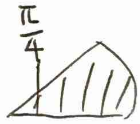
Problem 1:

(a) PDF of X: $f_X(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in (0,2) \\ 0, & \text{else} \end{cases}$

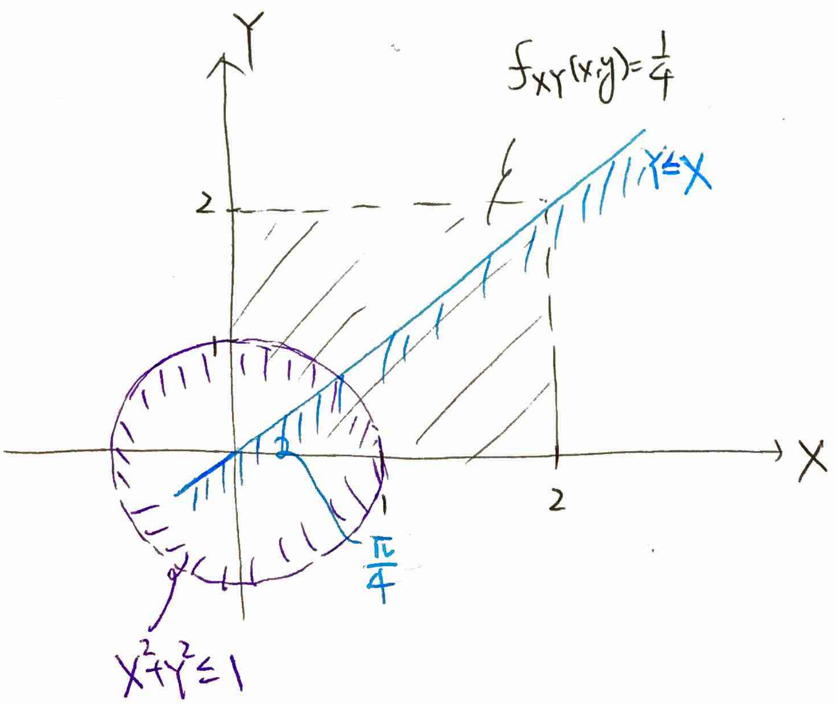
PDF of Y: $f_Y(y) = \begin{cases} \frac{1}{2}, & \text{if } y \in (0,2) \\ 0, & \text{else} \end{cases}$

Since X, Y are independent, the joint PDF of X and Y is:

$$f_{XY}(x,y) = \begin{cases} \frac{1}{4}, & \text{if } x \in (0,2) \text{ and } y \in (0,2) \\ 0, & \text{else} \end{cases}$$

$$P(Y \le X \text{ and } X^2 + Y^2 \le 1) = \frac{1}{4} \times (\text{area of}$$  $$) = \frac{1}{4} \times \frac{1}{2} \times 1^2 \times \frac{\pi}{4}$$

$$= \frac{\pi}{32}$$



(b) Want to show: $P(g(X)\in A, h(Y)\in B) = P(g(X)\in A)\cdot P(h(Y)\in B)$, for any sets A, B

Since X and Y independent, then $P(X\in A, Y\in B) = P(X\in A)\cdot P(Y\in B)$, for any sets A, B. $\boxed{P.2}$

Let us define the pre-images of $g(\cdot)$ and $h(\cdot)$ as: Let S be a set of real numbers.

$$g^{-1}(S) \triangleq \{x : g(x) \in S\}$$

$$h^{-1}(S) \triangleq \{y : h(y) \in S\}.$$

For any two sets A, B,

$$P(g(X)\in A, h(Y)\in B) \overset{\text{by definition of } g^{-1} \text{ and } h^{-1}}{=} P(X\in g^{-1}(A), Y\in h^{-1}(B))$$

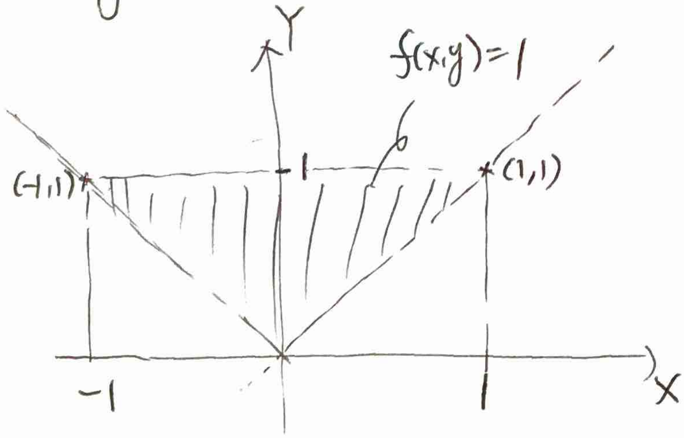
$$\overset{\text{by independence of } X,Y}{=} P(X\in g^{-1}(A))\cdot P(Y\in h^{-1}(B))$$

$$= P(g(X)\in A)\cdot P(h(Y)\in B).$$

Hence, $g(X)$ and $h(Y)$ are independent. $\square$

Problem 2

$$f(x,y)=\begin{cases}1, & \text{if } |x|<y,\ 0<y<1\\ 0, & \text{else}\end{cases}$$



(a).

$$E[XY]=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\cdot f(x,y)\,dx\,dy$$

$$=\int_0^1\left(\int_{-y}^{y} xy\cdot 1\ dx\right)dy$$

$$=\int_0^1 \underbrace{\left(\frac{1}{2}x^2y\Big|_{-y}^{y}\right)}_{=0} dy = 0.$$

Next, we find the marginal PDF of X and Y:

$$f_X(x)=\int_{-\infty}^{+\infty} f(x,y)\,dy \Rightarrow \text{If } x\in(-1,1):\ f_X(x)=\int_{|x|}^{1} 1\cdot dy = 1-|x|$$

$$\text{Otherwise} \ = \ f_X(x)=0$$

$$f_Y(y)=\int_{-\infty}^{+\infty} f(x,y)\,dx \Rightarrow \text{If } y\in(0,1):\ f_Y(y)=\int_{-y}^{y} 1\cdot dx = 2y$$

$$\text{Otherwise} \ = \ f_Y(y)=0.$$

(Conti.)

$$E[X] = \int_{-\infty}^{+\infty} x \cdot f_X(x)\,dx = \int_{-1}^{1} x \cdot (1-|x|)\,dx = 0.$$

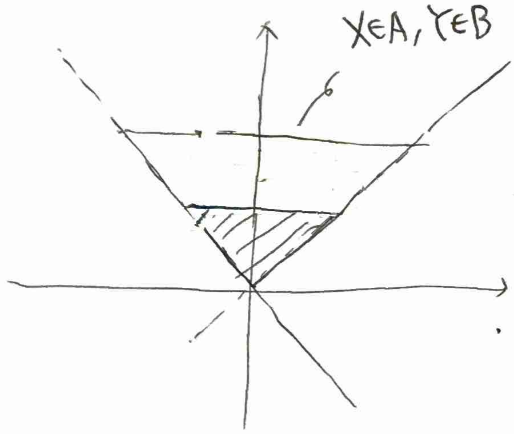
$$E[Y] = \int_{-\infty}^{+\infty} y \cdot f_Y(y)\,dy = \int_{0}^{1} y \cdot 2y = \frac{2}{3}$$

Therefore, we verify that $E[XY] = E[X] \cdot E[Y]$. $\square$

(b). Let's construct an example:

$A = (-0.5, 0.5)$

$B = (0, 0.5)$.



$$P(X \in A, Y \in B) = \frac{1}{4}$$

$$P(X \in A) = \frac{3}{4}$$

$$P(Y \in B) = \frac{1}{4}$$

Therefore, we know $P(X \in A, Y \in B) \neq P(X \in A) \cdot P(Y \in B)$.

Hence, X and Y are not independent.

$\square$

Problem 3

$X \sim N(0,1)$.

(a). $$E[X^3] = \int_{-\infty}^{+\infty} x^3 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 0 \text{ (by symmetry)}.$$

Remark: To be rigorous, we need to verify that $E[|X|^3]$ exists.

$$E[|X|^3] = \int_{-\infty}^{+\infty} |X|^3 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= 2 \cdot \int_0^{\infty} x^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

let $t = x^2$

then $\frac{dt}{dx} = 2x$

integration by parts

$$\overset{\curlyvee}{=} \int_0^{\infty} t \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{t}{2}} dt = \frac{1}{\sqrt{2\pi}} \left(\underbrace{t \cdot -2e^{-\frac{t}{2}} \Big|_0^{\infty}}_{=0} - \int_0^{\infty} -2e^{-\frac{t}{2}} dt \right) = \frac{4}{\sqrt{2\pi}} < \infty$$

$$E[X^4] = \int_{-\infty}^{+\infty} x^4 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 2 \cdot \int_0^{\infty} x^4 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

integration by parts

$$\overset{\curlyvee}{=} 2 \cdot \left(\underbrace{-x^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Big|_0^{\infty}}_{=0} + 3 \cdot \underbrace{\int_0^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx}_{= \frac{1}{2}E[X^2] = \frac{1}{2}(Var[X] + (E[X])^2)} \right)$$

$$= 3$$

(b). $Y = aX^2+bX+c, \quad X \sim N(0,1)$

$$E[aX^2+bX+c] = a\cdot E[X^2]+b\cdot E[X]+c = a+c$$

① $Var[X]=1.$

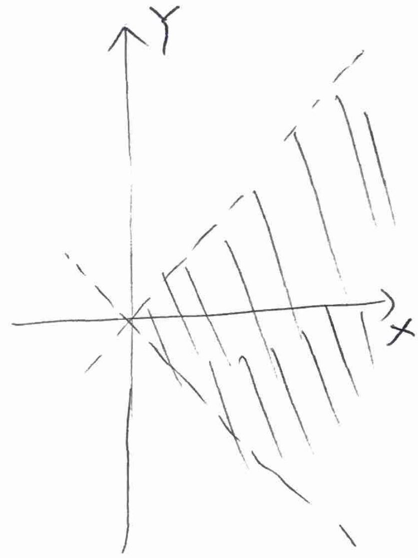
②
$$\begin{aligned}
Var[Y] &= Var(aX^2+bX+c) \\
&= E\left[\left(aX^2+bX+c - E[aX^2+bX+c]\right)^2\right] \\
&= E\left[\left(aX^2+bX+c-(a+c)\right)^2\right] \\
&= \underbrace{E[a^2X^4]}_{=3a^2} + \underbrace{E[b^2X^2]}_{=b^2} + \underbrace{E[(-a)^2]}_{=a^2} \\
&\quad + \underbrace{2\cdot E[-a^2X^2]}_{=-2a^2} + \underbrace{2E[abX^3]}_{=0} + \underbrace{2\cdot E[-abX]}_{=0} \\
&= 2a^2+b^2
\end{aligned}$$

③
$$\begin{aligned}
Cov(X,Y) &= E\left[(X-0)\cdot\left(aX^2+bX+\cancel{c}-(a+\cancel{c})\right)\right] \\
&= \underbrace{E[aX^3]}_{=0} + \underbrace{E[bX^2]}_{=b} + \underbrace{E[-aX]}_{=0} \\
&= b.
\end{aligned}$$

Therefore, $\rho(X,Y) = \dfrac{Cov(X,Y)}{\sqrt{Var[X]}\cdot\sqrt{Var[Y]}} = \dfrac{b}{\sqrt{1}\cdot\sqrt{2a^2+b^2}} = \dfrac{b}{\sqrt{2a^2+b^2}}$ //

Problem 4

$$f(x,y) = \begin{cases} C \cdot \exp(-x), & \text{if } x \geq 0, |y| < x \\ 0, & \text{else} \end{cases}$$



(a).

$$1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy$$

$$= \int_0^{\infty}\left(\int_{-x}^{x} C \cdot e^{-x}\,dy\right)dx$$

$$= \int_0^{\infty} C \cdot e^{-x} \cdot y \Big|_{-x}^{x}\,dx$$

$$= 2C \cdot \int_0^{\infty} x e^{-x}\,dx = 2C \cdot \left(\overbrace{-xe^{-x}\Big|_0^{\infty}}^{0} + \overbrace{\int_0^{\infty} e^{-x}\,dx}^{1}\right) = 2C$$

Therefore, $C = \frac{1}{2}$

(b).

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} \dfrac{\frac{1}{2}e^{-x}}{\frac{1}{2}e^{-|y|}}, & \text{if } x \geq 0 \text{ and } |y| < x \\ 0, & \text{else} \end{cases}$$

$$\left(f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)\,dx = \int_{|y|}^{\infty} \frac{1}{2}e^{-x}\,dx = -\frac{1}{2}e^{-x}\Big|_{|y|}^{\infty} = \frac{1}{2}e^{-|y|}\right)$$

Similarly,

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \begin{cases} \dfrac{\frac{1}{2}e^{-x}}{xe^{-x}}, & \text{if } x > 0, |y| < x \\ 0, & \text{else} \end{cases}$$

$$\left(f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy = \int_{-x}^{x} \frac{1}{2}e^{-x}\,dy = xe^{-x}, \quad \text{for all } x > 0\right)$$

(c). $E[Y|X=x]$ (Here, we only consider $x>0$).

$$= \int_{-\infty}^{+\infty} y \cdot f_{Y|X}(y|x)\, dy$$

$$= \int_{-x}^{x} y \cdot \frac{\frac{1}{2}e^{-x}}{xe^{-x}}\, dy$$

$$= \int_{-x}^{x} \frac{y}{2x}\, dy$$

$$= \frac{1}{4x} y^2 \Big|_{-x}^{x} = 0$$

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Problem 5

$$f_{XY}(x,y) = C \cdot e^{-8x^2-6xy-18y^2}$$

We observe that $f_{XY}(x,y)$ has the form of a bivariate normal r.v.:

$$f_{XY}(x,y) = C \cdot e^{-\frac{1}{2}\left(\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 16 & 6 \\ 6 & 36 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}\right)}$$

Then, we know $\Sigma^{-1} = \begin{bmatrix} 16 & 6 \\ 6 & 36 \end{bmatrix}$, and hence $\Sigma = \frac{1}{540}\begin{bmatrix} 36 & -6 \\ -6 & 16 \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{15} & -\frac{1}{90} \\ -\frac{1}{90} & \frac{4}{135} \end{bmatrix}$$

This implies that $\mathrm{Var}[X] = \frac{1}{15}$, $E[X] = 0$

$\mathrm{Var}[Y] = \frac{4}{135}$, $E[Y] = 0$.

$\mathrm{Cov}(X,Y) = -\frac{1}{90}$

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\cdot\sqrt{\mathrm{Var}(Y)}} = \frac{-\frac{1}{90}}{\sqrt{\frac{1}{15} \times \frac{4}{135}}} = \frac{-\frac{1}{90}}{\frac{2}{45}} = -\frac{1}{4}$$

$$C = \frac{1}{2\pi \cdot \sqrt{\det(\Sigma)}} = \frac{1}{2\pi\sqrt{\frac{1}{540}}} = \frac{\sqrt{135}}{\pi}$$