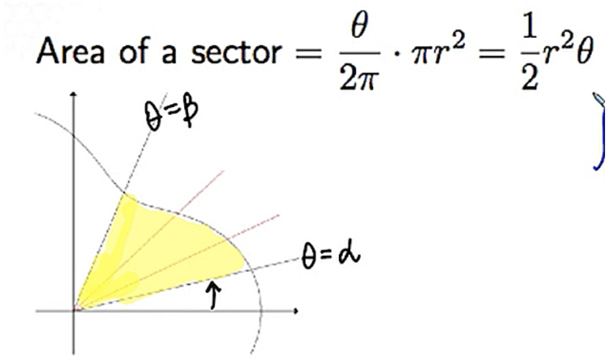
15.4: Double Integrals in Polar Form

The area of a polar region is based on the area of a sector of a circle.

- Area of a circle: πr^2
- Area of a sector: $\frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \theta$



$$\begin{aligned}\text{Area of polar region} &= \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta \\ &= \int_\alpha^\beta \int_a^b r \, dr \, d\theta \\ &= \iint_R r \, dr \, d\theta\end{aligned}$$

(Note that $\int_a^b r \, dr = \frac{1}{2} r^2 \big|_a^b$)

$$dA = r \, dr \, d\theta = dx \, dy$$

Given $F(r, \theta)$ is continuous on $\Gamma : a \leq r \leq b, \alpha \leq \theta \leq \beta$:

$$\iint_\Gamma F(r, \theta) r \, dr \, d\theta = \int_\alpha^\beta \int_a^b F(r, \theta) r \, dr \, d\theta$$

Given $F(r, \theta)$ is continuous on $\Omega : \alpha \leq \theta \leq \beta, \rho_1(\theta) \leq r \leq \rho_2(\theta)$:

$$\iint_\Omega F(r, \theta) r \, dr \, d\theta = \int_\alpha^\beta \int_{\rho_1(\theta)}^{\rho_2(\theta)} F(r, \theta) r \, dr \, d\theta$$

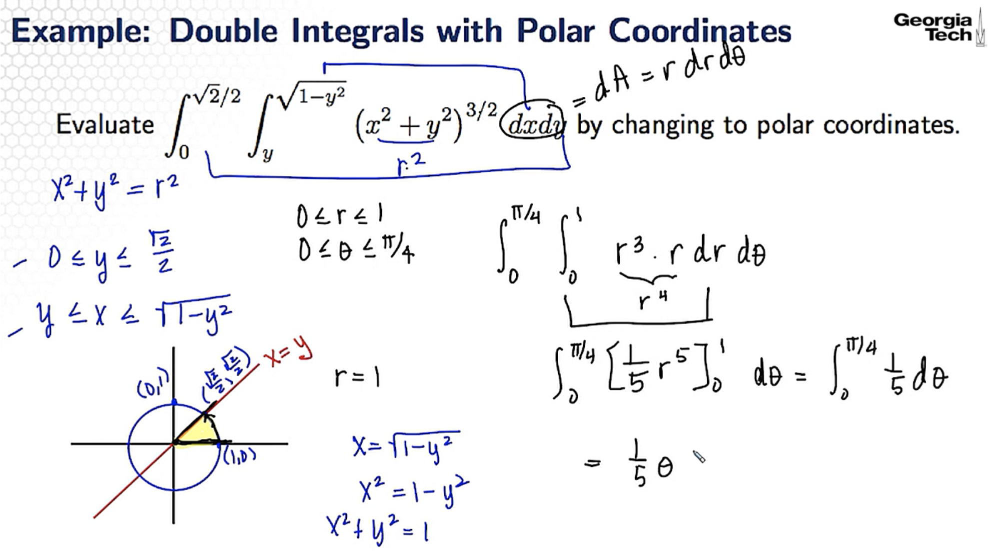
Fubini's theorem still applies, so these integration bounds can be rearranged as expected.

Volume

If $F(r,\theta) \geq 0$ over region R, then volume with R as base, bounded above by $F(r,\theta)$ is:

$$V = \iint_R F(r,\theta) r \, dr \, d\theta$$

(same as double integrals above)



#module3 #week8