Σ_n and Π_n

Recall that a set $A \subset \mathbb{N}_0$ is enumerable if and only if there exists a decidable set $B \subset \mathbb{N}_0 \times \mathbb{N}_0$ such that A is the projection of B. Let's identify sets and predicates so that we can say that a property $A(x)$ of natural numbers is enumerable if and only if it can be represented in the form

$$A(x) \Leftrightarrow \exists y B(x, y)$$

where $B(x, y)$ is some decidable property.

Q: What can be said about other combinations of quantifiers?

Q: What properties are representable in the form

$$A(x) \Leftrightarrow \exists y \exists z C(x, y, z)$$

where C is some decidable property?

We can replace consecutive quantifiers by using one with computable numbering of pairs:

$$C''(x, \sigma(y, z)) \Leftrightarrow C(x, y, z) \text{ and } A(x) \Leftrightarrow \exists w C''(x, w).$$

Q: What properties can be represented in the form

$$A(x) \Leftrightarrow \forall y B(x, y)$$

where $B(x, y)$ is a decidable property.

A: The properties with enumerable negations:

$$\begin{aligned} \neg A(x) &\Leftrightarrow \neg \forall y B(x, y) \\ &\Leftrightarrow \exists y (\neg B(x, y)) \end{aligned}$$

We are implicitly using the fact that decidability is preserved under negation.

We now progress to a general definition:

Definition 1. A property A belongs to the class Σ_n if it can be represented in the form

$$A(x) \Rightarrow \exists y_1 \forall y_2 \exists y_3 \cdots B(x_1, y_1, y_2, \ldots, y_n)$$

with n alternating quantifiers and B is a decidable property.

If the n alternating quantifiers starts with $\forall$, then we obtain the definition of the class Π_n.

Theorem 1. *(a) The class Σ_n and Π_n does not change if we allow groups of quantifiers of the same type instead of a single quantifier.*

(b) If a predicate belongs to Σ_n then its negation belongs to Π_n and vice versa.

Theorem 2. *(a) The intersection and union of two sets of the class Σ_n belongs to Σ_n.*

(b) The intersection and union of two sets of the class Π_n belong to Π_n.

Example 1.

$$A(x) \Leftrightarrow \exists y \forall z B(x, y, z)$$

$$C(x) \Leftrightarrow \exists u \forall v D(x, u, v)$$

Then

$$A(x) \wedge C(x) \Leftrightarrow \exists y \exists u \forall z \forall v [B(x, y, z) \wedge D(x, u, v)]$$

The property is decidable and we can compbine the quantifiers to get it to belong to Σ_n.

Theorem 3. *The classes Σ_n and Π_n are "hereditary downward" with respect to m-reducibility in the following way:*

If $A \leq_m B$ and $B \in \Sigma_n$ then $A \in \Sigma_n$ or $B \in \Pi_n$ then $A \in \Pi_n$.

Unviersal Sets in Σ_n and Π_n

Goal: Show that the classes Σ_n and Π_n are distinct for different n.

How? We find a universal set for Σ_n and show that it cannot belong to Σ_k for any $k < n$.

Theorem 4. *For any n, the class Σ_n contains a set universal for this class. The complement of this set will be universal for the class Π_n.*

Theorem 5. *Universal Σ_n-sets do not belong to the class Π_n. Similarly universal Π_n-sets do not belong to Σ_n.*

Remark 1. The hierarchy is strict.