1 General Facts about Trees

Lemma 1. *If T is a tree then T has a leaf.*

Proof. A leaf with vertex v has degree 1. Suppose that T does not have a leaf. Then the $\deg(v) \geq 2$ for every $v \in V$. Hence, we can enter and leave every vertex on a different edge. Now construct a walk starting at v_1. We can then exit v_1 to another vertex v_{i_1}. We can keep doing this to get a sequence

$$w = v_1 v_{i_1} v_{i_2} \cdots v_{i_k}$$

When we get to v_{i_k} we can leave and go to one of its neighbors. Because V is finite, the pigeonhole principle tells us that eventually we've gone through some vertex twice. Hence, there exists a cycle in T. This is a contradiction as a tree is acyclic. □

Lemma 2. *Every tree has $\#V - 1$ edges.*

Proof. We proceed by induction.

Base Case: Let T be a tree with two vertices. There is one edge incident to those two vertices as T contains no cycles; $\#V - 1 = 2 - 1 = 1$.

Inductive Hypothesis: (IH) Suppose a tree T with k vertices has $k - 1$ edges.

wts: A tree with $k+1$ vertices has k edges. Let T be a tree with $k+1$ vertices and v be a leaf. Remove the unique edge incident to v. This graph has k vertices. By the IH, we know that it has $k - 1$ edges. Now add v and its edge to conclude that T has k edges.

□

Proposition 1. *Let G be a graph. The following are all equivalent:*

1. *G is a tree.*
2. *G is connected and has $\#V - 1$ edges.*
3. *G has no cycles and has $\#V - 1$ edges.*
4. *There is a unique path between any two vertices.*

Proof. (1) $\Rightarrow$ (2) We've proven that a tree has $p - 1$ edges and G is connected by definition of a tree.
(2) $\Rightarrow$ (3) G has $\#V - 1$ edges by assumption. Suppose that G has a cycle C. Then there are at least three edges incident to two vertices. Then there are $\#V - 4$ edges for $\#V - 3$ vertices.By the pigeonhole principle some vertex is isolated which implies that the graph G is disconnected. We've arrived at a contradiction

from the assumption that G is connected.

(3) $\Rightarrow$ (4) If there are two different paths between vertices v_i and v_j, then there are vertices v_k and $v_{k'}$ that belong to one path but not both. For two paths

$$w = v_i v_{s_1} v_{s_2} \cdots v_k \cdots v_{s_k} v_{s_{k+1}} v_j$$

$$p = v_i v_{t_1} v_{t_2} \cdots v_{k'} \cdots v_{t_k} v_{t_{k+1}} v_j$$

If the v_{s_k} and v_{t_j} are all different then w and p form a cycle. Otherwise, there is a cycle containing the vertices v_k and $v_{k'}$. This cycle can be constructed by deleting the vertices that are repeated in p and w. This is a contradiction to our assumption that G contains no cycles. (4) $\Rightarrow$ (1) If there existed a cycle there would be vertices v_i and v_j that had two different paths between them. Hence, the graph is acyclic. That every pair of vertices has a unique path connecting them implies that the graph is connected. $\square$

2 Spanning Trees

Definition 1. For a graph $G = \langle V, E \rangle$, a spanning tree T for G is an acyclic subgraph of G that is connected.

A spanning tree for a graph G can be seen as:

1. A tree T which contains all of the vertices in V and each edge of T is also an edge of G.
2. A maximal set of edges from E that contains no cycle.
3. A minimal set of edges that connect all vertices.

Fix a spanning tree T. The *cotree*, which we will denote by $\bar{T}$, is the set of edges in E which do are not edges of T. Adding an edge from $\bar{T}$ to T will create a cycle C_e, where e is an edge in $\bar{T}$. The cycle C_e is a *fundamental cycle* of G. Because the edges in $\bar{T}$ are in one-to-one correspondence with the fundamental cycles, these edges can be seen as a basis for the *cycle space*.

On the other hand, if we delete an edge from T the tree T becomes disconnected. That is, every edge of a tree is a cut edge. The vertices are now partitioned into two sets corresponding to the connected components. A *fundamental cut* is the set of edges that need to be deleted from G to accomplish the same partition. Agalogous to what we did above, the fundamental cuts and the branches of the tree are in one-to-one correspondence. The fundamental cuts form a basis for the *bond space*.

3 Minimal Spanning Trees

Definition 2 (Minimal Spanning Trees)**.** Let G be a connected edge weighted graph. A minimum spanning tree (MST) of G is a subset of the edges traversing all vertices, with the minimal possible total edge weight, and is acyclic.

We now consider a weighted version of our graph:

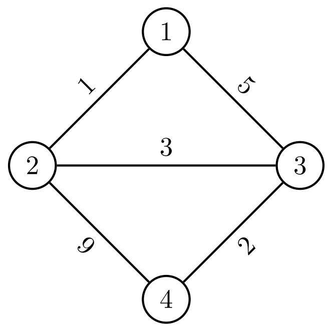

Figure 1: Weighted Graph

Proposition 2 (Cut Property)**.** *For any cut K of G, the edge in K with the least (strictly smaller than all others) weight must belong to all MST.*

Proposition 3 (Cycle Property)**.** *Given a cycle C in G, the edge in C with the largest weight cannot belong to an MST.*

Proposition 4 (Minimum Weight Edge Property)**.** *If the edge with the least weight is unique, then this edge belongs to every MST.*