Problem 1

$$f_X(x)=\begin{cases}\frac{1}{4}, & \text{if } x\in(-1,3)\\ 0, & \text{else}\end{cases}$$

(a) For $t\neq 0$:

$$M_X(t)=\int_{-1}^{3}\frac{1}{4}e^{tx}dx=\frac{1}{4t}e^{tx}\Big|_{-1}^{3}=\underline{\frac{1}{4t}(e^{3t}-e^{-t})}$$ #

For $t=0$:

$$M_X(t)=E[e^{0\cdot X}]=\underline{1.}$$ #

(b) For $t\neq 0$:

$$M_X'(t)=-\frac{1}{4t^2}(e^{3t}-e^{-t})+\frac{1}{4t}(3\cdot e^{3t}+e^{-t})$$

$$E[X]=M_X'(0)=\lim_{h\to 0}\frac{M_X(h)-M_X(0)}{h}=\lim_{h\to 0}\frac{\frac{1}{4h}(e^{3h}-e^{-h})-1}{h}$$

$$=\lim_{h\to 0}\frac{e^{3h}-e^{-h}-4h}{4h^2}$$

L'Hospital's rule

$$=\lim_{h\to 0}\frac{3e^{3h}+e^{-h}-4}{8h}$$

L'Hospital's rule

$$=\lim_{h\to 0}\frac{9e^{3h}-e^{-h}}{8}=\underline{1}$$ #

(Conti.)

$$E[X^2] = M_X''(0) = \lim_{h\to 0} \frac{M_X'(h) - M_X'(0)}{h}$$

$$= \lim_{h\to 0} \frac{-\frac{1}{4h^2}(e^{3h} - e^{-h}) + \frac{1}{4h}(3e^{3h} + e^{-h}) - 1}{h}$$

$$= \lim_{h\to 0} \frac{-(e^{3h} - e^{-h}) + h(3e^{3h} + e^{-h}) - 4h^2}{4h^3}$$

L'Hospital's

$$= \lim_{h\to 0} \frac{\cancel{-(3e^{3h} + e^{-h})} + \cancel{(3e^{3h} + e^{-h})} + h\cdot(9e^{3h} - e^{-h}) - 8h}{12h^2}$$

L'Hospital's

$$= \lim_{h\to 0} \frac{(9e^{3h} - e^{-h}) + h\cdot(27e^{3h} + e^{-h}) - 8}{24h}$$

L'Hospital's

$$= \lim_{h\to 0} \frac{(27e^{3h} + e^{-h}) + (27e^{3h} + e^{-h}) + h(81e^{3h} - e^{-h})}{24} = \frac{7}{3}$$

$$Var[X] = E[X^2] - (E[X])^2$$

$$= \frac{7}{3} - 1^2$$

$$= \frac{4}{3}$$

#

Problem 2

(a). Binomial (n,p) has an MGF as $(1-p+pe^t)^n$

If $M_X(t) = (\frac{1}{4}e^t + \frac{3}{4})^7$, then $X \sim \text{Binomial}(n=7, p=\frac{1}{4})$

(b). Geometric (p) has an MGF as $\dfrac{pe^t}{1-(1-p)e^t}$

If $M_X(t) = \dfrac{e^t}{2-e^t}$, then $X \sim \text{Geometric}(p=\frac{1}{2})$

(c). Poisson (λ) has an MGF as $e^{\lambda(e^t-1)}$.

If $M_X(t) = e^{3(e^t-1)}$, then $X \sim \text{Poisson}(\lambda=3)$.

Problem 3

(a). X_1, X_2 are independent uniform r.v.s on $(0,2)$. $\Rightarrow$ $f_{X_1}(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in (0,2) \\ 0 & \text{else} \end{cases}$

$f_{X_2}(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in (0,2) \\ 0, & \text{else} \end{cases}$

By convolution theorem:

$$f_{X_1+X_2}(x) = \int_{-\infty}^{+\infty} f_{X_1}(u) \cdot f_{X_2}(x-u)\, du$$

If $x \geq 4$:

$$f_{X_1+X_2}(x) = 0.$$

If $2 \leq x < 4$:

$$f_{X_1+X_2}(x) = \int_{-\infty}^{+\infty} f_{X_1}(u) f_{X_2}(x-u)\, du$$

$$= \int_{x-2}^{2} \frac{1}{2} \cdot \frac{1}{2}\, du = \frac{1}{4}(4-x).$$

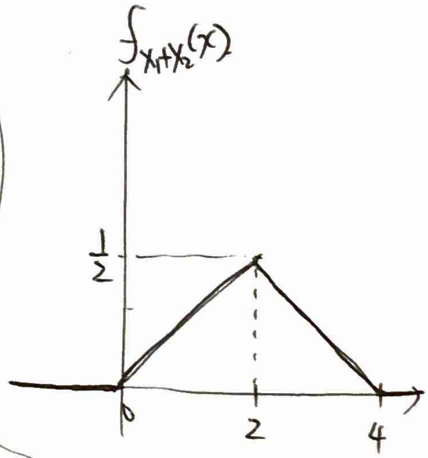
If $0 < x < 2$:

$$f_{X_1+X_2}(x) = \int_{-\infty}^{+\infty} f_{X_1}(u) \cdot f_{X_2}(x-u)\, du$$

$$= \int_{0}^{x} \frac{1}{2} \cdot \frac{1}{2}\, du = \frac{1}{4}x$$

If $x \leq 0$:

$$f_{X_1+X_2}(x) = 0.$$



(Cont.)

(b).

$Y_1 \sim N(1,4)$ and $Y_2 \sim N(4,9)$.

$\Rightarrow 3Y_1 \sim N(3,36)$ and $4Y_2 \sim N(16,144)$.

As Y_1, Y_2 are independent, $3Y_1$ and $4Y_2$ are also independent.

Then $3Y_1+4Y_2 \sim N(19,180)$. and $3Y_1+4Y_1$ has a CDF of $\Phi\left(\frac{y-19}{\sqrt{180}}\right)$.

Therefore, $P(3Y_1+4Y_2>20) = 1-\Phi\left(\frac{20-19}{\sqrt{180}}\right)$

$$= 1-\Phi\left(\frac{1}{\sqrt{180}}\right).$$

Problem 4

(a). $E[X] = Var[X] = \mu$.

Then, we know

$$P(X > \pi\mu) = P(X - \mu > (\pi - 1)\mu)$$

$$\leq P(|X - \mu| > (\pi - 1)\mu)$$

$$= P(|X - E[X]| > (\pi - 1)\mu) \leq \frac{Var[X]}{(\pi-1)^2\mu^2} = \frac{1}{(\pi-1)^2\mu}$$

(b). For each i, define X_i as a Bernoulli random variable for which:

$$X_i = \begin{cases} 1, & \text{if the algorithm returns the correct answer at the } i\text{-th trial} \\ 0, & \text{else.} \end{cases}$$

Under majority vote, If the final answer is not correct, then we must have

$$X_1 + X_2 + \cdots + X_N \leq \frac{N}{2}$$

Then, $P(\text{final answer is incorrect}) \leq P\left(X_1 + X_2 + \cdots + X_N \leq \frac{N}{2}\right)$

$$= P\left(\frac{X_1 + \cdots + X_N}{N} - \left(\frac{1}{2} + \delta\right) \leq -\delta\right)$$

By the negative part of Hoeffding's $\leq e^{-2N\delta^2}$

(Conti).

By choosing $N \geq \frac{1}{2\delta^2} \cdot \ln \frac{1}{\varepsilon}$, we have

$$P(\text{final answer is incorrect}) \leq e^{-2\cdot\left(\frac{1}{2\delta^2}\ln\frac{1}{\varepsilon}\right)\cdot\delta^2} = e^{-\ln\frac{1}{\varepsilon}} = \varepsilon,$$

which also implies that $P(\text{final answer is correct}) \geq 1-\varepsilon$.

$\square$

Problem 5

$X_1, X_2, \cdots, X_N$ are non-negative independent random variables.

Moreover, it is assumed that the PDFs of X_i's are uniformly bounded by 1.

(a). $E[e^{-tX_i}] = \int_0^{+\infty} f_{X_i}(x) \cdot \underset{\substack{\uparrow \\ \text{positive}}}{e^{-tx}}\, dx \le \int_0^{+\infty} 1 \cdot e^{-tx}\, dx = -\frac{1}{t} e^{-tx} \Big|_0^{\infty}$

$= \frac{1}{t}$, for all $t>0$.

(Note that $E[e^{-tX_i}]$ does not exist for any $t \le 0$)

(b).

For any $t>0$, we have

$$P\left(\sum_{i=1}^{N} X_i \le \epsilon N\right) \overset{\text{Chernoff technique}}{=} P\left(e^{t\sum_{i=1}^{N} X_i} \le e^{t\epsilon N}\right)$$

$$= P\left(e^{-t\sum_{i=1}^{N} X_i} \ge e^{-t\epsilon N}\right)$$

$$\overset{\text{Markov inequality}}{\le} \frac{E\left[e^{-t\sum_{i=1}^{N} X_i}\right]}{e^{-t\epsilon N}}$$

$$\overset{\text{independence of } X_1, X_2, \cdots, X_N}{=} \frac{\prod_{i=1}^{N} E[e^{-tX_i}]}{e^{-t\epsilon N}}$$

$$\le \left(\frac{1}{t} e^{t\epsilon}\right)^N \quad \text{———} \quad (*)$$

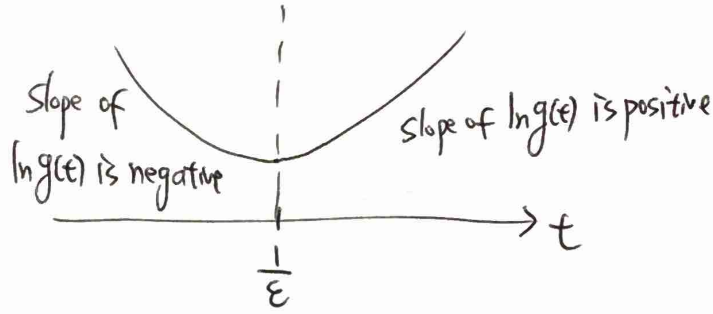
(Conti).

Finally, we shall minimize (✱) over $t>0$:

Define $g(t) = \left(\frac{1}{t} e^{t\varepsilon}\right)^N$.

Then, $\ln g(t) = N \cdot (-\ln t + t\varepsilon)$.

$$\frac{d(\ln g(t))}{dt} = N \cdot \left(-\frac{1}{t} + \varepsilon\right) \Rightarrow$$



Therefore, we know the minimizer of $\ln g(t)$ and $g(t)$ is $t = \frac{1}{\varepsilon}$,

Hence, we conclude that

$$P\left(\sum_{i=1}^{N} X_i \le \varepsilon N\right) \le \left(\frac{1}{\frac{1}{\varepsilon}} e^{\frac{1}{\varepsilon}\cdot\varepsilon}\right)^N = (e\varepsilon)^N$$

✱