

Let's begin with a definition:

**Definition 1.** A function  $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  is said to be computable if there exists an algorithm  $M$  that computes  $f$ . The algorithm  $M$ :

- (a) halts on the input  $n$  and outputs  $f(n)$  when  $f$  is defined for an  $n \in \mathbb{N}_0$ .
- (b) does not halt on the input  $n$  whenever  $f$  is undefined for  $n \in \mathbb{N}_0$ .

*Proof.* (Not Really)

Let  $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  be a partial computable function. Extend  $f$  to a total function

$$f(n) = \begin{cases} f(n) & \text{if } n \in \text{Dom } f \\ 0 & \text{otherwise} \end{cases}$$

The algorithm  $M$  does not halt on  $n \notin \text{Dom } f$ . However,  $M$  is the only things we have to decide if  $f(n)$  is defined. How do we know when to halt  $M$  and output  $g(n) = 0$ ? □

## Decidable Sets

**Definition 2.** A set  $X \subseteq \mathbb{N}_0$  is said to be decidable if there exists an algorithm  $M$  that determines if  $n \in \mathbb{N}_0$  belongs to  $X$

**Q:** What does this mean?

**A:** The characteristic function for  $X$

$$\chi_X(n) = \begin{cases} 1 & \text{if } n \in X \\ 0 & \text{otherwise} \end{cases}$$

is computable.

*Remark 1.*

- (a) Any finite set is decidable. We can make an algorithm that outputs 1 on a finite number of elements.
- (b) The collection of decidable sets is closed under intersection, union, and set difference.

## Enumerable Sets

Any set that satisfies any of the following equivalent conditions is said to be enumerable.

**Proposition 1.** Let  $X \subseteq \mathbb{N}_0$ . TFAE:

- (a) *There exists an algorithm  $M$  that outputs the elements of  $X$  and only those elements.*
- (b)  *$X$  is the domain of a computable function.*
- (c)  *$X$  is the range of a computable function.*
- (d) *The semicharacteristic function for  $X$*

$$f_X(n) = \begin{cases} 0 & \text{if } n \in X \\ \text{undefined} & \text{otherwise} \end{cases}$$

*Proof.* (a)  $\Rightarrow$  (b,d) Suppose that  $X$  is enumerated by an algorithm  $M$ . Then we can compute the semicharacteristic function for  $X$ :

- (i) Execute  $M$  until  $n$  is printed.
- (ii) Output 0 and terminate.

(c)  $\Rightarrow$  (b) Suppose  $X$  is in the domain of a computable function  $f$  and  $N$  is the algorithm that computes  $f$ . We'll run  $N$  in parallel on 0, 1, 2, ... gradually increasing the number of involved inputs:

- (i) Run one step of  $N$  on 0 and 1.
- (ii) Run two steps of  $N$  on 0, 1, and 2.
- (iii) Run three steps of  $N$  on each input 0, 1, 2, and 3.

All arguments on which  $N$  terminates are printed out as soon as they are detected. We avoid having  $N$  go into an infinite loop on an undefined element  $n \in \mathbb{N}_0$  by only performing a finite number of steps from the algorithm  $N$  each time. We can modify this algorithm to print the results returned rather than the arguments where it terminates to prove the converse (b)  $\Rightarrow$  (c).

(b)  $\Rightarrow$  (c) We've seen that  $X$  is the domain of a computable function. This function is computed by an algorithm  $N$ . Define

$$b(x) = \begin{cases} x & \text{if } M \text{ terminates on } X \\ \text{undefine} & \text{otherwise} \end{cases}$$

□

**Definition 3.** A set  $X \subseteq \mathbb{N}_0$  is enumerable if either:

- (i)  $X = \emptyset$
- (ii)  $X$  is in the range of a total computable function.

This definition turns out to be equivalent to all of the others. Suppose  $X$  is enumerated by an algorithm  $M$ . Let  $x_0 \in X$  be arbitrary. Define the total function

$$a(n) = \begin{cases} t & \text{if } M \text{ returns } t \text{ at the } n^{\text{th}} \text{ step} \\ x_0 & \text{otherwise} \end{cases}$$

The function  $a(n)$  has  $X$  in its range and is computable by its definition. Hence,  $X$  is enumerable.

**Theorem 1.** *The intersection and union of enumerable sets is enumerable.*

*Remark 2.* The complement of an enumerable set may not be enumerable.

## Enumerable & Decidable Sets

**Theorem 2** (Post). *Any decidable set of  $\mathbb{N}_0$  is enumerable. If a set  $A$  and its complement  $\mathbb{N}_0 \setminus A$  are enumerable, then  $A$  is enumerable.*

*Proof.* Suppose  $X \subseteq \mathbb{N}_0$  is decidable. Then the characteristic function is computable. If  $\chi_X(n) = 1$ , then we output  $n$ . Hence, we have an algorithm that will output exactly the elements of  $X$ .

" $\Rightarrow$ " Suppose  $X$  and  $X^c$  are enumerable. Then we can run two algorithms in parallel to determine if an element is in  $A$  or  $A^c$ . We conclude that the characteristic function is computable.

" $\Leftarrow$ " Suppose  $X$  is decidable. We've seen that  $X$  is enumerable. Let  $M$  be the algorithm that outputs the elements of  $X$ . If  $M$  outputs 0, print  $n$ . We conclude that there is an algorithm that enumerates  $A^c$ .  $\square$

*Remark 3.* We use  $\pi_1$  and  $\pi_2$  to be the usual projections:

$$\pi_1 : A \times B \rightarrow A \text{ where } \pi_1(a, b) = a$$

$$\pi_2 : A \times B \rightarrow B \text{ where } \pi_2(a, b) = b$$

**Theorem 3.** *A set  $Y \subseteq \mathbb{N}_0$  is enumerable if and only if  $Y$  is the projection of a decidable set  $Q \subseteq \mathbb{N}_0 \times \mathbb{N}_0$ .*

*Proof.* " $\Rightarrow$ " Suppose  $Y$  is enumerable. Then there exists an algorithm  $M$  which outputs the elements of  $Y$ . Let

$$Q = \{(y, n) \mid y \in Y \text{ and output after } n \text{ steps}\}$$

We can always decide if  $(y_0, n_0) \in Q$  by running  $M$  and therefore  $Q$  is decidable. By the definition of  $Q$   $\pi_1(Q) = Y$ .

" $\Leftarrow$ " Suppose  $Y$  is the projection of  $Q \subseteq \mathbb{N}_0 \times \mathbb{N}_0$  which is decidable. Hence  $\pi_1(Q)$  is also decidable and therefore  $\pi_1(Q)$  is enumerable by Post's theorem.  $\square$

## Enumerability & Computability

**Theorem 4.** *A function  $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  is computable if and only if its graph*

$$F = \{(x, y) \mid f(x) \text{ is defined and } f(x) = y\}$$

*is enumerable.*

*Proof.* " $\Rightarrow$ " Suppose that  $f$  is computable. Then there exists an enumeration for its domain. By also outputting the value  $f(x)$  we get an algorithm that enumerates  $F$ .

" $\Leftarrow$ " Suppose that  $F$  is enumerable. Let  $M$  be the algorithm that outputs  $F$ . Then

**Input:**  $n \in \mathbb{N}_0$

- (i) Run  $M$  until it outputs  $(n, f(n))$ .
- (ii) Output  $f(n)$
- (iii) Terminate the computation.

$\square$

**Theorem 5.** *The image and preimage of an enumerable set under a computable function are enumerable.*

*Proof.* Suppose that  $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$  is computable with  $X \subseteq \mathbb{N}_0$  enumerable and  $M$  the algorithm that outputs  $X$ . Then

$$\begin{aligned} f^{-1}(X) &= \{a \in \mathbb{N}_0 \mid f(a) \in X\} \\ &= \pi_1[(\mathbb{N}_0 \times X) \cap F] \end{aligned}$$

which is the projection of a decidable set.  $\square$