

14.9: Taylor's Formula for $f(x, y)$

Taylor Polynomial (recap)

If function f has n derivatives at point where $x = a$, then the n th Taylor Polynomial for f at a is:

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)(x-a)^k}{k!}$$

The theorem

If f has $n+1$ derivatives on an open interval containing a , then for every x in that open interval, we have:

$$f(x) = P_n(x) + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{\text{error term}}$$

for some (estimated) value c between a and x that maximizes that term.

The absolute value of last term is called the error when using $P_n(x)$ to approximate $f(x)$.

$$\text{error} = |f(x) - P_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!} |x-a|^{n+1}$$

Give an error estimate for the approximation of $\cos(2x)$ by $P_{10}(x)$ for an arbitrary x between 0 and $\pi/4$ centered at $x=0$.

$$\text{error} \leq \frac{|f^{(n+1)}(c)|}{(n+1)!} (x-a)^{n+1} \quad \leftarrow n+1=11$$

$$f'(x) = -2 \sin(2x)$$

$$f''(x) = -4 \cos(2x)$$

$$f'''(x) = 8 \sin(2x)$$

$$\leftarrow 2^3$$

$$|f^{(11)}(x)| = 2^{11} \cdot |(\sin(2x) \text{ or } \cos(2x))|$$

$$f^{(11)}(c) \leq 2048(1)$$

$$\text{error} \leq \frac{2048}{11!} (\pi/4)^{11}$$

(I believe this was done in BC)

Two Variables

Suppose $f(x, y)$ and its partials through order $n+1$ are continuous throughout open rectangular region R centered around (a, b) .

Then, throughout R :

$$f(a+h, b+k) = \sum_{i=0}^n \frac{1}{i!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^i f \Big|_{(a,b)} + E(a+h, b+k)$$

The last term $E(a+h, b+k)$ is the error term. It is evaluated at the point on the line segment connecting (a, b) and $(a+h, b+k)$ that maximizes the error term.

This error term is defined as:

$$E(a+h, b+k) = \frac{1}{(n+1)!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{(n+1)} f \Big|_{(a+ch, b+ck)}$$

Expanded form

$$\begin{aligned} f(a+h, b+k) &= f(a, b) + (hf_x + kf_y) \Big|_{(a,b)} \\ &\quad + \frac{1}{2!} (h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}) \Big|_{(a,b)} \\ &\quad + \frac{1}{3!} (h^3 f_{xxx} + 3h^2kf_{xxy} + 3hk^2f_{xyy} + k^3 f_{yyy}) \Big|_{(a,b)} \\ &\quad + \dots \\ &\quad + \frac{1}{n!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n f \Big|_{(a,b)} \end{aligned}$$

Wait, isn't the two-variable Taylor's formula an extension of single-variable Taylor's? Where's the $(x-a)$ term?

Note that in the single-variable Taylor's formula, x represents... just x .

However, two-variable Taylor's does not use x . It defines h and k , which represent a *relative* distance from a and b .

You can also write single-variable Taylor's in the form of the evaluation point $x = a$ and a relative distance h from that evaluation point (i.e. let $x = a + h$):

$$f(a+h) = \sum_{k=0}^n \frac{h^k f^{(k)}(a)}{k!} + \underbrace{\frac{h^{n+1} f^{(n+1)}(c)}{(n+1)!}}_{\text{error term}}$$

This aligns much more closely to the two-variable formula.

14.10: Partial Derivatives w/ Constraints

Sometimes, some of the variables are dependent on others.

The notation $\left(\frac{\partial w}{\partial y} \right)_{z,t}$ represents the partial derivative of w with respect to y , given that z and t are independent.

To evaluate a partial derivative with constraints:

1. Decide which variables are dependent & independent
2. Eliminate the other dependent variables
3. Differentiate and solve

Example

If $w = x^2 + y - z + \sin(t)$ and $x + y = t$, find $\left(\frac{\partial w}{\partial y}\right)_{z,t}$

Method 1: Eliminating other independent variables first.

Since $x + y = t$, substitute $x = t - y$ into w , resulting in:

$$w = (t - y)^2 + y - z + \sin(t)$$

All of the variables are now independent, so just compute the partial.

$$\frac{\partial w}{\partial y} = -2(t - y) + 1$$

Method 2: Deriving on the go.

We know x is dependent on y , so $\frac{\partial x}{\partial y}$ must be non-zero.

$$\frac{\partial w}{\partial y} = 2x \frac{\partial x}{\partial y} + 1$$

Then, since $x = t - y$,

$$\frac{\partial x}{\partial y} = -1$$

Substitute $\frac{\partial x}{\partial y}$ and x , you get the answer from before.

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