# Problem 1

(a). We know $k^* = \lceil (n+1)p \rceil$

$$P_X(k) = C^n_k \cdot p^k \cdot (1-p)^{n-k} = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}, \quad k = 0, 1, \cdots, n.$$

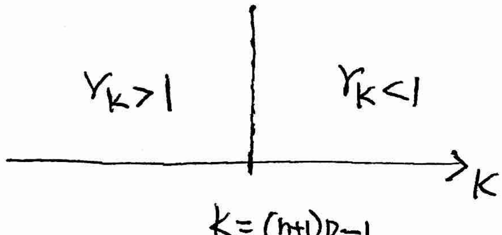
Now, we derive the following ratio

$$r_k := \frac{P_X(k+1)}{P_X(k)} = \frac{\frac{n!}{(k+1)!(n-(k+1))!} p^{k+1} (1-p)^{n-(k+1)}}{\frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}} = \frac{n-k}{k+1} \cdot \frac{p}{1-p}$$

Hence, we know $r_k$ is decreasing with $k$

$$r_k < 1 \iff \frac{n-k}{k+1} \frac{p}{1-p} < 1 \iff p(n-k) < (k+1)(1-p)$$

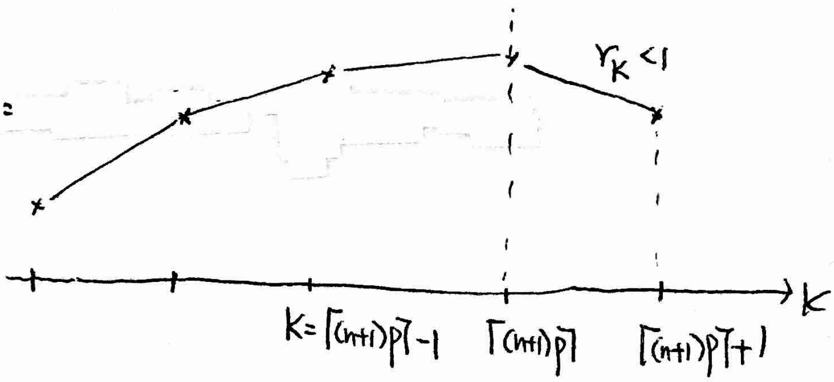
$$\iff (n+1)p < k+1 \quad \text{---} (1)$$




By (1) and the fact that $k$ is an integer,

We know $\begin{cases} r_k < 1, & k \geq \lceil (n+1)p \rceil = k^* \\ r_k \geq 1, & 1 \leq k \leq \lceil (n+1)p \rceil - 1 \end{cases}$

This implies:



(b). Define $Y = X + 7$.

Then, we know $Y \sim NB(p=0.05, r=7)$.

Therefore, the PMF of $X$, $p_X(k) = \begin{cases} C_6^{k+6} \cdot (0.05)^7 \cdot (0.95)^k & , k=0,1,2,\dots \\ 0 & , \text{otherwise} \end{cases}$

$\square$

(c). Under the condition $n=3$, the $\overbrace{\text{probability that the Astros win the series}}^{\text{For ease of notation, we denote this as } W_3}$ is :

$$W_3 = C_2^3 \cdot q^2 \cdot (1-q) + C_3^3 q^3 = -2q^3 + 3q^2$$

Similarly, for the case of $n=5$, we have

$$W_5 = C_3^5 q^3 \cdot (1-q)^2 + C_4^5 q^4 \cdot (1-q) + C_5^5 q^5$$

$$= 6q^5 - 15q^4 + 10q^3$$

Then, $W_5 > W_3 \iff 6q^5 - 15q^4 + 10q^3 > -2q^3 + 3q^2$

$\iff 6q^5 - 15q^4 + 12q^3 - 3q^2 > 0$

$\overset{\text{divide by } 3q^2 \text{ on both sides}}{\iff} 2q^3 - 5q^2 + 4q - 1 > 0$

$\iff (q-1)(2q-1)(q-1) > 0$

Therefore, we conclude that $W_5 > W_3$ if and only if $\frac{1}{2} < q < 1$.

$\square$

(d). $X_1 \sim \text{Geometric}(p_1)$, $X_2 \sim \text{Geometric}(p_2)$, $X = \min(X_1, X_2)$.

For any $k \geq 0$,

$$\begin{aligned}
P(X>k) &= P(\min(X_1,X_2)>k) \\
&= P(X_1>k \text{ and } X_2>k) \\
&= (1-p_1)^k \cdot (1-p_2)^k \\
&= ((1-p_1)(1-p_2))^k \\
&= [1-(p_1+p_2-p_1p_2)]^k
\end{aligned}$$

We know $X$ is a Geometric r.v. with parameter $(p_1+p_2-p_1p_2)$

Therefore, the PMF of $X$ is: $p_X(n) = \begin{cases} (1-(p_1+p_2-p_1p_2))^{n-1} \cdot (p_1+p_2-p_1p_2), & n \geq 1 \\ 0, & \text{otherwise} \end{cases}$ #

(e) Originally, Amy got 1236 votes, Bill got 1226 votes ) a margin of 10 votes.

Define a random variable $X$ = # of illegal votes that were count towards Amy.

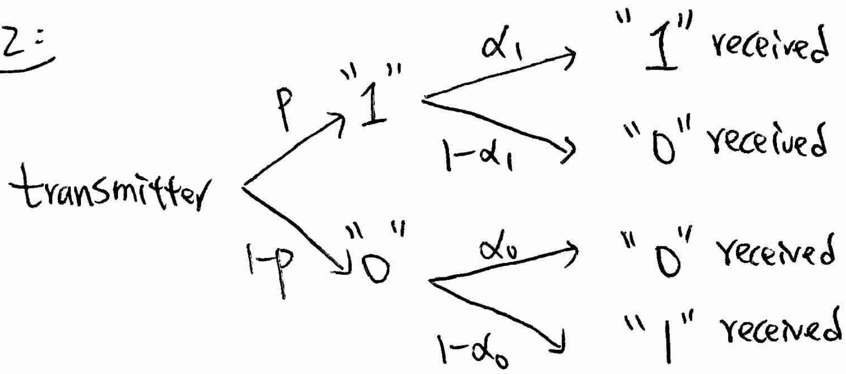
Then, $X \sim \text{HyperGeometric}(N=2462, D=1236, n=57)$

P( the result of the election will be changed)

$= P(X>33) \approx \underline{0.095}$ #

by Matlab's "hygecdf" function.

Problem 2:




(a). Define $X$ = # of 1's transmitted in the interval

$V$ = # of transmitted bits " "

$$P(X=k) = \sum_{n=0}^{\infty} P(X=k \mid V=k+\underbrace{n}_{\text{\# of 0's transmitted}}) \cdot P(V=k+n)$$

$$= \sum_{n=0}^{\infty} \left( C_k^{k+n} \cdot p^k \cdot (1-p)^n \right) \cdot \frac{e^{-\lambda T} \cdot (\lambda T)^{k+n}}{(k+n)!}$$

$$= \sum_{n=0}^{\infty} \frac{\cancel{(k+n)!}}{k!\,n!} p^k (1-p)^n \frac{e^{-\lambda T} (\lambda T)^{k+n}}{\cancel{(k+n)!}}$$

$$= \frac{e^{-\lambda p T}\, p^k \cdot (\lambda T)^k}{k!} \sum_{n=0}^{\infty} \frac{e^{-\lambda(1-p)T} \cdot (1-p)^n \cdot (\lambda T)^n}{n!}$$

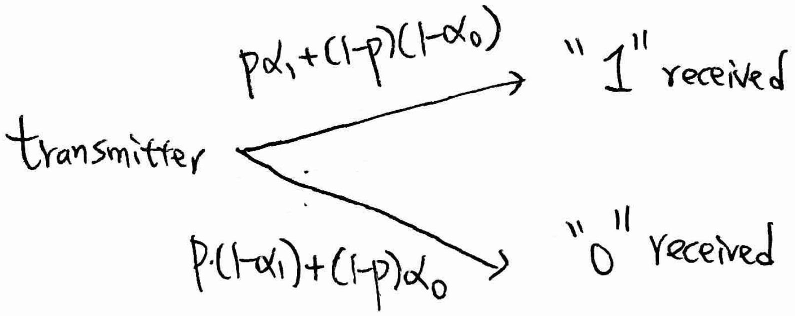
PMF of a Poisson( $\lambda(1-p), T$ )

This sum is 1.

$$= \frac{e^{-(\lambda p)T} ((\lambda p) \cdot T)^k}{k!}$$

Remark: The above procedure is usually called "Bernoulli splitting of a Poisson r.v."

(b). The communication chahell can be cast as follows:




By reusing the results of (a), we know $Y$ is also Poisson, with an average rate $\lambda\cdot[p\alpha_1+(1-p)(1-\alpha_0)]$.

Hence, the PMF of $Y$ =

$$\begin{cases} \dfrac{e^{-\lambda[p\alpha_1+(1-p)(1-\alpha_0)]T}\cdot(\lambda[p\alpha_1+(1-p)(1-\alpha_0)]T)^k}{k!}, & k\geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(c)

Define $\lambda_1^* = \lambda_1\cdot(p\alpha_1+(1-p)(1-\alpha_0))$, $\lambda_2^* = \lambda_2(p\alpha_2+(1-p)(1-\alpha_0))$

By (b): $T_1 \sim \text{Poisson}(\lambda_1^*, T)$, $T_2 \sim \text{Poisson}(\lambda_2^*, T)$.

For any $k\geq 0$:

$$P(Z=k) = \sum_{m=0}^{k} P(T_1=m)\cdot P(T_2=k-m).$$

$$= \sum_{m=0}^{k} \frac{e^{-\lambda_1^* T}\cdot(\lambda_1^* T)^m}{m!}\cdot\frac{e^{-\lambda_2^* T}\cdot(\lambda_2^* T)^{k-m}}{(k-m)!}$$

$$= \frac{e^{-(\lambda_1^*+\lambda_2^*)T}\cdot T^k}{k!}\cdot\sum_{m=0}^{k}\underbrace{\frac{k!}{m!(k-m)!}}_{=C_m^k}\lambda_1^{*m}\cdot(\lambda_2^*)^{k-m}$$

$$= \frac{e^{-(\lambda_1^*+\lambda_2^*)T}\cdot T^k}{k!}\underbrace{\sum_{m=0}^{k} C_m^k\cdot\lambda_1^{*m}\cdot(\lambda_2^*)^{k-m}}_{=(\lambda_1^*+\lambda_2^*)^k} = \frac{e^{-(\lambda_1^*+\lambda_2^*)T}\cdot((\lambda_1^*+\lambda_2^*)T)^k}{k!}$$

Remark:

Sum of 2 independent Poisson r.v.s is still Poisson.

For any $k<0$:

$$P(Z=k)=0.$$

Problem 3: $H(X) := -\sum_{i=1}^{n} P_i \ln P_i$

(a)

Method 1: Use Lagrangian to solve a constrained optimization problem

Note that $H(X)$ is a concave function since the Hessian of $(-H(X))$ is positive semi-definite

$$\frac{\partial H(X)}{\partial P_i} = -\ln P_i - 1, \quad \frac{\partial H(X)}{\partial P_j \partial P_i} = 0.$$

$$\frac{\partial H(X)}{\partial P_i^2} = -\frac{1}{P_i}$$

$$\text{Hessian of } H(X) = \begin{bmatrix} -\frac{1}{P_1} & & & 0 \\ & -\frac{1}{P_2} & & \\ & & \ddots & \\ 0 & & & -\frac{1}{P_n} \end{bmatrix}$$

Then, this problem is essentially a convex optimization problem with a linear constraint:

$$\boxed{\begin{aligned} &\max_{\{P_i\}} \; -\sum_{i=1}^{n} P_i \ln P_i \\ &\text{subject to} \quad \sum_{i=1}^{n} P_i = 1. \end{aligned}}$$

Next, we shall write down the Lagrangian as:

$$L(\lambda) := -\sum_{i=1}^{n} P_i \ln P_i - \lambda\left(\sum_{i=1}^{n} P_i - 1\right)$$

$$\frac{\partial L(\lambda)}{\partial P_i} = -\ln P_i - 1 - \lambda \quad \text{———} (*)$$

By the optimization theory, we know the maximum occurs when $\frac{\partial L(\lambda)}{\partial P_i} = 0, \ \forall i$

Therefore, by $(*)$, we conclude that the maximum of $H(X)$ is achieved at $P_1 = P_2 = \cdots = P_n = \frac{1}{n}$.

Then, the maximum possible $H(X)$ is $\ln n$.

#

## Method 2: Use Jensen's inequality

**Lemma (Jensen's inequality):**

If $f$ is a concave function and $Y$ is a discrete random variable,

then $f(E[Y]) \geq E[f(Y)]$.

$$H(X) := -\sum_{i=1}^{n} P_i \ln P_i = E\left[\ln\left(\frac{1}{P(X)}\right)\right]$$

$$\leq \ln\left(E\left[\frac{1}{P(X)}\right]\right) = \ln\left(\sum_{i=1}^{n} P_i \cdot \frac{1}{P_i}\right) = \ln n$$

$\ln(\cdot)$ is a concave function

It is easy to verify that this maximum is achieved at $P_1 = P_2 = \cdots = P_n = \frac{1}{n}$.

(b). We can observe that $H(X) \geq 0$ as $P_i \leq 1$, $\forall i$

Moreover, we know $H(X)$ is exactly zero if there exists a $j \in \{1, 2, \cdots, n\}$ such that $P_j = 1$ (and $P_k = 0$ for all $k \neq j$).

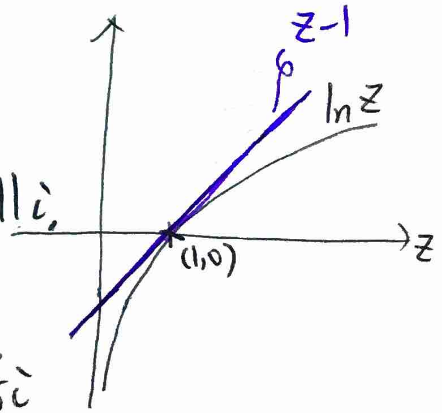
Hence, the minimum possible value of $H(X)$ is 0, and this can be achieved by any PMF $\{P_i\}$ that satisfies $P_j = 1$, for some $j \in \{1, 2, \cdots, n\}$.

$\square$

(c). $H(X) + \sum_{i=1}^{n} P_i \ln q_i$

$$= \sum_{i=1}^{n} P_i \ln\left(\frac{q_i}{P_i}\right) \overset{(*)}{\leq} \sum_{i=1}^{n} P_i \left(\frac{q_i}{P_i} - 1\right) = \sum_{i=1}^{n} q_i - P_i = 0$$

use the property that $\ln z \leq z - 1$, $\forall z > 0$




Moreover, note that the inequality $(*)$ is actually an equality if and only if $q_i = P_i$, for all $i$.

Therefore, we conclude that $H(X) = -\sum_{i=1}^{n} P_i \ln q_i$ if and only if $P_i = q_i$, for all $i \in \{1, 2, \cdots, n\}$.

$\square$

## Problem 4

(a). Since $X \sim Unif(a,b)$, then $P_X(k) = \begin{cases} \frac{1}{b-a+1}, & k=a, a+1, \cdots, b \\ 0, & \text{otherwise} \end{cases}$

Then, $E[X] = \sum_{i=a}^{b} \frac{i}{b-a+1} = \frac{1}{\cancel{b-a+1}} \cdot \frac{\cancel{(b-a+1)}(a+b)}{2} = \frac{a+b}{2}$ #

To find $Var[X]$, we define another random variable $Y := X-a$.

Then, we know $Y \sim Unif(0, b-a)$ and $Var[Y] = Var[X]$.

(as translation does not change the variance).

$$E[Y] = \frac{b-a}{2}$$

$$E[Y^2] = \sum_{i=0}^{b-a} \frac{i^2}{b-a+1} = \frac{1}{b-a+1}\left(\sum_{i=0}^{b-a} i^2\right) = \frac{1}{\cancel{b-a+1}} \cdot \frac{(b-a)\cancel{(b-a+1)}(2(b-a)+1)}{6}$$

$$= \frac{(b-a)\cdot(2(b-a)+1)}{6}$$

$$Var[Y] = E[Y^2] - (E[Y])^2$$

$$= \frac{(b-a)\cdot(2(b-a)+1)}{6} - \left(\frac{b-a}{2}\right)^2$$

$$= \frac{(b-a)^2 + 2(b-a)}{12}$$

$$= \frac{(b-a+1)^2 - 1}{12}$$ #

(b). $X \sim NB(p, r)$, then $p_X(k) = \begin{cases} C^{k-1}_{r-1} p^r (1-p)^{k-r}, & k \geq r \\ 0, & \text{else.} \end{cases}$

$$E[X] = \sum_{k=r}^{\infty} \frac{r \cdot k \cdot (k-1)!}{r \cdot (r-1)!(k-r)!} p^r (1-p)^{k-r}$$

$$= \frac{r}{p} \cdot \sum_{k=r}^{\infty} \underbrace{\frac{k!}{r!(k-r)!}}_{= C^k_r} p^{r+1} \cdot (1-p)^{k-r}$$

$$= \frac{r}{p} \sum_{k=r}^{\infty} C^{(k+1)-1}_{(r+1)-1} \cdot p^{r+1} \cdot (1-p)^{(k+1)-(r+1)}$$

let $m = k+1$

$$\overset{\downarrow}{=} \frac{r}{p} \underbrace{\sum_{m=r+1}^{\infty} C^{m-1}_{(r+1)-1} \cdot p^{r+1} (1-p)^{m-(r+1)}}_{\text{sum is } 1}$$

the PMF of $NB(p, r+1)$

$$= \frac{r}{p}.$$

#

$$E[X^2] = \sum_{k=r}^{\infty} \frac{\overset{= k(k+1)-k}{k^2} \cdot (k-1)!}{(r-1)!(k-r)!} p^r (1-p)^{k-r}$$

$$= \left( \sum_{k=r}^{\infty} \frac{r(r+1) k \cdot (k+1) \cdot (k-1)!}{(r+1) \cdot r (r-1)!(k-r)!} p^r \cdot (1-p)^{k-r} \right) - \left( \underbrace{\sum_{k=r}^{\infty} k \cdot \frac{(k-1)!}{(r-1)!(k-r)!} p^r \cdot (1-p)^{k-r}}_{= E[X]} \right)$$

$$= \frac{r(r+1)}{p^2} \cdot \sum_{k=r}^{\infty} \frac{(k+1)!}{(r+1)!(k-r)!} p^{r+2} \cdot (1-p)^{k-r} - E[X]$$

$$= \frac{r(r+1)}{p^2} \sum_{k=r}^{\infty} C^{(k+2)-1}_{(r+2)-1} \cdot p^{(r+2)} \cdot (1-p)^{(k+2)-(r+2)} - E[X]$$

$$= \frac{r(r+1)}{p^2} - \frac{r}{p}.$$

Therefore, $Var[X] = E[X^2] - (E[X])^2 = \left( \frac{r(r+1)}{p^2} - \frac{r}{p} \right) - \frac{r^2}{p^2} = \frac{r(1-p)}{p^2}$

#

(c). $X \sim \text{HyperGeometric}(N, D, n)$, then $P_X(k) = \begin{cases} \dfrac{C^D_k \cdot C^{N-D}_{n-k}}{C^N_n} & , k=0,1,\cdots,n \\ 0 & , \text{else} \end{cases}$

P.11

$$E[X] = \sum_{k=1}^{n} k \cdot \frac{C^D_k \cdot C^{N-D}_{n-k}}{C^N_n}$$

$$= \sum_{k=1}^{n} \frac{k \cdot \frac{D!}{k!(D-k)!} \cdot C^{N-D}_{n-k}}{C^N_n}$$

$$= D \cdot \sum_{k=1}^{n} \frac{\overbrace{\frac{(D-1)!}{(k-1)!(D-k)!}}^{C^{D-1}_{k-1}} \cdot C^{N-D}_{n-k}}{\underbrace{C^N_n}_{\frac{N}{n} C^{N-1}_{n-1}}}$$

$$= \frac{Dn}{N} \cdot \underbrace{\sum_{k=1}^{n} \frac{C^{D-1}_{k-1} C^{(N-1)-(D-1)}_{(n-1)-(k-1)}}{C^{N-1}_{n-1}}}_{\text{Sum is } 1}$$ the PMF of HyperGeometric$(N-1, D-1, n-1)$.

$$= \frac{Dn}{N}$$

#

Problem 5:

Define $P(X=a_i)=p_i$, $P(Y=a_i)=q_i$, and write $E[X]=E[Y]\triangleq C_1$

Since $E[X]=E[Y]$ and $Var[X]=Var[Y]$, then $E[X^2]=E[Y^2]\triangleq C_2$

For $X$, we have a system of linear equations as:

$$\begin{aligned} p_1+p_2+p_3 &= 1 \\ a_1p_1+a_2p_2+a_3p_3 &= C_1 \\ a_1^2p_1+a_2^2p_2+a_3^2p_3 &= C_2 \end{aligned} \xrightarrow{\text{written in matrix form}} \underbrace{\overbrace{\begin{bmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{bmatrix}}^{V}}_{\text{usually called ``Vandermonde matrix''}} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 \\ C_1 \\ C_2 \end{bmatrix}$$

Similarly, for $Y$, we have

$$\begin{bmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 \\ C_1 \\ C_2 \end{bmatrix}.$$

It is easy to verify that $V$ is invertible.

Therefore, we have $\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = V^{-1}\begin{bmatrix} 1 \\ C_1 \\ C_2 \end{bmatrix} \Rightarrow p_i=q_i$, for all $i=1,2,3$.

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = V^{-1}\begin{bmatrix} 1 \\ C_1 \\ C_2 \end{bmatrix}$$

$\square$