Textbook sections: 2.3

Water Tank Problem

Tank initially contains 40 lbs of salt dissolved in 600 gals of water. Starting at time $t = 0$, water that contains 1/2 lb salt per gal is poured into the tank at 4 gal/min and the mixture is drained from the tank at the same rate.

1. **Construct a differential equation for $Q(t)$, the lb of salt in tank at time $t \geq 0$.**

$$\frac{dQ}{dt} = \overbrace{\left(\frac{1 \text{ salt}}{2 \text{ gal}} \cdot \frac{4 \text{ gal}}{\text{min}}\right)}^{\text{rate in}} - \overbrace{\left(\frac{Q \text{ salt}}{600 \text{ gal}} \cdot \frac{4 \text{ gal}}{\text{min}}\right)}^{\text{rate out}} = 2 - \frac{Q}{150}$$

2. **Determine expression for $Q(t)$.**

$$Q' + \frac{1}{150}Q = 2$$

Integrating factor: $\mu = e^{\frac{1}{150}t}$:

$$\begin{aligned} e^{\frac{1}{150}t}Q' + \frac{1}{150}e^{\frac{1}{150}t}Q &= 2e^{\frac{1}{150}t} \\ \frac{d}{dt}(e^{\frac{1}{150}t}Q) &= 2e^{\frac{1}{150}t} \\ e^{\frac{1}{150}t}Q &= \int 2e^{\frac{1}{150}t}dt \\ e^{\frac{1}{150}t}Q &= 300e^{\frac{1}{150}t} + C \end{aligned}$$

Substituting `t = 0, Q = 40`:

$$\begin{aligned} 40 &= 300 + C \\ C &= -260 \end{aligned}$$

Then,

$$\begin{aligned} e^{\frac{1}{150}t}Q &= 300e^{\frac{1}{150}t} - 260 \\ Q &= 300 - \frac{260}{e^{\frac{1}{150}t}} \end{aligned}$$

3. What happens to the concentration of salt in the tank after a long period of time?

$$\lim_{t \to \infty} Q = 300$$