Guiding Questions

(a) Is the set of numbers of the empty function in a certain Gödel numbering decidable?

(b) Is it possible to determine whether a function is empty given its number in a Gödel numbering?

Remark 1. Any two Gödel numberings can be reduced to each other. Given a number of a function in one Gödel numbering it is possible to obtai a number of the same function in the other numbering. Conclusion? The answer to the above quesiton is independent of the Gödel numbering considered.

Theorem 1. *Let U be an arbitrary Gödel universal function. Then the set of all the numbers n such that the functions U_n is empty is undecidable.*

Proof. We use the method of reduction.

Claim: If this set were decidable, then any enumerable set would be decidable.

Let K be an arbitarary enumerable undecidable set. Consider the following binary computable function:

$$V(n,x) = \begin{cases} 0 & \text{if } n \in K \\ \text{undefined} & \text{if } n \notin K \end{cases}$$

Essentially this is the semicharacteristic function for K. The function V has the sections:

1. For $n \in K$, $V_n \equiv 0$.
2. For $n \notin K$, V_n is nowhere defined and is therefore the empty function.

Since U is a Gödel universal function, there exists a total computable function s such that

$$V(n,x) = U(s(n),x) \quad \forall n, x$$

that is $V_n = U_{s(n)}$. Hence, the value $s(n)$ is a number for the zero function where $n \notin K$. If the set of U-numbers for the empty function were decidable, we could apply that algorithm to $s(n)$ and K would be decidable. One of the things we can observed now: Any finite set is decidable so the cardinality of the numbers in any Gödel numbering is infinite.

Claim: The set of numbers of the empty function is not enumerable either.

The set of numbers for any numbering of nonempty functions is enumerable. How? If $U(n,x)$ is defined for some x then print n. We saw that Post's theorem said if the complement of an undecidable set is enumerable then the set itself is nonenumerable. □

Theorem 2 (Rice-Uspensky)**.** *Let $\mathscr{F}$ denote the class of all unary computable functions. Let $\mathscr{A} \subset \mathscr{F}$ be an arbitrary nontrivial property of computable functions with $\mathscr{A} \neq \varnothing$ and the inclusion is proper. Let U be* *a Gödel universal function. Then it is impossible to determine algorithmically wheter a computable function with a given U-number has the property $\mathscr{A}$.*

$\boxed{\text{The set } \{n \mid U_n \in \mathscr{A}\} \text{ is undecidable.}}$

Proof. WLOG we may assume that the empty function ζ belongs to $\mathscr{A}$. If it doesn't replace $\mathscr{A}$ with its complement. Let $\xi \in \mathscr{F} \setminus \mathscr{A}$ be arbitarary. We repeat the previous proof with

$$V(n,x) = \begin{cases} \xi(x) & \text{if } n \in K \\ \text{undefined} & \text{if } n \notin K \end{cases}$$

Again the function V is computable as an adjustment of the semicharacteristic function on K. For $n \in K$ then V_n conincides with ξ and $n \notin K$ it conincides with ζ. Hence, $V_n \in \mathscr{A}$ if and only if $n \notin K$.

If the set $\{n \mid U_n \in \mathscr{A}\}$ were decidable then $V_n \in \mathscr{A}$ is algorithmically decidable for a given n. Hence, we can decide if $n \in K$ or not which is contradictory to our assumption that K is undecidable.

Claim: If it were possible to recognize the property by U-numbers, then any two disjoint enumerable sets P and Q could be separated by a decidable set.

Choose any two functions ξ and η and consider the function:

$$V(n,x) = \begin{cases} \xi(x) & \text{if } n \in P \\ \eta(x) & \text{if } n \in Q \\ \text{undefined} & \text{if } n \notin P \cup Q \end{cases}$$

Note that for any n and x we can wait for n to appear in P or Q and then compute $\xi(x)$ or $\eta(x)$ as needed. So V is computable. If $n \in P$ then $V_n \equiv \xi$, if $n \in Q$ then $V_n \equiv \eta$. By verifying if $V_n \in \mathscr{A}$ we could decidably separate P from Q which is a contradiction. □

New Numbers of Old Functions

We've seen that the set of numbers of any specific funtion in a a Gödel numbering is undecidable and therefore infinite. We now prove a stronger statement.

Theorem 3. *Let U be a universal Gödel function. Then there exists a total binary function g such that for any i the values $g(i,0), g(i,1), \ldots$ are different U-numbers of the function U_i.*

Proof. Let h be an arbitrary function. We show that there exists an algorithm that finds infinitely many different U-numbers of the function h. Let P be an enumerable undecidable set. Consider a computable function

$$V(n,x) = \begin{cases} h(x) & \text{if } n \in P \\ \text{undefined} & \text{if } n \notin P \end{cases}$$

There are two functions in play for the sections, namely h and ζ.

Case 1: $h \neq \zeta$

Since U is a Gödel universal function, there exists a converter s that transforms V-numbers into U-numbers. That is

- $n \in P \Rightarrow U_{s(n)} = V_n = h$
- $n \notin P \Rightarrow U_{s(n)} = V_n = \zeta$

It follows that if $p(0), p(1), \ldots$ is a computable enumeration of the set P, then all of the U-numbers $s(p(0)), s(p(1)), \ldots$ are numbers of the function h.

Claim: The set $\{s(p(0)), s(p(1)), \ldots\}$ is infinite.

Suppose that this is not the case, and the set $X = \{s(n) \mid n \in P\}$ is finite. Then X is decidable and

- $n \in P \Rightarrow s(n) \in X$
- $n \notin P$ is a number of ζ and does not belong to X.

One now sees that $n \in P$ if and only if $s(n) \in X$ and the decidability of X impllies the decidability of P which is a contradiction.

Case 2: Suppose $h = \zeta$.

Let ξ be a computable function that is defined at least one point. Consider two enumerable inseparable sets P and Q and the computable function

$$V(n,x) = \begin{cases} h(x) & \text{if } n \in P \\ \xi(x) & \text{if } n \in Q \\ \text{undefined} & \text{if } n \notin P \cup Q \end{cases}$$

Let s be a converter from V-numbers to U-numbers. Then

- $n \in P \Rightarrow U_{s(n)} = h$
- $n \in Q \Rightarrow U_{s(n)} = \xi$
- $n \notin P \cup Q \Rightarrow U_{s(n)} = \zeta$

As before $s(p(0)), s(p(1)), \ldots$ are numbers of the function h.

Claim: If $h \neq \xi$ then the set X of the numbers $h, s(p(0)), s(p(1)), \ldots$ is undecidable.

If X were decidable, then we could separate P from Q by namely $P \subset \{n \mid s(n) \in X\}$ and $Q \cap X = \varnothing$. We conclude that we can produce numbers of h. If we don't know ehther $h = \zeta$ or not we may run these two constructions in parallel until one of them produces a new number.

We now consider the following computable functions:

$$V_1(\sigma(i,n),x) = \begin{cases} U(i,x) & \text{if } n \in P \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$V_2(\sigma(i,n),x) = \begin{cases} U(i,x) & \text{if } n \in P \\ 0 & \text{if } n \in Q \\ \text{undefined} & \text{if } n \notin P \cup Q \end{cases}$$

Since U is a universal Gödel function, we can find computable total functions s_1 and s_2 such that

$$V_1(\sigma(i,n),x) = U(s_1(\sigma(i,n)),x)V_2(\sigma(i,n),x) \qquad = U(s_2(\sigma(i,x)),x)$$

Let p be a total unary function such that $p = \{p(0), p(1), \ldots\}$. The desired function g is obtained as follows: $g(i,k)$ is the k^{th} number is the sequence $s_1(\sigma(i,p(0))), s_2(\sigma(i,p(0))), \ldots$. ☐

Isomorphism of Gödel Numbers

Theorem 4 (Rogers)**.** *Let U_1 and U_2 be two Gödel universal funtions for the class of unary computable functions. Then there exist two total mutually inverse computable function s_{12} and s_{21} such that*

$$U_1(n,x) = U_2(s_{12}(n),x)$$
$$U_2(n,x) = U_1(s_{21}(n),x)$$

for any n, x.

Remark 2. We're saying that any two Gödel numberings differ from each other only by a permutation of the numbers.