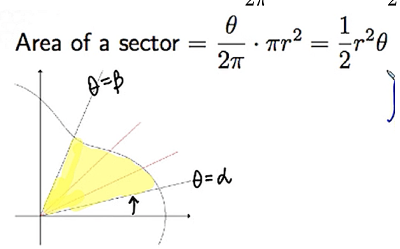
15.4: Double Integrals in Polar Form

Double integrals before were written in terms of $dx\,dy$, but they can be written in terms of $dr\,d\theta$, too.

Area of a polar region is based on the area of a sector of a circle.

- Area of a circle: πr^2
- Area of a sector: $\frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \theta$



$$\begin{aligned} \text{Area of polar region} &= \int_\alpha^\beta \frac{1}{2} r^2 \, d\theta \\ &= \int_\alpha^\beta \int_a^b r \, dr \, d\theta \\ &= \iint_R r \, dr \, d\theta \end{aligned}$$

(Note that $\int_a^b r\,dr = \frac{1}{2} r^2 \Big|_a^b$)

$$dA = r\,dr\,d\theta = dx\,dy$$

Given $F(r, \theta)$ is continuous on $\Gamma : a \le r \le b, \alpha \le \theta \le \beta$:

$$\iint_\Gamma F(r, \theta) r \, dr \, d\theta = \int_\alpha^\beta \int_a^b F(r, \theta) r \, dr \, d\theta$$

Given $F(r, \theta)$ is continuous on $\Omega : \alpha \le \theta \le \beta, \rho_1(\theta) \le r \le \rho_2(\theta)$:

$$\iint_\Omega F(r, \theta) r \, dr \, d\theta = \int_\alpha^\beta \int_{\rho_1(\theta)}^{\rho_2(\theta)} F(r, \theta) r \, dr \, d\theta$$

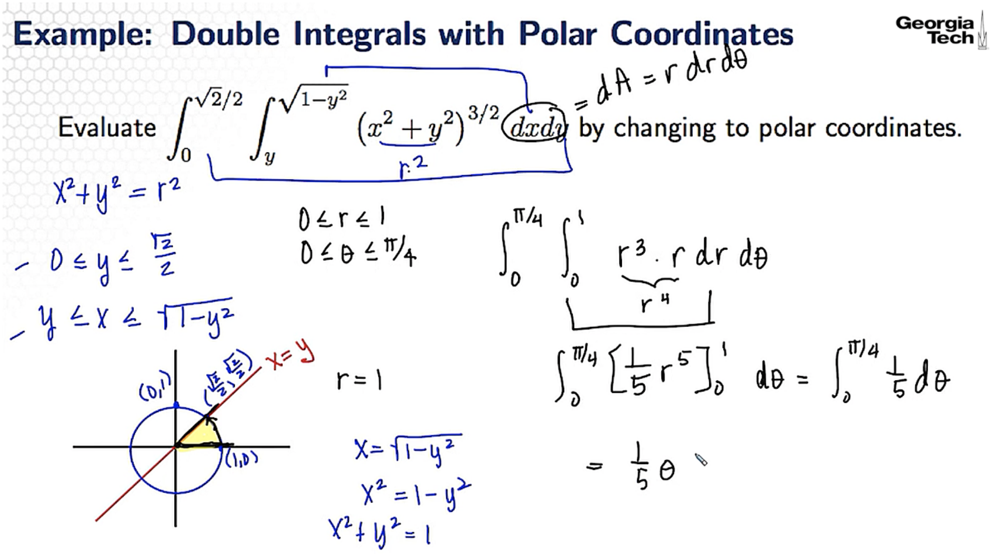
Fubini's Theorem applies here. (Bounds can be switched or rearranged as described in Fubini's.)

Volume

If $F(r, \theta) \ge 0$ over region R, then volume with R as base, bounded above by $F(r, \theta)$ is:

$$V = \iint_R F(r, \theta) r \, dr \, d\theta$$

(same as double integrals above)



#week8