DCP 1206: Probability
Lecture 28 — Markov Chain

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December 27, 2019

Announcements

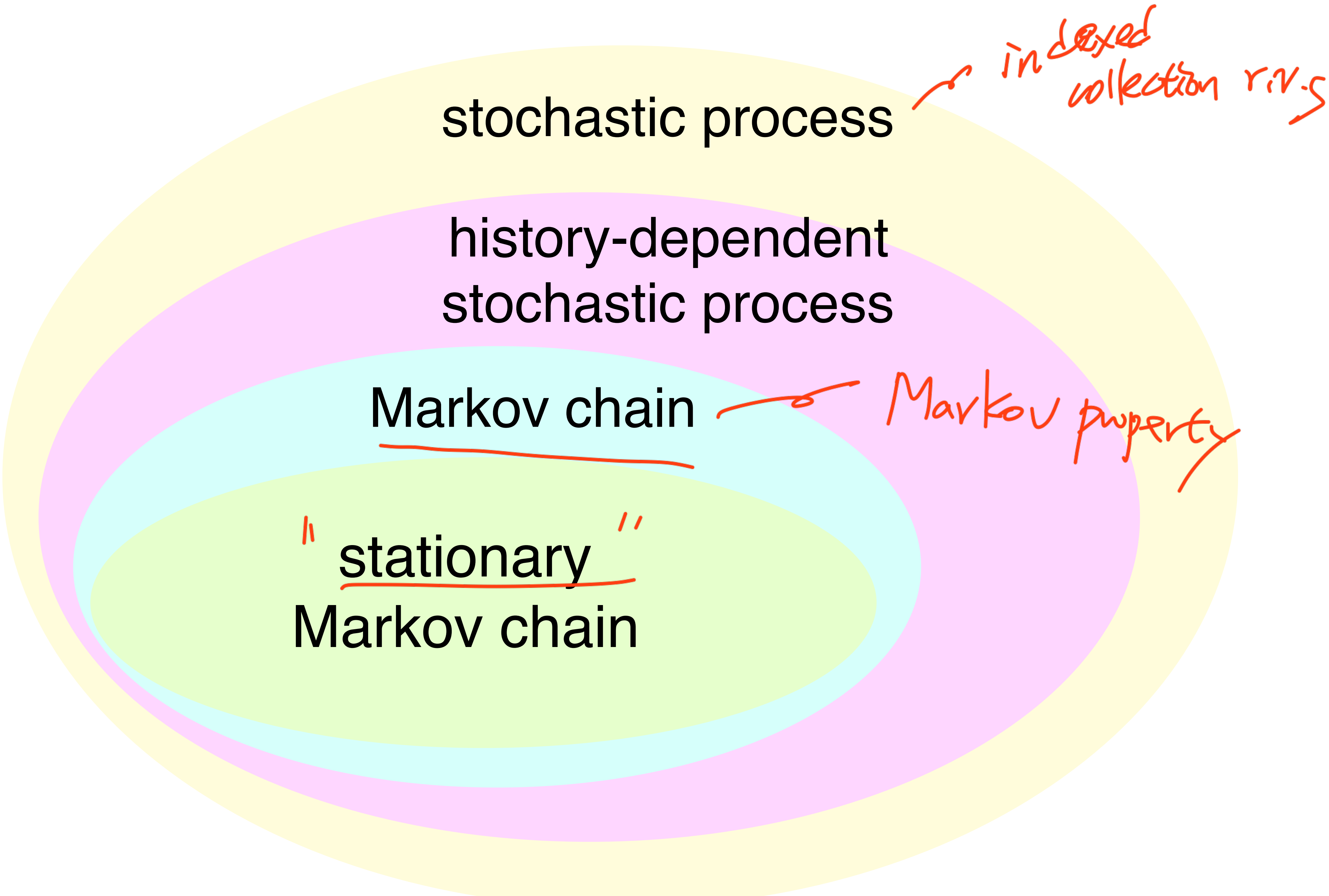
- Final exam
  - 1/8 (Wednesday), 10:10am-12pm
  - Location:
    - SA321 (科學一館321): Student number ending with 1,3,5,7,9
    - EC122: Student number ending with 0,2,4,6,8
  - Coverage: Lec 1 - Lec 29 (focus on Lec 14-29)
  - You are allowed to bring a cheat sheet (A4 size, 2-sided)

This Lecture

1. Markov Chain

- Reading material: Chapter 12.1-12.3

Quick Review

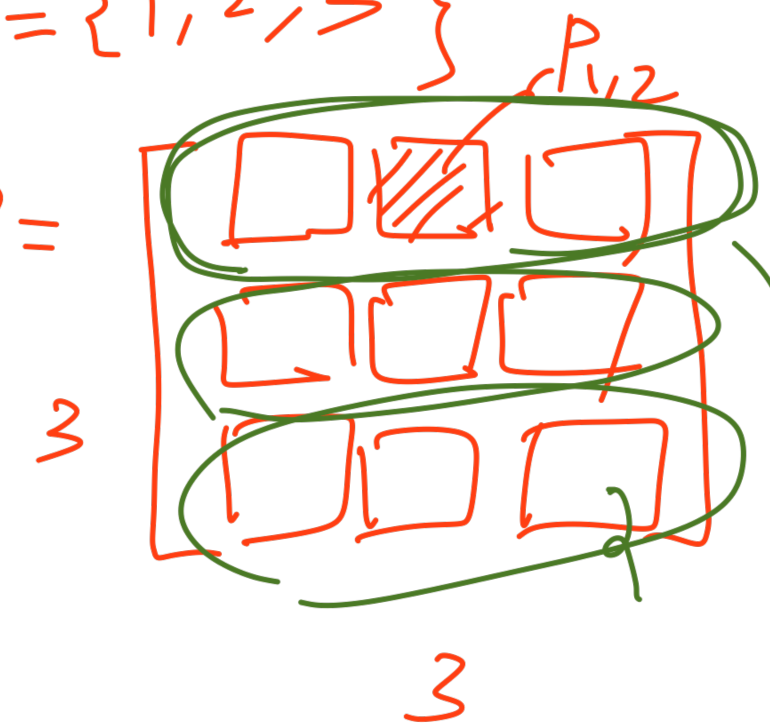


Stationary Markov Chain

- Question: How to specify a stationary Markov chain?

  1. State space S (with what condition for S?) $\Leftarrow$ finite or countably infinite
  2. Transition matrix P (with what condition for P?)

$S = \{1, 2, 3\}$

$P =$ 

$P_{1,2} = P(X_{t+1} = 2 \mid X_t = 1)$

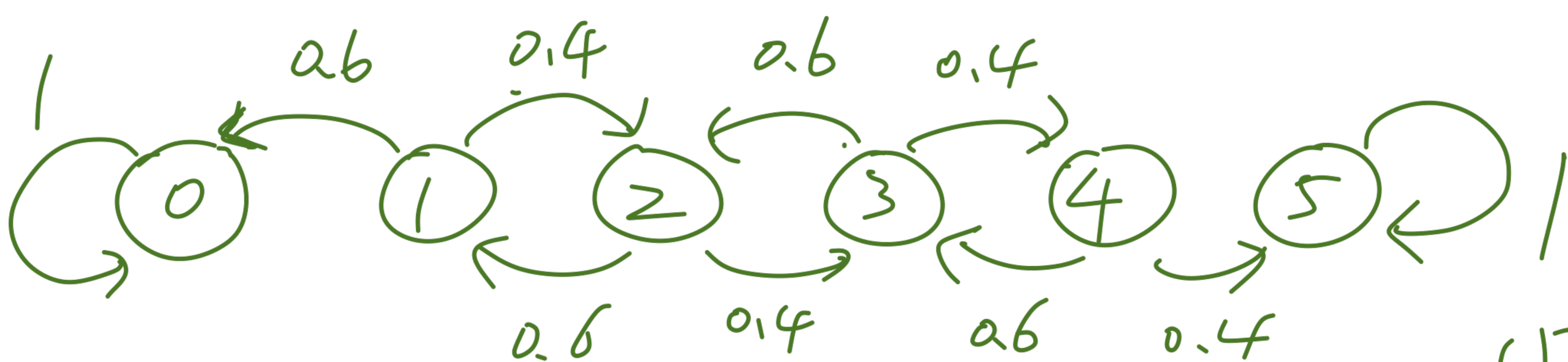
Sum $= 1$

$P_{1,1} + P_{1,2} + P_{1,3} = 1$

(Row sum is 1)

Markov Chain State Diagram

- Note: A Markov chain can be represented in a state diagram
- Example: Bill is playing with a bandit machine
  - In each round, he wins \$1 with probability 0.4 and loses \$1 with probability 0.6
  - He stops playing if he has \$0 or \$5



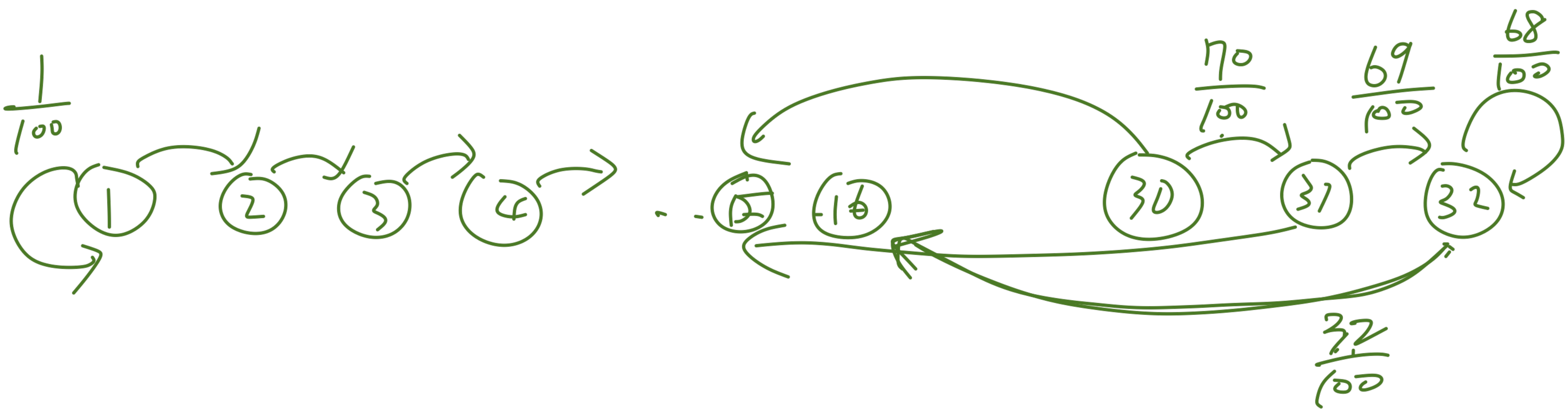
$P =$

Example: TCP Congestion Control

AIMD

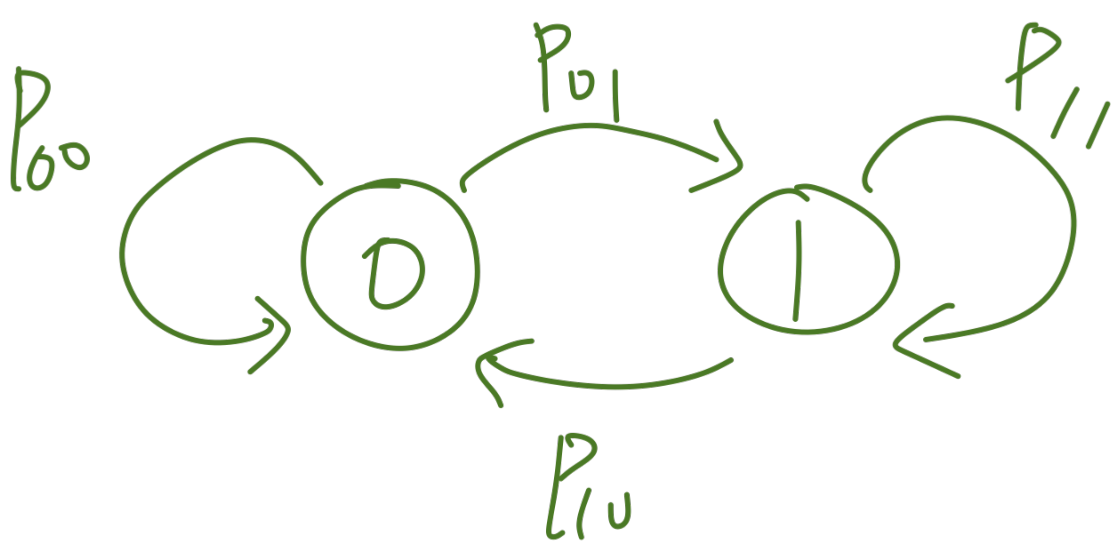
- Example: TCP enforces congestion control as follows
  - Available rates = $\{1, 2, 3, \cdots, 32\}$ Mbps
  - If congestion happens, decrease rate by half (with floor function)
  - Otherwise, increase the rate by 1 Mbps (unless already at max rate)
- Question: Suppose when rate = k Mbps, P(congestion occurs) = $k/100$. What is the Markov chain of interest? State space? Transition matrix? State diagram?

state = current TCP rate



Example: Gilbert-Elliot Model

- Example: A wireless channel is either ON or OFF
  - In Gilbert-Elliot model, the channel is modeled as a Markov chain
  - Let X_t denote the state
  - $X_t = 1$ if the channel is ON; Otherwise, $X_t = 0$
- Question: Plot the state diagram of the Markov chain?



GE model

A Historical Account of Markov Chain

- Markov chain was proposed by Andrey Markov

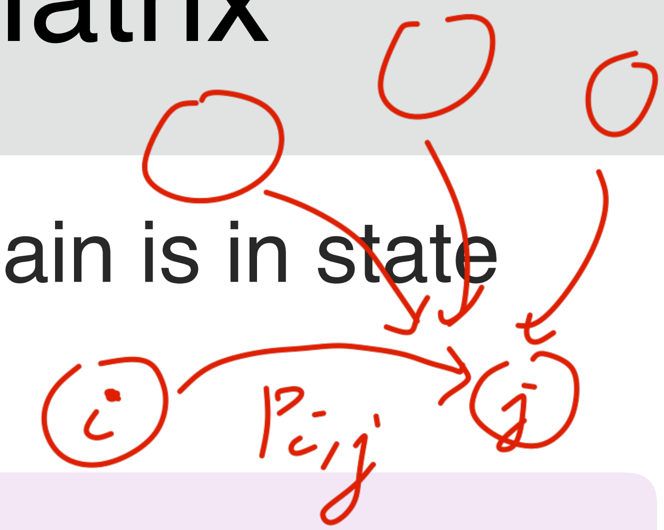
- Motivation: An open problem back in early 20-th century: “Can we have SLLN or WLLN for a sequence of non-i.i.d. random variables?”

Applications of Markov Chain

ITS

- Reinforcement learning (Markov decision process) $P_{i,j}(\text{action})$ reward $r_{i,j}(\text{action})$
- Sampling methods (e.g. Markov chain Monte Carlo)
- Queueing theory : queue evolution
- Natural language processing (e.g. n-gram)

State Distribution and Transition Matrix



- **Question**: Let $\pi_t(i)$ be the probability that the chain is in state $i \in S$ at time t. How to find $\pi_{t+1}(j)$?

- **Property**: Let S and $P = [p_{i,j}]_{i,j\in S}$ be the state space and transition matrix of the Markov chain, respectively. For any $t = 0,1,\cdots$ and $j \in S$, we have

$$\pi_{t+1}(j) = \sum_{i\in S} \pi_t(i) p_{i,j}$$

- **Remark**: Consider row vector $\pi_t = (\pi_t(0), \pi_t(1), \cdots, \pi_t(M))$

$$\pi_{t+1} = \pi_t P \quad \Rightarrow \quad \pi_{t+m} = \pi_t \cdot P^m, \ \forall m \in \mathbb{N}$$

Example: Gambler's Ruin (Cont.)

- Example: Bill is playing with a bandit machine
  - In each round, he wins \$1 with probability 0.4 and loses \$1 with probability 0.6
  - He stops playing if he has \$0 or \$5

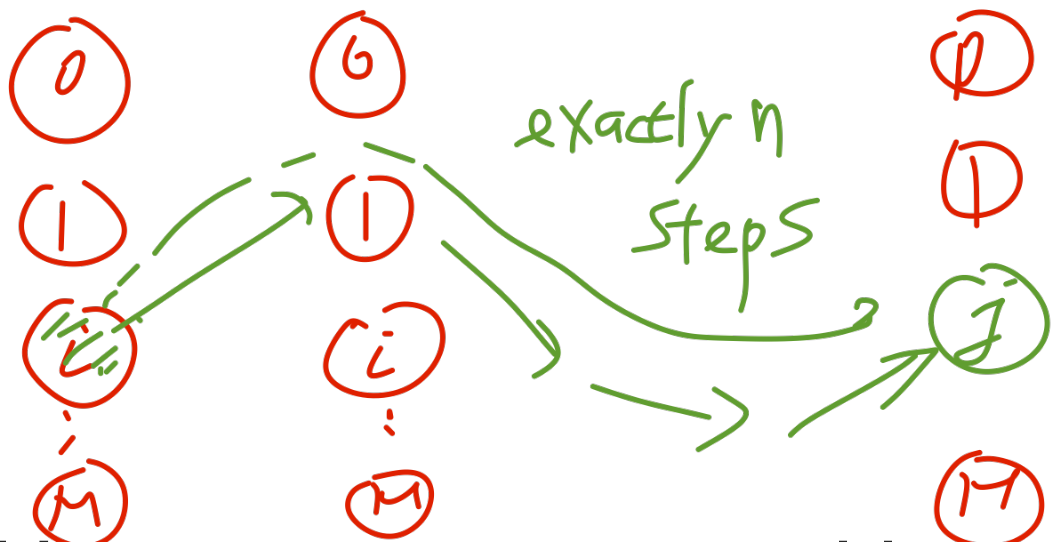
- Question: Suppose at time 0, $P(\text{Bill has } \$1) = P(\text{Bill has } \$3) = 0.5$. What's the distribution of the chain at time 1?

n-Step Transition Probabilities

- The n-step transition probabilities can be written as

$$\underline{P(X_{t+n} = j \mid X_t = i)} = P(X_n = j \mid X_0 = i) = p_{ij}^{(n)}$$

- Question: How to interpret this?



the (i,j)-th entry of P^n

- Question: How to connect transition matrix and $p_{i,j}^{(n)}$?

$$\pi_{t+n} = \pi_t P^n$$

$|S| \times |S|$

$P^n = |S| \left[\quad \right]$ (i,j)-th

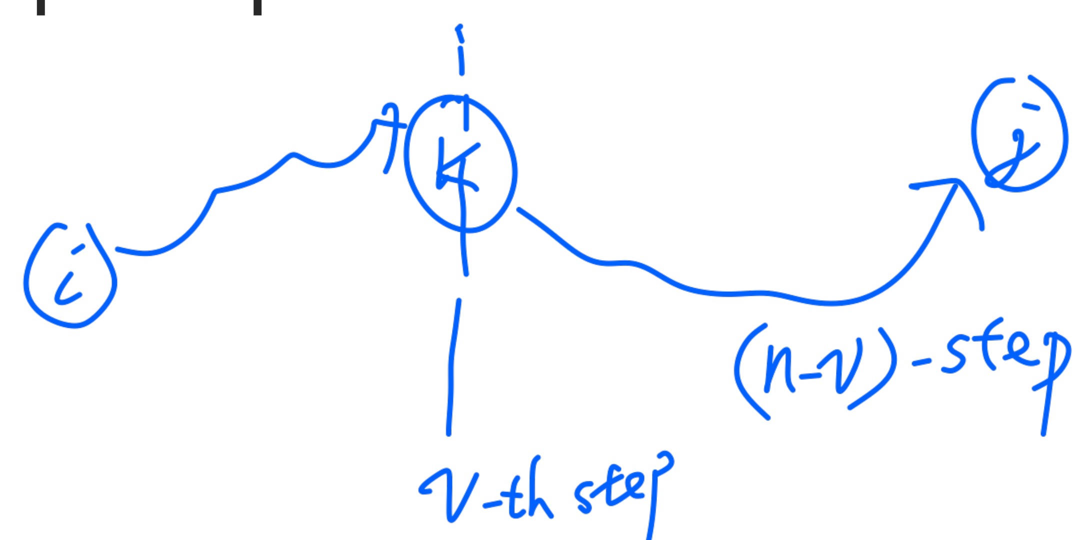
$|S|$

Chapman-Kolmogorov Equation

- **Chapman-Kolmogorov Equation**: Let $p_{i,j}^{(n)}$ be the n-step transition probability from state i to state j. For any $n = 0,1,\cdots$ and for any $i, j \in S = \{0,1,\cdots,M\}$

$$p_{i,j}^{(n)} = \underbrace{\sum_{k=0}^{M} p_{i,k}^{(v)} p_{k,j}^{(n-v)}}_{\text{divide-and-conquer}} \quad \text{for any } 0 \le v \le n$$

- Question: What's the probability principle behind this equation?



What happen to P^n if $n \to \infty$?

Do the steady-state probabilities

$$\pi_j^* \triangleq \lim_{n \to \infty} p_{i,j}^{(n)} \text{ exist?}$$

Steady-state probabilities exist if the Markov chain is "irreducible, positive recurrent, and aperiodic"

To explain this result, we need to consider classification of states

Concept #1: Accessibility

- Accessible: State j is accessible from state i if $p_{i,j}^{(n)} > 0$, for some $n \geq 0$ (denoted by $j \leftarrow i$)

- Example: Consider a Markov chain with transition matrix as

$$P = \begin{bmatrix} 0.4 & 0.6 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0.5 & 0.4 & 0.1 \\ 0 & 0 & 0 & 0.8 & 0.2 \end{bmatrix}$$

- Question: Which states are accessible from which other states?

Concept #2:Communicability and State Classes

- Communicate: State i and state j communicate if $j \leftarrow i$ and $i \leftarrow j$ (denoted by $i \leftrightarrow j$)
- Class: Two states are said to be in the same class if the two states communicate with each other
- Every Markov chain can be divided into disjoint classes

- Example: Consider a Markov chain with transition matrix as

$$P = \begin{bmatrix} 0.4 & 0.6 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0.5 & 0.4 & 0.1 \\ 0 & 0 & 0 & 0.8 & 0.2 \end{bmatrix}$$

- Question: Which states communicate with each other state? How many classes are there?

Concept #3: Irreducibility

- Irreducible: A Markov chain is irreducible if all states belong to one class

- Example: Consider a Markov chain with transition matrix as

$$P = \begin{bmatrix} 0.4 & 0.6 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0.5 & 0.4 & 0.1 \\ 0 & 0 & 0 & 0.8 & 0.2 \end{bmatrix}$$

- Question: Is this Markov chain irreducible?

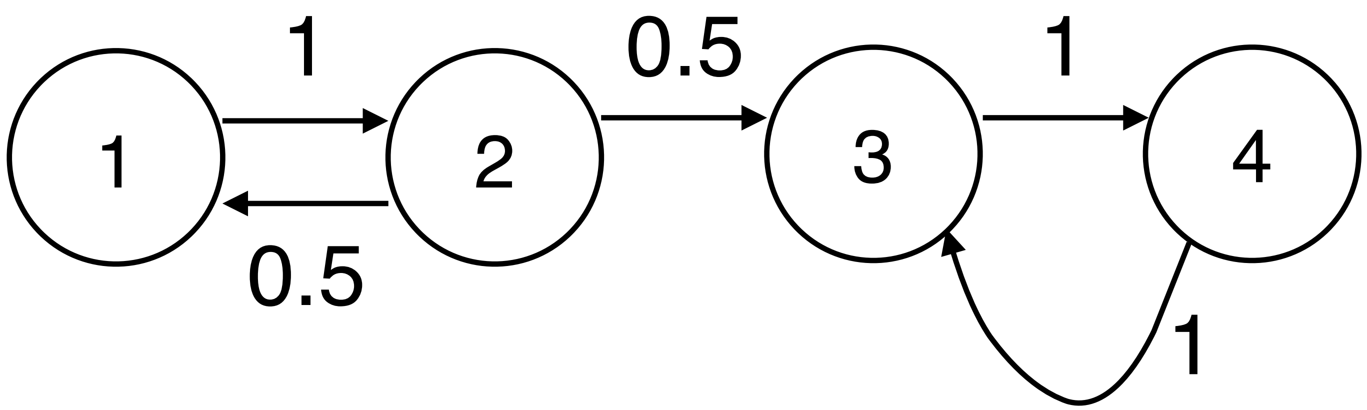
Example: Gambler's Ruin (Cont.)

- Example: Bill is playing with a bandit machine
  - In each round, he wins \$1 with probability 0.4 and loses \$1 with probability 0.6
  - He stops playing if he has \$0 or \$5

- Question: How many classes exist in this example?

Concept #4: Periodicity

- Example: Periodicity?



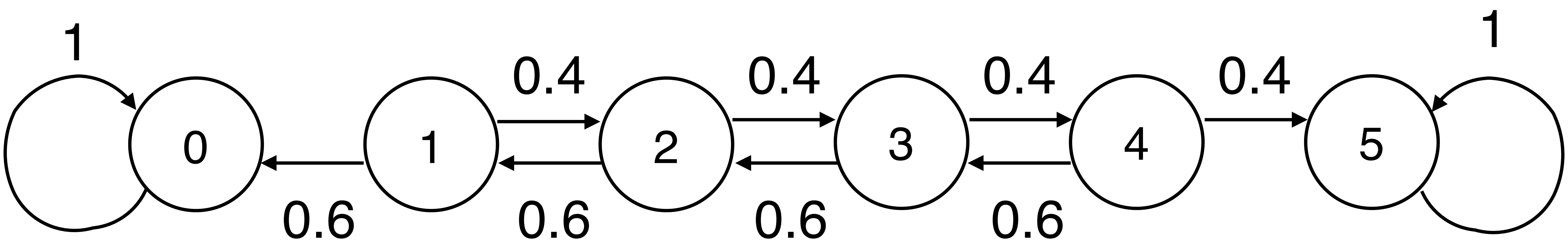
- The period of a state i is the largest integer k such that $p_{i,i}^{(n)} = 0$ for all n other than $k, 2k, 3k, \cdots$

- A state i is aperiodic if it has a period of 1

- Periodicity is a <u>class</u> property

Concept #5: Transient and Recurrent States

- A state i is
  - Transient : if there is a positive probability that the chain will leave state i and never return to state i
  - Recurrent: if state i is NOT transient
    - Absorbing: if state i is recurrent and $p_{i,i} = 1$
- Recurrence (and transience) is a class property
- Example: Find the transient (and recurrent) states



Concept #6: Positive Recurrence and Null Recurrence

- A recurrent state i is
  - Positive recurrent: if the expected return time to state i is finite
  - Null recurrent: if state i is NOT positive recurrent

1-Minute Summary

1. Markov Chain

- n-step transition probabilities and Chapman-Kolmogorov equation
- Accessibility / Communicate / Irreducibility
- Transient and recurrent states
- Positive recurrence and null recurrence
- Periodicity