TUTORIAL QUESTIONS
STA432 (BAYESIAN INFERENCE)

1. (a) Suppose we know that $\theta \sim N(\mu, \gamma^2)$ a priori, but have no other information about the parameter θ. We then sample n i.i.d. normal observations i.e. $X_i \sim Normal(\theta, \sigma^2), i = 1, \dots, n$, given the unknown parameter θ, where σ is known. Using the prior information about θ and the observed data $\boldsymbol{x} = (x_1, \dots, x_n)$, show that our updated knowledge about θ (i.e. the posterior distribution of θ) is
$$\theta|\boldsymbol{x} \sim Normal\left(\left[\frac{\mu}{\gamma^2} + \frac{\bar{x}}{\sigma^2/n}\right]\Big/\left[\frac{1}{\gamma^2} + \frac{1}{\sigma^2/n}\right], \left\{\frac{1}{\gamma^2} + \frac{1}{\sigma^2/n}\right\}^{-1}\right), \text{ where } \bar{x} = \sum_{i=1}^{n} x_i \ .$$
   (b) Express the mean of the posterior distribution in part (a) above as a weighted average of the sample mean $\bar{x}$ and the prior mean μ of θ in the form $\omega\bar{x} + (1-\omega)\mu$, where
$$\omega = \left(\frac{1}{\sigma^2/n}\right)\Big/\left(\frac{1}{\gamma^2} + \frac{1}{\sigma^2/n}\right).$$
   (c) Show that $\omega \to 1$ (i) as $n \to \infty$ (ii) as $\gamma \to \infty$ for a fixed n.

2. (a) Let $X_i \sim Bernoulli(\theta), i = 1, \dots, n$ have the likelihood $p(x_1, \dots, x_n|\theta)$.
      i. Obtain a prior distribution $\pi(\theta)$ for the likelihood $p(x_1, \dots, x_n|\theta)$ with Jeffreys' rule.
      ii. Ignoring the absence of the normalizing constant, state the prior distribution of θ obtained in part (a) i. in shorthand distribution notation. [Hint: $X \sim Normal(a, b^2)$ is a shorthand distribution notation for "X follows the normal distribution with parameters a and b"].

   (b) Using the prior and likelihood in part (a), show that
$$\theta|(x_1, \dots, x_n) \sim Beta(n + \tfrac{1}{2}, n - S_n + \tfrac{1}{2}) \text{ where } S_n = \sum_{i=1}^{n} x_i.$$
   (c) i. Suppose we use a non-informative flat prior $\pi(\theta) = c$, where c is a constant, for the likelihood $p(x_1, \dots, x_n|\theta)$. Obtain the posterior distribution of θ.
      ii. Given that a sample of $n = 12$ i.i.d. Bernoulli observations $x_1, \dots, x_{12}$ yields 7 successes (i.e. seven 1's), rewrite the posterior distribution of θ in part (c) i. using this information.

3. Let $X \sim Normal(\mu, 4)$, with likelihood $f(x|\mu)$ for a single observation $x = 2$. Assume that a prior distribution $\pi(\mu)$ is $\mu \sim Normal(4, 8)$.
   (a) Obtain a point estimate $\bar{\mu}$ of the posterior mean.
   (b) Show that $(-0.534, 5.867)$ is a 95% Bayesian credible interval for μ.
   (c) Suppose $X_i \sim Poisson(\theta)$, each with probability mass function
$$P(X_i = x_i) = \frac{\theta^{x_i} \mathrm{e}^{-\theta}}{x!}, \ \ x_i \in \{0, 1, 2, \dots\}, \ \ i = 1, \dots, n.$$
      i. Write down a conjugate prior distribution for the likelihood $p(x_1, \dots, x_n|\mu)$?
      ii. Show that the posterior distribution of θ is $Gamma(\alpha + n\bar{(x)}, \beta + n)$, given $\{x_1, \dots, x_n\}$ observations, where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and α, β are the parameters of the conjugate prior distribution in part (c) i.

4. Suppose we are interested in the prevalence of an infectious disease in a small town. The interest is in θ, the proportion of infected individuals in the town. Due to high cost of testing, only a small random sample of 20 individuals from the town will be tested for infection in a single round. Let Y be the number of infected individuals in the random sample of 20 individuals.

(a) i. Write down the parameter space Θ of θ.
    ii. Write down the sample space $\mathcal{Y}$ of Y.

(b) i. Write down a reasonable sampling model or distribution $f(y|\theta)$ for Y.
    ii. Based on your sampling model in part (b) i., what is the probability that there will be no infected individuals in the sample?

(c) i. Specify an appropriate conjugate prior distribution $\pi(\theta)$ for the sampling model $f(y|\theta)$ in part (b) i.
    ii. Obtain the posterior distribution of θ using the prior distribution and sampling model in part (c) i.
    iii. If the result of the single round of tests on the 20 individuals reveals that 4 individuals are infected, write down or obtain the exact posterior distribution of θ given $y = 3$.
    iv. What is the posterior mean from part (c) iii.?

5. (a) Suppose $H_1, H_2, \ldots, H_n$ are n mutually exclusive and exhaustive hypotheses about some unknown quantity. Let $P(H_i)$ be the prior probability assigned to the hypothesis $H_i, i =$

   $1, \ldots, n$. After we have observed some data D, write down the posterior probability of H_i given the data D. Define clearly any quantity introduced.

   (b) A secretary has two fountain pens in her purse. She only knows that, at least, one of them is blue but she is not sure if the other is blue or red.

   i. Formulate two mutually exclusive and exhaustive hypotheses, H_1 and H_2 to describe the secretary's uncertainty.
   ii. Suppose the two hypotheses stand a 50 - 50 chance to be true such that $P(H_1) = P(H_2) = 0.5$. To verify which hypothesis is more plausible, a pen is taken at random from the purse and checked. Which hypothesis is more plausible from Bayesian framework if the pen is blue? [Hint: Use Bayes Box if necessary].
   iii. If the pen taken at random from the purse is red instead, which hypothesis will be more plausible?
   iv. Suppose that the information the secretary had was that there are two fountains pens and a least one is blue or red. How many hypothesis will be needed in this case? State them.

6. Obtain the explicit form of the posterior distribution (i.e. state the complete pdf including normalizing constant) of the following pairs of conjugate priors $\pi(\lambda)$ and likelihood $p(x|\lambda)$ for a single observation $Y = y$.

   (a) $Y \sim Poisson(\lambda)$ and $\pi(\lambda) \sim Gamma(\alpha, \beta)$.
   (b) $Y \sim Bernoulli(\lambda)$ and $\pi(\lambda) \sim Beta(\alpha, \beta)$.
   (c) $Y \sim Binomial(\lambda)$, n is known and $\pi(\lambda) \sim Beta(\alpha, \beta)$.
   (d) $Y \sim Geometric(\lambda)$ and $\pi(\lambda) \sim Beta(\alpha, \beta)$.
   (e) $Y \sim Exponential(\lambda)$ and $\pi(\lambda) \sim Gamma(\alpha, \beta)$
   (f) $Y \sim NegBinomial(\lambda)$, r, the "stopping" parameter is known and $\pi(\lambda) \sim Beta(\alpha, \beta)$.

7. Following Jeffreys' rule, obtain a prior distribution $\pi(\theta)$for the likelihood $p(x_1, \ldots, x_n|\theta)$ for each of the following models.

   (a) $X_i \sim \; i.i.d. \; Poisson(\theta), \; i = 1, \ldots, n$.
   (b) $X_i \sim \; i.i.d. \; Bernoulli(\theta), \; i = 1, \ldots, n$.
   (c) $X_i \sim \; i.i.d. \; Benomial(\theta), \; i = 1, \ldots, n$.

(d) $X_i \sim$ *i.i.d. Geometric*(θ), $i = 1, \ldots, n$.

(e) $X_i \sim$ *i.i.d. Exponential*(θ), $i = 1, \ldots, n$.

(f) $X_i \sim$ *i.i.d. NegBinomial*(θ), $i = 1, \ldots, n$.

8. (a) Let $p(\theta|x)$ be the posterior distribution of θ given data x.

   i. Define the posterior mean $\bar{\theta}$ of θ.

   ii. Define the $(1-\alpha)100\%$ Bayesian Credible Interval for θ.

   (b) Let $X \sim Bernoulli(\theta)$ and $\theta \sim Beta(a, b)$.

   i. Obtain a point estimate of the posterior mean $\bar{\theta}$ given a single observation $X = x$.

   ii. Express the posterior mean $\bar{\theta}$ as a weighted average of the form $\bar{\theta} = \omega\bar{x} + (1-\omega)\theta_0$ where $\bar{x} = x$ is the mean of the single observation, and $\theta_0 = \dfrac{a}{a+b}$ is the prior mean of θ. Write down the expression for ω.

   (c) If n observations, $\{X_1, \ldots, X_n\} \sim$ *i.i.d. Bernoulli*(θ), are taken instead of just one observation,

   i. Obtain the point estimate of the posterior mean $\bar{\theta}_n$ given the observed data $\{x_1, \ldots, x_n\}$.

   ii. Show that $\bar{\theta}_n = \psi\bar{X} + (1-\psi)\theta_0$ where $\bar{x} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} x_i$ is the mean of the observed data $\{x_1, \ldots, x_n\}$ i.e. MLE of θ, $\psi = \dfrac{n}{a+b+n}$ and θ_0 remains as defined before.

9. Mr. A and Mr. B found an unusual coin on their journey through to meeting. They both agreed that the probabilityof heads θ should follow a $Beta(\alpha, \beta)$ distribution a priori. However, they disagreed on what experiment to carry out in order to investigate this belief further. While Mr. A suggested that they flip the coin some n times for the number of heads that would appear, Mr. B suggested that they keep flipping the coin until the first head appears.

   (a) Write down a probability model/distribution that correctly describes each of Mr. A's and Mr. B's experiment.

   (b) Using the agreed prior distribution of θ, determine the posterior distribution of θ under (i) Mr A's model and (ii) under Mr. B's model. (iii) Are the posterior distributions the same or from the same family of distibutions?