Derivation of Isotropic Gaussian from General Multivariate Gaussian

Chieh Wu

Jan/9/2025

Step 1: General Multivariate Gaussian

The probability density function (PDF) of a general multivariate Gaussian is:

$$p(x) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right),$$

where:

- $x \in \mathbb{R}^d$ is the random vector,
- $\mu \in \mathbb{R}^d$ is the mean vector,
- $\Sigma \in \mathbb{R}^{d\times d}$ is the covariance matrix,
- $|\Sigma|$ is the determinant of Σ,
- Σ^{-1} is the inverse of Σ.

Step 2: Assume Σ is Isotropic

An **isotropic Gaussian** is a special case where the covariance matrix Σ is a scalar multiple of the identity matrix. This means: - All dimensions have the same variance σ^2, - There are no correlations between dimensions.

Thus, the covariance matrix becomes:

$$\Sigma = \sigma^2 I,$$

where: - σ^2 is the common variance for all dimensions, - I is the identity matrix of size $d \times d$.

Step 3: Compute the Determinant of Σ

For an isotropic covariance matrix $\Sigma = \sigma^2 I$, the determinant is:

$$|\Sigma| = (\sigma^2)^d.$$

This is because the determinant of a diagonal matrix is the product of its diagonal elements, and all diagonal elements of $\sigma^2 I$ are σ^2.

Step 4: Compute the Inverse of Σ

The inverse of $\Sigma = \sigma^2 I$ is:

$$\Sigma^{-1} = \frac{1}{\sigma^2} I.$$

This is because the inverse of a scalar multiple of the identity matrix is the reciprocal of the scalar multiplied by the identity matrix.

Step 5: Simplify the Quadratic Form

The quadratic form $(x-\mu)^T\Sigma^{-1}(x-\mu)$ simplifies as follows:

1. Substitute $\Sigma^{-1} = \frac{1}{\sigma^2} I$:

$$(x-\mu)^T\Sigma^{-1}(x-\mu) = (x-\mu)^T\left(\frac{1}{\sigma^2} I\right)(x-\mu).$$

2. Simplify using the identity matrix I:

$$(x-\mu)^T\left(\frac{1}{\sigma^2} I\right)(x-\mu) = \frac{1}{\sigma^2}(x-\mu)^T(x-\mu).$$

3. The term $(x-\mu)^T(x-\mu)$ is the squared Euclidean distance between x and μ:

$$(x-\mu)^T(x-\mu) = \|x-\mu\|^2 = \sum_{i=1}^{d}(x_i-\mu_i)^2.$$

Thus, the quadratic form becomes:

$$(x-\mu)^T\Sigma^{-1}(x-\mu) = \frac{1}{\sigma^2}\|x-\mu\|^2.$$

Step 6: Substitute into the General Gaussian

Substitute the determinant and quadratic form into the general Gaussian equation:

$$p(x) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right).$$

1. Substitute $|\Sigma| = (\sigma^2)^d$:

$$p(x) = \frac{1}{(2\pi)^{d/2}(\sigma^2)^{d/2}} \exp\left(-\frac{1}{2}\cdot\frac{1}{\sigma^2}\|x-\mu\|^2\right).$$

2. Simplify the normalization constant:

$$\frac{1}{(2\pi)^{d/2}(\sigma^2)^{d/2}} = \frac{1}{(2\pi\sigma^2)^{d/2}}.$$

Step 7: Final Isotropic Gaussian Equation

The simplified isotropic Gaussian equation is:

$$p(x) = \frac{1}{(2\pi\sigma^2)^{d/2}} \exp\left(-\frac{1}{2\sigma^2}\|x-\mu\|^2\right).$$

Summary

The isotropic Gaussian is a special case of the multivariate Gaussian where: - The covariance matrix Σ is $\sigma^2 I$, - All dimensions have the same variance σ^2, - The quadratic form reduces to the scaled squared Euclidean distance $\frac{1}{\sigma^2}\|x-\mu\|^2$.