Textbook sections: 8.1, 8.2, 8.3

* Non-linear first order IVPs

We're approximating IVPs of the form...

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

Euler's Method (8.1)

To approximate the value of a DE at $t_0, t_1, t_2, \ldots$,

$$y_{n+1} = y_0 + f(t_n, y_n)(t_{n+1} - t_n).$$

In other words:

1. Given that we want to approximate y at $t = b$, starting at $t = a$, we break up the interval into n steps $t_0(= a),\ t_1,\ t_2,\ \ldots,\ t_n(= b)$.
2. Then, at step k, compute the tangent line at (t_k, y_k) to get y_{k+1}.
3. Repeat until you reach (t_n, y_n).

Error Analysis (8.2)

Characterizing errors by numerical approaches:

type of error	description
global truncation error	accounts for all errors made in approximating IVP solution
round off errors	arise from carrying computations with a finite num. of digits
local truncation error	error introduced in one step from use of approximate formula

Our analysis will only focus on local truncation errors.

Local truncation error

Let $\phi(t)$ = exact solution.
We seek an expression for the local truncation error at iteration $n + 1$:

$$e_{n+1} = \phi(t_{n+1}) - y_{n+1}.$$

We can use a Taylor expansion for $\phi(t)$ about t_n:

$$\phi(t_n + h) = \phi(t_n) + \phi'(t_n)h + \frac{1}{2}\phi''(c)h^2, \quad c \in (t_n, t_n + h)$$

* We can skip 3rd order + so on, because we're approximating error

Let $t_{n+1} = t_n + h$, then the error becomes:

$$\begin{aligned} e_{n+1} &= \phi(t_{n+1}) - y_{n+1} \\ &= \left(\phi(t_n) + \phi'(t_n)h + \frac{1}{2}\phi''(c)h^2\right) - (y_n + hf(t_n, y_n)) \end{aligned}$$

Since we're only considering error introduced in each step, we assume $y_n = \phi(n)$ (and consequently $y_n' = \phi'(n)$).

Thus, terms cancel out, leaving:

$$e_{n+1} = \frac{1}{2}\phi''(c)h^2, \quad c \in (t_n, t_n + h).$$

Note: error is proportional to h^2.

Example

Find the error for:

$$y' = y, \quad y(0) = 1$$

Solution

Note that $\phi' = y' = y$.

$$\begin{aligned} e_{n+1} &= \frac{1}{2}\phi''(c)h^2 \\ &= \frac{1}{2}y'(c)h^2 \end{aligned}$$

Example

Find the error for:

$$y' = 0.5 - t + 2y, \quad y(0) = 1$$

Solution

Note that $\phi' = y' = \frac{1}{2} - t + 2y$.

$$\begin{aligned} \phi'' &= -1 + 2\phi' \\ &= 1 - 2t + 4y - 1 \\ &= -2t + 4y \end{aligned}$$

Plugging this into the error equation:

$$e_{n+1} = \frac{1}{2}(-2c + 4y)h^2$$

Improved Methods (8.3)

Euler's method uses...

$$y_{n+1} = y_n + hf(t_n, y_n)$$

You can expand this into...

$$y_{n+1} = y_n + h(w_1k_1 + w_2k_2 + w_nk_n),$$

where $\sum w_i = 1$, k_i is $f(t, y)$ at specific points.

Improved Euler's Method

Instead of using the slope at $t = t_n$ ($f(t_n, y_n)$), Improved Euler's uses the average of the slopes at $t = t_n$ and $t = t_{n+1}$:

$$\frac{f(t_n, y_n) + f(t_{n+1}, \overbrace{y_n + hf(t_n, y_n)}^{\text{approx. of } \phi(n+1)})}{2}$$

Runge-Kutta (RK) Method

Instead of using $f(t_n, y_n)$, RK replaces with a weighted average of four values of $f(x, y)$:

$$\frac{1}{6}(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4})$$

weight	k_{ni}
1/6	$f(t_n, y_n)$
2/6	$f(t_n + \frac{h}{2}, y_n + \frac{1}{2}hk_{n1})$
2/6	$f(t_n + \frac{h}{2}, y_n + \frac{1}{2}hk_{n2})$
1/6	$f(t_n + h, y_n + hk_{n3})$

Example

Use one iteration of Improved Euler and RK methods to the estimate solution to the given IVP at specified point.
$y' = y, y(0) = 1$. Estimate the solution at $t = 1$.

(Note that $y(1) = e$.)

Euler's

$$\begin{aligned} y_1 &= y_0 + hf(t, y) \\ &= 1 + 1(0) \\ &= 1. \end{aligned}$$

Improved Euler's

$$\begin{aligned} m &= \frac{f(t, y) + f(t + h, y + hf(t, y))}{2} \\ &= \frac{1 + f(t, 1 + 1)}{2} \\ &= \frac{3}{2} \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + hm \\ &= 1 + 1\left(\frac{3}{2}\right) \\ &= \frac{5}{2} \end{aligned}$$

Runge-Kutta

$$m = \frac{1}{6}(k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4})$$

k_{ni}	value
k_{n1}	1
k_{n2}	$f\left(t + \frac{h}{2}, y + \frac{1}{2}hk_{n1}\right) = y_0 + \frac{1}{2}(1) = \frac{3}{2}$
k_{n3}	$f\left(t + \frac{h}{2}, y + \frac{1}{2}hk_{n2}\right) = y_0 + \frac{1}{2}\left(\frac{3}{2}\right) = \frac{7}{4}$
k_{n4}	$f(t + h, y + hk_{n3}) = y_0 + 1\left(\frac{7}{4}\right) = \frac{11}{4}$

etc.

Example

Use one iteration of Improved Euler and RK methods to the estimate solution to the given IVP at specified point.
$y' = \frac{1}{2} - t + 2y, y(0) = 1$. Estimate the solution at $t = 1$.

Euler's

$$\begin{aligned} y_1 &= y_0 + hf(t,y) \\ &= y_0 + h\left(\frac{1}{2} - t + 2y\right) \\ &= 1 + 1\left(\frac{1}{2} - 0 + 2\right) \\ &= \frac{7}{2} \end{aligned}$$

Improved Euler's

$$\begin{aligned} m &= \frac{f(t,y) + f(t+h, y + hf(t,y))}{2} \\ &= \frac{\frac{5}{2} + \left(\frac{1}{2} - (t-h) + 2\left(y + h(\frac{5}{2})\right)\right)}{2} \\ &= \frac{\frac{5}{2} + (\frac{1}{2} + 1 + 2(1 + \frac{5}{2}))}{2} \\ &= \frac{11}{2} \end{aligned}$$