DCP1206 Fall 2019: Probability **(Due: 2019/11/1 in class)**

Homework 3: Continuous Random Variables

Problem 1 (Normal Random Variables) (12+14=26 points)

(a) Determine the value(s) of k for which the following is the PDF of a normal random variable:

$$f(x) = \sqrt{k}\exp(-k^2x^2 - 2kx - 1), \ \ -\infty < x < \infty.$$

(b) A binary message is transmitted as a wireless signal X, which is either $+1$ or -1. The wireless channel corrupts the transmission with additive noise Y, which is a normal random variable with mean 0 and variance σ^2 (the noise Y is assumed to be independent of X). Therefore, the received signal (denoted by Z) is $Z = X + Y$. The receiver concludes that the signal -1 (or $+1$) was transmitted if $Z < 0$ (or $Z \geq 0$, respectively). What is the probability of error? (Hint: The probability of error is $P(X = +1 \text{ and } Z < 0) + P(X = -1 \text{ and } Z \geq 0)$. By using the multiplication rule, we know this can be written as $P(Z < 0|X = +1)P(X = +1) + P(Z \geq 0|X = -1)P(X = -1)$. Next, you may consider the CDF of a normal random variable and use the notation $\Phi(\cdot)$ to express your answer)

Problem 2 (Exponential Random Variables) (16 points)

Let X be an exponential random variable with parameter λ. Consider another random variable $Y = aX + b$, where a, b are real numbers and $a \neq 0$. Please write down the CDF and PDF of Y (Note: we assume that the PDF of Y is continuous everywhere, except at b). Under what condition is Y also an exponential random variable? (Hint: $a > 0$ and $a < 0$ may lead to different characteristics of CDF and PDF)

Problem 3 (PDF and Differential Entropy) (12 points)

Consider a continuous random variable X with PDF $f(\cdot)$. Similar to the notion of entropy in HW2, an information-theoretic metric called *differential entropy* can defined as:

$$h(X) := -\int_{f(x)>0} f(x)(\ln f(x))dx.$$

Suppose the random variable $X \sim \mathcal{N}(0, \sigma^2)$. Show that $h(X) = \frac{1}{2}\ln(2\pi e\sigma^2)$. (Hint: You may need to leverage the fact that $E[X^2] = \text{Var}[X] + (E[X])^2 = \sigma^2$)

Problem 4 (Variance of Continuous Random Variables) (10+12=22 points)

Please use the PDFs and the definitions of expected value and variance to show the following properties:

(a) Verify that a standard normal random variable X satisfies that $\text{Var}[X] = 1$. (Hint: Use integration by parts)

(b) Suppose $X \sim \text{Exp}(\lambda)$. Verify that $\text{Var}[X] = 1/\lambda^2$. (Hint: Use integration by parts)

Problem 5 (Moments of Continuous Random Variables) (10+14=24 points)

The random variable X is said to be a *Laplace random variable* or *double exponentially distributed* if its PDF is given by

$$f(x) = C \cdot \exp(-|x|), \ \ -\infty < x < \infty.$$

(a) Find the value of C. (Hint: leverage the symmetry of PDF and the fact that $\int_{-\infty}^{\infty} f(x)dx = 1$)

(b) Prove that $E[X^{2n}] = (2n)!$ and $E[X^{2n+1}] = 0$, for all $n \in \mathbb{N}$. (Hint: For $E[X^{2n}]$, you may want to use integration by parts. For $E[X^{2n+1}]$, in order to use symmetry, please explain whether $E[|X^{2n+1}|]$ exists or not by using the result that $E[X^{2n}] = (2n)! < \infty$)