DCP 1206: Probability
Lecture 17 — Covariance, Conditional Distributions, and Bivariate Normal

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Announcements

- HW4 is posted on E3
  - Part 1: due on 11/22 (Friday)
  - Part 2: due on 11/27 (Wednesday)

- Midterm exam booklet will be returned <u>today after class</u>
  - 4:30pm - 5:30pm @ EC122

Quick Review: Covariance

$E[XY] - E[X]E[Y]$

$Cov(X,Y) = 0$

another expression

definition

$E[(X - E[X])\cdot(Y - E[Y])]$

uncorrelated?

$\text{Cov}(X, Y)$

correlation coefficient?

independence?

$Cov(X,Y) = 0$

Connection with
$\text{Var}(X + Y)$, $\text{Var}(X)$, $\text{Var}(Y)$?

$Var(X+Y) = (Var(X)) + (Var(Y)) + (2\,Cov(X,Y))$

$\left(Cov(X,Y)\right)^2 \leq Var(X)\cdot Var(Y)$

This Lecture

1. Covariance and Correlation Coefficient

2. Conditional Distributions

3. Bivariate Normal Random Variables

- Reading material: Chapter 8.3,10.3, and 10.5

1. Covariance and Correlation Coefficient

Covariance is Sensitive to the Units

- Property: $\underline{\text{Cov}(aX, aY) = a^2 \cdot \text{Cov}(X, Y)}$
  - a: scaling factor due to change of unit

- Question: Any suggested solution?

Correlation Coefficient

- **Correlation Coefficient**: Let X, Y be two random variables with finite variance $\sigma_X^2 > 0$ and $\sigma_Y^2 > 0$. Then, the correlation coefficient of X and Y is defined as

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)} \cdot \sqrt{\mathrm{Var}(Y)}}$$

- **Question:** Do we have $\rho(X, Y) = \rho(aX, aY)$, for any $a \neq 0$?

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

$$\rho(aX, aY) = \frac{\mathrm{Cov}(aX, aY)}{\sigma_{aX} \cdot \sigma_{aY}} = \frac{a^2 \cdot \mathrm{Cov}(X, Y)}{(a \cdot \sigma_X)(a \cdot \sigma_Y)} = \rho(X, Y)$$

A Property of Correlation Coefficient

- **Property**:

$$-1 \leq \rho(X, Y) \leq 1$$

$$\frac{Cov(X,Y)}{\sqrt{Var(X)} \cdot \sqrt{Var(Y)}}$$

- Question: How to prove this?

$\rho(X,Y) = 0.8$

Read:

$$\left(Cov(X,Y)\right)^2 \leq Var(X) \cdot Var(Y)$$

$$\Longleftrightarrow \frac{|Cov(X,Y)|}{\sqrt{} \cdot \sqrt{}} \leq 1$$

Example: Correlation Coefficient

- Example: Let X be a continuous uniform r.v. on $[0,1]$.
  - Define $Y = X^2$
  - $\rho(X, Y) = ?$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\cdot\sqrt{Var(Y)}} = \frac{E[XY] - E[X]\cdot E[Y]}{\sqrt{E[X^2]-(E[X])^2}\cdot\sqrt{E[Y^2]-(E[Y])^2}}$$

$$= \frac{E[X^3] - E[X]\cdot E[X^2]}{\sqrt{E[X^2]-(E[X])^2}\cdot\sqrt{E[X^4]-(E[X^2])^2}}$$

2. Conditional Distributions

Example: Using Joint PMF to Find Conditional PMF

- Example: Bus #2 (NCTU - Mackay - Train Station)
  - $X =$ traveling time from NCTU to Mackay
  - $Y =$ traveling time from Mackay to Train Station
  - $P(X = 10 \mid Y = 15) = ?$

Joint PMF	X=10	X=15	X=20
Y=10	0.1	0.1	0.05
Y=15	0.1	0.3	0.1
Y=20	0.05	0.1	0.1

$$P(X=10 \mid Y=15) = \frac{P(X=10 \text{ and } Y=15)}{P(Y=15)}$$

$$= \frac{0.1}{0.1+0.3+0.1} = 0.2$$

$$P(X=15 \mid Y=15) = \frac{0.3}{0.5} = 0.6$$

$$P(X=20 \mid Y=15) = \frac{0.1}{0.5} = 0.2$$

Conditional PMF (Formally)

- **Conditional PMF**: Let X, Y be two discrete random variables with joint PMF $p_{XY}(x, y)$. When $P(Y = y) > 0$, the conditional PMF of X given $Y = \boxed{y}$ is

X=1[illegible], Y=15

$$p_{X|Y}(x \mid y) = \frac{p_{XY}(x, y)}{p_Y(y)}$$

Y=15

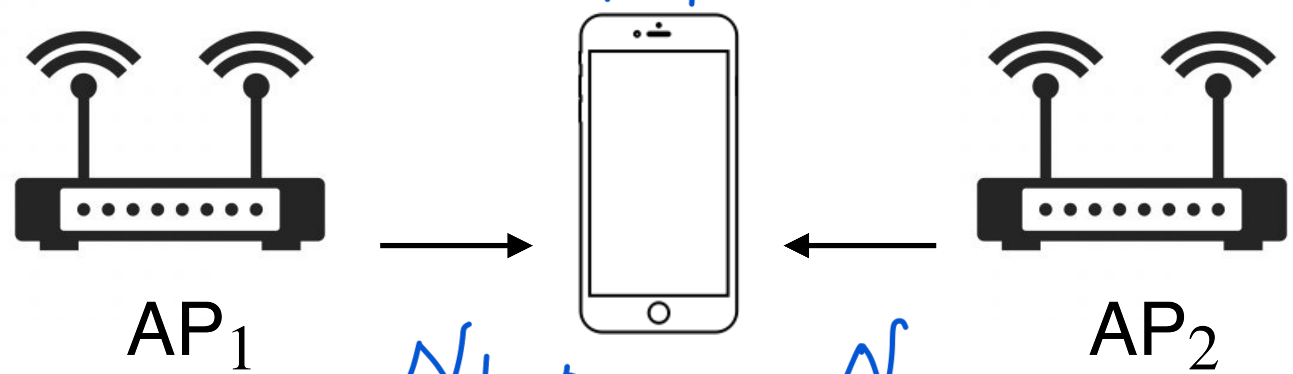
- Question: Conditional PMF of Y given $\boxed{X = x?}$

$$P_{Y|X}(y|x) = \frac{P_{XY}(x,y)}{P_X(x)}$$

- Question: $\sum_x p_{X|Y}(x \mid y) =$ 1

Example: Conditioning and Sum of Poisson

$M = N_1 + N_2$



$N_1 \sim \text{Poi}$ $\quad$ $N_2 \sim \text{Poi}$

- Let N_1 and N_2 be the # of bits transmitted by AP_1 and AP_2 in a time interval T, respectively
  - N_1 and N_2 are Poisson with rates λ_1 and λ_2, respectively.
  - Moreover, N_1 and N_2 are independent
  - Define $M = N_1 + N_2$
  - Question: Conditional PMF $p_{N_1|M}(n\,|\,m)$ = ?

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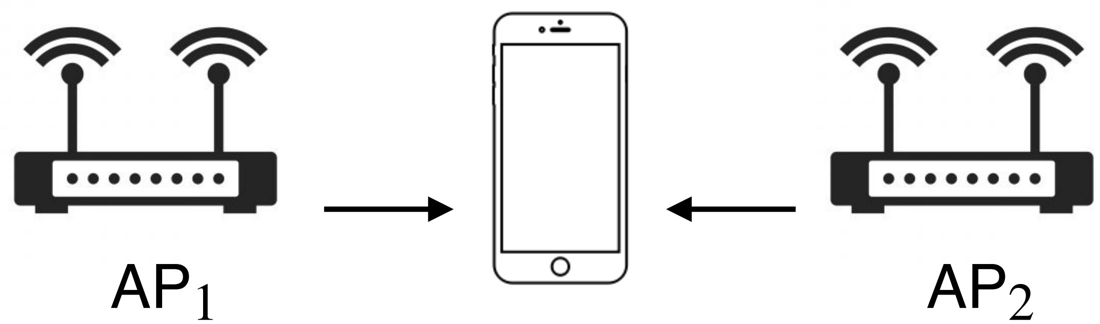
$$P(N_1 = n \mid M = m) = \frac{P(N_1 = n, M = m)}{P(M = m)}$$

Example: Conditioning and Sum of Poisson



- Conditional PMF $p_{N_1|M}(n \mid m)$

Conditional CDF: Discrete Case (Formally)

- Question: Given $p_{X|Y}(x \mid y)$, how to find $P(X \leq t \mid Y = y)$?

- **Conditional CDF**: Let X, Y be two discrete random variables with joint PMF $p_{XY}(x, y)$. When $P(Y = y) > 0$, the conditional CDF of X given $Y = y$ is defined as

$$F_{X|Y}(x \mid y) = P(X \leq x \mid Y = y) =$$

Conditional Expectation: Discrete Case (Formally)

- **Conditional Expectation**: Let X, Y be two discrete random variables. When $P(Y = y) > 0$, the conditional expected value of X given $Y = y$ is

$$E(X \mid Y = y) = \sum_{x} x \cdot P(X = x \mid Y = y) =$$

- Question: Conditional expectation of Y given $X = x$?

Conditional PDF (Formally)

- **Conditional PDF**: Let X, Y be two continuous random variables with joint PDF $f_{XY}(x, y)$. When $f_Y(y) > 0$, the conditional PDF of X given $Y = y$ is

$$f_{X|Y}(x \,|\, y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

- Question: Conditional PDF of Y given $X = x$?

Example: Find Conditional PDF From Joint PDF

- Example:

$$f(x,y) = \begin{cases} 2 & , \text{ if } 0 < y < x < 1 \\ 0 & , \text{ otherwise} \end{cases}$$

- $f_{X|Y}(x \mid y) = ?$

Conditional CDF: Continuous Case (Formally)

- **Conditional CDF**: Let X, Y be two continuous random variables and $f_{X|Y}(x \mid y)$ be the conditional PDF of X given $Y = y$. The conditional CDF of X given $Y = y$ is

  $F_{X|Y}(x \mid y) = P(X \leq x \mid Y = y) =$

Conditional Expectation: Continuous Case (Formally)

- **Conditional Expectation**: Let X, Y be two continuous random variables. When $f_Y(y) > 0$, the conditional expected value of X given $Y = y$ is

  $$E(X \mid Y = y) =$$

- Question: Conditional expectation of Y given $X = x$?

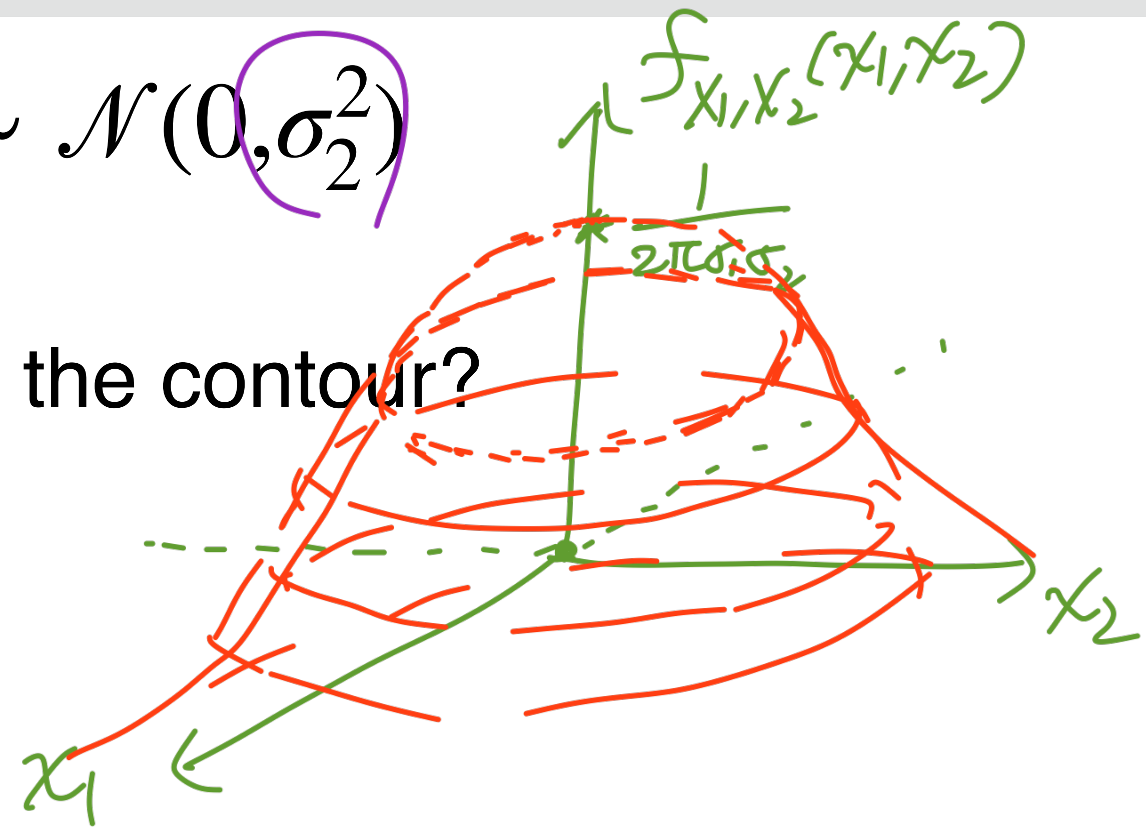
Example: Find Conditional Expectation

- Example:
$$f(x,y) = \begin{cases} 2 & , \text{ if } 0 < y < x < 1 \\ 0 & , \text{ otherwise} \end{cases}$$
- $E[X \mid Y = 0.5]$ = ?

3. Bivariate Normal Random Variables

Two Independent Normal Random Variables

- Example: $X_1 \sim \mathcal{N}(0,\sigma_1^2)$ and $X_2 \sim \mathcal{N}(0,\sigma_2^2)$
  - Suppose X_1, X_2 are independent.
  - What is the joint PDF? How to plot the contour?



PDF of X_1:

$$f_{X_1}(x) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(-\frac{1}{2}\frac{x^2}{\sigma_1^2}\right), \forall x \in \mathbb{R}$$

PDF of X_2:

$$f_{X_2}(x) = \frac{1}{\sqrt{2\pi}\sigma_2} \exp\left(-\frac{1}{2}\frac{x^2}{\sigma_2^2}\right), \forall x \in \mathbb{R}$$

Joint PDF of X_1, X_2:

$$f_{X_1X_2}(x_1,x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{1}{2}\left(\frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2}\right)\right)$$

Two Dependent Normal Random Variables

- Example: Suppose we want to construct X_1, X_2 such that
  - $X_1 \sim \mathcal{N}(0,\sigma_1^2)$ and $X_2 \sim \mathcal{N}(0,\sigma_2^2)$
  - $\rho(X_1, X_2) \neq 0$
  - How to achieve this?

Joint PDF of Bivariate Normal Random Variables

- **Joint PDF of Bivariate Normal (With Zero Mean)**:

$$\underbrace{f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\underbrace{\sqrt{1-\rho^2}}_{=1}} \exp\left[-\frac{1}{2\underbrace{(1-\rho^2)}_{=1}}\left(\frac{x_1^2}{\sigma_1^2} - \cancel{2\rho\frac{x_1x_2}{\sigma_1\sigma_2}} + \frac{x_2^2}{\sigma_2^2}\right)\right]}$$

$\underline{\rho = 0 =} \Rightarrow$

Two Dependent Normal Random Variables (Cont.)

- Goal: $X_1 \sim \mathcal{N}(0, \sigma_1^2)$, $X_2 \sim \mathcal{N}(0, \sigma_2^2)$, and $\rho(X_1, X_2) \neq 0$
- Idea: Let Z, W be 2 independent standard normal r.v.s

$$X_1 = \sigma_1 Z$$
$$X_2 = \sigma_2\left(\rho Z + \sqrt{1-\rho^2}\,W\right)$$

Linear Transformation of 2 Random Variables

- **Theorem**: Let U_1, U_2, V_1, V_2 be random variables that satisfy $V_1 = aU_1 + bU_2$ and $V_2 = cU_1 + dU_2$. Define the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then, we have

$$f_{V_1V_2}(v_1, v_2) = \frac{1}{|\det(A)|} f_{U_1U_2}(A^{-1}[v_1, v_2]^T)$$

For more details, please check: **https://www.stat.berkeley.edu/~aditya/resources/AllLectures2018Fall201A.pdf**

Joint PDF of X_1 and X_2

$$X_1 = \sigma_1 Z$$

$$X_2 = \sigma_2\left(\rho Z + \sqrt{1-\rho^2}W\right)$$

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{|\,\text{det}(A)\,|} f_{ZW}(A^{-1}[x_1, x_2]^T)$$

1-Minute Summary

1. Covariance and Correlation Coefficient

- $\rho(X, Y) =$ Cov$(X, Y)/(\sigma_X \sigma_Y)$, and $-1 \leq \rho(X, Y) \leq 1$

2. Conditional Distributions

- Conditional PMF / CDF / PDF

3. Bivariate Normal Random Variables

- Construction from 2 independent standard normal
- Joint PDF