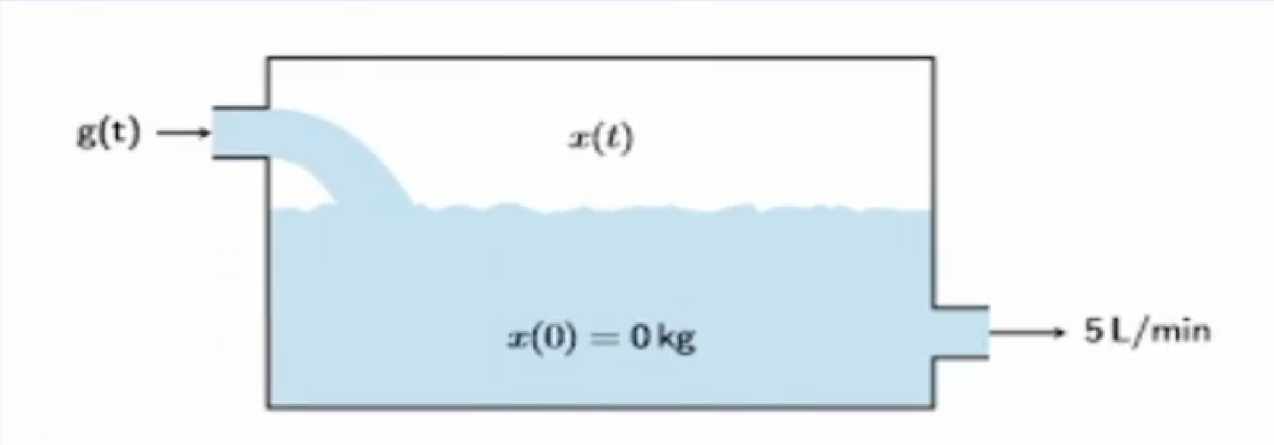
Textbook sections: 5.1, 5.2, 5.3, 5.4

ⓘ **Motivating Example**



Let the mass of salt (n ikg) in the tank at time t be $x(t)$. The tank holds 1000 L, and $x(0) = 0$.

The concentration of salt flowing into the tank is given by:

$$g(t) = \begin{cases} 0.04 \text{ kg/L} \cdot 6 \text{ L/min}, & 0 < t < 10 \\ 0.02 \text{ kg/L} \cdot 6 \text{ L/min}, & 10 < t \end{cases}$$

Solving

You could:

- Solve at $t \in (0, 10)$ to obtain $x(10)$,
- Use that to solve situation at $t \in (10, \infty)$.

However, as the piecewise gets increasingly complicated, this becomes unmaintainable.

The Laplace transform is useful as it lets us redefine our *differential* equations (which can be difficult to manipulate) into *algebraic* equations (which are relatively easier to manipulate).

If this process is possible (unfortunately, it is not always possible), then it can be an easier method for solving immensely complicated differential equations.

Definition of Laplace Transform (5.1)

Suppose $f(t)$ is defined on $[0, \infty)$.

Then, the **Laplace Transform** is

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

The domain of $\mathcal{L}\{f\}$ is the set of values where the integral exists.

Example: $\mathcal{L}\{1\}$

$$\begin{aligned}
\mathcal{L}\{1\}(s) &= \int_0^\infty e^{-st}\, dt \\
&= -\frac{1}{s} e^{-st}\Big|_{t=0}^{t=\infty} \\
&= -\frac{1}{s}(0-1) \\
&= \frac{1}{s}
\end{aligned}$$

Example: $\mathcal{L}\{e^{at}\}$

$$\begin{aligned}
\mathcal{L}\{e^{at}\}(s) &= \int_0^\infty e^{-st} e^{at}\, dt \\
&= \int_0^\infty e^{(a-s)t}\, dt \\
&= \frac{1}{a-s} e^{(a-s)t}\Big|_{t=0}^{t=\infty} \\
&= \frac{1}{a-s}(0-1), \quad s > a \\
&= \frac{1}{s-a}, \quad s > a
\end{aligned}$$

Example: $\mathcal{L}\{\sin at\}$

$$\begin{aligned}
\sin at &= \frac{1}{2i}(e^{iat} - e^{-iat}) \\
\mathcal{L}\{\sin at\} &= \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right) \\
&= \frac{a}{s^2+a^2}
\end{aligned}$$

$\mathcal{L}$ for a piecewise continuous function

Let

$$f(t)=\begin{cases} e^2t, & 0\le t<1 \\ 4, & 1\le t \end{cases}$$

Compute $\mathcal{L}\{f\}$.

$$\begin{aligned}\mathcal{L}\{f\} &= \int_0^\infty fe^{-st}\,dt \\ &= \int_0^1 e^{2t}e^{-st}\,dt + \int_1^\infty 4e^{-st}\,dt \\ &= \left(\frac{1}{2-s}e^{(2-s)t}\right)\Bigg|_{t=0}^{t=1} - \left(\frac{4}{s}e^{-st}\right)\Bigg|_{t=1}^{t=\infty} \\ &= \frac{e^{2-s}-1}{2-s}+\frac{4e^{-s}}{s} \quad s>0, s\neq 2\end{aligned}$$

Linearity

The Laplace transform is a **linear operator**:

$$\mathcal{L}\{f(t)+g(t)\}=\mathcal{L}\{f(t)\}+\mathcal{L}\{g(t)\}$$
$$\mathcal{L}\{kf(t)\}=k\mathcal{L}\{f(t)\}$$

Exponential Order

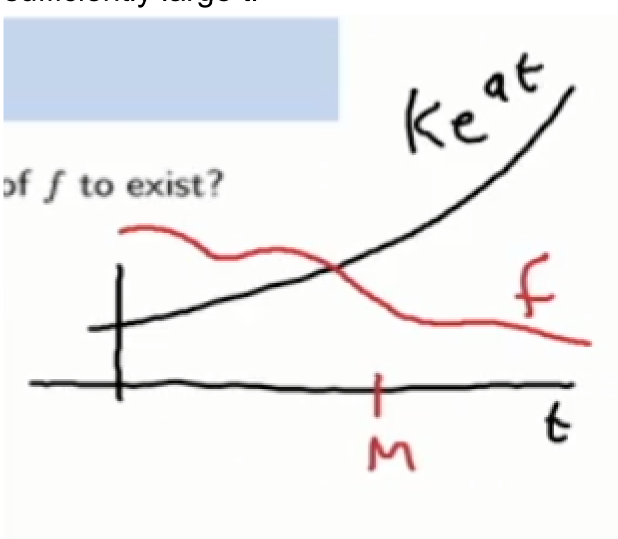
What conditions must be satisfied for the Laplace transform of f to exist?

f is of **exponential order** if there exists constants K, a, M such that

$$|f(t)|\le Ke^{at},$$

for all $t>M$.

To check whether a function f is of exponential order, we can show that $\frac{f(t)}{e^{at}}$ is bounded for sufficiently large t.



Properties of the Laplace Transform (5.2)

Suppose $F(s) = \mathcal{L}\{f(t)\}$.

Theorems

Translating Along s-domain

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$$

Derivation

$$\begin{aligned}\mathcal{L}\{e^{ct}f\} &= \int_0^\infty fe^{ct}e^{-st}\,dt \\ &= \int_0^\infty fe^{-(s-c)t}\,dt \\ &= F(s-c)\end{aligned}$$

Theorem: First Derivative

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

Derivation

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'e^{-st}\,dt$$

Using integration by parts:

$$\begin{aligned}u &= e^{-st} \qquad & du &= -se^{-st}dt \\ dv &= f'dt \qquad & v &= f\end{aligned}$$

Then,

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \left(e^{-st}f\right)\Big|_{t=0}^{t=\infty} - \int_0^\infty \left(-se^{-st}f\right)\,dt \\ &= -f(0) + sF(s)\end{aligned}$$

Theorem: Higher Order Derivatives

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$$

Theorem: Derivatives in the s-domain

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

ⓘ **Derivation**

Note that

$$\begin{aligned}\frac{d^n F}{ds^n} &= \frac{d^n}{ds^n}\int_0^\infty e^{-st} f(t)\,dt \\ &= (-1)^n \int_0^\infty t^n e^{-st} f\,dt \\ &= (-1)^n \mathcal{L}\{t^n f\}\end{aligned}$$

Using integration by parts:

$$\begin{aligned} u &= e^{-st} & du &= -se^{-st}dt \\ dv &= f'dt & v &= f \end{aligned}$$

Then,

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \left(e^{-st} f\right)\Big|_{t=0}^{t=\infty} - \int_0^\infty \left(-se^{-st} f\right)\,dt \\ &= -f(0) + sF(s)\end{aligned}$$

It follows from this theorem that

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

Computing Laplace Transforms of IVPs

ⓘ **Motivating Example: First Order DE**

Compute the Laplace Transform of:

$$y' = e^{-6t}\sin(t), \quad y(0) = 2$$

On the left-hand side:

$$\mathcal{L}\{y'\} = sY - y(0) = sY - 2$$

On the right-hand side:

$$\mathcal{L}\{e^{-6t}\sin t\} = \frac{1}{(s+6)^2 + 1}$$

Then:

$$sY - 2 = \frac{1}{(s+6)^2 + 1}$$
$$Y = \frac{2}{s} + \frac{\frac{1}{s}}{(s+6)^2 + 1}$$

Motivating Example: Second Order DE

Compute the Laplace Transform of:

$$y'' + 9y = e^{-t}\sin(4t), \quad y(0) = 0, y'(0) = 1$$

On the left-hand side:

$$\mathcal{L}\{y'' + 9y\} = (s^2Y - sy(0) - y'(0)) + 9Y = (s^2 + 9)Y - 1$$

On the right-hand side:

$$\mathcal{L}\{e^{-t}\sin 4t\} = \frac{4}{(s+1)^2 + 16}$$

Then,

$$(s^2 + 9)Y = 1 + \frac{4}{(s+1)^2 + 16}$$

To fully solve DEs with Laplace transforms, see Solving IVPs with Laplace Transforms.

Inverse Laplace Transform (5.3)

If f, g are piece-wise continuous and of exponential order, and

$$\mathcal{L}\{f\} = \mathcal{L}\{g\},$$

then $f = g$.

We can define $\mathcal{L}^{-1}$ to be the **inverse Laplace operator**, such that

$$\mathcal{L}^{-1}\{\mathcal{L}\{f\}\} = f.$$

Note that the Inverse Laplace transform is linear.

Examples

Example: $\mathcal{L}^{-1}\left\{\frac{4}{s^3}\right\}$

$$\frac{4}{s^3} = 2 \cdot \frac{2!}{s^3} = 2\mathcal{L}\{t^2\}.$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{4}{s^3}\right\} = 2t^2.$$

Example: $\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$

$$\frac{3}{s^2+9} = \mathcal{L}\sin 3t.$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \sin 3t.$$

Example: $\mathcal{L}^{-1}\left\{\frac{s-1}{s^2-2s+5}\right\}$

$$\frac{s-1}{s^2-2s+5} = \frac{(s-1)}{(s-1)^2+2^2} = \mathcal{L}\{e^t \cos 2t\}.$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{s-1}{s^2-2s+5}\right\} = e^t \cos 2t.$$

Example: $\mathcal{L}^{-1}\left\{\frac{5}{(s+2)^4}\right\} \stackrel{?}{=} t^n e^{kt}$

$$\frac{5}{(s+2)^4} = \frac{5}{6} \cdot \frac{3!}{(s+2)^4} = \mathcal{L}\left\{\frac{5}{6}t^3 e^{-2t}\right\}.$$

Example: Distinct Linear Partial Fractions — $\mathcal{L}^{-1}\left\{\frac{7s-1}{(s+1)(s+2)(s-3)}\right\}$

$$\frac{7s-1}{(s+1)(s+2)(s-3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-3}.$$

As such,

$$7s - 1 = A(s+2)(s-3) + B(s+1)(s-3) + C(s+1)(s+2).$$

By plugging in $s = -1, -2, 3$, we get...

$$\begin{aligned} -8 &= A(-1+2)(-1-3) \\ -15 &= B(-2+1)(-2-3) \\ 20 &= C(3+1)(3+2). \end{aligned}$$

Then, $A = 2, B = -3, C = 1$.

As such,

$$\mathcal{L}^{-1}\left\{\frac{7s-1}{(s+1)(s+2)(s-3)}\right\} = 2e^{-t} - 3e^{-2t} + e^{3t}.$$

Example: Repeated Linear Partial Fractions — $\mathcal{L}^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\}$

$$\frac{s^2+9s+2}{(s-1)^2(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3}.$$

As such,

$$s^2 + 9s + 2 = A(s-1)(s+3) + B(s+3) + C(s-1)^2.$$

By plugging in $s = -3$, we get...

$$-16 = C(-4)^2.$$

Then, $C = -1$.

By plugging in $s = 1$, we get:

$$12 = 4B.$$

Then $B = 3$.

We then have

$$\begin{aligned} s^2 + 9s + 2 &= A(s-1)(s+3) + 3(s+3) - (s-1)^2 \\ s^2 + 9s + 2 &= A(s^2 + 2s - 3) + 3s + 9 - s^2 + 2s - 1 \\ 2s^2 + 4s - 6 &= A(s^2 + 2s - 3). \end{aligned}$$

Then, $A = 2$.

As such,

$$\mathcal{L}^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\} = 2e^{t} + 3te^{t} - e^{3t}.$$

Solving IVPs with Laplace Transforms (5.4)

The process for solving IVPs with Laplace transforms is as follows:

1. Compute $\mathcal{L}$ for IVP.
2. Solve for $Y(s) = \mathcal{L}\{y\}$.
3. Compute $\mathcal{L}^{-1}$ of Y fo find y.

Example

$$y' + 3y = 13\sin 2t, \quad y(0) = 6.$$

LHS:

$$\begin{aligned}\mathcal{L}\{y' + 3y\} &= sY - y(0) + 3Y \\ &= (s+3)Y - 6\end{aligned}$$

RHS:

$$\mathcal{L}\{13\sin 2t\} = \frac{13 \cdot 2}{s^2 + 4}.$$

Therefore,

$$\begin{aligned}(s+3)Y &= 6 + \frac{26}{s^2+4} \\ Y &= \frac{6}{s+3} + \frac{26}{(s^2+4)(s+3)} \\ &= \frac{6s^2+50}{(s^2+4)(s+3)}\end{aligned}$$

Then, we can use partial fraction decomposition:

$$\begin{aligned}\frac{6s^2+50}{(s^2+4)(s+3)} &= \frac{A}{s+3} + \frac{Bs+C}{(s^2+4)} \\ 6s^2 + 50 &= A(s^2+4) + (Bs+C)(s+3).\end{aligned}$$

Plugging in $s = -3$:

$$6(-3)^2 + 50 = A(9+4).$$

Then, $A = 8$.

Plugging in $s = 0$:

$$50 = 32 + C(3).$$

Then, $C = 6$.

Then, by expanding, $B = -2$.

Then,

$$y = 8e^{-3t} - 2\cos 2t + 3\sin 2t.$$

Elementary Laplace Formulas

Table 5.3.1:

$f(t)$	$\mathcal{L}[f(t)]$	s**-domain**
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$t^n,\ n \in \mathbb{Z}^+$	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^p,\ p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$	$s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}$	$s > \lvert a \rvert$
$\cosh at$	$\frac{s}{s^2-a^2}$	$s > \lvert a \rvert$
$u_c(t)$, Unit step function	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
$e^{ct}f(t)$	$F(s-c)$	
$\int_0^t f(t-\tau)g(\tau)\,d\tau$, Convolutions	$F(s)G(s)$	
$\delta(t-c)$, Dirac delta function	e^{-cs}	
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$	
$t^nf(t)$	$(-1)^nf^{(n)}(s)$	