**Textbook sections**: 3.3, 3.4, 3.5

# Equilibrium Points

For an equilibrium point $P = (x, y)$, it can be categorized as the following:

| classification | intepretation |
|---|---|
| stable | points close to $P$ stay close |
| asymptotically stable | points close to $P$ stay close & converge to $P$ as $t \to \infty$ |
| unstable | points close to $P$ diverge |

(semi-stable is only applicable for autonomous equations)

See also: week 10 equilibrium points.

# Plotting via Wolfram Alpha

Given an equation

$$\vec{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x},$$

Wolfram Alpha can plot the phase portrait with...

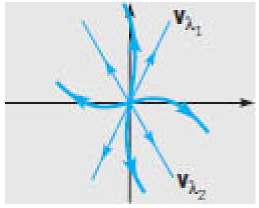
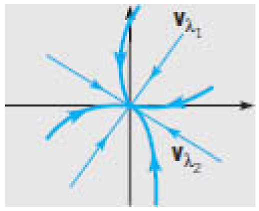
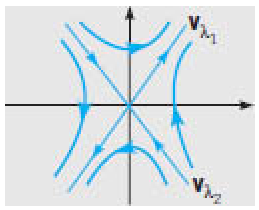
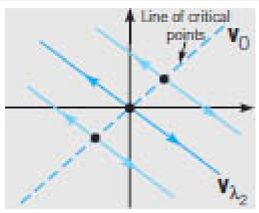
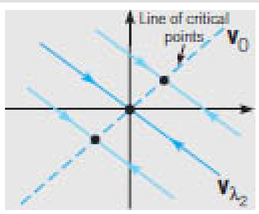
```
streamplot[{ax + by, cx + dy}, {x, -1, 1}, {y, -1, 1}]
```

# Phase Portraits

Phase portraits are $x_1$-$x_2$ graphs that indicate which points a system of DEs tend towards. The general structure of the phase portrait depends on the matrix's eigenvalues:

## Distinct Real Eigenvalues

See: Systems of 2 FOLDEs

| Eigenvalues | Sample Phase Portrait | Type of Critical Point | Stability |
|---|---|---|---|
| $\lambda_1, \lambda_2 > 0$ and $\lambda_1 \neq \lambda_2$ |   | $(0,0)$ is a *nodal source*. | unstable |
| $\lambda_1, \lambda_2 < 0$ and $\lambda_1 \neq \lambda_2$ |   | $(0,0)$ is a *nodal sink*. | asymptotically stable |
| $\lambda_1, \lambda_2$ opposite signs and $\lambda_1 \neq \lambda_2$ |   | $(0,0)$ is a *saddle*. | unstable |
| $\lambda_1 = 0,\, \lambda_2 > 0$ |   | | |
| $\lambda_1 = 0,\, \lambda_2 < 0$ |   | | |

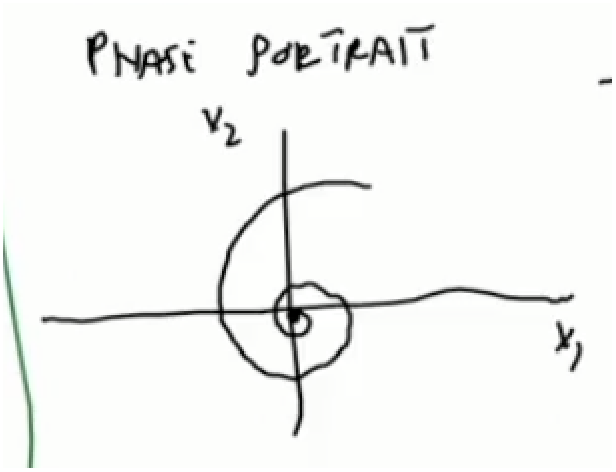
The curves for the nodal sink & source:

- Let $\lambda_1 > \lambda_2 > 0$.
- Farther from the origin, the curves are parallel to $\vec{v}_1$
- Closer to the origin, the curves are parallel to $\vec{v}_2$

# Complex Eigenvalues

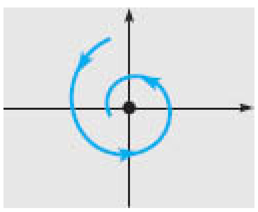
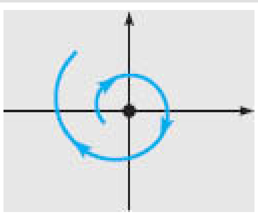
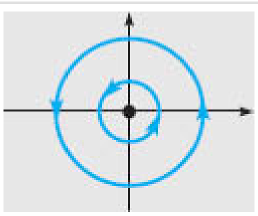
See: Complex Eigenvalues

Instead of directing in or out of a critical point, complex eigenvalues perform a spiral motion in or out:




To determine if that spiral is clockwise or counterclockwise, test a point (e.g. $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$) and see where the rotation points.

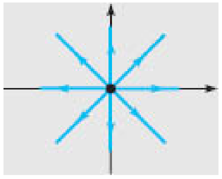
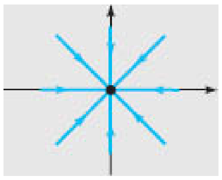
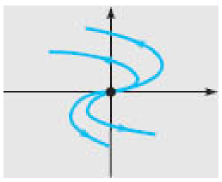
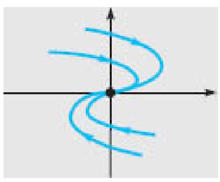
Let $\lambda = \alpha + i\beta$:

| Eigenvalues | Sample phase portrait | Type of critical point | stability |
|---|---|---|---|
| $\alpha < 0$ |  | (0, 0) is a *spiral sink* | asymptotically stable |
| $\alpha > 0$ |  | (0, 0) is a *spiral source* | unstable |
| a = 0 |  | (0, 0) is a *center* | stable |

## Repeated Eigenvalues

See: [Repeated Eigenvalues]

If there is a repeated eigenvalue, but two eigenvectors, the lines point towards/away the critical point like a star.
If there is one eigenvector, there are curves going into the eigenvector.

| Eigenvalues | Sample phase portrait | Type of critical point | stability |
|---|---|---|---|
| $A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, $\lambda > 0$ |  | (0, 0) is an *unstable proper node* (or *unstable star node*) | unstable |
| $A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, $\lambda < 0$ |  | (0, 0) is a *stable proper node* (or *stable star node*) | asymptotically stable |
| $A$ is not diagonal, $\lambda > 0$ |  | (0, 0) is an *unstable improper node* (or *unstable degenerate node*) | unstable |
| $A$ is not diagonal, $\lambda < 0$ |  | (0, 0) is a *stable improper node* (or *stable degenerate node*) | asymptotically stable |