Textbook sections: 3.2, 3.3
The term "FOLDE" in this text refers to "first-order linear differential equations."

Linear Systems of FOLDEs

Motivating Example

Consider population of two species (predator and prey).

Suppose:

- $x_1(t)$ is the number of foxes on the island
- $x_2(t)$ is the number of rabbits on the island
- k foxes are removed from the island per day ($k > 0$)
  Assume x_1 and x_2 are continuous and differentiable.

This can be represented as:

$$\begin{aligned}\frac{dx_1}{dt} &= ax_1 + bx_2 - k \\ \frac{dx_2}{dt} &= cx_1 + dx_2\end{aligned}$$

This can be represented as a matrix equation:

$$\frac{d\vec{x}(t)}{dt} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -k \\ 0 \end{bmatrix}.$$

Generally can be written as

$$\frac{d\vec{x}}{dt} = P(t)\vec{x} + \vec{g}(t)$$

Our example system has:

- dimension 2
- is non-homogeneous because $\vec{g}(t) \neq \vec{0}$.

Solutions

A **solution** to a system is a set of functions $\vec{x}(t)$ that satisfy the system.
The solutions to a **homogeneous** 2D system, where the eigenvalues are real and distinct:

$$\vec{u}_1(t) = e^{\lambda_1 t}\vec{v}_1$$
$$\vec{u}_2(t) = e^{\lambda_2 t}\vec{v}_2.$$

The general solution is the linear combination of these (principle of superposition):

$$\vec{x} = c_1\vec{u}_1 + c_2\vec{u}_2.$$

Example

Solve:

$$\frac{dx_1(t)}{dt} = 20x_1(t)$$
$$\frac{dx_2(t)}{dt} = -10x_1(t) + 30x_2(t)$$

Solving

This can be written as a system:

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 20 & 0 \\ -10 & 30 \end{bmatrix}\vec{x}$$

Computing eigenvalues:

$$\begin{bmatrix} 20-\lambda & 0 \\ -10 & 30-\lambda \end{bmatrix}$$

Case: λ_1 = 20

We get the matrix $\begin{bmatrix} 0 & 0 \\ -10 & 10 \end{bmatrix}$.
We can then find $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The solution is then: $\vec{x}(t) = \begin{bmatrix} e^{20t} \\ e^{20t} \end{bmatrix}$.

Case: λ_2 = 30

We get the matrix $\begin{bmatrix} -10 & 0 \\ -10 & 0 \end{bmatrix}$.
We can then find $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The solution is then: $\vec{x}(t) = \begin{bmatrix} 0 \\ e^{30t} \end{bmatrix}$.

ⓘ **Derivation of Solutions**

Proposition: Given $\vec{x} = e^{\lambda t}\vec{v}$, we want to show $\vec{x}'(t) = P\vec{x}$.

Since $\vec{v}$ is a constant vector:

$$\begin{aligned} \vec{x}'(t) &= \frac{d}{dt}(e^{\lambda t}\vec{v}) \\ &= \lambda e^{\lambda t}\vec{v}. \end{aligned}$$

We also see that

$$\begin{aligned} P\vec{x} &= P(e^{\lambda t}\vec{v}) \\ &= e^{\lambda t}(P\vec{v}) \\ &= \lambda e^{\lambda t}\vec{v}. \end{aligned}$$

Therefore, $\vec{x}' = P\vec{x} = \lambda e^{\lambda t}\vec{v}$.

Critical Points

The equilibrium solutions are the points where $\frac{d\vec{x}}{dt} = \vec{0}$.

For our example above,

$$\begin{aligned} \frac{d\vec{x}}{dt} &= \vec{0} \\ \begin{bmatrix} 20 & 0 \\ -10 & 30 \end{bmatrix}\vec{x} &= \vec{0} \end{aligned}$$

Matrix P is invertible, so $\vec{x}$ has to be the trivial solution $\vec{0}$.

2nd Order Linear DEs as Linear Systems

Second-order linear DEs can always be converted into a system of FOLDEs.

ⓘ **Motivating Example**

$$y'' - \sin(t)y' + 7y = e^t \cos t + 1.$$

Let $x_1 = y$, $x_2 = y'$.
Then:

$$x_1' = y' = x_2$$

$$\begin{aligned} x_2' = y'' &= e^t \cos t + 1 + \sin(t)y' - 7y \\ &= e^t \cos t + 1 + \sin(t)x_2 - 7x_1 \end{aligned}$$

In matrix form:

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -7 & \sin t \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ e^t \cos t \end{bmatrix}$$

Component Plots

A **component plot** of a system of FOLDEs describes a solution (by graphing x_1 and x_2 as functions of t).

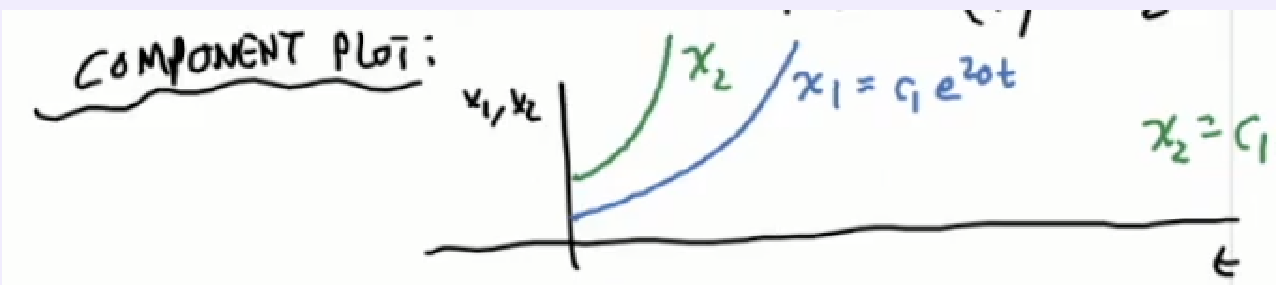
☰ Example

Using the original example:

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 20 & 0 \\ -10 & 30 \end{bmatrix} \vec{x},$$

a general solution is

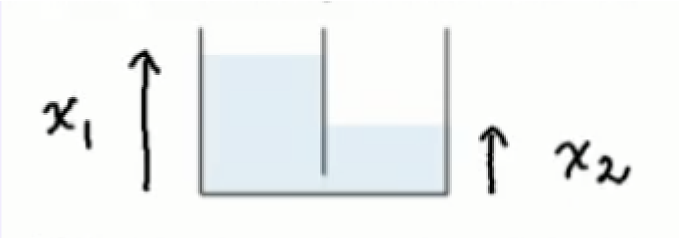
$$\begin{aligned} \vec{u}(t) &= c_1 \vec{u}_1 + c_2 \vec{u}_2 \\ &= c_1 e^{20t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{30t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} c_1 e^{20t} \\ c_1 e^{20t} + c_2 e^{30t} \end{bmatrix}. \end{aligned}$$



Practice Examples

☰ Tank Example

A tank is divided into 2 cells. Each cell is filled with a fluid & a small opening allows the fluid to flow freely between cells.



Assume:

- height of fluid in a cell changes at a rate proportional to the difference between fluid height of the two cells

Questions:

1. **What happens to the system after a long period of time?**
   We expect $x_1 = x_2$ as $t \to \infty$.
2. **Construct a linear system for the fluid level heights.**

$$\frac{dx_1}{dt} = k_1(x_1 - x_2)$$
$$\frac{dx_2}{dt} = k_2(x_2 - x_1)$$

Assume $k = k_1 = k_2$ and $k > 0$.

Then,

$$\frac{d\vec{x}}{dt} = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \vec{x}.$$

3. **Solve the linear systems.**

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix}$$

Eigenvalues:

$$(1-\lambda)^2 - 1 = 0$$
$$\lambda^2 - 2\lambda = 0$$

$\lambda = 0, 2.$
$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$

Solutions:

$$\begin{aligned}\vec{u}_1 &= \begin{bmatrix}1\\1\end{bmatrix}\\ \vec{u}_2 &= e^{2t}\begin{bmatrix}1\\-1\end{bmatrix}.\end{aligned}$$

Complete solution:

$$\vec{x} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2e^{2t}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}c_1 + c_2e^{2t}\\c_1 - c_2e^{2t}\end{bmatrix}.$$

4. **Determine whether the solution is unique.**

Given initial value $\vec{x}_0 = \vec{x}(t_0)$, there is a unique solution for $\begin{bmatrix}c_1\\c_2\end{bmatrix}$ if:

$$\begin{aligned}\vec{x}_0 &= c_1\vec{u}_1(t) + c_2\vec{u}_2(t)\\ &= [\vec{u}_1 \quad \vec{u}_2]\begin{bmatrix}c_1\\c_2\end{bmatrix}.\end{aligned}$$

$[\vec{u}_1 \quad \vec{u}_2]$ is invertible: $\vec{u}_1$ and > $\vec{u}_2$ are linearly independent.

Because of uniqueness, solution curves cannot cross.

Motion Example

Suppose that the position of a moving object is given by $y(t)$ satisfying

$$y'' + 2y' + \alpha y = 0, \quad \alpha \in \mathbb{R}$$

1. Make system of FOLDEs $\vec{x}' = A\vec{x}$.

Let $x_1 = y, x_2 = y'$. Then:

$$\begin{aligned}x_1' &= x_2\\ x_2' &= -2x_2 - \alpha x_1\end{aligned}$$

$$\vec{x}' = \begin{bmatrix}0 & 1\\-\alpha & -2\end{bmatrix}\vec{x}.$$

2. Express eigenvalues in terms of α.

$$\begin{aligned}-\lambda(-2-\lambda) + \alpha &= 0\\ \lambda^2 + 2\lambda + \alpha &= 0\end{aligned}$$

   $\lambda = -1 \pm \sqrt{1-\alpha}$.

3. Let $\alpha = -3$.
   - Determine the eigenvectors of A.
     $\lambda = -3, 1$.

- Write down the solution to system.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix},$$
$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$
$$\vec{x} = c_1 e^{-3t}\vec{v}_1 + c_2 e^{t}\vec{v}_2$$

- Sketch phase portraits & classify the critical points of the system.