$$p(X=\mathcal{D}\mid\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$$

$$= \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

$$p(\lambda\mid X=\mathcal{D}) \propto \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i} \frac{1}{\Gamma(k)\theta^k} \lambda^{k-1} e^{-\lambda/\theta}$$

$$\propto \frac{1}{\Gamma(k)\theta^k} \lambda^{k+n-1} e^{-\lambda \sum_{i=1}^{n} x_i - \lambda/\theta}$$

Let $\hat{k} = k+n$, $\hat{\theta} = \dfrac{1}{\sum_{i=1} x_i + 1/\theta}$

$$p(\lambda\mid X=\mathcal{D}) = \eta \lambda^{\hat{k}-1} e^{-\lambda/\hat{\theta}}$$

$$= \frac{1}{\Gamma(\hat{k})\hat{\theta}^{\hat{k}}} \lambda^{\hat{k}-1} e^{-\lambda/\hat{\theta}}$$

$$p(x\mid\mathcal{D}) = \int p(x\mid\lambda,\mathcal{D})\, p(\lambda\mid\mathcal{D})\, d\lambda$$

$$= \int \lambda e^{-\lambda x} \frac{1}{\Gamma(\hat{k})\hat{\theta}^{\hat{k}}} \lambda^{\hat{k}-1} e^{-\lambda/\hat{\theta}}\, d\lambda$$

$$= \frac{1}{\Gamma(\hat{k})\hat{\theta}^{\hat{k}}} \int \lambda^{\hat{k}} e^{-\lambda(x+1/\hat{\theta})}\, d\lambda$$

$$= \frac{1}{\Gamma(\hat{k})\hat{\theta}^{\hat{k}}} \int \lambda^{(\hat{k}+1)-1} e^{-\frac{\lambda}{1/(x+1/\hat{\theta})}}\, d\lambda$$

let $\bar{K} = \hat{K}+1$, $\bar{\theta} = \frac{1}{x + 1/\hat{\theta}}$, then

$$p(X|\mathcal{D}) = \frac{\Gamma(\bar{K})\,\bar{\theta}^{\bar{K}}}{\Gamma(\hat{K})\,\hat{\theta}^{\hat{K}}} \underbrace{\int \frac{1}{\Gamma(\bar{K})\,\bar{\theta}^{\bar{K}}} \lambda^{\bar{K}-1} e^{-\lambda/\bar{\theta}}\, d\lambda}_{\text{equals } = 1}$$

$$= \frac{\Gamma(\hat{K}+1)\,\bar{\theta}^{\bar{K}}}{\Gamma(\hat{K})\,\hat{\theta}^{\hat{K}}} = \frac{\Gamma(K+n+1)}{\Gamma(K+n)} \frac{\bar{\theta}^{\bar{K}}}{\hat{\theta}^{\hat{K}}}$$

- How do u sample $p(X|\mathcal{D})$?

→ Ancestral Sampling, and keep X samples.

- How do u get a point estimate?

$$p(X = x' | \mathcal{D})$$

$$p(X = x' | \mathcal{D}) = \int \lambda e^{-\lambda x'} p(\lambda)\, d\lambda \approx \frac{1}{n} \sum_{i=1}^{n} \lambda e^{-\lambda x'}$$

$$p(X|\mathcal{D}) = \frac{\Gamma(K+n+1)}{\Gamma(K+n)} \left[\theta \frac{(\theta \sum_{i=1} x_i - 1)^{K+n}}{(\theta X + \theta \sum X_i - 1)^{K+n+1}} \right]$$

The distribution generated pdf should be the same as from Sampling.

$$\hat{\theta} = \frac{1}{\Sigma X_i + \frac{1}{\theta}} = \frac{1}{\frac{\theta \Sigma X_i}{\theta} + \frac{1}{\theta}} = \frac{\theta}{\theta \Sigma X_i + 1}$$

$$\bar{\theta} = \frac{1}{X + 1/\hat{\theta}} = \frac{1}{X + \frac{\theta \Sigma X_i + 1}{\theta}} = \frac{1}{\frac{\theta X}{\theta} + \frac{\theta \Sigma X_i + 1}{\theta}}$$

$$= \frac{\theta}{\theta X + \theta \Sigma X_i + 1}$$

$$(K+n) \frac{\theta^{1003}}{(\theta X + \theta \Sigma X_i + 1)^{1003}} \frac{(\theta \Sigma X_i + 1)^{1002}}{\theta^{1002}}$$

$$(K+n)\,\theta \left[\frac{(\theta \Sigma X_i + 1)}{(\theta X + \theta \Sigma X_i + 1)}\right]^{1002} \frac{1}{\theta X + \theta \Sigma X_i + 1}$$

Let $\theta \Sigma X_i + 1 = \alpha$

$$\theta (K+n) \left[\frac{\alpha}{\theta x + \alpha}\right]^{1002} \frac{1}{\theta x + \alpha}$$