Relations

Murat Osmanoglu

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  (Ahmet, Physics) $\in R$, (Efe, Discrete) $\notin R$

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$R = \{(a, 1), (b, 2), (c, 2)\}$

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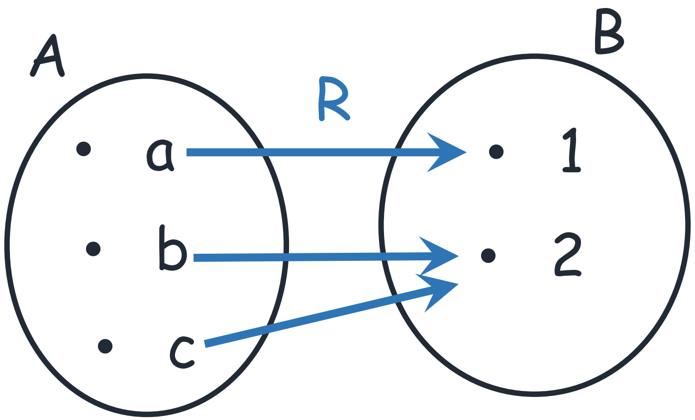

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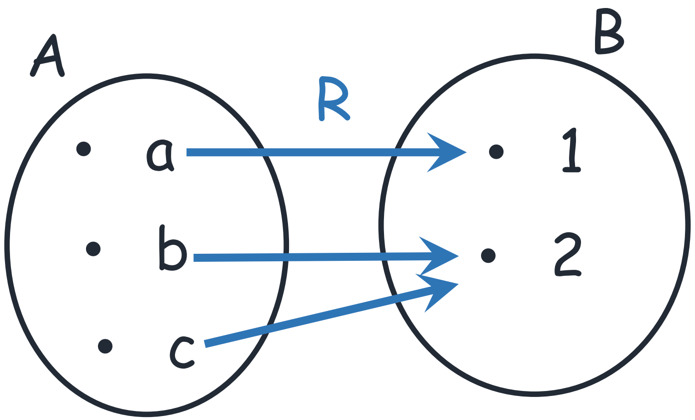

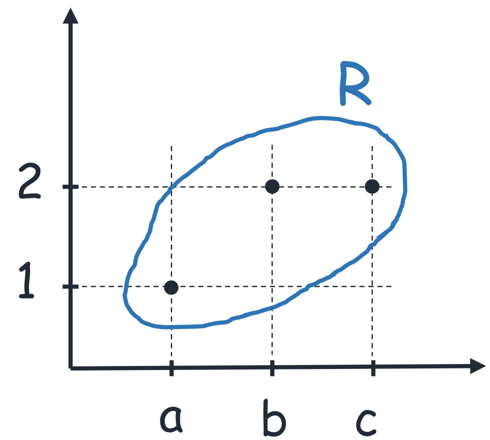

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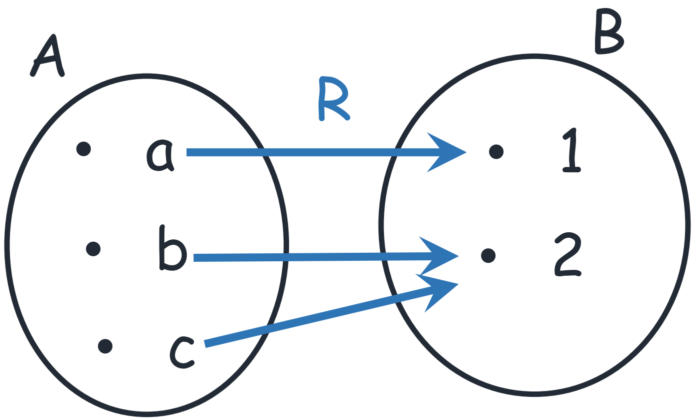

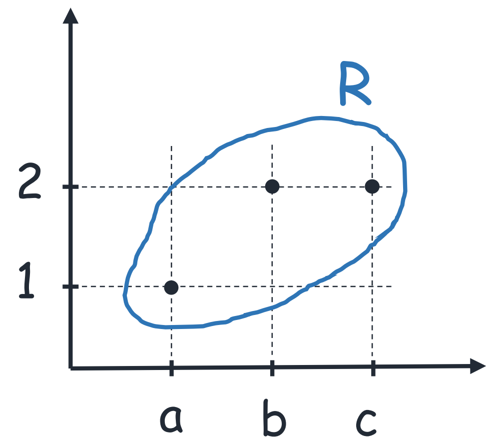

R	1	2
a	1	0
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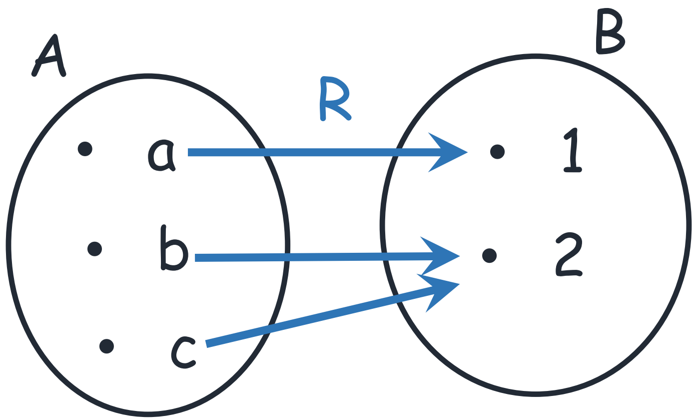

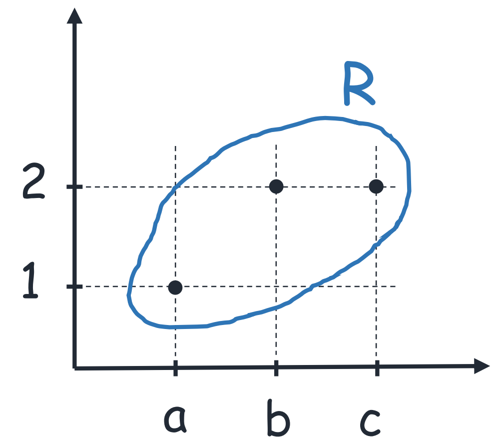

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- the number of relations that can be defined from A to B :

$$2^{|A||B|}$$

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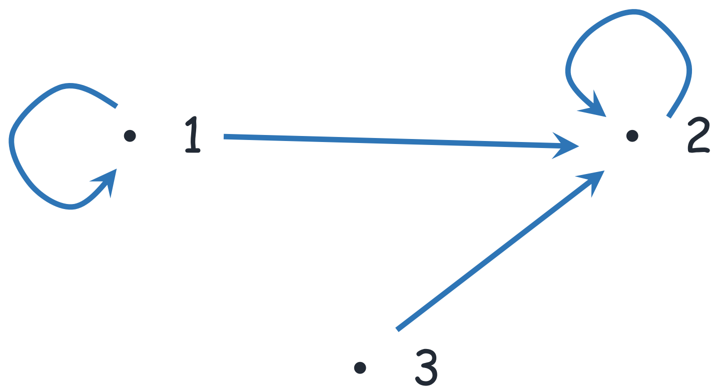

Functions as Relations

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domain

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Functions as Relations

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- for every element x of the domain, there is only one element y of the codomain such that (x,y) is an element of the relation

Functions as Relations

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  Let $R \subseteq A \times B$ be the relation, $\forall x[((x, y_1) \in R \wedge (x, y_2) \in R) \rightarrow (y_1 = y_2)]$

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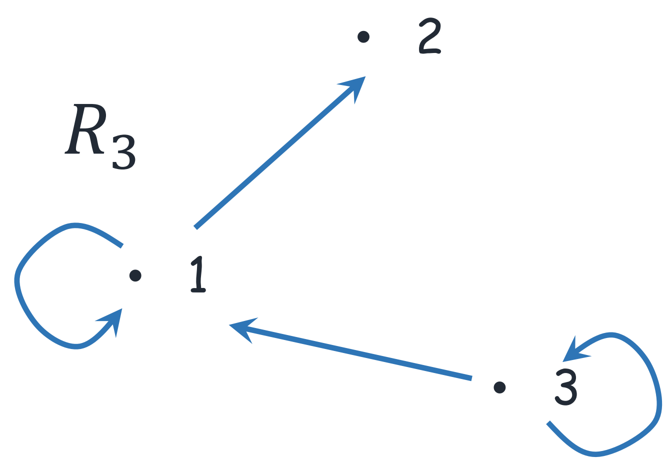
R_2	1	2	3
1	1	0	0
2	0	1	0
3	0	1	1

R_3



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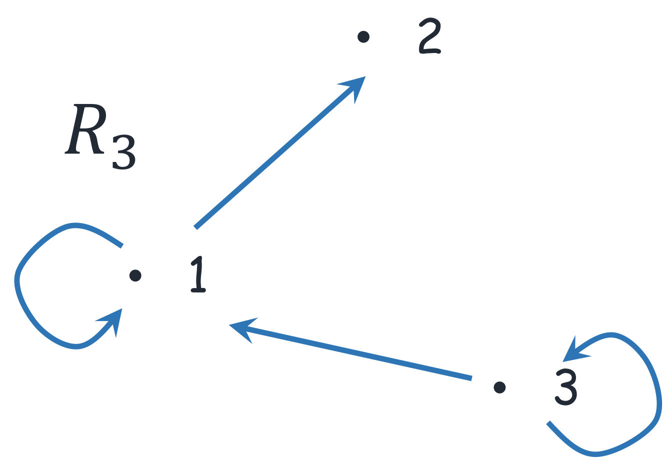

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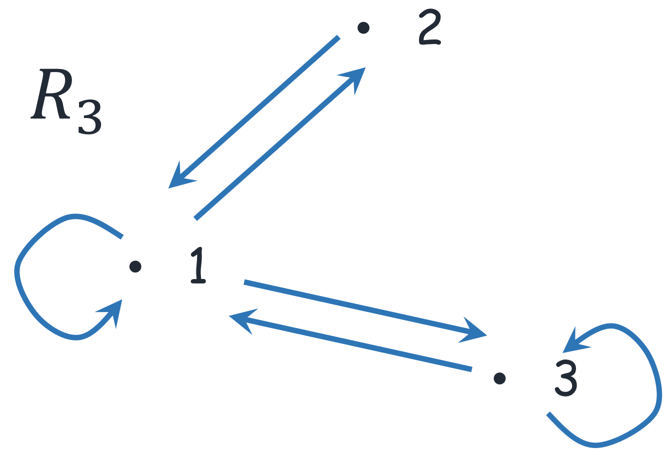

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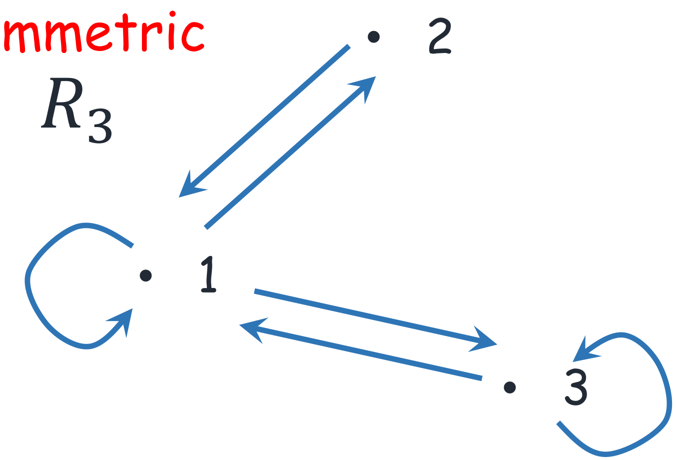

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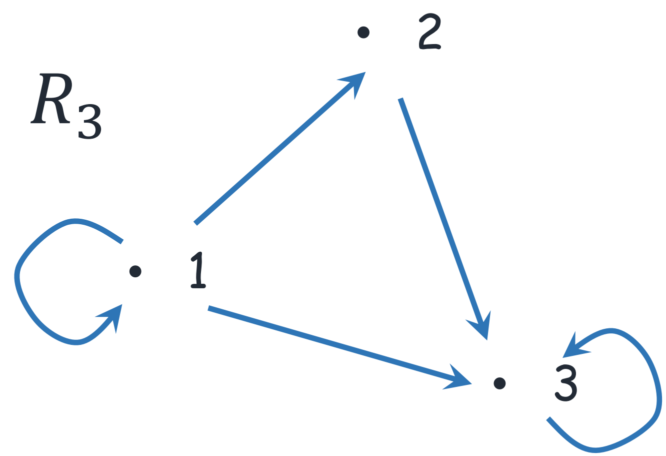

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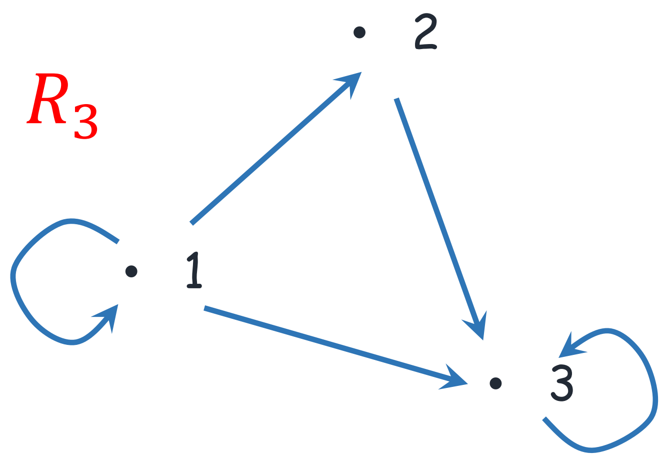

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Properties

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- add each of them the pairs (1, 1), . . ., (n, n) to make them reflexive

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  {a, . , .} $\binom{4}{2} = 6$

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$\{\ldots, \ldots\}$

$$\left(2^n . 2^{\frac{n^2-n}{2}}\right) = 2^{(n^2+n)/2}$$

Operations

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Union : Given $R, S \subseteq A \times B$,

$$\mathrm{T} = R \cup S = \{(x,y) | (x,y) \in R \ \vee \ (x,y) \in S\}$$

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$$T = \bar{R} = \{(x, y) | (x, y) \notin R \ \}$$

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Inverse : Given $R \subseteq A \times B$,

$$\mathrm{T} = R^{-1} = \{(y,x) \in B \times A | (x,y) \in R \ \}$$

Composition : Given $R \subseteq A \times B$ and $S \subseteq B \times C$

$$\mathrm{T} = S \circ R = \{(x,z) | (x,y) \in R \ \wedge \ (y,z) \in S\}$$

Operations

R	1	2
a	1	0
b	0	1
c	1	0

S	u	v
1	0	0
2	1	1

Operations

R	1	2
a	1	0
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c	1	0

$S \circ R$	u	v
a		
b		
c		

S	u	v
1	0	0
2	1	1

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R	1	2
a	1	0
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$S \circ R$	u	v
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c		

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c		

S	u	v
1	0	0
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Operations

R	1	2
a	1	0
b	0	1
c	1	0

S∘R	u	v
a	0	0
b	1	1
c		

S	u	v
1	0	0
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Operations

R	1	2
a	1	0
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R	1	2
a	1	0
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c	1	0

$S \circ R$	u	v
a	0	0
b	1	1
c	0	0

S	u	v
1	0	0
2	1	1

S^{-1}	1	2
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v	0	1

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- $A = \{ 1, 2, 3 \}$, $R = \{ (1, 1), (2, 1), (3, 2) \}$

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1	1	0	0
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1	1	0	0
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- $R^2 = R \circ R = \{(1,1),(2,1),(3,2)\}$
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- The relation R on a set A is transitive if and only if $R^n \subseteq R$ for some $n \in Z^+$

Equivalence Relations

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Definition : A relation R on a set A is called an equivalence relation if it's reflexive, symmetric, and transitive. If $(a, b) \in R$, then a and b are called equivalent, i.e. a~b.

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Equivalence Relations

- A given set S can be decomposed into disjoint subsets A_i. For a family of sets $A = \{A_i | i \in I\}$ such that $A_i \cap A_j \neq \emptyset$, a given set S can be written as

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- the poset $(\mathbb{Z}^+, \leq)$ is a totally ordered set.

> - For every $a, b \in \mathbb{Z}^+$, either $a \leq b$ or $b \leq a$. Thus, either $(a, b) \in (\leq)$ or $(b, a) \in (\leq)$

Partial Order

Definition : Consider a poset (S, R). An element a is called maximal if there is no $b \in S$ such that aRb. An element a is called minimal if there is no $b \in S$ such that bRa.

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Definition : An element a is called the greatest element if bRa for all $b \in S$. An element a is called the least element if aRb for all $b \in S$.

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- Consider the power set of a given set S.

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- Consider the power set of a given set S.
  - $\emptyset$ is the least element of $(P(S), \subseteq)$ since $\emptyset \subseteq T$ for any $T \in P(S)$
  - S is the greatest element of $(P(S), \subseteq)$ since $T \subseteq S$ for any $T \in P(S)$

Partial Order

- Let $A = \{0, 1, 2\}$, $B = A \times A$, R be a relation defined on B such that

Partial Order

- Let A = {0, 1, 2}, B = AxA, R be a relation defined on B such that $\big((a,b),(c,d)\big) \in R$

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$$((a,b),(c,d)) \in R \quad \text{if} \quad \begin{array}{l} a < c \ \text{ or} \\ a = c \ \text{ and } \ b \leq d \end{array}$$

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  $\big((a, b), (c, d)\big) \in R$ if $a < c$ or
  $a = c$ and $b \leq d$

  - $\big((0,1), (1,0)\big) \in R$ since $a < c$

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  - for all $(a,b) \in B$,

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- R is partial order relation ?
  - for all $(a, b) \in B$, since $a = a$ and $b \le b$, $((a, b), (a, b)) \in R$,

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- R is partial order relation ?
  - for all $(a, b) \in B$, since $a = a$ and $b \leq b$, $\big((a, b), (a, b)\big) \in R$, reflexive

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    either $a < c$, then $\left((c,d),(a,b)\right) \notin R$
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  - for all $(a,b) \in B$, since $a = a$ and $b \leq b$, $\big((a,b),(a,b)\big) \in R$, reflexive
  - for all $\big((a,b),(c,d)\big) \in R$ such that $(a,b) \neq (c,d)$, symmetric
    either $a < c$, then $\big((c,d),(a,b)\big) \notin R$
    or $a = c$ and $b < d$, then $\big((c,d),(a,b)\big) \notin R$

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    either $a < c$, then $\big((c,d),(a,b)\big) \notin R$
    or $a = c$ and $b < d$, then $\big((c,d),(a,b)\big) \notin R$
  - for all $\big((a,b),(c,d)\big) \in R$ and $\big((c,d),(e,f)\big) \in R$,

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    either $a < c$, then $\big((c,d),(a,b)\big) \notin R$
    or $a = c$ and $b < d$, then $\big((c,d),(a,b)\big) \notin R$
  - for all $\big((a,b),(c,d)\big) \in R$ and $\big((c,d),(e,f)\big) \in R$,
    either $a < c$ and $\mathrm{c} < e$,
    or $a < c$, and $c = e$ and $d \leq f$,
    or $a = c$ and $d \leq f$, and $c < e$,
    or $a = c$ and $b \leq d$, and $c = e$ and $d \leq f$,

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  - for all $\big((a,b),(c,d)\big) \in R$ such that $(a,b) \ne (c,d)$, symmetric
    either $a < c$, then $\big((c,d),(a,b)\big) \notin R$
    or $a = c$ and $b < d$, then $\big((c,d),(a,b)\big) \notin R$
  - for all $\big((a,b),(c,d)\big) \in R$ and $\big((c,d),(e,f)\big) \in R$,
    either $a < c$ and $c < e$, then $a < e$, $\big((a,b),(e,f)\big) \in R$
    or $a < c$, and $c = e$ and $d \le f$,
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  - for all $\big((a,b),(c,d)\big) \in R$ and $\big((c,d),(e,f)\big) \in R$,
    either $a < c$ and $c < e$, then $a < e$, $\big((a,b),(e,f)\big) \in R$
    or $a < c$, and $c = e$ and $d \leq f$, then $a < e$, $\big((a,b),(e,f)\big) \in R$
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    either $a < c$, then $\big((c,d),(a,b)\big) \notin R$
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  - for all $\big((a,b),(c,d)\big) \in R$ and $\big((c,d),(e,f)\big) \in R$,
    either $a < c$ and $c < e$, then $a < e$, $\big((a,b),(e,f)\big) \in R$
    or $a < c$, and $c = e$ and $d \leq f$, then $a < e$, $\big((a,b),(e,f)\big) \in R$
    or $a = c$ and $d \leq f$, and $c < e$, then $a < e$, $\big((a,b),(e,f)\big) \in R$
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  - for all $\big((a,b),(c,d)\big) \in R$ such that $(a,b) \neq (c,d)$, symmetric
    either $a < c$, then $\big((c,d),(a,b)\big) \notin R$
    or $a = c$ and $b < d$, then $\big((c,d),(a,b)\big) \notin R$
  - for all $\big((a,b),(c,d)\big) \in R$ and $\big((c,d),(e,f)\big) \in R$,
    either $a < c$ and $c < e$, then $a < e$, $\big((a,b),(e,f)\big) \in R$
    or $a < c$, and $c = e$ and $d \leq f$, then $a < e$, $\big((a,b),(e,f)\big) \in R$
    or $a = c$ and $d \leq f$, and $c < e$, then $a < e$, $\big((a,b),(e,f)\big) \in R$
    or $a = c$ and $b \leq d$, and $c = e$ and $d \leq f$, then $a = e$ and $b \leq f$,
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  - for all $\big((a,b),(c,d)\big) \in R$ such that $(a,b) \neq (c,d)$, symmetric
    either $a < c$, then $\big((c,d),(a,b)\big) \notin R$
    or $a = c$ and $b < d$, then $\big((c,d),(a,b)\big) \notin R$
  - for all $\big((a,b),(c,d)\big) \in R$ and $\big((c,d),(e,f)\big) \in R$, transitive
    either $a < c$ and $c < e$, then $a < e$, $\big((a,b),(e,f)\big) \in R$
    or $a < c$, and $c = e$ and $d \leq f$, then $a < e$, $\big((a,b),(e,f)\big) \in R$
    or $a = c$ and $d \leq f$, and $c < e$, then $a < e$, $\big((a,b),(e,f)\big) \in R$
    or $a = c$ and $b \leq d$, and $c = e$ and $d \leq f$, then $a = e$ and $b \leq f$,
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  - Is there a least element ?

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    (0, 0)

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- R is partial order relation ?
  - Is there a least element ?

    (0, 0)
  - Is there a greatest element ?

Partial Order

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  - $\big((0,1), (0,2)\big) \in R$ since $a = c$ and $b \leq d$

- R is partial order relation ?
  - Is there a least element ?

    (0, 0)
  - Is there a greatest element ?

    (2, 2)

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    (0, 0)
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  - Is it total order ?

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  $((a, b), (c, d)) \in R$ if $a < c$ or
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- R is partial order relation ?
  - Is there a least element ?

    (0, 0)
  - Is there a greatest element ?

    (2, 2)
  - Is it total order ?

    for all $a, b \in B$, $(a, b) \in R$ or $(b, a) \in R$

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- R is partial order relation ?
  - Is there a least element ?

    (0, 0)
  - Is there a greatest element ?

    (2, 2)
  - Is it total order ?

    for all $a, b \in B$, $(a,b) \in R$ or $(b,a) \in R$
  - How many elements are in R ?

Partial Order

- Let A = {0, 1, 2}, B = AxA, R be a relation defined on B such that

  $((a,b),(c,d)) \in R$ if $a < c$ or
  $a = c$ and $b \le d$

  - $((0,1),(1,0)) \in R$ since $a < c$
  - $((0,1),(0,2)) \in R$ since $a = c$ and $b \le d$

- R is partial order relation ?
  - Is there a least element ?

    (0, 0)
  - Is there a greatest element ?

    (2, 2)
  - Is it total order ?

    for all $a, b \in B$, $(a,b) \in R$ or $(b,a) \in R$
  - How many elements are in R ?

    $(0,0)R(0,1)R(0,2)R(1,0)R(1,1)R(1,2)R(2,0)R(2,1)R(2,2)$

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- a type of directed graph used to represent finite posets.

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- if $(x,y)\in(\leq)$, draw a line from x to y

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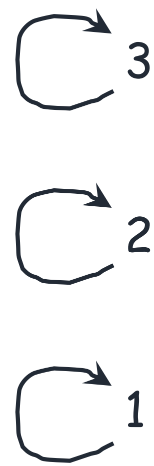

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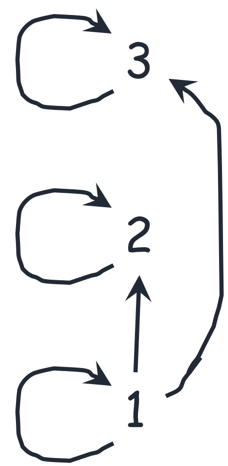

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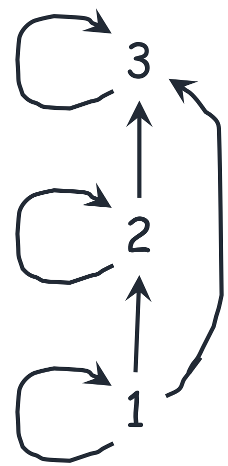

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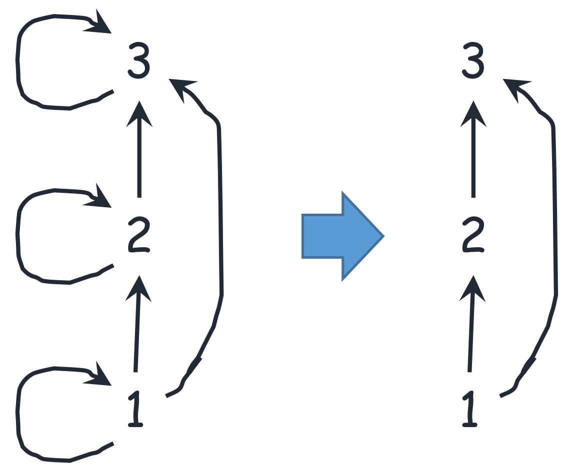

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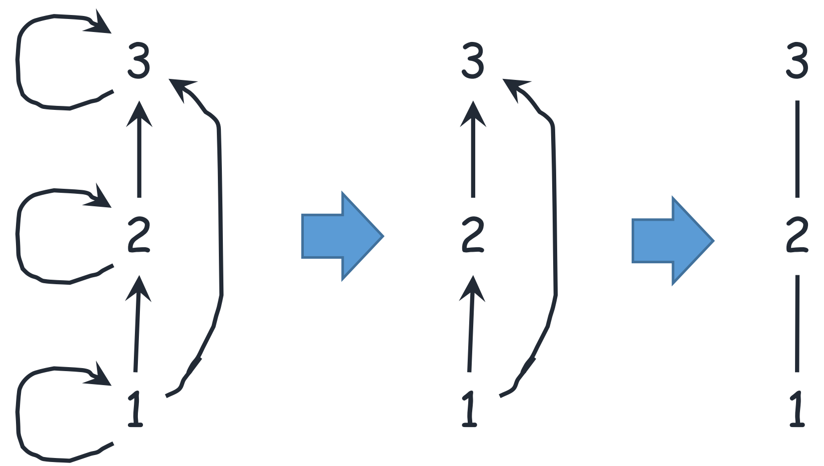

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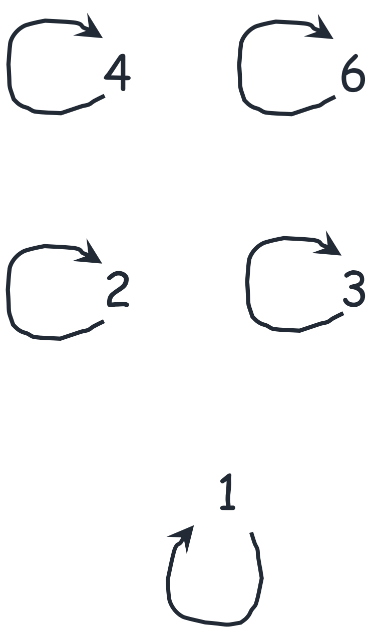

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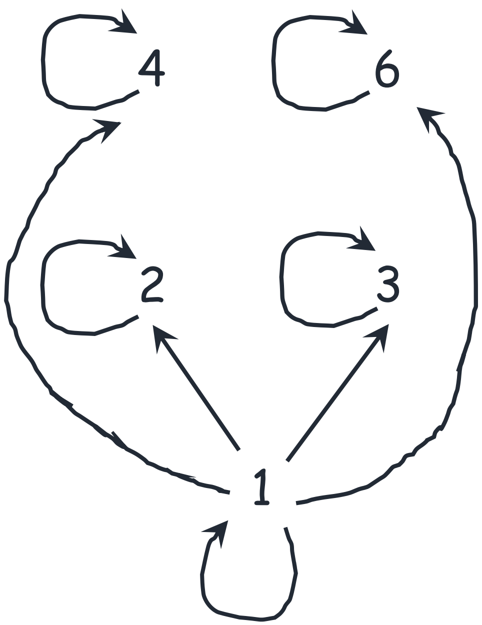

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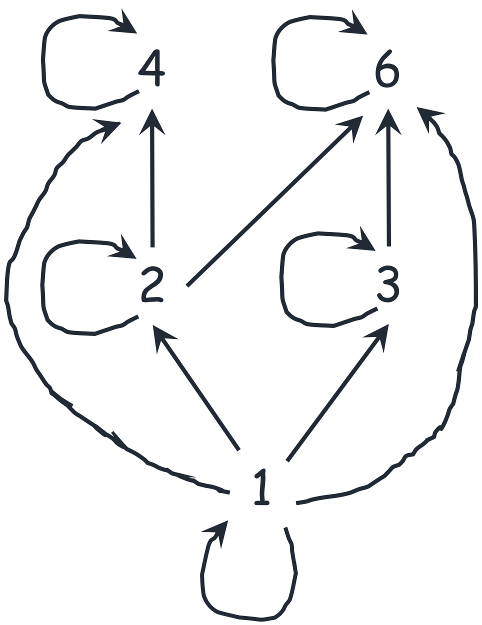

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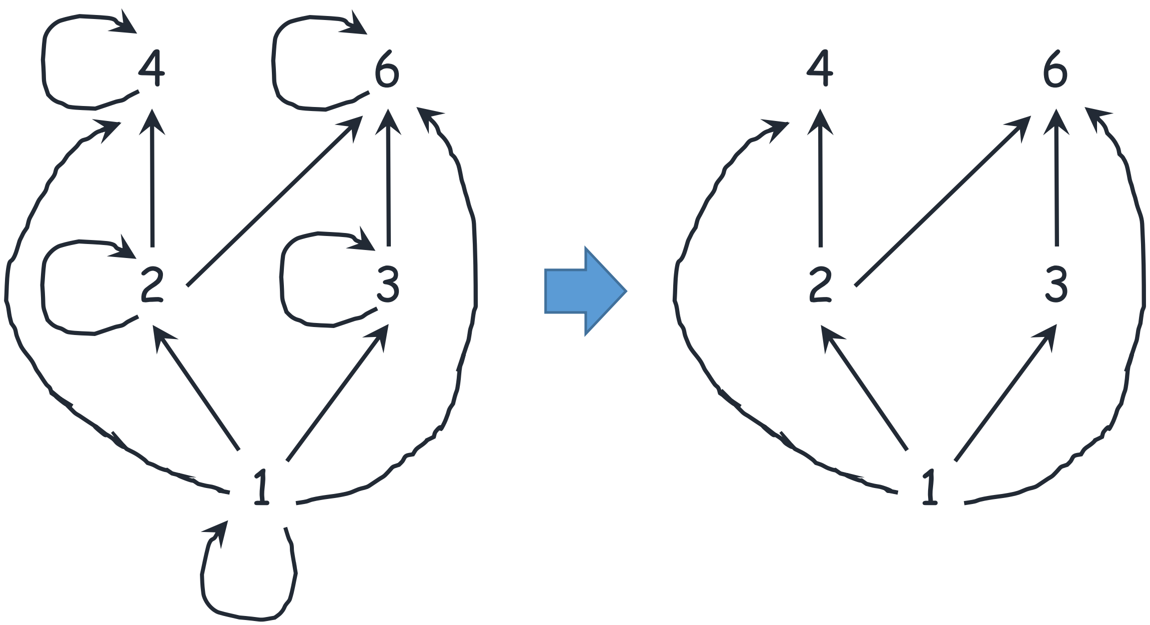

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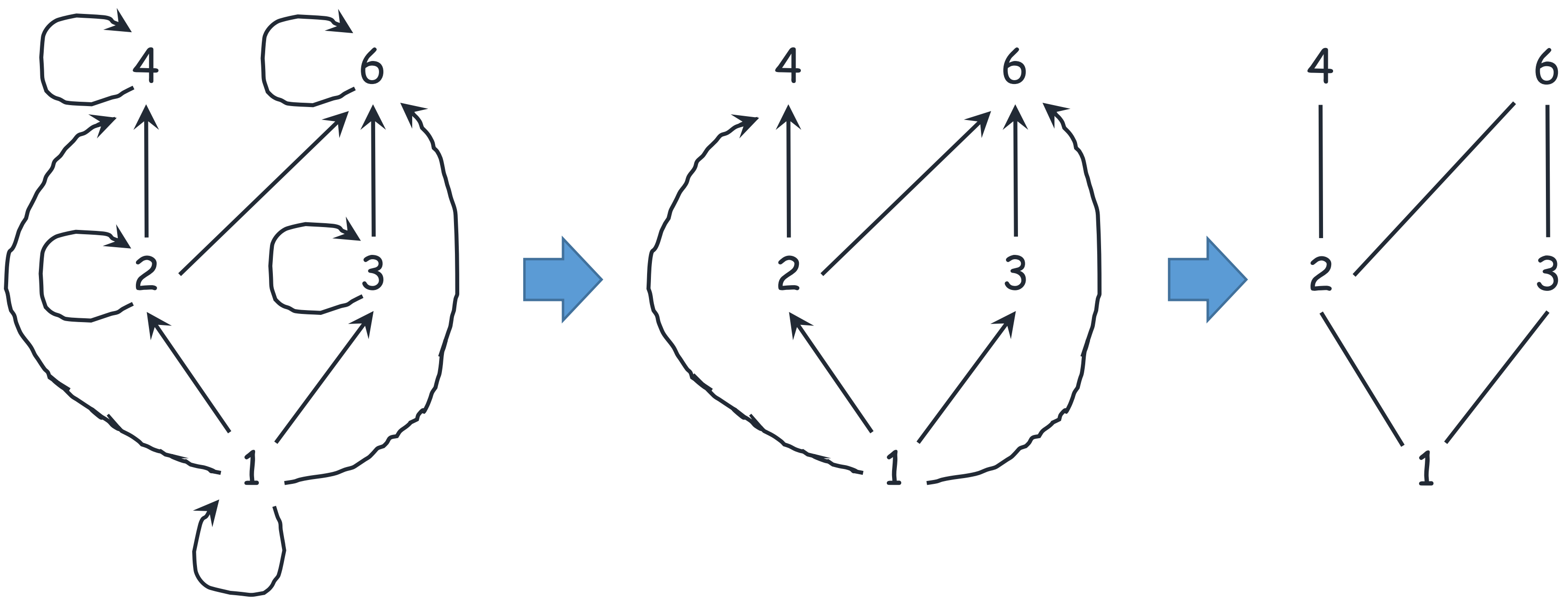

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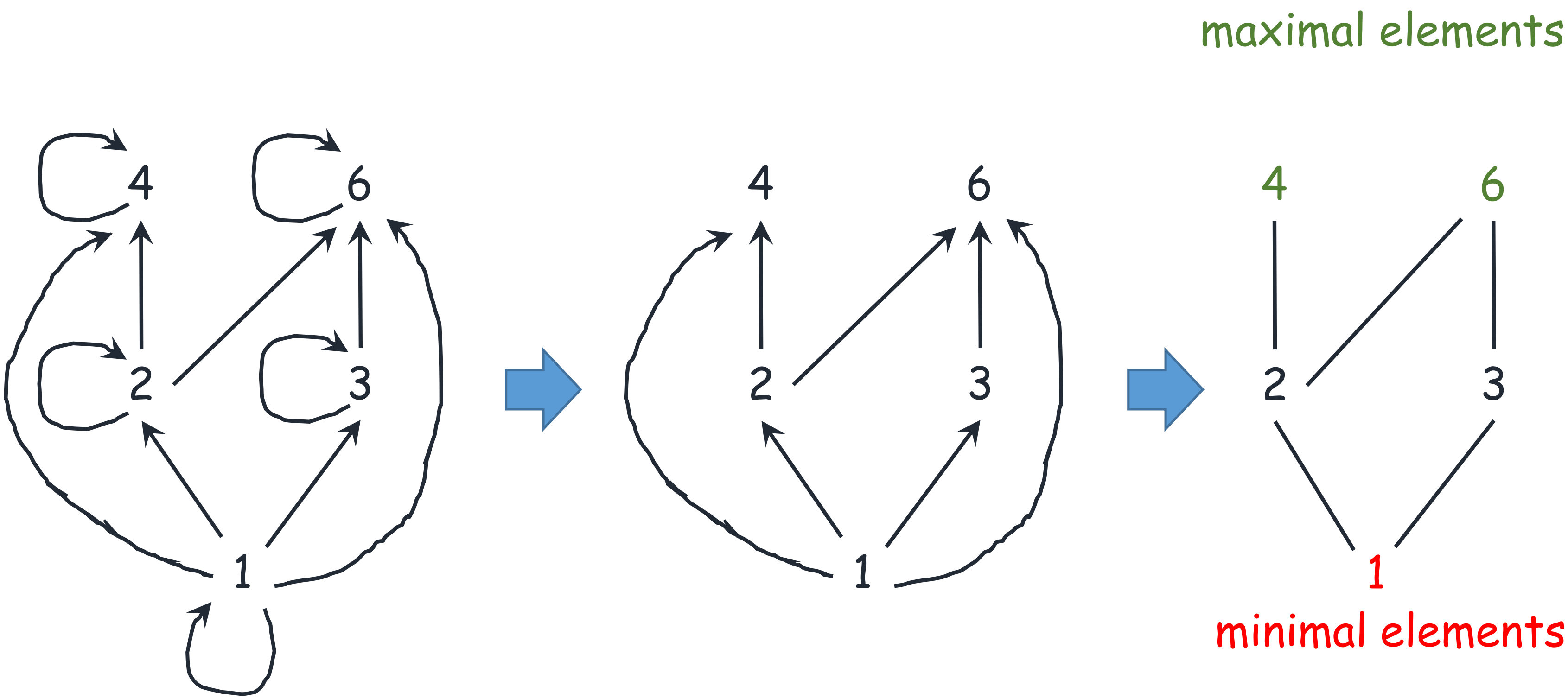

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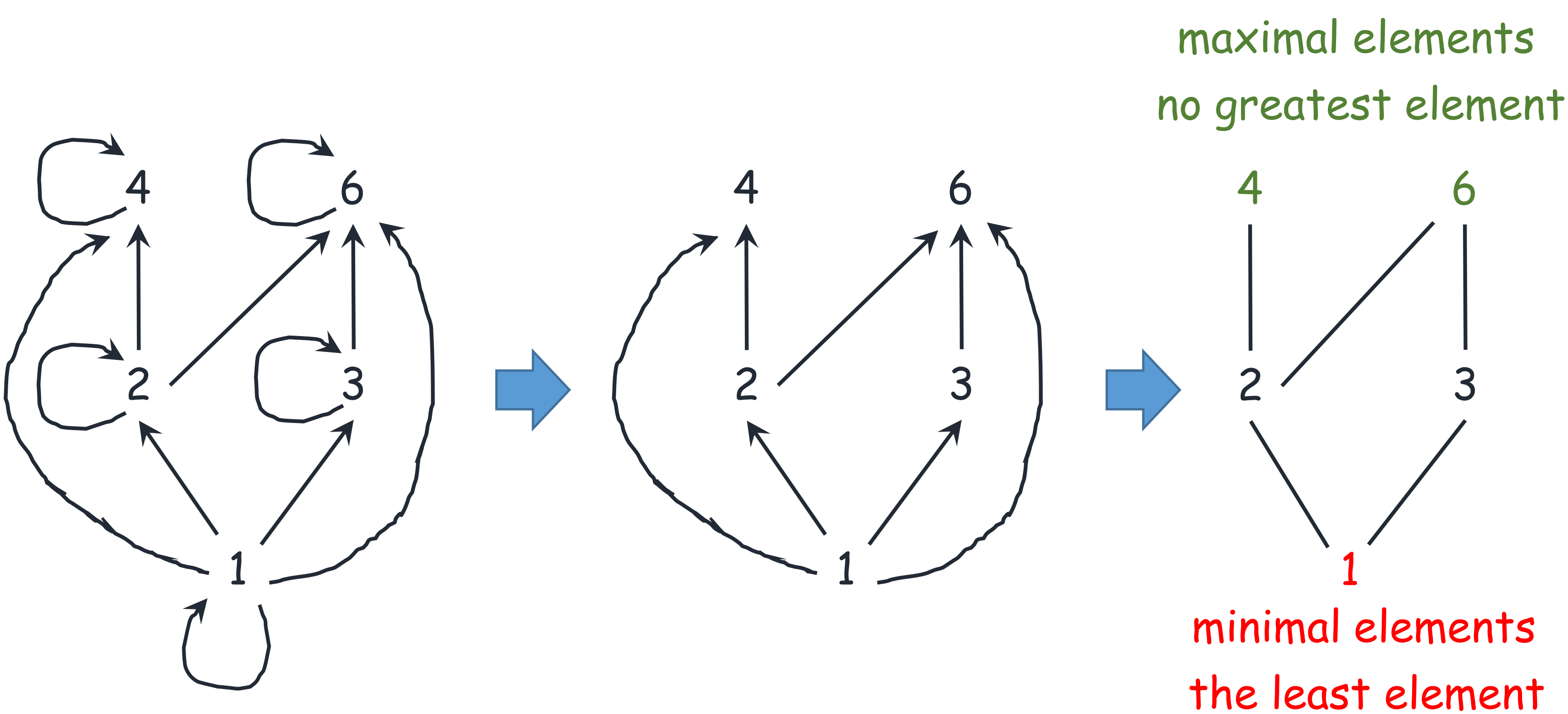

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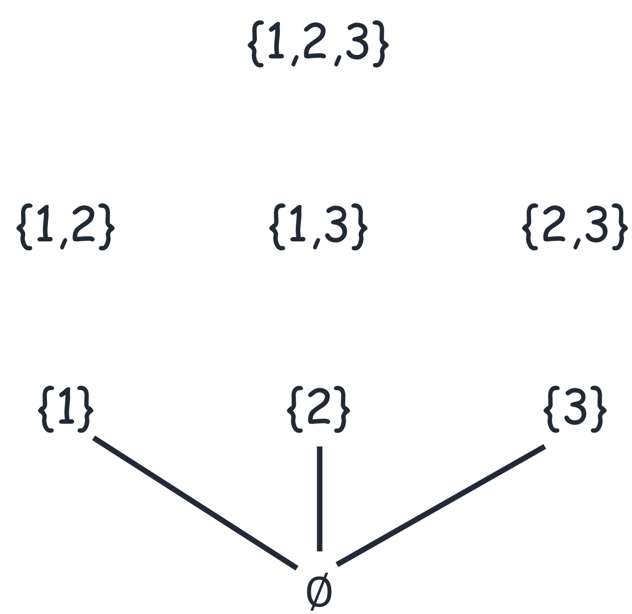

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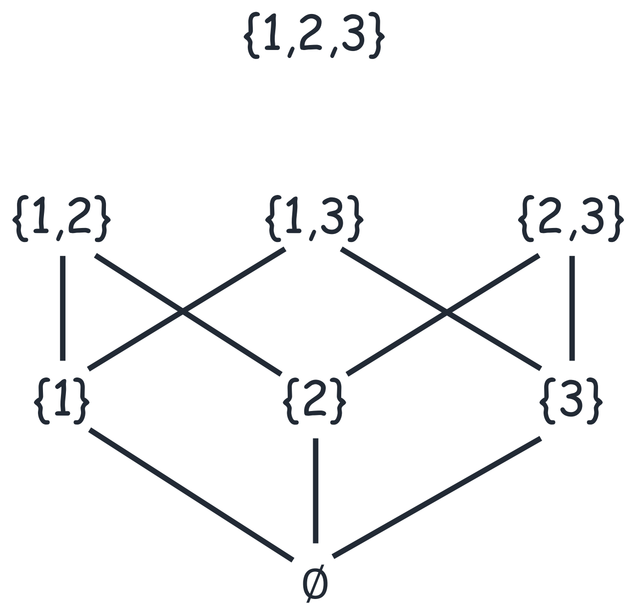

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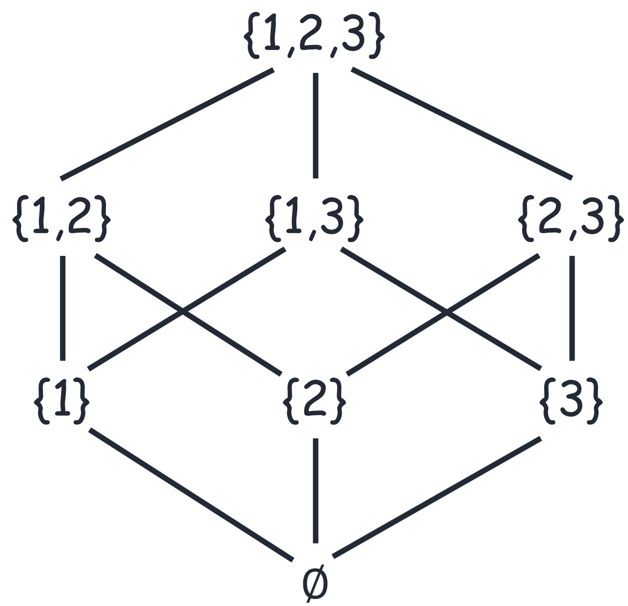

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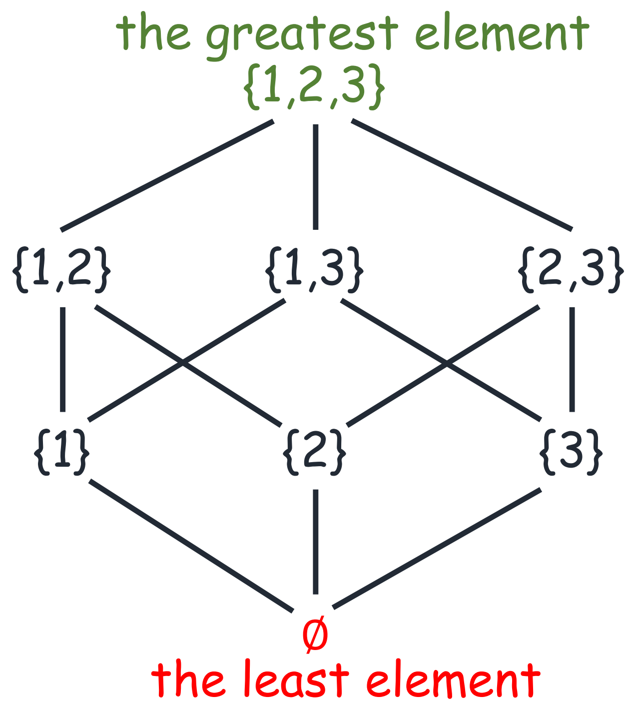

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Definition : Topological sorting of n elements from a poset (S, R) is $s_1 s_2 \ldots s_n$ such that there is no $(s_i, s_j) \in R$ where $j < i$

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input : a finite poset (S, R)
output : topological sorting of elements in S

initialize an empty queue Q
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  a = a minimal element of S
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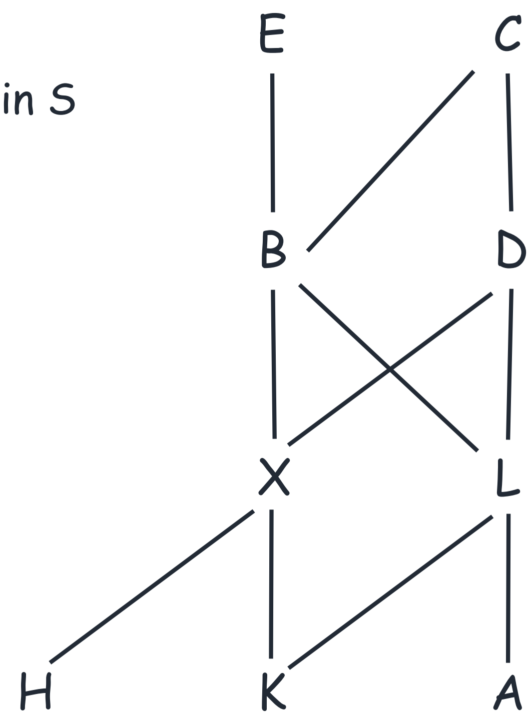

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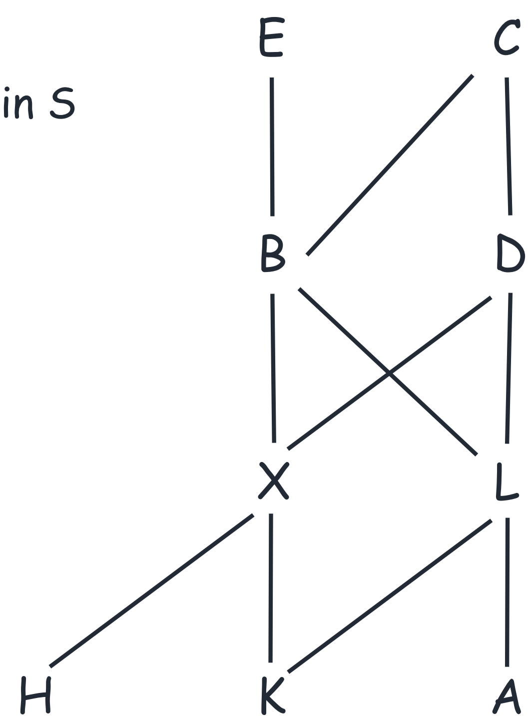
        S = S – {a}
        add a to Q
return Q



Q :

Topological Sorting

Definition : Topological sorting of n elements from a poset (S, R) is $s_1 s_2 \ldots s_n$ such that there is no $(s_i, s_j) \in R$ where $j < i$

input : a finite poset (S, R)
output : topological sorting of elements in S

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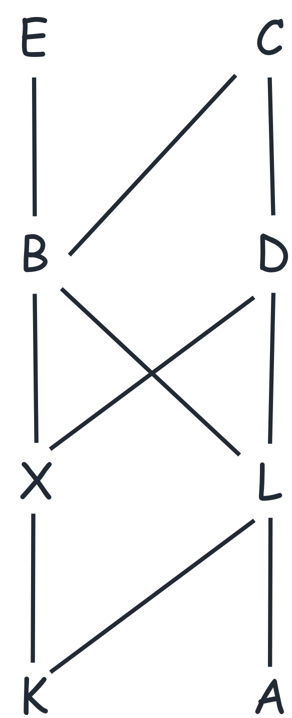

Q : H

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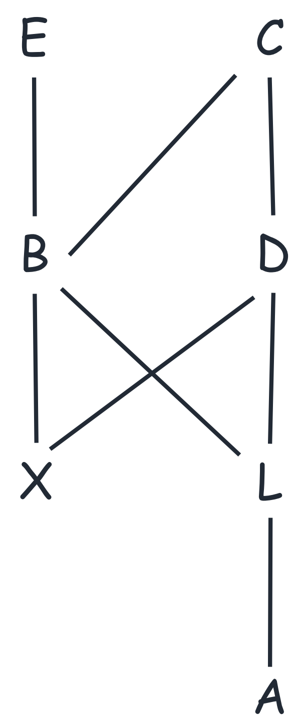

Q : H K

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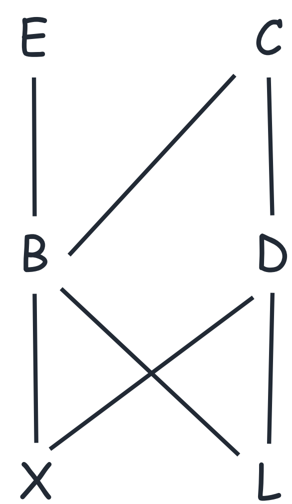

Q : H K A

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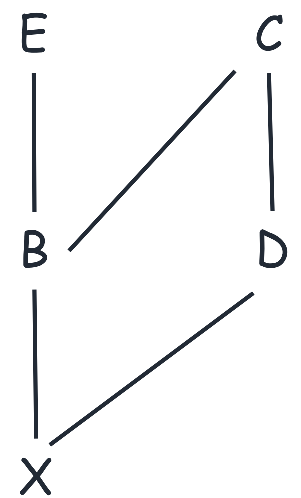

Q : H K A L

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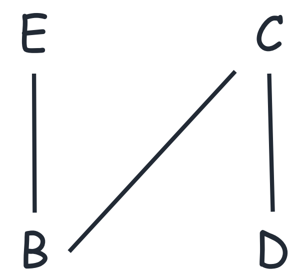

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```
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        a = a minimal element of S
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return Q
```

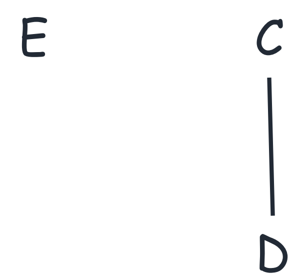

Q : H K A L X B

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```
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```

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C

input : a finite poset (S, R)
output : topological sorting of elements in S

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Q : H K A L X B E D

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Q : H K A L X B E D C