14.5: Directional Derivatives and the Gradient

Directional Derivatives

$f'_{\vec{u}}(x_0, y_0)$ or $D_{\vec{u}}f(P_0)$ gives the **directional derivative** of f in the direction of $\vec{u}$ at the point $P_0 = (x_0, y_0)$.

- The rate of change of f in the $\vec{u}$ direction.

If $\vec{u} = u_1\mathbf{i} + u_2\mathbf{j}$, then

$$D_{\vec{u}}f(P_0) = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

provided that the limit exists.

Example

Example: Find the directional derivative of $f(x, y) = x^2 + xy^2$ at the point P(1,1) in the direction of $\mathbf{i} - \mathbf{j}$. $\|\vec{i} - \vec{j}\| = \sqrt{2}$ $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$

$$D_u f(1,1) = \lim_{s \to 0} \frac{f(1 + s/\sqrt{2}, 1 - s/\sqrt{2}) - f(1,1)}{s}$$

$$= \lim_{s \to 0} \frac{(1 + s/\sqrt{2})^2 + (1 + s/\sqrt{2})(1 - s/\sqrt{2})^2 - 2}{s} = \lim_{s \to 0} \frac{s/\sqrt{2} + s^3/2\sqrt{2}}{s}$$

$$= \lim_{s \to 0} \left(1/\sqrt{2} + s^2/2\sqrt{2}\right) = 1/\sqrt{2}$$

Gradients

Gradient of a function $f(x, y)$ is vector

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

You can imagine how to extend this into 3+ dimensions.

Properties

$$\nabla(f(\vec{x})+g(\vec{x}))=\nabla f(\vec{x})+\nabla g(\vec{x})$$
$$\nabla(\alpha f(\vec{x}))=\alpha\nabla f(\vec{x})$$
$$\nabla(f(\vec{x})g(\vec{x}))=f(\vec{x})\nabla g(\vec{x})+\nabla f(\vec{x})g(\vec{x})$$

Redefinition of Directional Derivative

Directional derivative of f in the direction of $\vec{u}$ at point $P_0=(x_0,y_0)$ can be written as:

$$f'_{\vec{u}}(P_0)=\nabla f(P_0)\cdot\hat{u}=\|\nabla f(P_0)\|\cos\theta$$

(Note that this is essentially a projection of the gradient onto $\hat{u}$.)

Properties

1. At P_0, function f increases most rapidly in the direction of its gradient vector.
2. Function f decreases most rapidly in the direction of $-\nabla f$.
3. Any direction $\vec{u}$ orthogonal to gradient $\nabla f\neq 0$ is a direction of zero change in f.

Examples

Find the gradient of $f(x,y)=2e^x\sin(x^2+y)$

$$\nabla f(x,y)=(4xe^x\cos(x^2+y)+2e^x\sin(x^2+y))\mathbf{i} \\ +2e^x\cos(x^2+y)\mathbf{j}$$

Find a unit vector in the direction in which f increases most rapidly at P and give the rate of change of f in that direction.

$$f(x,y)=y^{-2}e^{2x} \text{ at } P(0,1)$$

$$\nabla f=\frac{2e^{2x}}{y^2}\vec{i}-\frac{2e^{2x}}{y^3}\vec{j}$$

$$\nabla f(0,1)=2\vec{i}-2\vec{j}$$

$$\|\nabla f(0,1)\|=\sqrt{8}=\underline{\underline{2\sqrt{2}}}$$

$$\vec{u}_{\nabla f}=\frac{1}{\sqrt{2}}\vec{i}-\frac{1}{\sqrt{2}}\vec{j}$$

Tangent Lines to Level Curves

The tangent line to level curve $f(x,y)=c$ at point (x_0,y_0) is

$$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)=0$$
$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}(x-x_0)+\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}(y-y_0)=0$$

(Can be derived from implicit differentiation rule)

Derivative Along a Path

if $\vec{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}$ is a smooth path C and $w=f(\vec{r}(t))$ is a scalar function evaluated along C, then the derivative along that path is

$$\frac{d}{dt}f(\vec{r}(t))=\nabla f(\vec{r}(t))\cdot\vec{r}'(t)$$

Note that this is equivalent to chain rule.

$$\begin{aligned}\frac{df}{dt}&=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}\\&=\left\langle\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right\rangle\cdot\left\langle\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right\rangle\\&=\nabla f\cdot r'\end{aligned}$$

Gradient Orthogonality to Level Curves and Surfaces

As an extension of the above (derivative along a path), ∇f must be normal to level curves and surfaces at every point (x_0,y_0) in differentiable function f.

From the definition of level curves and surfaces, $f(x,y,z)=c$.

Then,

$$\frac{d}{dt}f(x,y,z)=\frac{d}{dt}(c)$$
$$\frac{df}{dt}=0$$
$$\nabla f\cdot r'=0$$

Therefore, ∇f and r' (which is in the direction of the curve) must be orthogonal.

#module2 #week5