Textbook sections: 2.4

Initial Value Problem (IVPs)

Given an IVP, we may consider:

- **existence**: Does the IVP have a solution, and if so, where?
- **uniqueness**: Is the solution unique?

Theorems

Existence & Uniqueness of 1st Order Linear IVPs

If p and g are continuous on (α, β), $t_0 \in (\alpha, \beta)$, then there is a **unique** solution to the IVP

$$y' + p(t)y = g(t), \quad y(t_0) = y_0$$

Derivation

Consider the FOLDE solution.
Let μ be the integrating factor. We can solve the above IVP into...

$$\mu(t)y = \int \mu(t)g(t)dt + c$$

Our integration constant for μ's integral is arbitrary, so we can instead define μ as...

$$\mu(t) = e^{\int_{t_0}^{t} p(s)ds}$$

By doing so, $\mu(t_0) = 1$. We can integrate as long as the function is continuous, which means $p(s)$ must be continuous for the interval t_0 resides in... which is (α, β).

Solving for y:

$$y = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(s)g(s)ds + c\right)$$

When we substitute $t = t_0$, we find $y = y_0$. Hence,

$$y = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(s)g(s)ds + y_0\right)$$

Example

$$(9-t)^2 y' + 5ty = 3t^2,\ y(-1) = 1$$

In standard form:

$$y' - \frac{5t}{(3-t)(3+t)} y = \frac{3t^2}{9-t^2}$$

Then, $t_0 = -1$, an interval where a solution exists would be $(-3, 3)$.

- There will be an explicit expression.
- Points where there are discontinuities can be identified from the coefficients.

Existence & Uniqueness of 1st Order Non-Linear IVPs

If f and $\frac{\partial f}{\partial y}$ are continuous over $t \in (\alpha, \beta)$ and $y \in (\gamma, \delta)$, which contains (t_0, y_0), then there is a unique solution to the IVP:

$$y' = f(t, y), \quad y(t_0) = y_0$$

on $t \in (\alpha, \beta)$.

- Sufficient but not necessary.