14.9: Taylor's Formula for $f(x, y)$

Taylor Polynomial (recap)
If function f has n derivatives at point where $x = a$, then the nth Taylor Polynomial for f at a is:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!}$$

The theorem
If f has $n+1$ derivatives on an open interval containing a, then for every x in that open interval, we have:

$$f(x) = P_n(x) + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{\text{error term}}$$

for some (estimated) value c between a and x that maximizes that term.

The absolute value of last term is called the error when using $P_n(x)$ to approximate $f(x)$.

$$\text{error} = |f(x) - P_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}|x-a|^{n+1}$$

Give an error estimate for the approximation of $\cos(2x)$ by $P_{10}(x)$ for an arbitrary x between 0 and $\pi/4$ centered at $x = 0$.

$n+1 = 11$

$$\text{error} \le \frac{|f^{(n+1)}(c)|}{(n+1)!}(x-a)^{n+1}$$

$f'(x) = -2\sin(2x)$

$f''(x) = -4\cos(2x)$

$f'''(x) = 8\sin(2x)$

2^3

$|f^{(11)}(x)| = 2^{11} \cdot |(\sin(2x) \text{ or } \cos(2x))|$

$f^{(11)}(c) \le 2048(1)$

$$\text{error} \le \frac{2048}{11!}(\pi/4)^{11}$$

(I believe this was done in BC)

Two Variables

Suppose $f(x, y)$ and its partials through order $n+1$ are continuous throughout open rectangular region R centered around (a, b).
Then, throughout R:

$$f(a+h, b+k) = \sum_{i=0}^{n} \frac{1}{i!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{i} f\Bigg|_{(a,b)} + E(a+h, b+k)$$

The last term $E(a+h, b+k)$ is the error term. It is evaluated at the point on the line segment connecting (a, b) and $(a+h, b+k)$ that maximizes the error term.

This error term is defined as:

$$E(a+h, b+k) = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{(n+1)} f\Bigg|_{(a+ch, b+ck)}$$

If $n = 1$, this matches linearization.
If $n = 2$, the approximation is known as the *quadratic approximation*.
If $n = 3$, the approximation is known as the *cubic approximation*.
etc.

Expanded form

$$
\begin{aligned}
f(a+h, b+k) = f(a,b) &+ (hf_x + kf_y)\big|_{(a,b)} \\
&+ \frac{1}{2!}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})\big|_{(a,b)} \\
&+ \frac{1}{3!}(h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy})\big|_{(a,b)} \\
&+ \cdots \\
&+ \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f\Bigg|_{(a,b)}
\end{aligned}
$$

Wait, isn't the two-variable Taylor's formula an extension of single-variable Taylor's? Where's the $(x-a)$ term?

Note that in the single-variable Taylor's formula, x represents... just x.
However, two-variable Taylor's does not use x. It defines h and k, which represent a *relative* distance from a and b.

You can also write single-variable Taylor's in the form of the evaluation point $x = a$ and a relative distance h from that evaluation point (i.e. let $x = a + h$):

$$f(a+h) = \sum_{k=0}^{n} \frac{h^k f^{(k)}(a)}{k!} + \underbrace{\frac{h^{n+1} f^{(n+1)}(c)}{(n+1)!}}_{\text{error term}}$$

This aligns much more closely to the two-variable formula.

14.10: Partial Derivatives w/ Constraints

Previous partial derivatives assumed all variables were independent, but what if some of the variables have known relationships?

The notation $\left(\frac{\partial w}{\partial y}\right)_{z,t}$ represents the partial derivative of w with respect to y, given that z and t are independent of y.

To evaluate a partial derivative with constraints:

1. Decide which variables are dependent & independent

2. Eliminate the other dependent variables
3. Differentiate and solve

Example

If $w = x^2 + y - z + \sin(t)$ and $x + y = t$, find $\left(\frac{\partial w}{\partial y}\right)_{z,t}$

Method 1: Eliminating other independent variables first.

Since $x + y = t$, substitute $x = t - y$ into w, resulting in:

$$w = (t-y)^2 + y - z + \sin(t)$$

All of the variables are now independent, so just compute the partial.

$$\frac{\partial w}{\partial y} = -2(t-y) + 1$$

Method 2: Deriving on the go.

We know x is dependent on y, so $\frac{\partial x}{\partial y}$ must be non-zero.

$$\frac{\partial w}{\partial y} = 2x\frac{\partial x}{\partial y} + 1$$

Then, since $x = t - y$,

$$\frac{\partial x}{\partial y} = -1$$

Substitute $\frac{\partial x}{\partial y}$ and x, you get the answer from before.

#module2 #week6