Introduction to Algorithms

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10/17/2019

Announcements

Written Assignment 2 is due by Oct 31, 15:40. at https://e3.nctu.me

Programming Assignment 2 was extended, and is due by Nov 5, 23:59. at https://oj.nctu.me

Quiz 1 will be held in class on Oct 24.

Scope: slides 01 - 09, assignments, and their generalizations.

About Quiz 1

Asymptotic Bounds: was1-p1, was1-p2, was2-p1.

Basic DP: was2-p2, pas2-p1.

Reduction: was1-p5, was2-p6.

Don't forget the "I don't know" policy.

You may bring two cheating sheets in A4 size.

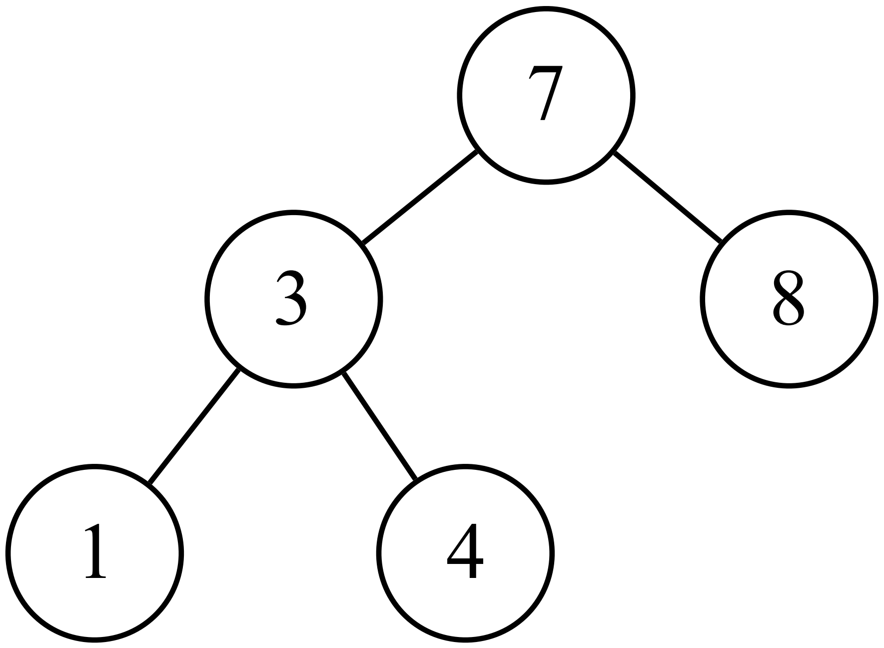
Binary Search Trees

What is a binary search tree?

A tree whose keys satisfy the *binary-search-tree property*:

For any node x, all nodes in the left subtree of x has key value at most x.key, and all nodes in the right subtree of x has key value at least x.key.

--- Example ---

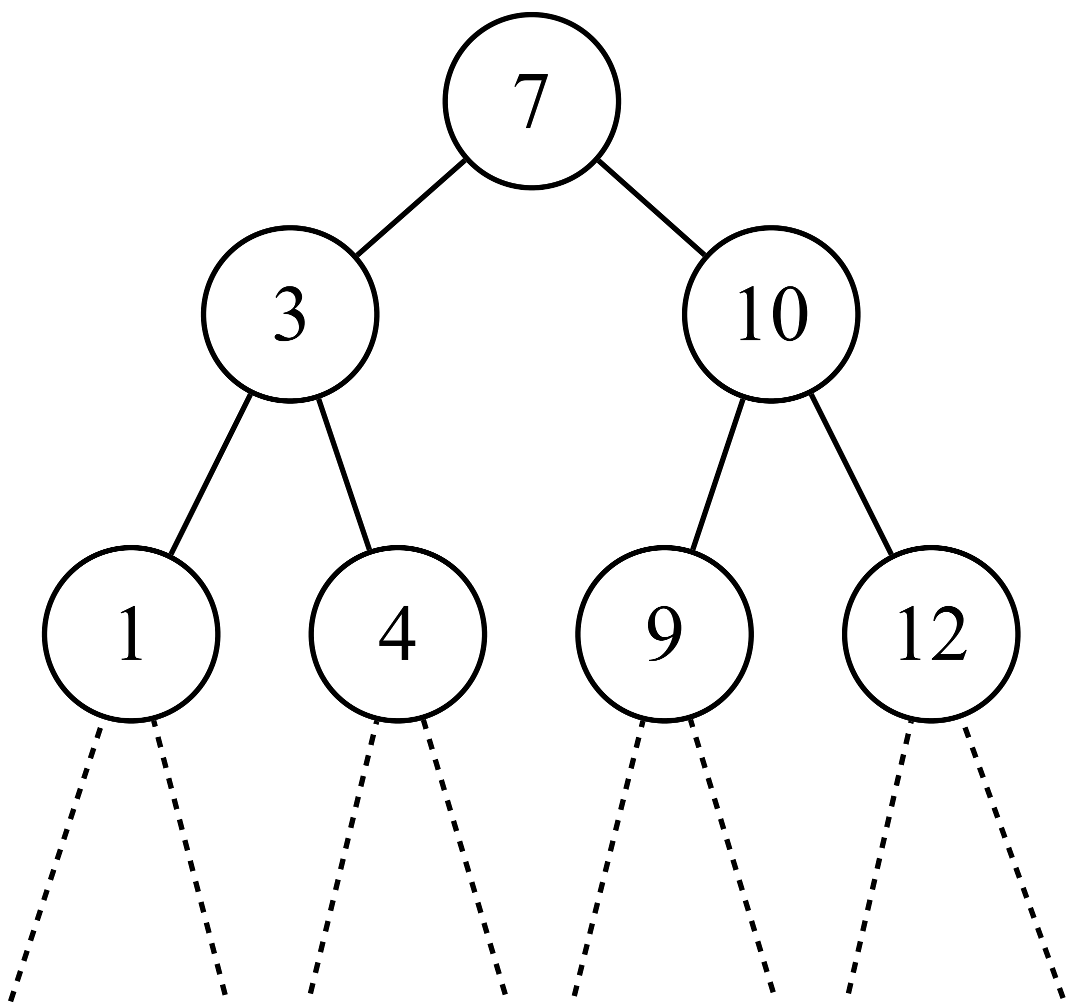


Every node x has 4 attributes:

x.key: the key value of x
x.left: a pointer to x's left subtree
x.right: a pointer to x's right subtree
x.p: a pointer to x's parent node

If a child or the parent is missing, the corresponding pointer is set as Null.

In-Order Tree Traversal



The in-order tree traversal visits the nodes in a tree in the order:

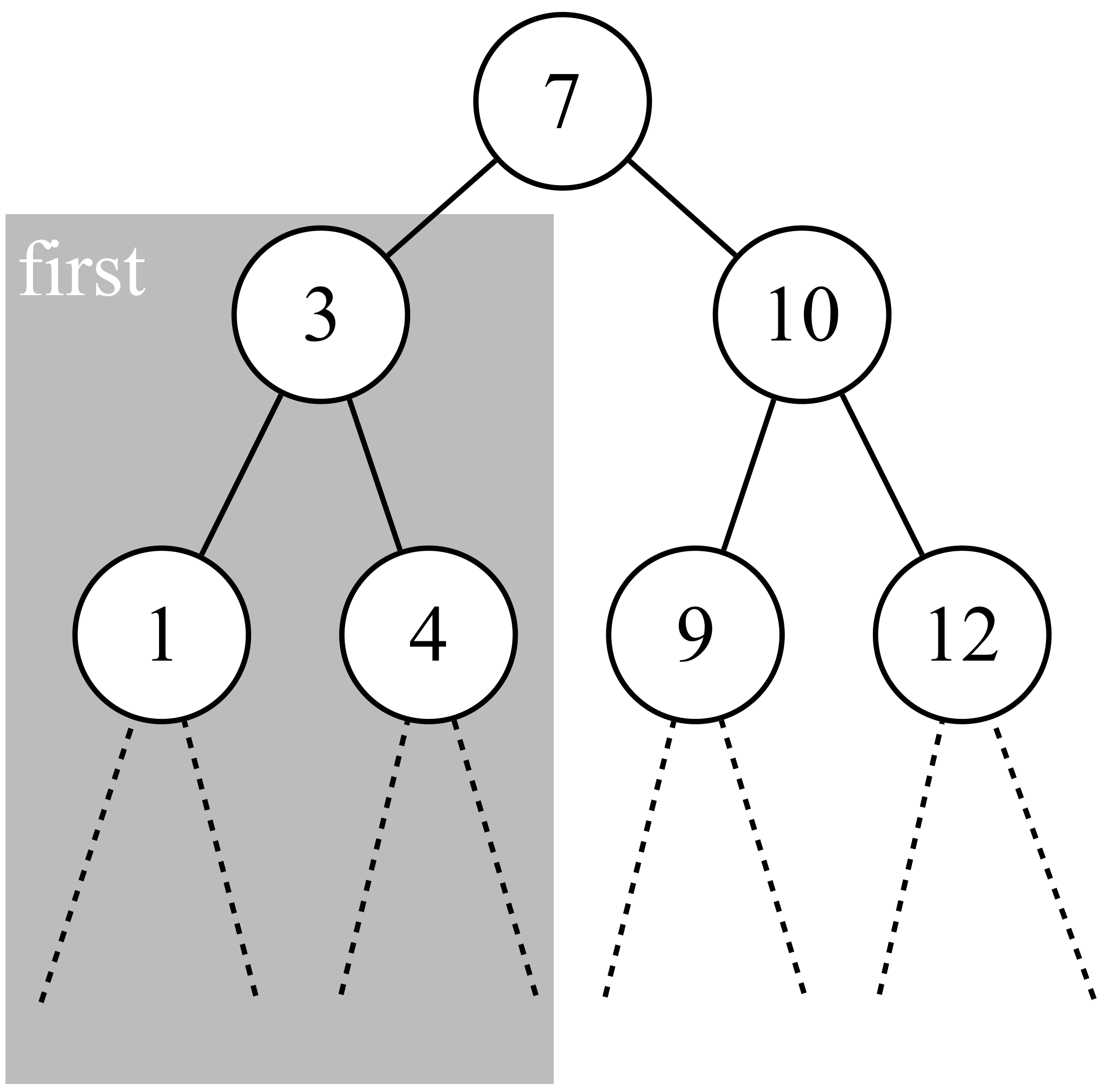
for any node x,

1. the nodes in x's left subtree are visited before x,

2. then x,

3. the nodes in x's right subtree are visited after x.

In-Order Tree Traversal



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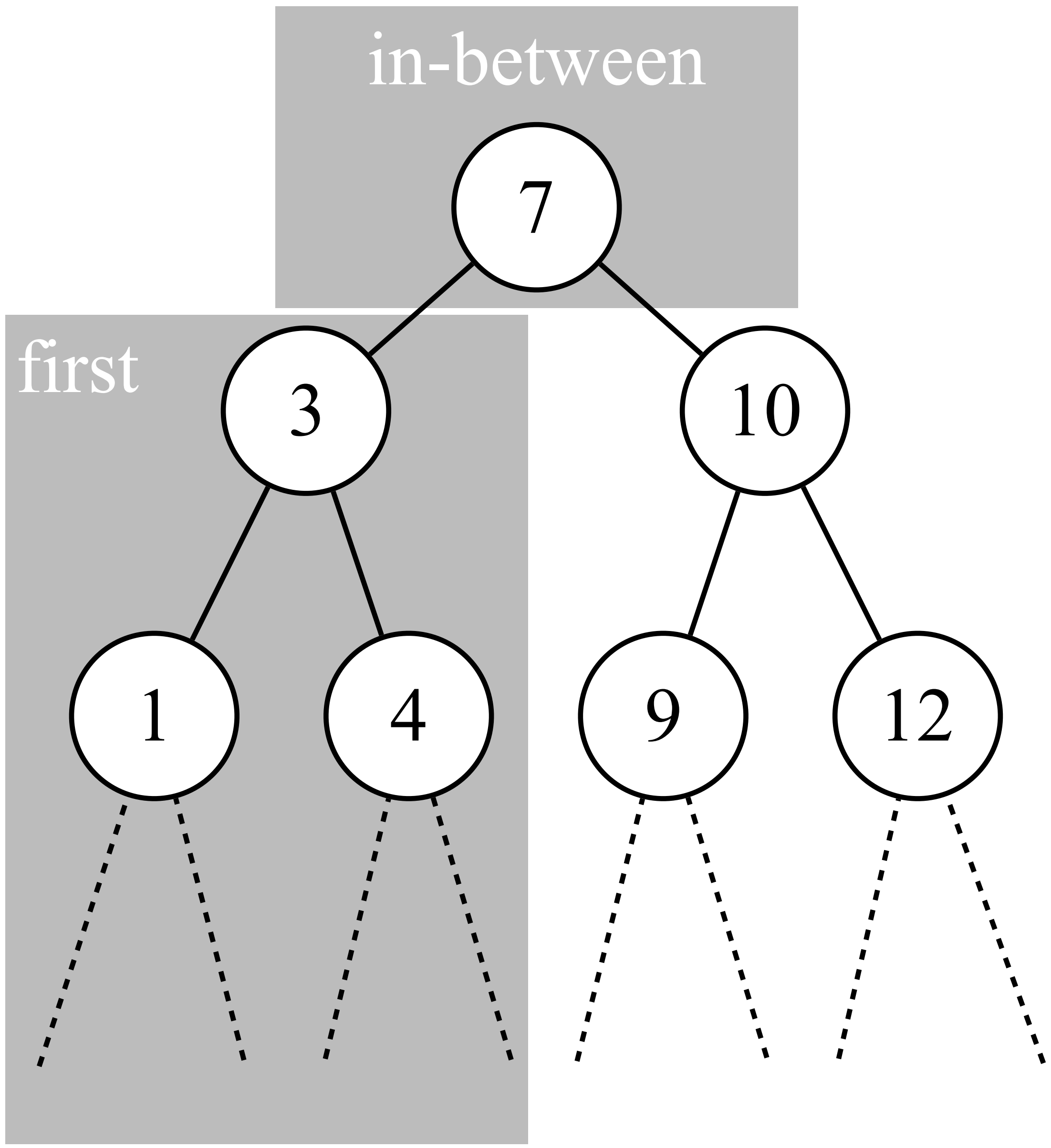

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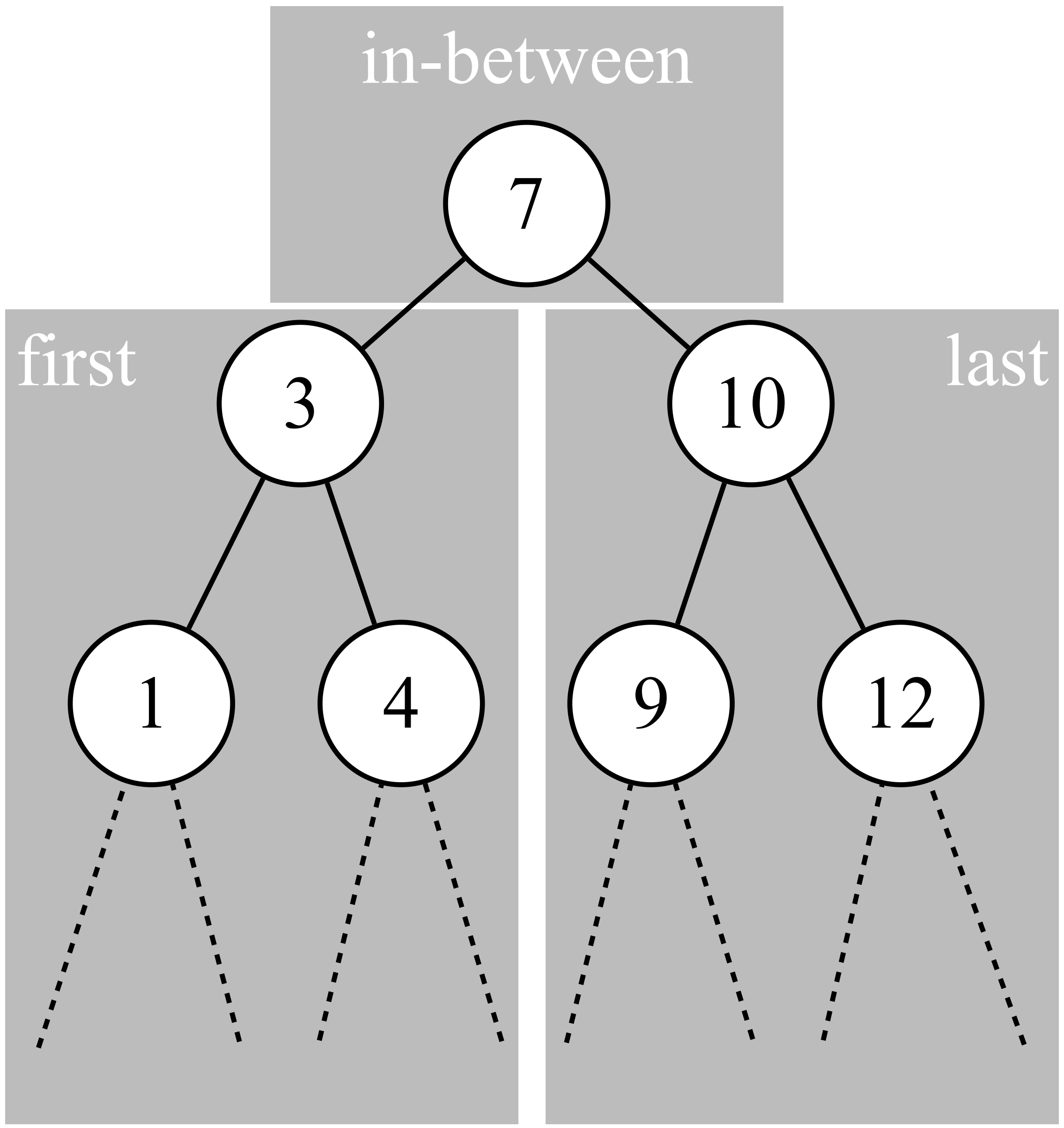

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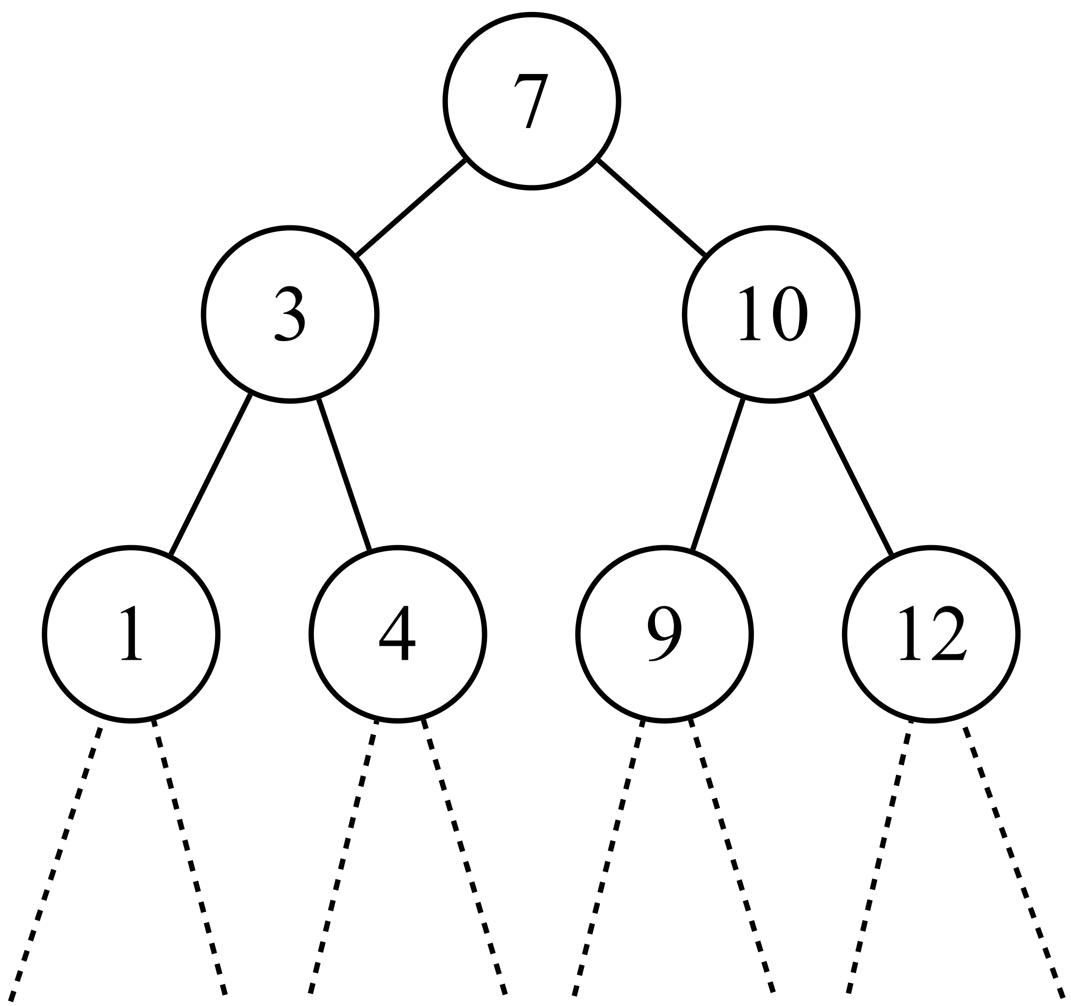
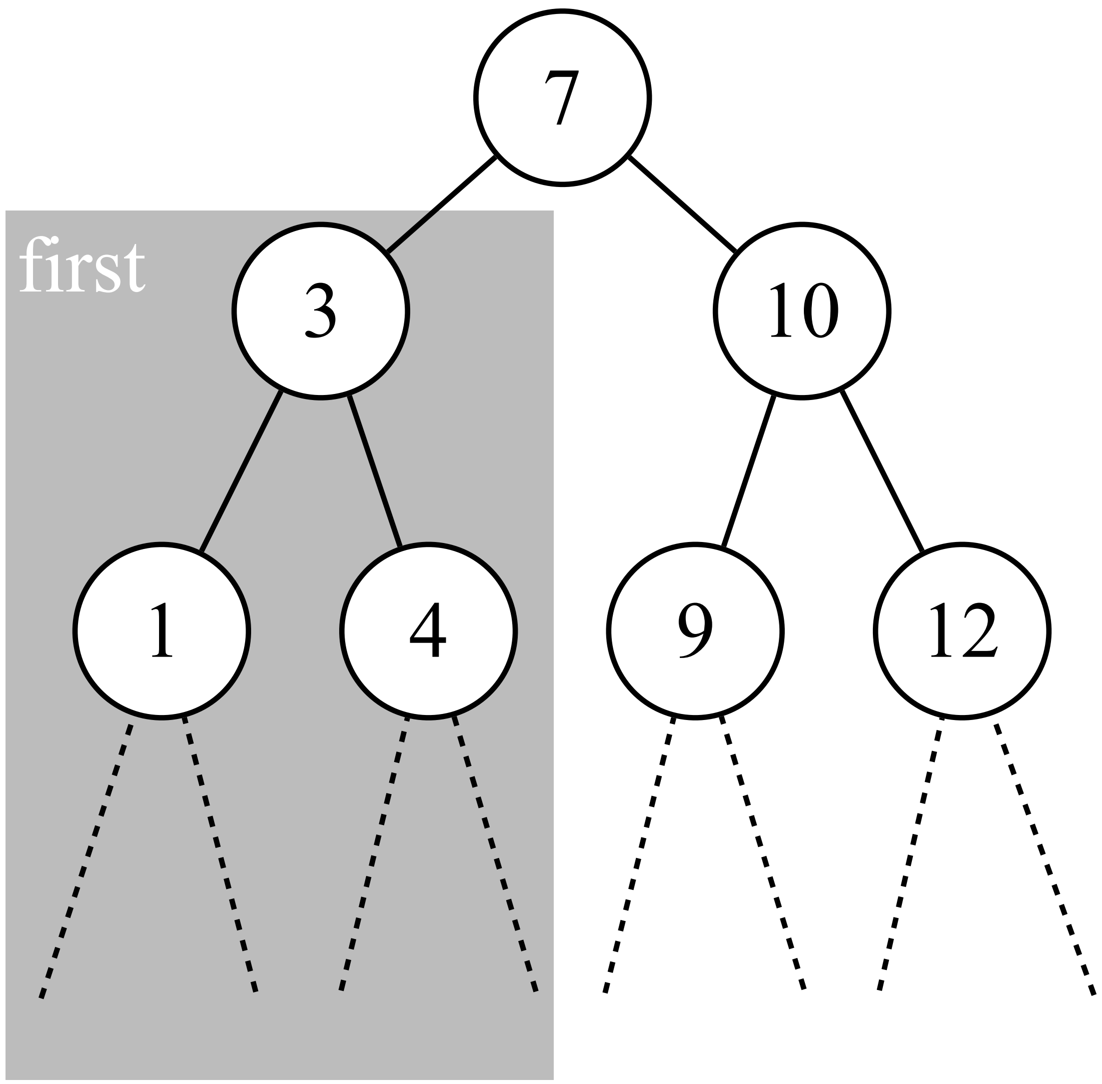
Output the keys on a BST in the sorted order




```
In-Order-Traversal(x){
    if(x ≠ NIL){
        In-Order-Traversal(x.left);
        print x.key;
        In-Order-Traversal(x.right);
    }
}
```

Q: Why does the above procedure output the keys in the sorted order?

Output the keys on a BST in the sorted order

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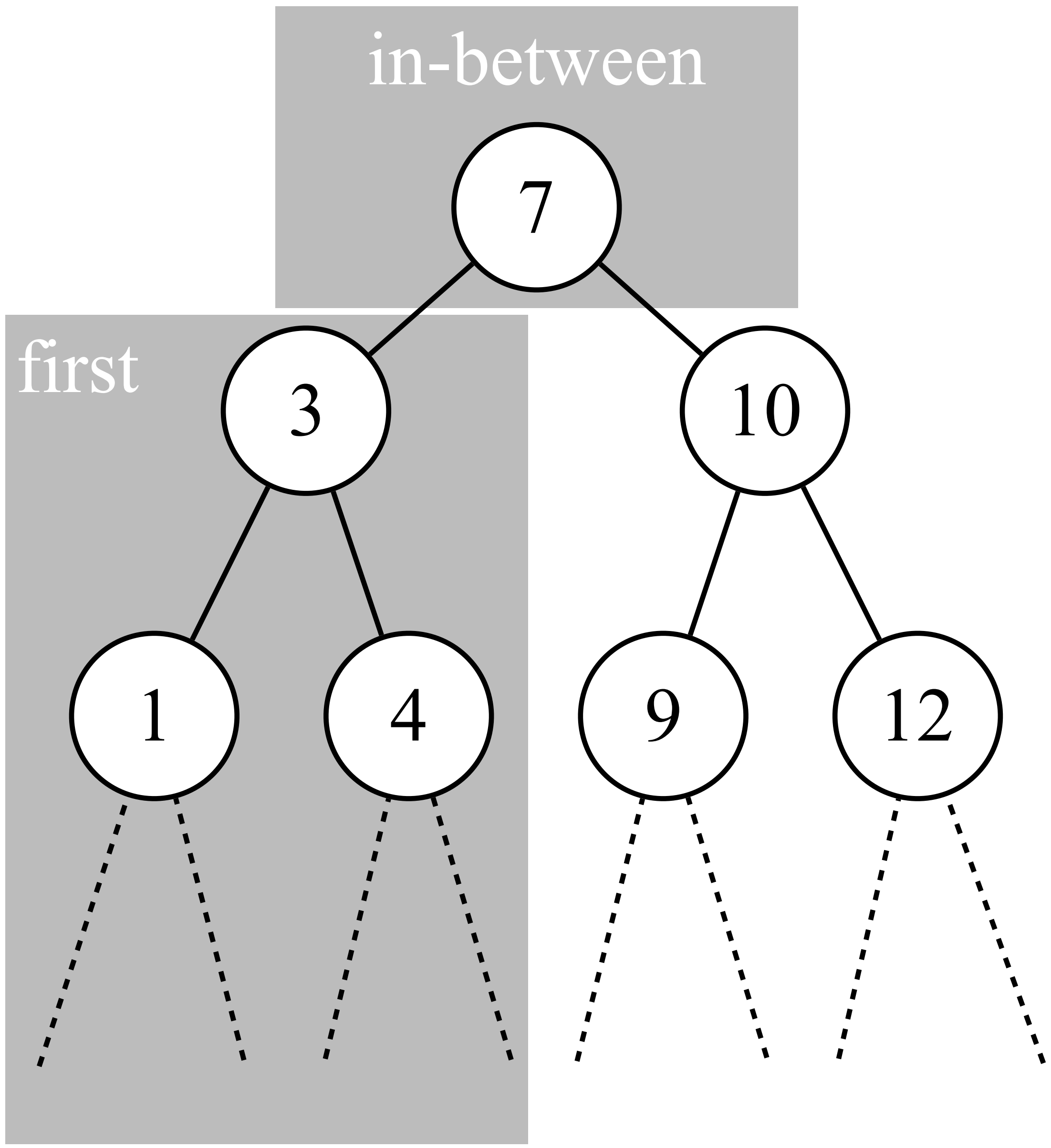


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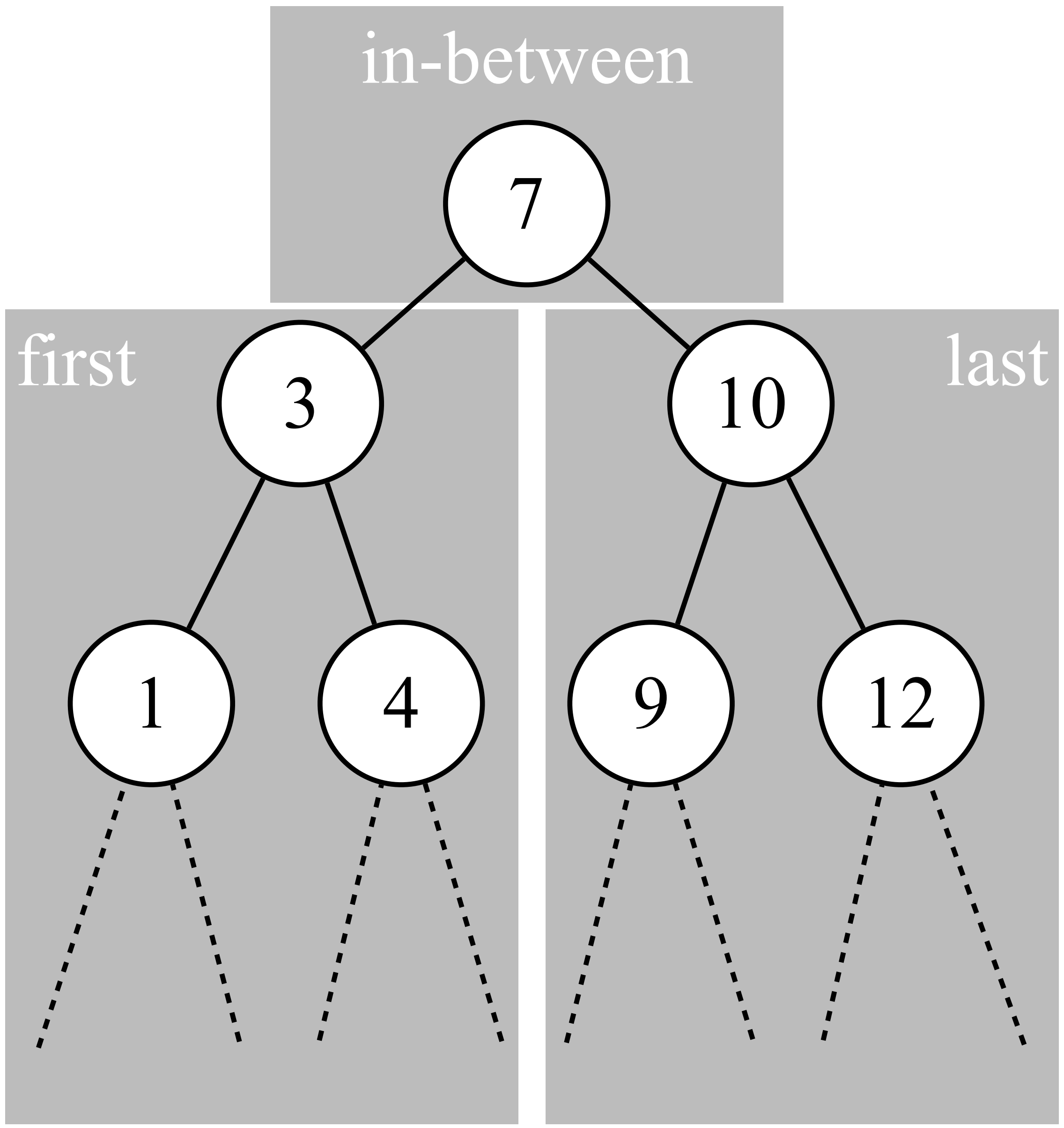


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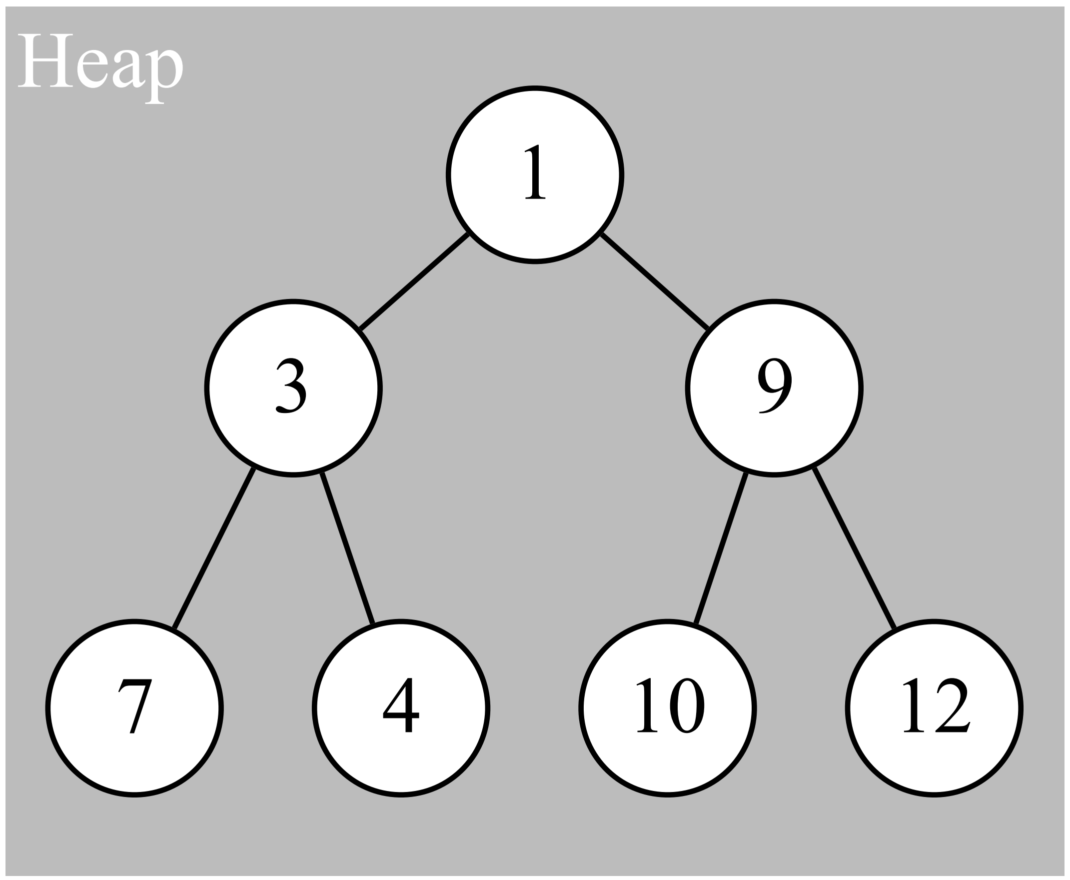
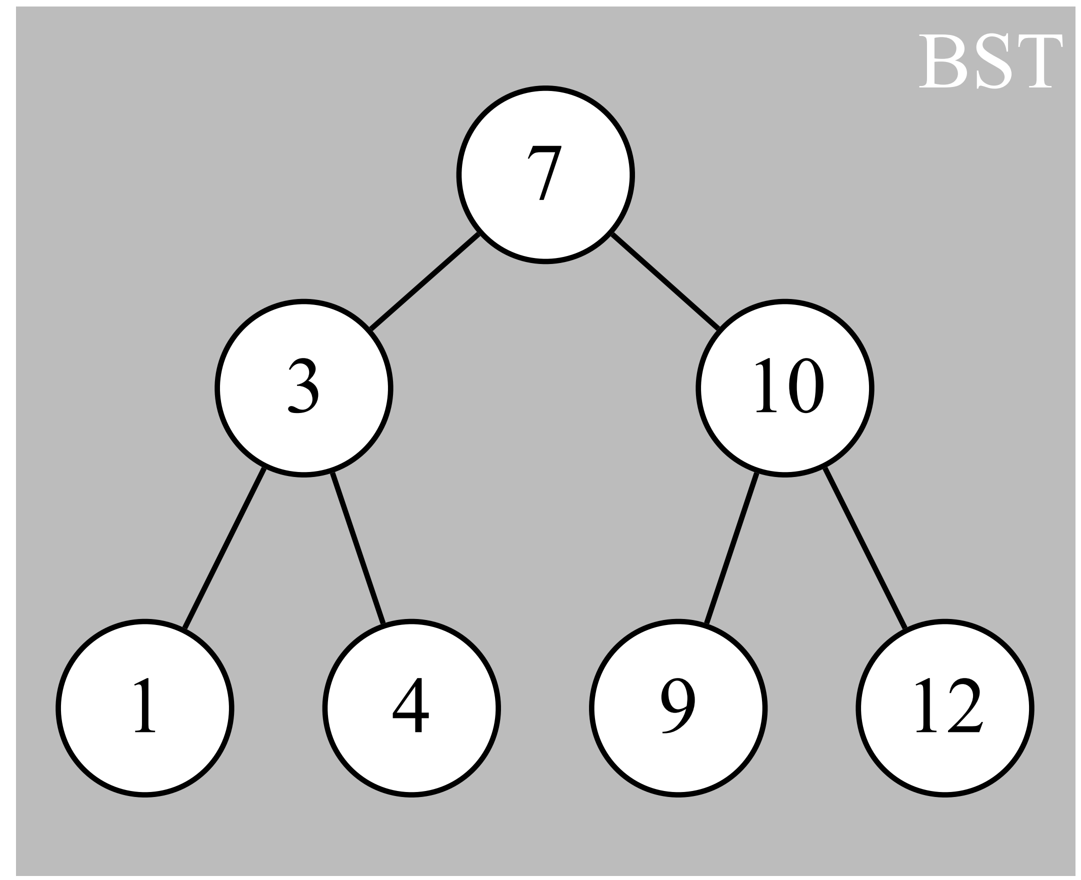
Q: Why does the above procedure output the keys in the sorted order?

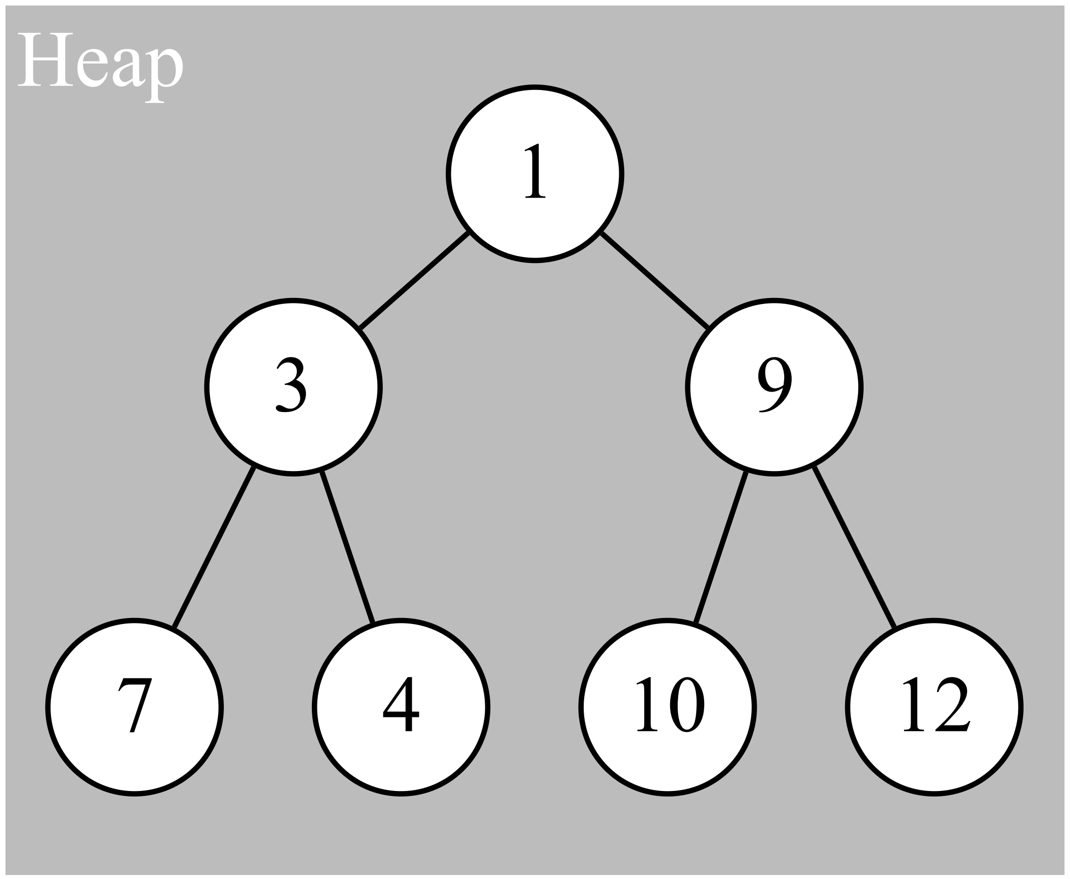
Differences between Heaps and BSTs

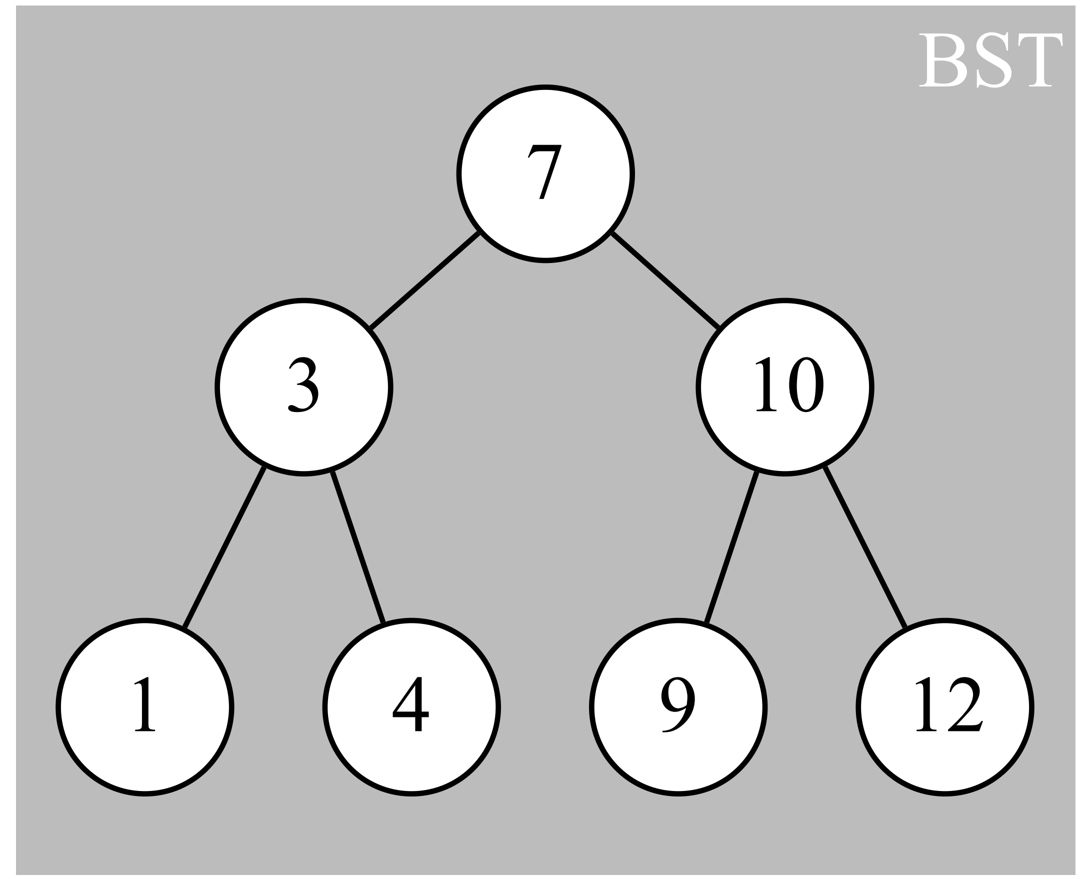




Q: Is it possible to output the keys in a heap in the sorted order by an O(n)-time traversal in the comparison-based model?

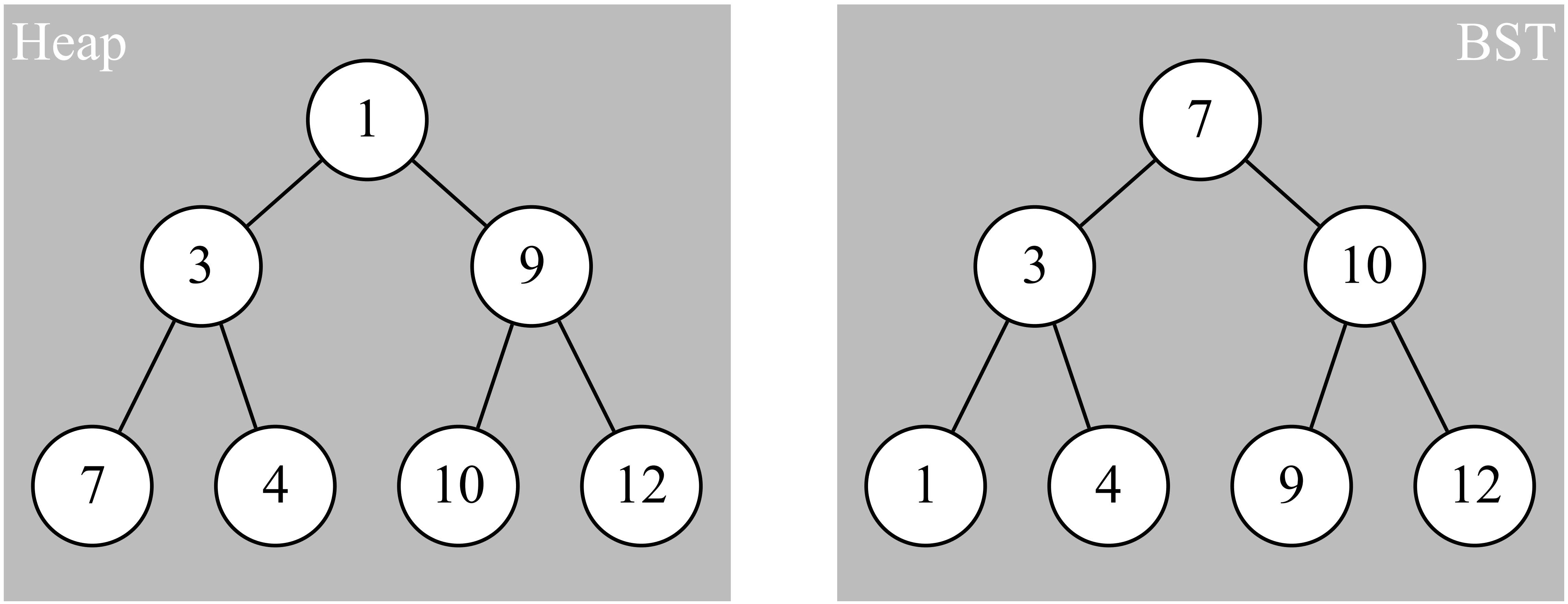
Differences between Heaps and BSTs





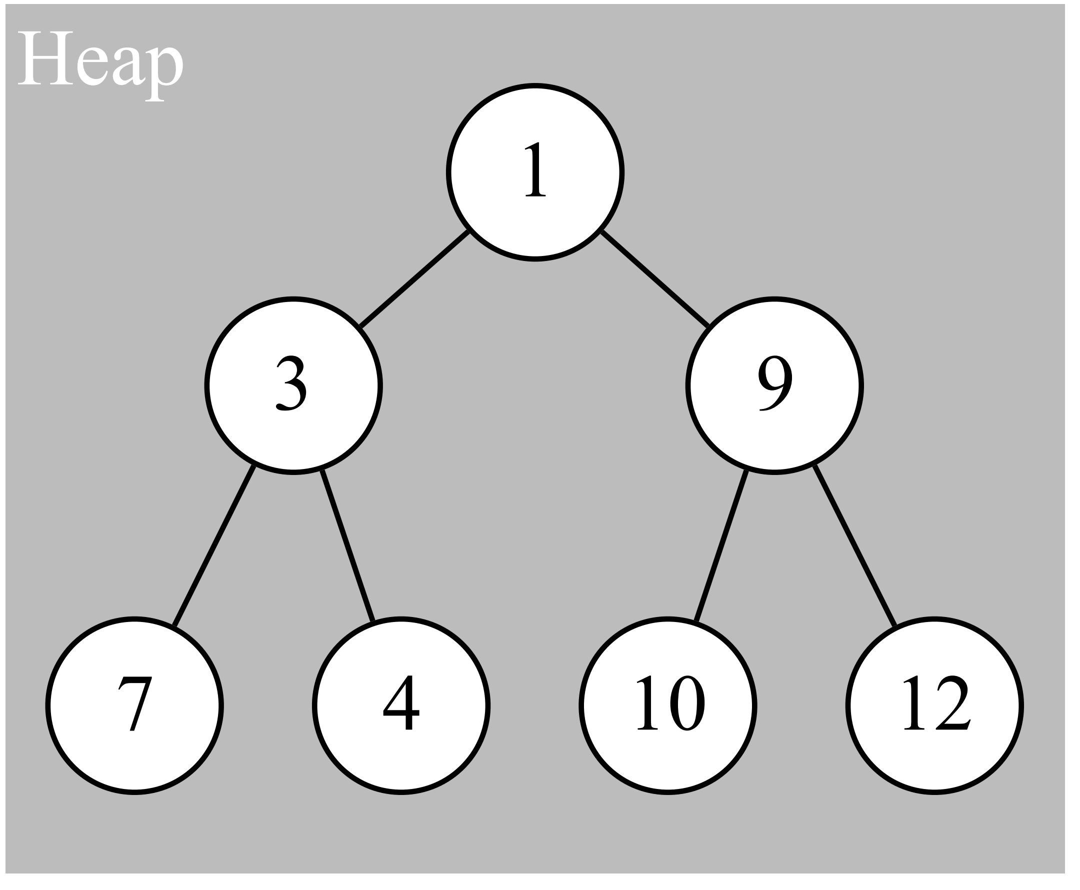
A: No. Otherwise build a heap (O(n) time) followed by the traversal (O(n) time) implements a sorting algorithm. →←

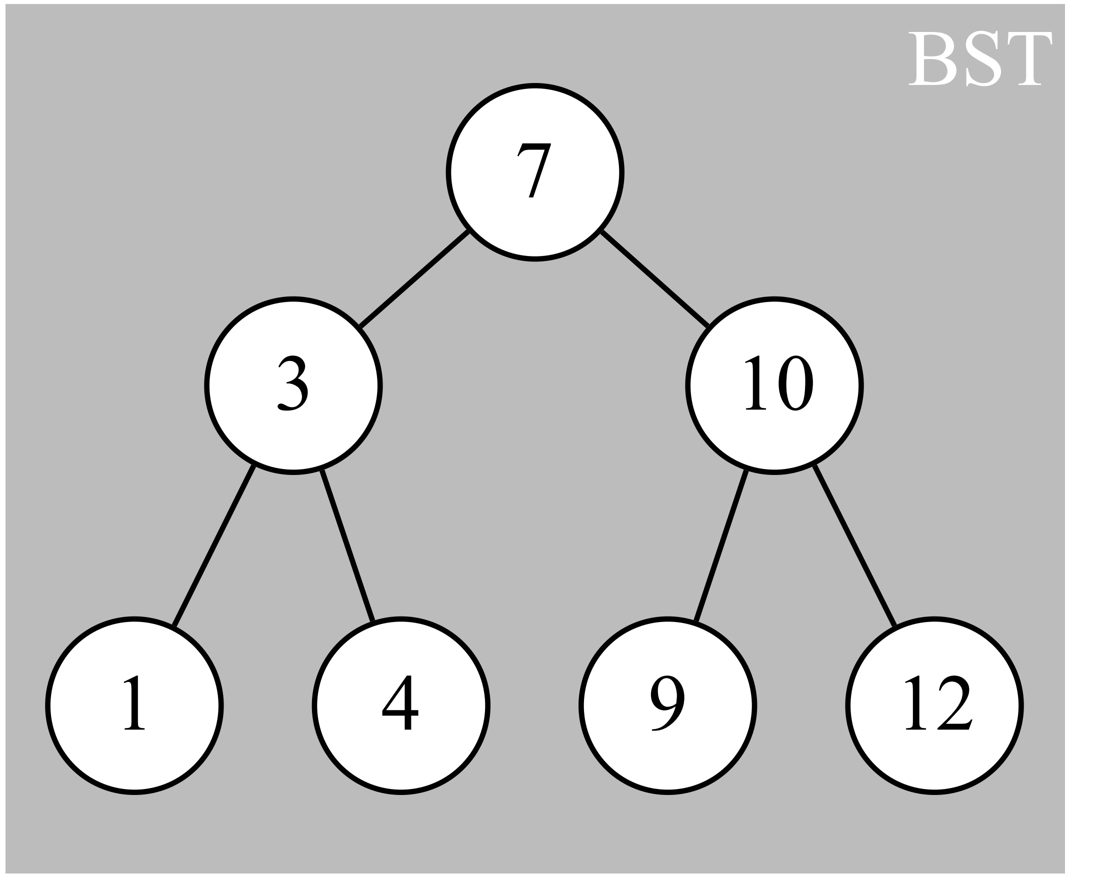
Differences between Heaps and BSTs



Q: Is it possible to build an n-node BST in $o(n \log n)$ time in the comparison-based model?

Differences between Heaps and BSTs





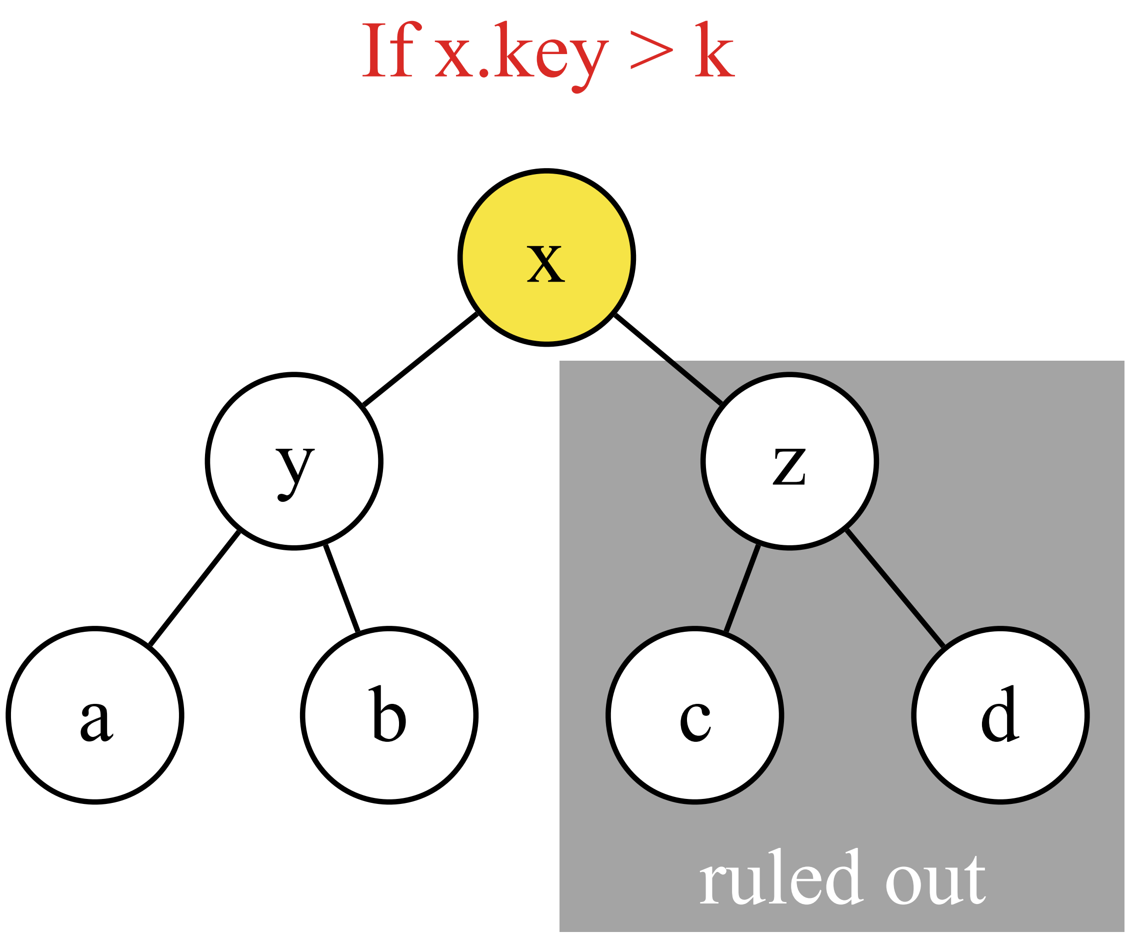
A: No. Otherwise build a BST ($o(n \log n)$ time) followed by the traversal ($O(n)$ time) implements a sorting algorithm. $\rightarrow\leftarrow$

Query Operations

Tree-Search(x, k)

Output "Yes" if some node y in the tree rooted at x has value k, or otherwise "No."

--- Example ---



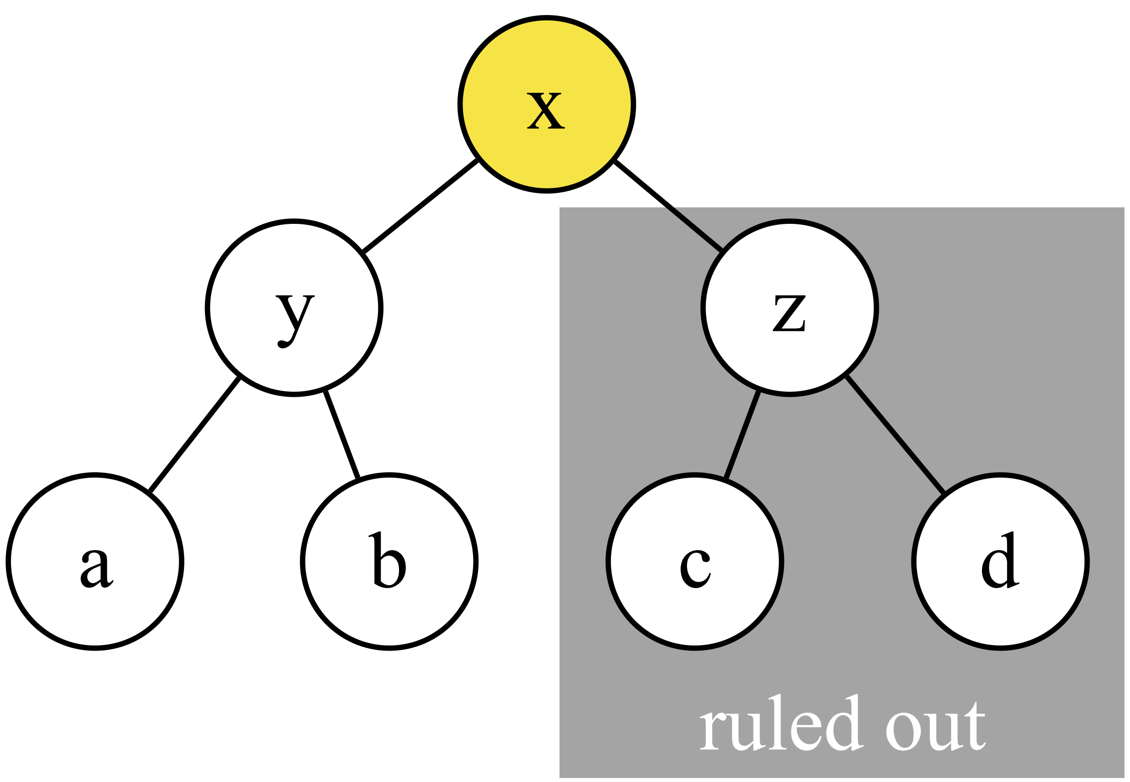

```
bool Tree-Search(x, k){

  if(x == Null) return No;

  if(x.key == k) return Yes;

  if(x.key < k)
    return Tree-Search(x.right, k);
  else
    return Tree-Search(x.left, k);
}
```

Tree-Search(x, k)

Output "Yes" if some node y in the tree rooted at x has value k, or otherwise "No."

--- Running Time ---

If x.key > k



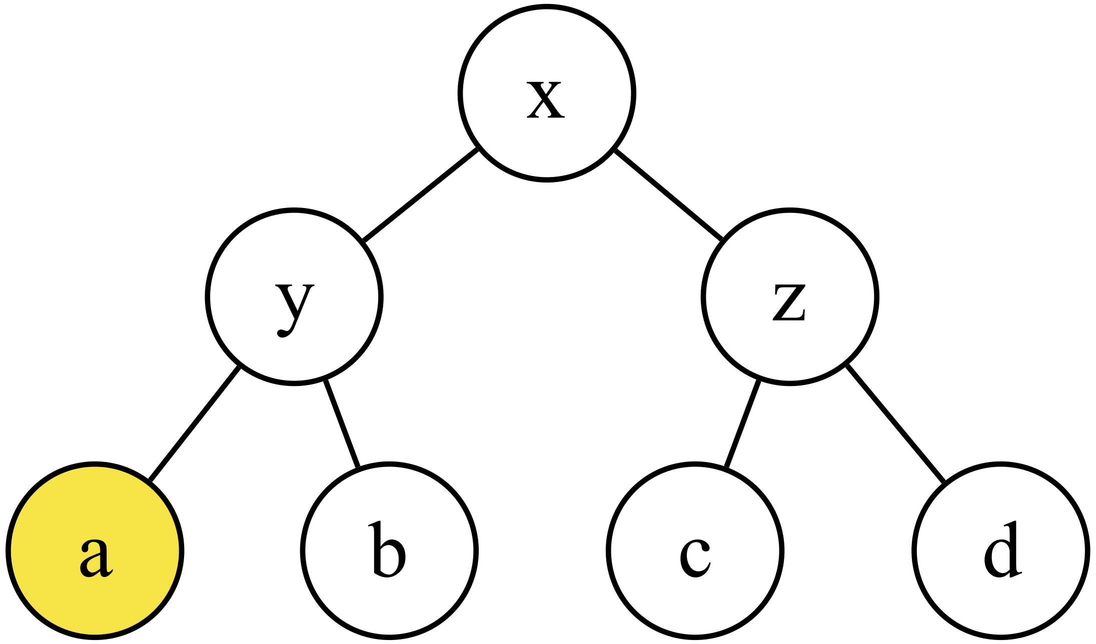
The runtime is O(height).

What is largest possible height of an n-node tree?

Tree-Min(x)

Output the minimum key value in the tree rooted at x has value k, and if no such a value exist, output "-∞."

--- Example ---




```
int Tree-Min(x){

  if(x == Null) return -∞;

  if(x.left != Null){
      return Tree-Min(x.left);
  }else{
      return x.key;
  }
}
```

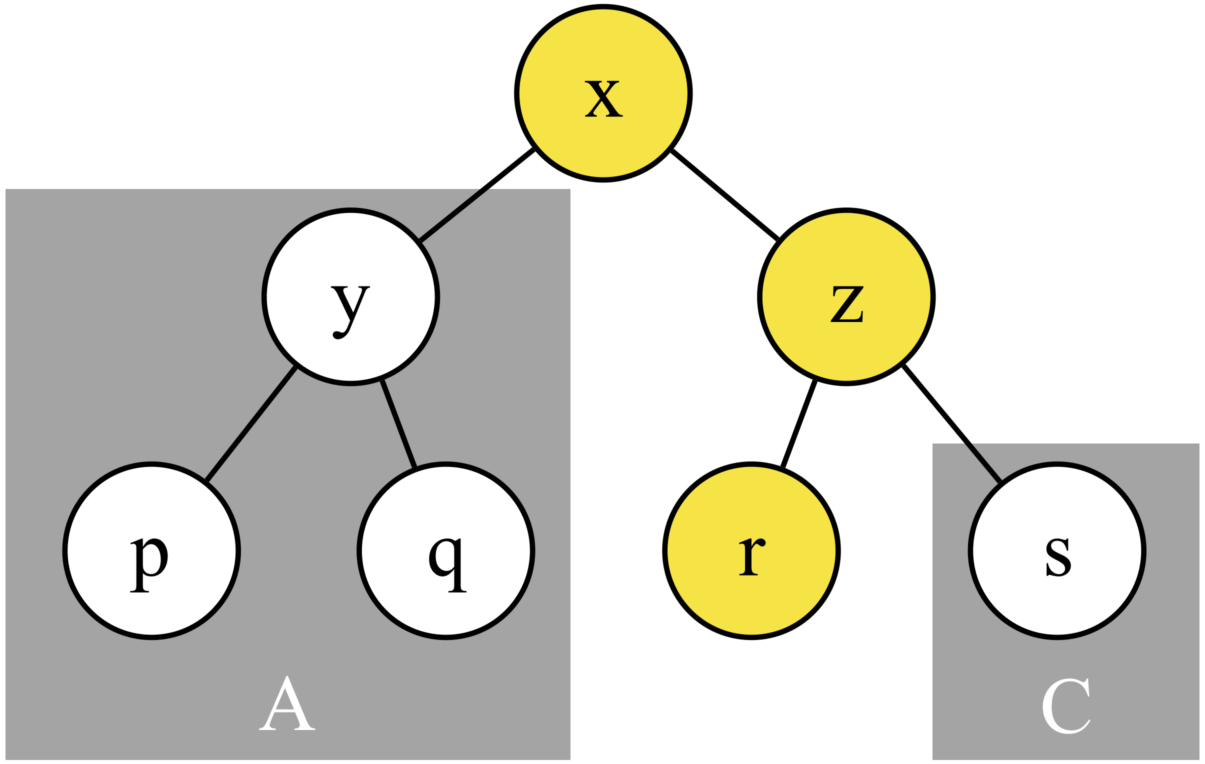
Exercise

Why is the implementation of Tree-Min correct?

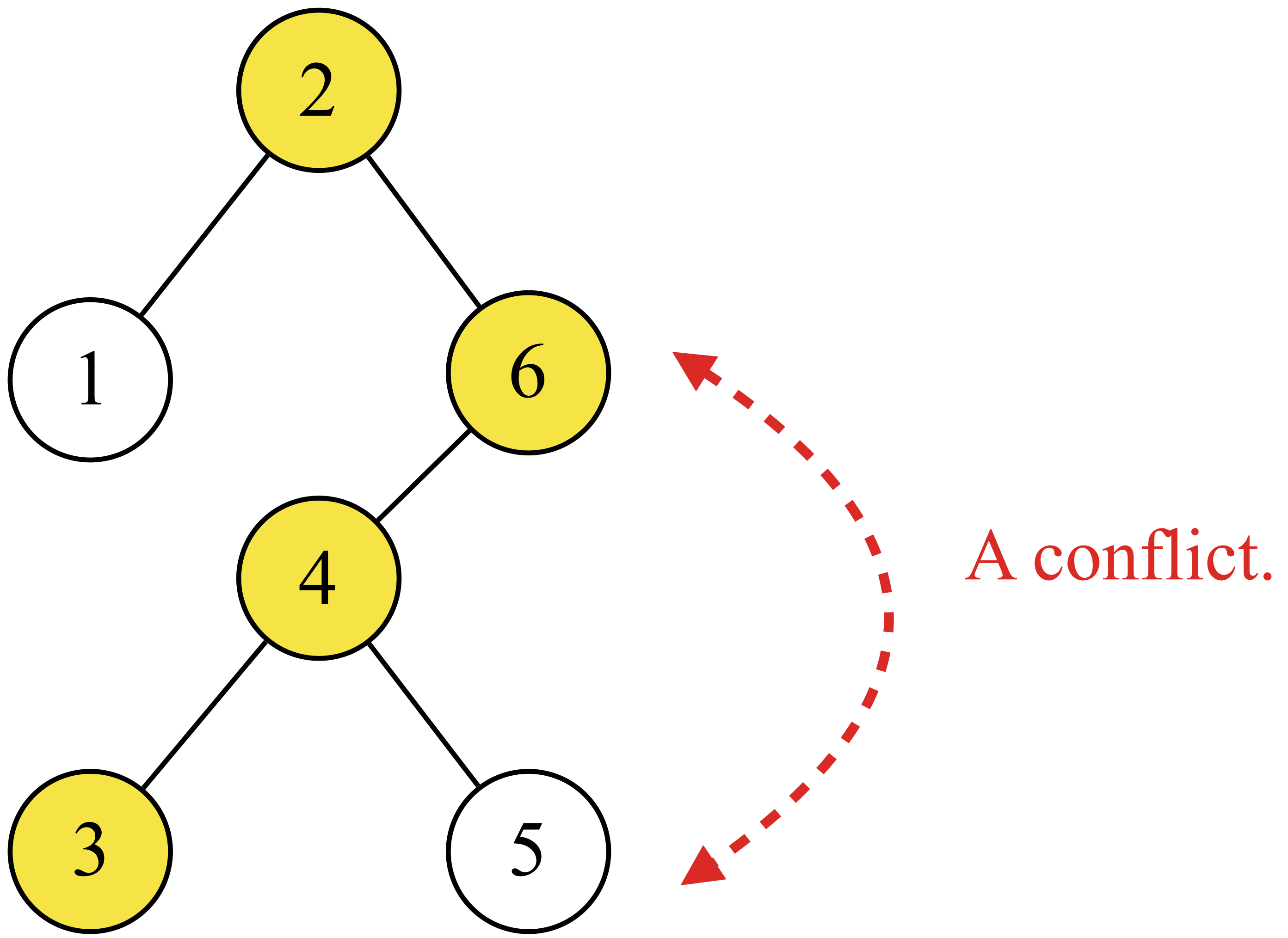
How to implement Tree-Max?

Exercise

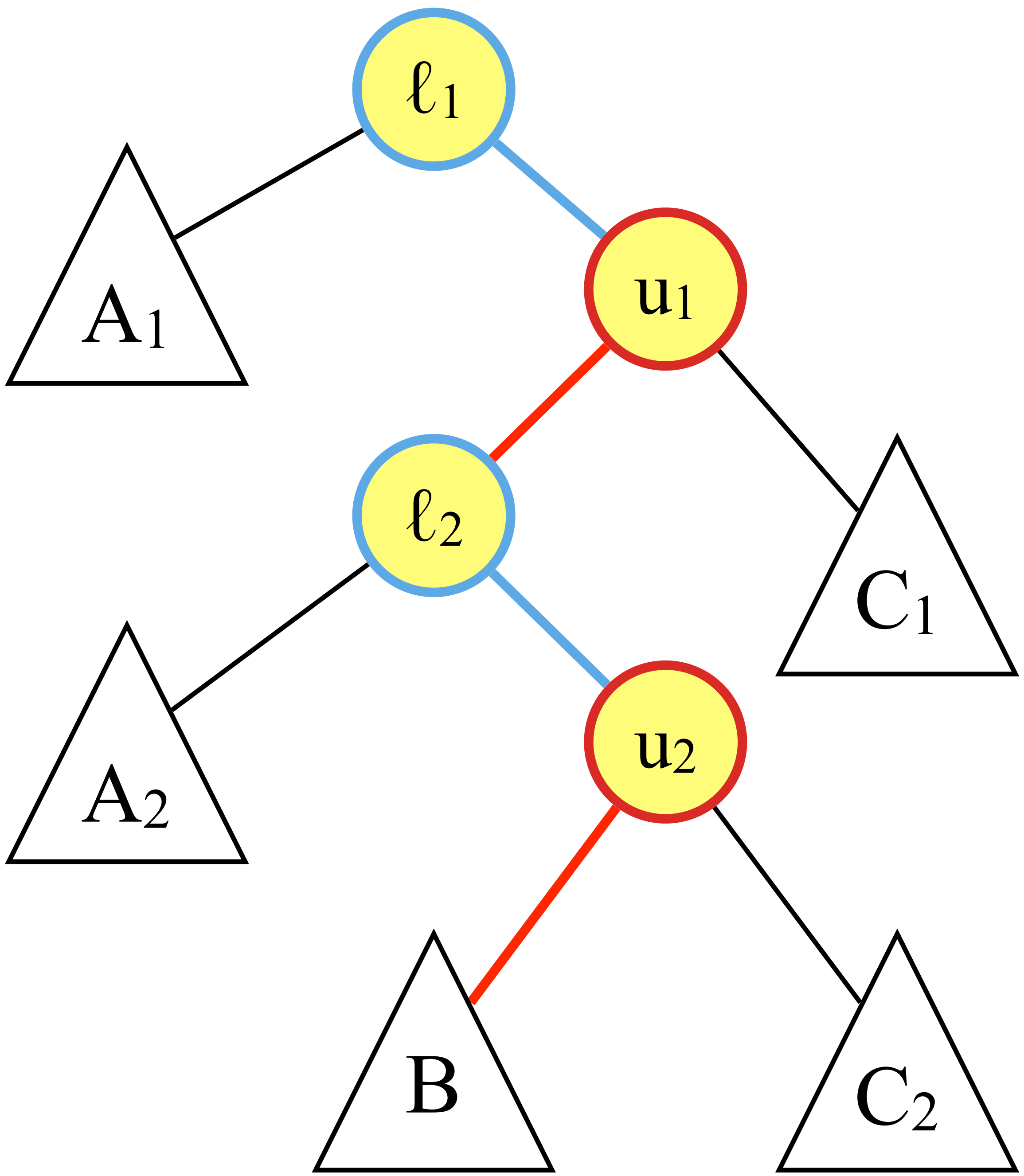
Let P be a root-to-leaf path of a binary search tree. Let A be the set of nodes to the left of P. Let B be the set of nodes on P. Let C be the set of nodes to the right of P. Prove or disprove that for any $a \in A$, $b \in B$, $c \in C$, we have $a.key \leq b.key \leq c.key$.



A Counter-Example



Property



$\max(\ell_1, \ell_2, \ldots) \leq B \leq \min(u_1, u_2, \ldots)$

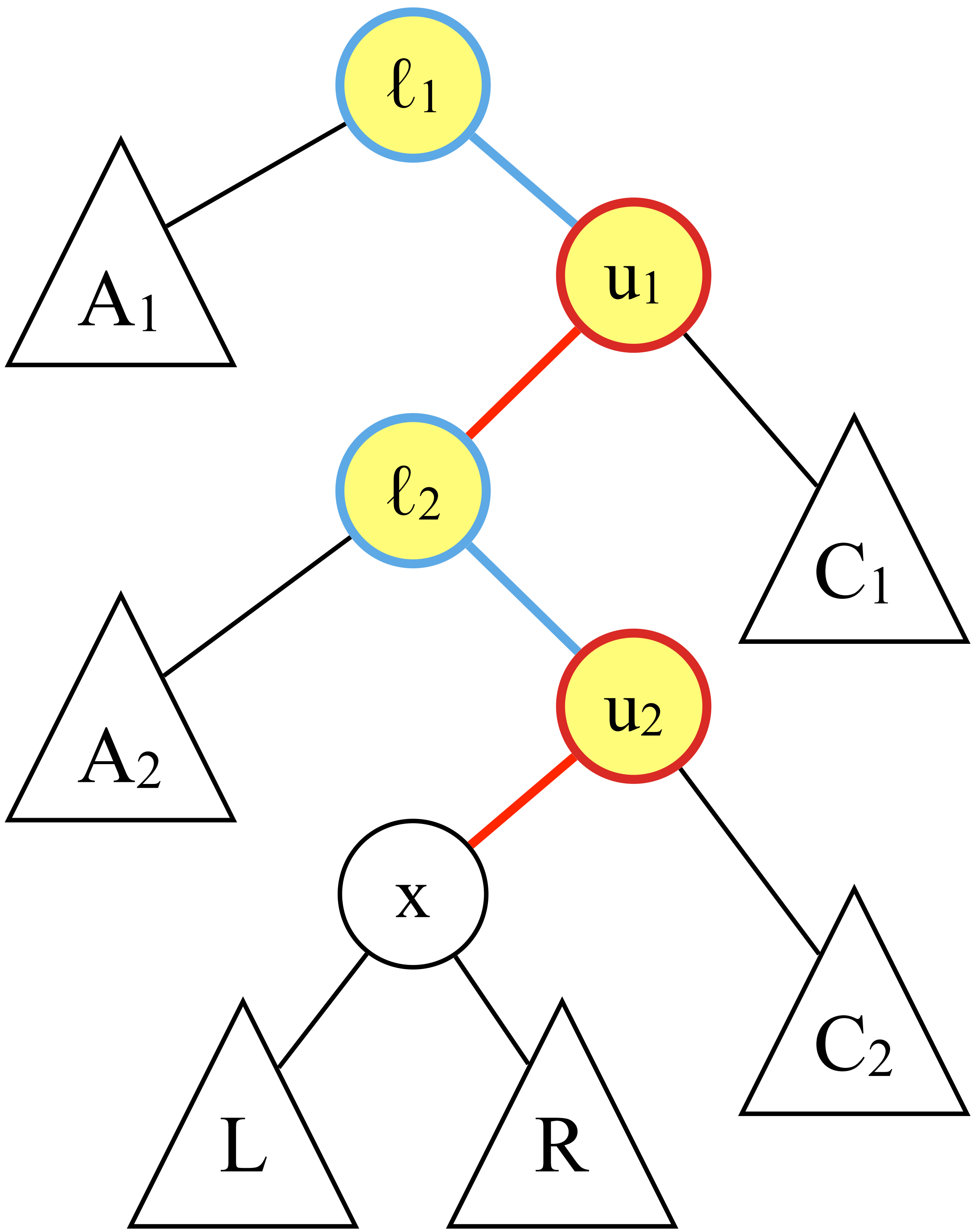
$\ell_1 \leq \ell_2 \leq \ldots \leq u_2 \leq u_1$

$A \leq \max(\ell_1, \ell_2, \ldots)$

$\min(u_1, u_2, \ldots) \leq C$

$A \leq B \leq C$

Tree-Successor(x)

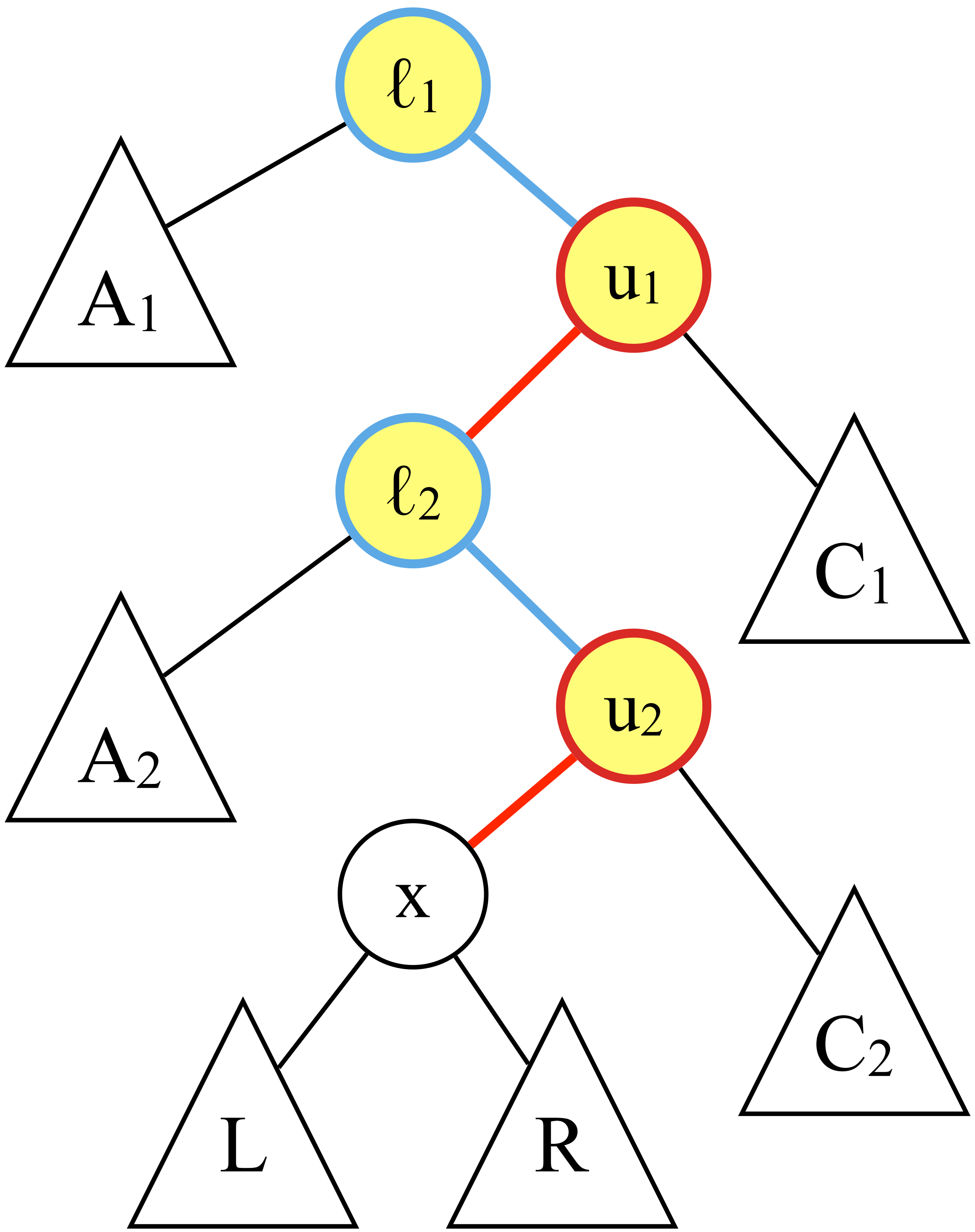


Output the smallest key ≥ x.key.

(Assume all keys are distinct, or tie-break arbitrarily.)

Tree-Successor(x) is in the tree rooted at x or some u_i. (Why?)

Tree-Predecessor(x)



Output the largest key ≤ x.key.

(Assume all keys are distinct, or tie-break arbitrarily.)

Tree-Predecessor(x) is in the tree rooted at x or some ℓ_i. (Why?)

Exercise

Give an implementation of Tree-successor(x) so that invoking the function k times can be done in O(k + height) time.

Optimal BST

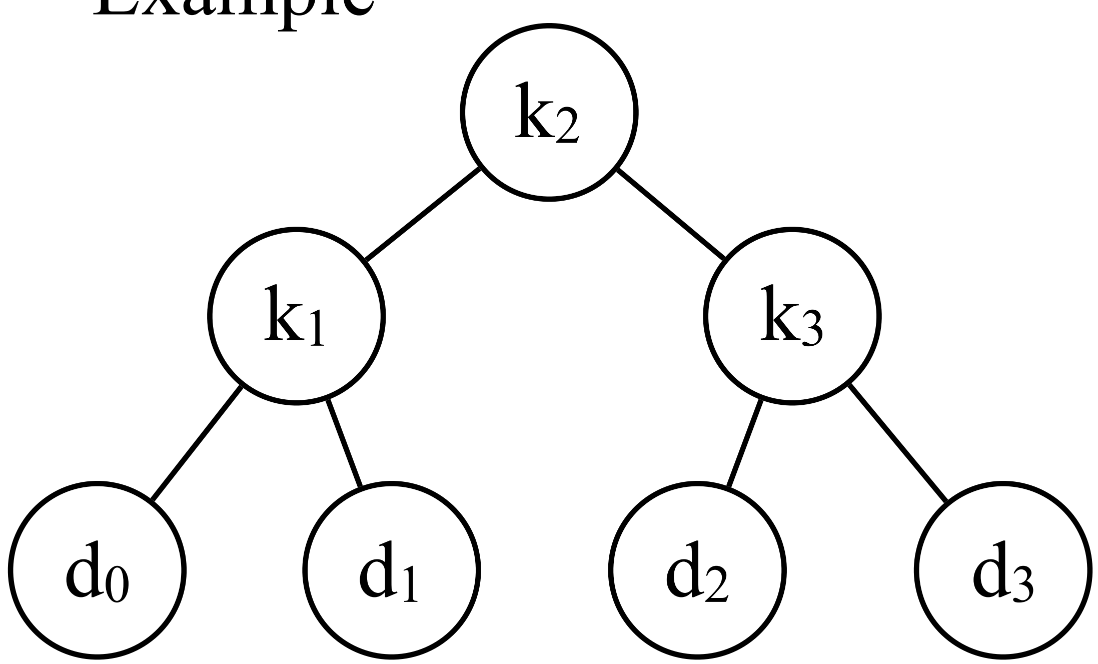
Optimal BST

Input: n keys $k_1 < k_2 < ... < k_n$ and a distribution of queried values:

- The queried value is k_i with probability p_i for each i in [1, n].
- The queried value is in-between (k_i, k_{i+1}) (assume it hits a dummy node) with probability q_i for each i in [1, n] where $k_0 = -\infty$ and $k_{n+1} = \infty$.

Output a BST so that the expected search cost is minimized.

--- Example ---



key	search cost
k_1	$2*p_1$
k_2	$1*p_2$
k_3	$2*p_3$
d_0	$3*q_0$
d_1	$3*q_1$
d_2	$3*q_2$
d_3	$3*q_3$

Optimal BST

Input: n keys $k_1 < k_2 < \ldots < k_n$ and a distribution of queried values:

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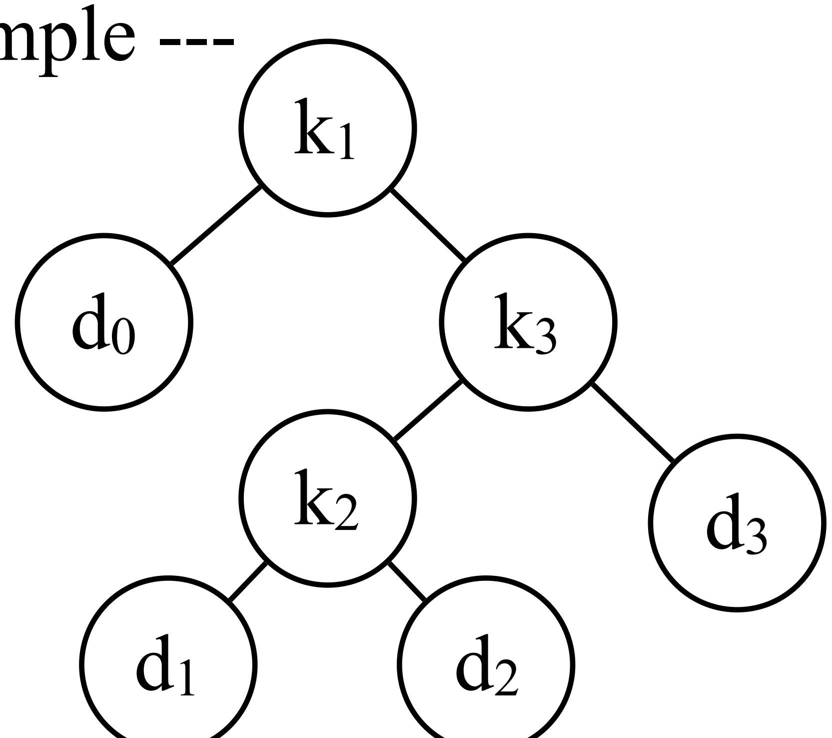
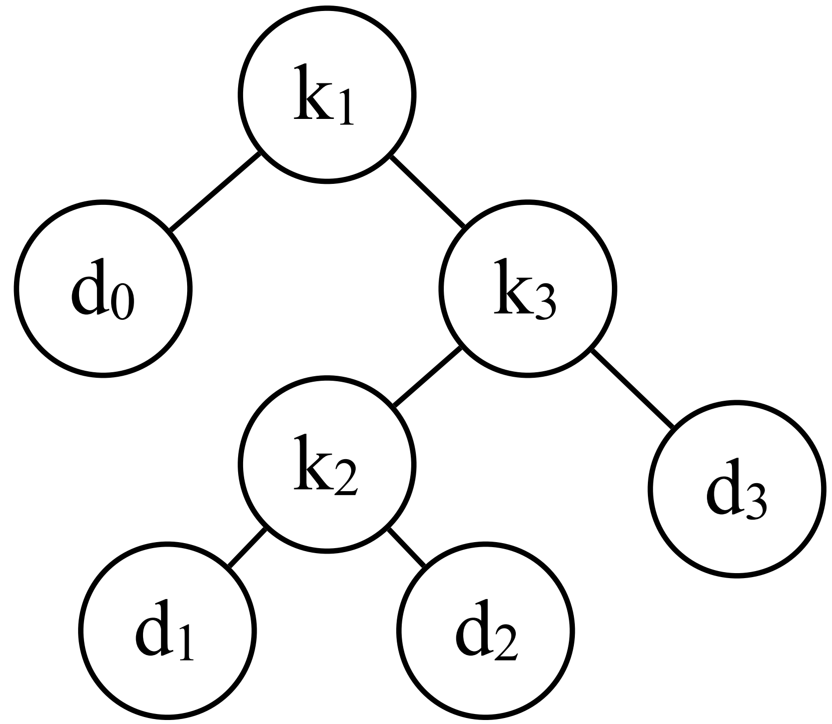


key	search cost
k_1	$1*p_1$
k_2	$3*p_2$
k_3	$2*p_3$
d_0	$2*q_0$
d_1	$4*q_1$
d_2	$4*q_2$
d_3	$3*q_3$

Optimal BST

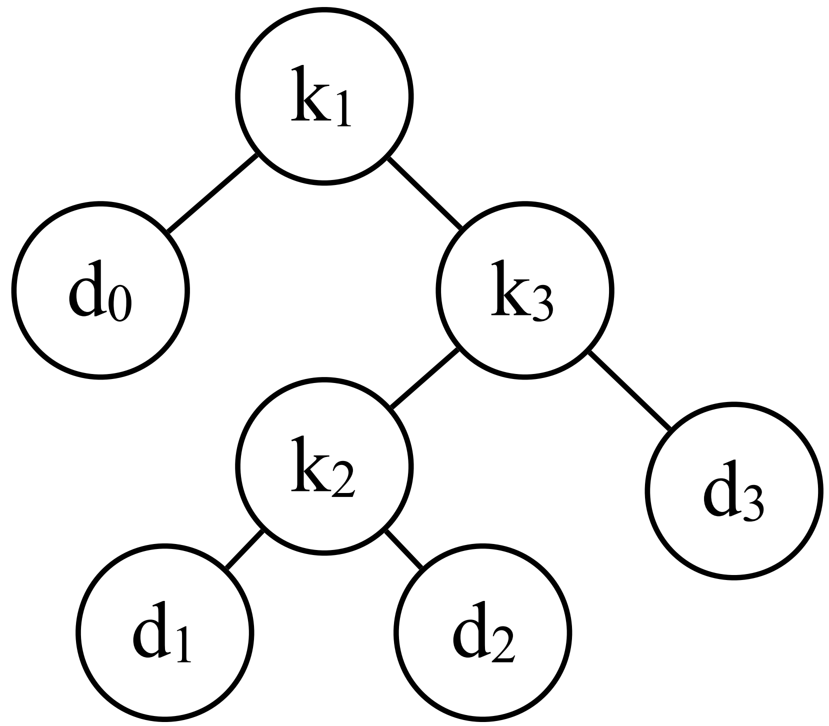


Claim. Every t-key subtree has t+1 dummy nodes.

count	a key node	a dummy node
parent	1 (except the root)	1
children	2	0

Let X be # dummy node.
By the double counting method on # (parent,child) pairs, we get:
$t - 1 + X = 2t + 0X$,
so $X = t+1$.

Optimal BST

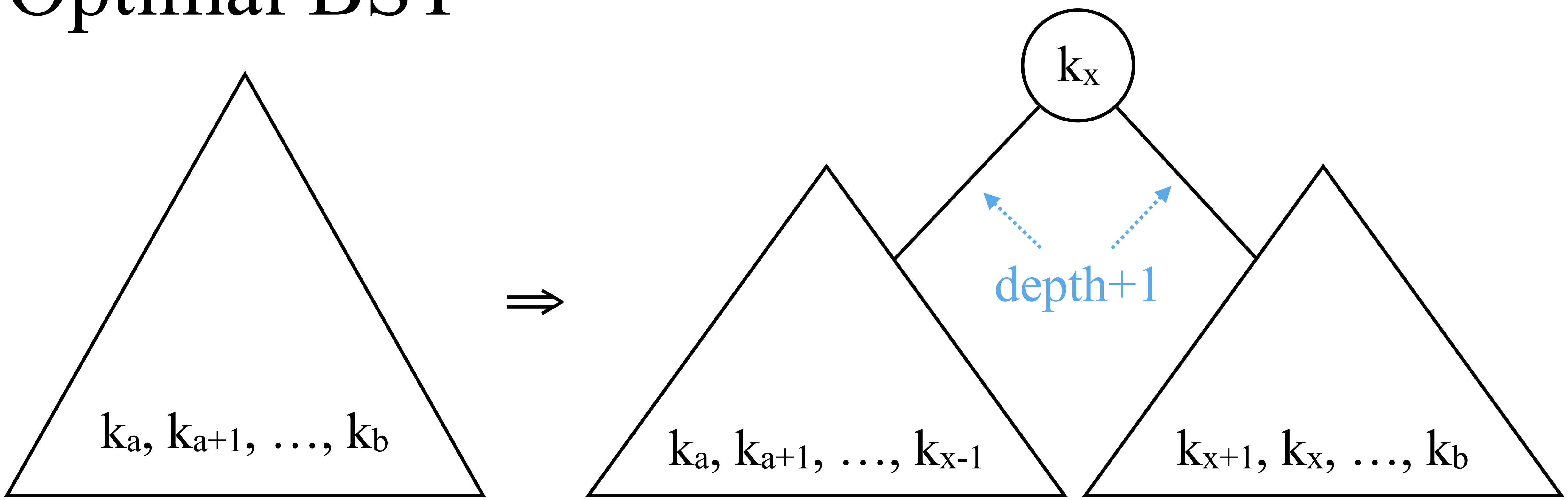


Claim. Every subtree that contains keys k_a, k_{a+1}, ..., k_b also contains dummy nodes d_{a-1}, d_a, ..., d_b.

Recall the property of BST: $A \leq \max(\ell_1, \ell_2, \ldots) \leq B \leq \min(u_1, u_2, \ldots) \leq C$,

so the keys and dummy nodes form a continous interval.

Optimal BST



Let $W(a, b)$ be the sum of probabilities associated with $k_a, k_{a+1}, \dots, k_b$ and $d_{a-1}, d_a, \dots, d_b$, and $Opt(a, b)$ be the smallest expected cost of an optimal BST that comprises the same keys and dummy nodes.

<u>Claim</u>. $Opt(a, b)$
$= \min_x Opt(a, x-1) + W(a, x-1) + p_x + Opt(x+1, b) + W(x+1, b)$
$= \min_x Opt(a, x-1) + Opt(x+1, b) + W(a, b)$.

Divide and Conquer

```
OBST(a, b){ // return the smallest expected cost of an optimal BST that comprises keys k_a, k_{a+1}, ..., k_b and dummy nodes d_{a-1}, d_a, ..., d_b

    if(a-1 == b) return q_b; // an empty subtree

    opt = ∞;

    for(x = a; x ≤ b; ++x){ // pick a root at k_x
        if(OBST(a, x-1)+OBST(x+1, b)+W(a, b) < opt){
            opt = OBST(a, x-1)+OBST(x+1, b)+W(a, b);
        }
    }
    return opt;
}
------
The initial call is OBST(1, n);
```

Dynamic Programming

```
OBST(a, b, sol[][]){ // return the smallest expected cost of an optimal
BST that comprises keys k_a, k_{a+1}, ..., k_b and dummy nodes d_{a-1}, d_a, ..., d_b

    if(a-1 == b) return q_b; // an empty subtree
    if(sol[a][b] < ∞) return sol[a][b];
    opt = ∞;

    for(x = a; x ≤ b; ++x){ // pick a root at k_x
        if(OBST(a, x-1, sol)+OBST(x+1, b, sol)+W(a, b) < opt){
            opt = OBST(a, x-1, sol)+OBST(x+1, b, sol)+W(a, b);
        }
    }
    return sol[a][b] = opt;
}
```

The initial call is OBST(1, n, sol = {∞});

Output the OBST

```
OBST(a, b, sol[][]){ // return the smallest expected cost of an optimal BST that comprises keys k_a, k_{a+1}, ..., k_b and dummy nodes d_{a-1}, d_a, ..., d_b

    if(a-1 == b) return q_b; // an empty subtree
    if(sol[a][b] < ∞) return sol[a][b];
    opt = ∞;

    for(x = a; x ≤ b; ++x){ // pick a root at k_x
        if(OBST(a, x-1, sol)+OBST(x+1, b, sol)+W(a, b) < opt){
            opt = OBST(a, x-1, sol)+OBST(x+1, b, sol)+W(a, b);
            root[a][b] = x; }
    }
    return sol[a][b] = opt;
}
------
The initial call is OBST(1, n, sol = {∞});
```

Running Time of OBST: $O(n^3)$.

Output the OBST

OutputOBST(a, b){ // output an OBST that comprises keys k_a, k_{a+1}, ..., k_b and dummy nodes d_{a-1}, d_a, ..., d_b (in in-order)

```
    if(a-1 == b){ output d_b; return; }

    OutputOBST(a, root[a][b]-1);

    output k_root[a][b];

    OutputOBST(root[a][b]+1, b);
}
```

Exercise

Knuth proves that there always exists an OBST so that

$$\text{root}[i][j-1] \le \text{root}[i][j] \le \text{root}[i+1][j] \text{ for every } i, j \text{ in } [1, n].$$

Use this fact to compute OBST in $O(n^2)$ time.

Running Time of OBST:
$O(n^2)$.