14.8: Lagrange Multipliers

- Can be used to help solve optimization problems that have constraints

Orthogonal Gradient Theorem

Suppose $f(x, y, z)$ is differentiable in region whose interior contains smooth curve:

$$C : \vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

If P_0 is a point on C where f has a local extremum relative to its values on C, ∇f is orthogonal to C at P_0.

Why?
f is at a local extremum on C when $\frac{df}{dt} = 0$.

For derivatives along a path, $\frac{df}{dt} = \nabla f \cdot \vec{r}'$.

As such, $\nabla f \cdot \vec{r}' = 0$, so if f is at a local extremum on C, ∇f must be orthogonal to the path of travel there.

Method of Lagrange Multipliers

Suppose $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq 0$ when $g(x, y, z) = 0$. To find local extremum of f subject to $g(x, y, z) = 0$, find x, y, z, λ satisfying:

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ g(x, y, z) &= 0 \end{aligned}$$

Why?
Let's say the point we're trying to find is $P_0 = (x, y, z)$, which meets the condition $g(x, y, z) = 0$ (note that this means P_0 is on a level surface).

If P_0 is a local extremum, then on every curve at P_0 on the level surface, it must be at a local extremum.

By the above Orthogonal Gradient Theorem, ∇f is orthogonal to the level surface. Since gradients are orthogonal to level surfaces, ∇g is orthogonal to the level surface.

So ∇f must be a multiple of ∇g, or in other words:

- If ∇f is at a local extremum and $g(x, y, z) = 0$, then there must be a constant λ such that $\nabla f = \lambda \nabla g$.

Two Constraints

$$\begin{aligned} \nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \\ g_1(x, y, z) = 0 \\ g_2(x, y, z) = 0 \end{aligned}$$

Why?
Let C be the smooth curve that describes the intersection between $g_1 = 0$ and $g_2 = 0$. ∇g_1 and ∇g_2 are both perpendicular to this curve since gradients are orthogonal to level surfaces.
Since ∇f is also perpendicular to this curve (since we're trying to find local extremum), then ∇f must be on the plane formed by ∇g_1 and ∇g_2 (and thus is a linear combination of the two).

Example

Maximize xy on ellipse $4x^2 + 9y^2 = 36$.

$$\begin{aligned} f(x, y) &= xy \\ \nabla f(x, y) &= y\mathbf{i} + x\mathbf{j} \\ \\ g(x, y) &= 4x^2 + 9y^2 - 36 \\ \nabla g(x, y) &= 8x\mathbf{i} + 18y\mathbf{j} \end{aligned}$$

Equations formed:

$$\begin{aligned} y = \lambda(8x) \\ x = \lambda(18y) \\ 4x^2 + 9y^2 - 36 = 0 \end{aligned}$$

We get values for x and y. This gives us points we can use to maximize xy.

#module2 #week6