Textbook sections: 5.7

Motivating Example

Consider the spring-mass system

$$my'' + \gamma y' + ky = g(t - t_0), \quad y(0) = 1, \quad y'(0) = 0,$$

where $g(t)$ represents a large force over a short duration at time t_0.
What can we use to model this force?

Dirac Delta Function

- Useful for modeling functions that have a large value over a short interval.
- Is not a function.

The **Dirac delta function** (δ) is characterized by two properties. For any function $f(t)$ continuous over an interval containing t_0:

1. $$\delta(t - t_0) = \begin{cases} 0, & t \neq t_0 \\ \text{undefined}, & t = t_0 \end{cases}$$

2. $$\int_a^b f(t)\delta(t - t_0)\,dt = f(t_0)$$

Laplace Transform of δ

$$\mathcal{L}\{\delta(t - c)\} = e^{-cs}$$

Derivation

$$\begin{aligned} \mathcal{L}\{\delta(t - t_0)\} &= \int_c^\infty e^{-st}\delta(t - t_0)\,dt \\ &= e^{-st_0}. \end{aligned}$$

Example

A mass attached to a spring is released from rest 1 m below equilibrium. After π seconds, the mass is struck by a hammer exerting an impulse.

The system is governed by

$$y'' + 9y = 3\delta(t-\pi), \quad y(0) = 1, \quad y'(0) = 0.$$

Solve this IVP.

Solving

$$\begin{aligned} \mathcal{L}\{y'' + 9y\} &= s^2Y - sy(0) - y'(0) + 9Y \\ &= s^2Y - s + 9Y \\ \mathcal{L}\{3\delta t - \pi\} &= 3e^{-\pi s}. \end{aligned}$$

Then,

$$\begin{aligned} s^2Y - s + 9Y &= 3e^{-\pi s} \\ (s^2+9)Y &= 3e^{-\pi s} + s \\ Y &= \frac{3e^{-\pi s}}{s^2+9} + \frac{s}{s^2+9} \\ &= \mathcal{L}\{u_\pi(t)\sin(3(t-\pi))\} + \mathcal{L}\{\cos 3t\} \end{aligned}$$

Then,

$$y = u_\pi(t)\sin(3(t-\pi)) + \cos 3t.$$