

16.1: Line Integrals

If f is defined on a curve C given parametrically by $\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, the line integral of f over C is:

$$\int_C f(x, y, z) ds = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k, y_k, z_k) \Delta s_k$$

(This is the form for a **line integral of a scalar field**)

To integrate a continuous function $f(x, y, z)$ over a curve C :

1. Find a smooth parametrization of C :

$$\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad (\text{where } a \leq t \leq b)$$

2. Evaluate the integral as:

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) |\vec{v}(t)| dt$$

(recall $\frac{ds}{dt} = |\vec{v}|$)

Mass and Moment Calculations

Suppose we need to find the mass & moment for coil springs and then rods lying along a smooth curve C in space.

Recall [physics definitions](#) from 15.6.

They apply here, too.

Mass

$$m = \int_C \lambda ds$$

(this is a pretty straightforward extension of 15.6 so I don't think there needs to be notes here)

16.2: Vector Fields & Line Integrals

Let \vec{F} be a vector field with continuous components defined along smooth curve C parametrized by $\vec{r}(t)$, $a \leq t \leq b$.

The **line integral of \vec{F} along C** is:

$$\int_C \vec{F} \cdot \vec{T} ds = \int_C \left(F \cdot \frac{d\vec{r}}{ds} \right) ds = \int_C \vec{F} \cdot d\vec{r}$$

(This is the form for a **line integral of a vector field**)

To evaluate, write \vec{F} and $d\vec{r}$ in terms of t and apply dot product.

Line integrals may also be written as:

$$\begin{aligned} & \int_C M dx + \int_C N dy + \int_C P dz \\ &= \int_C M(x, y, z) dx + \int_C N(x, y, z) dy + \int_C P(x, y, z) dz \end{aligned}$$

(same idea, write everything in terms of t)

Example

Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = \langle xy, x^2z, xyz \rangle$ along $y = x^2$ from $(0, 0, 0)$ to $(1, 1, 0)$ followed by the straight-line segment from $(1, 1, 0)$ to $(1, 1, 1)$.

$$\begin{aligned} C_1 : \vec{r}_1(t) &= \langle t, t^2, 0 \rangle \\ \vec{r}'_1(t) &= \langle 1, 2t, 0 \rangle \end{aligned}$$

$$\begin{aligned} C_2 : \vec{r}_2(t) &= \langle 1, 1, t \rangle \\ \vec{r}'_2(t) &= \langle 0, 0, 1 \rangle \end{aligned}$$

$$\begin{aligned} & \int_C \vec{F} \cdot d\vec{r} \\ &= \int_0^1 \langle (t)(t^2), 0, 0 \rangle \cdot \langle 1, 2t, 0 \rangle dt + \int_0^1 \langle (1)(1), (1)^2(t), (1)(1)(t) \rangle \cdot \langle 0, 0, 1 \rangle dt \\ &= \int_0^1 \langle t^3, 0, 0 \rangle \cdot \langle 1, 2t, 0 \rangle dt + \int_0^1 \langle 1, t, t \rangle \cdot \langle 0, 0, 1 \rangle dt \end{aligned}$$

Physics

Work

$$W = \int_C \vec{F} \cdot d\vec{r}$$

(\vec{F} is force)

Flow

$$\text{Flow} = \int_C \vec{F} \cdot \vec{T} ds$$

(\vec{F} is velocity)

This integral is called a **flow integral**. If the curve starts and ends at the same point, the flow is called the *circulation* around the curve.

Flux

$$\Phi = \int_C \vec{F} \cdot \vec{N} ds$$

(\vec{F} is velocity, C is a smooth simple closed curve (starts & ends at same place and does not cross itself))

Calculating flux across a smooth closed plane curve

Let $\vec{F} = M\mathbf{i} + N\mathbf{j}$ and $\vec{r} = x(t)\mathbf{i} + y(t)\mathbf{j}$,
then:

$$\text{Flux of } \mathbf{F} = M\mathbf{i} + N\mathbf{j} \text{ across } C = \oint M dy - N dx$$

(Integral is evaluated at any parametrization \vec{r} that traces C counterclockwise exactly once)

$$\left\| \vec{F} \times \frac{d\vec{r}}{ds} \right\| = \begin{vmatrix} M & N \\ \frac{dx}{ds} & \frac{dy}{ds} \end{vmatrix} = M \frac{dy}{ds} - N \frac{dx}{ds}$$

(Computing line integral with respect to ds gives you the above)

16.3: Path Independence, Conservative Fields, Potential Functions

Definitions

Let \vec{F} be a vector field defined on open region D in space.

Suppose that for any two points A and B in D , $\int_C \vec{F} \cdot d\vec{r}$ along path C from A to B is the same over all paths from A to B .

The integral is **path independent** and the field is **conservative on D** .

If \vec{F} is a vector field on D and $F = \nabla f$ for some scalar function f on D , f is called a **potential function for F** .

Example

Find a potential function f for $\vec{F} = (y \sin z)\mathbf{i} + (x \sin z)\mathbf{j} + (xy \cos z)\mathbf{k}$.

Let $\vec{F} = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$.

$$\frac{\partial f}{\partial x} = y \sin z$$

$$f = \int y \sin z \, dx = xy \sin z + \overbrace{g(y, z)}^{\text{remember } +C?}$$

$$\frac{\partial f}{\partial y} = x \sin z + \frac{\partial g}{\partial y}$$

$$\frac{\partial f}{\partial z} = xy \cos z + \frac{\partial g}{\partial z}$$

So,

$$x \sin z + \frac{\partial g}{\partial y} = x \sin z \implies \frac{\partial g}{\partial y} = 0$$

$$xy \cos z + \frac{\partial g}{\partial z} = xy \cos z \implies \frac{\partial g}{\partial z} = 0$$

Therefore,

$$f(x, y, z) = xy \sin z + C$$

Conservative Fields & Gradient Fields

Theorem

Let $\vec{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ (and M, N, P are continuous throughout open connected region D).
 F is conservative iff \vec{F} is a gradient field ∇f for a differentiable function f .

Component Test for Conservative Fields

Let $\vec{F} = X(x, y, z)\mathbf{i} + Y(x, y, z)\mathbf{j} + Z(x, y, z)\mathbf{k}$ on open simply connected domain (X, Y, Z have continuous first partial derivatives).

then \vec{F} is conservative iff:

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}$$

$$\frac{\partial X}{\partial z} = \frac{\partial Z}{\partial x}$$

$$\frac{\partial Y}{\partial z} = \frac{\partial Z}{\partial y}$$

If \vec{F} is a gradient field for f , then $F = \nabla f = f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k}$.

Then, second mixed partials need to be equivalent ($f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$, $f_{yz} = f_{zy}$) for this to be true.

Fundamental Theorem of Line Integrals

Let C be a smooth curve joining points A and B , parametrized by $\vec{r}(t)$.

Let f be a differentiable function with continuous gradient vector $\vec{F} = \nabla f$ on domain D containing C .

$$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

Loop Property of Conservative Fields

Equivalent statements

(pretend that is the ccw integral)

1. $\oint_C \vec{F} \cdot d\vec{r} = 0$ around every loop (every closed curve C) in D .
2. The field F is conservative on D .

Exactness

Differential form: Expression of the form $M(x, y, z) dx + N(x, y, z) dy + P(x, y, z) dz$.

A differential form is **exact** on domain D if:

$$M dx + N dy + P dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = df$$

for some scalar function f throughout D .

In other words, a differential form is exact iff $\langle M, N, P \rangle = \nabla f$ (iff $\vec{F} = \langle M, N, P \rangle$ is conservative).