15.7: Triple Integrals in Cylindrical & Spherical Coordinates

Triple integrals can be integrated through different coordinate systems other than rectangular ($dx\,dy\,dz$), which may be easier when there is radial or spherical symmetry.

Cylindrical Coordinates

- Represent point P in space by ordered triples (r, θ, z) $(r \geq 0)$

1. r and θ are polar coordinates for the projection of P onto the xy-plane
2. z is the rectangular vertical coordinate

Usage

Should be used when...

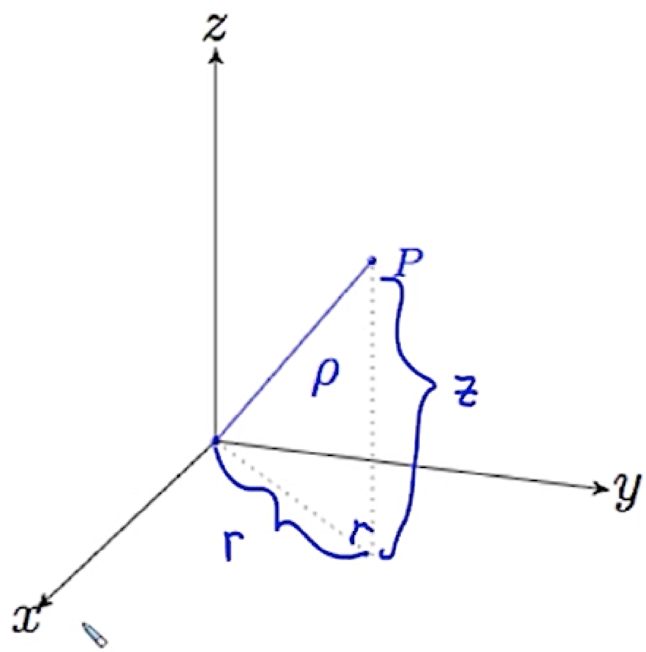
- there is an axis of symmetry
- an integrand involves $x^2 + y^2$
- we're integrating over a circle (or part of) in the xy-plane

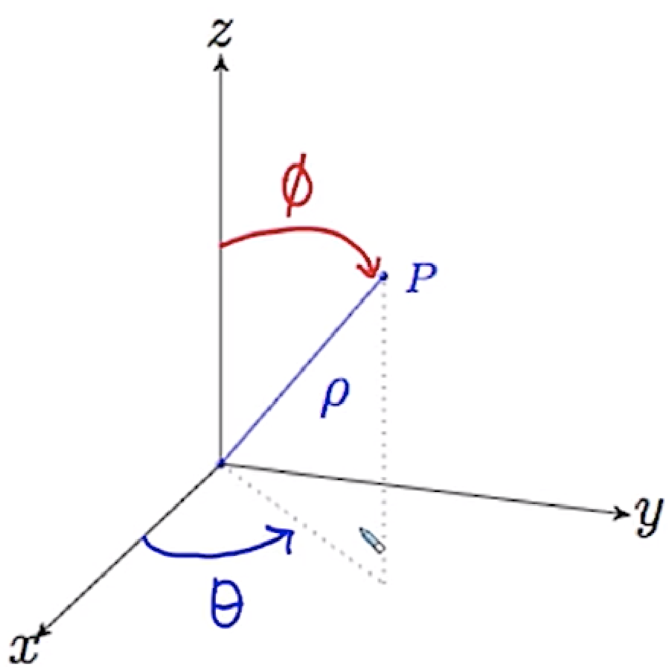
Very similar to using polar coordinates w/ double integrals, but with an added z component for triple integrals.

Spherical Coordinates

- Represent point P in space by ordered triples (ρ, ϕ, θ)

1. ρ is distance from P to the origin $(\rho \geq 0)$
2. ϕ is the angle $\overrightarrow{OP}$ makes with the $+z$-axis $(0 \leq \phi \leq \pi)$
3. θ is the angle from cylindrical coordinates





Converting Rectangular to Spherical

$$\rho^2 = x^2 + y^2 + z^2 = r^2 + z^2$$

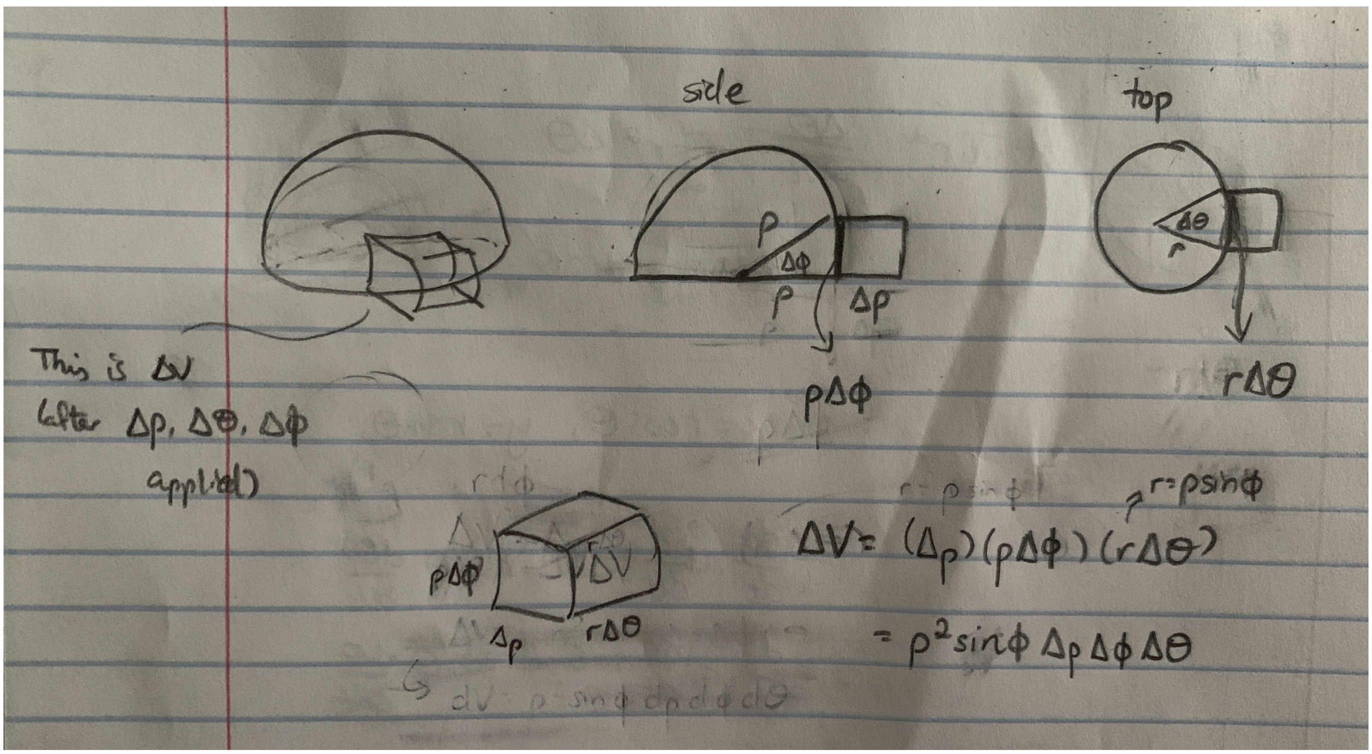
$$r = \rho \sin \phi$$
$$z = \rho \cos \phi$$

$$x = r \cos \theta = \rho \sin \phi \cos \theta$$
$$y = r \sin \theta = \rho \sin \phi \sin \theta$$

Triple Integral Definition

$$\iiint_T dV = \iiint_T \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

Why?



ΔV is the curved box above. Assuming ΔV is a rectangular prism (when ΔV is very small, it's essentially a rectangular prism),

$$\Delta V = (\Delta\rho)\overbrace{(\rho\,\Delta\phi)}^{\text{arclength from the side}}\underbrace{(r\,\Delta\theta)}_{\text{arclength from the top}}$$

$$\begin{aligned} &= \rho r\,\Delta\rho\,\Delta\phi\,\Delta\theta \\ &= \rho(\rho\sin\phi)\,\Delta\rho\,\Delta\phi\,\Delta\theta \\ &= \rho^2\sin\phi\,\Delta\rho\,\Delta\phi\,\Delta\theta \end{aligned}$$

$$dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Related: purely algebraic derivation

#module3 #week9