Textbook sections: 5.5

Motivating Example

Suppose that at time $t = 0$, we place a pie into an oven whose temperature, $y(t)$, is 20° C.

- When $t = 0$, the pie and oven are both 20° C.
- $y(t)$ increases linearly at rate 50°C/min until $t = 4$.
- For $t \geq 4 \text{ min}, y = 220$.

Construct an IVP that represents this situation. How can we solve it?

Construction

Let $\Pi(t)$ be the temperature of the pie. Then, from Newton's Law of Cooling,

$$\Pi' = -k(\Pi - y).$$

However, y is not constant, and is defined as:

$$y = \begin{cases} 20 + 50t, & 0 \leq t < 4 \\ 220, & t \geq 4 \end{cases}$$

Step and Indicator Functions

The **unit step function** is defined as:

$$u_c(t) = \begin{cases} 0, & 0 \leq t < c \\ 1, & c \leq t \end{cases}$$

(The unit step function starts at 0, and increments to 1 at $t = c$.)

The **indicator function** is defined as:

$$u_{bc}(t) = u_b(t) - u_c(t) = \begin{cases} 0, & 0 \leq t < b \\ 1, & b \leq t < c \\ 0, & c \leq t \end{cases}$$

(The indicator function starts at 0, increments to 1 at $t = b$, and drops to 0 at $t = c$.)

Converting Piecewise Functions

Example

Express the following function in terms of step functions.

$$f(t) = \begin{cases} 2, & 0 \le t < 3 \\ -2, & 3 \le t \end{cases}$$

In step functions,

$$\begin{aligned} f(t) &= 2u_{03} - 2u_3 \\ &= 2(u_0 - u_3) - 2u_3 \\ &= 2u_0 - 4u_3. \end{aligned}$$

Example

Express the following function in terms of step functions.

$$g(t) = \begin{cases} t, & 0 \le t < 2 \\ t^2, & 2 \le t < 4 \\ t^3, & 4 \le t \end{cases}$$

In step functions, this can be written as

$$\begin{aligned} g(t) &= tu_{02} + t^2u_{24} + t^3u_4 \\ &= tu_0 - tu_2 + t^2u_2 - t^2u_4 + t^3u_4 \\ &= tu_0 + (t^2 - t)u_2 + (t^3 - t^2)u_4. \end{aligned}$$

Laplace of Step Function

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}.$$

Derivation

$$\begin{aligned} \mathcal{L}\{u_c(t)\} &= \int_c^\infty e^{-st}\, dt \\ &= -\frac{1}{s}e^{-st}\Big|_{t=c}^{t=\infty} \\ &= \frac{1}{s}e^{-sc} \end{aligned}$$

Example

Solve the IVP:

$$y' + y = f, \quad y(0) = 0, \quad f = \begin{cases} 1, & 0 \le t < 1 \\ -1, & 1 \le t \end{cases}$$

Note that

$$\begin{aligned} f &= u_{01} - u_1 \\ &= (u_0 - u_1) - u_1 \\ &= u_0 - 2u_1. \end{aligned}$$

Then,

$$\begin{aligned} \mathcal{L}\{y' + y\} &= sY + y(0) + Y \\ &= (s+1)Y. \\ \mathcal{L}\{f\} &= \frac{1}{s} - \frac{2e^{-s}}{s}. \end{aligned}$$

Setting them equal, we find

$$Y = \frac{1}{s(s+1)} - \frac{2e^{-s}}{s(s+1)}.$$

Note that

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}.$$

Then,

$$\begin{aligned} Y &= \frac{1}{s} - \frac{1}{s+1} - 2e^{-s}\left(\frac{1}{s} - \frac{1}{s+1}\right) \\ &= \frac{1}{s} - \frac{1}{s+1} - 2\left(\frac{e^{-s}}{s} - \frac{e \cdot e^{-s-1}}{s+1}\right) \\ &= \mathcal{L}\{1 - e^{-t} - 2u_1(t) + 2e^{-t}u_1(t)\}. \end{aligned}$$

Therefore,

$$y = 1 - e^{-t} - 2u_1(t) + 2ee^{-t}u_1(t).$$

Shift in t-Domain

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$$

Derivation

$$\mathcal{L}\{u_c(t)f(t-c)\} = \int_c^\infty f(t-c)e^{-st}\,dt.$$

Let $G = t - c, dG = dt$.
When $t \to c,\ G \to 0$, and when $t \to \infty,\ G \to \infty$.

Then,

$$\begin{aligned}\mathcal{L}\{u_c(t)f(t-c)\} &= \int_0^\infty f(G)e^{-s(G+c)}\,dG \\ &= e^{-sc}\mathcal{L}\{f\}.\end{aligned}$$

Example

Compute

$$\mathcal{L}^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\}.$$

Note that $c=2$ and $\mathcal{L}\{e^{4t}\}=\frac{1}{s-4}$. Then,

$$\mathcal{L}^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\} = u_2 e^{4(t-2)}.$$

Periodic Functions

Suppose $f(t)$ is a periodic function with period T. Then,

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)\,dt$$

Derivation

$$\begin{aligned}\mathcal{L}\{f\} &= \int_0^\infty e^{-st}f\,dt \\ &= \int_0^T e^{-st}f\,dt + \int_T^\infty e^{-st}f\,dt.\end{aligned}$$

Let $u=t-T,\ du=dt$.
When $t\to T,\ u\to 0$, and when $t\to\infty,\ u\to\infty$. Then,

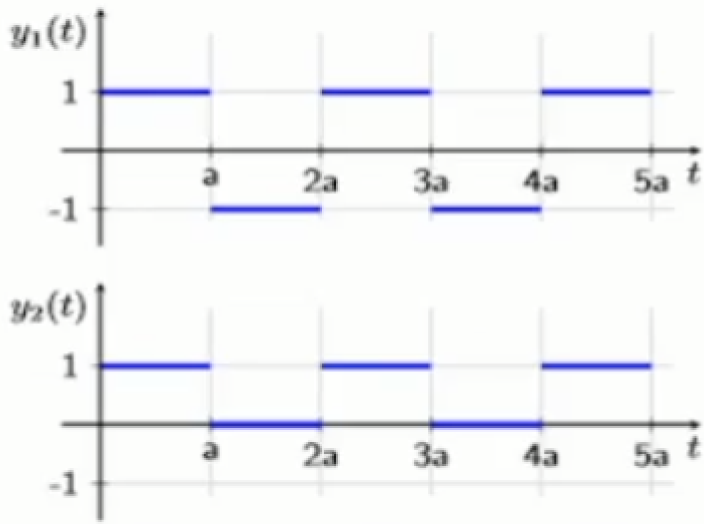
$$\begin{aligned}\mathcal{L}\{f\} &= \int_0^T e^{-st}f\,dt + \int_0^\infty e^{-s(u+T)}\overbrace{f(u+T)}^{f(u)}\,du \\ &= \int_0^T e^{-st}f\,dt + e^{-Ts}\int_0^\infty e^{-su}f\,du.\end{aligned}$$

Then,

$$
\begin{aligned}
\int_0^\infty e^{-st} f\,dt &= \int_0^T e^{-st} f\,dt + e^{-Ts}\int_0^\infty e^{-su} f\,du \\
(1-e^{-Ts})\int_0^\infty e^{-st} f\,dt &= \int_0^T e^{-st} f\,dt \\
\int_0^\infty e^{-st} f\,dt &= \frac{1}{1-e^{-Ts}}\int_0^T e^{-st} f\,dt.
\end{aligned}
$$

Examples

Compute the Laplace Transform of the following periodic functions.



Example: y_1

This function repeats every $2a$. As such,

$$
\begin{aligned}
\mathcal{L}\{y_1\} &= \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt \\
&= \frac{1}{1-e^{-2as}}\left(\int_0^a e^{-st}\,dt - \int_a^{2a} e^{-st}\,dt\right) \\
&= \frac{1}{1-e^{-2as}}\left(-\frac{1}{s}(e^{-as}-1) + \frac{1}{s}(e^{-2as}-e^{-as})\right)
\end{aligned}
$$