14.6: Tangent Planes & Differentials

Tangent Planes & Normal Lines

Tangent plane to level surface $f(x, y, z) = c$ of a differentiable function f at point $P_0(x_0, y_0, z_0)$ where the gradient is not zero is the plane through P_0 normal to $\nabla f(x_0, y_0, z_0)$.

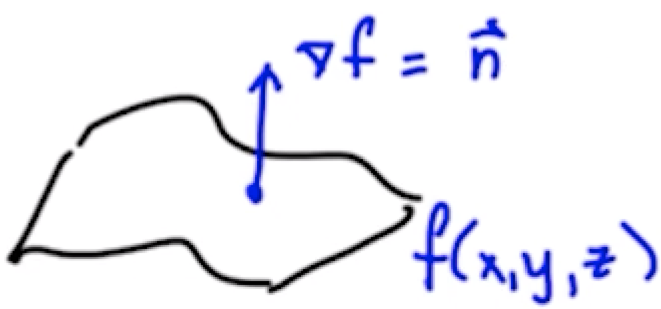
$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$
$$\nabla f(P_0) \cdot \overrightarrow{P_0 P} = 0$$

The **normal line** to level surface $f(x, y, z) = c$ is the line through P_0 parallel to $\nabla f(x_0, y_0, z_0)$.

$$x = x_0 + f_x(P_0)t$$
$$y = y_0 + f_y(P_0)t$$
$$z = z_0 + f_z(P_0)t$$

or

$$\vec{r}(t) = P_0 + t\nabla f(P_0)$$



(Recall from 14.5 Directional Derivatives & Gradient > Gradient Orthogonality to Level Curves and Surfaces that ∇f must be orthogonal to level curves and surfaces.)

Differentials

Linearization

The **linearization** of differentiable function $f(x, y)$ at (x_0, y_0) is:

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$
$$L(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

The approximation $f(x, y) \approx L(x, y)$ is called the **standard linear approximation** of f at the point.

The **total differential of** f is the resulting change from (x_0, y_0) to $(x_0 + dx, y_0, dy)$

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

Error in standard linear approximation when using L to approximate f:

$$|E| \leq \frac{1}{2}M(|x - x_0| + |y - y_0|)^2$$

M represents the upper bound of the second partials on the rectangle centered at P_0.

Extension of above formulas to more dimensions is trivial.

More Differentials

They also help in estimating change in a function in a particular direction.
To estimate the change in value of a differentiable function f when moving a small distance, ds, from point P_0 in the direction of the unit vector $\hat{u}$,

$$df = f'_{\hat{u}}(P_0)ds = (\nabla f(P_0) \cdot \hat{u})ds$$

#module2 #week5