Introduction to Algorithms

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Approximation Algorithms

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In practice, we may not have sufficient computational resources to compute the exact (optimal) solution.

Computation resources could be:
(1) computation time
(2) memory space
(3) communication cost
...

Why do we need approximation algorithms?

Problem A is NP-hard, and thus it needs superpolynimal time unless P = NP.

If an approximate solution is good enough,
we may reduce the computation time.

Why do we need approximation algorithms?

Problem B (a two-party game) has communcation complexity $\Omega(n^2)$, and thus solving B in the streaming model requires memory space of $\Omega(n^2)$ bits.

If an approximate solution is good enough, we may reduce the amount of memory space.

Approximation ratio

We assume that the feasible solutions to our optimization problems are positive values. Let C* be the optimal solution, and let C be the found approximate solution.

For a maximization problem, the (multiplicative) approximation ratio is defined to be $(C^*/C) \geq 1$.

For a minimization problem, the (multiplicative) approximation ratio is defined to be $(C/C^*) \geq 1$.

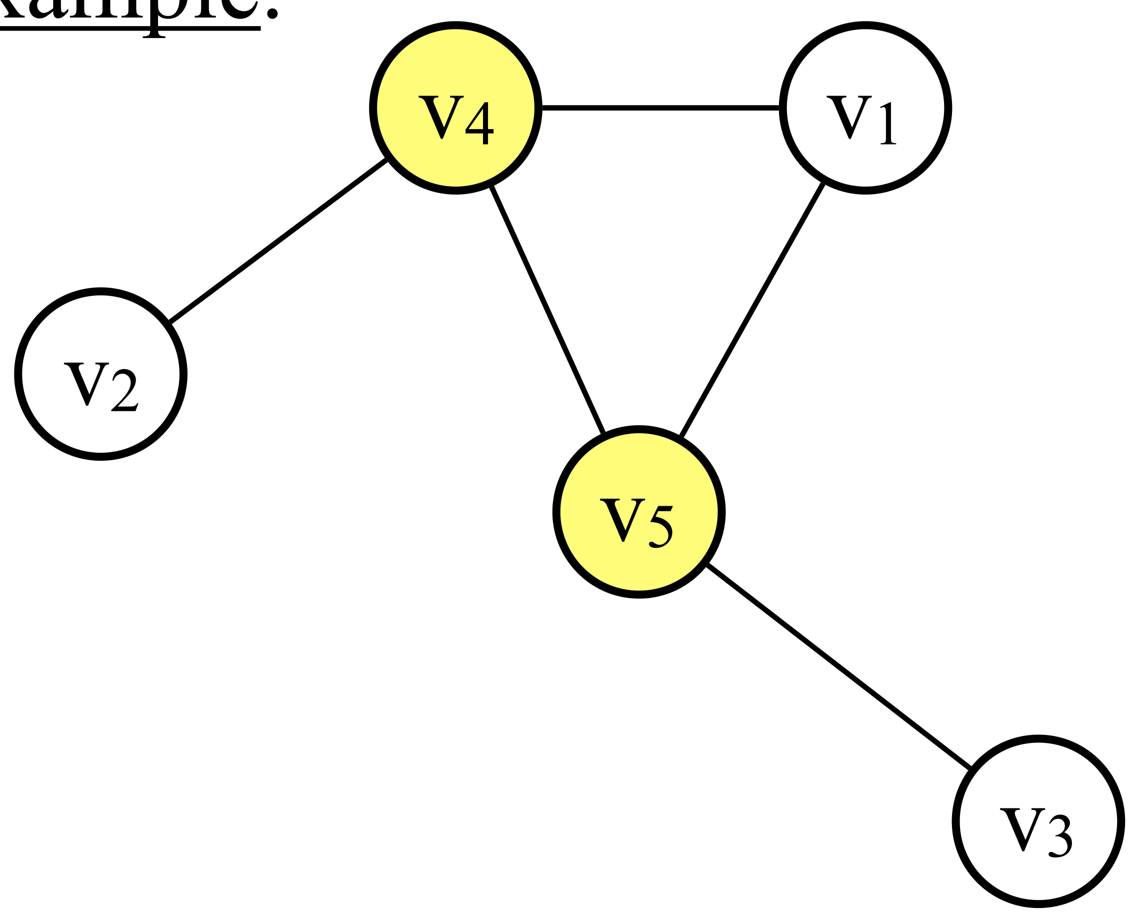
Examples

Minimum Vertex Cover

Input: an undirected graph G = (V, E).

Output: the size of a minimum-cardinality subset S of V so that for every edge (u, v) in E, node u or node v (maybe both) is in S.

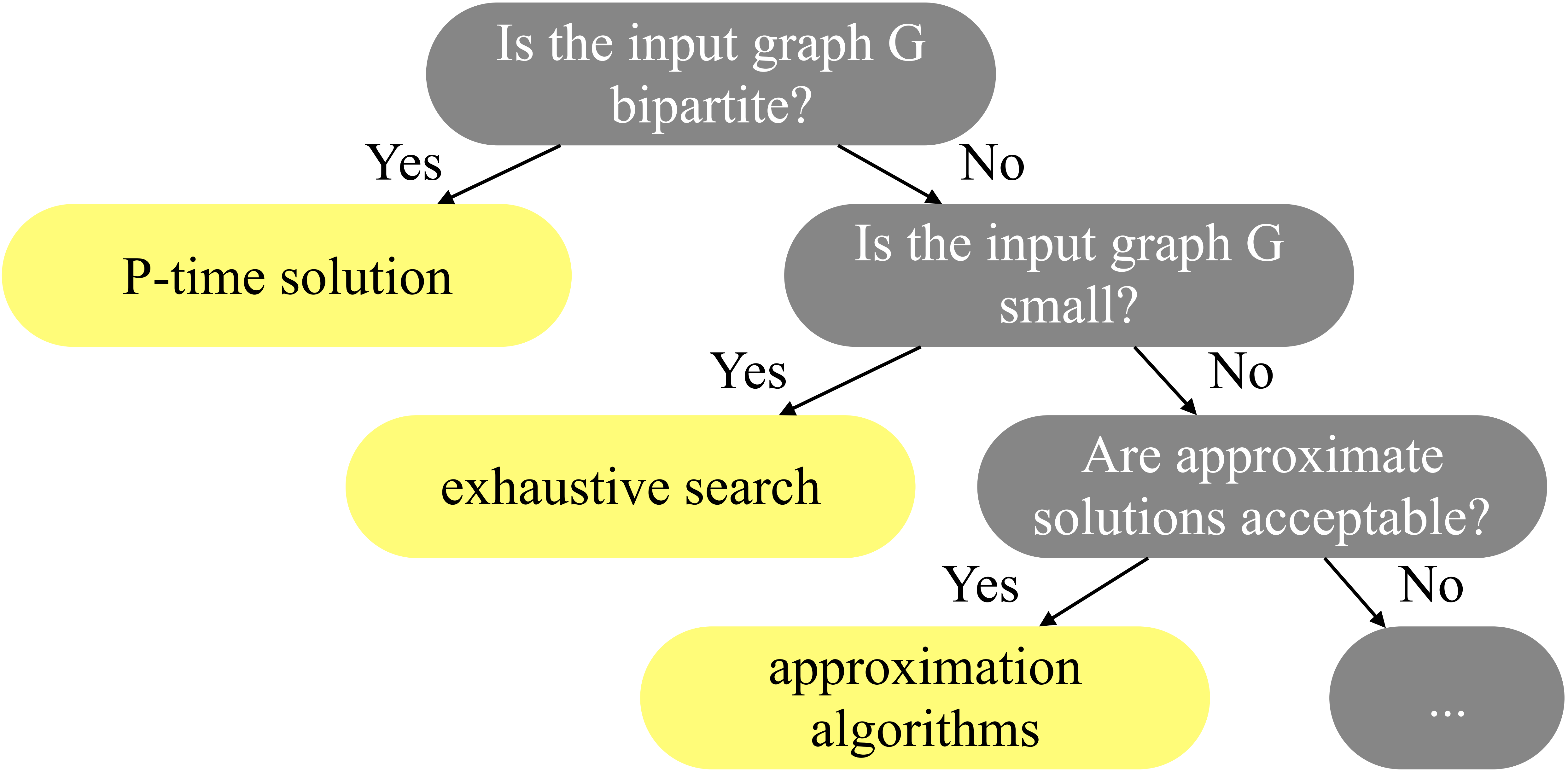
<u>Example</u>.



A vertex cover of cardinality (size) 2.

Minimum Vertex Cover

Recall that for general graphs Minimum Vertex Cover is NP-hard, and for bipartite graphs it has a P-time solution.



Minimum Vertex Cover

The following greedy algorithm has the approximation ratio 2. Alternatively, we could say, it is a 2-approximation algorithm.

```
S ← ∅, X ← ∅; // S is the set cover

while (there exists an edge eᵢ = (u, v) so that u ∉ S and v ∉ S){
    X ← X ∪ {eᵢ}; // a dummy operation
    S ← S ∪ {u} ∪ {v};
}

return |S|;
```

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Note that X is a matching.

Minimum Vertex Cover

The greedy algorithm has the approximation ratio 2.
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Proof.

Let S^* be the minimum vertex cover. For each edge (u, v) in X, u in S^* or v in S^*. Since X is a matching, no element in S^* can cover more than one edges in X. Therefore, $|S^*| \geq |X|$.

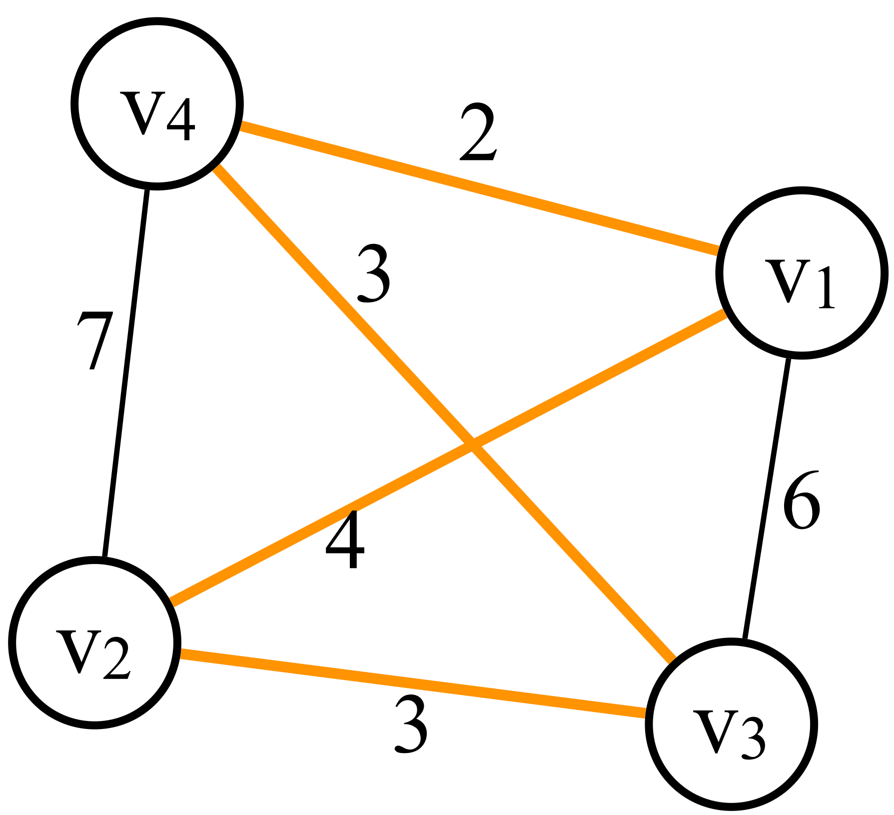
Together with $|S| = 2|X|$, we have $|S| \leq 2|S^*|$. Thus, the approximation ratio is at most 2.

Travelling Salesman Problem

Input: a complete undirected graph $G = (V, E)$, in which each edge e is associated with a non-negative weight $w(e) \geq 0$.

Output: a simple cycle C that traverses all nodes in G so that $\sum_{e \in C} w(e)$ is minimized.

Example.



A TS tour of weight 12.

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We assume that the input graph satisfies the triangle inequality; i.e., for all $u, v, x \in V$, $w(u, v) + w(v, x) \geq w(u, x)$.

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This assumption is satisfied for many applications on real maps.

Travelling Salesman Problem

The following algorithm is a 2-approximation algorithm.

$T \leftarrow \mathrm{MST}(G)$; // minimum spanning tree
$E_{tour} \leftarrow$ Eulerian tour of T;
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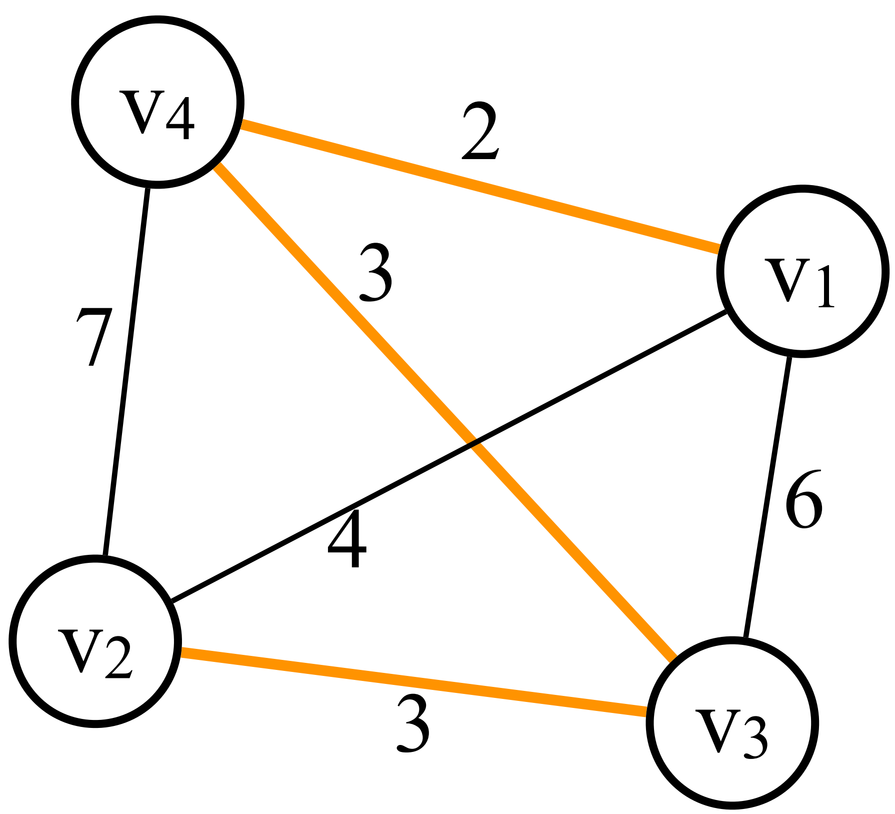

MST(G)

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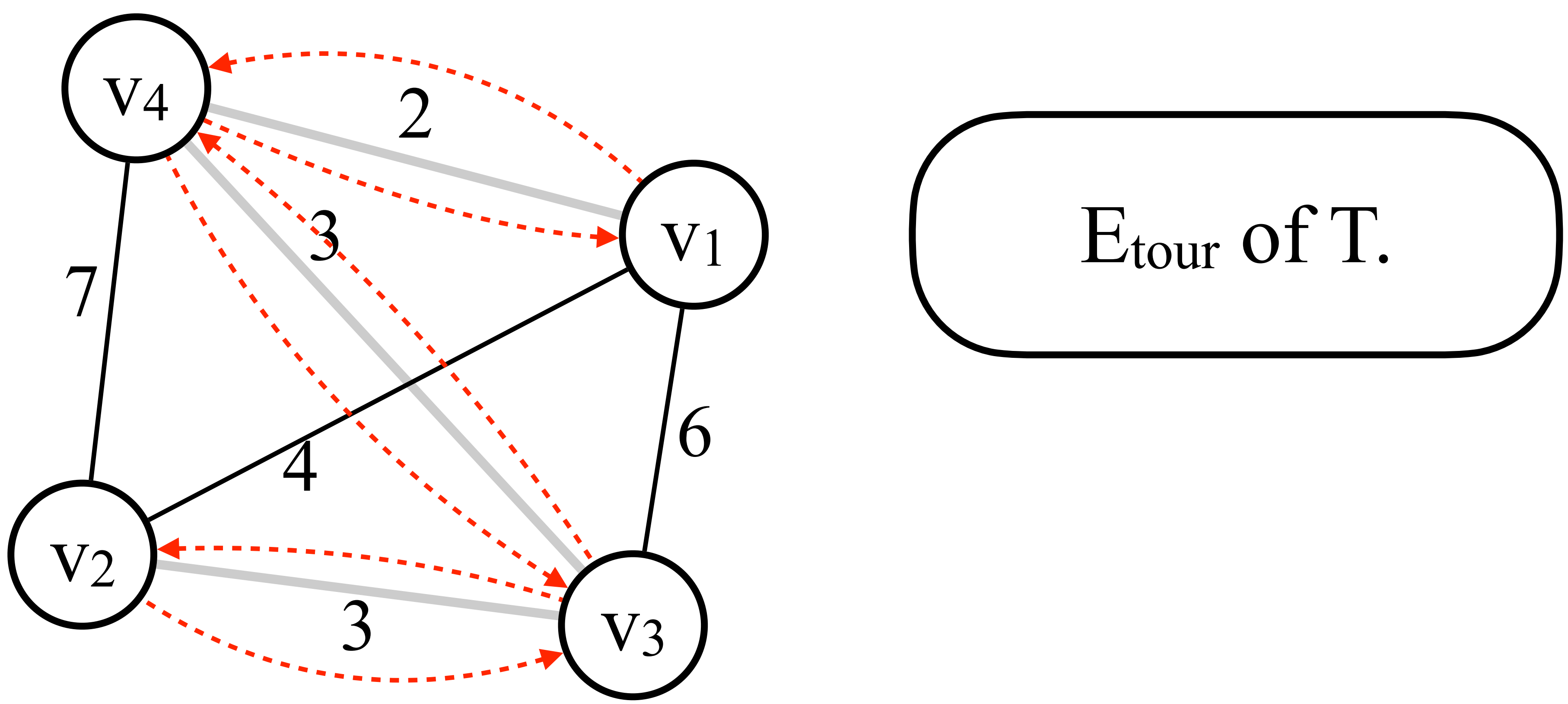

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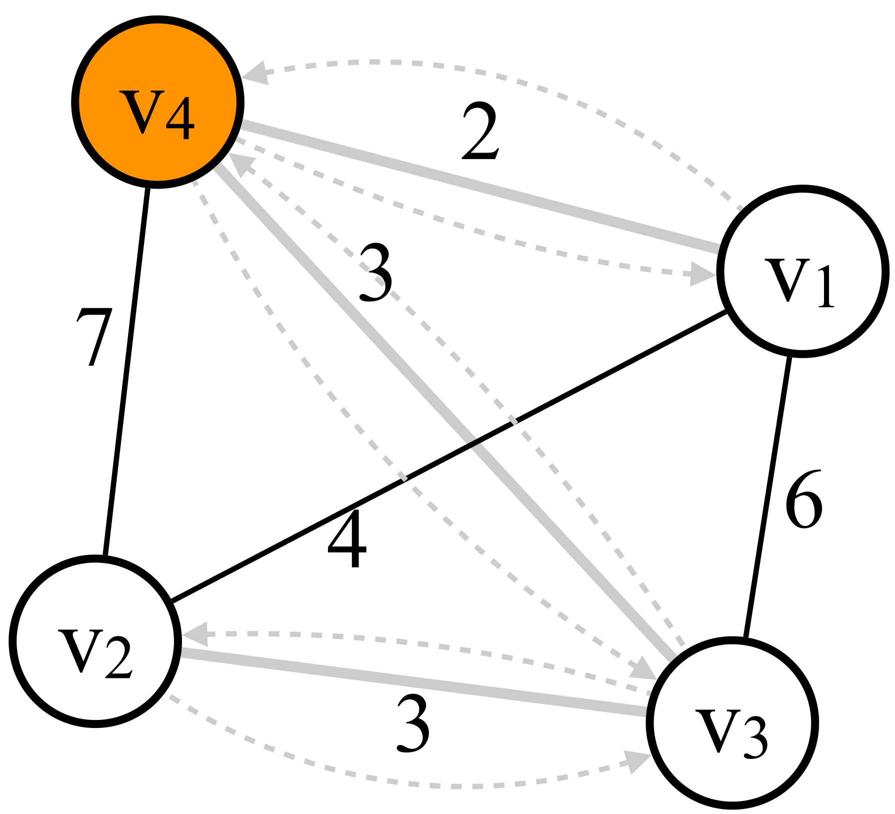

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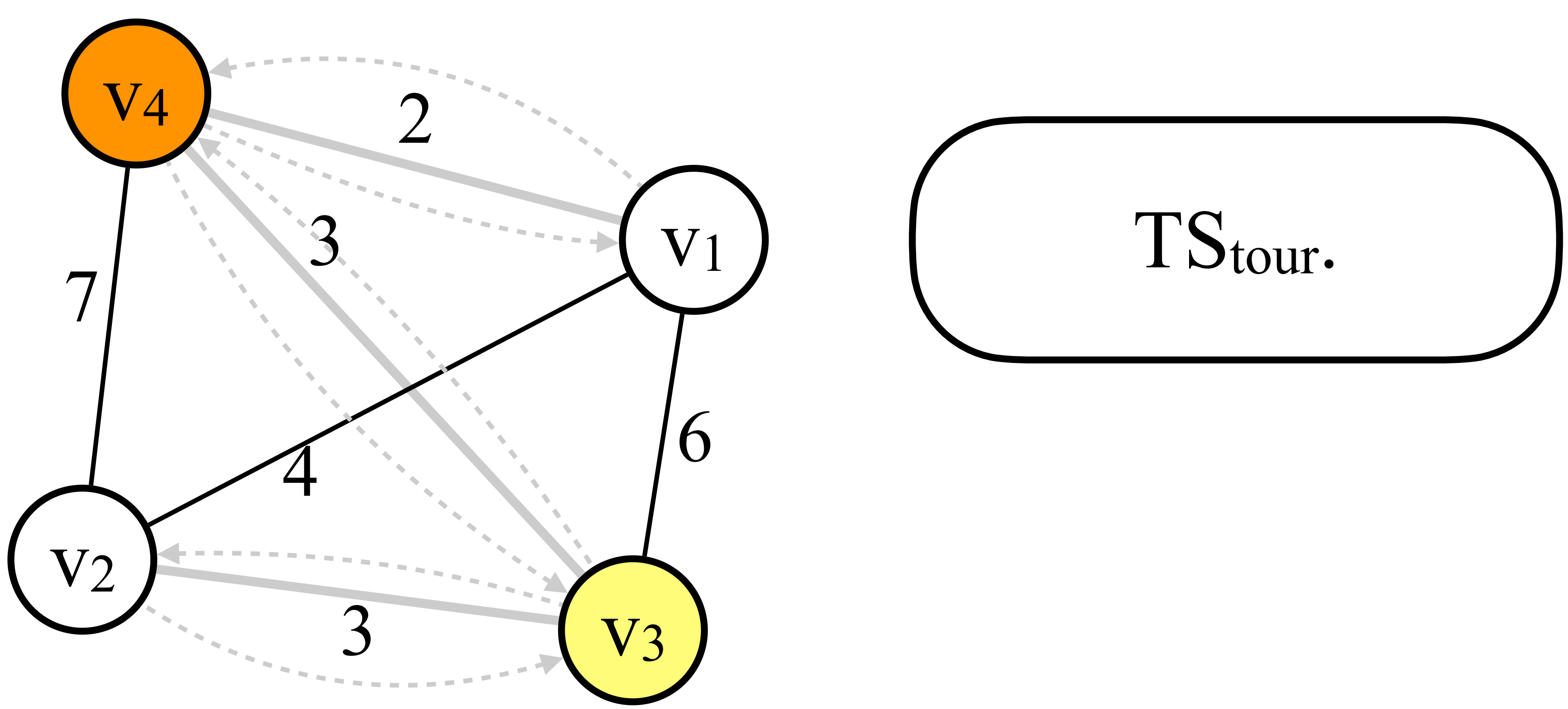

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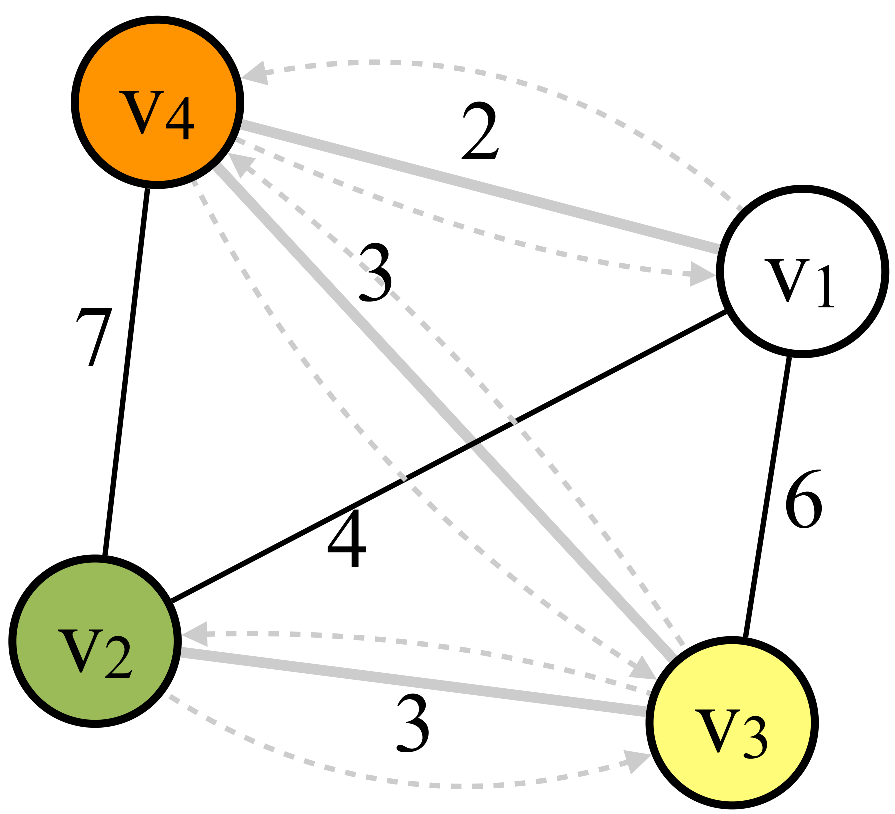

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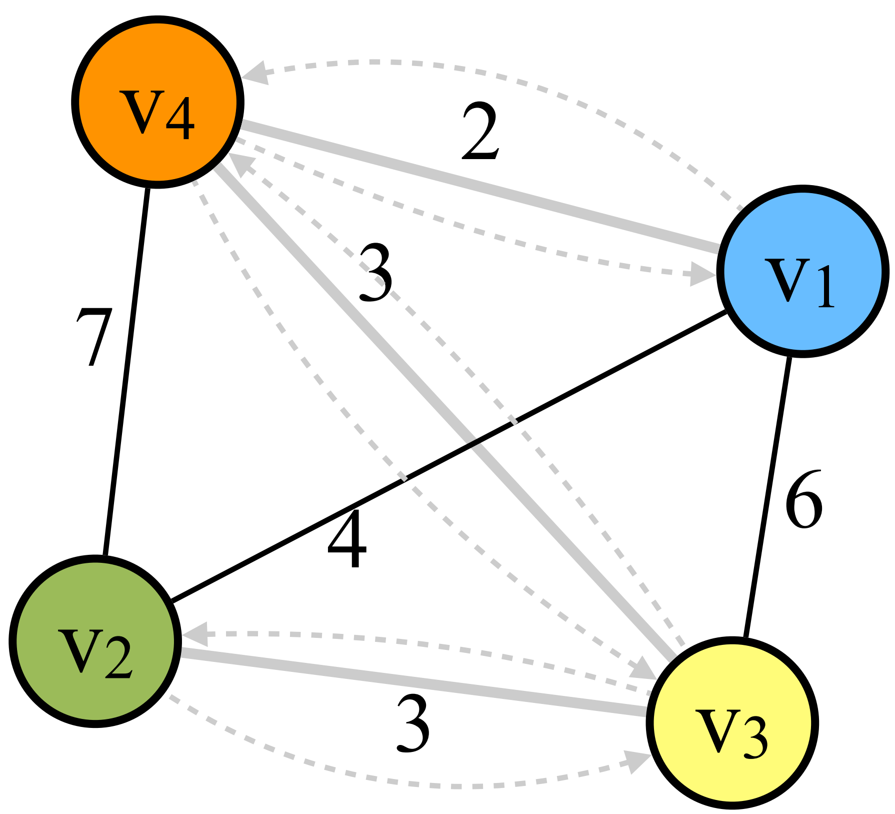

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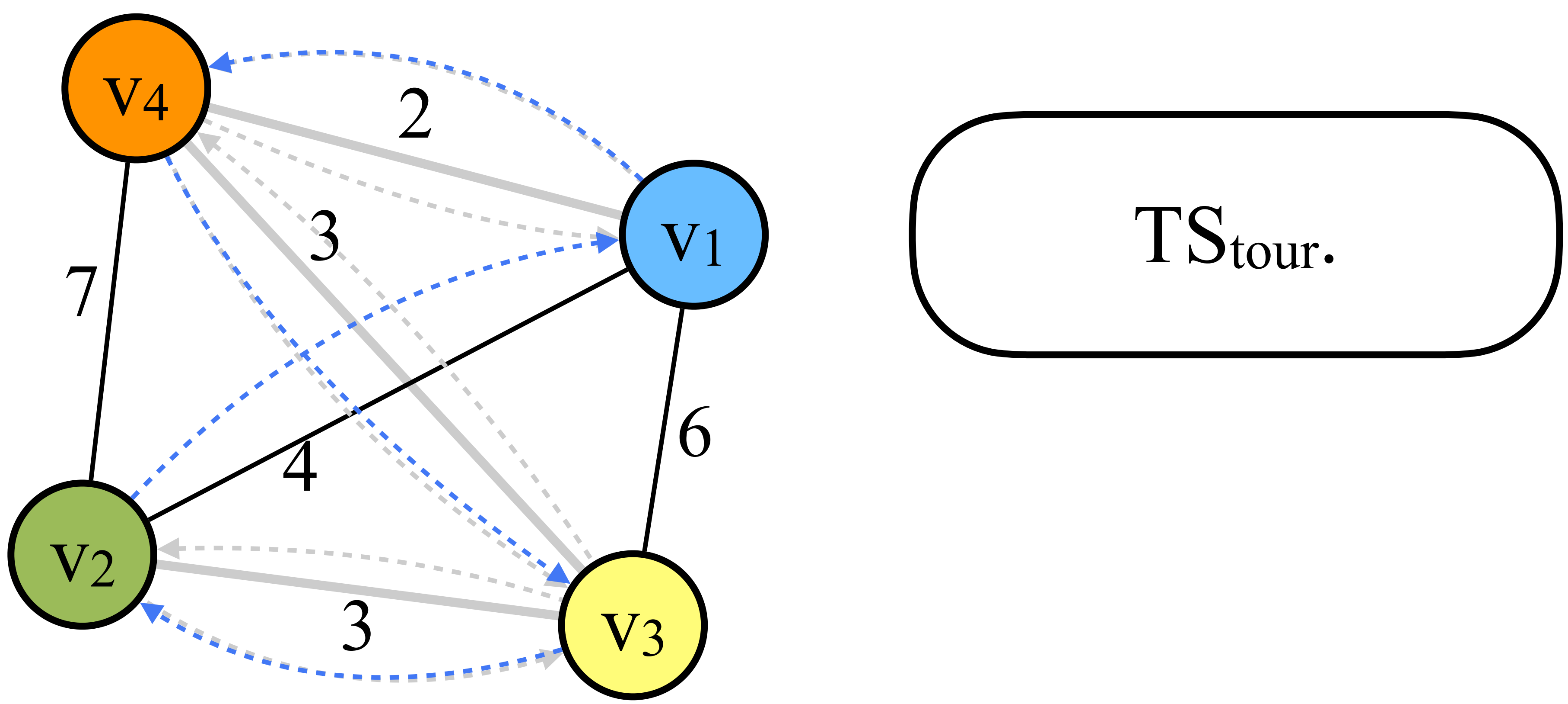

Travelling Salesman Problem

The algorithm is a 2-approximation algorithm.

Proof.

Let TS* be the minimum-weight travelling salesman tour. TS* can be seen as a spanning tree + an edge. Hence,

$$w(TS^*) \geq w(T)$$

because T is the minimum spanning tree and all $w(e) \geq 0$.

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By the trianlge inequality, shortcutting the tour cannot increase the weight and thus

$$2w(T) = w(E_{tour}) \geq w(TS_{tour}).$$

Since TS* is the optimal TS-tour, $w(TS_{tour}) \geq w(TS^*)$. As a result, $w(TS_{tour}) \in [w(T), 2w(T)]$ and $w(TS_{tour}) \in [w(TS^*), 2w(TS^*)]$.

Maximum Matching

Input: an undirected graph G = (V, E)

Output: the size of a maximum-cardinality subset F of E so that no two edges in F share an endpoint.

This problem can be solved in $O(n^{1/2} m)$ time (i.e. has a P-time solution). If the running time is not affordable, one can obtain a 2-approximation of this problem in $O(n+m)$ time.

Maximum Matching

<u>Claim</u>. Let M* be a maximum matching, and let M be any maximal matching. Then, $|M| \geq |M^*|/2$.

Proof.

Let (u, v) be an edge in M*. Because M is a maximal matching, at least one of nodes u, v is an endpoing of some edge in M. Otherwise, M shall include edge (u, v). This implies that $|M| \geq |M^*|/2$.

Maximum Matching

The following greedy algorithm can return a maximal matching.

```
M ← ∅, X ← ∅;
Maximal-matching(G){
    foreach(edge (u, v) in G){
        if(u ∉ X and v ∉ X){
            X ← X ∪ {u} ∪ {v};
            M ← M ∪ {(u, v)};
        }
    }
}
return |M|;
```

The above procedure runs in $O(|V|+|E|)$ time.

Exercise: 2/3-approximation of Matching

```
Matching(G){

  M ← ∅;
  foreach(e){
      if(M ∪ {e} is a matching){
          M ← M ∪ {e};
      }
  }
  while(there exists a length-3 augmenting path P w.r.t. M){
      M ← M ⊕ P;
  }

  return M;
}
```

Exercise: 2/3-approximation of Matching

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```

$|M| \geq (2/3)\mathrm{OPT}$. (Why?)