A Baby Step to Spatial Semantics

Shimpei Endo*

Master of Logic, Institute for Logic, Language and Computation, Universteit van Amsterdam,
Amsterdam, The Netherlands
**E-mail: endoshimpeiendo@gmail.com*

This article presents a brand-new semantics for propositional modal logic named spatial semantics. This semantics features spatial concepts such as dimensions and projection to capture the meaning of modal operators $\Box$ and $\Diamond$. A few observations toward soundness are given to appeal its utility as a tool for characterizing many different logics including (modal logics based on) non-classical logics.

Keywords: semantics, propositional modal logics, metaphysics

1. Introduction

This article presents a novel formal system of semantics for propositional modal logic. The new semantics system named *spatial semantics* captures logical reasonings in terms of *space* namely *area* and *dimensions*.

We already have several semantics for propositional modal logics. The most prominent one should be, without question, the *relational semantics*, also known as *Kripke* semantics[1]. This semantics has almost become the *de facto* standard semantic framework for propositional modal logic thanks to its user-friendly and intuitive nature and expressive power. You can characterize most popular modal logics in terms of conditions over the relation; For example, *reflexivity* for the logic T *reflexivity* and *transitivity* for the logic $S4$ and so on (see a standard textbook[2] for modal logics).

Furthermore, there are alternative options with more spatial components such as *topological semantics*[3]. [a] Such spatial flavored semantics are expected to give proper and smooth accounts to some phenomena which

[a]As for semantics with more spatial flavor, we also have *neighborhood semantics*[4]. With respect to expressive power to make distinction between logics, it is a further strong competitor to us since it is possible to characterize non-standard (i.e. weaker than minimal modal logic K) modal logics.

cause troubles in the standard Kripke semantics.

As shown above, there seems to exist strong competitors. For what do I challenge them? Firstly, Kripke semantics faces a problem when we consider *applied* semantics out of this *pure* semantics. Kripke semantics itself is a stream of formal expressions – therefore *pure*, for Copeland's sense[5]. Our most popular framework disappoint a certain class of philosophers who expect mathematical formalization to represent or describe some metaphysical nature because Kripke semantics leaves out explanation of *what this very thing called relation is.* Formally, rather, *pure-semantically* speaking, each possible world in Kripke semantics is understood as the (maximally consistent) set of propositions. Plus, the relation which bridges among these worlds is written as a set of tuples: $W \times W$, given W is a set of worlds. However, this formalization does not provide sufficient (mathematical) description of what is this very thing called relation. In other words, it leaves a question what makes this relation among worlds hold. In a Kripke model, the relation is given and certain worlds are determined whether connected and others not without detailed information on what makes such relation hold.

Facing this problem of applied semantics, topological semantics seems to work better for providing metaphysical correspondence to these formal expressions at the first glance. Topological semantics does not impose such mysterious relations but it induces topology as its name tells, which mathematically determines (abstract) closeness among point in a set (or a universe, in a topological model). We can relatively easily grasp what topological semantics expresses as an applied semantics; it depicts topological (spatial) allocation of worlds spreading a certain space where every world sits.

Nevertheless, as relational semantics does, topological options still dismiss several non-classical logical systems from the beginning. They are both based on classical logic. Kripke semantics sees a world as a maximally consistent set of propositions. Topological semantics follows classicality immdediately because worlds are points and negation is understood as the compliment of P. Any model is sound and complete with respect to classical logic in these frameworks. [b] However, we still do not have many

[b]Some may disagree with this point by claiming the fact that Kripke semantics can mock and characterize non-classical systems well. In particular, famously, via Goedel translation, $S4$ is known to be the logic for intuitionistic logic and $S5$ be for classical. For a contemporary reference, see Mints' work[6].

semantical systems for modal logics based on non-classical logics. It would be useful if we have a system which can capture not only classical but also non-classical logics in the unified framework.

The construction of this article is as follows. The section 2 formulates the language of propositional modal logics which our semantics gives a meaning to. Then, we will define the structure and model of the new semantics (section 3). Before moving to the truth conditions, we will introduce an operation *squeezing* which plays a central role to define the truth conditions of implication ($\rightarrow$) and modal operators ($\Box$, $\Diamond$) in the section 4. Then the section 5 introduces the truth conditions and some graphical demonstrations to help the reader to grasp this brand-new system. The next section 6 overviews several results on soundness.

2. Language

We employ the standard notation for the language of propositional modal logics.

Definition 2.1 (Language of propositional modal logics). *Given a set of propositions* $PROP = \{p_1, p_2, ..., p_n, ...\}$*, a sentence of propositional modal logic is in the form of:*

$$\phi ::= p_i|\phi \wedge \phi|\phi \vee \phi|\phi \rightarrow \phi|\neg\phi|\Box\phi|\Diamond\phi$$

3. Structure and model

Definition 3.1 (Structure of spatial semantics). *Let* I *is an index set of at most countable. The structure of spatial semantics is called the* locus*:* $L = \prod_{i \in I}\langle D_i, \tau_i\rangle$*, while each* $\langle D_i, \tau_{i \in I}\rangle$ *forms a topology. An element of* L *is called a* world*.*

We get the model of spatial semantics once we add the valuation on this structure.

Definition 3.2 (Model of spatial semantics). *A model of spatial semantics* M *is the form of* $\langle L, V\rangle$ *with* L *a locus defined just above and a function valuation as follows.* $V : PROP \mapsto \mathcal{P}L$*; with* $p \in PROP$*,* $V(p) \subseteq L$*.*

4. Key operation: squeezing

This spatial semantics requires several operations to provide truth conditions (especially for modal operators $\Box$ and $\Diamond$).

This central operation is called *squeezing*, which generates new models from a given model via its *projection*, a well-known operation on product sets (or topologies). This operation forces the model to go *one step down, in a dimensional sense* in the following manner.

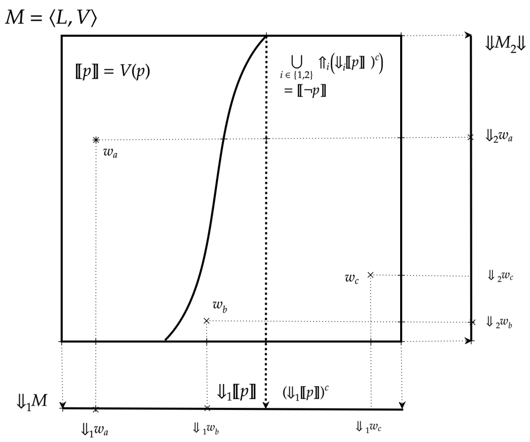
Definition 4.1 (Projection). *Let I, J be index sets. Write X_I for $X_I = \Pi_{i \in I} X_i$. A projection on X_I with $J \subset I$ is a function $\pi_J : X_I \mapsto X_J$, $x_{i \in I} \mapsto x_{j \in J}$. Write $\overrightarrow{x} = (x_1, x_2, ..., x_i, ...)$, with $x_i \in X_i$.*

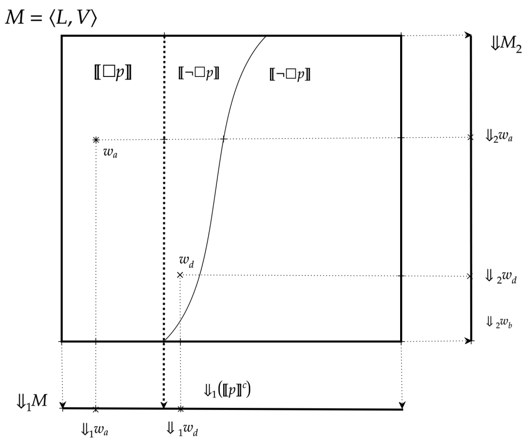
Our operation squeezing is based on a very simple type of projection: just eliminating one axis out of a given coordinate.

Definition 4.2 (Squeezing and unsqueezing). *Given $i \in I$ and $\overrightarrow{x} = (x_1, x_2, ..., x_{i-1} x_i, x_{i+1} ...)$, squeezing is a function $\Downarrow_i$ which gives $\Downarrow_i \overrightarrow{x} = \overrightarrow{x} = (x_1, x_2, ..., x_{i-1}, x_{i+1} ...)$ For a subset X of L, write $\Downarrow_i X = \{\Downarrow_i \overrightarrow{x} \overrightarrow{x} \in X\}$. Unsqueezing is defined as its* inverse. *Write $\Downarrow_i^{-1} := \Uparrow_i$.*

Let us observe examples to see how squeezing and unsqueezing work. $M, w_a \vDash p$ because $w_a \in [\![p]\!]$. Where does it make $\neg p$ true? It does *not* have to be the compliment of $[\![p]\!]$ in fact $w_b \notin [\![p]\!]$ but $w_b \nvDash \neg p$ since $w_b \notin [\![\neg p]\!]$. $M, w_c \vDash \neg p$ because $w_c \in [\![\neg p]\!]$.

To see modality, observe w_d (in a different picture but the same model M). $M, w_a \vDash \Box p$ since it has a direction to squeeze (namely $\Downarrow_1$) which makes $w_a \in \Uparrow ((\Downarrow [\![p]\!])^c)$. In contrast, $M, w_d \nvDash \Box p$ since in any direction $i \in I = \{1, 2\}$ to squeeze $\Downarrow_i w_d \notin (\Downarrow_i ([\![p]\!]^c)^c)$





There are two types of models in my framework: squeezed and original. This distinction will play a crucial role to distinguish between minimal and intuitionistic logic (under singleton conditions).

Definition 4.3 (Squeezed and original). *If a model is made by squeezing, it is a squeezed model. Otherwise, it is called the original model.*

5. Truth conditions

Truth conditions are defined by the following "area" constructed not only set-theoritical operation such as union and disjunction but also *dimensional* operation featuring squeezed models. This *truth-making area*, as its name tells, specifies a location in which a sentence becomes true within a locus. See the inductive definition as follows.

Definition 5.1 (Truth-making area). *Consider a spatial model* $M = \langle L, V \rangle$. *The truth-making area of a sentence* ϕ *is defined in the following inductive manner.*

- $[\![p]\!]_M = V(p)$
- $[\![\bot]\!]_M = [\![\phi]\!]_M \cap [\![\neg\phi]\!]_M t$
- $[\![\phi \wedge \psi]\!]_M = [\![\phi]\!]_M \cap [\![\psi]\!]_M$
- $[\![\phi \vee \psi]\!]_M = [\![\phi]\!]_M \cup [\![\psi]\!]_M$
- $[\![\phi \to \psi]\!]_M = \cup_{i\in I} \Uparrow_i^{(} (\Downarrow_i [\![\phi]\!]_M)^c \cup (\Downarrow_i ([\![\psi]\!]^c_M))^c)$
- $[\![\neg\phi]\!]_M = \cup_{i\in I} \Uparrow_i ((\Downarrow_i [\![\phi]\!])^c)$
- $[\![\Box\phi]\!]_M = \cup_{i\in I} \Uparrow_i ((\Downarrow_i ([\![\phi]\!]_M{}^c))^c)$
- $[\![\Diamond\phi]\!]_M = \cap_{i\in I} \Uparrow_i (\Downarrow_i [\![\phi]\!]_M)$

For the sake of simplicity, omit subscript $_M$ when obvious which model M is under consideration.

The most confusing clause would be implication. The notation looks entangling. But it just limits where $\phi \to \psi$ true from the simple disjunction $[\![\neg\phi]\!] \cup [\![\psi]\!]$ to a more narrowed area. It needs to secure $\neg\phi$ and ψ more rigidly. That is, you need to find some squeezed model $\neg\phi$ or ψ hold *without causing a contradiction in any squeezed models.* [c]

It requires some unfolding labor to check that $\neg\phi$ is the shortened expression of $\phi \to \bot$. Given a valuation V for a propositional letter p, where is $[\![\neg p]\!]$? Differently from the previous semantics, $[\![\neg p]\!]$ does not have to be the compliment of $[\![p]\!]$. There can appear some *gap* between $[\![p]\!]$ and $[\![\neg p]\!]$ when considering models with curved valuation $V(p)$.

Areas for modal operators $\Box$ and $\Diamond$ are defined to keep up with the following metaphysical idea. "Necessarily p" holds when p holds *safely, i.e., even given information is more limited.* Such situations of limited information are described as models generated by squeezing in my framework. The *necessarily-p* area is defined via the combination of squeezing and unsqueezing. That is to say, it specifies area which not only validates p but also avoids p and $\neg p$ holding at the same time in *some* model squeezed from there.

[c] It may make some intuitive sense once you recall that implication of intuitionisitic logic is defined topologically as $[\![\phi \to \psi]\!] = int([\![\phi]\!]^c \cup [\![\psi]\!])$. This makes $[\![\phi \to \psi]\!] \subseteq [\![\phi]\!]^c \cup [\![\psi]\!]$. This is a spatial (or dimensional) way of imposing such idea.

Now, we can define the truth condition for any sentence ϕ of the language of propositional modal logic using this truth-making area.

Definition 5.2 (Truth-condition). *$M, w \vDash \phi$ iff $w \in [\![\phi]\!]_M$.*

Let us check how these truth conditions work in the following examples.

Example 5.1 (Square on $\mathbb{R}^2$). *Consider the following model, which forms a square on $\mathbb{R}^2$. We consider a very small set of propositions $PROP = p$ and an only subset of the locus $\{w_a, w_b, w_c, w_d\} \subseteq [0,1] \times [0,1]$ for the sake of simplicity. Settle $M = \langle L, V \rangle$ as follows:*

- *$L = \langle D_1, \tau_1 \rangle \times \langle D_2, \tau_2 \rangle$ with each dimension of an interval of real numbers. $D_1, D_2 = [0,1]$ and both of τ_1 and τ_2 is the usual topology for the real.*
- *$V(p)$ is given as the figure shows.*

As for making p true, this semantics requires the world is included in $[\![p]\!] = V(p)$. Since $w_a, w_b \in [\![p]\!]$, $w_a \vDash p$, $w_b \vDash p$. Similarly, for $w_c, w_d \notin [\![p]\!]$, $w_c \nvDash p$, and $w_d \nvDash p$.

Let us see the negation $\neg p$. Note that, although $w_c \nvDash p$ (i.e., w_c fails to make p true), this does not promise $w_c \nvDash \neg p$ (i.e., w_c makes $\neg p$ true). This is due to how our semantics calculates $[\![\neg p]\!]$ out of the given $[\![p]\!]$; it does not work with a simple compliment but it requires to inspect the squeezed models and specify areas which do not p true there and *unsqueeze* such areas. As a result, it is only w_d (as for out current consideration) makes $\neg p$ true.

How about a modal sentence $\Box p$? It needs a recipe similar to a case of $\neg p$, which sees the squeezed models. The following explanation will help to make it understandable to unfold these complicated brackets in the definition. The concept *necessity* can be understood as mode of truth is *safe*: p is true *safely.* [d] Firstly, pick an arbitrary squeezed model $\Downarrow_i M$. Next, take the compliment of $\Downarrow_i [\![p]\!]$ in that squeezed model. This very operation mocks the thought that we can say p is *safely* true if it holds provided less information. Then, unsqueeze this area back to the original model and finally take the union of each squeezed dimension. This explains why $w_a \vDash \Box p$ while $w_b \nvDash \Box p$. On one hand, w_a has a dimension (namely $\Downarrow_1 M$) where w_a is not only in $\Downarrow_1 M$ but also shares location wiht *no* other

[d] Recall that this idea is imposed in topological semantics by taking $[\![\Box p]\!]$ to be *interior* of $[\![p]\!]$, excluding the boarder line of $[\![p]\!]$.

w_j such that $w_j \not\models p$ in the original M. On the other hand, w_b shares some w_j such that $w_j \not\models p$.

6. A few remarks on soundness

My semantics would disappoint many users since almost nothing they want is promised under it. In particular, our semantics, by itself, cannot validate even the weakest modal logic K because it fails *classicality* (for any classical tautology ϕ, $\models \phi$). Nevertheless, from the perspective of characterizing many logics, this is the strength rather than the weakness. Why? Because it makes possible to characterize propositional modal logics based on non-classical (propositional) logics in an unified framework. The logical weakness of this semantics turns out to be rather an advantage to embrace very weak logics (without bivalence for instance) and it will tell what spatial nature of each logic. Stronger logics will be written or expressed by putting certain conditions over our structure or models.

Our spatial semantics can promise few exceptionally fundamental axioms such as *duality* ($\Box p \leftrightarrow \neg\Diamond\neg p$ equivalently $\Diamond p \leftrightarrow \neg\Box\neg p$) and *necessitation* ($\models \phi$ entails $\models \Box\phi$). Other than them, including classicality, which is contained in the starting point of most semantics in the market, a

The following observations will lead characterization results and pinpoint (difficulties to reach) what we need to gain desired soundness results.

6.1. *Toward classical logic*

We will see how to *upgrade* it to more moderate non-classical logics namely minimal and intuitionisitc and, in turn, up to classical logic. Since our purpose is to characterize many (weak) logics so we do not start with much constraits over models. However, there is a model where most logicians have no use of: a model with -1 dimension, i.e., *empty* model.

Claim 6.1 (Empty model). *$\emptyset \not\models \phi$ for any sentence ϕ.*

Proof. Because $\emptyset \not\in \emptyset$. $\square$

Who wants such weakest logic, which proves nothing at all? However, we will keep this empty model in our storage since we will need it as the squeezed model of 0-dimensional model (i.e. singleton model).

Even eliminating this most crazy case, our framework drops law of explosion as minimal logic does.

Claim 6.2 (Failure of explosion). *Given ϕ a sentence of propositional modal logic and M^m is not empty, $M^m \not\models \bot \to \phi$.*

Proof. For instance, consider a squeezed model $\Downarrow_2 M$ in the previous example. $\Downarrow_2 M \neq [\![\bot \to \phi]\!]$ since $\Downarrow_1 \Downarrow_2 [\![\bot]\!] = \Downarrow_1 \Downarrow_2 [\![]\!] = \Downarrow_1 \Downarrow_2 M$. So its complement of singleton is $\emptyset$. $\Uparrow_1 \emptyset = \emptyset$. So $[\![\bot \to \phi]\!]$ is calculated in effect as $\Uparrow_1 \Downarrow_1 [\![\phi]\!]$, which does *not* have to equal to the entire $\Downarrow_2 M$. $\square$

By putting certain conditions over models, explosion should be recovered. A condition is to be original.

Claim 6.3 (Recovery of explosion). *If we consider any non-empty model M^i which is original, $M^i \models \bot \to \phi$ for a sentence ϕ.*

Proof. Observe that $[\![\bot]\!] = emptyset$ in any original model M^i. So is any squeezed model (except for empty one) $\Downarrow_j M^i$, $\Downarrow_j [\![\bot]\!] = \emptyset$, implying that $(\Downarrow_j [\![\bot]\!])^c = \Downarrow_j M$. This leads that $\Uparrow_j (\Downarrow_j [\![\bot]\!])^c = M$. Therefore, no matter what $\Uparrow_j \Downarrow_j [\![\phi]\!]$ takes, $[\![\bot \to \phi]\!] = M^i$. $\square$

This condition (or weaker) seems to be models for intuitionistic logic since it excludes law of excluded middle (bivalence).

Claim 6.4 (Bivalence fails). *$M^i \not\models p \vee \neg p$ for some M^i.*

Proof. Just find out a truth value gap between $[\![p]\!]$ and $[\![\neg p]\!]$. $\square$

Claim 6.5 (Bivalence recovers). *$M^c \models p \vee \neg p$ if an non-empty M^c satisfies the following condition: for every $p \in PROP$, there exists $i \in I$ such that $\Uparrow_i \Downarrow_i [\![p]\!] = [\![p]\!] M^c$.*

Proof. The condition tells that unsqueezing of squeezed valuation returns to the valuation itself. Notice that calculation defined in the truth condition makes compliment to be the truth-making area of negation: $[\![\neg p]\!] = [\![p]\!]^c$ (and $[\![\neg \phi]\!] = [\![\phi]\!]^c$ by induction on construction of ϕ). Apparently, there appears no truth value gap. $\square$

So far so good toward soundness and characterization; We seem to have found some conditions on spatial models to distinguish different non-classical logics. However, the story does not go that easy. Even minimal logic fails in our framework due to the tricky *implication* ($\to$).

Claim 6.6. *Let ϕ be a sentence of minimal logic. $\models \phi$ does not hold.*

Let us take natural deduction as syntax to observe this failure. When we try to prove soundness via induction on construction of proof, the introduction rule of implication ($\rightarrow: I$) causes trouble. Suppose we have a deduction D^* which derives q from an assumption p. As inductive hypothesis, assume that any model M satisfying $M \vDash p$ also satisfies $M \vDash q$. What to show is such $M \vDash p \rightarrow q$. We can get inclusive relation $[\![p]\!] \subseteq [\![q]\!]$ in the model M, but it does *not* suffice to make $[\![\rightarrow]\!]_M = M$ since $[\![p \rightarrow q]\!] \subset [\![p]\!]^c \cup [\![q]\!]$.

6.2. *Popular modal logics*

We have not settled even minimal logic yet. But before finding out perfect conditions for minimal logic, let us see what happens to some famous theorem of popular modal logics hereafter.

A few exceptionally fundamental theorems are promised in spatial semantics. Duality and neccesiation holds without conditions.

Theorem 6.1 (Duality). *If M^m is not empty, then $M^m \vDash \Box p \leftrightarrow \neg\Diamond\neg p$. Also, $\vDash \Diamond\phi \leftrightarrow \neg\Box\neg p$.*

Proof. By definition. It just requires some labor of unfolding the definitions. As for $\Box p$, notice that $[\![\Box\neg p]\!] = [\![\neg p]\!]$. Then calculate the union of unsqueezed of compliment of $[\![\neg p$ in any squeezed model: $\cup_{i \in I} \Downarrow_i$ A similar argument for the other one. $\Box$

Lemma 6.1. *If $[\![\phi]\!] = L$ and $[\![\neg\phi]\!] = \emptyset$, then $\vDash \phi$ and $\vDash \Box\phi$.*

Proof. For ϕ, observe that any $w \subseteq [\![\phi]\!] L$. For $\Box\phi$, check any squeezed direction, $\Downarrow_i L = \Downarrow_i [\![\phi]\!]$. So each w has $\Downarrow_i w$ such that $\Downarrow_i w \subseteq L$. $\Box$

Theorem 6.2. $\vDash \phi$ *implies* $\vDash \Box\phi$.

[e]

Proof. Using the lemma shown just above, it suffices to show that $\vDash \phi$ leads to $[\![\phi]\!] = L$ and $[\![\neg\phi]\!] = \emptyset$. Suppose $\vDash \phi$ and $[\![\phi]\!] \subsetneq L$. Then if one modifies a model M^* by putting an extra world $w^* \in L \setminus [\![\phi]\!]$ to M. $M^*, w^* \nvDash \phi$ apparently. This contradicts the assumption that $\vDash \phi$. The similar argument applies to lead a contradiction from $[\![\neg\phi]\!] \neq \emptyset$, by creating a model M^{**} with a new model $w^{**} \in [\![\neg\phi]\!]$ $\Box$

[e] Necessiation can be distinguished by a quick remedy. That is, to allow a locus to have space where no world sits. This blocks the argument employed in the proof of this article by accepting a model $M \vDash \phi$ but $[\![\phi]\!] \neq M$. Intuitively, the condition for necessiation is if a locus is *filled* with worlds.

Theorem 6.3 (K: distriburtion). *M^m is a non-empty model. $M^m \vDash \Box(p \to q) \to (\Box p \to \Box q)$.*

Proof. Induction on dimension. *Base case*: $M = \{\bullet\}$. $\Downarrow \{\bullet\} = \emptyset$ hence its compliment $(\Downarrow \{\bullet\})^c = \emptyset$. This entails K holds at $\{\bullet\}$ trivially. *Inductive step.* When there is no $i \in I$ into which you can squeeze p and q, i.e., i such that $\Uparrow_i \Downarrow_i [\![p \wedge q]\!] = \emptyset$, it trivially holds. Hence, only consider cases with such i. Fix such direction $i \in I$ to squeeze. To show is $(\Downarrow_i [\![\Box(p \wedge q)]\!])^c \cup \Downarrow_i [\![\Box p \wedge \Box q]\!] = \Downarrow_i M$ because it entails that $\Uparrow_i ((\Downarrow_i [\![\Box(p \wedge q)]\!])^c) \cup \Downarrow_i [\![\Box p \wedge \Box q]\!] = \Uparrow_i \Downarrow_i M = M$. Apparently,

$$\Downarrow_i [\![\Box(p \wedge q)]\!] \subseteq \Downarrow_i [\![\Box p]\!] \text{ and } \Downarrow_i [\![\Box(p \wedge q)]\!] \subseteq \Downarrow_i [\![\Box q]\!]$$

Recall

$$\Downarrow_i]\!]\Box p \wedge \Box q]\!] = \Downarrow_i ([\![\Box p]\!] \cap [\![\Box q]\!]) = \Downarrow_i [\![\Box p]\!] \cap \Downarrow_i [\![\Box q]\!]$$

Immediately,

$$\Downarrow_i [\![\Box(p \wedge q)]\!] \subseteq \Downarrow_i [\![\Box p \wedge \Box q]\!]$$

. □

To close this part, let us see another very popular theorem of propositional modal logics: T. In order to secure this theorem, non-emptieness is not enough.

Theorem 6.4 (T: $\Box p \to p$ fails). *Even M non-emtpy, $M \vDash \Box p \to p$ does not hold necessarily.*

Proof. Observe a model M_2 as a counter-model with a valuation which has two directions to squeeze to make $\Box p$. While $(\Downarrow_1 [\![\Box p]\!])^c = \Downarrow_2 [\![\Box p]\!])^c = \emptyset$, $[\![\Box p]\!]_{M_2} \subset M_2$. □

Once we greatly restrict our models into only with -1, 0 and 1, we will recover the T axiom.

Claim 6.7 (T recovers). *Consider a non-empty model L. If $I = 0, 1$, $M \vDash \Box p \to p$.*

Proof. Consider $\{\bullet\}$ as a zero-dimensional model. Since $[\![\Box p]\!]_{\{\bullet\}} = \{\bullet\}$, it leads $\Uparrow ((\Downarrow ([\![\Box p]\!]^c_{\{\bullet\}}))^c) = \{\bullet\}$ too. Hence, $[\![\Box p \to p]\!]_{\{\bullet\}} = \{\bullet\}$. Consider a structure $\langle L \rangle$ with a single dimension $L_1 = \langle D, \tau \rangle$. Its unique direction to squeeze is to its zero dimensional $\{\bullet\}$. Since $\Uparrow X = L_1$ for any $\Downarrow X \subseteq \Downarrow L_1$, it also trivially holds that $[\![\Box p \to p]\!]$. Confirm it by unpacking $\Uparrow (\Downarrow [\![\Box p]\!])^c = L_1$. □

7. Conclusive remarks

We have observed that the new formal semantics for propositional modal logics. This new semantics called spatial semantics features spatial concepts such as dimension. It works to capture non-classical reasonings and captures how modal operators play in modal reasoning. This semantics just by itself is very useless for most since it cannot even satisfy a common formal systems such as the minimal modal logics K. However, taken it as a formal *platform* where you can compare more logics and describe their characteristics in a unified framework, it benefits formal and philosophical debates among different logics. To demonstrate this function of spatial semantics, we have observed several segments of results for soundness by putting certain conditions.

7.1. *Impact*

The original motivation is metaphysical but it gives a formal benefit independently of philosophical aim. Formal benefit is obvious. It is who wants to model many sorts of logic (including non-classical ones) with the single framework. This semantics has space for distinguishing classical and non-classical logic in some spatial terms.

This formal benefit can go beyond pure semantics. This semantics is better (or, hopefully, is the one) applied semantics which provides the mathematical description of what universe (if any) stands. In particular, this semantics seems to work perfectly for formal description of Yagisawa's modal dimensionalism[7], which takes modality as a sort of dimension(s) called modal dimension(s) in addition to space and time. More over, this spatial framework would provide a more rigid and rich description on

7.2. *Further tasks*

Toward soundness, our tasks are two-folds. The first we need to find out not only necessary but sufficient conditions to make more certain logic, if any. The second concerns the general heuristic scheme to find the first. Conditions given in this paper were all found by much of try-and-error attempts and some inspiration. We want a machine-like routine to find and determine the characteristics, if any. Or, it is still open that which logic can characterize spatial models and be characterized by spatial models.

Any logician would be curious of *completeness*. The most plausible strategy for the proof is to build a sort of conversion (one-to-one correspondence) between my semantics and other systems with well-known completeness results (see[3] for this type of proof). Preferably for metaphysical and applied-semantical sake, it is desired to have a more direct proof with some *spatial* twist (as , Aiello[8], for instance, does with *topological* twist).

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