**Textbook sections**: 3.3

**Motivating Example**

Consider $\vec{x}' = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$ ($\lambda_1 \to \vec{v}_1$, $\lambda_2 \to \vec{v}_2$).

1. Write down a solution to the system, assuming $\lambda_1$ and $\lambda_2$ are real and distinct.

$$\vec{x}_0 = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

2. For some $t_0 \in \mathbb{R}$, suppose $\vec{x}(t_0) = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Construct a matrix equation that can be used to solve for $c_1, c_2$.

$$\begin{aligned} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} &= e^{\lambda_1 t_0}\vec{v}_1 + e^{\lambda_2 t_0}\vec{v}_2 \\ &= \begin{bmatrix} e^{\lambda_1 t_0}\vec{v}_1 & e^{\lambda_2 t_0}\vec{v}_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \end{aligned}$$

3. Is matrix in (2) invertible?

   If the matrix is invertible, then there's a unique solution for $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

   The matrix is only invertible if the matrix's determinant (the Wronskian) is zero.

$$\begin{vmatrix} e^{\lambda_1 t_0}\vec{v}_1 & e^{\lambda_2 t_0}\vec{v}_2 \end{vmatrix} = e^{\lambda_1 t_0} e^{\lambda_2 t_0} \begin{vmatrix} \vec{v}_1 & \vec{v}_2 \end{vmatrix}$$

   Since $\vec{v}_1$ and $\vec{v}_2$ are linearly independent and the coefficients are positive, the det cannot be non-zero.

# The Wronskian & Linear Independence

Given vector functions $\vec{x}_1(t)$ and $\vec{x}_2(t)$, the **Wronskian** of $\vec{x}_1$ and $\vec{x}_2$ is:

$$W[\vec{x}_1, \vec{x}_2](t) = \begin{vmatrix} \vec{x}_1 & \vec{x}_2 \end{vmatrix} = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}$$

If $W[\vec{x}_1, \vec{x}_2](t) \neq 0$ on interval of $t$, $\vec{x}_1$ and $\vec{x}_2$ are linearly independent everywhere on that interval.

Vectors are linearly independent if there is no linear combination that results in $\vec{0}$ except for $\vec{0}$.

- e.g $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ are linearly dependent because $\begin{bmatrix} 5 & 3 & -1 \end{bmatrix}$.

# Fundamental Set of Solutions

The fundamental set is $\vec{x}_1$ and $\vec{x}_2$, the two linearly independent solutions to $\vec{x}' = A\vec{x}$ where $A \in \mathbb{R}^{2\times 2}$.