1 Cauchy-Binet Theorem

Here is our setup:

- A is an $m \times n$ matirx and B is an $n \times m$ matrix.
- We let S be an m element subset of $\{1, 2, \ldots, n\}$. We define $A[S]$ to be the $m \times m$ submatrix of A formed by the m columns of A enumerated by S.
- The analogous definition for $B[S]$ is the $m \times m$ submatrix formed by the m rows of B indexed by S.

Goal: Compute $\det(AB)$.

Theorem 1 (Cauchy - Binet)**.** *Let $A = (a_{ij})$ be an $m \times n$ matrix and $B = (b_{ij})$ be an $n \times m$ matrix. Then*

1. *If $m > n$, then* $\det(AB) = 0$.
2. *If $m \geq n$, then*

$$\det(AB) = \sum_S (\det A[S] \cdot \det B[S])$$

where S ranges over every m-element subset of $\{1, 2, \ldots, n\}$.

Proof. (Sketch) Suppose $m > n$. As a consequence of the definition of rank, we know that

$$\operatorname{rk} AB \leq \operatorname{rk} A \leq n < m.$$

Since AB is an $m \times m$ matrix, and has rank strictly less than m we conclude that the rows / columns of AB cannot be linearly independent. One concludes that $\det AB = 0$ as claimed.

We now assume that $m \leq n$. We denoted by M_{rs} to be a matrix of size $r \times s$. We make some remarks regarding block multiplication of matrices below:

1. $$\begin{pmatrix} R_{mm} & S_{mn} \\ T_{nm} & U_{nn} \end{pmatrix} \begin{pmatrix} V_{mn} & W_{mm} \\ X_{nm} & Y_{nm} \end{pmatrix} = \begin{pmatrix} RV + SX & RW + SY \\ TV + UX & TW + UY \end{pmatrix}$$

2. Letting $S = A$, $T = 0_{nm}$, $U = I_n$, $V = A$, $W = 0_{mn}$, $X = I_n$ and $Y = B$ one can easily verify that

$$\begin{pmatrix} I_m & A \\ 0_{nm} & I_n \end{pmatrix} \begin{pmatrix} A & 0_{mn} \\ -I_n & B \end{pmatrix} = \begin{pmatrix} 0_{mn} & AB \\ -I_n & B \end{pmatrix}$$

3. Taking determinants, we note that

$$\begin{vmatrix} A & 0_{mn} \\ -I_n & B \end{vmatrix} = \begin{vmatrix} 0_{mn} & AB \\ -I_n & B \end{vmatrix}$$

The determinant on the RHS is $\pm \det AB$ the sign being determined by n being even or odd.

We now consider the definition of the determinant taken over the sum of the symmetric group given below:

$$\det(M) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, m_{1\sigma(1)} m_{2\sigma(2)} \cdots m_{n\sigma(n)}$$

where the m_{ij} are the entries of M.

Now consider the determinant on the LHS above. Any nonzero summand in the determinant is a product of nonzero entries each of which lies in one column and one row. In choosing these entries, we must avoid the zeros in the upper RH side. Hence the entries chosen in the last m columns must come from B and we take entries from A both of which are indexed by S. For the remaining entries, we must select the minus ones in the lower left hand block. There are $n - m$ of these rows with -1.

Q: What is the contribution of the expansion from the terms which use exactly the -1s from the rows $m + i$ and columns i for $i \notin S$?

We form the matrix M_S by taking the block matrix with A in the upper left and B in the lower right. We then proceed to delete the row $m + i$ and column i. This yields a block diagonal matrix where $A[S]$ is in one block and $B[S]$ in the second block. Further,

$$\det(M_S) = \det(A[S]) \det(B[S]).$$

We get the claimed formula by summing over all subsets of S. □

2 Eigenvalues & Eigenvectors

We begin by reviewing some results regarding the eigenvalues and eigenvectors of a linear transformation. Let T be a linear transformation $V \to V$. We say that $\lambda \in \mathbb{F}$ is an *eigenvalue* if there exists a nonzero $v \in V$ such that $Tv = \lambda v$. The vector v is called an *eigenvector*.

Lemma 1. *Let M be the $n \times n$ matrix with entries all ones. The eigenvalues of M are n with multiplicity 1 and 0 with multiplicity $n - 1$*

Proof. All entries of M are equal and nonzero and therefore there is exactly one independent row which gives us that $\operatorname{rk} M = 1$. The rank-nullity theorem states that:

$$\dim \operatorname{Null}(M) + \operatorname{rk}(M) = n$$

$$\dim \operatorname{Null}(M) = n - 1$$

If $Mv = \lambda v = 0$ and $\lambda = 0$ then $Mv = 0$ and $v \in \mathrm{Null}(M)$. Further, $\sum \lambda_i = \mathrm{Tr}(M) = n$ and $\mathrm{rk}(M) = 1$, we conclude that the remaining eigenvalue of M is n. $\square$

Lemma 2. *If the eigenvalues of P are $\lambda_1, \ldots, \lambda_n$ then the eigenvalues of $P + cI$, for any constant c are $\lambda_1 + c, \ldots, \lambda_n + c$.*

Proof. First we note that when v_i is an eigenvector of P that

$$(P + cI)v_i = Pv_i + cIv_i = \lambda_i v + cv = (\lambda_i + c)v$$

and therefore λ_i and eigenvalue of P we have that $\lambda_i + c$ is an eigenvalue of $P + cI$. Now suppose that w is an eigenvector of $P + cI$ and λ its associated eigenvalue. Then

$$(P + cI)w = \lambda w = Pw + cw$$

and therefore $Pw = (\lambda - c)w$. Hence all eigenvalues arise as in the claimed statement. $\square$

Lemma 3. *Suppose we can write $S = A - D$ where D is a diagonal matrix with entries $d \in \mathbb{R}$ and λ, v are an eigenvalue and an associated eigenvector of A respectively. Then $\lambda - d$ is an eigenvalue of S.*

Proof. This falls out from using the definitions:

$$\begin{aligned} Sv &= (A - D)v \\ &= Av - Dv \\ &= \lambda v - dv \\ &= (\lambda - d)v \end{aligned}$$

$\square$

Note that if λ is an eigenvalue of S and v an associated eigenvector, then $Sv = \lambda v$ and $(A - D)v = Av - Dv = \lambda v$ which yields $Av = (\lambda + d)v$ and therefore all eigenvalues of S arise in this way.

Theorem 2. *If S is a symmetric matrix, then S has only real eigenvalues.*

Proof. Suppose that λ is an eigenvalue and v is an associated eigenvector. Since S has real entries, taking complex conjugates, one gets that $\bar{Sv} = \bar{\lambda v}$ yields $S\bar{v} = \bar{\lambda}\bar{v}$. Then

$$\lambda(v \cdot \bar{v}) = \bar{v}^T(\lambda v) = \bar{v}^T S v$$
$$\bar{\lambda}(v \cdot \bar{v}) = (\bar{\lambda}\bar{v})^T \cdot v = (S\bar{v})^T v$$

We see that the two lines synthesize together to see that $(S\bar{v})^T v = \bar{v}^T S v$. An eigenvector is nonzero by definition, and therefore $\bar{v} \cdot v \neq 0$. We now have that $\lambda = lam\bar{b}da$ and conclude that $\lambda \in \mathbb{R}$. $\square$

Proposition 1. *A symmetric $n \times n$ matrix with real entries S has n linearly independent real eigenvectors. One can normalize and choose them to be orthogonal.*

3 Graphs & Eigenvalues

Lemma 4. *Given a graph G, fix two vertices v_i and v_j. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the adjacency matrix A.*

(a) There exist real numbers $c_1, c_2, \ldots, c_n$ such that

$$(A^l)_{ij} = c_1\lambda_1^l + \cdots + c_n\lambda_n^l$$

(b) Let U be the real orthogonal matrix that diagonalizes A; that is $U^{-1}AU = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Then $c_k = u_{ik}u_{jk}$

Proof. Let $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Then

$$D^l = (U^{-1}AU)^l = U^{-1}A^lU$$

and therefore

$$(A^l)_{ij} = \sum u_{ik}\lambda_k^l u_{jk}$$

where we've used the fact that $U^{-1} = U^T$ because U is orthogonal. $\square$

Proposition 2. *The eigenvalues of the complete graph K_n are*

(a) -1 with multiplicity $n-1$

(b) $n-1$ with multiplicity 1.

Proof. The adjacency matrix A of K_n can be written $A = M - I_n$ where is the matrix with entries all ones. Apply the lemmas above to conclude that an eigenvalue of a is an eigenvalue of M minus 1. Use the lemma regarding the eigenvalues of M to conclude the statement given. $\square$

Corollary 1. *The number of closed walks of length l in K_n from some vertex v_i to itself is*

$$\begin{aligned}(A^l)_{ij} &= \frac{1}{n}[(n-1)^l + (-1)^l(n-1)] \\ &= \frac{n-1}{n}((n-1)^{l-1} + (-1)^l)\end{aligned}$$

Proof. We've seen that

$$\begin{aligned}(A^l)_{ij} &= \lambda_1^l + \cdots + \lambda_n^l \\ &= (n-1)^l + (-1)^l(n-1)\end{aligned}$$

and all vertices have the same degree so we divide out by n. $\square$

4 Matrix Tree Theorem

Lemma 5. *Let S be a set of $p-1$ edges of G. If S does not form the set of edges of a spanning tree, then* $\det M_0[S] = 0$*. If on the other hand, S is the set of edges of a spanning tree then* $\det M_0[S] = \pm 1$*.*

Proof. If S is not the edges of a spanning tree then some subset $R \subseteq S$ forms a cycle C in G. Suppose the cycle C has edges $f_1, \ldots, f_r$ in that order. Multiply the column indexed by f_i by 1 if in going around C we traverse in the direction of the arrow and multiply by -1 otherwise. Add these modified columns and note that we get the zero column. The multiplication has made the orientation so that the arrows point in the direction of the cycle. These edges have a linear dependence relation and therefore det $M_0[S] = 0$.

Now suppose S is the set of edges of a spanning tree T. Let e be an edge of T which is connected to v_p; the vertex that indexed the last row of M. We removed this row to get M_0. The column of $M_0[S]$ indexed by e has exactly one nonzero entry (we deleted the other one) which is ± 1. Remove the row and column containing this nonzero entry. We get a $(p-2) \times (p-2)$ matrix M_0' and $\det M_0[S] = \pm \det M_0'$ by cofactor expansion. One can finish by induction. $\square$

We define the *complexity* of a graph G, denoted $\kappa(G)$, to be the number of spanning trees of G.

Theorem 3 (Matrix Tree Theorem)**.** *Let G be a finite connected graph without loops, L its Laplacian and L_0 denote L with its last column and last row removed. Then*

$$\begin{aligned}\det L_0 &= \kappa(G)\\ &= \{\#\ \textit{of spanning trees of}\ G\}\end{aligned}$$

Proof. Since $L = MM^T$, where M is the incidence matrix for G. We then have that $L_0 = M_0 M_0^T$. Apply the Cauchy-Binet theorem to get

$$\det L_0 = \sum_S (\det M_0[S]) \left(\det M_0^T[S]\right)$$

where S ranges over all $p-1$ element subsets of the set of edges of G. Since in general $A^T[S] = A[S]^T$ we have that

$$\det L_0 = \sum_S (\det M_0[S])^2 .$$

Since $\det M_0[S] = \pm 1$ if S forms the set of edges of a spanning tree of G and is zero if and only if S corresponds to a set of edges for a spanning tree, we conclude that the sum yields $\kappa(G)$ as desired. ☐

Corollary 2.

(a) Let G be a connected loopless graph with p vertices. Suppose that the eigenvalues of $L(G)$ are $\lambda_1, \lambda_2, \ldots, \lambda_{p-1}$ with $\lambda_p = 0$. Then

$$\kappa(G) = \frac{1}{p}\lambda_1 \cdot \lambda_2 \cdots \lambda_{p-1}.$$

(b) Suppose that G is also regular of degree d and the eigenvalues of $A(G)$ are $\mu_1, \mu_2, \ldots, \mu_{p-1}$ and $\mu_p = d$. Then

$$\kappa(G) = \frac{1}{p}(d-\mu_1)(d-\mu_2)\cdots(d-\pm_{p-1})$$

Proof. (a) Well

$$\det(L - xI) = (\lambda_1 - x)\cdots(\lambda_{p-1} - x)(\lambda_p - x) = -x(\lambda_1 - x)\cdots(\lambda_{p-1} - x)$$

and one sees that the coefficient of x is $-\mu_1\mu_2\cdots\mu_{p-1} = -p\det L_0 = -\kappa(G)$ and the claim follows.

(b) This is immediate from (a) and the eigenvalues of $A(G)$ when G is regular.

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