Textbook sections: 7.1

This extends autonomous equations & population dynamics, focusing on autonomous non-linear systems.

Autonomous Systems

An **autonomous system** is a system

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y)$$

(Not connected to t).

This can be non-linear.

Critical Points

Critical points occur when...

$$\frac{dx}{dt} = 0, \text{ and } \frac{dy}{dt} = 0$$

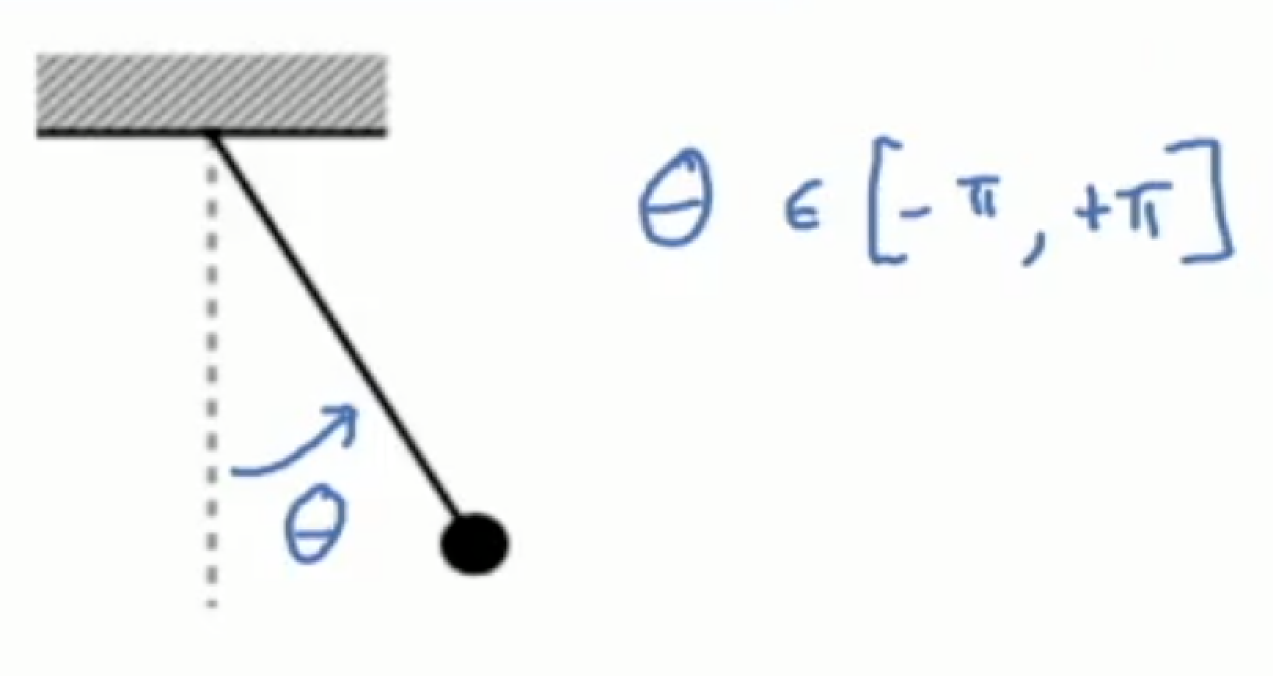
Let $\vec{x}_0$ be a given critical point, and let $\vec{x}(t)$ be a trajectory that starts sufficiently close to $\vec{x}_0$. Then, $\vec{x}_0$ can be classified as one of the following.

classification	intepretation
stable	$\|\vec{x} - \vec{x}_0\|$ is bounded
asymptotically stable	stable and $\vec{x} \to \vec{x}_0$
unstable	not bounded

Example: Pendulums

Under a gravitational force, the angle $\theta(t)$ makes with the vertical axis can be described with

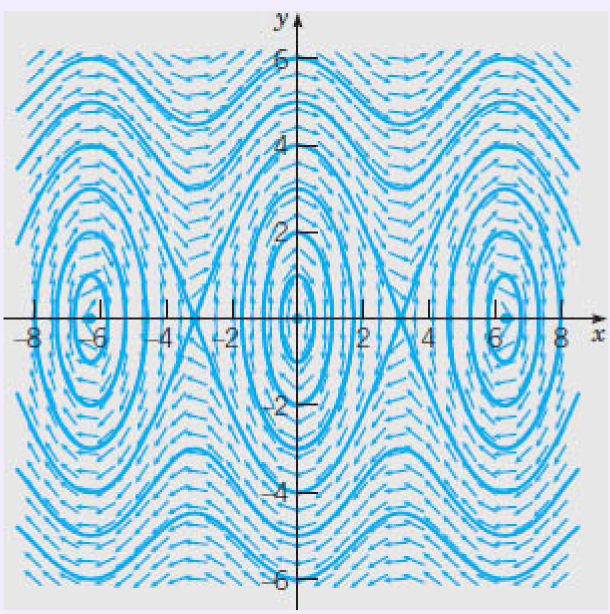
$$\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega^2 \sin\theta = 0$$



Sketching

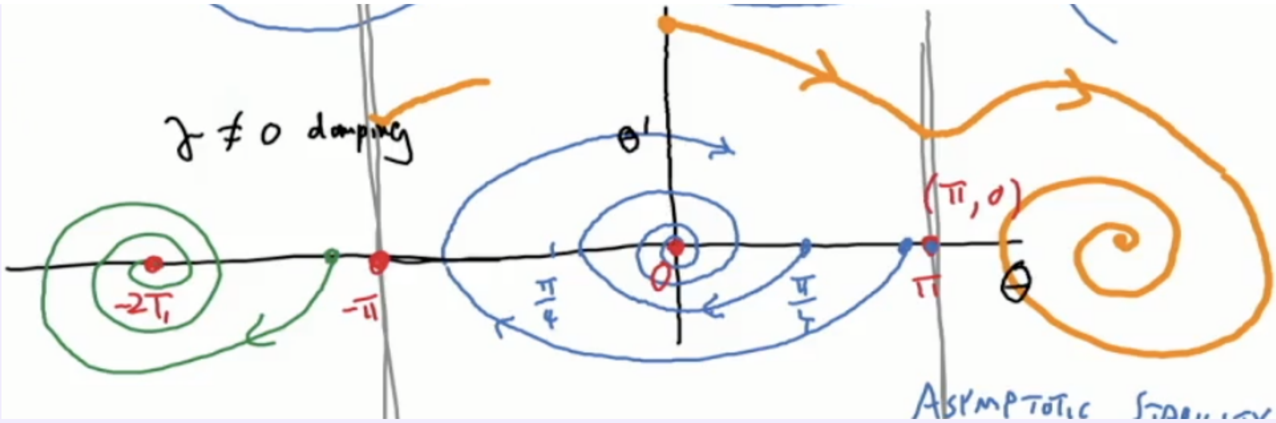
Sketch the phase portrait θ' vs. θ for a few different initial conditions. Where are the critical points located?

$\gamma = 0$, no damping



At $(0, 0)$, there is a stable equilibrium. At $(\pm k\pi, 0)$, the curve oscillates through $(+)$ and $(-)$.

$\gamma \neq 0$, damping



At $(\pm 2k\pi, 0)$, there are asympotically stable equilibria. At $(\pm 2k\pi + \pi, 0)$, there are unstable(?) equilibria.

Nullclines

An **x-nullcline** is a line along which $\frac{dx}{dt} = 0$.
A **y-nullcline** is a line along which $\frac{dy}{dt} = 0$.
A **nullcline** is a line along which $x' = y' = 0$.

Example

Identify all the critical points of the autonomous system

$$\frac{dx}{dt} = 2x - x^2 - xy$$
$$\frac{dy}{dt} = 3y - 2y^2 - 3xy.$$

x-nullclines

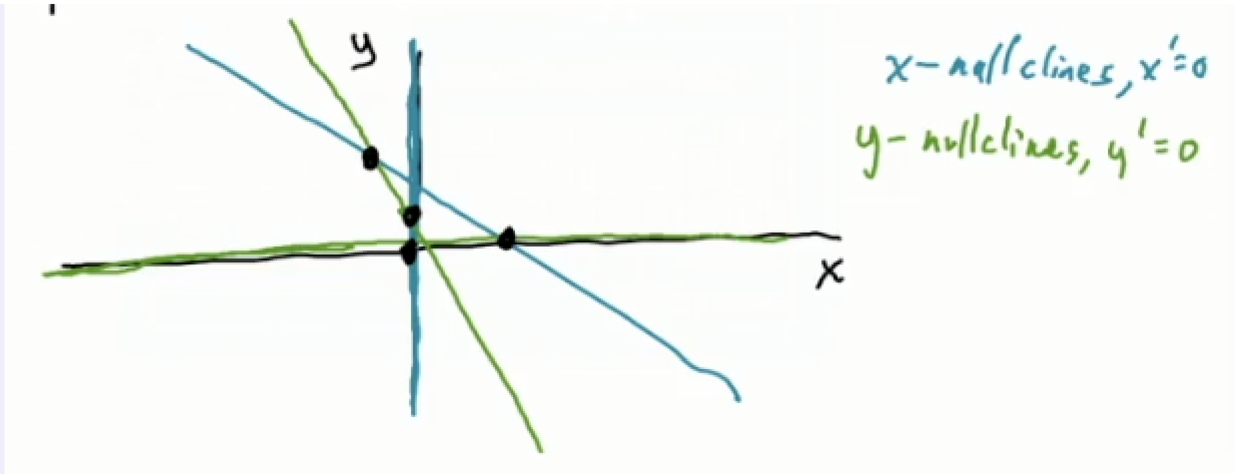
$x' = 0$ when $2x - x^2 - xy = x(2 - x - y) = 0$.
Hence, $x' = 0$ when $x = 0$ or $y = 2 - x$.

y-nullclines

$y' = 0$ when $3y - 2y^2 - 3xy = y(3 - 2y - 3x) = 0$.
Hence, $y' = 0$ when $y = 0$ or $y = \frac{3}{2}x - \frac{3}{2}$.

Points

The points occur where a x-nullcline intersects a y-nullcline.


y
x
x-nullclines, x'=0
y-nullclines, y'=0