We begin with a definition:

Definition 1. Let U be a universal function for the class of computable one varialbe functions. We can specify a numbering of this class: The n^{th} function is the section U_n. More generally, a nautral numbering of an arbitrary set $\mathscr{F}$ is a total map $\nu : \mathbb{N}_0 \to \mathscr{F}$ which has range $\mathscr{F}$.

Goal: Give an accurate formulation and proof for the statement: There exists an algorithm that assigns to a pair of numbers of any two computable functions a number for their composition.

Require: The universal function doing the numbering to be computable.

Need: A Gödel numbering.

We let U be a binary computable universal function for the class of unary computable functions. It is called a Gödel universal function if for any computable binary function V there exists a total computable unary function s such that $V(m,x) = U(s(m),x) \quad \forall m,x$. In other words the sections V_m and $U_{s(m)}$ coincide.

Theorem 1. *A Gödel universal function exists.*

Proof. Let T be a ternary computable function that is universal for all binary computable functions. One can construct such a thing as follows:

Fix an arbitrary computable numbering of pairs $\sigma : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ which is a 1-to-1 correspondence. We assign $\sigma(u,v)$ to be the number of the pair (u,v). Let R be a universal computable function for the class of all unary computable functions. We then define $T(n,u,v) = R(n, \sigma(u,v))$.

claim: T is universal for binary computable functions.

Let F be an arbitrary binary computable function. Let $f(\sigma(u,v)) = F(u,v)$ which is a unary computable function. Since R is universal, there exists an $n \in \mathbb{N}_0$ such that $R_n = f$ when applied to the input x. For this n, we have that

$$T(n,u,v) = R(n,\sigma(u,v)) = f(\sigma(u,v)) = F(u,v).$$

Since T_n coincides with F we've shown that T is universal.

We now endeavor to use T to define a binary Gödel universal function U. We set $U(\sigma(n,u),v) = T(n,u,v)$. Any binary computable function V occurs among the sections of T. Hence there is some n such that $V(u,n) = T(n,u,v)$ for every u,v. Then $V(u,v) = U(\sigma(n,u),v)$. We define $s(u) = \sigma(n,u)$. We conclude that the constructed universal function is Gödel. $\square$

Definition 2. The numberings of computable functions that correspond to Gödel universal functions are called Gödel numberings.

Theorem 2. *Let U be a Gödel universal function for the class of unary computable functions. Then there exists a total function c that assigns to numbers p,q of two unary function a number $c(p,q)$ of their composition $U_{c(p,q)} = U_p \circ U_q$ or $U_{c(p,q)}(x) = U(p, U_q(x))$.*

Proof. Consider a binary computable function V defined by the equation $V(\sigma(p,q),x) = U(p, U(q,x))$. By the definition of a Gödel universal function, there exists a unary total computable function s such that $V(m,x) = U(s(m),x) \quad m,x$. In which case

$$V(\sigma(p,q),x) = U(s(\sigma(p,q)),x).$$

Hence $c(p,q) = s(\sigma(p,q))$ is the desired relation. □

Remark 1. The converse is also true.

Q: Does there exist a computable universal function that is not Gödel?

A: It turns out that the answer is yes.

Computable Sequences of Computable Functions

Let $f_0, f_1, \ldots$ be a sequence of computable functions of one variable. We want to assign meaning to a sequence $i \mapsto f_i$ is computable. Here are two possible definitions:

1. This sequence is called computable if the binary function F defined by $F(i,n) = f_i(n)$ is computable.

2. This sequence is called computable if there exists a computable sequence of nautral numbers $c_0, c_1, c_2, \ldots$ such that such that c_i is one of the numbers of the function f_i.

The definition (b) depends on a numbering of computable functions. It turns out that these two definitions are equivalent if it is a Gödel numbering.

Theorem 3. *If the numbering is computable, then (b) $\Rightarrow$ (a). If the number is a Gödel numbering, then (a) $\Rightarrow$ (b).*

Proof. If U is a computable universal function and the sequence $i \mapsto c_i$ is computable then

$$F(i,x) = f_i(x) = U_{c_i}(x)$$

is computable as the composition of computable functions.

Coversely, if a function F is computable and U is Gödel universal function, then the coverter that exists by definition of a U being Gödel, is the function that takes i into one of the numbers of f_i. □

Gödel Universal Sets

Definition 3. An enumerable set $W \subset \mathbb{N}_0 \times \mathbb{N}_0$ is called a Gödel universal enumerable set if for every enumerable set $V \subset \mathbb{N}_0 \times \mathbb{N}_0$ there exists a total computalbe function $s : \mathbb{N}_0 \to \mathbb{N}_0$ such that $(n,x) \in V$ if and only if $(s(n), x) \in W$ for every n, x.

- Each set $U \subset \mathbb{N}_0 \times \mathbb{N}_0$ defines a numbering of a certain family of subsets in the following way: n is a number of the n^{th} section U_n.
- An enumerable subset of $\mathbb{N}_0 \times \mathbb{N}_0$ specifies a numbering of a certain family of enumerable subsets of $\mathbb{N}_0$. such numberings are computable.
- An enumerable subset $W \subset \mathbb{N}_0 \times \mathbb{N}_0$ is universal if and only if any enumerable subset $\mathbb{N}_0$ has a W-number, and W is a Gödel set if and only if any computable numbering V is computably reducible to the W numbering in the sense that $V_n = W_{s(n)}$ for some computable function s.

Theorem 4. *A Gödel universal enumerable set $W \subset \mathbb{N}_0 \times \mathbb{N}_0$ exists.*

Proof. This is a consequence of the following lemma. □

Lemma 1. *The domain of a Gödel universal function for the class of unary computable function is a Gödel universal set for the class of enumerable subsets of $\mathbb{N}_0$.*

Proof. Let U be a Gödel universal function, and let W be its domain. Consider an arbitrary enumerable set $V \subset \mathbb{N}_0 \times \mathbb{N}_0$ and a computable function G with domain V. Since U is Gödel there exists a total computable function $s : \mathbb{N}_0 \to \mathbb{N}_0$ such that $G_n = U_{s(n)}$ and therefore their domains V_n and $W_{s(n)}$ also coincide. □

Theorem 5. *Let $W \subset \mathbb{N}_0 \times \mathbb{N}_0$ be a Gödel universal enumerable set. Then a number of the intersection of two enumerable sets can be algorithmically computed from W-numbers of these sets: there exists a binary total computable function s such that $W_{s(m,n)} = W_m \cap W_n$.*