I'm a Big Fan of Modeling
Group 4

Problem Summary: The goal of this project was to determine the feasibility of using a fan to separate falling paper and cardboard. We aimed for 60-70% of the paper to fall into a separate bin from the cardboard while minimizing the power of the fan and the height from which the materials drop. This setup is visualized in Figure 1. The equations used to model the acceleration of the objects used aerodynamics and were integrated using numerical methods. The x position of each object when it left the windy area created by the fan was noted and used to estimate the locations of collection boxes. The model returned the optimal values of wind speed as 13.8 m/s and initial height as 15.3 m. While there were some unexpected results produced, overall the model seemed to match our expectations and illuminated new questions.

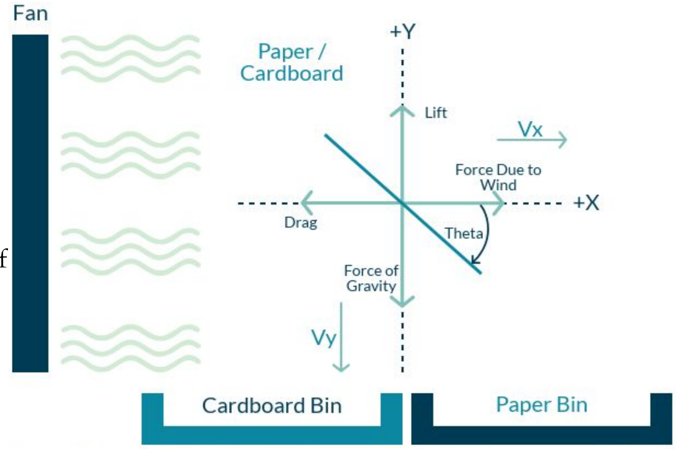

Figure 1: Problem Setup. The diagonal line represents the falling paper or cardboard. The forces acting on this object are shown. The object will execute a semi-parabolic arc and land in one of the boxes. At least 60% of the paper will end up in the paper bin.

Assumptions, Equations, and Rationale: Some of our assumptions were made in order to define variables. For the fan we ran preliminary tests on, the wind speed did not appear to vary by production location, so we assumed that the fan propels the air at the same speed throughout its diameter. Also, the materials are dropped at exactly the top of the fan and the collection bins are just below the fan. As given in the problem, the paper and the cardboard are uniformly intermixed. The size of the paper and cardboard was generated uniformly and randomly within a range of common sizes.

Many assumptions were made to simplify the aerodynamics. We know intuitively that as the distance from the fan increases the airspeed of the wind decreases, so we modeled airspeed as decreasing exponentially with distance. Two big assumptions we made were that the paper and cardboard were approximately flat and stayed rigid as they fell. These assumptions were necessary since the aerodynamics of an object depends greatly on its shape, and trying to model a changing shape would take more time, knowledge of aerodynamics, and resources than we had. Another assumption was that objects were dropped at time intervals such that they didn't interfere with each other through collisions or disrupt each other's airflow. If a machine slowly shifted paper and cardboard into this machine this condition could be satisfied.

We assumed that the drag of the paper as it falls was proportional to v_x^2, not its speed relative to the moving air, $(v_a - v_x)^2$. This means we assumed that the air on the +X side of the material (see Figure 1) was stationary because once the object falls, it causes eddies in the airflow that create turbulence, leading us to assume the average velocity of the air on the +X side was 0 m/s. Without this assumption, we obtained nonsensical results, so this assumption is necessary for our model.

$$a_x = \frac{1}{2m}\left(1 - \frac{\theta^2}{2}\right) \varrho_a\, C_d\, A\; v_a\,(v_a - 2\,v_x)$$

$$a_y = \frac{1}{m}\pi\,\theta\, A\, \varrho_a\, v_y^2\left(1 + \frac{A\,2\theta}{s^2}\right)^{-1} - g$$

Figure 2a. Acceleration on the X axis.
Figure 2b. Acceleration on the Y axis. Both equations are derived from the principles of lift, gravitational force, and non-linear air resistance.

Using these assumptions and the concepts of drag, lift, and the gravitational force, we modeled acceleration in the x and y directions as seen in Figure 2. In these equations, m is mass of the object, A is the cross sectional area of the object, ϱ_a is the air density at sea level, θ is the angle the paper makes with the horizontal (less than 0.2 radians due to the small angle regime). The acceleration on the Y axis, a_y, is described by Figure 2a, where v_y is the Y velocity, s is the wingspan(diagonal length of the paper), and g is the gravitational acceleration. The acceleration on the X axis, a_x, is described by Figure 2b, where C_d is the drag coefficient, set to 1.17 to account for the geometry of a rectangular sheet. The x velocity of the material is v_x and the airspeed of the wind is v_a. These equations were integrated numerically in Python.

Modeling Techniques: After finding an equation for the x position as a function of time, we developed a process to simulate a variable number of pieces of paper and cardboard falling according to the above equations. Intersections, denoted by $\boldsymbol{\chi}$, were defined as any point at which the final x position of the paper is less than or equal to the final x position of the cardboard. We simulated numerous falling objects of various sizes and masses. The number of intersections, $\boldsymbol{\chi}$, is a function of y_0 and v_a, and was optimized to find the minimum y_0 and v_a. The function exists in three-dimensional space, however it can be expressed as a two-dimensional projection on the v_a and y_0 plane to return the minimum values of the variables that produce a 30-40% intersection rate. The function's values at a given intersection percentage is shown in Figure 3. For y_0, the bounds of the area of interest were 2m and 20m. For v_a the bounds were 0.1 m/s and 20 m/s.

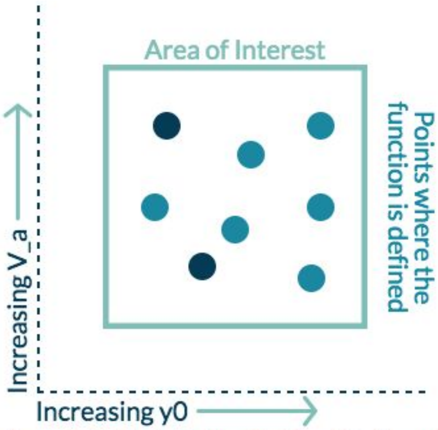

Figure 3: 3D Optimization Plot. Combinations of air velocity and initial height are tested, and if 30-40% of the pieces of paper intersect with the cardboard, the point is plotted. Darker colors indicate higher percentages of intersection.

The lower bound of y_0 was set to 2m and paper and cardboard of less than two meters were dropped. The lower bound of v_a was set to be 0.1m/s because the value needs to be nonzero and we are considering precision to 0.1 SI units. The upper bounds of y_0 and v_a are arbitrary. Using Python, we checked each point at intervals of 0.1 SI units, noted the point if $\boldsymbol{\chi}$ was in the desired interval, and checked to see if it was the closest point to (0,0) that satisfied the conditions. By running numerous simulations and finding the average of the values that produced minimums, we obtained the point (15.3 m, 13.8 m/s). A simulation of 30 falling objects with this initial height and wind speed produced Figure 4.

Verification: After creating this model, we tested the extreme cases to check if our model would match physical reality. Setting initial y position to zero or v_a to zero made everything intersect, as we would expect. Many instances in which the initial height was less than the optimal height resulted in paper intersecting with cardboard at a lower rate than 30%. This would indicate a higher efficiency for separation when the initial height was smaller, which is contrary to our expectations. We would need more time and equipment to physically test this result.

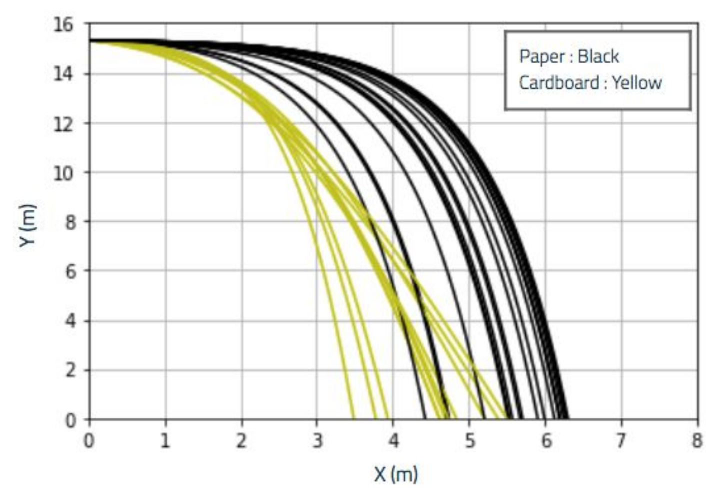

Figure 4: Optimal Airspeed and Initial Position. Approximately 30-40% of the paper ends up in the cardboard bin.

Conclusion: Using simplifying assumptions, this model should accurately predict the separation of paper from cardboard. Based on our calculations, we conclude that by using an initial y position of 15.3 m and an air speed of 13.8 m/s, 30% to 40% of the falling paper would consistently separate from the cardboard. It would be interesting to examine the changing geometry of the falling paper further and use that to inform a better model. Some conclusions of the model, such as the low percentage of intersections at small heights, need to be examined further and tested physically if possible.