DCP 1206: Probability
Lecture 14 — Joint PDF and Expected Value Regarding 2 Random Variables

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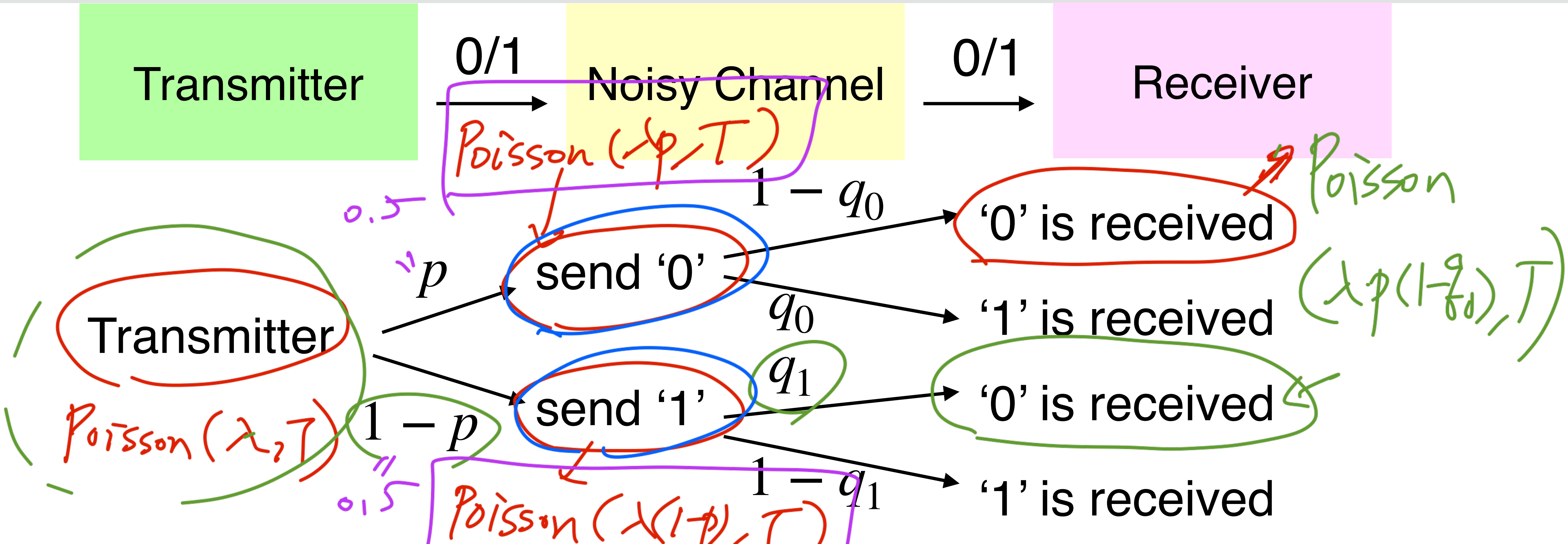
Announcements

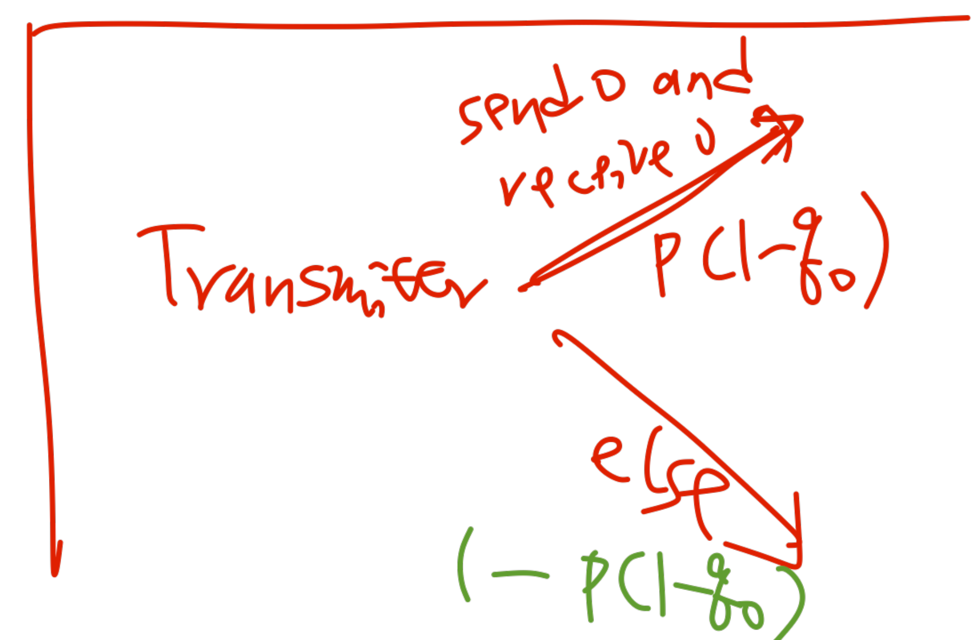
- TA hour
  - 11/4 (Monday), 7pm-8pm @ EC345
- Midterm
  - 11/6 (Wednesday), 10:10am-12pm
  - EC122: Student number ending with 1,3,5,7,9
  - SA321 (科學一館321): Student number ending with 0,2,4,6,8

HW2 Problem 2: Poisson and Binary Channel

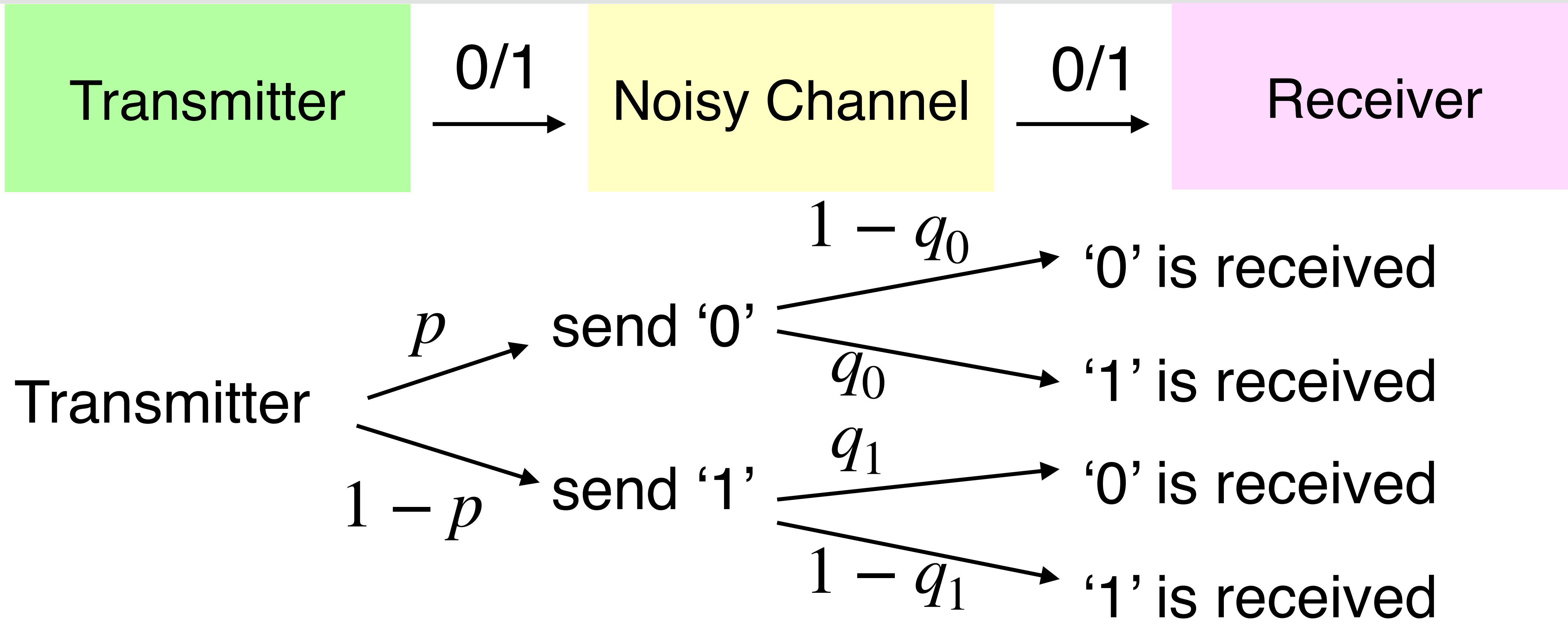


- Let N_T be the number of total transmitted bits in a time interval T
- Suppose N_T is Poisson with rate = λ

* Bernoulli splitting of a Poisson



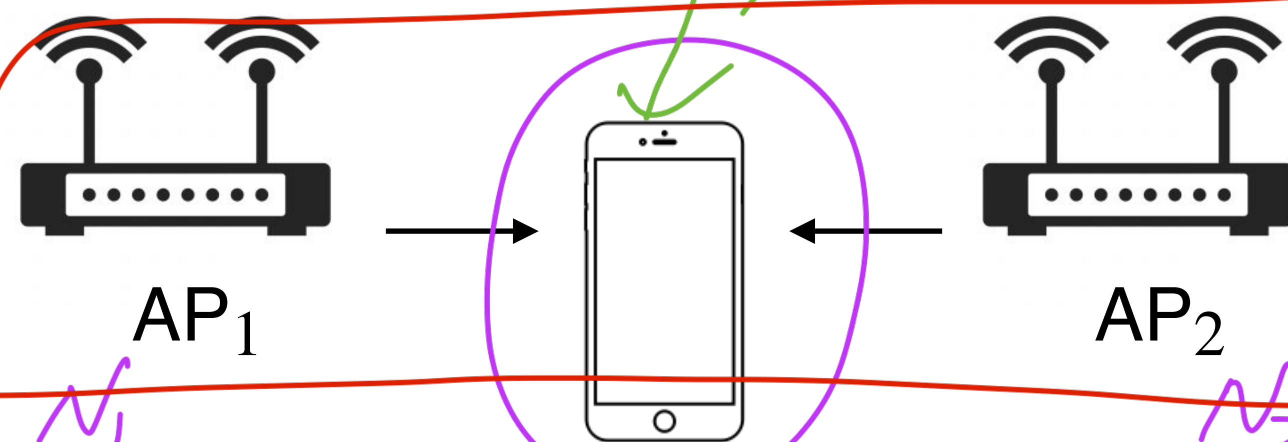
HW2 Problem 2: Poisson and Binary Channel



- Let N_T be the number of total transmitted bits in a time interval T
- Suppose N_T is Poisson with rate = λ

HW2 Problem 2(c): Sum of Poisson

AP3 $\quad N_3 \sim \text{Poisson}(\lambda_3, T)$



$N_1 \qquad N_2$

- Let N_1 and N_2 be the number of total bits transmitted by AP$_1$ and AP$_2$ in a time interval T, respectively
  - Suppose N_1 and N_2 are independent
  - Total received bits? (Y)

$N_1 \sim \text{Poisson}(\lambda_1, T)$

$N_2 \sim \text{Poisson}(\lambda_2, T)$

N_1, N_2 independence

$Y \sim \text{Poisson}(\lambda_1 + \lambda_2, T)$

$$\underbrace{P(Y=k) = \sum_{m=0}^{k} P(N_1 = m) \cdot P(N_2 = k-m)}$$

HW2 Problem 5:

$$Var[X] = E[X^2] - (E[X])^2$$

$$E[Y^2]$$

Problem 5 (Expected Value and Variance) (8 points)

Let X and Y be two discrete random variables with the identical set of possible values $\{a_1, a_2, a_3\}$ (a_1, a_2, and a_3 are different). Show that if $E[X] = E[Y]$ and $\text{Var}[X] = \text{Var}[Y]$, then X and Y are identically distributed.

$$P(X=a_i) = x_i$$

$$P(Y=a_i) = y_i$$

$$P(X=a_i) = P(Y=a_i) \quad \forall i = 1,2,3$$

$$x_1 + x_2 + x_3 = 1 \qquad y_1 + y_2 + y_3 = 1$$

$$a_1x_1 + a_2x_2 + a_3x_3 = E[X] \qquad a_1y_1 + a_2y_2 + a_3y_3 = E[Y]$$

$$a_1^2x_1 + a_2^2x_2 + a_3^2x_3 = E[X^2] \qquad a_1^2y_1 + a_2^2y_2 + a_3^2y_3 = E[Y^2]$$

$$A \begin{bmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} x = \begin{bmatrix} 1 \\ E[X] \\ E[X^2] \end{bmatrix} = b$$

$$Ax = b \Rightarrow x = A^{-1}b$$

HW2 Problem 1(b):

(b) For certain software, independently of other users, the probability that a user encounters a fault is 0.05. Suppose the users arrive sequentially. Let X be the number of users who do not encounter a fault before the 7th user who encounters a fault. What is the PMF of X?

fault : F

no fault : N

F N F F F F N F F $\longrightarrow$ $X = 2$

n-th fault

$Y = X + 7$

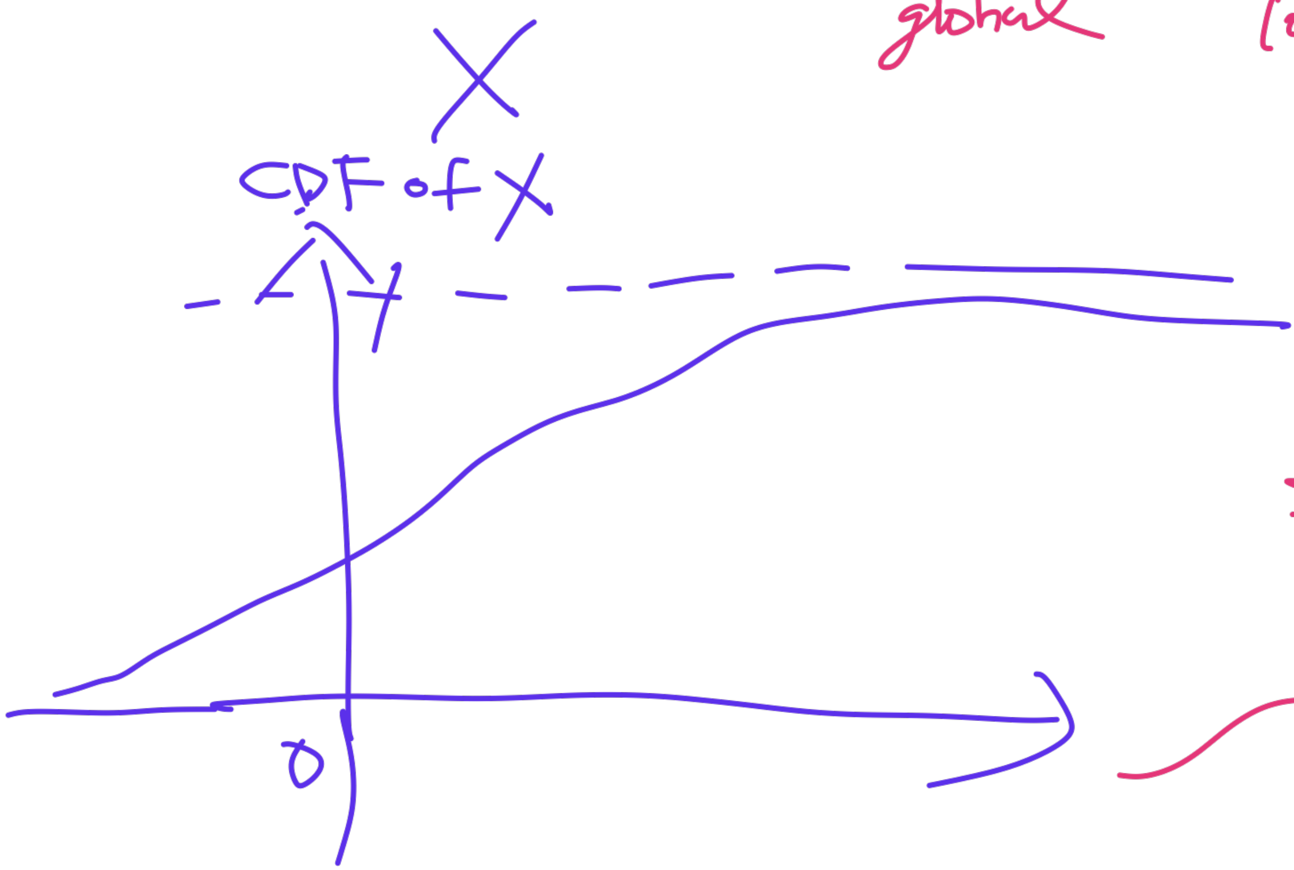
$\sim$

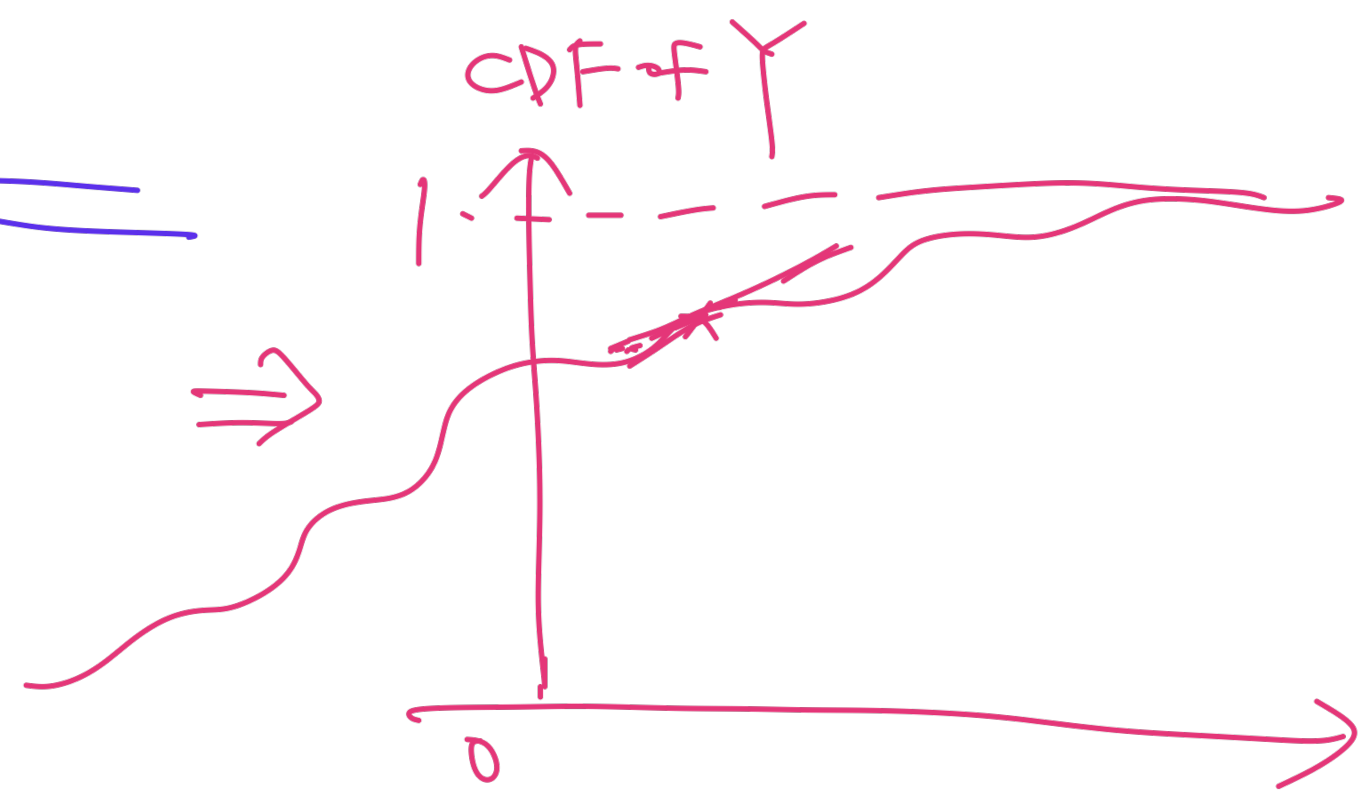
$NB(r=7, p=0.05)$

HW3 Problem 2: $Y = aX + b$

- Let X be an exponential random variable with parameter λ
    - Define $Y = aX + b \quad (a \neq 0)$
    - Find the CDF and PDF of Y
- Q: Why always go from CDF to PDF? Is it possible to directly get PDF?

global local





HW3 Problem 5: Existence of Moments

Problem 5 (Moments of Continuous Random Variables) (10+14=24 points)

The random variable X is said to be a *Laplace random variable* or *double exponentially distributed* if its PDF is given by

$$f(x) = C \cdot \exp(-|x|), \quad -\infty < x < \infty.$$

(a) Find the value of C. (Hint: leverage the symmetry of PDF and the fact that $\int_{-\infty}^{\infty} f(x)dx = 1$)

(b) Prove that $E[X^{2n}] = (2n)!$ and $E[X^{2n+1}] = 0$, for all $n \in \mathbb{N}$. (Hint: For $E[X^{2n}]$, you may want to use integration by parts. For $E[X^{2n+1}]$, in order to use symmetry, please explain whether $E[|X^{2n+1}|]$ exists or not by using the result that $E[X^{2n}] = (2n)! < \infty$)

Existence of Moments (Formally)

Existence of Moments:
Let X be a random variable. Then, the n-th moment of X (i.e. $E[X^n]$) is said to exist if $E[\,|X^n|\,] < \infty$

absolute

finite

Rearrangement of Series

$2k$

- Example: Consider a series $\{a_n\}$: $1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \cdots$
  - What is $\sum_{n=1}^{\infty} a_n$? $= 0$

$\sum_{n=1}^{\infty} |a_n| = \infty$

$\ln M - 1$ $\qquad$ $\ln 2M - 0$

- Example: Rearrange $\{a_n\}$ as $\{b_n\}$:

$\ln M \approx \underbrace{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{M}}_{M}, -1, \underbrace{\frac{1}{M+1}, \cdots, \frac{1}{2M}}_{M}, \underbrace{-\frac{1}{2}}\cdots, \quad \cdots$

  - What is $\sum_{n=1}^{\infty} b_n$? $\approx \ln M$

Riemann Rearrangement Theorem

Riemann Rearrangement Theorem:
Let $\{a_n\}$ be a sequence of numbers. If $\{a_n\}$ satisfies that

1. $\sum_{n=1}^{\infty} a_n$ converges

2. $\sum_{n=1}^{\infty} |a_n| = \infty$

1, 100, 10000, ... ∞

Then, for any $B \in \mathbb{R} \cup \{\infty\}$, there exists a rearrangement $\{b_n\}$ of $\{a_n\}$ such that $\sum_{n=1}^{\infty} b_n = B$

Quick Review

$$
\begin{tikzcd}
\begin{array}{c}\text{Joint CDF}\\ F_{XY}(x,y)\end{array} \arrow[d, shift right] & \begin{array}{c}\text{Marginal CDF}\\ F_X(x), F_Y(y)\end{array} \\
\begin{array}{c}\text{Joint PDF}\\ f_{XY}(x,y)\end{array} \arrow[u, shift right] \arrow[ur] \arrow[r] & \begin{array}{c}\text{Marginal PDF}\\ f_X(x), f_Y(y)\end{array}
\end{tikzcd}
$$

Marginal PDF to Joint PDF?

- Question: Could we get joint PDF from marginal PDF?

Joint CDF $F_{XY}(x, y)$ ← Marginal CDF $F_X(x), F_Y(y)$

In general: NO

"Independent": Yes

Joint PDF $f_{XY}(x, y)$ ← Marginal PDF $f_X(x), f_Y(y)$

This Lecture

1. Marginal PDF → Joint PDF

2. Expected Value Regarding 2 Random Variables

- Reading material: Chapter 8.1-8.2

1. Marginal PDF $\rightarrow$ Joint PDF

Property: Independence of 2 Continuous Random Variables

Joint PDF is the product of the marginal PDFs under independence:

If two continuous random variables X, Y are **independent**, then the joint PDF satisfies that

$$\underline{f_{XY}(t, u) = f_X(t) \cdot f_Y(u)}$$

Similar property:

$$\left(F_{XY}(t,u) = F_X(t) \cdot F_Y(u) \right)$$

- Proof:

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$$

$$= \frac{\partial^2 F_X(x) \cdot F_Y(y)}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F_X(x) \cdot F_Y(y)}{\partial y}\right)$$

$$= \frac{\partial}{\partial x} F_X(x) \cdot \left(\underbrace{\frac{\partial F_Y(y)}{\partial y}}_{= f_Y}\right) = f_X(x) f_Y(y)$$

Example: Uniform and Exponential

Unif(a,b) $\downarrow$ PDF $\frac{1}{b-a}$

- Example: $X \sim$ Unif$(0,1)$ and $Y \sim$ Exp$(\lambda = 1)$ be two independent continuous uniform random variables.
  - What is the joint PDF?

$$f_{XY} = f_X \cdot f_Y$$

$$f_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \le 0 \\ 0, & x \ge 1 \end{cases}$$

$$f_Y(y) = \begin{cases} 0, & y \le 0 \\ e^{-y}, & y > 0 \end{cases}$$

	$y \le 0$	$y > 0$
$0 < x < 1$	.	$1 \cdot e^{-y} = e^{-y}$
$x \le 0$	.	.
$x \ge 1$	.	.

Example: 2 Uniform Random Variables

- Example: $X \sim$ Unif$(0{,}2)$ and $Y \sim$ Unif$(0{,}3)$ be two independent continuous uniform random variables.
  - Joint PDF?
  - Find the probability that $X \geq Y$?

2. Expected Value Regarding Two Random Variables

LOTUS: 2 Discrete Random Variables

Expected Value of a Function of 2 Discrete RVs:

1. Let X, Y be 2 discrete random variables with sets of possible values S_X, S_Y and joint PMF $p(x, y)$
2. Let $g(\cdot, \cdot)$ be a function from $\mathbb{R}^2 \to \mathbb{R}$

The expected value of $g(X, Y)$ is

$$E[g(X,Y)] = \sum_{x \in S_X} \sum_{y \in S_Y} \left(g(x,y) \cdot p(x,y) \right)$$

- Question: $E[X + Y] = ?$

Proof: Expected Value of $X + Y$

- Property: $E[X+Y] = E[X] + E[Y]$

LOTUS: 2 Continuous Random Variables

Expected Value of a Function of 2 Continuous RVs:

1. Let X, Y be 2 continuous random variables with joint PDF $f_{XY}(x, y)$

2. Let $g(\cdot, \cdot)$ be a function from $\mathbb{R}^2 \to \mathbb{R}$

The expected value of $g(X, Y)$ is

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)p(x,y)dxdy$$

- Question: $E[X+Y]$ =

Example

- Example: X and Y be

$$f(x, y) = \begin{cases} \frac{3}{2}(x^2 + y^2) & \text{, if } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{, otherwise} \end{cases}$$

- $E[X + Y] = $?

How About $E[XY]$?

- Question: Do we have $E[XY] = E[X]E[Y]$?

- Example: Let X be a continuous uniform r.v. on $[-1,1]$.
  - Define $Y = X$
  - $E[X] =$? $E[Y] =$?
  - $E[XY] =$?

$E[XY] = E[X]E[Y]$ If X, Y Are Independent

Theorem: Let X, Y be 2 independent continuous random variables with joint PDF $f_{XY}(x, y)$. Then,

$$E[XY] = E[X]E[Y]$$

- Remark: This result also holds for discrete random variables.
- Proof:

1-Minute Summary

1. Marginal PDF → Joint PDF

$f_{XY} = f_X \cdot f_Y$

$F_{XY} = F_X \cdot F_Y$

- Under independence: Joint PDF is the product of marginal PDFs

2. Expected Value Regarding 2 Random Variables

- LOTUS for 2 discrete / continuous random variables
- $E[XY] = E[X]E[Y]$ if X, Y are independent