Textbook sections: 4.5, 4.6

Nonhomogeneous SOLDEs (4.5)

We seek solutions to the *nonhomogeneous problem*:

$$y'' + p(t)y' + q(t)y = g(t).$$

This corresponds to the homogeneous problem:

$$y'' + p(t)y' + q(t)y = 0.$$

Suppose Y_1 and Y_2 are solutions to the nonhomo problem. Then,

$$\begin{aligned} Y_1'' + pY_1' + qY_1 = g \\ Y_2'' + pY_2' + qY_2 = g \end{aligned}$$

And therefore, $Y_2 - Y_1$ must be a solution to the homo problem.

General Solution

We recognize that

$$Y_2 - Y_1 = c_1 y_1 + c_2 y_2,$$

and therefore

$$\overbrace{Y_2}^{\text{general solution}} = \overbrace{c_1 y_1 + c_2 y_2}^{\text{homo solution}} + \overbrace{Y_1}^{\text{particular solution}}$$

See: [Particular solutions to nonhomogeneous equations].

Forced Vibrations (4.6)

Damped, forced systems

$$y'' + 3y' + 2y = \sin t$$

It has the solution:

$$y = \overbrace{c_1 e^{-2t} + c_2 e^{-t}}^{\text{homo/transient}} + \overbrace{0.1 \sin t - 0.3 \cos t}^{\text{particular/steady-state}}$$

Undamped oscillator

$y'' + \omega_0^2 y = F_0 \cos(\omega t)$ (where $y(0) = 0, y'(0) = 0, F_0 > 0$)

Solve DE for $\omega \neq \omega_0$ and $\omega = \omega_0$.

1. $\omega \neq \omega_0$

Homogeneous is: $y_h = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$.

Test a particular solution: $y_p = A \cos \omega t + B \sin \omega t$.
We find
$F_0 \cos(\omega t) = (\omega_0^2 - \omega^2)(A \cos \omega t + B \sin \omega t)$.
Then, $A = \frac{F_0}{\omega_0^2 - \omega^2}, B = 0$.

2. $\omega = \omega_0$

Test a particular solution: $y_p = At \cos(\omega t) + Bt \sin(\omega t)$.
We find
$y = c_1 \cos(\omega t) + c_2 \sin(\omega t) + \frac{F}{2\omega} t \sin(\omega t)$.

As $t \to \infty$, $|y| \to \infty$, this is **resonance**!