

Algorithms

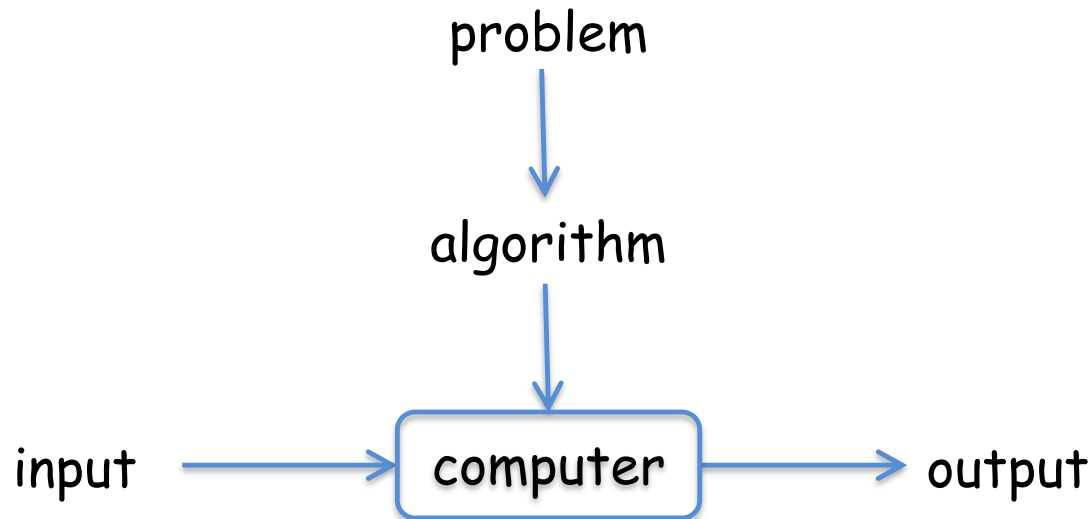
Murat Osmanoglu

Introduction

- 'a sequence of **unambiguous** instructions for solving a given problem' (Levitin) - **obtaining a required output for any legitimate input in a finite amount of time**

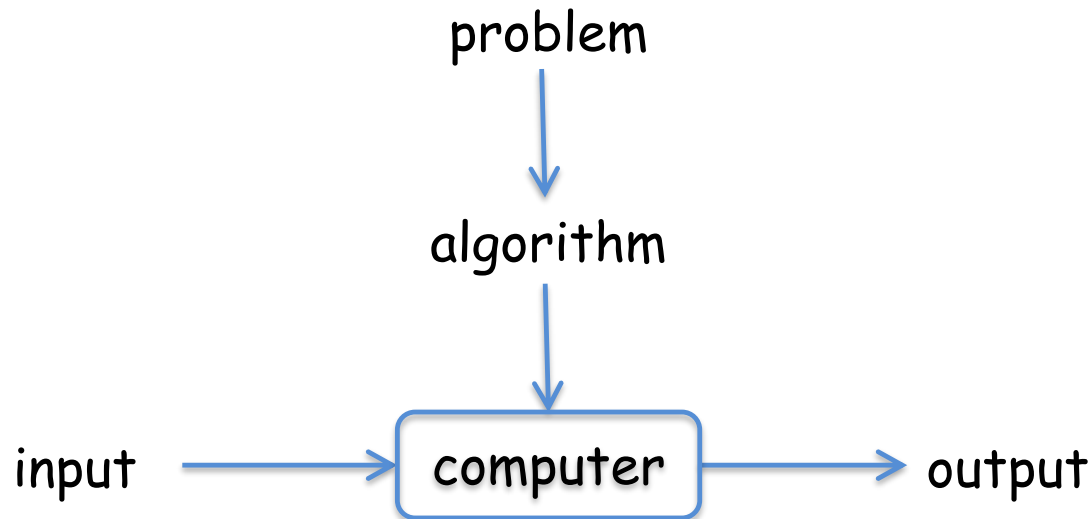
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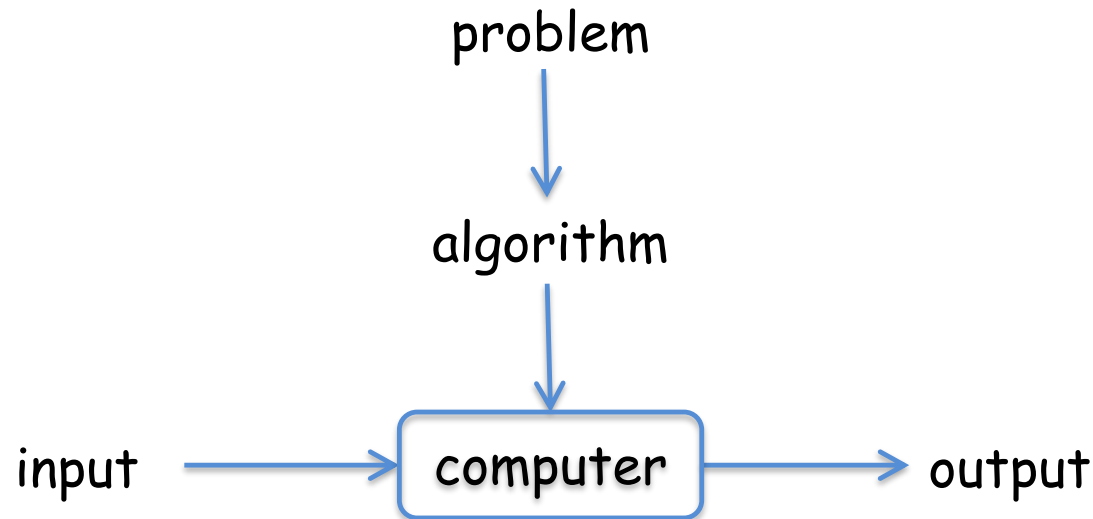
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- each step should be expressed in a clear way
- the nature of the input should be specified carefully

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- **Euclid's Algorithm for the integers m, n**
 - Step 1** : if $n = 0$, return m ; otherwise go to Step 2
 - Step 2** : divide m by n and set the variable r to the remainder
 - Step 3** : set m as n and n as r , go to Step 1

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- **Euclid(m, n)**

input : two non-negative, not-both-zero integers m and n

output: the greatest common divisor of m and n

while $n \neq 0$

$r \leftarrow m \pmod{n}$

$m \leftarrow n$

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return m

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- high-level description of algorithms that combines a natural language and familiar structures from a programming language

- use ' \leftarrow ' for the assignments and ' $//$ ' for the comments

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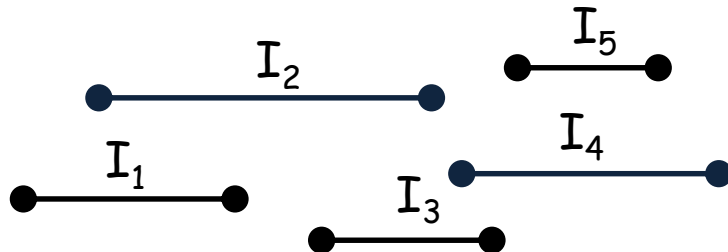
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- prove that the algorithm always returns the desired output for every legitimate input in a finite amount of time **in a formal way**
- use mathematical proof techniques such as proof by contradiction, induction, etc.
- suppose there are n meetings requests for a meeting room. Each meeting i has a starting time s_i and an ending time t_i . We have a constraint : no two meetings can be scheduled at same time. Design an algorithm that schedules as many meetings as possible to the room

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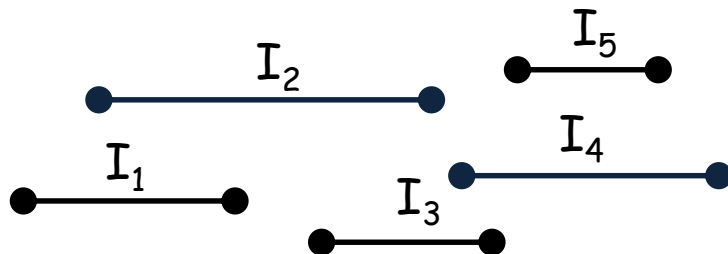
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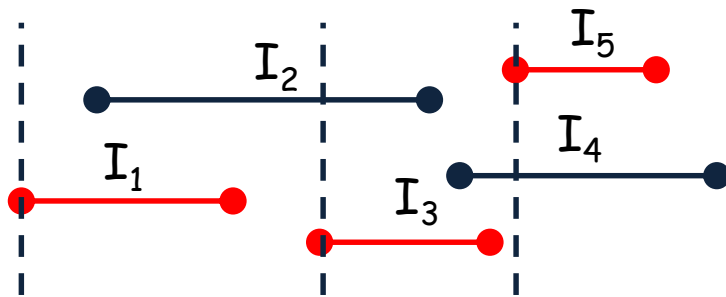


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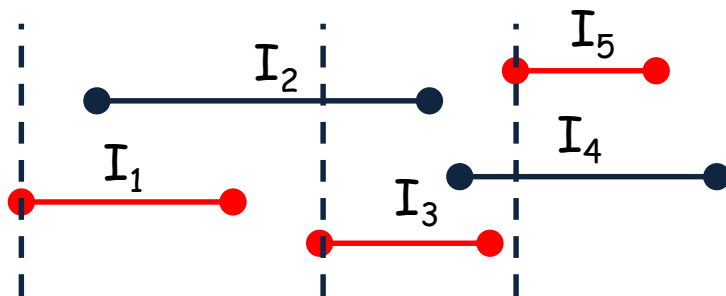


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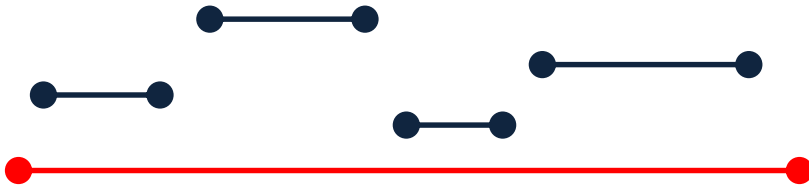


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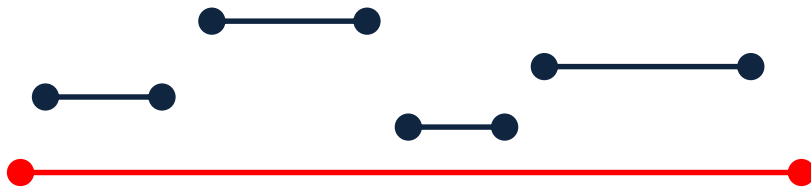


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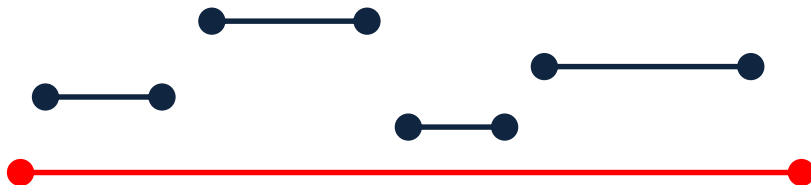


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- choose the first interval as the one having the earliest start time
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- one instance of inputs for which the algorithm fails would be enough to show the algorithm is incorrect
(however, failure to find such instance does not mean 'it is obvious' that the algorithm is correct)

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Evaluating the efficiency of the Algorithm

- investigate algorithm's efficiency with respect to two resources: running time and memory space (time complexity and space complexity)
- how long does the algorithm take to generate a desired output as a function of input size ?
- how much working memory (typically RAM) required for the algorithm to terminate as a function of input size ?

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 - c_{op} and the multiplicative constant $\frac{1}{2}$ can be removed from the formula (we can answer such question without knowing them)
 - focus on **rate of growth** of the function $T(n)$

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- For small inputs, a difference in running times can be ignored (it does not actually distinguish efficient algorithms from inefficient ones)
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- Growth rates of common functions measured in nanoseconds (assume each operation takes one nanosecond)

n	$f(n)$	$\lg n$	n	$n \lg n$	n^2	2^n	$n!$
10		0.003 μs	0.01 μs	0.033 μs	0.1 μs	1 μs	3.63 ms
20		0.004 μs	0.02 μs	0.086 μs	0.4 μs	1 ms	77.1 years
30		0.005 μs	0.03 μs	0.147 μs	0.9 μs	1 sec	8.4×10^{15} yrs
40		0.005 μs	0.04 μs	0.213 μs	1.6 μs	18.3 min	
50		0.006 μs	0.05 μs	0.282 μs	2.5 μs	13 days	
100		0.007 μs	0.1 μs	0.644 μs	10 μs	4×10^{13} yrs	
1,000		0.010 μs	1.00 μs	9.966 μs	1 ms		
10,000		0.013 μs	10 μs	130 μs	100 ms		
100,000		0.017 μs	0.10 ms	1.67 ms	10 sec		
1,000,000		0.020 μs	1 ms	19.93 ms	16.7 min		
10,000,000		0.023 μs	0.01 sec	0.23 sec	1.16 days		
100,000,000		0.027 μs	0.10 sec	2.66 sec	115.7 days		
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- any algorithm with $n!$ running time becomes useless for $n \geq 20$

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- Growth rates of common functions measured in nanoseconds (assume each operation takes one nanosecond)

n	$f(n)$	$\lg n$	n	$n \lg n$	n^2	2^n	$n!$
10		0.003 μ s	0.01 μ s	0.033 μ s	0.1 μ s	1 μ s	3.63 ms
20		0.004 μ s	0.02 μ s	0.086 μ s	0.4 μ s	1 ms	77.1 years
30		0.005 μ s	0.03 μ s	0.147 μ s	0.9 μ s	1 sec	8.4×10^{15} yrs
40		0.005 μ s	0.04 μ s	0.213 μ s	1.6 μ s	18.3 min	
50		0.006 μ s	0.05 μ s	0.282 μ s	2.5 μ s	13 days	
100		0.007 μ s	0.1 μ s	0.644 μ s	10 μ s	4×10^{13} yrs	
1,000		0.010 μ s	1.00 μ s	9.966 μ s	1 ms		
10,000		0.013 μ s	10 μ s	130 μ s	100 ms		
100,000		0.017 μ s	0.10 ms	1.67 ms	10 sec		
1,000,000		0.020 μ s	1 ms	19.93 ms	16.7 min		
10,000,000		0.023 μ s	0.01 sec	0.23 sec	1.16 days		
100,000,000		0.027 μ s	0.10 sec	2.66 sec	115.7 days		
1,000,000,000		0.030 μ s	1 sec	29.90 sec	31.7 years		

- for $n = 10$, all such algorithms take roughly the same time
- any algorithm with $n!$ running time becomes useless for $n \geq 20$
- any algorithm with 2^n running time becomes impractical for $n > 40$

Algorithm Efficiency

Order of Growth (Rate of Growth)

- For small inputs, a difference in running times can be ignored (it does not actually distinguish efficient algorithms from inefficient ones)
- For large inputs, a difference in running times becomes clear and remarkable
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- any algorithm with $n!$ running time becomes useless for $n \geq 20$
- any algorithm with 2^n running time becomes impractical for $n > 40$
- quadratic-time algorithms are practical up to $n = 1 \text{ million}$
- by analyzing the order of growth of the function $T(n)$ that counts the algorithm's basic operation (simply considering the leading term of the function), we can evaluate whether a given algorithm is practical for a problem of a given size

Algorithm Efficiency

Worst-Case, Best-Case, Average-Case Analysis

Algorithm Efficiency

Worst-Case, Best-Case, Average-Case Analysis

Linear-Search(list, x)

input : $\{a_1, a_2, \dots, a_n; x\}$

output: location

for $i = 1$ to n

if $x = a_i$

return i

return 0

Algorithm Efficiency

Worst-Case, Best-Case, Average-Case Analysis

Linear-Search(list, x)

input : $\{a_1, a_2, \dots, a_n; x\}$

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 if $x = a_i$ ———

 return i ——— 1 op

return 0 ———

Algorithm Efficiency

Worst-Case, Best-Case, Average-Case Analysis

Worst Case :

- consider the worst-case input of size n for which the algorithm runs the longest among all possible inputs of same size

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input : $\{a_1, a_2, \dots, a_n; x\}$

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for $i = 1$ **to** n  n steps

if $x = a_i$ 
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Worst-Case, Best-Case, Average-Case Analysis

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input : $\{a_1, a_2, \dots, a_n; x\}$

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 return i ——— 1 op

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Worst Case :

- consider the worst-case input of size n for which the algorithm runs the longest among all possible inputs of same size
(the element x matches the last one in the list, or the list does not contain the element x)
- $T(n) = n + 3$

Algorithm Efficiency

Worst-Case, Best-Case, Average-Case Analysis

Linear-Search(list, x)

input : $\{a_1, a_2, \dots, a_n; x\}$

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Best Case :

- consider the best-case input of size n for which the algorithm runs the fastest among all possible inputs of same size

Algorithm Efficiency

Worst-Case, Best-Case, Average-Case Analysis

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input : $\{a_1, a_2, \dots, a_n; x\}$

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if $x = a_i$ ———
 return i ——— 1 op

return loc ———

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Best Case :

- consider the best-case input of size n for which the algorithm runs the fastest among all possible inputs of same size
(the element x matches the first one in the list)
- $T(n) = 2$

Algorithm Efficiency

Worst-Case, Best-Case, Average-Case Analysis

Linear-Search(list, x)

input : $\{a_1, a_2, \dots, a_n; x\}$

output: location

for $i = 1$ **to** n ——— n steps

if $x = a_i$ ———
 return i ——— 1 op

return loc

Average Case :

- neither the worst-case nor the best-case analysis gives us the necessary information about how the algorithm behaves on a random input
- it's the expected value for the number of operations

Worst Case :

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- if $x = a_1$, then the algorithm terminates after 2 operations

if $x = a_2$, then the algorithm terminates after 3 operations

⋮

if $x = a_i$, then the algorithm terminates after $i + 1$ operation

⋮

if $x = a_n$, then the algorithm terminates after $n + 1$ operations

if $x \notin L$, then the algorithm terminates after $n + 1$ operations

Average Case :

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for $i = 1$ **to** n $\text{———— } n \text{ steps}$
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- if $x = a_n$, then the algorithm terminates after $n + 1$ operations
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Average Case :

- let p be the probability that $x \in L$, and $q = 1 - p$ be the probability that $x \notin L$

Algorithm Efficiency

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Average Case :

- let p be the probability that $x \in L$, and $q = 1 - p$ be the probability that $x \notin L$
- for each element a_i , the probability that $x = a_i$ is p/n

Algorithm Efficiency

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$$E(X) = \sum p(s).X(s)$$

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- the expected value for the number of operations

$$E(X) = \sum p(s).X(s)$$

$$= 2 \cdot \frac{p}{n} + 3 \cdot \frac{p}{n} + \dots + (n+1) \cdot \frac{p}{n} + (n+1) \cdot q = p \frac{(n+3)}{2} + q \cdot (n+1)$$

Algorithm Efficiency

Worst-Case, Best-Case, Average-Case Analysis

Linear-Search(list, x)

input : $\{a_1, a_2, \dots, a_n; x\}$

output: location

for $i = 1$ to n ——— n steps

 if $x = a_i$ ——— 1 op

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- for $p = 1$ and $q = 0$
 $E(X) = (n+3)/2$

- for $p = 0$ and $q = 1$
 $E(X) = n+1$

- for $p = q = 1/2$
 $E(X) = (3n+5)/4$

Algorithm Efficiency

Worst-Case, Best-Case, Average-Case Analysis

Linear-Search(list, x)

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return loc

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⋮

if $x = a_i$, then the algorithm terminates after $i + 1$ operation

- average-case analysis is more difficult than worst-case and best-case analysis

- applying the corresponding values to formula is easy, but probabilistic assumption for each particular case is hard to verify

- mostly deal with worst-case analysis

terminates

terminates

Average Case :

- let p be the prob
- $q = 1 - p$ be the
- for each element a_i , the probability that $x = a_i$ is p/n
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$$= 1 \text{ and } q = 0$$

$$= (n+3)/2$$

- for $p = 0$ and $q = 1$
 $E(X) = n + 1$

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 $E(X) = (3n + 5)/4$

Asymptotic Notations

- for the algorithm's efficiency, we focus on the order of growth of the function that counts the algorithm's basic operations

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- for the algorithm's efficiency, we focus on the order of growth of the function that counts the algorithm's basic operations
- to compare and rank such orders of growth, three common tools will be employed:
 - O (big-oh), asymptotic upper bound
 - Ω (big-omega), asymptotic lower bound
 - Θ (big-theta), asymptotic tight bound

Asymptotic Notations

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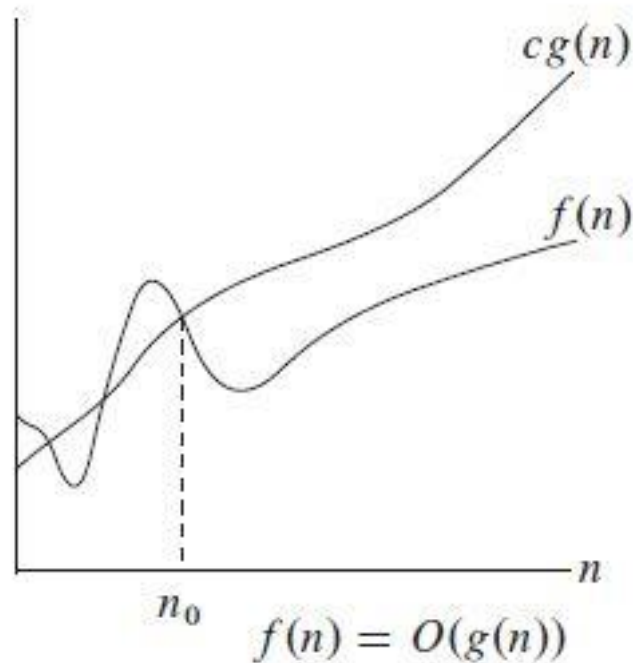
- $\Theta(g(n))$ is the class of all functions with the same order of growth as $g(n)$

$$24n^2 + 17n \in \Theta(n^2), n^2 + 17 \log n \in \Theta(n^2), 27n + 100 \notin \Theta(n^2), n^3 \notin \Theta(n^2)$$

Big-Oh Notation

Definition : Let $f, g : \mathbb{Z}^+ \rightarrow \mathbb{R}$ be two functions. If there are constants C and n_0 such that $|f(n)| \leq C \cdot |g(n)|$ for all $n \in \mathbb{Z}$ where $n \geq n_0$, we say that f is big-oh of g ,

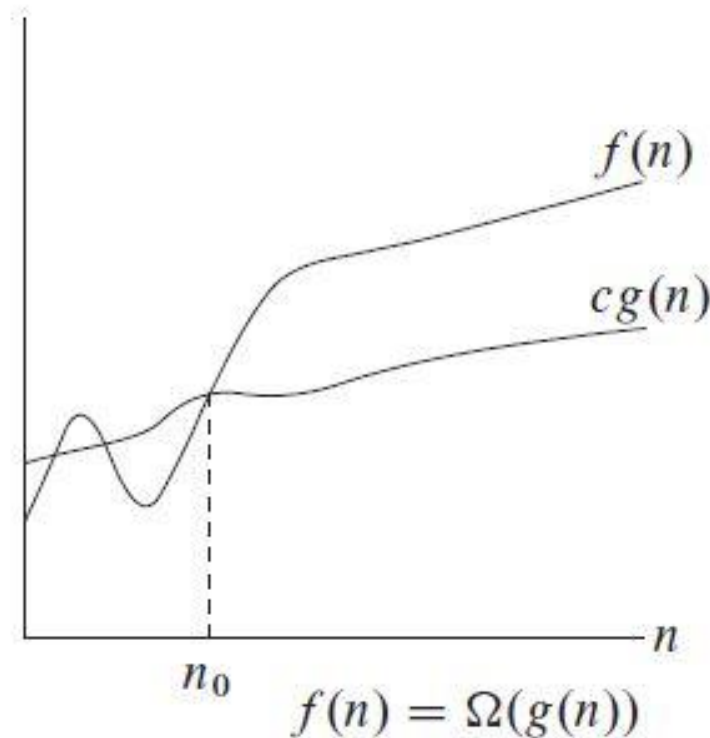
$$f(n) = O(g(n))$$



Big-Omega Notation

Definition : Let $f, g : \mathbb{Z}^+ \rightarrow \mathbb{R}$ be two functions. If there are constants C and n_0 such that $|f(n)| \geq C \cdot |g(n)|$ for all $n \in \mathbb{Z}$ where $n \geq n_0$, we say that f is big-omega of g ,

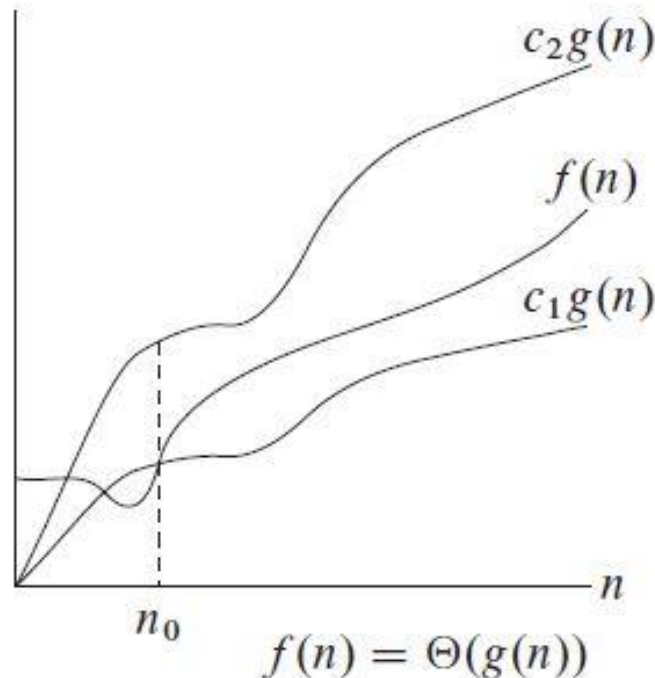
$$f(n) = \Omega(g(n))$$



Big-Theta Notation

Definition : Let $f, g : \mathbb{Z}^+ \rightarrow \mathbb{R}$ be two functions. If there are constants C_1, C_2 , and n_0 such that $C_1 \cdot |g(n)| \leq |f(n)| \leq C_2 \cdot |g(n)|$ for all $n \in \mathbb{Z}$ where $n \geq n_0$, we say that f is big-theta of g ,

$$f(n) = \Theta(g(n))$$



Big-Oh Notation

- $f, g : \mathbb{Z}^+ \rightarrow \mathbb{R}, f(n) = 5n$ and $g(n) = n^2$.

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 - $f(1) = 5, f(2) = 10, f(3) = 15, f(4) = 20, f(5) = 25, \dots$
 $g(1) = 1, g(2) = 4, g(3) = 9, g(4) = 16, g(5) = 25, \dots$

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 - for $n \geq 5, n^2 \geq 5n \rightarrow |f(n)| \leq |g(n)|$

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 - for $n \geq 5, n^2 \geq 5n \rightarrow |f(n)| \leq |g(n)|$
 - for $C = 1$ and $n_0 = 5,$
 $|f(n)| \leq C \cdot |g(n)|$ for all $n \geq n_0$. Thus, $f(n) = O(g(n))$.

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 - for $n \geq 5, n^2 \geq 5n \rightarrow |f(n)| \leq |g(n)|$
 - for $C = 1$ and $n_0 = 5,$
 $|f(n)| \leq C \cdot |g(n)|$ for all $n \geq n_0$. Thus, $f(n) = O(g(n))$.
 - C and n_0 don't have to be unique

Big-Oh Notation

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$$\begin{aligned} |f(n)| &= |5n^2 + 3n + 1| = 5n^2 + 3n + 1 \\ &\leq 5n^2 + 3n^2 + n^2 \end{aligned}$$

Big-Oh Notation

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$$|f(n)| = |5n^2 + 3n + 1| = 5n^2 + 3n + 1$$

$$\leq 5n^2 + 3n^2 + n^2 = 9n^2$$

Big-Oh Notation

- $f, g : \mathbb{Z}^+ \rightarrow \mathbb{R}, f(n) = 5n^2 + 3n + 1$ and $g(n) = n^2$.

$$\begin{aligned} |f(n)| &= |5n^2 + 3n + 1| = 5n^2 + 3n + 1 \\ &\leq 5n^2 + 3n^2 + n^2 = 9n^2 = 9|g(n)| \end{aligned}$$

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for $C = 9$ and $n_0 = 1$,

$|f(n)| \leq C \cdot |g(n)|$ for all $n \geq n_0$. Thus, $f(n) = O(g(n))$.

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for $C = 1$ and $n_0 = 1$,

$|g(n)| \leq C \cdot |f(n)|$ for all $n \geq n_0$. Thus, $g(n) = O(f(n))$.

Big-Oh Notation

- $f, g : \mathbb{Z}^+ \rightarrow \mathbb{R}, f(n) = 7n^2$ and $g(n) = n^3$.

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Big-Oh Notation

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$$|f(n)| = |7n^2| = 7n^2 \leq 7n^3 = 7|g(n)|$$

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$$|g(n)| = |n^3| = n^3 \leq C \cdot 7 \cdot n^2 = C \cdot |f(n)| \rightarrow n \leq C \cdot 7 \text{ for all } n \geq n_0$$

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$$|g(n)| = |n^3| = n^3 \leq C \cdot 7 \cdot n^2 = C \cdot |f(n)| \rightarrow n \leq C \cdot 7 \text{ for all } n \geq n_0$$

there cannot be any C and n_0 that satisfy this inequality.

Big-Oh Notation

- $f : \mathbb{Z}^+ \rightarrow \mathbb{R}, f(n) = a_t n^t + a_{t-1} n^{t-1} + \dots + a_1 n + a_0$

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for $C = |a_t| + \dots + |a_1| + |a_0|$ and $n_0 = 1$,

$|f(n)| \leq C.n^t$ for all $n \geq n_0$. Thus, $f(n) = O(n^t)$

Big-Oh Notation

- $f : \mathbb{Z}^+ \rightarrow \mathbb{R}, f(n) = 1 + 2 + \dots + n$

$$|f(n)| = |1 + 2 + \dots + n| = 1 + 2 + \dots + n \leq n + n + \dots + n = |n^2|$$

for $C = 1$ and $n_0 = 1$,

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- $f : \mathbb{Z}^+ \rightarrow \mathbb{R}, f(n) = 1^2 + 2^2 + \dots + n^2$

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$$|f(n)| \leq C \cdot |n^{t+1}| \text{ for all } n \geq n_0. \text{ Thus, } f(n) = O(n^{t+1})$$

Big-Oh Notation

- Basic Efficiency Classes

1 : constant

$\log n$: logarithmic

n : linear

$n \log n$: linearithmic (loglinear)

n^2 : quadratic

n^t : polynomial

2^n : exponential

$n!$: factorial

Big-Oh Notation

- $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$

$$|f_1(n) + f_2(n)| \leq |f_1(n)| + |f_2(n)|$$

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$$\begin{aligned} |f_1(n) + f_2(n)| &\leq |f_1(n)| + |f_2(n)| \\ &\leq C_1|g_1(n)| + C_2|g_2(n)| \end{aligned}$$

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$$f_1(n) + f_2(n) = O(\max \{g_1(n), g_2(n)\})$$

$$f_1(n) \cdot f_2(n) = O(g_1(n) \cdot g_2(n))$$

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$$\begin{array}{ccc} \swarrow & & \swarrow \\ O(n) & & O(n^2) \end{array}$$

Big-Oh Notation

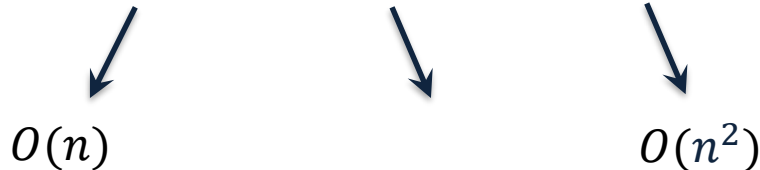
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$$\log(n^2 + 1) \leq \log(2n^2)$$

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$O(n)$ $O(1)$ $O(n^2)$

$$\begin{aligned} \log(n^2 + 1) &\leq \log(2n^2) \\ &= \log 2 + \log n^2 \\ &= \log 2 + 2 \log n \end{aligned}$$

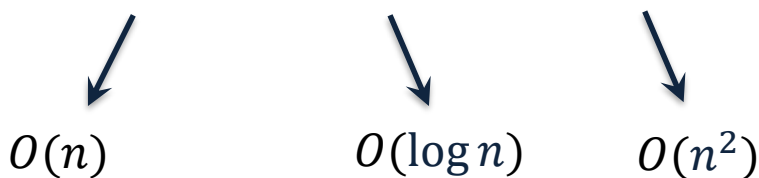
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$O(n)$

$O(\log n)$

$O(n^2)$

$$\begin{aligned} \log(n^2 + 1) &\leq \log(2n^2) \\ &= \log 2 + \log n^2 \\ &= \log 2 + 2 \log n \\ &\leq 3 \log n \end{aligned}$$

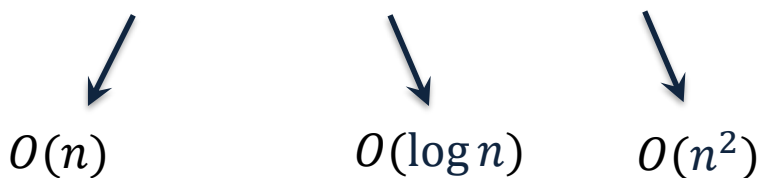
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$$\begin{aligned} \log(n^2 + 1) &\leq \log(2n^2) \\ &= \log 2 + \log n^2 \\ &= \log 2 + 2 \log n \\ &\leq 3 \log n \end{aligned}$$

$$f(n) = O(n^2)$$

Worst-Case Analysis

Max-Integer(list)

input : $\{a_1, a_2, \dots, a_n\}$

output: max of $\{a_1, a_2, \dots, a_n\}$

$\text{max} \leftarrow a_1$

for $i = 2$ to n

if $\text{max} < a_i$

$\text{max} \leftarrow a_i$

return max

Worst-Case Analysis

Max-Integer(list)

input : $\{a_1, a_2, \dots, a_n\}$

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- $T(n)$: the number of operations the algorithm performs

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 2 op

- $T(n)$: the number of operations the algorithm performs
- the algorithm performs 2 operations on each execution of the loop

Worst-Case Analysis

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 2 op

- $T(n)$: the number of operations the algorithm performs
- the algorithm performs 2 operations on each execution of the loop
- loop's variable increases from 2 to n (use sum formula)

Worst-Case Analysis

Max-Integer(list)

input : $\{a_1, a_2, \dots, a_n\}$

output: max of $\{a_1, a_2, \dots, a_n\}$

max $\leftarrow a_1$ _____ 1 op

for $i = 2$ to n

if $\text{max} < a_i$

max $\leftarrow a_i$

return max

 _____ 2 op

- $T(n)$: the number of operations the algorithm performs
- the algorithm performs 2 operations on each execution of the loop
- loop's variable increases from 2 to n (use sum formula)

$$C(n) = \sum_{i=2}^n 2 + 1$$

Worst-Case Analysis

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output: max of $\{a_1, a_2, \dots, a_n\}$

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 _____ 2 op

- $T(n)$: the number of operations the algorithm performs
- the algorithm performs 2 operations on each execution of the loop
- loop's variable increases from 2 to n (use sum formula)

$$C(n) = \sum_{i=2}^n 2 + 1 = \sum_{i=1}^{n-1} 2 + 1$$

Worst-Case Analysis

Max-Integer(list)

input : $\{a_1, a_2, \dots, a_n\}$

output: max of $\{a_1, a_2, \dots, a_n\}$

max $\leftarrow a_1$ _____ 1 op

for $i = 2$ to n

if $\text{max} < a_i$

max $\leftarrow a_i$

return max

 _____ 2 op

- $T(n)$: the number of operations the algorithm performs
- the algorithm performs 2 operations on each execution of the loop
- loop's variable increases from 2 to n (use sum formula)

$$C(n) = \sum_{i=2}^n 2 + 1 = \sum_{i=1}^{n-1} 2 + 1 = 2 \cdot (n - 1) + 1 \in O(n)$$

Worst-Case Analysis

UniqueElements(list)

input : $\{a_1, a_2, \dots, a_n\}$

output: return 'true' if all the elements are distinct; 'false' otherwise

for $i = 1$ to $n - 1$

for $j = i+1$ to n

if $a_i = a_j$

return false

return true

Worst-Case Analysis

UniqueElements(list)

input : $\{a_1, a_2, \dots, a_n\}$

output: return 'true' if all the elements are distinct; 'false' otherwise

for $i = 1$ to $n - 1$

for $j = i+1$ to n

if $a_i = a_j$

return false 1 op

return true

- the algorithm performs 1 operation on each execution of the innermost loop

Worst-Case Analysis

UniqueElements(list)

input : $\{a_1, a_2, \dots, a_n\}$

output: return 'true' if all the elements are distinct; 'false' otherwise

```
for i = 1 to n - 1
    for j = i+1 to n
        if  $a_i = a_j$ 
            return false
return true
```

1 op

- the algorithm performs 1 operation on each execution of the innermost loop
- loop's variable increases from 1 to $n - 1$ for the outer loop, and from $i + 1$ to n for the innermost loop

Worst-Case Analysis

UniqueElements(list)

input : $\{a_1, a_2, \dots, a_n\}$

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for i = 1 to n - 1
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1 op

- the algorithm performs 1 operation on each execution of the innermost loop
- loop's variable increases from 1 to $n - 1$ for the outer loop, and from $i + 1$ to n for the innermost loop

$$T(n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^n 1$$

Worst-Case Analysis

UniqueElements(list)

input : $\{a_1, a_2, \dots, a_n\}$

output: return 'true' if all the elements are distinct; 'false' otherwise

```
for i = 1 to n - 1
  for j = i+1 to n
    if  $a_i = a_j$ 
      return false
return true
```

1 op

- the algorithm performs 1 operation on each execution of the innermost loop
- loop's variable increases from 1 to $n - 1$ for the outer loop, and from $i + 1$ to n for the innermost loop

$$T(n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^n 1 = \sum_{i=1}^{n-1} [n - (i + 1) + 1]$$

Worst-Case Analysis

UniqueElements(list)

input : $\{a_1, a_2, \dots, a_n\}$

output: return 'true' if all the elements are distinct; 'false' otherwise

```
for i = 1 to n - 1
    for j = i+1 to n
        if  $a_i = a_j$ 
            return false
return true
```

1 op

- the algorithm performs 1 operation on each execution of the innermost loop
- loop's variable increases from 1 to $n - 1$ for the outer loop, and from $i + 1$ to n for the innermost loop

$$T(n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^n 1 = \sum_{i=1}^{n-1} [n - (i + 1) + 1] = \sum_{i=1}^{n-1} (n - i)$$

Worst-Case Analysis

UniqueElements(list)

input : $\{a_1, a_2, \dots, a_n\}$

output: return 'true' if all the elements are distinct; 'false' otherwise

```
for i = 1 to n - 1
    for j = i+1 to n
        if  $a_i = a_j$ 
            return false
return true
```

1 op

- the algorithm performs 1 operation on each execution of the innermost loop
- loop's variable increases from 1 to $n - 1$ for the outer loop, and from $i + 1$ to n for the innermost loop

$$\begin{aligned} T(n) &= \sum_{i=1}^{n-1} \sum_{j=i+1}^n 1 = \sum_{i=1}^{n-1} [n - (i + 1) + 1] = \sum_{i=1}^{n-1} (n - i) \\ &= \sum_{i=1}^{n-1} n - \sum_{i=1}^{n-1} i \end{aligned}$$

Worst-Case Analysis

UniqueElements(list)

input : $\{a_1, a_2, \dots, a_n\}$

output: return 'true' if all the elements are distinct; 'false' otherwise

```
for i = 1 to n - 1
    for j = i+1 to n
        if  $a_i = a_j$ 
            return false
return true
```

1 op

- the algorithm performs 1 operation on each execution of the innermost loop
- loop's variable increases from 1 to $n - 1$ for the outer loop, and from $i + 1$ to n for the innermost loop

$$\begin{aligned} T(n) &= \sum_{i=1}^{n-1} \sum_{j=i+1}^n 1 = \sum_{i=1}^{n-1} [n - (i + 1) + 1] = \sum_{i=1}^{n-1} (n - i) \\ &= \sum_{i=1}^{n-1} n - \sum_{i=1}^{n-1} i = n(n - 1) - \frac{n(n - 1)}{2} \end{aligned}$$

Worst-Case Analysis

UniqueElements(list)

input : $\{a_1, a_2, \dots, a_n\}$

output: return 'true' if all the elements are distinct; 'false' otherwise

```
for i = 1 to n - 1
    for j = i+1 to n
        if  $a_i = a_j$ 
            return false
return true
```

1 op

- the algorithm performs 1 operation on each execution of the innermost loop
- loop's variable increases from 1 to $n - 1$ for the outer loop, and from $i + 1$ to n for the innermost loop

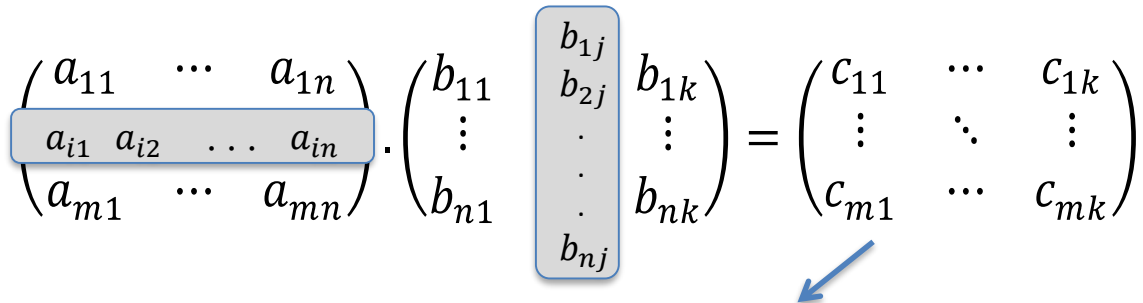
$$\begin{aligned} T(n) &= \sum_{i=1}^{n-1} \sum_{j=i+1}^n 1 = \sum_{i=1}^{n-1} [n - (i + 1) + 1] = \sum_{i=1}^{n-1} (n - i) \\ &= \sum_{i=1}^{n-1} n - \sum_{i=1}^{n-1} i = n(n - 1) - \frac{n(n - 1)}{2} = \frac{1}{2}n(n - 1) \in O(n^2) \end{aligned}$$

Worst-Case Analysis

m×n matrix A

n×k matrix B

m×k matrix C

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{1j} & b_{1k} \\ \vdots & b_{2j} & \vdots \\ b_{n1} & b_{nj} & b_{nk} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1k} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mk} \end{pmatrix}$$


$$c_{ij} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \cdots + a_{in} \cdot b_{nj}$$

MatrixMultiplication(A, B)

input : two n×n matrices A, B

output: C = A.B

for i = 1 to n

for j = 1 to n

 C[i, j] ← 0

for k = 1 to n

 C[i, j] ← C[i, j] + A[i, k] * B[k, j]

return C

Worst-Case Analysis

MatrixMultiplication(A,B)

input : two nxn matrices A,B

output: A.B

for i = 1 to n

 for j = 1 to n

$C[i,j] \leftarrow 0$

 for k = 1 to n

$C[i,j] \leftarrow C[i,j] + A[i,k] * B[k,j]$

return C

Worst-Case Analysis

MatrixMultiplication(A,B)

input : two nxn matrices A,B

output: A.B

for i = 1 to n

 for j = 1 to n

$C[i,j] \leftarrow 0$ ————— 1 op

 for k = 1 to n

$C[i,j] \leftarrow C[i,j] + A[i,k] * B[k,j]$

return C

————— 2 op

Worst-Case Analysis

MatrixMultiplication(A,B)

input : two nxn matrices A,B

output: A.B

for i = 1 to n

 for j = 1 to n

$C[i,j] \leftarrow 0$ _____ 1 op

 for k = 1 to n

$C[i,j] \leftarrow C[i,j] + A[i,k] * B[k,j]$

return C

_____ 2 op

$$T(n) = \sum_{i=1}^n \sum_{j=1}^n (1 + \dots)$$

Worst-Case Analysis

MatrixMultiplication(A,B)

input : two nxn matrices A,B

output: A.B

for i = 1 to n

 for j = 1 to n

$C[i,j] \leftarrow 0$ _____ 1 op

 for k = 1 to n

$C[i,j] \leftarrow C[i,j] + A[i,k] * B[k,j]$

return C

_____ 2 op

$$T(n) = \sum_{i=1}^n \sum_{j=1}^n \left(1 + \sum_{k=1}^n 2 \right)$$

Worst-Case Analysis

MatrixMultiplication(A,B)

input : two nxn matrices A,B

output: A.B

for i = 1 to n

 for j = 1 to n

$C[i,j] \leftarrow 0$ _____ 1 op

 for k = 1 to n

$C[i,j] \leftarrow C[i,j] + A[i,k] * B[k,j]$

return C

_____ 2 op

$$T(n) = \sum_{i=1}^n \sum_{j=1}^n \left(1 + \sum_{k=1}^n 2 \right) = \sum_{i=1}^n \sum_{j=1}^n (2n + 1)$$

Worst-Case Analysis

MatrixMultiplication(A,B)

input : two nxn matrices A,B

output: A.B

for i = 1 to n

 for j = 1 to n

$C[i,j] \leftarrow 0$ ————— 1 op

 for k = 1 to n

$C[i,j] \leftarrow C[i,j] + A[i,k] * B[k,j]$

return C

2 op

$$T(n) = \sum_{i=1}^n \sum_{j=1}^n \left(1 + \sum_{k=1}^n 2 \right) = \sum_{i=1}^n \sum_{j=1}^n (2n + 1) = \sum_{i=1}^n n(2n + 1)$$

Worst-Case Analysis

MatrixMultiplication(A,B)

input : two nxn matrices A,B

output: A.B

for i = 1 to n

 for j = 1 to n

$C[i,j] \leftarrow 0$ ————— 1 op

 for k = 1 to n

$C[i,j] \leftarrow C[i,j] + A[i,k] * B[k,j]$

return C

2 op

$$\begin{aligned} T(n) &= \sum_{i=1}^n \sum_{j=1}^n \left(1 + \sum_{k=1}^n 2 \right) = \sum_{i=1}^n \sum_{j=1}^n (2n + 1) = \sum_{i=1}^n n(2n + 1) \\ &= n.n.(2n + 1) \end{aligned}$$

Worst-Case Analysis

MatrixMultiplication(A,B)

input : two nxn matrices A,B

output: A.B

for i = 1 to n

 for j = 1 to n

$C[i,j] \leftarrow 0$ ————— 1 op

 for k = 1 to n

$C[i,j] \leftarrow C[i,j] + A[i,k] * B[k,j]$

return C

2 op

$$\begin{aligned} T(n) &= \sum_{i=1}^n \sum_{j=1}^n \left(1 + \sum_{k=1}^n 2 \right) = \sum_{i=1}^n \sum_{j=1}^n (2n + 1) = \sum_{i=1}^n n(2n + 1) \\ &= n.n.(2n + 1) = 2n^3 + n^2 \in O(n^3) \end{aligned}$$

Worst-Case Analysis

MatrixMultiplication(A,B)

input : two nxn matrices A,B

output: A.B

for i = 1 to n

for j = 1 to n

$C[i,j] \leftarrow 0$ 1 op

for k = 1 to n

$C[i,j] \leftarrow C[i,j] + A[i,k] * B[k,j]$

can be ignored

since constant number of operations can be counted as 1 operation, it can be ignored

return C

2 op

$$\begin{aligned} T(n) &= \sum_{i=1}^n \sum_{j=1}^n \left(1 + \sum_{k=1}^n 2 \right) = \sum_{i=1}^n \sum_{j=1}^n (2n + 1) = \sum_{i=1}^n n(2n + 1) \\ &= n.n.(2n + 1) = 2n^3 + n^2 \in O(n^3) \end{aligned}$$

$$T(n) = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n 1 = n^3 \in O(n^3)$$