Textbook sections: 4.5, 4.7

There are two methods of finding a particular solution.

Method of Undetermined Coefficients (4.5)

Motivating Examples

Motivating Example, $e^{\lambda t}$ (is not a solution to homogeneous)

Find the particular solution to $y'' + 3y' + 2y = 10e^{3t}$.

Testing $y = Ae^{3t}$, we get

$$\begin{aligned} 9Ae^{3t} + 9Ae^{3t} + 2e^{3t} &= 10e^{3t} \\ 20Ae^{3t} &= 10e^{3t} \\ A &= \frac{1}{2}. \end{aligned}$$

Therefore, $\frac{1}{2}e^{3t}$ is a particular solution.

Motivating Example, $\sin t$

Find the particular solution to $y'' + 3y' + 2y = \sin t$.

Testing $y = A\cos t + B\sin t$,

$$\begin{aligned} (-A\cos t - B\sin t) + 3(-A\sin t + B\cos t) + 2(A\cos t + B\sin t) &= \sin t \\ (-3A + B)\sin t + (A + 3B)\cos t &= \sin t \end{aligned}$$

We can solve, getting $A = -\frac{3}{10}, B = \frac{1}{10}$.

Therefore, $(-\frac{3}{10}\cos t + \frac{1}{10}\sin t)$ is a particular solution.

Motivating Example, $e^{\lambda t}$ (is a solution to homogeneous)

Find the particular solution to $y'' - 6y' + 9y = e^{3t}$.

Testing $y = Ae^{3t}$, we get

$$\begin{aligned} 9Ae^{3t} - 18Ae^{3t} + 9Ae^{3t} &= e^{3t} \\ 0 &= e^{3t} \end{aligned}$$

e^{3t} is a solution to the homogeneous problem, so cannot be used.

Testing $y = Ate^{3t}$, the solution fails for the same reason.

Testing $y = At^2e^{3t}$.

$$y' = 2Ate^{3t} + 3At^2e^3t$$
$$y'' = 2Ae^{3t} + 12Ate^{3t} + 9At^2e^{3t}$$

After some math, we see this does work and provides a particular solution to our heterogeneous equation.

Strategy for Undetermined Coefficients

To solve

$$ay'' + by' + cy = g(t),$$

1. Obtain the general solution of the homo sol'n.
2. Determine if undetermined coefficients can be used.*
3. If RHS is a sum, do the problem for the individual terms.
   4. Find a particular solution for each problem. Assume particular solution form, determine coefficients.
   5. Repeat for all terms.
4. Form general solution & solve IVP

*Undetermined coefficients can only be used for polynomials, sin/cos, exponentials.
Let:

- P_n be some polynomial of deg n
- Q_n, R_n be polynomials of deg n (with undetermined coefficients)
- t^* designate the smallest exponent of t such that the particular solution is not a solution in the homogeneous counterpart

$g(t)$	**particular solution** $Y(t)$
P_n	t^*Q_n
$P_ne^{\alpha t}$	$t^*e^{\alpha t}Q_n$
$P_ne^{\alpha t}\sin(\beta t)$, $P_ne^{\alpha t}\cos(\beta t)$	$t^*e^{\alpha t}(\cos(\beta t)Q_n + \sin(\beta t)R_n)$

Limitations

- Does not give an explicit expression for the particular solution

- Can only be applied for sine, cosine, exponentials, and polynomials

These are resolved with variation of parameters.

Variation of Parameters (4.7)

Strategy for Variation of Parameters (for SOLDEs)

We seek a solution to nonhomogeneous problem:

$$y'' + p(t)y' + q(t)y = g(t).$$

(Note: WLOG, y'' coefficient is 1.)

The solution to corresponding homo problem is:

$$y_h = c_1 y_1(t) + c_2 y_2(t)$$

To find a particular solution, replace c_1, c_2 with functions v_1, v_2, and try to find those functions.

$$y_p = v_1(t)y_1(t) + v_2(t)y_2(t)$$

Procedure for Variation of Parameters (for SOLDEs)

1. Solve homogeneous problem to find y_1, y_2.
2. Solve the system of nonlinear equations:

$$\begin{aligned} y_1 v_1' + y_2 v_2' &= 0 \\ y_1' v_1' + y_2' v_2' &= g \end{aligned}$$

3. Integrate v_1', v_2' to get v_1, v_2.
4. $y_p = v_1 y_1 + v_2 y_2$.

Derivation

We are trying to solve the nonhomogeneous problem

$$y'' + py' + qy = g.$$

Let y_1, y_2 be solutions to the homogeneous problem.
Let $y_p = v_1 y_1 + v_2 y_2$ be the general solution to the nonhomogeneous problem. Then,

$$y_p' = v_1 y_1' + v_2 y_2' + v_1' y_1 + v_2' y_2.$$

For simplicity, assume $v_1' y_1 + v_2' y_2 = 0$ (eq. 1).

Using this assumption:

$$y_p' = v_1 y_1' + v_2 y_2'$$
$$y_p'' = v_1' y_1' + v_2' y_2' + v_1 y_1'' + v_2 y_2''$$

We can substitute this into the nonhomogeneous problem:

$$\begin{aligned} y_p'' + p y_p' + q y_p &= (v_1' y_1' + v_2' y_2' + v_1 y_1'' + v_2 y_2'') + p(v_1 y_1' + v_2 y_2') + q(v_1 y_1 + v_2 y_2) \\ &= v_1 \overbrace{(y_1'' + p y_1' + q y_1)}^{0} + v_2 \overbrace{(y_2'' + p y_2' + q y_2)}^{0} + v_1'(y_1') + v_2'(y_2') \\ &= v_1' y_1' + v_2' y_2' \end{aligned}$$

Thus, our two constraints for variation of parameters are...

$$y_1 v_1' + y_2 v_2' = 0$$
$$y_1' v_1' + y_2' v_2' = g$$

Variation of parameters can be computed explicitly using the explicit formula:

$$y_p = -y_1 \int \frac{y_2 g}{W[y_1, y_2]}\, dt + y_2 \int \frac{y_1 g}{W[y_1, y_2]}\, dt$$

This is derived from the systems of FOLDEs formula.

Example

Determine a particular solution to

$$t^2 y'' - 4ty' + 6y = 4t^3, \quad t > 0$$

given that $y_1 = t^2, y_2 = t^3$ are solutions to the homogeneous equation.

Solving

In standard form,

$$y'' - \frac{4}{t} y' + \frac{6}{t^2} y = 4t.$$

$$t^2 v_1' + t^3 v_2' = 0$$
$$2t v_1' + 3t^2 v_2' = 4t$$

Note that $(2) \times t$:

$$\begin{aligned} 2t^2 v_1' + 3t^3 v_2' &= 4t^2 \\ t^3 v_2' &= 4t^2 \\ v_2' &= \frac{4}{t} \end{aligned}$$

Then, substituting into (1):

$$
\begin{aligned}
t^2 v_1' + 4t^2 &= 0 \\
t^2 v_1' &= -4t^2 \\
v_1' &= -4
\end{aligned}
$$

Integrating, we get:

$$
\begin{aligned}
v_1 &= -4t \\
v_2 &= 4 \ln t
\end{aligned}
$$

So our particular solution is:

$$y_p = -4t^3 + 4t^3 \ln t$$

Strategy for Variation of Parameters (for Systems of FOLDEs)

We seek a solution to nonhomogeneous problem:

$$\vec{x}_1' = P\vec{x} + \vec{g}(t)$$

(P being some matrix function, $\vec{g}$ being some vector function)

If the solution to the corresponding homogeneous problem is

$$\vec{x}_h = c_1\vec{x}_1(t) + c_2\vec{x}_2(t),$$

we define the **fundamental matrix**:

$$X(t) = [\vec{x}_1 \quad \vec{x}_2]$$

There is a particular solution:

$$\vec{x}_p = X(t) \int \overbrace{X^{-1}(t)\vec{g}(t)}^{\text{solution to } [X|\vec{g}]} \, dt$$

Derivation

Let v_1 and v_2 be some scalar function of t, and assume

$$
\begin{aligned}
\vec{x}_p &= v_1(t)\vec{x}_1(t) + v_2(t)\vec{x}_2(t) \\
&= [\vec{x}_1 \quad \vec{x}_2] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\
&= X\vec{v}.
\end{aligned}
$$

By differentiating $\vec{x}_p$, we find that

$$\vec{x}_p' = X'\vec{v} + X\vec{v}'.$$

We can substitute $\vec{x}_p$ and $\vec{x}_p'$ into our nonhomogeneous problem:

$$\begin{aligned} \vec{x}_p' &= P\vec{x}_p + \vec{g} \\ X'\vec{v} + X\vec{v}' &= P(X\vec{v}) + \vec{g} \end{aligned}$$

Note that $PX = X'$.

$$\begin{aligned} X'\vec{v} + X\vec{v}' &= X'\vec{v} + \vec{g} \\ X\vec{v}' &= \vec{g} \end{aligned}$$

Then,

$$\begin{aligned} \vec{v}' &= X^{-1}\vec{g} \\ \vec{v} &= \int X^{-1}\vec{g}\,dt \\ \vec{x}_p = X\vec{v} &= X\int X^{-1}\vec{g}\,dt \end{aligned}$$

Example

Determine a particular solution to

$$\vec{x}_1 = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}\vec{x}(t) + \begin{bmatrix} e^{2t} \\ 1 \end{bmatrix}$$

given the solutions to the homogeneous equation are

$$\begin{aligned} \vec{x}_1 &= e^t\begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ \vec{x}_2 &= e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

The fundamental matrix:

$$X = \begin{bmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{bmatrix}$$

Then, solving $[X|\vec{g}]$:

$$\begin{bmatrix} 3e^t & e^{-t} & | & e^{2t} \\ e^t & e^{-t} & | & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3e^{2t} & 1 & | & e^{3t} \\ 3e^{2t} & 3 & | & 3e^t \end{bmatrix}$$

$$\begin{bmatrix} 3e^{2t} & 1 & | & e^{3t} \\ 0 & 2 & | & 3e^t - e^{3t} \end{bmatrix}$$

$$\begin{bmatrix} 3e^{2t} & 1 & | & \frac{3e^{3t}}{2} \\ 0 & 1 & | & \frac{3e^t}{2} - \frac{e^{3t}}{2} \end{bmatrix}$$

$$\begin{bmatrix} 3e^{2t} & 0 & | & \frac{3e^{3t}}{2} - \frac{3e^t}{2} \\ 0 & 1 & | & \frac{3e^t}{2} - \frac{e^{3t}}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & \frac{e^t}{2} - \frac{e^{-t}}{2} \\ 0 & 1 & | & \frac{3e^t}{2} - \frac{e^{3t}}{2} \end{bmatrix}$$

Then,

$$\begin{aligned}
\vec{x}_p &= X(t) \int X^{-1}(t)\vec{g}(t)\, dt \\
&= \frac{1}{2} X \int \begin{bmatrix} e^t - e^{-t} \\ 3e^t - e^{3t} \end{bmatrix} dt \\
&= \frac{1}{2} \begin{bmatrix} 3e^t & e^{-t} \\ e^t & e^{-t} \end{bmatrix} \begin{bmatrix} e^t + e^{-t} \\ 3e^t - \frac{1}{3}e^{3t} \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} (3e^{2t} + 3) + (3 - \frac{1}{3}e^{2t}) \\ (e^{2t} + 1) + (3 - \frac{1}{3}e^{2t}) \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} \frac{8}{3}e^{2t} + 6 \\ \frac{2}{3}e^{2t} + 4 \end{bmatrix} \\
&= \begin{bmatrix} \frac{4}{3}e^{2t} + 3 \\ \frac{1}{3}e^{2t} + 2 \end{bmatrix}
\end{aligned}$$