13.1, 13.2: Vector Functions

A **vector function** is a function that takes one or more variables and returns a vector.

- Given real valued functions f_1, f_2, f_3,
    - $\vec{f}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$
- f_1, f_2, f_3 are the **components** of $\vec{f}$
- Parametrization should be obvious

Limits of Vector Functions

Let $\vec{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$,

$$\lim_{t \to t_0} \vec{r}(t) = \vec{L}$$

if for every $\epsilon > 0$, there exists corresponding $\delta > 0$ such that for all $t \in D$,

$$\|\vec{r}(t) - \vec{L}\| < \epsilon \text{ whenever } 0 < \|t - t_0\| < \delta$$

To evaluate a limit, find the limit of each component.

$\vec{f}$ is **continuous** at point $t = t_0$ if $\lim\limits_{t \to t_0} \vec{f}(t) = \vec{f}(t_0)$
($\vec{f}$ is continuous if all points are continuous)

Limit rules

Given $\vec{f}(t) \to \vec{L}, \vec{g}(t) \to \vec{M}, u(t) \to U$, then:

1. $\vec{f}(t) + \vec{g}(t) \to \vec{L} + \vec{M}$
2. $\alpha\vec{f}(t) \to \alpha\vec{L}$
3. $u(t)\vec{f}(t) \to U\vec{L}$
4. $\vec{f}(t) \cdot \vec{g}(t) \to \vec{L} \cdot \vec{M}$
5. $\vec{f}(t) \times \vec{g}(t) \to \vec{L} \times \vec{M}$

Derivatives of Vector Functions

$$\vec{r}'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}$$

If limit exists, $\vec{r}$ is **differentiable** at t.
($\vec{r}$ is differentiable if all points are differentiable)

$\vec{r}'$ is the same as the sum of the derivatives of each component

Derivative rules

1. $(\vec{f}+\vec{g})'(t) = \vec{f}'(t) + \vec{g}'(t)$
2. $(\alpha\vec{f})'(t) = \alpha\vec{f}'(t)$
3. $(\vec{f}\cdot\vec{g})'(t) = \vec{f}'(t)\cdot\vec{g}(t) + \vec{f}(t)\cdot\vec{g}'(t)$ (product rule, applies for *dot product*, *cross product*, *scalar multiplication*)
4. $(\vec{f}\circ u)'(t) = \vec{f}'(u(t))u'(t)$ (chain rule)

Tangent Lines, Velocity, Acceleration

$\vec{r}(t)$ is **smooth** if $\frac{dr}{dt}$ is continuous & never zero

The **tangent line** to smooth curve $\vec{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \text{ at } t = t_0$ is the line that passes $\vec{r}(t_0)$ & is parallel to $\vec{r}'(t_0)$.

If $\vec{r}$ is position,

- $\vec{v} = \frac{d\vec{r}}{dt}$ (velocity)
- $\vec{a} = \frac{d\vec{v}}{dt}$ (acceleration)
- direction of motion = direction of $\vec{v}$ = $\vec{v}/\|\vec{v}\|$ (see unit tangent vector)
- speed = $\|\vec{v}\|$

Integrals of Vector Functions

$\vec{R}(t)$ is an antiderivative of $\vec{r}(t)$ on interval I if $\frac{d\vec{R}}{dt} = \vec{r}$ at each point in I.

$$\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{C}$$

To calculate integrals, componentize $\vec{r}$ and solve for each component.

Integral rules

1. $\int_a^b (\vec{f}+\vec{g})(t)\,dt = \int_a^b \vec{f}(t)\,dt + \int_a^b \vec{g}(t)\,dt$
2. $\int_a^b (\alpha\vec{f})(t)\,dt = \alpha\int_a^b \vec{f}(t)\,dt$
3. $\int_a^b (\vec{c}\vec{f})(t)\,dt = \vec{c}\int_a^b \vec{f}(t)\,dt$
4. $\left\|\int_a^b (\vec{f})(t)\,dt\right\| \leq \int_a^b \left\|\vec{f}(t)\right\|dt$

Projectile Motion

bro use kinematics

#week2