DCP1206 Fall 2019: Probability **(Due: 2019/12/27 in class)**

Homework 6, Part I: Law of Large Numbers and Central Limit Theorem

Problem 1 (Convergence in Probability) (12+12=24 points)

A sequence of random variables $X_1, X_2, \cdots$ is said to converge to a number c **in the mean square**, if

$$\lim_{n\to\infty} E[(X_n - c)^2] = 0.$$

(a) Show that convergence in the mean square implies convergence in probability. (Hint: For every $\varepsilon > 0$, consider $P(|X_n - c| \geq \varepsilon)$ and use Markov inequality)

(b) Give an example that shows that convergence in probability does not imply convergence in the mean square.

Problem 2 (Strong Law of Large Numbers) (12 points)

Consider two sequences of random variables $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ defined on the same sample space. Suppose that X_n converges to a and Y_n converges to b, almost surely. Show that $X_n + Y_n$ converges to $a + b$, almost surely. (Hint: Consider two events A, B defined as $A = \{\omega : X_n(\omega) \text{ does not converge to } a\}$ and $B = \{\omega : Y_n(\omega) \text{ does not converge to } b\}$)

Problem 3 (Normal Approximation) (12 points)

Let $\bar{X}$ denote the mean of a random sample of size 28 from a distribution with mean 1 and variance 4. Approximate $P(0.95 < \bar{X} < 1.05)$. (Hint: Central Limit Theorem)

Problem 4 (Almost-Sure Convergence) (12 points)

Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables drawn from a continuous uniform distribution on (a, b) with $a < b$. For $n \geq 1$, define $Y_n := \max(X_1, X_2, \cdots, X_n)$. Show that Y_n converges to a constant b, almost surely. (Hint: Slides of Lecture 23. Please carefully justify every step of your proof)