

## 16.7: Stokes' Theorem

### $\nabla$ , div, and curl

The del operator:

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$$

Two formulas use the  $\nabla$  operator:

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F}$$

### Curl Identity

$$\operatorname{curl} \operatorname{grad} f = 0$$

$$\nabla \times \nabla f = 0$$

Why?

$$\nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

Mixed partials have to be equal if continuous, so everything here evals to 0.

### Example

Find the div and curl for  $\vec{F} = (x^2 - yz)\mathbf{i} + ye^x\mathbf{j} + (xy + z)\mathbf{k}$ .

$$\begin{aligned} \operatorname{div} \vec{F} &= \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(ye^x) + \frac{\partial}{\partial z}(xy + z) \\ &= 2x + e^x + 1 \end{aligned}$$

$$\begin{aligned}
 \text{curl } \vec{F} = \nabla \times \vec{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & ye^x & xy + z \end{vmatrix} \\
 &= \mathbf{i} \left( \frac{\partial}{\partial y}(xy + z) - \frac{\partial}{\partial z}(ye^x) \right) \\
 &\quad - \mathbf{j} \left( \frac{\partial}{\partial x}(xy + z) - \frac{\partial}{\partial z}(x^2 - yz) \right) \\
 &\quad + \mathbf{k} \left( \frac{\partial}{\partial x}(ye^x) - \frac{\partial}{\partial y}(x^2 - yz) \right) \\
 &= x\mathbf{i} - 2y\mathbf{j} + (ye^x + z)\mathbf{k}
 \end{aligned}$$

## Stokes' Theorem

Let  $S$  be a piecewise smooth oriented surface with piecewise smooth boundary curve  $C$ .

Let  $\vec{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$  be a vector field with continuous 1st partial derivatives on open region containing  $S$ .

Then the circulation of  $\vec{F}$  around  $C$  in the CCW dir:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} d\sigma = \iint_S (\text{curl } \vec{F}) \cdot \hat{n} d\sigma$$

( $\hat{n}$  is the unit normal vector with respect to the surface)

### Closed Loop Property:

If  $\text{curl } \vec{F} = 0$  at every point of a simply connected open region  $D$  in space, then on any piecewise-smooth closed path  $C$  in  $D$ :

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

(pretty straightforward extension of [the loop property of conservative fields](#))

## 16.8: The Divergence Theorem and a Unified Theory

### Divergence Theorem

Let  $S$  be a piecewise smooth oriented surface.

Let  $\vec{F}$  be a vector field whose components have continuous 1st partial derivatives.

Then, the flux of  $\vec{F}$  across  $S$  in the direction of the surface's outward unit normal field  $\hat{n}$ :

$$\iint_S \vec{F} \cdot \vec{n} d\sigma = \iiint_D \nabla \cdot \vec{F} dV$$

**Corollary:**

The outward flux across a piecewise smooth oriented closed surface is 0 for any vector field  $F$  with 0 divergence at every point of the region.

**Divergence & Curl**

$$\text{div} (\text{curl } \vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = 0$$

## Unifying Fundamental Theorem of Vector Integral Calculus

The integral of a differential operator ( $\nabla$ ) acting on a field over a region = the sum of the field components (appropriate to the operator) over the boundary of the region

Example with Stokes' theorem:

$$\underbrace{\oint_C \vec{F} \cdot d\vec{r}}_{\Sigma \text{ of field components over boundary } C} = \underbrace{\iint_S (\nabla \times \vec{F}) \cdot \hat{n} d\sigma}_{\int \text{ of } \nabla \text{ acting on } \vec{F} \text{ over } S}$$

## Generalizations of Green's Theorem

Recall [Green's Theorem](#). Stokes' theorem and the divergence theorem are both extensions of Green's.

**Tangential Form**

$$\oint_C \vec{F} \cdot \vec{T} ds = \iint_R (\nabla \times \vec{F}) \cdot \mathbf{k} dA \text{ (tangential form)}$$

$$\oint_C \vec{F} \cdot \vec{T} ds = \iint_R (\nabla \times \vec{F}) \cdot \hat{n} dA \text{ (Stokes' theorem)}$$

**Normal Form**

$$\oint_C \vec{F} \cdot \hat{n} ds = \iint_R \nabla \cdot \vec{F} dA \text{ (normal form)}$$

$$\iint_S \vec{F} \cdot \hat{n} d\sigma = \iiint_D \nabla \cdot \vec{F} dV \text{ (Divergence theorem)}$$

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