Textbook sections: 6.1, 6.2

This lesson extends Systems of 2 FOLDEs.

Terminology

Matrix functions

Matrices who elements are functions:

$$P(t) = \begin{bmatrix} p_{11}(t) & \dots & p_{1n}(t) \\ \vdots & & \vdots \\ p_{n1}(t) & \dots & p_{nn}(t) \end{bmatrix}$$

Differentiation and integration are elementwise.

Expressing nth Order DE as Linear System

Extends 2nd Order Linear DEs as Linear Systems.

$$y^{(4)} + y = \sin(t)$$

Set the variables:

$$\begin{aligned} x_1 &= y \\ x_2 &= y' \\ x_3 &= y'' \\ x_4 &= y^{(3)} \end{aligned}$$

Then, construct the differential identities:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \end{aligned}$$

And the DE:

$$x_4' = -x_1 + \sin(t)$$

Altogether:

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin(t) \end{bmatrix}$$

Uniqueness of Solution to Linear System

This section extends Existence & Uniqueness of 1st Order Linear IVPs.
If P and $\vec{g}$ are continuous on (α, β), $t_0 \in (\alpha, \beta)$, there is a unique solution to the IVP

$$\vec{x}' = P(t)\vec{x} + \vec{g}(t), \quad \vec{x}(t_0) = \vec{x}_0$$

Proof goes beyond the scope of this course.

Example

Identify an interval on which a unique solution will exist for

$$(t-2)y'' + 3y = t, \quad y(0) = 0, y'(0) = 1$$

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{3}{t-2} & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{t}{t-2} \end{bmatrix}$$

So, there must be a unique solution for $(-\infty, 2)$.

Linear Independence of Scalar and Vector Functions

of Scalar Functions

This section extends The Wronskian and Fundamental Set of Solutions.
Let $y_1, y_2, \ldots, y_n$ be the solutions to $y^{(n)} + p_1 y^{(n-1)} + \ldots + p_n y = 0$.
If these solutions are linearly independent in given interval I, they form the **fundamental set of solutions** for this equation in interval I, and every solution is some linear combination of these solutions.

To determine if they are linearly dependent. Note that for 3 solutions:

$$c_1 y_1 + c_2 y_2 + c_3 y_3 = 0$$

However, this is 1 equation, 3 unknowns. We also know that...

$$\begin{gathered} c_1 y_1' + c_2 y_2' + c_3 y_3' = 0 \\ c_1 y_1'' + c_2 y_2'' + c_3 y_3'' = 0 \end{gathered}$$

This can be expressed as:

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The Wronskian, for n solutions is:

$$W[y_1, y_2, \dots, y_n] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

If this computes to 0 for a given interval I, then there must be a linear combination and therefore linear dependence in that interval.

Example

Determine whether the functions are linearly independent.

1. $y_1 = e^t, y_2 = e^{-2t}, y_3 = 3e^t - 2e^{-2t}$

$y_3 = 3y_1 - 2y_2$. So these are linearly dependent.

2. $y_1 = t, y_2 = t^2, y_3 = 1 - 2t^2$

$$\begin{bmatrix} t & t^2 & 1-2t^2 \\ 1 & 2t & -4t \\ 0 & 2 & -4 \end{bmatrix} \vec{c} = \vec{0}$$

If the determinant is 0, then the functions must be linearly dependent.

$$\det A = t(-8t + 8t) - 1(-4t^2 - 2 + 4t^2) = 2$$

# of Vector Functions

Let $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ be solutions to

$$\vec{x} = P(t)\vec{x} + \vec{g}(t)$$

on interval I. These are **the fundamental set of solutions** if the Wronskian ($W[\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n]$) is non-zero.

Example

$$\vec{y}_1 = \begin{bmatrix} t \\ 1-t \\ 0 \end{bmatrix} \quad \vec{y}_2 = \begin{bmatrix} 0 \\ 1 \\ t \end{bmatrix} \quad \vec{y}_3 = \begin{bmatrix} t \\ 3-t \\ 2t \end{bmatrix}$$

If these vectors are linearly independent then:

$$[\vec{y}_1 \quad \vec{y}_2 \quad \vec{y}_3]\vec{c} = \vec{0}$$

only if $\vec{c} = \vec{0}$.

($\det{[\vec{y}_1 \quad \vec{y}_2 \quad \vec{y}_3]} = 0$, you can figure out $\vec{c}$ by inspection.)