DCP1206 Fall 2019: Probability **(Due: 2019/10/02 in class)**

Homework 1: Probability Model, Conditioning, and Combinatorics

Problem 1 (Probability Axioms) (5+5=10 points)

(a) Use induction to prove the union bound, i.e. for any sequence of events $A_1, A_2 \cdots, A_N$, we have

$$P\Big(\bigcup_{n=1}^{N} A_n\Big) \leq \sum_{n=1}^{N} P(A_n).$$

(b) Consider an experiment with a sample space $\Omega = \{1,2,3,4,5\}$. Suppose we know $P(\{1,5\}) = 0.3$, $P(\{1,2,4\}) = 0.45$, and $P(\{3,4\}) = 0.2$. Among all the possible valid probability assignments, what is the maximum possible value of $P(\{2,3,5\})$? Please explain your answer.

Problem 2 (Set Operations) (8+6=14 points)

(a) For any sequence of sets $S_1, S_2, \cdots$, prove the general form of the De Morgan's laws: (i) $\Big(\bigcup_{n=1}^{\infty} S_n\Big)^c = \bigcap_{n=1}^{\infty} S_n^c$; and (ii) $\Big(\bigcap_{n=1}^{\infty} S_n\Big)^c = \bigcup_{n=1}^{\infty} S_n^c$.

(b) Let $\{A_1, A_2, \cdots\}$ be a sequence of events of a sample space Ω. Find a sequence of mutually exclusive events $\{B_1, B_2, \cdots\}$ such that $\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$, for all $n \geq 1$. Please justify your answer.

Problem 3 (Countable and Uncountable Sets) (6+6=12 points)

(a) Show that there are *uncountably* infinite many real numbers in the interval $(0,1)$. (Hint: Prove this by contradiction. Specifically, (i) assume that there are countably infinite real numbers in $(0,1)$ and denote them as $x_1, x_2, x_3, \cdots$; (ii) express each real number x_1 between 0 and 1 in decimal expansion; (iii) construct a number y whose digits are either 1 or 2. Can you find a way to choose 1 or 2 such that y is different from all the x_is?)

(b) Is the set of all irrational numbers in $(0,1)$ countably infinite? Please explain your answer.

Problem 4 (Continuity of Probability Function and Probability Axioms) (8+8=16 points)

Consider an infinite sequence of coin tosses. The probability of having a head at the i-th toss is p_i, with $p_i \in [0,1]$ (Note: different tosses may not be independent and can potentially have different head probabilities). We use I to denote the event of having infinite number of heads.

(a) Suppose $\sum_{i=1}^{\infty} p_i$ is finite. Show that $P(I) = 0$. (Hint: Define $A_n :=$ {the n-th toss is a head}. Then, $B_k := \bigcup_{n=k}^{\infty} A_n$ is the event that there is at least one head after the k-th toss (including the k-th toss). The event I (i.e. we observe infinitely many heads) is equivalent to saying that B_k occurs, for every $k = 1,2,3,\cdots$. Therefore, $I = \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n$. Consider the continuity of probability function for $\bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty} A_n$ and then apply the union bound)

(b) Suppose all the tosses are independent and that $\sum_{i=1}^{\infty} p_i = \infty$. Show that $P(I) = 1$. (Hint: Construct a sequence of events E_n, each of which denotes that there is only a finite number of heads and the last head occurs at the n-th toss. Moreover, define an event $N :=$ {there is no head at all}. Then, we know $I^c = \big(\bigcup_{i\geq 1} E_i\big) \cup N$. As E_i's and N are mutually exclusive, we can use the third probability axiom. Could you figure out $P(E_i)$ and $P(N)$ using the independence assumption?)

Problem 5 (Communication Over a Binary Erasure Channel) (5+5+5+5=20 points)

At each time, the transmitter sends a bit (either 0 or 1), and the receiver either successfully receives the bit or it receives a message that the bit was not received ("erased"). If a 0 is sent, the erasure probability is ε_0;

similarly, the erasure probability is ε_1 if a 1 is sent. Moreover, at each transmission, the transmitter chooses to send a 0 with probability p. Each transmission is assumed to be independent from other transmissions.

(a) What is the probability that a transmitted bit is erased on the receiver side?

(b) Suppose three bits have been transmitted. What is the probability that at most 1 bit is erased?

(c) Suppose the receiver gets two consecutive erasure messages. What is the probability that 10 is sent by the transmitter?

(d) Suppose $p = 1/3$, $\varepsilon_0 = 1/4$, and $\varepsilon_1 = 1/5$, and four bits have been transmitted. Given that the first two received bits are 01 and the last two received bits are erased, what is the most probable sequence of bits sent by the transmitter?

Problem 6 (Inference via Bayes' Rule) (6+6+6=18 points)

Suppose we are given a coin with an unknown head probability $\theta \in \{0.3, 0.5, 0.7\}$. In order to infer the value θ, we experiment with the coin and consider Bayesian inference as follows: Define events $A_1 = \{\theta = 0.3\}$, $A_2 = \{\theta = 0.5\}$, $A_3 = \{\theta = 0.7\}$. Since initially we have no further information about θ, we simply consider the prior probability assignment to be $P(A_1) = P(A_2) = P(A_3) = 1/3$.

(a) Suppose we toss the coin once and observe a head (for ease of notation, we define the event $B = \{\text{the first toss is a head}\}$). What is the posterior probability $P(A_1|B)$? How about $P(A_2|B)$ and $P(A_3|B)$? (Hint: use the Bayes' rule)

(b) Suppose we toss the coin for 10 times and observe HHTHHHTHHH (for ease of notation, we define the event $C = \{\text{HHTHHHTHHH}\}$). Moreover, all the tosses are known to be independent. What is the posterior probability $P(A_1|C)$, $P(A_2|C)$, and $P(A_3|C)$? Given the experimental results, what is the most probable value for θ?

(c) Given the same setting as (b), suppose we instead choose to use a different prior probability assignment $P(A_1) = 2/5, P(A_2) = 2/5, P(A_3) = 1/5$. What is the posterior probabilities $P(A_1|C)$, $P(A_2|C)$, and $P(A_3|C)$? Given the experimental results, what is the most probable value for θ?

Problem 7 (Combinatorial Methods) (5+5=10 points)

(a) Applying the binomial expansion, find the coefficient of the term x^n in the expansion of $(1+x)^{2n} = (1+x)^n(1+x)^n$ to prove that $C_n^{2n} = \sum_{i=0}^{n} \left(C_i^n\right)^2$

(b) For any integers $n \geq 1$ and $r \geq 1$, show that $\sum_{i=0}^{r} C_i^{n+i} = C_r^{n+r+1}$ (Hint: $C_r^{n+1} = C_r^n + C_{r-1}^n$).