Numerical Differentiation & Integration

Romberg Integration

Numerical Methods (4th Edition)

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Beamer Presentation Slides
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Outline

1. Composite Trapezoidal Rule & Richardson Extrapolation

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2. Romberg Integration: Basic Construction

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3. Romberg Integration: Recursive Calculation

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Numerical Integration: Basic Romberg Method

Composite Trapezoidal Rule: Error Term

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Numerical Integration: Basic Romberg Method

Composite Trapezoidal Rule: Error Term

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- We have seen that the Composite Trapezoidal rule has a truncation error of order $O(h^2)$.

Numerical Integration: Basic Romberg Method

Composite Trapezoidal Rule: Error Term

- We will illustrate how Richardson extrapolation applied to results from the Composite Trapezoidal rule can be used to obtain high accuracy approximations with little computational cost.
- We have seen that the Composite Trapezoidal rule has a truncation error of order $O(h^2)$. Specifically, we showed that for $h = (b-a)/n$ and $x_j = a + jh$ we have

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{(b-a)f''(\mu)}{12}h^2$$

for some number μ in(a, b).

Numerical Integration: Basic Romberg Method

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Composite Trapezoidal Rule: Error Term (Cont'd)

Numerical Integration: Basic Romberg Method

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By an alternative method, it can be shown that if $f \in C^\infty[a,b]$,

Numerical Integration: Basic Romberg Method

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{(b-a)f''(\mu)}{12}h^2$$

Composite Trapezoidal Rule: Error Term (Cont'd)

By an alternative method, it can be shown that if $f \in C^\infty[a,b]$, the Composite Trapezoidal rule can also be written with an error term in the form

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] + K_1h^2 + K_2h^4 + K_3h^6 + \cdots$$

where each K_i is a constant that depends only on $f^{(2i-1)}(a)$ and $f^{(2i-1)}(b)$.

Numerical Integration: Basic Romberg Method

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] + K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots$$

Applying Richardson Extrapolation

Numerical Integration: Basic Romberg Method

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] + K_1h^2 + K_2h^4 + K_3h^6 + \cdots$$

Applying Richardson Extrapolation

- We have seen that Richardson extrapolation can be performed on any approximation procedure whose truncation error is of the form

$$\sum_{j=1}^{m-1} K_j h^{\alpha_j} + O(h^{\alpha_m})$$

for a collection of constants K_j and when $\alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_m$.

Numerical Integration: Basic Romberg Method

Applying Richardson Extrapolation (Cont'd)

- In particular, we have seen demonstrations to illustrate how effective this techniques is when the approximation procedure has a truncation error with only even powers of h, that is, when the truncation error has the form:

$$\sum_{j=1}^{m-1} K_j h^{2j} + O(h^{2m})$$

Numerical Integration: Basic Romberg Method

Applying Richardson Extrapolation (Cont'd)

- In particular, we have seen demonstrations to illustrate how effective this techniques is when the approximation procedure has a truncation error with only even powers of h, that is, when the truncation error has the form:

$$\sum_{j=1}^{m-1} K_j h^{2j} + O(h^{2m})$$

- Because the Composite Trapezoidal rule has this form, it is an obvious candidate for extrapolation. This results in a technique known as Romberg integration.

Outline

1. Composite Trapezoidal Rule & Richardson Extrapolation
2. Romberg Integration: Basic Construction
3. Romberg Integration: Recursive Calculation
4. Romberg Integration: The Recursive Algorithm

Numerical Integration: Basic Romberg Method

Applying Richardson Extrapolation (Cont'd)

- To approximate the integral $\int_a^b f(x)\,dx$ we use the results of the Composite Trapezoidal Rule with $n = 1, 2, 4, 8, 16, \ldots$, and denote the resulting approximations, respectively, by $R_{1,1}$, $R_{2,1}$, $R_{3,1}$, etc.

Numerical Integration: Basic Romberg Method

Applying Richardson Extrapolation (Cont'd)

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- We then apply extrapolation in the manner seen before, that is, we obtain $O(h^4)$ approximations $R_{2,2}$, $R_{3,2}$, $R_{4,2}$, etc, by

$$R_{k,2} = R_{k,1} + \frac{1}{3}(R_{k,1} - R_{k-1,1}), \quad \text{for } k = 2, 3, \ldots$$

Numerical Integration: Basic Romberg Method

Applying Richardson Extrapolation (Cont'd)

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- We then apply extrapolation in the manner seen before, that is, we obtain $O(h^4)$ approximations $R_{2,2}$, $R_{3,2}$, $R_{4,2}$, etc, by

$$R_{k,2} = R_{k,1} + \frac{1}{3}(R_{k,1} - R_{k-1,1}), \quad \text{for } k = 2, 3, \ldots$$

and $O(h^6)$ approximations $R_{3,3}$, $R_{4,3}$, $R_{5,3}$, etc, by

$$R_{k,3} = R_{k,2} + \frac{1}{15}(R_{k,2} - R_{k-1,2}), \quad \text{for } k = 3, 4, \ldots.$$

Numerical Integration: Basic Romberg Method

Romberg Integration

In general, after the appropriate $R_{k,j-1}$ approximations have been obtained, we determine the $O(h^{2j})$ approximations from

$$R_{k,j} = R_{k,j-1} + \frac{1}{4^{j-1}-1}(R_{k,j-1} - R_{k-1,j-1}), \quad \text{for } k = j, j+1, \ldots$$

Numerical Integration: Basic Romberg Method

Example: Composite Trapezoidal & Romberg

- Use the Composite Trapezoidal rule to find approximations to $\int_0^\pi \sin x \, dx$ with $n = 1, 2, 4, 8,$ and 16.

Numerical Integration: Basic Romberg Method

Example: Composite Trapezoidal & Romberg

- Use the Composite Trapezoidal rule to find approximations to $\int_0^\pi \sin x \, dx$ with $n = 1, 2, 4, 8,$ and 16.
- Then perform Romberg extrapolation on the results.

Numerical Integration: Basic Romberg Method

Solution (1/6): Composite Trapezoidal Rule Approximations

The Composite Trapezoidal rule for the various values of n gives the following approximations to the true value 2.

$$R_{1,1} = \frac{\pi}{2}[\sin 0 + \sin \pi] = 0$$

Numerical Integration: Basic Romberg Method

Solution (1/6): Composite Trapezoidal Rule Approximations

The Composite Trapezoidal rule for the various values of n gives the following approximations to the true value 2.

$$
\begin{aligned}
R_{1,1} &= \frac{\pi}{2}[\sin 0 + \sin \pi] = 0 \\
R_{2,1} &= \frac{\pi}{4}\left[\sin 0 + 2\sin\frac{\pi}{2} + \sin \pi\right] = 1.57079633
\end{aligned}
$$

Numerical Integration: Basic Romberg Method

Solution (1/6): Composite Trapezoidal Rule Approximations

The Composite Trapezoidal rule for the various values of n gives the following approximations to the true value 2.

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\begin{aligned}
R_{1,1} &= \frac{\pi}{2}[\sin 0 + \sin \pi] = 0 \\
R_{2,1} &= \frac{\pi}{4}\left[\sin 0 + 2\sin\frac{\pi}{2} + \sin \pi\right] = 1.57079633 \\
R_{3,1} &= \frac{\pi}{8}\left[\sin 0 + 2\left(\sin\frac{\pi}{4} + \sin\frac{\pi}{2} + \sin\frac{3\pi}{4}\right) + \sin \pi\right] \\
&= 1.89611890
\end{aligned}
$$

Numerical Integration: Basic Romberg Method

Solution (2/6): Composite Trapezoidal Rule Approximations

$$R_{4,1} = \frac{\pi}{16}\left[\sin 0 + 2\left(\sin\frac{\pi}{8} + \sin\frac{\pi}{4} + \cdots + \sin\frac{3\pi}{4} + \sin\frac{7\pi}{8}\right) + \sin\pi\right] = 1.97423160$$

Numerical Integration: Basic Romberg Method

Solution (2/6): Composite Trapezoidal Rule Approximations

$$R_{4,1} = \frac{\pi}{16}\left[\sin 0 + 2\left(\sin\frac{\pi}{8} + \sin\frac{\pi}{4} + \cdots + \sin\frac{3\pi}{4} + \sin\frac{7\pi}{8}\right) + \sin\pi\right] = 1.97423160$$

$$R_{5,1} = \frac{\pi}{32}\left[\sin 0 + 2\left(\sin\frac{\pi}{16} + \sin\frac{\pi}{8} + \cdots + \sin\frac{7\pi}{8} + \sin\frac{15\pi}{16}\right) + \sin\pi\right] = 1.99357034$$

Numerical Integration: Basic Romberg Method

Solution (3/6): Romberg Extrapolation

The $O(h^4)$ approximations are

Numerical Integration: Basic Romberg Method

Solution (3/6): Romberg Extrapolation

The $O(h^4)$ approximations are

$$R_{2,2} = R_{2,1} + \frac{1}{3}(R_{2,1} - R_{1,1}) = 2.09439511$$

Numerical Integration: Basic Romberg Method

Solution (3/6): Romberg Extrapolation

The $O(h^4)$ approximations are

$$\begin{aligned} R_{2,2} &= R_{2,1} + \frac{1}{3}(R_{2,1} - R_{1,1}) = 2.09439511 \\ R_{3,2} &= R_{3,1} + \frac{1}{3}(R_{3,1} - R_{2,1}) = 2.00455976 \end{aligned}$$

Numerical Integration: Basic Romberg Method

Solution (3/6): Romberg Extrapolation

The $O(h^4)$ approximations are

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\begin{aligned}
R_{2,2} &= R_{2,1} + \frac{1}{3}(R_{2,1} - R_{1,1}) = 2.09439511 \\
R_{3,2} &= R_{3,1} + \frac{1}{3}(R_{3,1} - R_{2,1}) = 2.00455976 \\
R_{4,2} &= R_{4,1} + \frac{1}{3}(R_{4,1} - R_{3,1}) = 2.00026917
\end{aligned}
$$

Numerical Integration: Basic Romberg Method

Solution (3/6): Romberg Extrapolation

The $O(h^4)$ approximations are

$$\begin{aligned}
R_{2,2} &= R_{2,1} + \frac{1}{3}(R_{2,1} - R_{1,1}) = 2.09439511 \\
R_{3,2} &= R_{3,1} + \frac{1}{3}(R_{3,1} - R_{2,1}) = 2.00455976 \\
R_{4,2} &= R_{4,1} + \frac{1}{3}(R_{4,1} - R_{3,1}) = 2.00026917 \\
R_{5,2} &= R_{5,1} + \frac{1}{3}(R_{5,1} - R_{4,1}) = 2.00001659
\end{aligned}$$

Numerical Integration: Basic Romberg Method

Solution (4/6): Romberg Extrapolation

The $O(h^6)$ approximations are

Numerical Integration: Basic Romberg Method

Solution (4/6): Romberg Extrapolation

The $O(h^6)$ approximations are

$$
\begin{aligned}
R_{3,3} &= R_{3,2} + \frac{1}{15}(R_{3,2} - R_{2,2}) = 1.99857073 \\
R_{4,3} &= R_{4,2} + \frac{1}{15}(R_{4,2} - R_{3,2}) = 1.99998313 \\
R_{5,3} &= R_{5,2} + \frac{1}{15}(R_{5,2} - R_{4,2}) = 1.99999975
\end{aligned}
$$

Numerical Integration: Basic Romberg Method

Solution (5/6): Romberg Extrapolation

The two $O(h^8)$ approximations are

Numerical Integration: Basic Romberg Method

Solution (5/6): Romberg Extrapolation

The two $O(h^8)$ approximations are

$$\begin{aligned} R_{4,4} &= R_{4,3} + \frac{1}{63}(R_{4,3} - R_{3,3}) = 2.00000555 \\ R_{5,4} &= R_{5,3} + \frac{1}{63}(R_{5,3} - R_{4,3}) = 2.00000001 \end{aligned}$$

and the final $O(h^{10})$ approximation is

$$R_{5,5} = R_{5,4} + \frac{1}{255}(R_{5,4} - R_{4,4}) = 1.99999999$$

These results are shown in the following table.

Numerical Integration: Basic Romberg Method

Solution (6/6): Tabulated Extrapolation Results

0				
1.57079633	2.09439511			
1.89611890	2.00455976	1.99857073		
1.97423160	2.00026917	1.99998313	2.00000555	
1.99357034	2.00001659	1.99999975	2.00000001	1.99999999

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Romberg Integration Recursive Calculation

A More Efficient Implementation

- Notice that when generating the approximations for the Composite Trapezoidal Rule approximations in the last example, each consecutive approximation included all the functions evaluations from the previous approximation.

Romberg Integration Recursive Calculation

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- That is, $R_{1,1}$ used evaluations at 0 and π, $R_{2,1}$ used these evaluations and added an evaluation at the intermediate point $\pi/2$.

Romberg Integration Recursive Calculation

A More Efficient Implementation

- Notice that when generating the approximations for the Composite Trapezoidal Rule approximations in the last example, each consecutive approximation included all the functions evaluations from the previous approximation.
- That is, $R_{1,1}$ used evaluations at 0 and π, $R_{2,1}$ used these evaluations and added an evaluation at the intermediate point $\pi/2$.
- Then $R_{3,1}$ used the evaluations of $R_{2,1}$ and added two additional intermediate ones at $\pi/4$ and $3\pi/4$.

Romberg Integration Recursive Calculation

A More Efficient Implementation

- Notice that when generating the approximations for the Composite Trapezoidal Rule approximations in the last example, each consecutive approximation included all the functions evaluations from the previous approximation.
- That is, $R_{1,1}$ used evaluations at 0 and π, $R_{2,1}$ used these evaluations and added an evaluation at the intermediate point $\pi/2$.
- Then $R_{3,1}$ used the evaluations of $R_{2,1}$ and added two additional intermediate ones at $\pi/4$ and $3\pi/4$.
- This pattern continues with $R_{4,1}$ using the same evaluations as $R_{3,1}$ but adding evaluations at the 4 intermediate points $\pi/8$, $3\pi/8$, $5\pi/8$, and $7\pi/8$, and so on.

Romberg Integration: Recursive Calculation

A More Efficient Implementation (Cont'd)

- This evaluation procedure for Composite Trapezoidal Rule approximations holds for an integral on any interval $[a, b]$.

Romberg Integration: Recursive Calculation

A More Efficient Implementation (Cont'd)

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- In general, the Composite Trapezoidal Rule denoted $R_{k+1,1}$ uses the same evaluations as $R_{k,1}$ but adds evaluations at the 2^{k-2} intermediate points.

Romberg Integration: Recursive Calculation

A More Efficient Implementation (Cont'd)

- This evaluation procedure for Composite Trapezoidal Rule approximations holds for an integral on any interval $[a, b]$.
- In general, the Composite Trapezoidal Rule denoted $R_{k+1,1}$ uses the same evaluations as $R_{k,1}$ but adds evaluations at the 2^{k-2} intermediate points.
- Efficient calculation of these approximations can therefore be done in a recursive manner.

Romberg Integration: Recursive Calculation (Cont'd)

Formulating a Recursive Algorithm

To obtain the Composite Trapezoidal Rule approximations for $\int_a^b f(x)\,dx$, let $h_k = (b-a)/m_k = (b-a)/2^{k-1}$.

Romberg Integration: Recursive Calculation (Cont'd)

Formulating a Recursive Algorithm

To obtain the Composite Trapezoidal Rule approximations for $\int_a^b f(x)\,dx$, let $h_k = (b-a)/m_k = (b-a)/2^{k-1}$. Then

$$R_{1,1} = \frac{h_1}{2}[f(a)+f(b)] = \frac{(b-a)}{2}[f(a)+f(b)]$$

and

Romberg Integration: Recursive Calculation (Cont'd)

Formulating a Recursive Algorithm

To obtain the Composite Trapezoidal Rule approximations for $\int_a^b f(x)\,dx$, let $h_k = (b-a)/m_k = (b-a)/2^{k-1}$. Then

$$
\begin{aligned}
R_{1,1} &= \frac{h_1}{2}[f(a)+f(b)] = \frac{(b-a)}{2}[f(a)+f(b)] \\
\text{and} \qquad R_{2,1} &= \frac{h_2}{2}[f(a)+f(b)+2f(a+h_2)]
\end{aligned}
$$

Romberg Integration: Recursive Calculation (Cont'd)

Formulating a Recursive Algorithm

To obtain the Composite Trapezoidal Rule approximations for $\int_a^b f(x)\,dx$, let $h_k = (b-a)/m_k = (b-a)/2^{k-1}$. Then

$$
\begin{aligned}
R_{1,1} &= \frac{h_1}{2}[f(a)+f(b)] = \frac{(b-a)}{2}[f(a)+f(b)] \\
\text{and} \qquad R_{2,1} &= \frac{h_2}{2}[f(a)+f(b)+2f(a+h_2)]
\end{aligned}
$$

By re-expressing this result for $R_{2,1}$ we can incorporate the previously determined approximation $R_{1,1}$

$$
R_{2,1} = \frac{(b-a)}{4}\left[f(a)+f(b)+2f\left(a+\frac{(b-a)}{2}\right)\right] = \frac{1}{2}[R_{1,1}+h_1 f(a+h_2)]
$$

Romberg Integration: Recursive Calculation (Cont'd)

Formulating a Recursive Algorithm

In a similar manner we can write

$$R_{3,1} = \frac{1}{2}\{R_{2,1} + h_2[f(a+h_3) + f(a+3h_3)]\}$$

and, in general See Diagram, we have

$$R_{k,1} = \frac{1}{2}\left[R_{k-1,1} + h_{k-1}\sum_{i=1}^{2^{k-2}} f\left(a+(2i-1)h_k\right)\right]$$

for each $k = 2, 3, \ldots, n$.

Romberg Integration: Recursive Calculation (Cont'd)

Extrapolation then is used to produce $O(h_k^{2j})$ approximations by

Romberg Method

$$R_{k,j} = R_{k,j-1} + \frac{1}{4^{j-1} - 1}(R_{k,j-1} - R_{k-1,j-1})$$

for $k = j, j+1, \ldots$.

as shown in the following table.

Romberg Integration: Recursive Calculation (Cont'd)

$$R_{k,j} = R_{k,j-1} + \frac{1}{4^{j-1}-1}(R_{k,j-1} - R_{k-1,j-1})$$

for $k = j, j+1, \ldots$.

The Romberg Table

k	$O\left(h_k^2\right)$	$O\left(h_k^4\right)$	$O\left(h_k^6\right)$	$O\left(h_k^8\right)$		$O\left(h_k^{2n}\right)$
1	$R_{1,1}$					
2	$R_{2,1}$	$R_{2,2}$				
3	$R_{3,1}$	$R_{3,2}$	$R_{3,3}$			
4	$R_{4,1}$	$R_{4,2}$	$R_{4,3}$	$R_{4,4}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
n	$R_{n,1}$	$R_{n,2}$	$R_{n,3}$	$R_{n,4}$	$\cdots$	$R_{n,n}$

Romberg Integration: Recursive Calculation

Constructing the Romberg Table: One Row at a Time

- The effective method to construct the Romberg table makes use of the highest order of approximation at each step.

Romberg Integration: Recursive Calculation

Constructing the Romberg Table: One Row at a Time

- The effective method to construct the Romberg table makes use of the highest order of approximation at each step.
- That is, it calculates the entries row by row, in the order $R_{1,1}$, $R_{2,1}$, $R_{2,2}$, $R_{3,1}$, $R_{3,2}$, $R_{3,3}$, etc.

Romberg Integration: Recursive Calculation

Constructing the Romberg Table: One Row at a Time

- The effective method to construct the Romberg table makes use of the highest order of approximation at each step.
- That is, it calculates the entries row by row, in the order $R_{1,1}$, $R_{2,1}$, $R_{2,2}$, $R_{3,1}$, $R_{3,2}$, $R_{3,3}$, etc.
- This also permits an entire new row in the table to be calculated by doing only one additional application of the Composite Trapezoidal rule.

Romberg Integration: Recursive Calculation

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- This also permits an entire new row in the table to be calculated by doing only one additional application of the Composite Trapezoidal rule.
- It then uses a simple averaging on the previously calculated values to obtain the remaining entries in the row.

Romberg Integration: Recursive Calculation

Constructing the Romberg Table: One Row at a Time

- The effective method to construct the Romberg table makes use of the highest order of approximation at each step.
- That is, it calculates the entries row by row, in the order $R_{1,1}$, $R_{2,1}$, $R_{2,2}$, $R_{3,1}$, $R_{3,2}$, $R_{3,3}$, etc.
- This also permits an entire new row in the table to be calculated by doing only one additional application of the Composite Trapezoidal rule.
- It then uses a simple averaging on the previously calculated values to obtain the remaining entries in the row.
- Calculate the Romberg table one complete row at a time.

Romberg Integration: Recursive Calculation

Example: Extending the Romberg Table

Add an additional extrapolation row to the Romberg table of the previous example:

0				
1.57079633	2.09439511			
1.89611890	2.00455976	1.99857073		
1.97423160	2.00026917	1.99998313	2.00000555	
1.99357034	2.00001659	1.99999975	2.00000001	1.99999999

to approximate $\int_0^{\pi} \sin x \, dx$.

Romberg Integration: Recursive Calculation

Solution (1/4): Generate Additional Row of the Table

To obtain the additional row we need the trapezoidal approximation

$$R_{6,1} = \frac{1}{2}\left[R_{5,1} + \frac{\pi}{16}\sum_{k=1}^{2^4} \sin\frac{(2k-1)\pi}{32}\right] = 1.99839336$$

Solution (2/4): Generate New Row Values of the Romberg Table

The values of the new row (See Table) are as follows:

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The values of the new row (See Table) are as follows:

$$R_{6,2} = R_{6,1} + \frac{1}{3}(R_{6,1} - R_{5,1})$$

Solution (2/4): Generate New Row Values of the Romberg Table

The values of the new row (See Table) are as follows:

$$\begin{aligned}
R_{6,2} &= R_{6,1} + \frac{1}{3}(R_{6,1} - R_{5,1}) \\
&= 1.99839336 + \frac{1}{3}(1.99839336 - 1.99357035) = 2.00000103
\end{aligned}$$

Solution (2/4): Generate New Row Values of the Romberg Table

The values of the new row (See Table) are as follows:

$$
\begin{aligned}
R_{6,2} &= R_{6,1} + \frac{1}{3}(R_{6,1} - R_{5,1}) \\
&= 1.99839336 + \frac{1}{3}(1.99839336 - 1.99357035) = 2.00000103 \\
R_{6,3} &= R_{6,2} + \frac{1}{15}(R_{6,2} - R_{5,2})
\end{aligned}
$$

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The values of the new row (See Table) are as follows:

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\begin{aligned}
R_{6,2} &= R_{6,1} + \frac{1}{3}(R_{6,1} - R_{5,1}) \\
&= 1.99839336 + \frac{1}{3}(1.99839336 - 1.99357035) = 2.00000103 \\
R_{6,3} &= R_{6,2} + \frac{1}{15}(R_{6,2} - R_{5,2}) \\
&= 2.00000103 + \frac{1}{15}(2.00000103 - 2.00001659) = 2.00000000
\end{aligned}
$$

Solution (2/4): Generate New Row Values of the Romberg Table

The values of the new row (See Table) are as follows:

$$
\begin{aligned}
R_{6,2} &= R_{6,1} + \frac{1}{3}(R_{6,1} - R_{5,1}) \\
&= 1.99839336 + \frac{1}{3}(1.99839336 - 1.99357035) = 2.00000103 \\
R_{6,3} &= R_{6,2} + \frac{1}{15}(R_{6,2} - R_{5,2}) \\
&= 2.00000103 + \frac{1}{15}(2.00000103 - 2.00001659) = 2.00000000 \\
R_{6,4} &= R_{6,3} + \frac{1}{63}(R_{6,3} - R_{5,3}) = 2.00000000
\end{aligned}
$$

Solution (2/4): Generate New Row Values of the Romberg Table

The values of the new row (See Table) are as follows:

$$
\begin{aligned}
R_{6,2} &= R_{6,1} + \frac{1}{3}(R_{6,1} - R_{5,1}) \\
&= 1.99839336 + \frac{1}{3}(1.99839336 - 1.99357035) = 2.00000103 \\
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$$

Romberg Integration: Recursive Calculation

Solution (3/4): The Final Extrapolation Table

0					
1.57079633	2.09439511				
1.89611890	2.00455976	1.99857073			
1.97423160	2.00026917	1.99998313	2.00000555		
1.99357034	2.00001659	1.99999975	2.00000001	1.99999999	
1.99839336	2.00000103	2.00000000	2.00000000	2.00000000	2.00000000

Romberg Integration: Recursive Calculation

Solution (4/4): Comments on the Numerical Results

- Notice that all the extrapolated values except for the first (in the first row of the second column) are more accurate than the best composite trapezoidal approximation (in the last row of the first column).

Romberg Integration: Recursive Calculation

Solution (4/4): Comments on the Numerical Results

- Notice that all the extrapolated values except for the first (in the first row of the second column) are more accurate than the best composite trapezoidal approximation (in the last row of the first column).
- Although there are 21 entries in the table, only the six in the left column require function evaluations since these are the only entries generated by the Composite Trapezoidal rule; the other entries are obtained by an averaging process.

Romberg Integration: Recursive Calculation

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- In fact, because of the recurrence relationship of the terms in the left column, the only function evaluations needed are those to compute the final Composite Trapezoidal rule approximation.

Romberg Integration: Recursive Calculation

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- Although there are 21 entries in the table, only the six in the left column require function evaluations since these are the only entries generated by the Composite Trapezoidal rule; the other entries are obtained by an averaging process.
- In fact, because of the recurrence relationship of the terms in the left column, the only function evaluations needed are those to compute the final Composite Trapezoidal rule approximation.
- In general, $R_{k,1}$ requires $1 + 2^{k-1}$ function evaluations, so in this case $1 + 2^5 = 33$ are needed.

Outline

The Romberg Algorithm

To approximate the integral $I = \int_a^b f(x)\,dx$, select an integer $n > 0$.

INPUT endpoints a, b; integer n.

OUTPUT an array R (compute R by rows; only the last 2 rows are saved in storage).

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Steps 3 to 9 are on the next slide

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$$\text{Set } R_{2,1} = \frac{1}{2}\left[R_{1,1} + h\sum_{k=1}^{2^{i-2}} f(a + (k - 0.5)h)\right]$$

(Approximation from the Trapezoidal method)

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Step 5 For $j = 2, \ldots, i$

$$\text{set } R_{2,j} = R_{2,j-1} + \frac{R_{2,j-1} - R_{1,j-1}}{4^{j-1} - 1} \qquad (\textit{Extrapolation})$$

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(*Update row 1 of R*)

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(*Update row 1 of R*)

Step 9 STOP

The Romberg Algorithm

Comments on the Algorithm (1/2)

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- The algorithm requires a preset integer n to determine the number of rows to be generated.

The Romberg Algorithm

Comments on the Algorithm (1/2)

- The algorithm requires a preset integer n to determine the number of rows to be generated.
- We could also set an error tolerance for the approximation and generate n, within some upper bound, until consecutive diagonal entries $R_{n-1,n-1}$ and $R_{n,n}$ agree to within the tolerance.

The Romberg Algorithm

Comments on the Algorithm (2/2)

The Romberg Algorithm

Comments on the Algorithm (2/2)

- To guard against the possibility that two consecutive row elements agree with each other but not with the value of the integral being approximated, it is common to generate approximations until not only

$$\left|R_{n-1,n-1} - R_{n,n}\right|$$

is within the tolerance, but also

$$\left|R_{n-2,n-2} - R_{n-1,n-1}\right|$$

The Romberg Algorithm

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is within the tolerance, but also

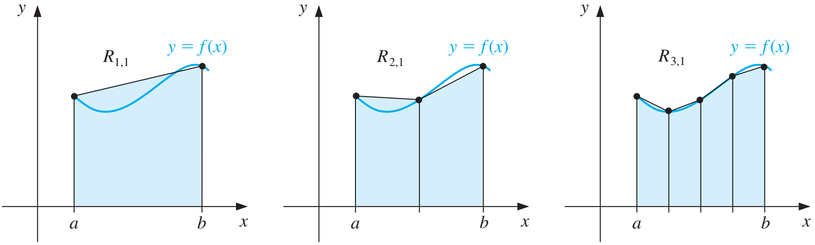
$$\left|R_{n-2,n-2} - R_{n-1,n-1}\right|$$

- Although not a universal safeguard, this will ensure that two differently generated sets of approximations agree within the specified tolerance before $R_{n,n}$, is accepted as sufficiently accurate.

Questions?

Reference Material

The Romberg Method



$$R_{k,1} = \frac{1}{2}\left[R_{k-1,1} + h_{k-1}\sum_{i=1}^{2^{k-2}} f\left(a + (2i-1)h_k\right)\right]$$

for each $k = 2, 3, \ldots, n$.

Return to Recursive Formulation of Romberg

Romberg Table

The Romberg Table

k	$O\left(h_k^2\right)$	$O\left(h_k^4\right)$	$O\left(h_k^6\right)$	$O\left(h_k^8\right)$		$O\left(h_k^{2n}\right)$
1	$R_{1,1}$					
2	$R_{2,1}$	$R_{2,2}$				
3	$R_{3,1}$	$R_{3,2}$	$R_{3,3}$			
4	$R_{4,1}$	$R_{4,2}$	$R_{4,3}$	$R_{4,4}$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
n	$R_{n,1}$	$R_{n,2}$	$R_{n,3}$	$R_{n,4}$	$\cdots$	$R_{n,n}$

Return to Romberg Integration Example

$$R_{k,j} = R_{k,j-1} + \frac{1}{4^{j-1} - 1}\left(R_{k,j-1} - R_{k-1,j-1}\right)$$

for $k = j, j+1, \ldots$.