14.8: Lagrange Multipliers

- Can be used to help solve optimization problems that have constraints

Orthogonal Gradient Theorem

Suppose $f(x, y, z)$ is differentiable in region whose interior contains smooth curve:

$$C : \vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

If P_0 is a point on C where f has a local extremum relative to its values on C, ∇f is orthogonal to C at P_0.

Why?
f is at a local extremum on C when $\frac{df}{dt} = 0$.

For derivatives along a path, $\frac{df}{dt} = \nabla f \cdot \vec{r}'$.

As such, $\nabla f \cdot \vec{r}' = 0$, so if f is at a local extremum on C, ∇f must be orthogonal to the path of travel there.

Method of Lagrange Multipliers

Suppose $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq 0$ when $g(x, y, z) = 0$. To find local extremum of f subject to $g(x, y, z) = 0$, find x, y, z, λ satisfying:

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ g(x, y, z) &= 0 \end{aligned}$$

Why?
Let's say the point we're trying to find is $P_0 = (x, y, z)$, which meets the condition $g(x, y, z) = 0$ (note that this means P_0 is on a level surface).

If P_0 is a local extremum, then on every curve at P_0 on the level surface, it must be at a local extremum.

By the above Orthogonal Gradient Theorem, ∇f is orthogonal to the level surface.
Since gradients are orthogonal to level surfaces, ∇g is orthogonal to the level surface.

So ∇f must be a multiple of ∇g, or in other words:

- If ∇f is at a local extremum and $g(x, y, z) = 0$, then there must be a constant λ such that $\nabla f = \lambda \nabla g$.

Example

Maximize xy on ellipse $4x^2 + 9y^2 = 36$.

$$\begin{aligned} f(x,y) &= xy \\ \nabla f(x,y) &= y\mathbf{i} + x\mathbf{j} \end{aligned}$$

$$\begin{aligned} g(x,y) &= 4x^2 + 9y^2 - 36 \\ \nabla g(x,y) &= 8x\mathbf{i} + 18y\mathbf{j} \end{aligned}$$

Equations formed:

$$\begin{aligned} y &= \lambda(8x) \\ x &= \lambda(18y) \\ 4x^2 + 9y^2 - 36 &= 0 \end{aligned}$$

We get values for x and y. This gives us points we can use to maximize xy.
λ is unused.

14.9: Taylor's Formula for $f(x, y)$

Taylor Polynomial (recap)
If function f has n derivatives at point where $x = a$, then the nth Taylor Polynomial for f at a is:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!}$$

The theorem
If f has $n+1$ derivatives on an open interval containing a, then for every x in that open interval, we have:

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some (estimated) value c between a and x that maximizes that term.

The absolute value of new term is called the error when using $P_n(x)$ to approximate $f(x)$.

$$\text{error} = |f(x) - P_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}|x-a|^{n+1}$$

Give an error estimate for the approximation of $\cos(2x)$ by $P_{10}(x)$ for an arbitrary x between 0 and $\pi/4$ centered at $x = 0$.

$n+1 = 11$

$$\text{error} \leq \frac{|f^{(n+1)}(c)|}{(n+1)!}(x-a)^{n+1}$$

$f'(x) = -2\sin(2x)$

$f''(x) = -4\cos(2x)$

$f'''(x) = 8\sin(2x)$

2^3

$|f^{(11)}(x)| = 2^{11} \cdot |(\sin(2x) \text{ or } \cos(2x))|$

$f^{(11)}(c) \leq 2048(1)$

$$\text{error} \leq \frac{2048}{11!}(\pi/4)^{11}$$

(I believe this was done in BC)

Two Variables

Suppose $f(x, y)$ and its partials thru order $n+1$ are continuous throughout open rectangular region R centered around (a, b). Then, throughout R:

$$f(a+h, b+k) = \sum_{i=0}^{n+1} \frac{1}{i!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^i f\Bigg|_{(a,b)}$$

Error term is the last one, last is also an approximate error term.

14.10: Partial Derivatives w/ Constraints

Steps

1. Decide which variables are dependent & independent
2. Eliminate the other dependent variables
3. Differentiate and solve

Example

If $w = x^2 + y - z + \sin(t)$ and $x + y = t$, find $\left(\frac{\partial w}{\partial y}\right)_{z,t}$

(notation designates that z, t are independent)

$$
\begin{aligned}
x &= t - y \\
w &= (t-y)^2 + y - z + \sin(t) \\
\frac{\partial w}{\partial y} &= -2(t-y) + 1
\end{aligned}
$$

#module2 #week6