Textbook sections: 3.4, 3.5

Complex Eigenvalues (3.4)

For a real matrix with complex eigenvalues, we are given a complex conjugate pair of eigenvalues.

Select one of these eigenvalues & its corresponding eigenvector:

$$\lambda = \alpha + i\beta \quad \vec{v} = \vec{a} + i\vec{b}$$

The general solution is

$$\begin{aligned} \vec{x}_1 &= e^{\alpha t}(\vec{a}\cos(\beta t) - \vec{b}\sin(\beta t)) \\ \vec{x}_2 &= e^{\alpha t}(\vec{a}\sin(\beta t) + \vec{b}\cos(\beta t)) \\ \vec{x} &= c_1\vec{x}_1 + c_2\vec{x}_2. \end{aligned}$$

Derivation

Note that a solution to the system is

$$\vec{u} = e^{\lambda t}\vec{v}.$$

Using our definitions above, we can expand:

$$\begin{aligned} \vec{u} &= e^{\lambda t}\vec{v} \\ &= e^{\alpha t}e^{i\beta t}(\vec{a} + i\vec{b}) \\ &= e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))(\vec{a} + i\vec{b}) \\ &= e^{\alpha t}((\vec{a}\cos(\beta t) - \vec{b}\sin(\beta t)) + i(\vec{a}\sin(\beta t) + \vec{b}\cos(\beta t))) \\ &= \overbrace{e^{\alpha t}(\vec{a}\cos(\beta t) - \vec{b}\sin(\beta t))}^{\vec{x}_1} + i\overbrace{e^{\alpha t}(\vec{a}\sin(\beta t) + \vec{b}\cos(\beta t))}^{\vec{x}_2} \\ &= \vec{x}_1 + i\vec{x}_2 \end{aligned}$$

Hence, the general solution is

$$\begin{aligned} \vec{x}_1 &= e^{\alpha t}(\vec{a}\cos(\beta t) - \vec{b}\sin(\beta t)) \\ \vec{x}_2 &= e^{\alpha t}(\vec{a}\sin(\beta t) + \vec{b}\cos(\beta t)) \\ \vec{x} &= c_1\vec{x}_1 + c_2\vec{x}_2 \end{aligned}$$

Note that the conjugate eigenvalue and eigenvector will result in the same $\vec{x}_1$ and $\vec{x}_2$.

See: phase portraits.

Example

Determine general solution to

$$\vec{x}' = \begin{bmatrix} -1 & 2 \\ -1 & -3 \end{bmatrix} \vec{x}$$

Computing eigenthings

Eigenvectors:

$$\begin{aligned} (-1-\lambda)(-3-\lambda)+2 &= 0 \\ \lambda^2+4\lambda+5 &= 0 \\ \lambda &= -2 \pm \sqrt{4-5} \\ &= -2 \pm i \end{aligned}$$

Eigenvalues:

$$(A-\lambda I)\vec{v} = \begin{bmatrix} -1-\lambda & 2 \\ -1 & -3-\lambda \end{bmatrix} \vec{v} = 0$$

$\lambda_1 = -2 - i$:

$$A - \lambda I = \begin{bmatrix} 1+i & 2 \\ -1 & -1+i \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$$

$\lambda_2 = -2 + i$:

$$A - \lambda I = \begin{bmatrix} 1-i & 2 \\ -1 & -1-i \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}$$

General solution

Using $\lambda_2 = -2 + i$:

$$\begin{aligned} &\alpha = -2, \\ &\beta = 1 \\ &\vec{a} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &\vec{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \end{aligned}$$

So the general solution:

$$\vec{x}_1 = e^{-2t}\left(\begin{bmatrix}-1\\1\end{bmatrix}\cos t - \begin{bmatrix}-1\\0\end{bmatrix}\sin t\right)$$
$$\vec{x}_2 = e^{-2t}\left(\begin{bmatrix}-1\\1\end{bmatrix}\sin t + \begin{bmatrix}-1\\0\end{bmatrix}\cos t\right)$$
$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$$

Repeated Eigenvalues (3.5)

In a situation where there's a repeated eigenvalue but not two eigenvectors, the solution found will be:

$$\vec{r}(t) = c_1 e^{\lambda t}\vec{v} + c_2 e^{\lambda t}(\vec{v}t + \vec{w})$$

$\vec{w}$ can be found by finding independent solutions to $(A - \lambda I)\vec{w} = \vec{v}$ (this is known as a **generalized eigenvector**).

See: phase portraits.

Motivating Example 1

Given an object moving in the plane with motion $\vec{r}(t) = \begin{bmatrix}x(t)\\y(t)\end{bmatrix}$, such that the velocity is defined as:

$$x' = -x + ky$$
$$y' = -y$$

and such that $\vec{r}(0) = \begin{bmatrix}1\\2\end{bmatrix}$.

$k = 0$

$$\frac{d\vec{r}}{dt} = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\vec{r}$$

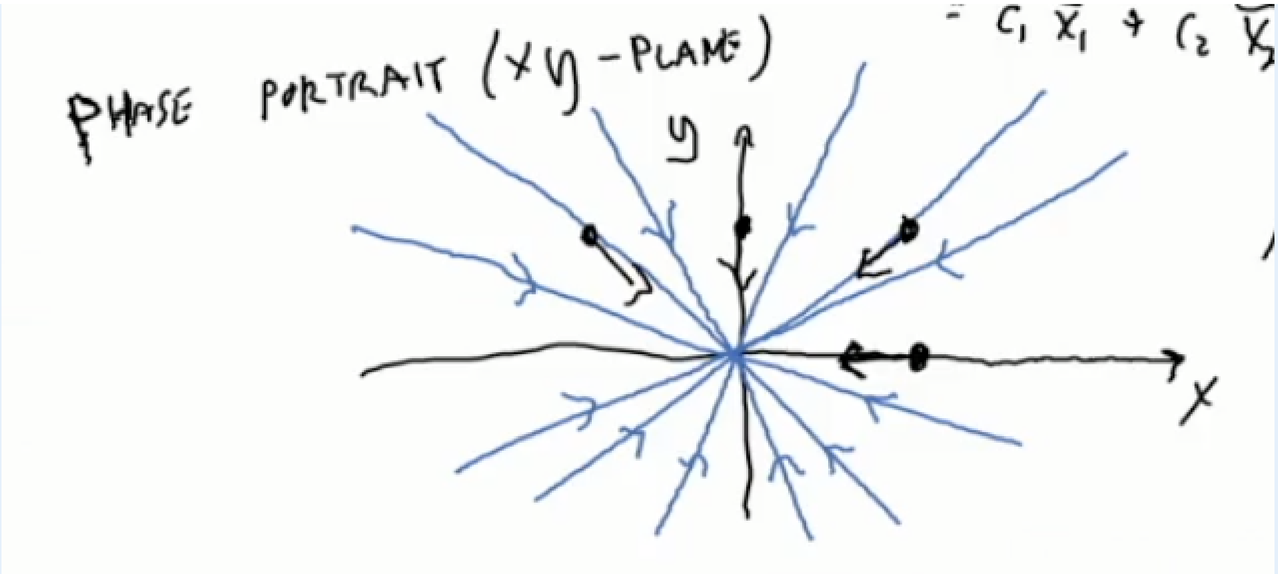
We can compute the eigenvalues as

$$(-1-\lambda)^2 = 0$$

So $\lambda_1 = \lambda_2 = -1$.

The eigenvectors are: $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$.

Phase Portrait



$k \neq 0$

There aren't two eigenvectors so we can't use the matrix method.

Since $y' = -y$, we know that $y = c_2 e^{-t}$.

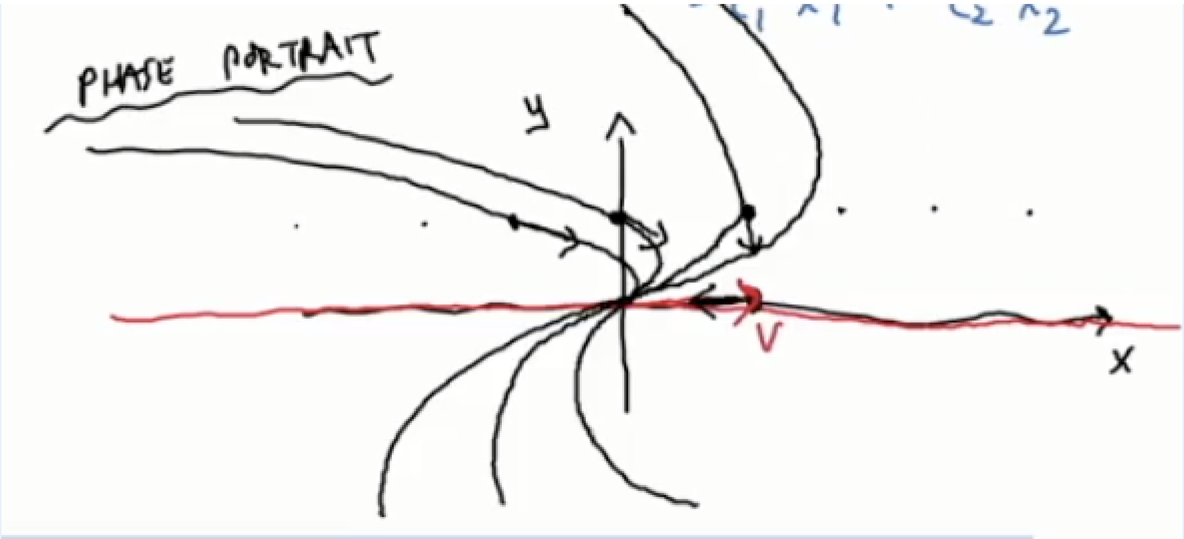
We can then substitute and solve for $x' + x = ky$:

$$\begin{aligned} e^t x' + e^t x &= e^t ky \\ e^t x &= \int k c_2 \, dt \\ x &= \frac{k c_2 t}{e^t} + \frac{c_1}{e^t} \end{aligned}$$

Hence:

$$\begin{aligned} \vec{r} &= \begin{bmatrix} c_1 e^{-t} + k c_2 t e^{-t} \\ c_2 e^{-t} \end{bmatrix} \\ &= c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-t} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} kt + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \end{aligned}$$

Phase Portrait



Motivating Example 2

Find the general solution to the given system:

$$\vec{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \vec{x}.$$

First Solution

In computing the eigenvalues and eigenvectors, we find $\lambda_1 = \lambda_2 = 2$, and $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Hence, one of our solutions is $\vec{x}_1 = e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Second Solution

Is there another linearly independent solution?

Motivating Example 1 indicates that our 2nd solution is of the form $e^{2t}(t\vec{v} + \vec{w})$. As such, assume

$$\vec{x}_2 = e^{2t}(t\vec{v} + \vec{w}).$$

Note that:

$$\begin{aligned} \vec{x}_2' &= \frac{d}{dt}(te^{2t}\vec{v} + e^{2t}\vec{w}) \\ &= e^{2t}\vec{v} + 2te^{2t}\vec{v} + 2e^{2t}\vec{w}, \end{aligned}$$

and:

$$\begin{aligned}\vec{x}_2' &= A\vec{x}_2 \\ &= A(te^{2t}\vec{v} + e^{2t}\vec{w}).\end{aligned}$$

Setting these equal, we have

$$\begin{aligned}e^{2t}\vec{v} + 2te^{2t}\vec{v} + 2e^{2t}\vec{w} &= A(te^{2t}\vec{v} + e^{2t}\vec{w}) \\ \vec{v} + 2t\vec{v} + 2\vec{w} &= A(t\vec{v}) + A\vec{w}\end{aligned}$$

Aligning the t and non-t terms:

$$\begin{aligned}A(t\vec{v}) &= 2t\vec{v} \\ \vec{v} + 2\vec{w} &= A\vec{w}\end{aligned}$$

We are trying to find $(A - 2I)\vec{w} = \vec{v}$, so solve:

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \vec{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

In solving this, we find $w_1 + w_2 = 1$, leaving the solution $\vec{w} = \begin{bmatrix} w_1 \\ 1 - w_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + w_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Thus, our second solution is $\vec{x}_2 = e^{2t}\left(t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

(Note: the extra $w_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ from our derivation of $\vec{w}$ would join with $\vec{x}_1$.)