Algebraic Groups

We wish to build up our knowledge about groups acting on varieties. We begin by asking the question: What is an affine variety? Each definition becomes a little more abstract, as we progress. We then define a projective variety. There are many ways to view the objects used in algebraic geometry, and each viewpoint contributes to developing intuition.

Once we have a notion of what an affine variety is, we proceed to considering algebraic groups. Again, we progress from an intuitive notion through the definition of a group scheme. We do this rather quickly in order to develop the idea of a geometric quotient.

What is an affine variety?

(1) In a first course, one defines an affine variety to be the zero locus of a finite number of polynomials. This is an easy definition to understand. However, one encounters the need to broaden the definition rather quickly.

(2) Identify $\mathrm{Mat}_{n\times n}(k)$ with $\mathbb{A}^{n^2}$. Then $\mathrm{GL}_n(k)$ is the complement of the closed subset defined by the vanishing of the determinant. Hence, $\mathrm{GL}_n(k)$ is a Zariski open subset of $\mathbb{A}^{n^2}$.

On the other hand, if we consider

$$\left\{(p,y)\in\mathbb{A}^{n^2}\times\mathbb{A}^1 \mid \det(p)y=1\right\}$$

is a Zariski closed subset of $\mathbb{A}^{n^2}\times\mathbb{A}^1$ and is in bijection with $\mathrm{GL}_n(k)$. So, $\mathrm{GL}_n(k)\subseteq\mathbb{A}^{n^2}$ is open and is closed in $\mathbb{A}^{n^2}\times\mathbb{A}^1$.

Form this point of view, X is an affine variety if it is isomorphic to some zero locus.

(3) The Steinberg definition of an affine variety is as follows:

The pair (X,A) is said to be an affine variety if the follwing hold:

(a) A is a finitely generated k-algebra with 1_A.

(b) The functions in A separate points in X. That is, for distinct points $x,y\in X$, there exists $f\in A$ such that $f(x)\neq f(y)$.

(c) For every k-algebra homomorphism $\varphi : A\to k$ there is some $x\in X$ such that $\varphi=\varepsilon_x$.

(4) A *ringed space* is a pair $(X,\mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. An *affine scheme* is a locally ringed space $(X,\mathcal{O}_X)$ which is isomorphic to the spectrum of some ring.

Each of these definitions is useful depending on the context that is being considered.

What is a projective variety?

Algebraic Groups. We are interested in varieties which are equipped with a group structure.

(1) An algebraic group is a variety which has a group structure.

Definition 1. An algebraic group G is a variety with regular maps

$$\begin{aligned} * : G \times G &\to G \\ (g, h) &\mapsto g * H \end{aligned}$$

and

$$\begin{aligned} i : G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

One can show that an affine algebraic group is a Zariski closed subset of $\mathrm{GL}_n(k)$, for some n.

(2) A *group variety* over a field k is a smooth k-scheme G of finite type equipped with k-morphisms

$$\begin{aligned} \mu : G \times G &\to G \\ \iota : G &\to G \end{aligned}$$

and a distinguished rational point $\mathbf{e}$ such that the following diagrams commmute:

$$\begin{tikzcd}
G \times G \times G \arrow[r, "1_G \times \mu"] \arrow[d, "\mu \times 1_G"'] & G \times G \arrow[d, "\mu"] \\
G \times G \arrow[r, "\mu"'] & G
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[rr, "(1_G, \iota)"] \arrow[dd, "(\iota, 1_G)"'] \arrow[dr, dash] & & G \times G \arrow[dd, "\mu"] \\
& \operatorname{Spec} \mathbb{F} \arrow[dr, "\mathbf{e}"] & \\
G \times G \arrow[rr, "\mu"'] & & G
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r, "(1_G, e)"] \arrow[d, "(e, 1_G)"'] \arrow[dr, equal] & G \times G \arrow[d, "\mu"] \\
G \times G \arrow[r, "\mu"'] & G
\end{tikzcd}$$

Examples.

- Finite Groups
- We mentioned above that $\mathrm{GL}_n(k)$ can be viewed as the complement of the hypersurface determined by $\det(A) = 0$. Hence, a prinicipal open subset of $\mathbb{A}^{n^2}$ and therefore an affine variety.
- $\mathrm{SL}_n(k)$ is a subgroup of $\mathrm{GL}_n(k)$ determined by $\det -1$. Hence, it is closed and is an affine variety. This is an example of an algebraic subgroup.
- Let E be an elliptic curve and $E(k)$ the points in k which satisfy the equation given by the curve. It is well known there is a group law on $E(k)$. This is an example of a projective algebraic group.

Group Actions. Let G be an algebraic group and X a variety. A *regular action* of G on X is a regular map

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto g \cdot x \end{aligned}$$

Suppose that G is a finite group which acts on an affine variety X, which is the zero locus of $\{f_\alpha(x_1, \ldots, x_n)\}$ in $\mathbb{A}^n$. The claim is that the quotient of X by G, denoted X/G, exists. We outline the program to show this below:

- Show that the coordinate ring of X/G is finitely generated.
- Show that the points of the quotient variety correspond to the orbits of the action.
- This is a special case. We usually have to use the fact that the points of the quotient variety correspond to the closed orbits.

Quotients by Finite Groups. In the following, we consider a finite group G actiong on an affine variety X. We're going to suppose that X is the zero locus of polynomials $\{f_\alpha(x_1, \ldots, x_n)\}$. Let $\mathcal{I}(x) = I$ be the ideal of X and $k[X] = k[x_1, \ldots, x_n]/I$ the coordinate ring.

A quotient Y exists if there is a G-invariant surjective morphism $\pi : X \twoheadrightarrow Y$ such that given any other morphism $f : X \to Z$ there is a unique morphism that factors through π making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, two heads, "\pi"] \arrow[dr, "f"'] & Y \arrow[d, dashed, "\phi"] \\ & Z \end{tikzcd}$$

Taking Z to be $\mathbb{A}^1$, one gets the idea that the coordinate ring for Y should be the G-invariant regular functions on X

$$k[x_1, \ldots, x_n]^G = \{f \in k[x_1, \ldots, x_n] \mid (g \cdot f)(p) = f(p) \forall g \in G, \forall p \in X\}$$

We need to see that the ring of invariants is finitely generated.

To see this, we remind ourselves that there xists a permutation represenation $\rho : G \hookrightarrow S_n$, where n is the order of G. So,

$$k[e_1, \ldots, e_n] = k[x_1, \ldots, x_n]^{S_n} \subseteq k[x_1, \ldots, x_n]^G \subseteq k[x_1, \ldots, x_n]$$

where $e_1, \ldots, e_n$ are the symmetric polynomials. Since $k[e_1, \ldots, e_n] \subseteq k[x_1, \ldots, x_n]$ is module-finite, we conlcude that $k[x_1, \ldots, x_n]^G$ is finitely generated.

Constructing a variety from an action of G on X, one would first attempt to take the points to correspond to the orbits. This is what we wish to see. Suppose that p, q are two points of X which do not lie in the same orbit. Since G is a finite group, the orbit $G \cdot p$ is a finite set. Hence, the orbit is a closed set, and there exists a regular function which vanishes on the orbit $G \cdot p$ and $f(y) = 1$. We define

$$\varphi := \prod_{g \in G} g \cdot f$$

which is a G-invariant regular function on Y when taken modulo I. We conclude that the regular functions on Y separate points as required.

Show that the map is surjective.