## Problem 1

(a) Prove by induction:

N=2: $$P(A_1 \cup A_2) = P(A_1) + P(A_2) - \underbrace{P(A_1 \cap A_2)}_{\geq 0, \text{ by Axiom } 1} \leq \sum_{n=1}^{2} P(A_n) \quad \text{—— (1)}$$

Suppose when N=K, the following holds:

$$P\left(\bigcup_{n=1}^{K} A_n\right) \leq \sum_{n=1}^{K} P(A_n) \quad \text{—— (2)}$$

For N=K+1:

$$P\left(\bigcup_{n=1}^{K+1} A_n\right) = P\left(\left(\bigcup_{n=1}^{K} A_n\right) \cup A_{K+1}\right) \overset{\text{by (1)}}{\leq} P\left(\bigcup_{n=1}^{K} A_n\right) + P(A_{K+1})$$

$$\overset{\text{by (2)}}{\leq} \sum_{n=1}^{K+1} P(A_n)$$

Therefore, by induction, we know $P\left(\bigcup_{n=1}^{N} A_n\right) \leq \sum_{n=1}^{N} P(A_n)$ holds for all $N \in \mathbb{N}$.

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(b)

Method 1: It is easy to check that

$$P(\Omega) = P(\{1,2,3,4,5\}) \leq P(\{1,5\}) + P(\{1,2,4\}) + P(\{3,4\}) = 0.95$$

Hence, there is no valid assignment.

Method 2: Try to find some other contradiction

Since $P(\{1,5\}) = 0.3$ and $P(\{3,4\}) = 0.2$, then $P(\{2\}) = 0.5$

However, $P(\{1,2,4\}) = P(\{2\}) + P(\{1,4\}) = 0.45$

contradiction.

Therefore, there is no valid probability assignment.

Problem 2

(a) (i): $\left(\bigcup_{n=1}^{\infty} S_n\right)^c = \bigcap_{n=1}^{\infty} S_n^c \Rightarrow$ prove this by showing $\left(\bigcup_{n=1}^{\infty} S_n\right)^c \subseteq \bigcap_{n=1}^{\infty} S_n^c$ and $\left(\bigcup_{n=1}^{\infty} S_n\right)^c \supseteq \bigcap_{n=1}^{\infty} S_n^c$

- Suppose that $x \in \left(\bigcup_n S_n\right)^c$. Then, $x \notin \bigcup_n S_n$, which implies that we must have $x \notin S_n$, for every $n$. Hence, $x \in \bigcap_n S_n^c \Rightarrow \left(\bigcup_{n=1}^{\infty} S_n\right)^c \subseteq \bigcap_{n=1}^{\infty} S_n^c$
- The converse inclusion can be established by reversing the above argument.

(ii) $\left(\bigcap_{n=1}^{\infty} S_n\right)^c = \bigcup_{n=1}^{\infty} S_n^c \Leftrightarrow \bigcap_{n=1}^{\infty} S_n = \left(\bigcup_{n=1}^{\infty} S_n^c\right)^c \Leftrightarrow \underbrace{\bigcap_{n=1}^{\infty} (S_n^c)^c = \left(\bigcup_{n=1}^{\infty} (S_n^c)\right)^c}$

It is easy to verify that this is true based on (i).

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(b). Choose $B_1 = A_1$

$B_2 = A_2 - A_1 = A_2 \cap A_1^c$

$\vdots$

$B_n = A_n - \left(\bigcup_{i=1}^{n-1} A_i\right) = A_n \cap \left(\bigcup_{i=1}^{n-1} A_i\right)^c \overset{\text{De Morgan's Law}}{=} A_n \cap \left(\bigcap_{i=1}^{n-1} A_i^c\right)$

Check: (i) For any $j \neq k$, we have $B_j \cap B_k = \phi$:

Without loss of generality, let $j < k$.

Then, $B_j = A_j \cap \left(\bigcap_{i=1}^{j-1} A_i^c\right)$ and $B_k = A_k \cap \left(\bigcap_{i=1}^{k-1} A_i^c\right)$

It is easy to verify that $B_j \cap B_k = \phi$, for all pairs $j \neq k$.

(ii) Check $\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$, for all $n$

Prove by induction: We already know $B_1 = A_1$

Suppose $\bigcup_{i=1}^{k} B_i = \bigcup_{i=1}^{k} A_i$.

Then $\bigcup_{i=1}^{k+1} B_i = B_{k+1} \cup \bigcup_{i=1}^{k} B_k$

$$= B_{k+1} \cup \bigcup_{i=1}^{k} A_i$$

$$= \left(A_{k+1} \cap \left(\bigcap_{i=1}^{k} A_i^c\right)\right) \cup \bigcup_{i=1}^{k} A_i$$

$$= \left(A_{k+1} \cap \left(\bigcup_{i=1}^{k} A_i\right)^c\right) \cup \bigcup_{i=1}^{k} A_i$$

$$= \bigcup_{i=1}^{k+1} A_i$$

Therefore, we conclude that $\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$, for all $n \in \mathbb{N}$.

# Problem 3

(a). Prove by contradiction:

Suppose there are countably infinite real numbers in $(0,1)$.

We denote these numbers as $x_1, x_2, x_3, \cdots$

For each $x_i$, we express the number in decimal expansion, i.e.

$$x_i = 0.\, a_i^{(1)} a_i^{(2)} a_i^{(3)} a_i^{(4)} \cdots$$

Now we construct a real number $y = 0.\, b^{(1)} b^{(2)} b^{(3)} b^{(4)} \cdots$,

where $b^{(k)} = \begin{cases} 1, & \text{if } a_k^{(k)} = 2 \\ 2, & \text{otherwise} \end{cases} \Rightarrow y \in (0,1)$

Therefore, we know $y \neq x_i$, for all $i \Rightarrow$ Contradiction!

Hence, there are uncountably infinite real numbers in $(0,1)$.

(b). We know $\underbrace{\text{all irrational numbers in } (0,1)}_{\text{denoted by } \mathbb{I}_{(0,1)}} = \mathbb{R}_{(0,1)} - \mathbb{Q}_{(0,1)}$

We show that $\mathbb{I}_{(0,1)}$ is an uncountably infinite set by contradiction:

Suppose $\mathbb{I}_{(0,1)}$ is countably infinite and denote $\mathbb{I}_{(0,1)} = \{ y_1, y_2, y_3 \cdots \}$

We already know $\mathbb{Q}_{(0,1)}$ is countable, i.e. $\mathbb{Q}_{(0,1)} = \{ z_1, z_2, z_3, \cdots \}$

Then, we can write $\mathbb{R}_{(0,1)} = \{ y_1, z_1, y_2, z_2, y_3, z_3, \cdots \}$ and $\mathbb{R}_{(0,1)}$ is also countable

$\Rightarrow$ contradiction!

## Problem 4:

(a) Define $A_n$ to be the event that the n-th toss is a head.

It is easy to verify that $I = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$.

Then, $P(I) = P\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n\right) = P\left(\lim_{m\to\infty} \bigcap_{k=1}^{m} \bigcup_{n=k}^{\infty} A_n\right)$

$$\overset{\text{by continuity}}{=} \lim_{m\to\infty} P\left(\bigcap_{k=1}^{m} \bigcup_{n=k}^{\infty} A_n\right)$$

$$= \lim_{m\to\infty} P\left(\bigcup_{n=m}^{\infty} A_n\right)$$

$$\overset{\text{union bound}}{\leq} \lim_{m\to\infty} \sum_{n=m}^{\infty} P(A_n)$$

$$\overset{\sum_{i=1}^{\infty} P_i \text{ is finite}}{=} 0 \qquad \square$$

(b). Define $N$ to be the event that there is no head at all

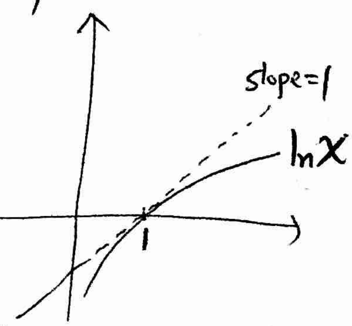
Also, define $E_n$ to be the event that there are only a finite number of heads and the last head occurs at the n-th toss.

Then, it is easy to verify that $I^c = \left(\bigcup_{n\geq 1} E_n\right) \cup N$ and that $E_n$'s and $N$ are mutually exclusive.

By the third probability axiom, $P(I^c) = \left(\sum_{n\geq 1} P(E_n)\right) + P(N)$

Note that $P(N) \leq \prod_{i=1}^{M} (1-P_i)$ for all $M$.

$\Rightarrow \quad \ln P(N) \leq \sum_{i=1}^{M} \ln(1-P_i) \leq \sum_{i=1}^{M} -P_i$, for all $M$




As $\sum_{i=1}^{\infty} P_i = \infty$, then $\ln P(N) = -\infty$ and equivalently $P(N) = 0$.

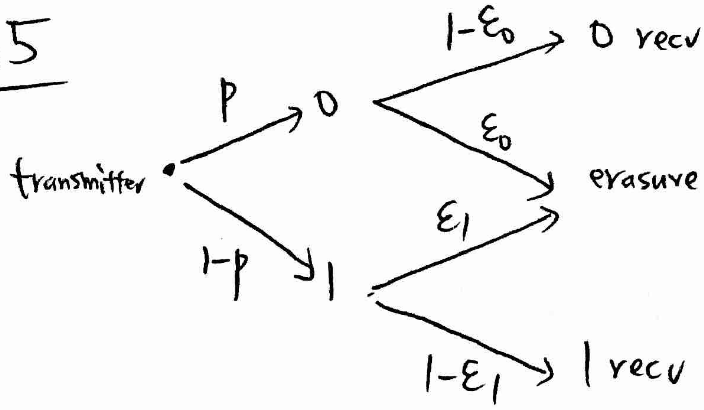
Similarly, we can show that $P(E_n)=0$, for all $n$.

Hence, $P(I^c) = \left(\sum_{n\geq 1} P(E_n)\right) + P(N) = 0$.

Therefore, we conclude that $P(I)=1$.

$\square$

## Problem 5




(a). P(a transmitted bit is erased) = P(0 sent)$\cdot$P(erased | 0 sent) + P(1 sent)$\cdot$P(erased | 1 sent)

total probability theorem

$$= p\cdot\varepsilon_0 + (1-p)\cdot\varepsilon_1$$

(b)

P(at most 1 bit erased in 3 sent bits)

$$= \sum_{\substack{x_1\in\{0,1\}\\ x_2\in\{0,1\}\\ x_3\in\{0,1\}}} P(\text{at most 1 bit erased and } x_1x_2x_3 \text{ sent})$$

$$= \sum_{\substack{x_1\in\{0,1\}\\ x_2\in\{0,1\}\\ x_3\in\{0,1\}}} \Big\{ P(x_1 \text{ is erased and } x_1x_2x_3 \text{ sent}) + P(x_2 \text{ is erased and } x_1x_2x_3 \text{ sent}) + P(x_3 \text{ is erased and } x_1x_2x_3 \text{ sent}) + P(\text{no erasure and } x_1x_2x_3 \text{ sent}) \Big\}$$

$$= \sum_{\substack{x_1\in\{0,1\}\\ x_2\in\{0,1\}\\ x_3\in\{0,1\}}} \left\{ \left( \prod_{i=1}^{3} (1-p)^{x_i} p^{1-x_i} \right) \cdot \Big[ \varepsilon_{x_1}\cdot(1-\varepsilon_{x_2})(1-\varepsilon_{x_3}) + (1-\varepsilon_{x_1})\varepsilon_{x_2}\cdot(1-\varepsilon_{x_3}) + (1-\varepsilon_{x_1})(1-\varepsilon_{x_2})\cdot\varepsilon_{x_3} + (1-\varepsilon_{x_1})(1-\varepsilon_{x_2})(1-\varepsilon_{x_3}) \Big] \right\}$$

(c).

$$P(10 \text{ sent} \mid 2 \text{ erasures recv})$$

Bayes' rule

$$\overset{\downarrow}{=} \frac{P(10 \text{ sent and both are erased})}{\displaystyle\sum_{\substack{x_1 \in \{0,1\} \\ x_2 \in \{0,1\}}} P(x_1 x_2 \text{ sent and both are erased})}$$

$$= \frac{(1-p)\cdot p \cdot \varepsilon_1 \cdot \varepsilon_0}{2(1-p)\cdot p \cdot \varepsilon_1 \cdot \varepsilon_0 + (1-p)^2 \varepsilon_1^2 + p^2 \varepsilon_0^2}$$

(d). As the first two received bits are 01, then the first two sent bits must be 01.

For the last two bits, we can follow the similar argument as (c):

$$P(00 \text{ sent} \mid 2 \text{ erasures}) = \frac{p^2 \varepsilon_0^2}{2(1-p)\cdot p \cdot \varepsilon_1 \varepsilon_0 + (1-p)^2 \varepsilon_1^2 + p^2 \varepsilon_0^2} = \frac{225}{1521}$$

$$\underbrace{P(01 \text{ sent} \mid 2 \text{ erasures})}_{\shortparallel} = \frac{p(1-p)\cdot \varepsilon_0 \varepsilon_1}{2(1-p)\cdot p \varepsilon_1 \varepsilon_0 + (1-p)^2 \varepsilon_1^2 + p^2 \varepsilon_0^2} = \frac{360}{1521}$$

$$P(10 \text{ sent} \mid 2 \text{ erasures})$$

$$P(11 \text{ sent} \mid 2 \text{ erasures}) = \frac{(1-p)^2 \varepsilon_1^2}{2(1-p)\cdot p \varepsilon_1 \varepsilon_0 + (1-p)^2 \varepsilon_1^2 + p^2 \varepsilon_0^2} = \frac{576}{1521}$$

Therefore, <u>0111</u> is the most probable sequence of transmitted bits. $\square$

Problem 6

(a). $P(A_1|B)=\frac{P(A_1)\cdot P(B|A_1)}{P(B)}=\frac{P(A_1)\cdot P(B|A_1)}{P(A_1)\cdot P(B|A_1)+P(A_2)\cdot P(B|A_2)+P(A_3)\cdot P(B|A_3)}$

$$=\frac{\frac{1}{3}\times 0.3}{\frac{1}{3}\times 0.3+\frac{1}{3}\times 0.5+\frac{1}{3}\times 0.7}=\frac{1}{5}$$

Similarly, $P(A_2|B)=\frac{1}{3}$, $P(A_3|B)=\frac{7}{15}$.

(b). $P(A_1|C)=\frac{P(A_1)\cdot P(C|A_1)}{P(C)}=\frac{P(A_1)\cdot P(C|A_1)}{P(A_1)\cdot P(C|A_1)+P(A_2)\cdot P(C|A_2)+P(A_3)\cdot P(C|A_3)}$

$$=\frac{\frac{1}{3}\times(0.3)^8\times(0.7)^2}{\frac{1}{3}\times(0.3)^8\times(0.7)^2+\frac{1}{3}\times(0.5)^8\times(0.5)^2+\frac{1}{3}\times(0.7)^8\times(0.3)^2}$$

$\cong 0.52\%$

Similarly, by Baye's rule, we have:

$P(A_2|C)\cong 15.76\%$

$P(A_3|C)\cong 83.72\%$ $\Rightarrow$ Hence, the most probable value for $\theta$ is 0.7

(c). Suppose $P(A_1)=\frac{2}{5}$, $P(A_2)=\frac{2}{5}$, $P(A_3)=\frac{1}{5}$.

$$P(A_1|C)=\frac{\frac{2}{5}\times(0.3)^8\times(0.7)^2}{\frac{2}{5}\times(0.3)^8\times(0.7)^2+\frac{2}{5}\times(0.5)^8\times(0.7)^2+\frac{1}{5}\times(0.7)^8\times(0.3)^2}\cong 0.89\%$$

$P(A_2|C)\cong 27.11\%$

$P(A_3|C)\cong 72.00\%$. $\Rightarrow$ Hence, the most probable value for $\theta$ is 0.7

## Problem 7

(a). $(1+x)^{2n} = (1+x)^n (1+x)^n$

The coefficient of the term $x^n$ in $(1+x)^{2n} = C^{2n}_n$.

〃 〃 $x^n$ in $(1+x)^n(1+x)^n$

$$= \sum_{i=0}^{n} \left(\text{the coefficient of the term } x^i \text{ in } (1+x^n)\right) \cdot \left(\text{the coefficient of the term } x^{n-i} \text{ in } (1+x)^n\right)$$

$$= \sum_{i=0}^{n} C^n_i \cdot C^n_{n-i}$$

$$= \sum_{i=0}^{n} \left(C^n_i\right)^2$$

$\square$

(b). Since $C^{n+i}_i = C^{n+i+1}_i - C^{n+i}_{i-1}$ for $i \geq 1$,

then $$\sum_{i=0}^{r} C^{n+i}_i = C^n_0 + \sum_{i=1}^{r}\left(C^{n+i+1}_i - C^{n+i}_{i-1}\right)$$

$$= \underbrace{C^n_0}_{=1} + \left(C^{n+r+1}_r - \underbrace{C^{n+1}_0}_{=1}\right) = C^{n+r+1}_r$$

$\square$