Problem 1:

(a) The PDF of a normal r.v. must have the form of

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right), \quad \forall x \in \mathbb{R}$$

Next, we rearrange $f(x)$ as

$$f(x) = \sqrt{K} \exp\left(-K^2\left(x+\frac{1}{K}\right)^2\right)$$

(Note that $\sqrt{K}$ suggests that $K \geq 0$)

$$= \sqrt{K} \exp\left(-\frac{\left(x+\frac{1}{K}\right)^2}{2\cdot\left(\frac{1}{\sqrt{2}K}\right)^2}\right)$$

$$\Rightarrow \begin{cases} \sqrt{K} = \dfrac{1}{\sigma\sqrt{2\pi}} \Rightarrow \sigma = \dfrac{1}{\sqrt{2\pi K}} & \text{—— (1)} \\ \dfrac{1}{\sqrt{2}K} = \sigma & \text{—— (2)} \end{cases}$$

By (1)-(2), we have $\dfrac{1}{\sqrt{2}K} = \dfrac{1}{\sqrt{2\pi K}} \Rightarrow \underline{K = \pi}$ #

(b)

$$P(Z<0 \mid X=+1) = P(Y<-1 \mid X=+1)$$

$$= P(Y<-1)$$

X, Y are independent

$$= \Phi\left(\frac{-1}{\sigma}\right) = 1-\Phi\left(\frac{1}{\sigma}\right)$$

Similarly, $P(Z \geq 0 \mid X=-1) = P(Y \geq 1 \mid X=-1)$

$$= P(Y \geq 1)$$

X, Y are independent

$$= 1-\Phi\left(\frac{1}{\sigma}\right)$$

Therefore, the error probability is $1-\Phi\left(\frac{1}{\sigma}\right)$ for both $X=+1$ and $X=-1$.

Hence, the overall error probability is also $\underline{1-\Phi\left(\frac{1}{\sigma}\right)}$ ✱

Problem 2: $Y = aX + b$, is a linear transformation of X, and $X \sim Exp(\lambda)$

CDF:

$$F_Y(t) = P(Y \leq t) = P(aX + b \leq t)$$

Note that the CDF of X is

$$F_X(t) = \begin{cases} 1 - e^{-\lambda t}, & \text{if } t \geq 0 \\ 0, & \text{else.} \end{cases}$$

To find $F_Y(t)$, we need to discuss two cases, namely $a > 0$ and $a < 0$.

① $a > 0$:

$$F_Y(t) = P(aX + b \leq t) = P\left(X \leq \frac{t-b}{a}\right) = \begin{cases} 1 - e^{-\lambda\left(\frac{t-b}{a}\right)}, & \text{if } t \geq b \\ 0, & \text{else} \end{cases}$$

② $a < 0$:

$$F_Y(t) = P(aX + b \leq t) = P\left(X \geq \frac{t-b}{a}\right) = \begin{cases} e^{-\lambda\left(\frac{t-b}{a}\right)}, & \text{if } t \leq b \\ 1, & \text{else} \end{cases}$$

since $a < 0$

From the above discussion, we know for Y to be an exponential r.v., we need to have $a > 0$ and $b = 0$.

(Conti.)

PDF:

As it is assumed that the PDF of Y is continuous, we have $f_Y(t) = F_Y'(t)$.

Again, we consider the following two cases:

① $a>0$:

$$f_Y(t) = F_Y'(t) = \begin{cases} \frac{\lambda}{a} e^{-\lambda\left(\frac{t-b}{a}\right)} , & \text{if } t>b \\ 0 , & \text{else} \end{cases}$$

② $a<0$:

$$f_Y(t) = F_Y'(t) = \begin{cases} -\frac{\lambda}{a} e^{-\lambda\left(\frac{t-b}{a}\right)} , & \text{if } t<b \\ 0 , & \text{else} \end{cases}$$

Problem 3 :

$$h(X) := -\int_{f(x)>0} f(x) \cdot \ln(f(x))\, dx$$

As $X \sim N(0, \sigma^2)$, we know $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$, $\forall x \in \mathbb{R}$.

Then, $$h(X) = -\int_{-\infty}^{+\infty} f(x) \cdot \left[\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{x^2}{2\sigma^2}\right] dx$$

a constant

$$= -\left[\underbrace{\int_{-\infty}^{+\infty} \ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) f(x)\, dx}_{= \ln\frac{1}{\sigma\sqrt{2\pi}}} - \frac{1}{2\sigma^2}\underbrace{\int_{-\infty}^{+\infty} x^2 f(x)\, dx}_{= E[X^2] = Var[X] + (E[X])^2 = \sigma^2}\right]$$

$$= \ln\left(\sigma\sqrt{2\pi}\right) + \frac{1}{2}$$

$$= \frac{1}{2}\ln\left(2\pi e \sigma^2\right).$$

#

Problem 4 :

(a). $X \sim N(0,1) \Rightarrow$ the PDF of X: $f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$, $\forall x \in \mathbb{R}$.

Since $Var[X] = E[X^2] - (E[X])^2$ and $E[X] = 0$,

all we need is to show that $E[X^2] = 1$.

Recall: Integration by parts

$$\int u\,dv = uv - \int v\,du$$

$$\frac{d\left(\frac{-1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\right)}{dx} = \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 \cdot f_X(x)\,dx$$

$$= \int_{-\infty}^{+\infty} x \cdot \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

$$= \underbrace{x \cdot -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \Bigg|_{-\infty}^{+\infty}}_{=0} - \int_{-\infty}^{+\infty} -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

$$= \int_{-\infty}^{+\infty} \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)}_{\text{PDF of } X} dx = 1$$

#

(b). $X \sim Exp(\lambda)$. then $f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{else} \end{cases}$

$$E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x}\, dx$$

$$= \underbrace{x^2 \cdot (-e^{-\lambda x}) \Big|_0^\infty}_{=0} - \int_0^\infty -e^{-\lambda x} \cdot 2x\, dx$$

$$= \frac{2}{\lambda} \underbrace{\boxed{\int_0^\infty \lambda e^{-\lambda x} \cdot x\, dx}}_{E[X] = \frac{1}{\lambda}}$$

$$= \frac{2}{\lambda^2}.$$

Recall: integration by parts

$$\int u\,dv = uv - \int v\,du$$

$$Var[X] = E[X^2] - (E[X])^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{1}{\lambda^2}$$

#

Problem 5:

(a).

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{+\infty} C \cdot \exp(-|x|)\,dx$$

$$= 2 \cdot \int_0^{\infty} C \cdot \exp(-x)\,dx$$

$$= 2C \cdot \left(-\exp(-x)\right)\Big|_0^{\infty} = 2C = 1.$$

Hence, $C = \frac{1}{2}$ #

(b). $$E[X^{2n}] = \int_{-\infty}^{+\infty} x^{2n} \cdot \frac{1}{2} \exp(-|x|)\,dx$$

$$= \cancel{2} \cdot \int_0^{\infty} x^{2n} \cdot \cancel{\tfrac{1}{2}} \exp(-x)\,dx$$

$$= \int_0^{\infty} x^{2n} \exp(-x)\,dx = (2n)!$$

↑ by Lemma 1 in the next page

Lemma 1: $\int_0^\infty x^m \exp(-x)\,dx = m!$, for all $m \in \mathbb{N}$

Pf: Prove by induction

m=1: $\int_0^\infty x \exp(-x)\,dx \underset{\substack{\uparrow \\ \text{integration} \\ \text{by parts}}}{=} \underbrace{x \cdot -\exp(-x)\Big|_0^\infty}_{= 0} - \int_0^\infty 1 \cdot -\exp(-x)\,dx$

$$= \int_0^\infty \exp(-x)\,dx = -\exp(-x)\Big|_0^\infty = 1.$$

Assume for m=k, we have $\int_0^\infty x^k \exp(-x)\,dx = k!$

Then, for m=k+1,

$$\int_0^\infty x^{k+1} \exp(-x)\,dx \underset{\substack{\uparrow \\ \text{integration} \\ \text{by parts}}}{=} \overbrace{x^{k+1} \cdot -\exp(-x)\Big|_0^\infty}^{= 0} - \int_0^\infty (k+1) \cdot x^k \cdot -\exp(-x)\,dx$$

$$= (k+1) \cdot \int_0^\infty x^k \cdot \exp(-x)\,dx = (k+1) \cdot k! = (k+1)!$$

(Conti).

As we found that $E[X^{2n}] = (2n)!$, we know the existence of $E[|X^{2n+2}|]$ implies that $E[|X^{2n+1}|]$ also exists, for all $n \in \mathbb{N}$.

Now, we can use the symmetry and show that for any $n \in \mathbb{N}$,

$$E[X^{2n+1}] = \int_{-\infty}^{+\infty} x^{2n+1} \cdot \frac{1}{2} \exp(-|x|)\, dx$$

$$= \int_{0}^{\infty} x^{2n+1} \cdot \frac{1}{2} \exp(-x)\, dx + \int_{-\infty}^{0} x^{2n+1} \cdot \frac{1}{2} \cdot \exp(x)\, dx$$

$$= 0$$

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