Textbook sections: 7.2

Almost Linear Systems

Almost linear systems are a method of creating linear approximations to non-linear systems.

> **Use-Case**
>
> For a given system
>
> $$\vec{x}' = A\vec{x} + \vec{g},$$
>
> if the system is almost linear near a given critical point, we can approximate the system as a linear system
>
> $$\vec{x}' \approx A\vec{x}$$
>
> around that critical point.
>
> By doing so, we can apply our [analysis of stability of linear systems] to understand the behavior of the non-linear system.

Isolated Critical Points (ICPs)

Note that almost linear systems can only be applied around an **isolated critical point**. A critical point is considered *isolated* if there is some circle around the given critical point such that there are no other critical points.
(i.e., there are no lines or curves of critical points)

We can describe a non-linear autonomous system as

$$\vec{x}' = A\vec{x} + \vec{g}.$$

(Note that $A\vec{x}$ is the linear part of $\vec{x}'$, $\vec{g}$ is the non-linear part of $\vec{x}'$.)

If $\vec{x}_0$ is an ICP of this system, it is **also** an ICP of $\vec{x}' = A\vec{x}$, assuming $\det A \neq 0$.

Definition of Almost Linear System

Suppose

$$\vec{x}' = A\vec{x} + \vec{g},$$

where A is a matrix with constant coefficients (and $\det A \neq 0$), and $\vec{g}$ is a non-linear function of $\vec{x}$.

We can consider the system almost linear near $\vec{x} = \vec{0}$ if:

- $\vec{x} = \vec{0}$ is an isolated critical point
- $\frac{\|\vec{g}\|}{\|\vec{x}\|} \to \vec{0}$, as $\vec{x} \to \vec{x}_0$. (i.e., $\vec{g}$ approaches $\vec{0}$ faster than $\vec{x}$ approaches $\vec{0}$.)

We can also extend this arbitrarily. A system is almost linear near $\vec{x} = \vec{x}_0$ if:

- $\vec{x} = \vec{x}_0$ is an isolated critical point
- $\frac{\|\vec{g}\|}{\|\vec{x}-\vec{x}_0\|} \to \vec{0}$, as $\vec{x} \to \vec{x}_0$. (i.e., $\vec{g}$ approaches $\vec{0}$ faster than $\vec{x}$ approaches $\vec{x}_0$.)

Almost Linear Systems using Jacobian

Consider a given autonomous system

$$\begin{aligned} x' &= F(x, y) \\ y' &= G(x, y), \end{aligned}$$

with an ICP at $\vec{x}_0 = (x_0, y_0)$.

Then, we can describe the system as

$$\vec{x}' = J(\vec{x} - \vec{x}_0) + \vec{\eta},$$

where:

- J is the **Jacobian matrix** $\begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}$, computed at $\vec{x} = \vec{x}_0$,
- $\vec{\eta}$ is a vector describing the quadratic (and beyond) terms for the Taylor expansions of F and G.

Near the ICP $\vec{x}_0$, this system can be approximated as

$$\vec{x}' \approx J(\vec{x} - \vec{x}_0).$$

(Assuming $\det J \neq 0$ at $\vec{x} = \vec{x}_0$.)

Derivation

We can find the Taylor expansion of F around (x_0, y_0):

$$F(x, y) = F(x_0, y_0) + F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0) + \eta_1,$$

where η_1 represents the quadratic (and beyond) terms of the Taylor expansion.

Since (x_0, y_0) is a critical point, $F(x_0, y_0) = 0$. We can then rewrite F's Taylor expansion as

$$F(x,y) = \begin{bmatrix} F_x & F_y \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \eta_1.$$

In a similar process, we can find the Taylor expansion of G to be

$$G(x,y) = \begin{bmatrix} G_x & G_y \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \eta_2.$$

By combining these, we find

$$\begin{aligned} \vec{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \vec{\eta} \\ &= J(\vec{x} - \vec{x}_0) + \vec{\eta} \end{aligned}$$

Proving Almost Linearity

Because $\vec{\eta}$ only consists of quadratic (and beyond) terms $\vec{\eta}$ will shrink quadratically towards 0 as $\vec{u}$ shrinks linearly towards 0, so we see that

$$\lim_{\vec{x} \to \vec{x}_0} \frac{\|\vec{\eta}\|}{\|\vec{x} - \vec{x}_0\|} = 0.$$

Example

Identify all critical points and classify the critical points (according to stability) of

$$\begin{aligned} \frac{dx}{dt} &= x + x^2 + y^2 \\ \frac{dy}{dt} &= y - xy. \end{aligned}$$

Critical Points

When $\frac{dx}{dt} = 0$,

$$\begin{aligned} x + x^2 + y^2 &= 0 \\ \left(x + \frac{1}{2}\right)^2 + y^2 &= \frac{1}{4}. \end{aligned}$$

When $\frac{dy}{dt} = 0$,

$$\begin{aligned} y - xy &= 0 \\ y(1 - x) &= 0. \end{aligned}$$

By drawing the graph and solving, we find the critical points $(-1, 0)$ and $(0, 0)$.

Classifying

We first find the Jacobian:

$$J = \begin{bmatrix} 1+2x & 2y \\ -y & 1-x \end{bmatrix}.$$

At $(-1, 0)$, the Jacobian is $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$.
The eigenvalues of this matrix are $\lambda = -1, 2$, making the system an unstable saddle.

At $(0, 0)$, the Jacobian is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
The eigenvalue of this matrix is $\lambda = 1$, making the system an unstable source.

Test for Almost Linear Systems

Consider a given autonomous system

$$\begin{aligned} x' &= F(x, y) \\ y' &= G(x, y), \end{aligned}$$

with an ICP at $\vec{x}_0 = (x_0, y_0)$.

If F and G are twice differentiable, the system is almost linear (because it can be Jacobian-ified).

Effect of Small Perturbations

Suppose

$$\vec{x}' = A\vec{x} + \vec{g},$$

where $\vec{x}_0$ is an ICP and $\vec{g} \neq 0$.

Because the linear system only *approximates* the linear system, the eigenvalues of the almost linear system only *approximate* the linear system's eigenvalues.

For most systems, this difference is negligible. However, there are two cases where it can result in a difference in behavior or stability: pure imaginary λ and real repeated λ.

Pure Imaginary λ

Ideally, if a linear system has $\lambda = \pm i\beta$, there is a center around the critical point.
However due to perturbations, the approximate $\lambda = \pm i\beta$ shifts to $\lambda = \alpha' \pm i\beta'$.

Here, $\alpha' \approx 0, \beta' \approx \beta$.
This can cause a change in stability AND behavior.

- If $\alpha' > 0$, then the almost linear system has an *unstable* spiral source.
- If $\alpha' = 0$, then the almost linear system has a *stable* center.
- If $\alpha' < 0$, then the almost linear system has an *asymptotically stable* spiral sink.

Real Repeated λ

Ideally, if a linear system has repeated eigenvalues $\lambda = \alpha$, then there is a proper or improper node. However due to perturbations, the approximate $\lambda = \alpha$ shifts to $\lambda = \alpha' \pm i\beta'$.

Here, $\alpha' \approx \alpha, \beta' \approx 0$.
This does not change the stability, but it does change the behavior.

Table of Changes

λ_1, λ_2	**linear system behavior**	**stability**	**almost linear system behavior**	**stability**
$\lambda_1 = \lambda_2 > 0$	proper/improper node	unstable	nodal or spiral source	unstable
$\lambda = \lambda_2 < 0$	proper/improper node	a. stable	nodal or spiral sink	a. stable
$\lambda_1, \lambda_2 = \pm i\beta$	center	stable	center, spiral source, or spiral sink	indeterminate

For systems with eigenvalues that don't match these two conditions, the linear system & almost linear systems maintain the same behavior. Those tables can be found here.