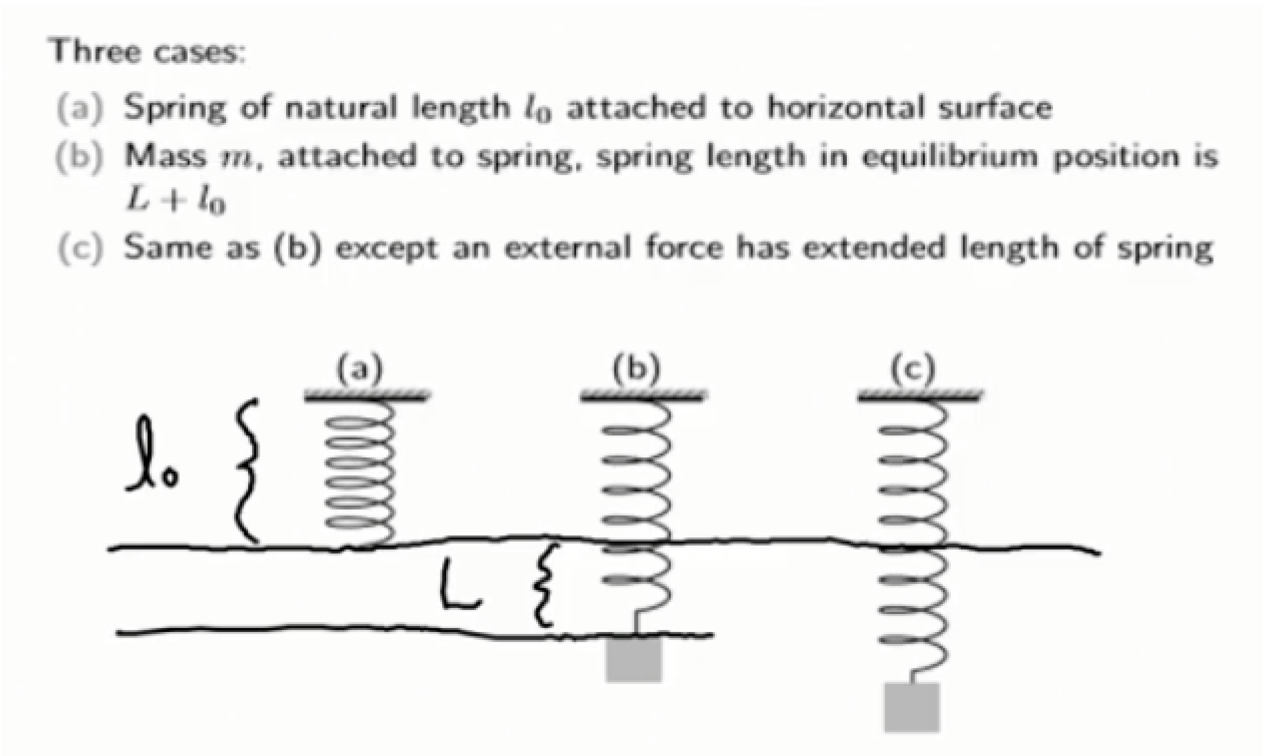
Textbook sections: 4.1, 4.4

Examples of 2nd Order LDEs (4.1)

Spring-Mass Systems

1. Spring of natural length l_0 attached to horizontal surface
2. Mass m, attached to spring, spring length in equilibrium position is $L + l_0$
3. Same as (2), but external force extended length of spring



For case (2),

$$
\begin{aligned}
F_g - F_s &= 0 \\
mg - kL &= 0
\end{aligned}
$$

For case (3),
if external force is $f(t)$, then

$$
\begin{aligned}
ma &= F_g - F_s + F \\
my'' &= mg - k(L + y) + F \\
my'' &= -ky + F
\end{aligned}
$$

Forces

forces	equation
gravitational	mg
spring	kL
external	$f(t)$
damping	$\gamma y'$

Mechanical Vibrations (4.4)

Undamped Motion

Consider a vertical spring with undamped motion:

$$mg - kx = ma$$

or

$$\begin{aligned} ma + kx &= 0 \\ mx'' + kx &= 0 \end{aligned}$$

Solve the equation, draw a component plot, and characterize the motion in terms of amplitude, frequency, and phase.

Solving

Then, $x = e^{\lambda t}$, where $m\lambda^2 + k = 0$. Then, $\lambda = \pm i\sqrt{\frac{k}{m}}$.

Thus, the complete solution is $x = c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$.

Characterization

Let $c_1 = R\cos\delta$, $c_2 = R\sin\delta$.
Then,

$$\begin{aligned} x &= c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right) \\ &= R\cos\delta\cos\left(\sqrt{\frac{k}{m}}t\right) + R\sin\delta\sin\left(\sqrt{\frac{k}{m}}t\right) \\ &= R\cos\left(\sqrt{\frac{k}{m}}t - \delta\right) \end{aligned}$$

R = amplitude
$\sqrt{\frac{k}{m}}$ = frequency

d = phase shift

Damped Motion

Adding damping to the equation:

$$my'' + \gamma y' + ky = 0$$

Determine an expression for the roots of the characteristic equation as a function of k, m, γ.

Characteristic equation is $m\lambda^2 + \gamma\lambda + k = 0$.
Then,

$$\lambda = -\frac{\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4mk}}{2m}$$

motion	**occurs when**	**motion**
no damping	$\gamma = 0$	underdamped, with $\alpha = 0$
underdamped	$\gamma - 4mk < 0$	$e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)$ where $\lambda = \alpha + i\beta$
critically damped	$\gamma - 4mk = 0$	$e^{\lambda t}, te^{\lambda t}$
overdamped	$\gamma - 4mk > 0$	$e^{\lambda_1 t}, e^{\lambda_2 t}$