DCP 1206: Probability
Lecture 12 — Joint Distribution of Two Random Variables

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October 25, 2019

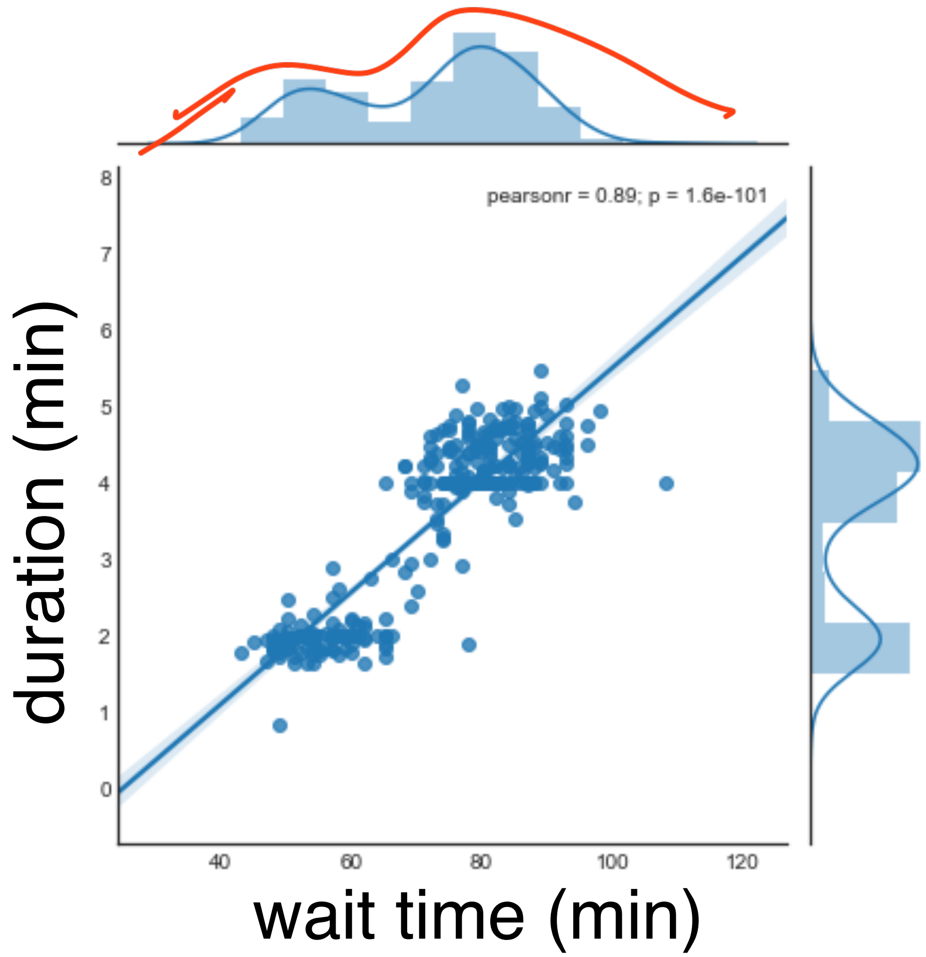
Announcements

- Midterm on 11/6 (on Wednesday)
  - 10:10am - 12pm
  - Coverage: Lec 1 - Lec 13
  - You are allowed to bring a cheat sheet (A4 size)

- HW3 is on E3 (Due: 11/1 in class)

Why Jointly Study 2 Random Variables?

- Example: Old Faithful Geyser

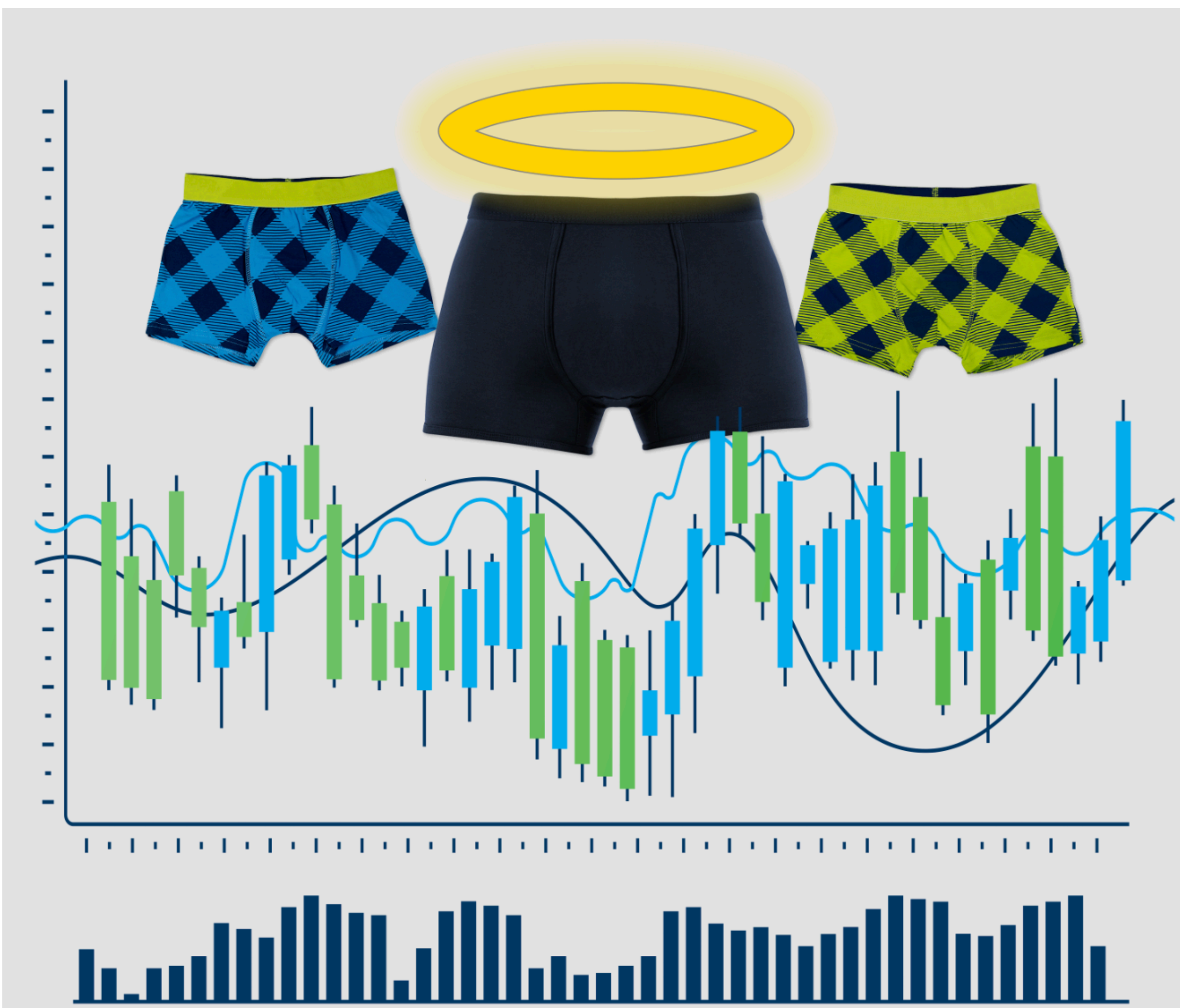


- Eruption duration could help predict the next wait time

http://www.stat.cmu.edu/~larry/all-of-statistics/=data/faithful.dat

Men’s Underwear Index (MUI)

- “MUI is an economic index that can supposedly detect the beginnings of a recovery during an economic slump”… (Wiki)



Alan Greenspan

X = $ sales of mens underwear

Y = % of growth of GDP

http://www.washingtonpost.com/wp-dyn/content/article/2009/08/30/AR2009083002761.html
https://medium.com/@nadexexchange/the-me-too-movement-says-the-underwear-index-stinks-a34d7c5c1430

Hemline Index

- “ …hemlines on women's dresses rise along with stock prices.”

X = avg length of skirt

Y = % of GDP growth

https://imarknaden.se/2018/02/26/258/

This Lecture

1. Joint CDF and Marginal CDF

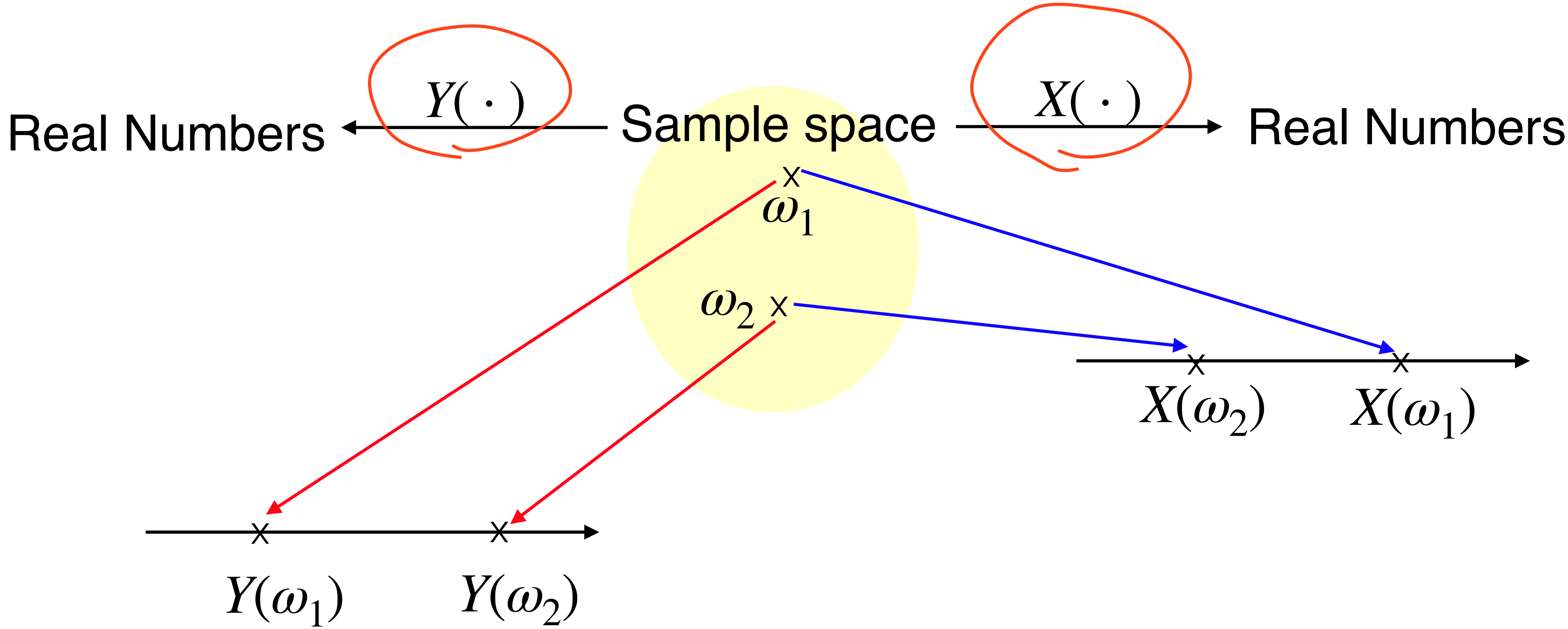
2. Joint PMF and Marginal PMF

3. Joint PDF and Marginal PDF

- Reading material: Chapter 8.1

1. Joint CDF and Marginal CDF

Recall: Random Variables Defined on Ω



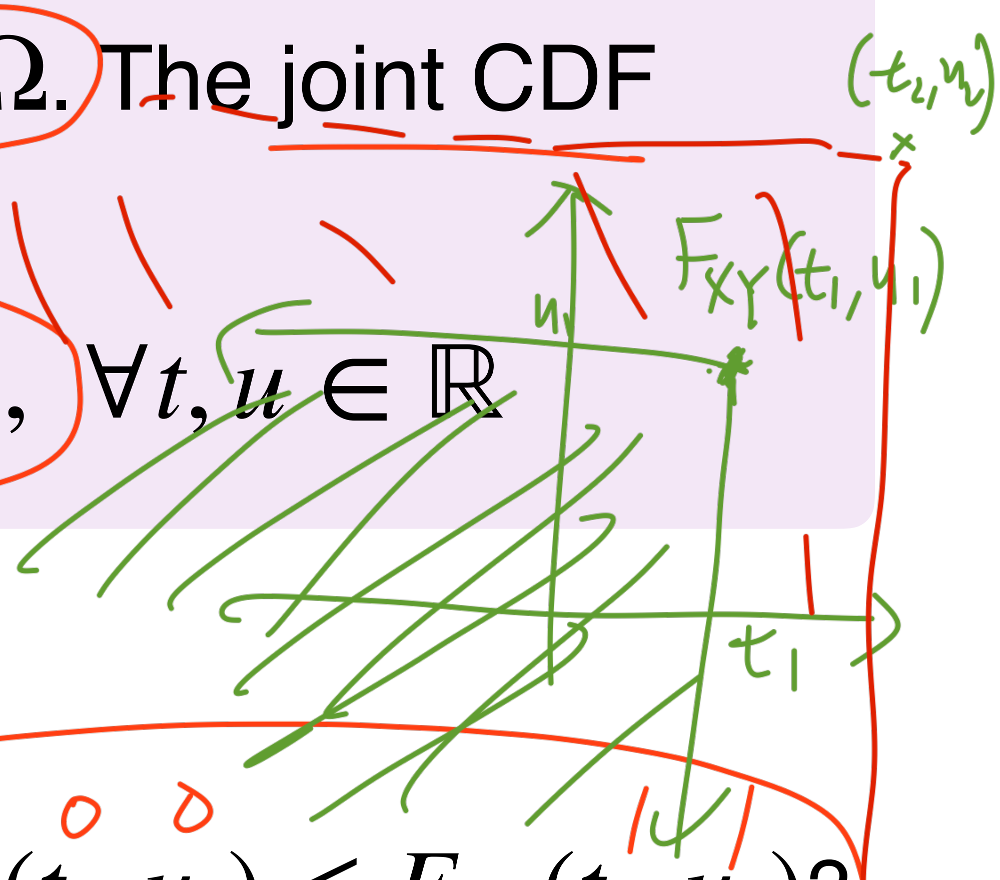
- Could we study the CDF regarding both X and Y?

Joint CDF

Joint CDF: Let X and Y be two random variables defined on the same sample space Ω. The joint CDF $F_{XY}(t,u)$ is defined as

$$F_{XY}(t,u) = P(X \le t, Y \le u), \quad \forall t, u \in \mathbb{R}$$

and



- $0 \le F_{XY}(t,u) \le 1$? ✓
- Suppose $t_1 \le t_2$ and $u_1 \le u_2$, then $F_{XY}(t_1,u_1) \le F_{XY}(t_2,u_2)$?
- What is $F_{XY}(\infty,\infty)$? How about $F_{XY}(-\infty,-\infty)$?

$P(X \le \infty, Y \le \infty) = 1$ $\quad$ $P(X \le -\infty, Y \le -\infty) = 0$

Event Probabilities and Joint CDF (I)

$$F_{XY}(t, u) = P(X \leq t, Y \leq u), \ \forall t, u \in \mathbb{R}$$

- $P(X \leq t)$ = ? $= P(X \leq t, Y \leq \infty) = F_{XY}(t, \infty)$

- $P(Y \leq u)$ = ? $= P(X \leq \infty, Y \leq u) = F_{XY}(\infty, u)$

Marginal CDF

Marginal CDF: Let X and Y be two random variables defined on the same sample space Ω, and the joint CDF is $F_{XY}(t, u)$. The marginal CDF of X and Y are

$$F_X(t) = P(X \leq t) = F_{XY}(t, \infty)$$

$$F_Y(t) = P(Y \leq t) = F_{XY}(\infty, t)$$

Event Probabilities and Joint CDF (II)

$$F_{XY}(t, u) = P(X \leq t, Y \leq u), \ \forall t, u \in \mathbb{R}$$

- $\underbrace{P(t_1 < X \leq t_2) = ?}$ $\quad P(\underbrace{t_1 < X \leq t_2}, Y \leq \infty)$

  $= P(X \leq t_2, Y \leq \infty) - P(X \leq t_1, Y \leq \infty)$

  $= F_{XY}(t_2, \infty) - F_{XY}(t_1, \infty)$

- $\underbrace{P(u_1 < Y \leq u_2) = ?}$

  $= F_{XY}(\infty, u_2) - F_{XY}(\infty, u_1)$

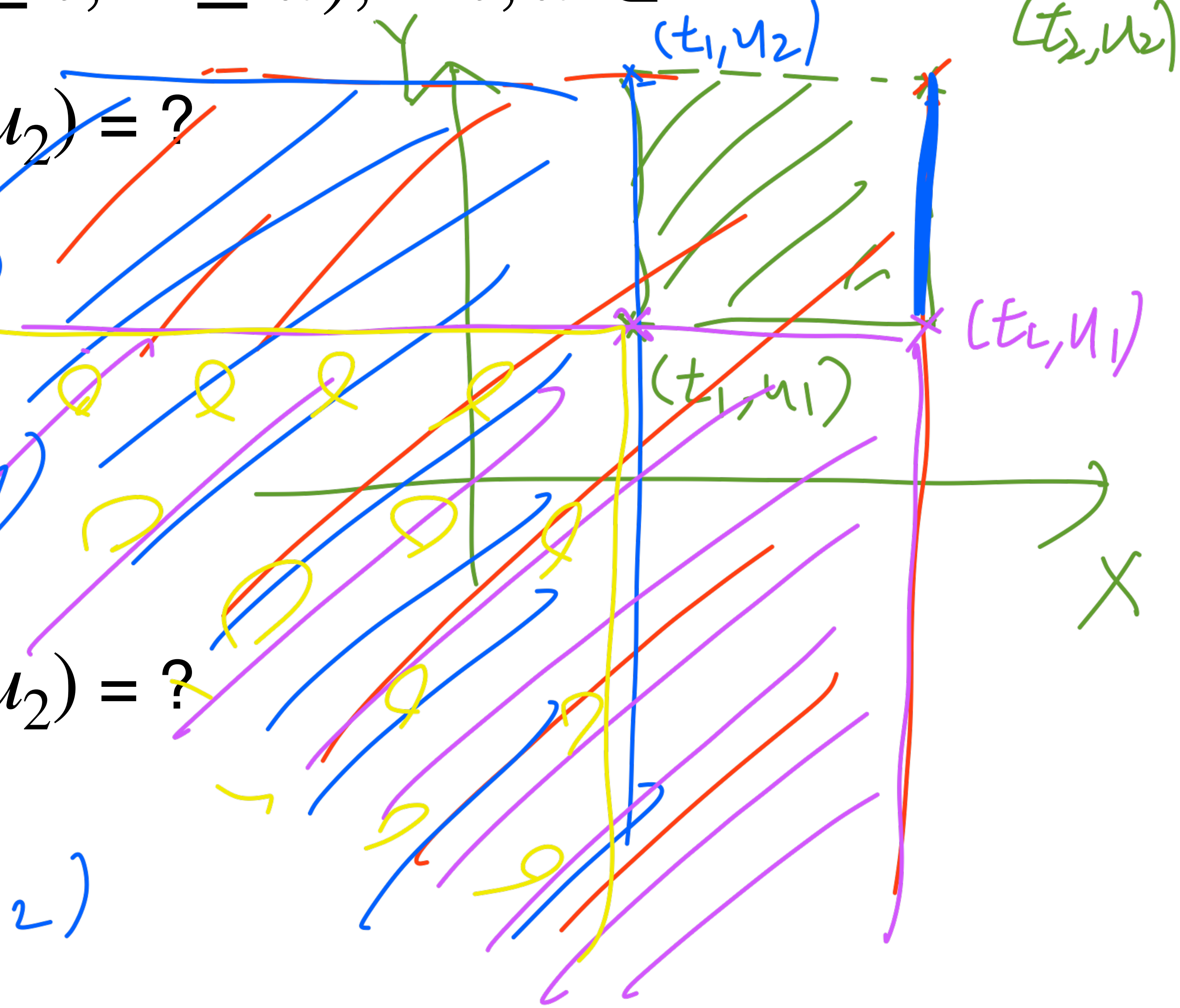
Event Probabilities and Joint CDF (III)

$$F_{XY}(t,u) = P(X \leq t, Y \leq u), \;\; \forall t, u \in \mathbb{R}$$

- $P(t_1 < X \leq t_2, u_1 < Y \leq u_2) = ?$

$F_{XY}(t_2,u_2) - F_{XY}(t_1,u_2)$

$- F_{XY}(t_2,u_1) + F_{XY}(t_1,u_1)$



- $P(t_1 < X < t_2, u_1 < Y \leq u_2) = ?$

$F_{XY}(t_2^-,u_2) - F_{XY}(t_1,u_2)$

$- F_{XY}(t_2^-,u_1) + F_{XY}(t_1,u_1)$

2. Joint PMF and Marginal PMF

Joint PMF of 2 Discrete Random Variables

Joint PMF: Let X and Y be two discrete random variables defined on the same sample space Ω. The joint PMF $p(x, y)$ is defined as

$$p(x, y) = P(X = x, Y = y)$$

fair

toss 2 coins

H → 0

T → 1

$X =$ 1st toss

$Y =$ 2nd toss

- Let the sets of possible values of X and Y be S_X and S_Y
  - $P(X = x) = P(X=x, Y \in S_Y) = \sum_{y \in S_Y} p(x,y)$
  - $P(Y = y) = P(X \in S_X, Y=y) = \sum_{x \in S_X} p(x,y)$

Y \ X	0	1
0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{4}$

Marginal PMF

Marginal PMF: Let X and Y be two discrete random variables defined on the same sample space Ω, and the joint PMF is $p(x, y)$. The marginal PMF of X and Y are

$$P(X = x) = \sum_{y \in S_Y} p(x, y)$$

$$P(Y = y) = \sum_{x \in S_X} p(x, y)$$

where S_X and S_Y are the sets of possible values of X and Y

Example: From Joint PMF to Marginals

- Example: Let the joint PMF of X and Y be

$$p(x,y) = \begin{cases} \frac{1}{25}(x^2+y^2) & \text{, if } x = 1,2,\ y = 0,1,2 \\ 0 & \text{, otherwise} \end{cases}$$

- What is the marginal PMF of X and Y?

$$P(X=x) = \sum_{y=0,1,2} p(x,y) = \sum_{y=0,1,2} \frac{1}{25}(x^2+y^2) = \frac{3x^2}{25} + \frac{1}{5}$$

$$P(Y=y) = \sum_{x=1,2} p(x,y) = \sum_{x=1,2} \frac{1}{25}(x^2+y^2) = \left(\frac{2}{25}\right)y^2 + \left(\frac{1}{5}\right)$$

Example: Bernoulli and Poisson

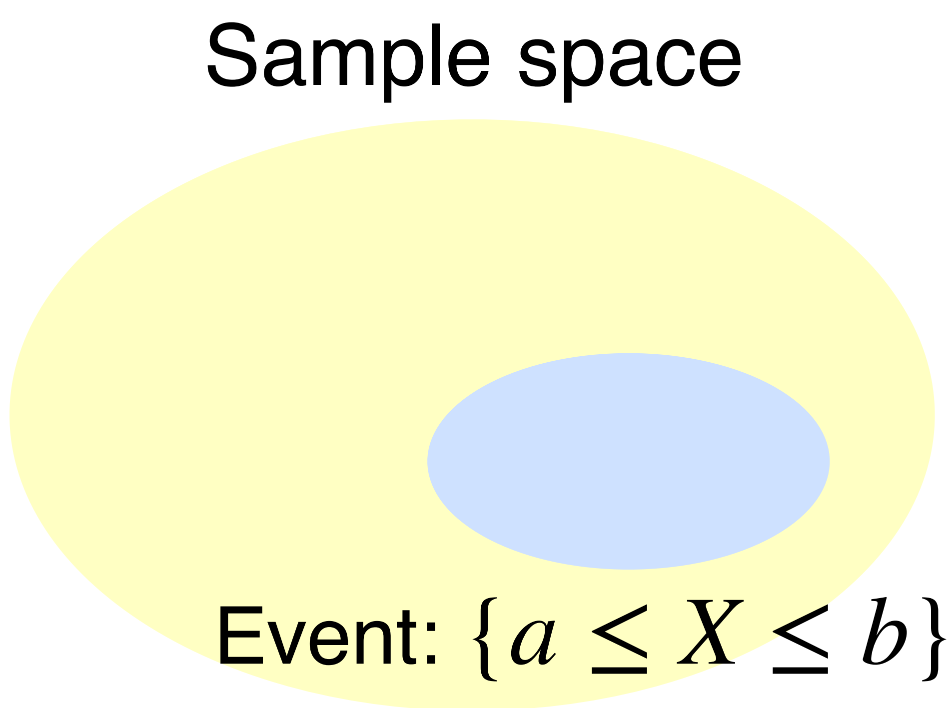
- Example: Consider $X \sim$ Bernoulli(p), $Z_0 \sim$ Poisson$(\lambda_0 T)$ and $Z_1 \sim$ Poisson$(\lambda_1 T)$ (all are independent). Suppose $Y = Z_0$ if $X = 0$, and $Y = Z_1$ if $X = 1$.
  - What is the joint PMF of X and Y?
  - What is the marginal PMF of Y?

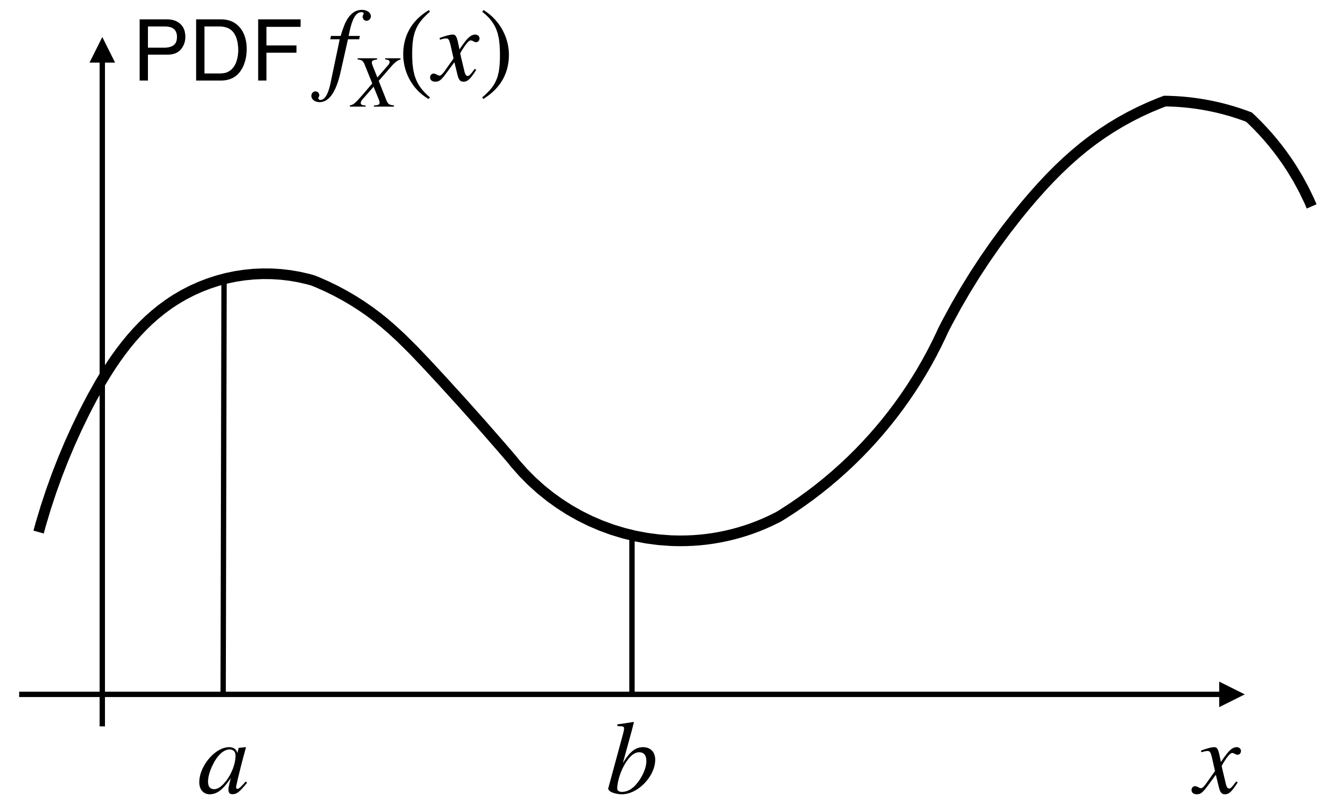
$$p(x,y) = \begin{cases} (1-p)\dfrac{e^{-\lambda_0 T}\cdot(\lambda_0 T)^y}{y!} & x=0,\ y=0,1,2,\cdots \\ p\cdot\dfrac{e^{-\lambda_1 T}\cdot(\lambda_1 T)^y}{y!} & x=1,\ y=0,1,2,\cdots \\ 0 & \text{otherwise} \end{cases}$$

$$p(Y=y) = \sum_{x=0,1} p(x,y) = \underbrace{\qquad}_{x=0} + \underbrace{\qquad}_{x=1}$$

3. Joint PDF and Marginal PDF

Recall: Probability Density Function (PDF)





Probability Density Function (PDF):
Let X be a random variable. Then, $f_X(x)$ is the PDF of X if for every subset B of the real line, we have

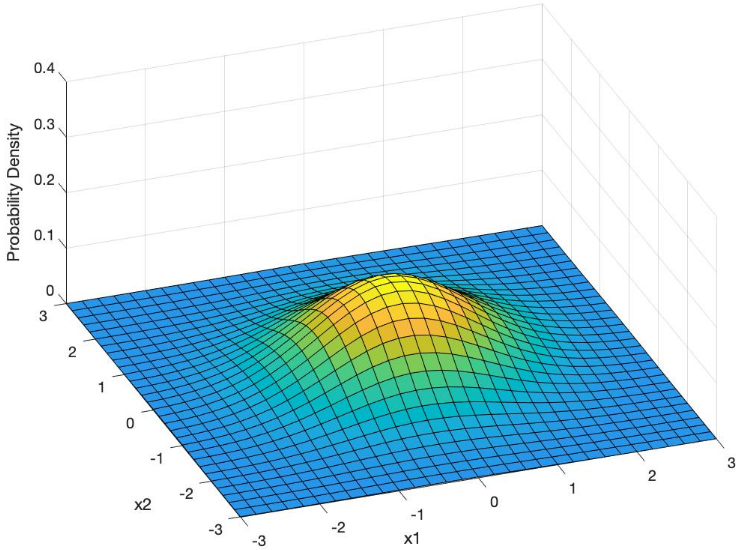
$$P(X \in B) = \int_B f_X(x)dx$$

Joint PDF

Joint PDF: Let X and Y be two continuous random variables. Then, $f_{XY}(x, y)$ is the joint PDF of X and Y if for every subset B of $\mathbb{R}^2$, we have

$$P((X, Y) \in B) = \iint_B f_{XY}(x, y)dxdy$$

- $P(X \in B_X, Y \in B_Y) =$
- $P(X \in \mathbb{R}, Y \in \mathbb{R}) =$



Joint PDF and Event Probabilities

$f_{XY}(x, y)$ is the joint PDF of X and Y

- $P(X \leq t) =$

- $P(Y \leq u) =$

- $f_X(x) =$

- $f_Y(y) =$

Marginal PDF

Marginal PDF: Let X and Y be two continuous random variables, and $f_{XY}(x, y)$ is the joint PDF of X and Y. The marginal PDF of X and Y are

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx$$

Example: Joint PDF

- Example: Let the joint PDF of X and Y be

$$f(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2) & \text{, if } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{, otherwise} \end{cases}$$

- Joint CDF of X and Y? The marginal CDF of X?
- How about the marginal PDF of Y?

1-Minute Summary

Joint PMF
$P(X = x, Y = y)$

$\sum_x P(\ )$

Marginal PMF
$P(X = x), P(Y = y)$

Joint CDF
$F_{XY}(x, y)$

$+\infty$

Marginal CDF
$F_X(x), F_Y(y)$

$= P(X \le x, Y \le y)$

Joint PDF
$f_{XY}(x, y)$

Marginal PDF
$f_X(x), f_Y(y)$

Next Lecture

1. Conditional distributions & Independent random variables

2. Bivariate Joint Normal Distributions