

# 14.9: Taylor's Formula for $f(x, y)$

## One variable (recap)

If function  $f$  has  $n$  derivatives at point where  $x = a$ , then the  $n$ th Taylor Polynomial for  $f$  at  $a$  is:

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)(x-a)^k}{k!}$$

### The theorem

If  $f$  has  $n+1$  derivatives on an open interval containing  $a$ , then for every  $x$  in that open interval, we have:

$$f(x) = P_n(x) + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{\text{error term}}$$

for some (estimated) value  $c$  between  $a$  and  $x$  that maximizes that term.

The absolute value of last term is called the error when using  $P_n(x)$  to approximate  $f(x)$ .

$$\text{error} = |f(x) - P_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!} |x-a|^{n+1}$$

Give an error estimate for the approximation of  $\cos(2x)$  by  $P_{10}(x)$  for an arbitrary  $x$  between 0 and  $\pi/4$  centered at  $x = 0$ .

$$\text{error} \leq \frac{|f^{(n+1)}(c)|}{(n+1)!} (x-a)^{n+1} \quad \leftarrow n+1 = 11$$

$$f'(x) = -2 \sin(2x)$$

$$f''(x) = -4 \cos(2x)$$

$$f'''(x) = 8 \sin(2x)$$

$$\leftarrow 2^3$$

$$|f^{(11)}(x)| = 2^{11} \cdot |(\sin(2x) \text{ or } \cos(2x))|$$

$$f^{(11)}(c) \leq 2048 (1)$$

$$\text{error} \leq \frac{2048}{11!} (\pi/4)^{11}$$

(I believe this was done in BC)

## Two variables

Suppose  $f(x, y)$  and its partials through order  $n + 1$  are continuous throughout open rectangular region  $R$  centered around  $(a, b)$ .

Then, throughout  $R$ :

$$f(a + h, b + k) = \sum_{i=0}^n \frac{1}{i!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^i f \Big|_{(a,b)} + E(a + h, b + k)$$

The last term  $E(a + h, b + k)$  is the error term. It is evaluated at the point on the line segment connecting  $(a, b)$  and  $(a + h, b + k)$  that maximizes the error term.

This error term is defined as:

$$E(a + h, b + k) = \frac{1}{(n+1)!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{(n+1)} f \Big|_{(a+ch, b+ck)}$$

If  $n = 1$ , this matches [linearization](#).

If  $n = 2$ , the approximation is known as the *quadratic approximation*.

If  $n = 3$ , the approximation is known as the *cubic approximation*.

etc.

### Expanded form

$$\begin{aligned} f(a + h, b + k) &= f(a, b) + (hf_x + kf_y)|_{(a,b)} \\ &+ \frac{1}{2!}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})|_{(a,b)} \\ &+ \frac{1}{3!}(h^3 f_{xxx} + 3h^2kf_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy})|_{(a,b)} \\ &+ \dots \\ &+ \frac{1}{n!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n f \Big|_{(a,b)} \end{aligned}$$

**Wait, isn't the two-variable Taylor's formula an extension of single-variable Taylor's? Where's the  $(x - a)$  term?**

Note that in the single-variable Taylor's formula,  $x$  represents... just  $x$ .

However, two-variable Taylor's does not use  $x$ . It defines  $h$  and  $k$ , which represent a *relative* distance from  $a$  and  $b$ .

You can also write single-variable Taylor's in the form of the evaluation point  $x = a$  and a relative distance  $h$  from that evaluation point (i.e. let  $x = a + h$ ):

$$f(a + h) = \sum_{k=0}^n \frac{h^k f^{(k)}(a)}{k!} + \underbrace{\frac{h^{n+1} f^{(n+1)}(c)}{(n+1)!}}_{\text{error term}}$$

This aligns much more closely to the two-variable formula.

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