14.5: Directional Derivatives and the Gradient

Directional Derivatives

$f'_{\vec{u}}(x_0, y_0)$ or $D_{\vec{u}}f(P_0)$ gives the **directional derivative** of f in the direction of $\vec{u}$ at the point $P_0 = (x_0, y_0)$.

- The rate of change of f in the $\vec{u}$ direction.

If $\vec{u} = u_1\mathbf{i} + u_2\mathbf{j}$, then

$$D_{\vec{u}}f(P_0) = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

provided that the limit exists.

Example

Example: Find the directional derivative of $f(x,y) = x^2 + xy^2$ at the point P(1,1) in the direction of $\mathbf{i} - \mathbf{j}$. $\|\vec{i} - \vec{j}\| = \sqrt{2}$ $\quad \vec{u} = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$

$$D_u f(1,1) = \lim_{s \to 0} \frac{f(1 + s/\sqrt{2}, 1 - s/\sqrt{2}) - f(1,1)}{s}$$

$$= \lim_{s \to 0} \frac{(1 + s/\sqrt{2})^2 + (1 + s/\sqrt{2})(1 - s/\sqrt{2})^2 - 2}{s} = \lim_{s \to 0} \frac{s/\sqrt{2} + s^3/2\sqrt{2}}{s}$$

$$= \lim_{s \to 0} \left(1/\sqrt{2} + s^2/2\sqrt{2} \right) = 1/\sqrt{2}$$

Gradients

Gradient of a function $f(x, y)$ is vector

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

You can imagine how to extend this into 3+ dimensions.

Properties

$$\nabla(f(\vec{x}) + g(\vec{x})) = \nabla f(\vec{x}) + \nabla g(\vec{x})$$
$$\nabla(\alpha f(\vec{x})) = \alpha \nabla f(\vec{x})$$
$$\nabla(f(\vec{x})g(\vec{x})) = f(\vec{x})\nabla g(\vec{x}) + \nabla f(\vec{x})g(\vec{x})$$

Directional Derivative

Directional derivative of f in the direction of $\vec{u}$ at point $P_0 = (x_0, y_0)$ can be written as:

$$f'_{\vec{u}}(P_0) = \nabla f(P_0) \cdot \hat{u} = \|\nabla f(P_0)\| \cos\theta$$

Properties

1. At P, function f increases most rapidly in the direction of its gradient vector.
2. Function f decreases most rapidly in the direction of $-\nabla f$.
3. Any direction $\vec{u}$ orthogonal to gradient $\nabla f \neq 0$ is a direction of zero change in f.

Examples

Find the gradient of $f(x,y) = 2e^x \sin(x^2 + y)$

$$\begin{aligned}\nabla f(x,y) = &(4xe^x \cos(x^2+y) + 2e^x \sin(x^2+y))\mathbf{i} \\ &+ 2e^x cos(x^2+y)\mathbf{j}\end{aligned}$$

Find a unit vector in the direction in which f increases most rapidly at P and give the rate of change of f in that direction.

$$f(x,y) = y^{-2}e^{2x} \text{ at } P(0,1)$$

$$\nabla f = \frac{2e^{2x}}{y^2}\vec{i} - \frac{2e^{2x}}{y^3}\vec{j}$$

$$\nabla f(0,1) = 2\vec{i} - 2\vec{j}$$

$$\|\nabla f(0,1)\| = \sqrt{8} = 2\sqrt{2}$$

$$\vec{u}_{\nabla f} = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$$

Tangent Lines to Level Curves

The tangent line to level curve $f(x,y) = c$ at point (x_0, y_0) is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0$$
$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}(x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}(y - y_0) = 0$$

(Can be derived from implicit differentiation rule)

Derivative Along a Path

if $\vec{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is a smooth path C and $w = f(\vec{r}(t))$ is a scalar function evaluated along C, then the derivative along that path is

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$$

14.6: Tangent Planes & Differentials

Tangent Planes & Normal Lines

Tangent plane to level surface $f(x, y, z) = c$ of a differentiable function f at point $P_0(x - 0, y_0, z_0)$ where the gradient is not zero is the plane through P_0 normal to $\nabla f(x_0, y_0, z_0)$.

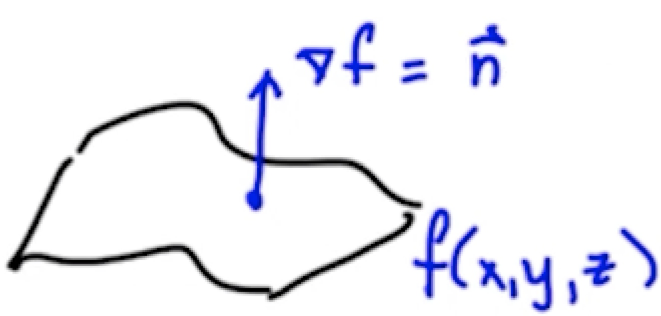
$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$
$$\nabla f(P_0) \cdot \overrightarrow{P_0 P} = 0$$

Normal line to level surface $f(x, y, z) = c$ is the line through P_0 parallel to $\nabla f(x_0, y_0, z_0)$.

$$x = x_0 + f_x(P_0)t$$
$$y = y_0 + f_y(P_0)t$$
$$z = z_0 + f_z(P_0)t$$

or

$$\vec{r}(t) = P_0 + t\nabla f(P_0)$$



Differentials

Linearization

The **linearization** of differentiable function $f(x, y)$ at (x_0, y_0) is:

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$
$$L(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

The approximation $f(x, y) \approx L(x, y)$ is called the **standard linear approximation** of f at the point.

The **total differential of** f is the resulting change from (x_0, y_0) to $(x_0 + dx, y_0, dy)$

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

Error in standard linear approximation when using L to approximate f:

$$|E| \leq \frac{1}{2}M(|x - x_0| + |y - y_0|)^2$$

M represents the upper bound of the second partials on the rectangle centered at P_0.

Extension of above formulas to more dimensions is trivial.

More Differentials

They also help in estimating change in a function in a particular direction.
To estimate the change in value of a differentiable function f when moving a small distance, ds, from point P_0 in the direction of the unit vector $\hat{u}$,

$$df = f'_{\hat{u}}(P_0)ds = (\nabla f(P_0) \cdot \hat{u})ds$$

14.7: Extreme Values

Local Extrema

Let $f(x, y)$ be defined on a region R containing point (a, b). Then:

1. $f(a, b)$ is a **local maximum** of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk around (a, b).
2. $f(a, b)$ is a **local minimum** of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk around (a, b).

First Derivative Test: If $f(x, y)$ has a local extremum at interior point (a, b), then $\nabla f(a, b) = \vec{0}$ (all the partial derivatives are 0).

- **Critical Points**: remember single variable calc?

Saddle point: Critical point that isn't a local extremum (some points are greater, some are less)

Second partials test (analogous to the 2nd derivative test):

- Used to determine if a critical point is a saddle point or a local min or max

$$\begin{aligned}
A &= f_{xx}(x_0, y_0) \\
B &= f_{xy}(x_0, y_0) \\
C &= f_{yy}(x_0, y_0) \\
D &= \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2
\end{aligned}$$

1. If $D < 0$, (x_0, y_0) is a saddle point.
2. If $D > 0$ and $A > 0$, local minimum.
3. If $D > 0$ and $A < 0$, local minimum.
4. If $D = 0$, test is inconclusive.

Absolute Extrema

Absolute maximum: Greatest value $f(x, y)$ for all $(x, y) \in D$
Absolute maximum: Smallest value $f(x, y)$ for all $(x, y) \in D$

Process for finding absolute extrema:

1. Find critical points in D.
2. Find extreme points on boundary of D.
3. Evaluate f at candidates.
4. Yeah.