Derivation of Diagonal Gaussian from General Multivariate Gaussian

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Step 1: General Multivariate Gaussian
The probability density function (PDF) of a general multivariate Gaussian is:

$$p(x) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right),$$

where:

- $x \in \mathbb{R}^d$ is the random vector,
- $\mu \in \mathbb{R}^d$ is the mean vector,
- $\Sigma \in \mathbb{R}^{d\times d}$ is the covariance matrix,
- $|\Sigma|$ is the determinant of Σ,
- Σ^{-1} is the inverse of Σ.

Step 2: Assume Σ is Diagonal
If Σ is diagonal, it has the form:

$$\Sigma = \begin{pmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_d^2 \end{pmatrix},$$

where σ_i^2 is the variance of the i-th dimension. A diagonal covariance matrix implies that the dimensions are uncorrelated.
Step 3: Compute the Determinant of Σ
For a diagonal matrix, the determinant is the product of the diagonal elements:

$$|\Sigma| = \prod_{i=1}^{d} \sigma_i^2.$$

Step 4: Compute the Inverse of Σ
The inverse of a diagonal matrix is also diagonal, with each diagonal element inverted:

$$\Sigma^{-1} = \begin{pmatrix} \frac{1}{\sigma_1^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_d^2} \end{pmatrix}.$$

Step 5: Simplify the Quadratic Form
The quadratic form $(x-\mu)^T\Sigma^{-1}(x-\mu)$ is a key component of the Gaussian PDF. Let's expand and simplify it step by step.
Step 5.1: Define the Difference Vector
Let $z = x - \mu$, where z is the difference between the random vector x and the mean vector μ. In component form:

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_d \end{bmatrix} = \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \\ \vdots \\ x_d - \mu_d \end{bmatrix}.$$

Step 5.2: Write the Quadratic Form
The quadratic form is:

$$z^T\Sigma^{-1}z.$$

Substituting Σ^{-1} and z:

$$z^T \Sigma^{-1} z = \begin{bmatrix} z_1 & z_2 & \cdots & z_d \end{bmatrix} \begin{pmatrix} \frac{1}{\sigma_1^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_d^2} \end{pmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_d \end{bmatrix}.$$

Step 5.3: Perform Matrix Multiplication
Multiply the matrices step by step:
1. Multiply z^T and Σ^{-1}:

$$z^T \Sigma^{-1} = \begin{bmatrix} \frac{z_1}{\sigma_1^2} & \frac{z_2}{\sigma_2^2} & \cdots & \frac{z_d}{\sigma_d^2} \end{bmatrix}.$$

2. Multiply the result by z:

$$z^T \Sigma^{-1} z = \frac{z_1^2}{\sigma_1^2} + \frac{z_2^2}{\sigma_2^2} + \cdots + \frac{z_d^2}{\sigma_d^2}.$$

Step 5.4: Final Simplified Quadratic Form
Thus, the quadratic form simplifies to:

$$(x - \mu)^T \Sigma^{-1} (x - \mu) = \sum_{i=1}^{d} \frac{(x_i - \mu_i)^2}{\sigma_i^2}.$$

This is a weighted sum of squared differences, where each term is weighted by the inverse of the variance in that dimension.
Step 6: Substitute into the General Gaussian
Substitute the determinant and quadratic form into the general Gaussian equation:

$$p(x) = \frac{1}{(2\pi)^{d/2} \left(\prod_{i=1}^{d} \sigma_i^2 \right)^{1/2}} \exp\left(-\frac{1}{2} \sum_{i=1}^{d} \frac{(x_i - \mu_i)^2}{\sigma_i^2} \right).$$

Step 7: Simplify the Normalization Constant
The normalization constant can be rewritten as:

$$\frac{1}{(2\pi)^{d/2} \left(\prod_{i=1}^{d} \sigma_i^2 \right)^{1/2}} = \frac{1}{(2\pi)^{d/2} \prod_{i=1}^{d} \sigma_i}.$$

Final Diagonal Gaussian Equation
The simplified diagonal Gaussian equation is:

$$p(x) = \frac{1}{(2\pi)^{d/2} \prod_{i=1}^{d} \sigma_i} \exp\left(-\frac{1}{2} \sum_{i=1}^{d} \frac{(x_i - \mu_i)^2}{\sigma_i^2} \right).$$