\pi)$

### Example

Find the parametrization for $z = 4 - x^2 - y^2, z \ge 0$.

$$\begin{aligned} z &= 4 - x^2 - y^2 \\ &= 4 - r^2 \\ \therefore r &\le 2 \end{aligned}$$

The parametrization:

$$\vec{r}(r,\theta) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + (4 - r^2)\mathbf{k}$$

$(0 \le r \le 2, 0 \le \theta < 2\pi)$

## Surface Area

A parametrized surface $\vec{r}(u,v) = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}$ is **smooth** if $\vec{r}_u$ and $\vec{r}_v$ are continuous and $\vec{r}_u \times \vec{r}_v \ne 0$ on the interior of the parameter domain.

### SA of Parametrized Surfaces

Given smooth surface $\vec{r}(u,v) = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}, a \le u \le b, c \le v \le d$:

$$\sigma = \iint_R |\vec{r}_u \times \vec{r}_v| \, dA = \int_c^d \int_a^b |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

**Why?**
For a small rectangular area $\Delta\sigma$ on the surface,

$$\begin{aligned} \Delta\sigma &= |(\vec{r}_u \cdot \Delta u) \times (\vec{r}_v \cdot \Delta v)| \\ &= |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v \end{aligned}$$

So, $d\sigma = |\vec{r}_u \times \vec{r}_v|\, du\, dv$.

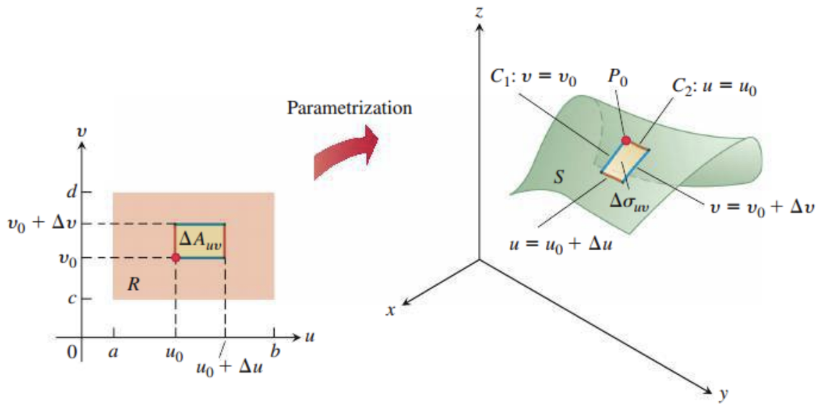


**FIGURE 16.43** A rectangular area element $\Delta A_{uv}$ in the $uv$-plane maps onto a curved patch element $\Delta\sigma_{uv}$ on $S$.

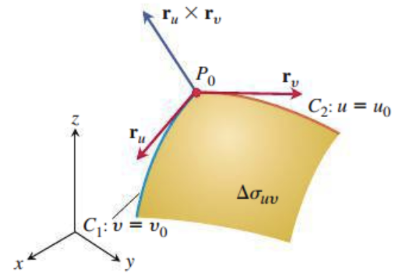


**FIGURE 16.44** A magnified view of a surface patch element $\Delta\sigma_{uv}$.

## SA of Implicit Surfaces

Area of surface $F(x, y, z) = c$ over closed & bounded region $R$:

$$\sigma = \iint_R \frac{\|\nabla F\|}{|\nabla F \cdot \hat{p}|}\, dA$$

(where $\hat{p} = \mathbf{i}, \mathbf{j}, \text{ or } \mathbf{k}$ is normal to $R$ and $\nabla F \cdot \hat{p} \neq 0$)

**Why?**

Create $\mathfrak{shadow\ region}$ $R$ (a projection of the surface onto a coordinate plane), and let $\hat{p}$ be the normal vector of $R$

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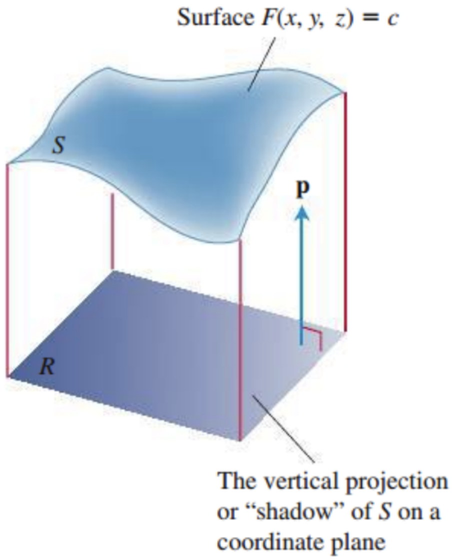


**FIGURE 16.47** As we soon see, the area of a surface $S$ in space can be calculated by evaluating a related double integral over the vertical projection or "shadow" of $S$ on a coordinate plane. The unit vector **p** is normal to the plane.

Assume surface is smooth and require $\nabla \cdot \hat{p} \neq 0$.

Let $R$ be the xy-plane (then $\hat{p} = \mathbf{k}$). The curve is parametrized as:

$$\vec{r}(x, y) = x\mathbf{i} + y\mathbf{j} + z(x, y)\mathbf{k}$$

(note that $z(x, y)$ is not explicitly known)

$$\vec{r}_x = \mathbf{i} + \frac{\partial z}{\partial x}\mathbf{k} = \mathbf{i} - \frac{F_x}{F_z}\mathbf{k}$$
$$\vec{r}_y = \mathbf{j} + \frac{\partial z}{\partial y}\mathbf{k} = \mathbf{j} - \frac{F_y}{F_z}\mathbf{k}$$

(recall $F(x, y, z(x, y)) = 0$, so implicit chain rule can be applied)

Then:

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -\frac{F_x}{F_z} \\ 0 & 1 & -\frac{F_y}{F_z} \end{vmatrix}$$

$$\begin{aligned} &= \frac{F_x}{F_z}\mathbf{i} + \frac{F_y}{F_z}\mathbf{j} + \mathbf{k} \\ &= \frac{1}{F_z}(F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}) \\ &= \frac{\nabla F}{F_z} \\ &= \frac{\nabla F}{\nabla F \cdot \hat{p}} \end{aligned}$$

### SA for $z = f(x, y)$

$$\sigma = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy$$

**Why?**

$z = f(x, y)$ can be parametrized as $\vec{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$.

Then:

$$\begin{aligned} \vec{r}_x &= \mathbf{i} + f_x\mathbf{k} \\ \vec{r}_y &= \mathbf{j} + f_y\mathbf{k} \\ \vec{r}_x \times \vec{r}_y &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k} \end{aligned}$$

So:

$$\sigma = \iint_R |\vec{r}_u \times \vec{r}_v|\, dA = \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy$$

# 16.6: Surface Integrals

## Definition

**Surface area differential**:

$$d\sigma = |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

**Surface integral of $G$ over surface $S$**

$$\iint_S G(x, y, z)\, d\sigma = \lim_{n \to \infty} \sum_{k=1}^{n} G(x_k, y_k, z_k)\Delta\sigma_k$$

## Surface integrals of scalar functions

To evaluate a surface integral, substitute $d\sigma$ for [the surface area formulas above](#) depending on which type of surface is being evaluated against.

**Parametrized surface**
Given smooth surface $\vec{r}(u,v) = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}, (u,v) \in R$

$$\iint_S G(x,y,z)d\sigma = \iint_R G(f(u,v), g(u,v), h(u,v))|\vec{r}_u \times \vec{r}_v| \, du \, dv$$

**Implicit surface**

$$\iint_S G(x,y,z)d\sigma = \iint_R G(x,y,z) \frac{\|\nabla F\|}{|\nabla F \cdot \hat{p}|} \, dA$$

($S$ lies above its closed & bounded shadow region $R$ in the coordinate plane beneath it)
(where $\hat{p} = \mathbf{i}, \mathbf{j}, \text{ or } \mathbf{k}$ is normal to $R$ and $\nabla F \cdot \hat{p} \neq 0$)

**For** $z = f(x,y)$

$$\iint_S G(x,y,z)d\sigma = \iint_R G(x,y,f(x,y))\sqrt{f_x^2 + f_y^2 + 1} \, dx \, dy$$

($R$ is the region on the $xy$-plane)

## Surface integrals of vector fields

Let $\vec{F}$ be a vector field in 3D space with continuous components defined over a smooth surface $S$, with normal unit vectors $\hat{n}$ orienting $S$.
The **surface integral of $\vec{F}$ over $S$**:

$$\iint_S \vec{F} \cdot \hat{n} \, d\sigma$$

This integral is also called the **flux** of the vector field $\vec{F}$ across $S$.

### Evaluation across level surface

If the surface being integrated over can be written as $g(x,y,z) = c$,

$$\hat{n} = \pm \frac{\nabla g}{\|\nabla g\|}$$

(since [the gradient is normal to level surfaces](#))

### Evaluation across parametrized surface

If the surface being integrated over can be parametrized as $\vec{r}(u,v)$,

$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$$

(since the axes of the area on the surface are in the direction of $\vec{r}_u$ and $\vec{r}_v$).

Because of this, the integral can be simplified as:

$$\iint_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

## Mass & Moment

Same as 15.6 Apps of Double & Triple Integrals > Physics Definitions, but with:

$$dm = \delta d\sigma$$

#module4 #week12