Textbook sections: 5.8

ⓘ **Motivating Example**

Consider the spring-mass system

$$y'' + \omega^2 y = g(t), \quad y(0) = 0, \quad y'(0) = 0,$$

such that $w^2 = \frac{k}{m}$.

If $g(t)$ is unknown, can we express $y(t)$ in terms of $g(t)$? How?

1. Variation of Parameters
2. Convolutions and $\mathcal{L}$

Using Convolutions & $\mathcal{L}$

Take $\mathcal{L}$ of both sides of our DE:

$$\begin{aligned} \mathcal{L}\{y'' + \omega^2 y\} &= \mathcal{L}\{g\} \\ s^2 Y - sy(0) - y'(0) + \omega^2 Y &= G \\ (s^2 + \omega^2) Y &= G \\ Y &= \frac{1}{s^2 + \omega^2} G \\ &= \frac{1}{\omega} \frac{\omega}{s^2 + \omega^2} G. \end{aligned}$$

This could be computed if we had a way to compute

$$\mathcal{L}^{-1}\{F(s)G(s)\}$$

for arbitrary F and G.

Convolutions

The **convolution** between piecewise continuous functions f and g is

$$f * g = \int_0^t f(t - \tau) g(\tau)\, d\tau.$$

Some useful properties:

property	identity
commutative	$f * g = g * f$
distributive	$f * (g_1 + g_2) = f * g_1 + f * g_2$
associative	$(f * g) * h = f * (g * h)$
zero	$f * 0 = 0 * f = 0$

Convolution Theorem

Let f and g be functions such that $\mathcal{L}\{f\} = F$ and $\mathcal{L}\{g\} = G$. Then:

$$\mathcal{L}\{f * g\} = F(s)G(s).$$

Derivation

Note that

$$\begin{aligned} F(s) &= \int_0^\infty e^{-s\xi} f(\xi)\, d\xi \\ G(s) &= \int_0^\infty e^{-s\tau} g(\tau)\, d\tau. \end{aligned}$$

Then,

$$F(s)G(s) = \int_0^\infty e^{-s\xi} f(\xi)\, d\xi \int_0^\infty e^{-s\tau} g(\tau)\, d\tau.$$

Since these integrals are not dependent on each other, we can merge them into a double integral:

$$\begin{aligned} F(s)G(s) &= \int_0^\infty e^{-s\tau} g(\tau) \left(\int_0^\infty e^{-s\xi} f(\xi)\, d\xi \right) d\tau \\ &= \int_0^\infty g(\tau) \left(\int_0^\infty e^{-s(\xi+\tau)} f(\xi)\, d\xi \right) d\tau. \end{aligned}$$

Let $t = \xi + \tau$ for select τ. Then, $dt = d\xi$.
When $\xi \to 0,\ t \to \tau$, and when $\xi \to \infty,\ t \to \infty$.

$$F(s)G(s) = \int_0^\infty g(\tau) \left(\int_\tau^\infty e^{-st} f(t - \tau)\, dt \right) d\tau.$$

We can use change of variable to find that

$$\int_0^\infty \int_\tau^\infty (\ldots)\, dt\, d\tau \to \int_0^\infty \int_0^t (\ldots)\, d\tau\, dt.$$

Hence,

$$
\begin{aligned}
F(s)G(s) &= \int_0^\infty e^{-st}\left(\int_\tau^\infty f(t-\tau)g(\tau)\,d\tau\right)dt \\
&= \mathcal{L}\{f*g\}.
\end{aligned}
$$

Continuing the Motivating Example

$$Y = \frac{1}{\omega}\frac{\omega}{s^2+\omega^2}G.$$

Using the Convolution Theorem, we see that

$$
\begin{aligned}
Y &= \frac{1}{\omega}\mathcal{L}\{\sin\omega t\}\cdot\mathcal{L}\{g(t)\} \\
&= \frac{1}{\omega}\mathcal{L}\{\sin\omega t * g(t)\}.
\end{aligned}
$$

Therefore,

$$y = \frac{1}{\omega}\int_0^t \sin(\omega(t-\tau))g(\tau)\,d\tau.$$

Example

Compute the inverse Laplace Transform of the function:

$$\frac{1}{(s^2+1)^2}$$

We see that

$$
\begin{aligned}
\frac{1}{(s+1)^2} &= \frac{1}{s^2+1}\cdot\frac{1}{s^2+1} \\
&= \mathcal{L}\{\sin t\}\cdot\mathcal{L}\{\sin t\} \\
&= \mathcal{L}\{\sin t * \sin t\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} &= \sin t * \sin t \\
&= \int_0^t \sin(t-\tau)\sin\tau\,d\tau
\end{aligned}
$$

Example

Compute the inverse Laplace Transform of the function:

$$\frac{14}{(s+2)(s-6)}$$

$$\begin{aligned}
\mathcal{L}\left\{\frac{14}{(s+2)(s-6)}\right\} &= 14\int_0^t e^{-2(t-\tau)}e^{6\tau}\,d\tau \\
&= 14e^{-2t}\int_0^t e^{8\tau}\,d\tau \\
&= 14e^{-2t}\cdot\frac{1}{8}(e^{8t}-1)
\end{aligned}$$