16.1: Line Integrals over Scalar Fields

If f is defined on a curve C given parametrically by $\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, the line integral of f over C is:

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k$$

To integrate a continuous function $f(x,y,z)$ over a curve C:

1. Find a smooth parametrization of C:

   $$\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

   (where $a \le t \le b$)
2. Evaluate the integral as:

   $$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t))|\vec{v}(t)|\,dt$$

   (recall $\frac{ds}{dt} = |\vec{v}|$)

Mass and Moment Calculations

Suppose we need to find the mass & moment for coil springs and then rods lying along a smooth curve C in space.

Recall physics definitions from 15.6.
They apply here, too.
Mass

$$m = \int_C \lambda\,ds$$

(this is a pretty straightforward extension of 15.6 so I don't think there needs to be notes here)

16.2: Line Integrals over Vector Fields

Let $\vec{F}$ be a vector field with continuous components defined along smooth curve C parametrized by $\vec{r}(t), a \le t \le b$.

The **line integral of $\vec{F}$ along** C is:

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \left(F\cdot\frac{d\vec{r}}{ds}\right)ds = \int_C \vec{F}\cdot d\vec{r}$$

To evaluate, write $\vec{F}$ and $d\vec{r}$ in terms of t and apply dot product.

Line integrals may also be written as:

$$\int_C M\,dx + \int_C N\,dy + \int_C P\,dz$$
$$= \int_C M(x,y,z)\,dx + \int_C N(x,y,z)\,dy + \int_C P(x,y,z)\,dz$$

(same idea, write everything in terms of t)

Example

Evaluate $\int_C \vec{F}\cdot dr$, where $\vec{F} = \langle xy, x^2z, xyz\rangle$ along $y = x^2$ from $(0,0,0)$ to $(1,1,0)$ followed by the straight-line segment from $(1,1,0)$ to $(1,1,1)$.

$$C_1 : \vec{r}_1(t) = \langle t, t^2, 0\rangle$$
$$\vec{r}_1'(t) = \langle 1, 2t, 0\rangle$$

$$C_2 : \vec{r}_2(t) = \langle 1, 1, t\rangle$$
$$\vec{r}_2'(t) = \langle 0, 0, 1\rangle$$

$$\int_C \vec{F}\cdot d\vec{r}$$
$$= \int_0^1 \langle (t)(t^2), 0, 0\rangle \cdot \langle 1, 2t, 0\rangle\,dt + \int_0^1 \langle (1)(1), (1)^2(t), (1)(1)(t)\rangle \cdot \langle 0,0,1\rangle\,dt$$
$$= \int_0^1 \langle t^3, 0, 0\rangle \cdot \langle 1, 2t, 0\rangle\,dt + \int_0^1 \langle 1, t, t\rangle \cdot \langle 0,0,1\rangle\,dt$$

Applications to Physics

Work

$$W = \int_C \vec{F}\cdot d\vec{r}$$

- $\vec{F}$ is force

Flow

$$\text{Flow} = \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T}\, ds$$

- $\vec{F}$ is velocity

This integral is called a **flow integral**. If the curve starts and ends at the same point, the flow is called the *circulation* around the curve.

Flux

$$\Phi = \int_C \vec{F} \cdot \hat{n}\, ds$$

- $\vec{F}$ is a vector field in the plane, $M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$
- C is a smooth simple closed curve (starts & ends at same place and does not cross itself)
- $\hat{n}$ is the outward-pointing unit vector normal to C

Flux across a planar curve

$$\Phi \text{ across } C = \oint M\, dy - N\, dx$$

(Integral is evaluated at any parametrization $\vec{r}$ that traces C counterclockwise exactly once)

Why?

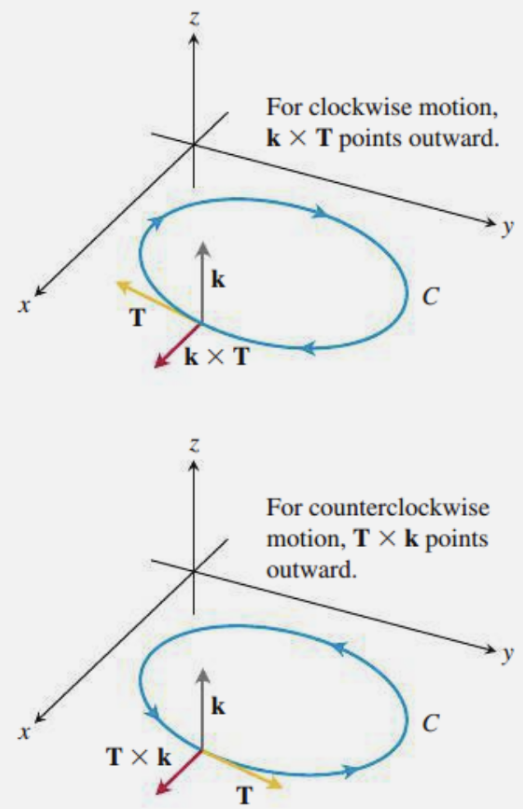

FIGURE 16.24 To find an outward unit normal vector for a smooth simple curve C in the xy-plane that is traversed counterclockwise as t increases, we take $\mathbf{n} = \mathbf{T} \times \mathbf{k}$. For clockwise motion, we take $\mathbf{n} = \mathbf{k} \times \mathbf{T}$.

Assuming counterclockwise,

$$\hat{n} = \vec{T} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}$$

Then:

$$\begin{aligned} \Phi &= \int_C \vec{F} \cdot \hat{n}\, ds \\ &= \int_C \langle M, N \rangle \cdot \left\langle \frac{dy}{ds}, \frac{-dx}{ds} \right\rangle ds \\ &= \oint M\, dy - N\, dx \end{aligned}$$

#module4 #week10