Solving A Rayleigh Quotion

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The goal is to solve

$$\max_{w} \frac{w^T S_1 w}{w^T S_2 w} \qquad (1)$$

- Given the assumption that S_1, S_2 are symmetric positive semi-definite (SPSD)

- First note that SPSD matrices can "always" be decomposed via eigendecomposition where

$$S_i = U \Sigma U^T$$

This also implies that S_i can also be split into just 2 matrices where

$$S_i = U \Sigma U^T$$
$$= \underbrace{U \Sigma^{1/2}}_{R_i^T} \underbrace{\Sigma^{1/2} U^T}_{R_i}$$

$$= R_i^T R_i \qquad (2)$$

Using Eq (2) in (1), we get

$$\max_{w} \frac{w^T R_1^T R_1 w}{w^T R_2^T R_2 w} \qquad (3)$$

We next let

$$v = R_2 w \qquad (4)$$

which implies

$$R_2^{-1} v = w \qquad (5)$$

Using (4), (5) in (3) we get

$$\max_{w} \frac{v^T (R_2^{-1})^T R_1^T R_1 (R_2^{-1}) v}{v^T v}$$

This is the same as

$$\max_{w} \frac{\left[R_1 (R_2^{-1}) v \right]^2}{\|v\|^2}$$

$$\left[R_1 (R_2^{-1}) v \right]^2$$

$$\max_{w} \left[\frac{R_1(R_2^{-1})\, v}{\|v\|} \right]^2$$

$$\max_{w} \underbrace{\left[R_1(R_2^{-1}) \frac{v}{\|v\|} \right]^2}_{Q}$$

Note $\frac{v}{\|v\|}$ is a unit vector projecting onto $R_1(R_2^{-1})$.

To make Q as large as possible we want

$$[R_1(R_2^{-1})]^T = \frac{v}{\|v\|}$$

Therefore the v that maximize Q is some constant multiple α with

$$\alpha [R_1(R_2^{-1})]^T = v \qquad (6)$$

From Eq (5) we know that

$$R_2^{-1} v = w \qquad (5)$$

$$R_2^{-1} V = W \qquad (5)$$

Since we know the best V we can plug (6) into (5)

$$\alpha R_2^{-1}\left[R_1 (R_2^{-1})\right]^T = W$$

$$\alpha R_2^{-1}(R_2^{-1})^T R_1^T = W$$

$$\alpha R_2^{-1}(R_2^T)^{-1} R_1^T = W$$

$$\alpha \left[R_2^T R_2\right]^{-1} R_1^T = W$$

$$\alpha S_2^{-1} R_1^T = W^*$$

This is the optimal solution for Rayleigh Quotion.

In addition, note that if we plug the optimal W^* into the original objective, we get

$$\max_W \frac{W^T S_1 W}{W^T S_2 W}$$

$$\max \; (\alpha S_2^{-1} R_1^T)^T S_1 \; (\alpha S_2^{-1} R_1^T)$$

$$\max_{w} \frac{(\alpha S_2^{-1} R_1^T)^T S_1 (\alpha S_2^{-1} R_1^T)}{(\alpha S_2^{-1} R_1^T)^T S_2 (\alpha S_2^{-1} R_1^T)}$$

$$\max_{w} \frac{\cancel{\alpha^2} (S_2^{-1} R_1^T)^T S_1 (S_2^{-1} R_1^T)}{\cancel{\alpha^2} (S_2^{-1} R_1^T)^T S_2 (S_2^{-1} R_1^T)}$$

Notice that α can be cancelled so any number for α, we would get the same objective value.

It is then easiest to just set it to 1, leading us to the final solution

$$S_2^{-1} R_1^T = w^*$$