Consider an irreducible, aperiodic, and recurrent Markov chain with transition matrix P and state space $S = \{0, 1, \cdots\}$ (denoted by $\{X_t\}_{t \geq 0}$)

Suppose a steady-state distribution $\pi = (\pi_0, \pi_1, \cdots)$ exists (That is, π satisfies $\pi = \pi P$ and $\sum_{i \in S} \pi_i = 1$)

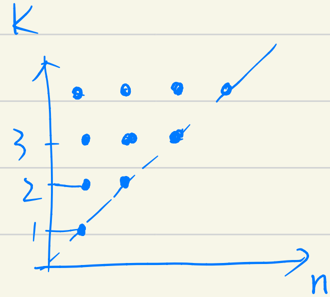
Define T_{ii} to be the time between two consecutive visits to state i

Show that:

$$\pi_i = \frac{1}{E[T_{ii}]}$$

Pf: Without loss of generality, let the initial distribution of X_0 be π.

Define T_i to be the first time that the chain visits i, starting from time 1



Consider $E[T_i \mid X_0 = i]$

$$= \sum_{k=1}^{\infty} k \cdot P(T_i = k \mid X_0 = i)$$

$$= \sum_{k=1}^{\infty} \left(\sum_{n=1}^{k} 1 \right) \cdot P(T_i = k \mid X_0 = i)$$

$$= \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} P(T_i = k \mid X_0 = i)$$

$$= \sum_{n=1}^{\infty} P(T_i \geq n \mid X_0 = i)$$

Then, we have

$$\pi_i \cdot E[T_i \mid X_0 = i] = \sum_{n=1}^{\infty} P(T_i \geq n \mid X_0 = i) \cdot \pi_i$$

$$= \sum_{n=1}^{\infty} P(T_i \geq n \mid X_0 = i) \cdot P(X_0 = i)$$

$$= \sum_{n=1}^{\infty} P(T_i \geq n \text{ and } X_0 = i)$$

Then:

$$\pi_i \cdot E[T_i \mid X_0 = i]$$

$$= \sum_{n=1}^{\infty} P(T_i \geq n \text{ and } X_0 = i)$$

$$= \sum_{n=1}^{\infty} P(X_{n-1} \neq i, X_{n-2} \neq i, \cdots, X_1 \neq i, X_0 = i)$$

$$= P(X_0 = i) + \sum_{n=2}^{\infty} \Big(P(X_{n-1} \neq i, \cdots, X_1 \neq i)$$

$$- P(X_{n-1} \neq i, \cdots, X_1 \neq i, X_0 \neq i) \Big)$$

since the initial distribution is the steady-state distribution π

$$= P(X_0 = i) + \sum_{n=2}^{\infty} \Big(P(X_{n-2} \neq i, \cdots, X_0 \neq i)$$

$$- P(X_{n-1} \neq i, \cdots, X_1 \neq i, X_0 \neq i) \Big)$$

$$= P(X_0 = i) + \Big[P(X_0 \neq i) - \underbrace{\lim_{n\to\infty} P(X_{n-1} \neq i, \cdots, X_1 \neq i, X_0 \neq i)}_{=0} \Big]$$

(given the chain is recurrent)

$$= P(X_0 = i) + P(X_0 \neq i)$$

$$= 1$$

Hence, we have $\pi_i \cdot E[T_i \mid X_0 = i] = 1$ —— $(*)$

Define T_{ii} to be the time between two consecutive visits to state i

By $(*)$ and the Markov property,

we have $\pi_i \cdot E[T_{ii}] = 1$

$\square$