1 Set Theory

We begin with some reminders from set theory. We make no attempt to be exacting and take the intuitive approach as demonstrated in Robert Stoll's book. We let A and B be two arbitrary sets.

Definition 1. The relative complement of B from A is the set

$$A - B = \{a \in A \mid a \notin b\}$$

Notice that if the two sets are disjoint ($A \cap B = \emptyset$) then $A - B = A$. We now use this concept to define the *symmetric difference*.

Definition 2. The symmetric difference $A + B$ is defined to be the set

$$A + B = (A - B) \cup (B - A)$$

After pondering this a bit, one realizes that $A + B$ is simply the set of objects that belong to one of the sets but not both.

Remark 1.

- Some books use the notation $A \triangle B$ for the symmetric difference.
- If you're hoping that the notation $A + B$ is going to lead to an abelian group, you're in luck.
- $A + B = (A \cup B) - (A \cap B)$.

We now recall the *power set* $\mathcal{P}(A)$ of a set A. The power set consists of all subsets of A. This is in bijection with all functions $A \to \mathbb{F}_2$, where $\mathbb{F}_2 \cong \mathbb{Z}/2\mathbb{Z}$ the field of two elements. This is often denoted by $\mathbb{F}_2^A$ or $\{0,1\}^A$. Each subset of A say $S = \{a_1, a_2, \dots, a_k\}$ is identified with the function $1_S : A \to \mathbb{F}_2$ which assigns 1 to each $a_i \in S$ and zero to all other elements of A. This is referred to as the *indicator function* of S.

Proposition 1. *Let A, B and C be three arbitrary sets that live in a universe U. Then*

(a) ***Closure:*** *$A + B$ is a subset of U.*

(b) ***Associative:*** $(A + B) + C = A + (B + C)$.

(c) ***Identity:*** $A + \emptyset = A = \emptyset + A$.

(d) ***Inverse:*** $A + A = \emptyset$.

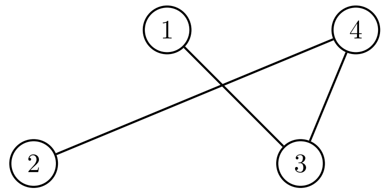


Figure 1: Graph with no cycle

Remark 2.

- Using the proposition one gets what was eluded to above; that the power set is an abelian group under the operation of the symmetric difference.
- One can extend this to obtain a *boolean ring* by using $\cap$ as the multiplication.
- The property in (d) yields that every element is nilpotent.

Now let $S \in \mathcal{P}(A)$. One can define an action of $\mathbb{F}_2$ on $\mathcal{P}(A)$ by $0 \cdot S = \emptyset$ and $1 \cdot S = S$. It is a standard exercise to show that this action is compatible with the operation of symmetric difference.

Conclusion: The power set can be viewed as a vector space over $\mathbb{F}_2$.

2 Vector Spaces & Graphs

Let $G = (V, E)$ be a graph where V is the set of vertices and E is the set of edges. We are considering undirected graphs where the cardinality of V and E are finite. One defines $\mathcal{V}(V) = \{0,1\}^V$ and $\mathcal{V}(E) = \{0,1\}^E$ to be the *vertex space* and *edge space*. Given that any subset of V can be written as the sum (symmetric difference) of the individual unit subsets

$$\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} = \{v_{i_1}\} + \{v_{i_2}\} + \cdots + \{v_{i_k}\}$$

we conclude that the set of $\{\{v_i\} \mid v_i \in V\}$ is a basis for the vertex space and that it is finite-dimensional with dimension $\#V$. Everything done here for the vertex space can be done analagously for the edge space.

3 Matrices

We will be using the following graph in our discussion:

Notice that the vertex space is four dimensional and has standard basis $v_1 = \{1\}, v_2 = \{2\}, v_3 = \{3\}, v_4 = \{4\}$ and the edge space is three dimensional and has standard basis $e_1 = \{(1,3)\}, e_2 = \{(2,4)\}, e_3 = \{(3,4)\}$.

3.1 The incidence matrix B

We now define the *incidence matrix* $B = (b_{ij})$ to be

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \in e_j \\ 0 & \text{otherwise} \end{cases}$$

Notice that this will yield a linear map $B : \mathcal{V}(E) \to \mathcal{V}(V)$. For our graph we have the following incidence matrix:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Example 1. The vector $(0, 1, 0)^T$ corresponds to e_2 and

$$Be_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

The image vector corresponds to the 2nd and 4th vertices, which are the vertices incident to e_2. We now understand the action of B on any basis element of the edge space: we get the vertices that for then end points of the edge.

Example 2. Now notice that e_1 and e_2 are a *matching* of the given graph. That is the edges e_1 and e_2 are independent in the sense that the edges account for every vertex and no two edges are adjacent to each other.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

The image of B in this case yields every vertex.

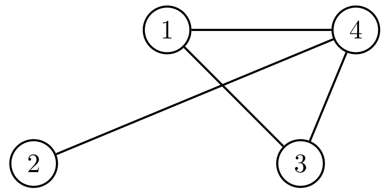


Figure 2: Graph with a cycle; edge e_4 added

Example 3. Now we consider the path from $v_1 = \{1\}$ to $v_4 = \{4\}$ via e_1 and e_3. Then

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Remember we are doing the multiplication mod 2. We are unsurprised that vertex $\{2\}$ is left our because it is not in our path. Notice that the other vertex that was left out ($\{3\}$) has even degree (in our path) as we entered and left this vertex.

Remark 3. In our example, B has full rank and therefore the map associated to B has trivial kernel.

Let's modify our graph by adding an edge $e_4 = (1, 4)$.

We now have the following incidence matrix:

$$B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Example 4. We are going to consider the cycle beginning and ending at $v_1 = \{1\}$ via the simple path $e_1, e_3,$ and e_4.

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

In this example, the incidence matrix has sent our cycle to zero. We note that every node in our cycle has even degree.

Example 5. Here we take a path from vertex $\{2\}$ to vertex $\{4\}$ using all of the edges:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

The image here yields vertex 2 and vertex 4. The two vertices in the graph with odd degree.

Proposition 2. *The linear map $B : \mathcal{V}(E) \to \mathcal{V}(V)$ takes an edge set to the set of vertices incident to an odd number of edges in the preimage. In other words, if $F \subset E$ then the image of B on F picks out the vertices which are incident to an odd number of vertices in F.*

3.2 The dual B^T

Given a matrix and its associated taking the transpose gives us the dual map. We ask: what role does it play? We begin by noting that B^T will define a linear map $\mathcal{V}(V) \to \mathcal{V}(E)$.

Example 6. Let's consider the vertex $\{1\}$ and compute $B^T v_1$:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The edge we've labeled e_1 is the only edge with $v_1 = \{1\}$ as an endpoint.

Example 7. Now let's look at $v_4 = \{4\}$.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Here the image has selected the two edges we've labeled e_2 and e_3. These are the two edges incident to v_4.

Example 8. Let's now consider $U = \{v_3, v_4\}$ and find the image of U under B^T

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

The edges e_1 and e_2 are the edges in the image. Notice that e_3 has been left out as both of the vertices in U form that edge.

Proposition 3. *The matrix B^T defines a map $\mathcal{V}(V) \to \mathcal{V}(E)$ which maps a subset of vertices $U \subset V$ to the edges with exactly one vertex in U.*

3.3 The Quadratic form BB^T

We return to our original incidence matrix. We now compute BB^T:

$$BB^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 2 \end{pmatrix}$$

We can reduce the entries to mod 2 as we have been doing. However, we want to make a couple of observations here:

(a) The diagonal matrix has entries d_{ii} the degree of v_i, $i = 1, 2, 3, 4$.

(b) The left summand is the adjacency matrix as defined below.

(c) BB^T is a symmetric matrix and diagonally dominant and is therefore positive semidefinite..

3.4 Adjacency Matrix

We now formally define the adjacency matrix for a graph.

Definition 3. The adjacency matrix $A = (a_{ij})$ for a graph $G = (V, E)$ is defined in the following manner:

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$$

The adjacency matrix for the graph in figure 1 as stated in the previous section:

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Remark 4. The adjacency matrix for an undirected graph will always be symmetric.

Example 9. Let $U = \{v_1, v_3\}$. Then

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

The vertex v_2 gets left out because it is not a neighbor to any of the vertices in U. Otherwise,

- v_1 is a neighbor to v_3 with one edge.
- v_3 is a neighbor to v_1 with one edge.
- v_4 is a neighbor to v_3 with one edge.

Proposition 4. *Viewed as a map $\mathcal{V}(V) \to \mathcal{V}(V)$ the adjacency matrix maps a set of vertices U to those vertices with an odd number of neighbors in U.*

4 Subspaces

We are going to take a look at two subspaces of the edge space. We will be using the following graph for our discussion.

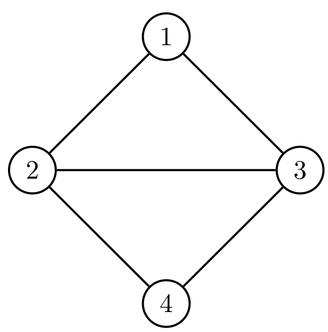
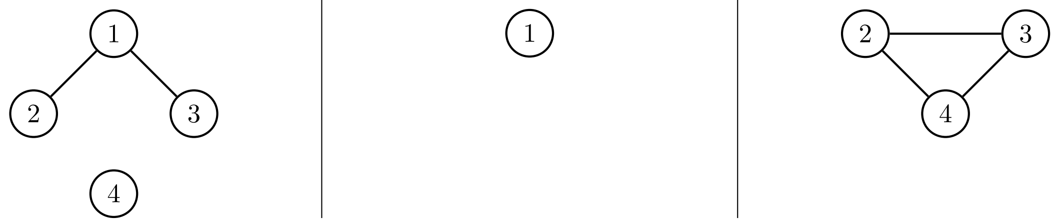


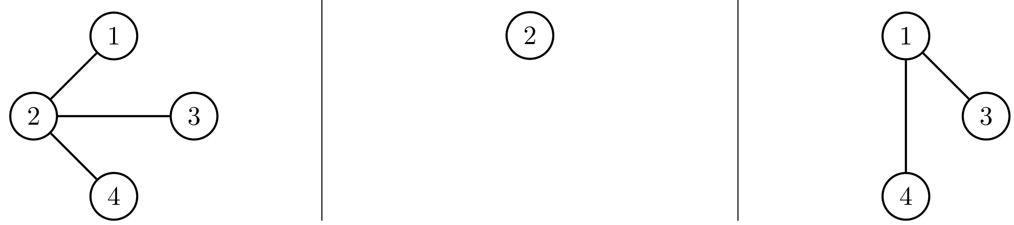
Figure 3: Graph with multiple cycles

A set F of edges is said to be a *cut* in G if there exists a partition $\{V_1, V_2\}$ of V such that $F = E(V_1, V_2)$. The edges in F are said to *cross* this partition. The sets V_1 and V_2 are the *sides* of the cut. A minimal non-empty cut in G is a *bond*.

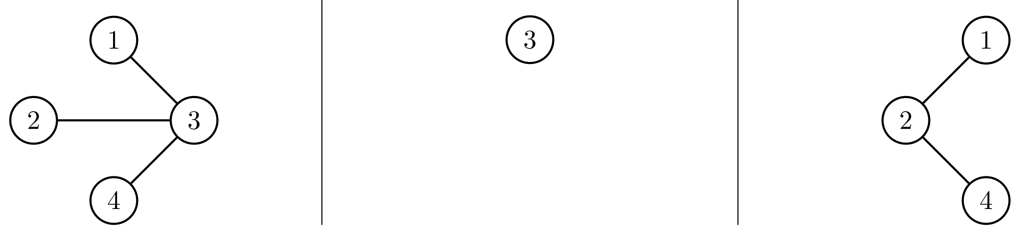
$X = \{1\}$ $E(1)$ | $G[X]$ | $G[V \setminus X]$



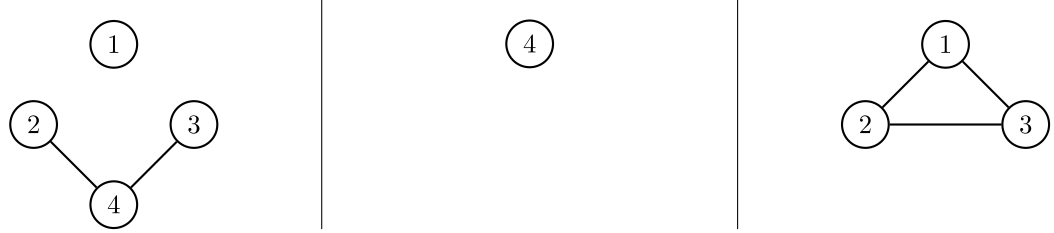
$X = \{2\}$ $E(2)$ | $G[X]$ | $G[V \setminus X]$



$X = \{3\}$ $E(3)$ | $G[X]$ | $G[V \setminus X]$

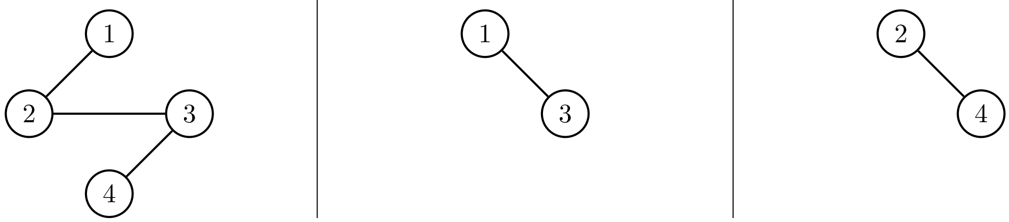


$X = \{4\}$ $E(4)$ | $G[X]$ | $G[V \setminus X]$

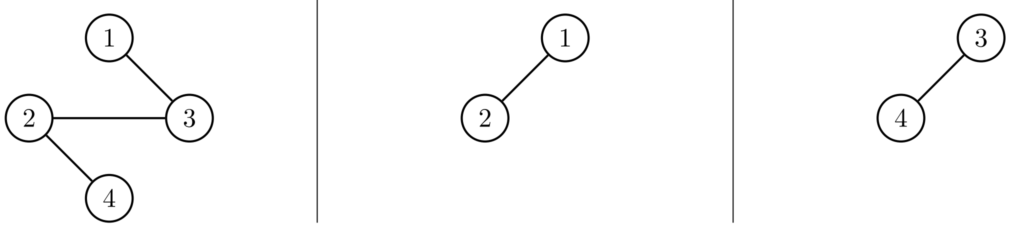


Remark 5. A basis for the *bond space* is $\{E(1), E(2), E(4)\}$.

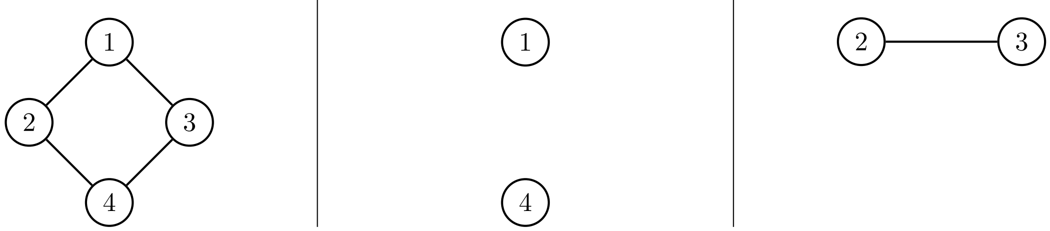
$X = \{1, 3\}$ $E(1) + E(3)$ $G[X]$ $G[V \setminus X]$



$X = \{1, 2\}$ $E(1) + E(2)$ $G[X]$ $G[V \setminus X]$



$X = \{1, 4\}$ $E(1) + E(4)$ $G[X]$ $G[V \setminus X]$



Remark 6.

- The last cut $E(1) + E(4)$ is the only example of our graph that is not a minimal cut. Notice that this cut is the disjoint union $E(1)\dot{\cup}E(4)$.
- The induced graph $G[X]$ is disconnected and $G[V \setminus X]$ is connected.
- The edges that cross the cut form a cycle.

Lemma 1. *Every non-minimal cut is a disjoint union of bonds.*

Theorem 1. *In a connected graph G, a nonempty edge cut is a bond if and only if $G[X]$ and $G[V \setminus X]$ are connected.*

Proposition 5 (Cut Space / Bond Space)**.** *Together with the $\emptyset$, the cuts in G from a subspace $\mathcal{B}(G) \subseteq \mathcal{V}(E)$. This space is generted by cuts of the form $E(v)$, where $v \in V$.*

We define the cycle space $\mathcal{C}$ to be the subspace spanned by the edge sets of all cycles in G. The dimension of the cycle space is referred to as the *cyclomatic number* of G.

Proposition 6. *TFAE for $D \subset E$:*

(a) $D \in \mathcal{C}(G)$

(b) *D is a possible empty disjoint union of edge sets of cycles in G.*

(c) *All vertex degrees of the graph (V, D) are even.*

5 Minimal Spanning Trees

Definition 4 (Minimal Spanning Trees)**.** Let G be a connected edge weighted graph. A minimum spanning tree (MST) of G is a subset of the edges traversing all vertices, with the minimal possible total edge weight, and is acyclic.

We now consider a weighted version of our graph:

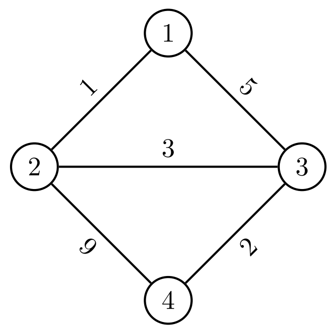


Figure 4: Weighted Graph

Proposition 7 (Cut Property)**.** *For any cut K of G, the edge in K with the least (strictly smaller than all others) weight must belong to all MST.*

Proposition 8 (Cycle Property)**.** *Given a cycle C in G, the edge in C with the largest weight cannot belong to an MST.*

Proposition 9 (Minimum Weight Edge Property)**.** *If the edge with the least weight is unique, then this edge belongs to every MST.*