14.5: Directional Derivatives and the Gradient

Directional Derivatives

$f'_{\vec{u}}(x_0, y_0)$ or $D_{\vec{u}}f(P_0)$ gives the **directional derivative** of f in the direction of $\vec{u}$ at the point $P_0 = (x_0, y_0)$.

- The rate of change of f in the $\vec{u}$ direction.

If $\vec{u} = u_1\mathbf{i} + u_2\mathbf{j}$, then

$$D_{\vec{u}}f(P_0) = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

provided that the limit exists.

Example

Example: Find the directional derivative of $f(x,y) = x^2 + xy^2$ at the point P(1,1) in the direction of $\mathbf{i} - \mathbf{j}$. $\|\vec{i} - \vec{j}\| = \sqrt{2}$ $\quad \vec{u} = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$

$$D_u f(1,1) = \lim_{s \to 0} \frac{f(1 + s/\sqrt{2}, 1 - s/\sqrt{2}) - f(1,1)}{s}$$

$$= \lim_{s \to 0} \frac{(1 + s/\sqrt{2})^2 + (1 + s/\sqrt{2})(1 - s/\sqrt{2})^2 - 2}{s} = \lim_{s \to 0} \frac{s/\sqrt{2} + s^3/2\sqrt{2}}{s}$$

$$= \lim_{s \to 0} \left(1/\sqrt{2} + s^2/2\sqrt{2}\right) = 1/\sqrt{2}$$

Gradients

Gradient of a function $f(x, y)$ is vector

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

You can imagine how to extend this into 3+ dimensions.

The gradient represents the direction and rate of fastest increase in f. (This is also stated below in the redefinition of a directional derivative.)

Properties

$$\nabla(f(\vec{x}) + g(\vec{x})) = \nabla f(\vec{x}) + \nabla g(\vec{x})$$
$$\nabla(\alpha f(\vec{x})) = \alpha \nabla f(\vec{x})$$
$$\nabla(f(\vec{x})g(\vec{x})) = f(\vec{x})\nabla g(\vec{x}) + \nabla f(\vec{x})g(\vec{x})$$

Redefinition of Directional Derivative

Directional derivative of f in the direction of $\vec{u}$ at point $P_0 = (x_0, y_0)$ can be written as:

$$f'_{\vec{u}}(P_0) = \nabla f(P_0) \cdot \hat{u} = \|\nabla f(P_0)\| \cos\theta$$

where $\hat{u}$ is the unit vector in the $\vec{u}$ direction.

(Note that this is essentially a projection of the gradient onto $\hat{u}$.)

Properties

1. At P_0, function f increases most rapidly in the direction of ∇f.
2. Function f decreases most rapidly in the direction of $-\nabla f$.
3. Any direction $\vec{u}$ orthogonal to gradient $\nabla f \neq 0$ is a direction of zero change in f.

Examples

Find the gradient of $f(x, y) = 2e^x \sin(x^2 + y)$

$$\nabla f(x, y) = (4xe^x \cos(x^2 + y) + 2e^x \sin(x^2 + y))\mathbf{i} + 2e^x \cos(x^2 + y)\mathbf{j}$$

Find a unit vector in the direction in which f increases most rapidly at P and give the rate of change of f in that direction.

$$f(x, y) = y^{-2}e^{2x} \text{ at } P(0, 1)$$

$$\nabla f = \frac{2e^{2x}}{y^2}\vec{i} - \frac{2e^{2x}}{y^3}\vec{j}$$

$$\nabla f(0,1) = 2\vec{i} - 2\vec{j}$$

$$\|\nabla f(0,1)\| = \sqrt{8} = \underline{\underline{2\sqrt{2}}}$$

$$\vec{u}_{\nabla f} = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$$

Tangent Lines to Level Curves

The tangent line to level curve $f(x, y) = c$ at point (x_0, y_0) is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0$$

Why?
The equation for a tangent line is:

$$y - y_0 = \left.\frac{dy}{dx}\right|_{(x_0, y_0)} (x - x_0)$$

[The implicit differentiation rule] can be applied here, so $\frac{dy}{dx} = -\frac{f_x}{f_y}$.

Hence,

$$y - y_0 = -\frac{f_x(x_0, y_0)}{f_y(x_0, y_0)}(x - x_0) = 0$$

(which can be solved into the formula above)

Derivative Along a Path

if $\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is a smooth path C and $w = f(\vec{r}(t))$ is a scalar function evaluated along C, then the derivative along that path is

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$$

Note that this is equivalent to chain rule.

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\ &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \\ &= \nabla f \cdot r' \end{aligned}$$

Gradient Orthogonality to Level Curves and Surfaces

Let point $P_0(x_0, y_0, z_0)$ be on [level surface] $f(x, y, z) = c$.
For all directions $\hat{u}$ tangent to the level surface, $f'_{\hat{u}}(P_0) = 0$ (the directional derivative in the direction $\hat{u}$ at point P_0 must be zero).

- Since c is constant, traveling in the direction of the level surface will not change f, so the directional derivative is 0.

From [the properties of directional derivatives], when $f'_{\hat{u}}(P_0) = 0$, ∇f must be orthogonal to $\hat{u}$.

So, ∇f must be orthogonal to every direction tangent to the level surface and thus must be normal to the level surface.

#module2 #week5