Tutorial on Probability and Statistics

COMP 64101 - Weeks 1 and 2

October 2 and 7, 2025

1 Probability

(P1) Answer the following.

- Under what circumstances is an event A independent of itself?
- By considering events concerned with independent tosses of a red die and a blue die, give examples of events A, B and C which are not independent, but nevertheless are such that every pair of them is independent.

(P2) Whether certain mice are black or brown depends on a pair of genes, each of which is either B or b. If both members of the pair are alike, the mouse is said to be homozygous, and if they are different it is said to be heterozygous. The mouse is brown only if it is homozygous bb. The offspring of a pair of mice have two such genes, one from each parent, and if the parent is heterozygous, the inherited gene is equally likely to be B or b. Suppose that a black mouse results from a mating between two heterozygotes.

- What are the probabilities that this mouse is homozygous and that it is heterozygous? Now suppose that this mouse is mated with a brown mouse, resulting in seven offspring, all of which turn out to be black.
- Use Bayes' Theorem to find the probability that the black mouse was homozygous BB.

(P3) Suppose that $X \sim \mathrm{Bin}(N, \mu)$, where N is large and μ is small, but $N\mu = \lambda$ has an intermediate value. Use the exponential limit $\left(1 + \frac{k}{n}\right)^n \to e^k$ to show that $\mathbb{P}(X = 0) \cong e^{-\lambda}$ and $\mathbb{P}(X = 1) \cong \lambda e^{-\lambda}$. Extend this result to show that X is such that

$$p(x) \cong \frac{\lambda^x}{x!} e^{-\lambda},$$

that is, X is approximately distributed as a Poisson variable of mean λ.

(P4) Let X and Y be continuous random variables defined on the same probability space. Call p_X and p_Y their respective marginal densities, $p_{X,Y}$ the joint density, and $p_{X|Y}$ the conditional density of X given Y. Let $\mathbb{E}(X) := \int x p_X(x) \mathrm{d}x$, $\mathbb{E}(Y) := \int y p_Y(y) \mathrm{d}y$, and $\mathbb{E}(X \mid Y) := \int x p_{X|Y}(x \mid y) \mathrm{d}x$. Assume that they all exist and are integrable. Then, show the tower property, that is,

$$\mathbb{E}\left[\mathbb{E}(X \mid Y)\right] = \mathbb{E}(X).$$

2 Statistics

(S1) Suppose we are given the following twelve observations from a Normal distribution:

15.644, 16.437, 17.287, 14.448, 15.308, 15.169, 18.123, 17.635, 17.259, 16.311, 15.390, 17.252.

and we are told that the variance $\sigma^2 = 1$. Find a 90% HDR for the posterior distribution of the mean assuming the usual reference prior, i.e. a Uniform.

(S2) With the same data as in the previous question, what is the predictive distribution for a possible future observation $\tilde{x}$?

(S3) Compute the MLE for a univariate Normal likelihood $\mathcal{N}(\mu, \sigma^2)$. Now, put on your Bayesian hat and assume that σ^2 is known, but μ is not. Define the conjugate prior, obtain the posterior, and find the MAP estimator.

(S4) Suppose the data is modeled as i.i.d. $\mathrm{Exp}(\theta)$, and the prior is $\theta \sim \mathrm{Gamma}(a, b)$, i.e.

$$p(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}.$$

We know that the posterior is

$$p(\theta \mid x_1, \ldots, x_n) = \mathrm{Gamma}(a', b'),$$

where $a' = a + n$ and $b' = b + \sum_{i=1}^n x_i$. What is the posterior predictive density $p(x_{n+1}|x_1, \ldots, x_n)$? What is the marginal likelihood $p(x_1, \ldots, x_n)$?