Textbook sections: 4.2, 4.3

Theory of SOLDEs (4.2)

Existence & Uniqueness of 2nd Order Linear DE

See: [Existence & Uniqueness of 1st Order IVPs]

If p, q, and g are continuous on open interval I ($t_0 \in I$), then there is a unique solution to the IVP

$$y'' + p(t)y' + q(t) = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_1$$

Fundamental Set of Solutions

See: [Fundamental Set of Solutions]

There can be two lin indep. solutions to a SOLDE, and these are the fundamental set.

Reduction of Order

Given one solution y_1 to a homogeneous SOLDE, the **reduction of order** is used to derive a y_2 to form a fundamental set.

To find a second solution, test $y_2 = y_1 v(t)$ in the original system, and solve for v to get a second solution y_2.

Motivating Example

$y_1(t) = t$ is a solution to

$$y'' - \frac{1}{t}y' + \frac{1}{t^2}y = 0$$

Try $y_2 = y_1 v(t)$ (v is unknown):
Then,

$$\begin{aligned} y_2' &= \frac{d}{dt}(tv) = v + tv' \\ y_2'' &= \frac{d}{dt}(v + tv') = 2v' + tv'' \end{aligned}$$

Then,

$$
\begin{aligned}
y'' - \frac{1}{t}y' + \frac{1}{t^2}y &= 0 \\
(2v' + tv'') - \frac{1}{t}(v + tv') + \frac{1}{t^2}(tv) &= 0 \\
tv'' + v' &= 0 \\
\frac{d}{dt}(v't) &= 0
\end{aligned}
$$

We can integrate to get:

$$
\begin{aligned}
v't &= k_1 \\
\frac{k_1}{t} &= v'
\end{aligned}
$$

Solving this, we get $v = k_1 \ln t + k_2$.
Then, our two solutions are:

$$
\begin{aligned}
y_1 &= t \\
y_2 &= t \ln t
\end{aligned}
$$

If $y_2 = y_1 v$, then we recognize

$$
\begin{aligned}
y_2' &= v'y_1 + vy_1' \\
y_2'' &= v''y_1 + 2v'y_1' + vy_1''.
\end{aligned}
$$

Plugging this into our original system, we find

$$
\begin{aligned}
(v''y_1 + 2v'y_1' + y_1'') + p(v'y_1 + vy_1') + qy_1 &= g \\
v''y_1 + v'(2y_1' + py_1) + v(y_1'' + py_1' + qy_1) &= g \\
v''y_1 + v'(2y_1' + py_1) &= g.
\end{aligned}
$$

SOLDEs with Linear Homogeneous Constant Coefficients (4.3)

See: System of 2 FOLDEs.

These are of the form:

$$
ay'' + by' + cy = 0.
$$

The eigenvalues for a SOLDE are the solutions to

$$
a\lambda^2 + b\lambda + c = 0.
$$

As a system, each eigenvalue has a corresponding eigenvector of the form $\begin{bmatrix} 1 \\ \lambda \end{bmatrix}$.

ⓘ **Derivation**

We first convert the SOLDE into a linear system.

Let $x_1 = y$, $x_2 = y'$. Then:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{c}{a}x_1 - \frac{b}{a}x_2 \end{aligned}$$

And:

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & \frac{-b}{a} \end{bmatrix} \vec{x}$$

We can then compute the eigenvalues:

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} -\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{vmatrix} \\ &= \lambda\left(\frac{b}{a} + \lambda\right) + \frac{c}{a} \\ &= \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} \end{aligned}$$

Alternatively, this can be written as:

$$a\lambda^2 + b\lambda + c = 0.$$

By plugging in λ in $A - \lambda I$, we see the eigenvectors to this system are of the form $\begin{bmatrix} 1 \\ \lambda \end{bmatrix}$.

To find the general solution, there are 3 different cases to observe:

Distinct Real Eigenvalues

The general solution to a system with distinct real eigenvalues is

$$\vec{y} = \begin{bmatrix} y \\ y' \end{bmatrix} = c_1 e^{\lambda_1 t}\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda_2 t}\begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

From the top row, the general solution to the SOLDE is

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

Repeated Real Eigenvalues

The general solution to a system with repeated real eigenvalues is

$$\vec{y} = c_1 e^{\lambda t}\vec{v} + c_1 e^{\lambda t}(t\vec{v} + \vec{w}).$$

$\vec{w}$ must be a solution to $[A - \lambda I | \vec{v}]$, we can create the augmented matrix

$$\begin{bmatrix} -\lambda & 1 & | & 1 \\ \dots & \dots & | & \dots \end{bmatrix}$$

to find $-\lambda w_1 + w_2 = 1$.
We then see that

$$\vec{w} = \begin{bmatrix} w_1 \\ 1 + \lambda w_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + w_1 \begin{bmatrix} 1 \\ \lambda \end{bmatrix}.$$

Using $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we find the general solution to the SOLDE:

$$y = c_1 e^{\lambda t} + t c_2 e^{\lambda t}.$$

Complex Eigenvalues

As we did to find the general solution to a system with complex conjugate eigenvalues, we recognize that a possible solution to the system of FOLDEs is

$$\vec{y}_1 = e^{(\alpha + i\beta)t} \begin{bmatrix} 1 \\ \alpha + i\beta \end{bmatrix},$$

where $\lambda_1 = \alpha + i\beta$.

Thus, a possible solution to the SOLDE is

$$\begin{aligned} y_1 &= e^{(\alpha + i\beta)t} \\ &= e^{\alpha t}(\cos \beta t + i \sin \beta t). \end{aligned}$$

Thus, we can find a general solution:

$$y_1 = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t.$$

Cauchy-Euler Equation

A DE of the form

$$ax^2 \frac{d^2 y}{dx^2} + bx \frac{dy}{dx} + cy = 0$$

is a 2nd-order **Cauchy-Euler** equation. To solve, we try solutions of the form $y = x^m$.

Substituting, we get:

$$\begin{aligned} ax^2(m(m-1)x^{m-2}) + bx(mx^{m-1}) + cx^m &= 0 \\ am(m-1) + bm + c &= 0 \\ am^2 + (b-a)m + c &= 0 \end{aligned}$$

Let λ_1 and λ_2 be the solutions to this equation. Then,

- If λ_1 and λ_2 are distinct reals, the fundamental set is x^{λ_1} and x^{λ_2}.
- If λ_1 and λ_2 are repeated reals, the fundamental set is x^{λ} and $x^{\lambda}\ln x$.
- If λ_1 and λ_2 are complex, then the fundamental set is:
  - $x^{\alpha}\cos(\beta\ln x)$ and $x^{\alpha}\sin(\beta\ln x)$,
  - where $\lambda = \alpha + i\beta$.