

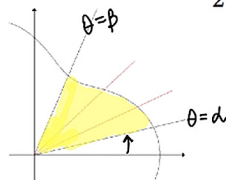
15.4: Double Integrals in Polar Form

[Double integrals before](#) were written in terms of $dx dy$, but they can be written in terms of $dr d\theta$, too.

Area of a polar region is based on the area of a sector of a circle.

- Area of a circle: πr^2
- Area of a sector: $\frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \theta$

$$\text{Area of a sector} = \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \theta$$



$$\begin{aligned} \text{Area of polar region} &= \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta \\ &= \int_{\alpha}^{\beta} \int_a^b r dr d\theta \\ &= \iint_R r dr d\theta \end{aligned}$$

(Note that $\int_a^b r dr = \frac{1}{2} r^2 \Big|_a^b$)

$$dA = r dr d\theta = dx dy$$

Given $F(r, \theta)$ is continuous on $\Gamma : a \leq r \leq b, \alpha \leq \theta \leq \beta$:

$$\iint_{\Gamma} F(r, \theta) r dr d\theta = \int_{\alpha}^{\beta} \int_a^b F(r, \theta) r dr d\theta$$

Given $F(r, \theta)$ is continuous on $\Omega : \alpha \leq \theta \leq \beta, \rho_1(\theta) \leq r \leq \rho_2(\theta)$:

$$\iint_{\Omega} F(r, \theta) r dr d\theta = \int_{\alpha}^{\beta} \int_{\rho_1(\theta)}^{\rho_2(\theta)} F(r, \theta) r dr d\theta$$

Fubini's Theorem applies here. (Bounds can be switched or rearranged as described in Fubini's.)

Volume

If $F(r, \theta) \geq 0$ over region R , then volume with R as base, bounded above by $F(r, \theta)$ is:

$$V = \iint_R F(r, \theta) r \, dr \, d\theta$$

(same as double integrals above)

Example: Double Integrals with Polar Coordinates

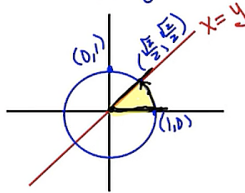
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Evaluate $\int_0^{\sqrt{2}/2} \int_y^{\sqrt{1-y^2}} (x^2 + y^2)^{3/2} \, dx \, dy$ by changing to polar coordinates.

$$x^2 + y^2 = r^2$$

$$- 0 \leq y \leq \frac{\sqrt{2}}{2}$$

$$- y \leq x \leq \sqrt{1-y^2}$$



$$0 \leq r \leq 1$$

$$0 \leq \theta \leq \pi/4$$

$$\int_0^{\pi/4} \int_0^1 \underbrace{r^3 \cdot r}_{r^4} \, dr \, d\theta$$

$$\int_0^{\pi/4} \left[\frac{1}{5} r^5 \right]_0^1 \, d\theta = \int_0^{\pi/4} \frac{1}{5} \, d\theta$$

$$= \frac{1}{5} \theta \Big|_0^{\pi/4}$$

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