Problem 1:

(a) ○ (Union bound)

(b) × (Y is still exponential only if $a>0$)

(c) ○ (HW2, Problem 1(d))

(d). ○

(e) × ($P(\{1,2\})=0.3$ and $P(\{2,3\})=0.5$ imply that $P(\{3\})\geq 0.2$.
Therefore, the maximum possible value of $P(\{4\})$ is 0.5)

(f) ○ (Similar to HW1, Problem 7(a)).

(g) × (By definition, A, B, C are independent if

$$\begin{cases} P(A\cap B)=P(A)P(B) \\ P(A\cap C)=P(A)P(C) \\ P(B\cap C)=P(B)P(C) \\ P(A\cap B\cap C)=P(A)P(B)P(C) \end{cases}$$).

Problem 2:

$$f(x) = \sqrt{2K} \cdot \exp\left(-K^2x^2 + 2Kx - 1\right)$$

$$= \sqrt{2K} \cdot \exp\left(-(Kx-1)^2\right) = \sqrt{2K} \cdot \exp\left(-\frac{(x-\frac{1}{K})^2}{(\frac{1}{K})^2}\right)$$

$$\Rightarrow \begin{cases} \sqrt{2K} = \dfrac{1}{\sigma\sqrt{2\pi}} \Rightarrow \dfrac{1}{\sigma^2} = 4K\pi \\ \dfrac{1}{K^2} = 2\sigma^2 \Rightarrow \dfrac{1}{K^2} = 2 \cdot \dfrac{1}{4K\pi} = \dfrac{1}{2K\pi} \Rightarrow K = 2\pi \\ \dfrac{1}{K} = \mu \end{cases}$$

Therefore, we conclude that $\begin{cases} \mu = \dfrac{1}{2\pi} \\ \sigma^2 = \dfrac{1}{8\pi^2} \end{cases}$

Problem 3:

For any integer $k \geq 0$:

$$P(Y=k) = \sum_{m=0}^{k} P(X_1=m) \cdot P(X_2=k-m)$$

$$= \sum_{m=0}^{k} \frac{e^{-\lambda_1 T} \cdot (\lambda_1 T)^m}{m!} \cdot \frac{e^{-\lambda_2 T} \cdot (\lambda_2 T)^{k-m}}{(k-m)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)T} \cdot T^k}{k!} \cdot \sum_{m=0}^{k} \underbrace{\frac{\lambda_1^m \cdot \lambda_2^{k-m}}{m!(k-m)!} \cdot k!}_{= C_m^k \lambda_1^m \lambda_2^{k-m}}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)T} \cdot T^k}{k!} \cdot \underbrace{\sum_{m=0}^{k} C_m^k \cdot \lambda_1^m \cdot \lambda_2^{k-m}}_{= (\lambda_1+\lambda_2)^k}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)T} \cdot ((\lambda_1+\lambda_2)T)^k}{k!}$$

Therefore, the PMF of Y: $P(Y=k) = \begin{cases} \frac{e^{-(\lambda_1+\lambda_2)T} \cdot ((\lambda_1+\lambda_2)T)^k}{k!}, & k \geq 0 \\ 0, & \text{else} \end{cases}$

Remark: Y is a Poisson random variable with parameters $(\lambda_1+\lambda_2, T)$.

Problem 4: $X \sim Unif(0,5)$, $Y \sim Exp(2)$, and X, Y are independent

(a).

PDF of X: $f_X(x) = \begin{cases} \frac{1}{5}, & 0<x<5 \\ 0, & \text{else} \end{cases}$

$$E[e^X] = \int_{-\infty}^{+\infty} e^x \cdot f_X(x)dx = \int_0^5 e^x \cdot \frac{1}{5}dx = \frac{1}{5}e^x\Big|_0^5 = \underline{\frac{1}{5}(e^5-1)} \ \#$$

$$E[(e^X)^2] = E[e^{2X}] = \int_{-\infty}^{+\infty} e^{2X} \cdot f_X(x)dx$$

$$= \int_0^5 e^{2X} \cdot \frac{1}{5}dx = \frac{1}{10}e^{2X}\Big|_0^5 = \frac{1}{10}(e^{10}-1).$$

$$Var[e^X] = E[(e^X)^2] - (E[e^X])^2$$

$$= \frac{1}{10}(e^{10}-1) - \left(\frac{1}{5}(e^5-1)\right)^2$$

$$= \underline{\frac{3}{50}e^{10} + \frac{2}{25}e^5 - \frac{7}{50}} \ \#$$

(b). Let $F_{XY}(t,u)$ denote the joint CDF of X and Y.

The CDF of X: $F_X(t) = \begin{cases} 0 & , \text{ if } t<0 \\ \frac{t}{5} & , \text{ if } 0 \le t \le 5 \\ 1 & , \text{ if } t>5 . \end{cases}$

The CDF of Y: $F_Y(u) = \begin{cases} 1-e^{-2u} & , \text{ if } u \ge 0 \\ 0 & , \text{ if } u<0 \end{cases}$

Since X and Y are independent, then we know $F_{XY}(t,u) = F_X(t) \cdot F_Y(u)$.

$F_{XY}(t,u)$	$t<0$	$0 \le t \le 5$	$t>5$
$u \ge 0$	0	$\frac{t}{5}(1-e^{-2u})$	$1-e^{-2u}$
$u<0$	0	0	0

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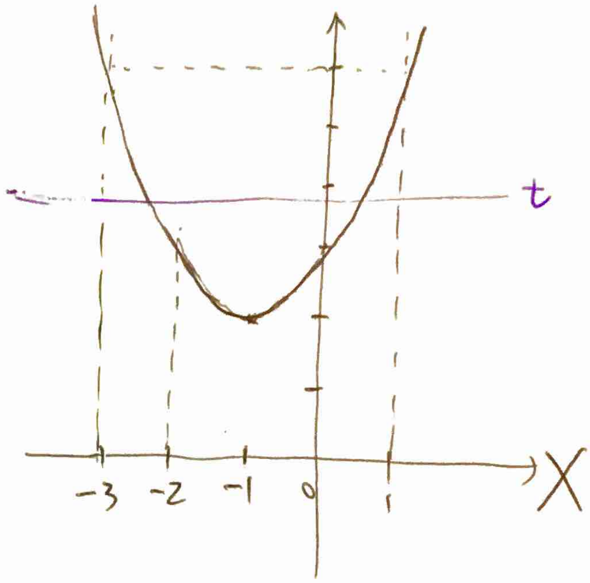
(c). $W = X^2 + 2X + 3, \quad X \sim \mathrm{Unif}(0,5)$



The CDF of W:

$$F_W(t) = P(W \le t)$$
$$= P(X^2 + 2X + 3 \le t)$$
$$= P((X+1)^2 + 2 \le t)$$

We discuss the following cases:

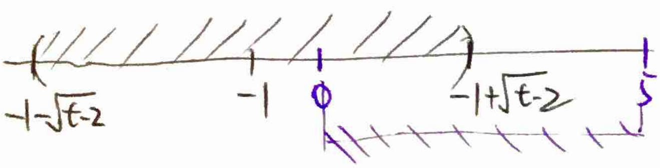
① $t < 2$: The event $\{(X+1)^2 + 2 \le t\}$ is an empty set.

② $t \ge 2$: $P((X+1)^2 + 2 \le t) = P((X+1)^2 \le t-2)$

$$= P(|X+1| \le \sqrt{t-2})$$

mutually exclusive

$$= P(X \le -1+\sqrt{t-2} \text{ or } X \ge -1-\sqrt{t-2})$$



$$= P(X \le -1+\sqrt{t-2}) + P(X \ge -1-\sqrt{t-2})$$

$$= \begin{cases} 0, & 2 \le t < 3 \\ \frac{1}{5}(-1+\sqrt{t-2}), & 3 \le t \le 38 \\ 1, & t > 38 \end{cases}$$

To conclude:

$$F_W(t) = \begin{cases} 0, & t < 3 \\ \frac{1}{5}(-1+\sqrt{t-2}), & 3 \le t \le 38 \\ 1, & t > 38 \end{cases}$$

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Problem 5

(a). $$P(Z<0 \mid X=+1) = P(Y<-1 \mid X=+1)$$

$$= P(Y<-1)$$

the noise is independent from X

$$= \Phi\left(\frac{-1}{\sigma}\right). \quad \left(\text{or } 1-\Phi\left(\frac{1}{\sigma}\right)\right)$$

Similarly, $$P(Z\geq 0 \mid X=-1) = P(Y\geq 1 \mid X=-1)$$

$$= P(Y\geq 1)$$

$$= \Phi\left(\frac{-1}{\sigma}\right). \quad \left(\text{or } 1-\Phi\left(\frac{1}{\sigma}\right)\right)$$

(b). $$P(\text{"+" is sent} \mid \text{"+" is received}) = \frac{P(\text{"+" is sent and "+" is received})}{P(\text{"+" is received})}$$

Bayes' rule

$$= \frac{P(\text{"+" is sent})\cdot P(\text{"+" is received} \mid \text{"+" is sent})}{P(\text{"+" is sent})P(\text{"+" is received} \mid \text{"+" is sent}) + P(\text{"−" is sent})\cdot P(\text{"+" is recv} \mid \text{"−" is sent})}$$

$$= \frac{p\cdot\left(1-\Phi\left(\frac{-1}{\sigma}\right)\right)}{p\cdot\left(1-\Phi\left(\frac{-1}{\sigma}\right)\right) + (1-p)\cdot\left(\Phi\left(\frac{-1}{\sigma}\right)\right)}.$$

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(c). For ease of notation, use S and R to denote the sent bits and the received bits, respectively.

$$P(S=\text{"}+-\text{"} \mid R=\text{"}+-\text{"})$$

Bayes' rule

$$\stackrel{\downarrow}{=} \frac{P(S=\text{"}+-\text{"})\cdot P(R=\text{"}+-\text{"} \mid S=\text{"}+-\text{"})}{P(S=\text{"}+-\text{"})\cdot P(R=\text{"}+-\text{"} \mid S=\text{"}+-\text{"}) + P(S=\text{"}++\text{"})\cdot P(R=\text{"}+-\text{"} \mid S=\text{"}++\text{"}) + P(S=\text{"}-+\text{"})\cdot P(R=\text{"}+-\text{"} \mid S=\text{"}-+\text{"}) + P(S=\text{"}--\text{"})\cdot P(R=\text{"}+-\text{"} \mid S=\text{"}--\text{"})}$$

$$= \frac{p\cdot(1-p)\cdot\left(1-\Phi\left(\frac{-1}{\sigma}\right)\right)^2}{p(1-p)\cdot\left(1-\Phi\left(\frac{-1}{\sigma}\right)\right)^2 + p^2\cdot\left(1-\Phi\left(\frac{-1}{\sigma}\right)\right)\cdot\Phi\left(\frac{-1}{\sigma}\right) + (1-p)\cdot p\left(\Phi\left(\frac{-1}{\sigma}\right)\right)^2 + (1-p)^2\cdot\Phi\left(\frac{-1}{\sigma}\right)\cdot\left(1-\Phi\left(\frac{-1}{\sigma}\right)\right)}$$

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Problem 6 : Let X be the number of red lights, and $X \sim \text{Binomial}(6, \frac{1}{3})$.

① $T = 15 + X \Rightarrow$ The PMF of T:

$$P(T=k) = \begin{cases} C^{6}_{k-15} \left(\frac{1}{3}\right)^{k-15} \cdot \left(\frac{2}{3}\right)^{21-k}, & 15 \leq k \leq 21 \\ 0, & \text{else.} \end{cases}$$

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② $E[T] = 15 + E[X]$ (by the linearity of expected value).

$$= 15 + 6 \times \frac{1}{3}$$

$$= 17.$$

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③ $Var[X] = Var[X]$ (since translation does not change the variance)

$$= 6 \times \frac{1}{3} \times \left(1 - \frac{1}{3}\right)$$

$$= \frac{4}{3}$$

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