Convolution theorem & Sum of 2 independent normal r.v.s

- Consider $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$
- X, Y are assumed to be independent
- Define $Z = X + Y$

The PDF of Z can be derived by using "Convolution theorem":

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) \cdot f_Y(z-x)\,dx$$

$$= \int_{-\infty}^{+\infty} \left(\frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}\right)\left(\frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{(z-x-\mu_2)^2}{2\sigma_2^2}}\right) dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{2\pi\sigma_1\sigma_2} \exp\left[-\frac{\sigma_2^2(x-\mu_1)^2 + \sigma_1^2(z-x-\mu_2)^2}{2\sigma_1^2\sigma_2^2}\right] dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{2\pi\sigma_1\sigma_2} \exp\left[-\frac{(\sigma_1^2+\sigma_2^2)x^2 + \left(-2\mu_1\sigma_2^2 - 2\sigma_1^2(z-\mu_2)\right)x + \left(\sigma_2^2\mu_1^2 + \sigma_1^2(z^2+\mu_2^2-2\mu_2 z)\right)}{2\sigma_1^2\sigma_2^2}\right] dx$$

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$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}} \cdot \frac{1}{\sqrt{2\pi}\,\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}} \cdot \exp\left[-\frac{\left(x - \frac{\mu_1\sigma_2^2+\sigma_1^2(z-\mu_2)}{\sigma_1^2+\sigma_2^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2}\right]$$

$$\cdot \underbrace{\exp\left[-\frac{\left(\sigma_2^2\mu_1^2 + \sigma_1^2(z^2+\mu_2^2-2\mu_2 z)\right)}{2\sigma_1^2\sigma_2^2}\right] \cdot \exp\left[\frac{\left(\frac{\mu_1\sigma_2^2+\sigma_1^2(z-\mu_2)}{\sigma_1^2+\sigma_2^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2}\right]}_{(*)} dx \quad \text{—(1)}$$

$$(*) = \exp\left[-\frac{\left(\sigma_2^2\mu_1^2+\sigma_1^2(z^2+\mu_2^2-2\mu_2 z\right)}{2\sigma_1^2\sigma_2^2}\right]\cdot\exp\left[\frac{\left(\frac{\mu_1\sigma_2^2+\sigma_1^2(z-\mu_2)}{\sigma_1^2+\sigma_2^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2}\right]$$

$$= \exp\left[-\frac{(\sigma_1^2+\sigma_2^2)\left(\sigma_2^2\mu_1^2+\sigma_1^2(z^2+\mu_2^2-2\mu_2 z)\right)-\left(\mu_1\sigma_2^2+\sigma_1^2(z-\mu_2)\right)^2}{2\sigma_1^2\sigma_2^2\cdot(\sigma_1^2+\sigma_2^2)}\right]$$

$$= \exp\left[-\frac{(\sigma_2^2\sigma_1^2)z^2+\left(-2\mu_2\sigma_1^2(\cancel{\sigma_1^2}+\sigma_2^2)-2\sigma_1^2(\mu_1\sigma_2^2-\cancel{\mu_2\sigma_1^2})\right)z+\left((\sigma_1^2+\sigma_2^2)(\sigma_2^2\mu_1^2+\sigma_1^2\mu_2^2)-(\mu_1\sigma_2^2-\mu_2\sigma_1^2)^2\right)}{2\sigma_1^2\sigma_2^2(\sigma_1^2+\sigma_2^2)}\right]$$

$$= \exp\left[-\frac{\cancel{\sigma_2^2\sigma_1^2}z^2-2\cancel{\sigma_2^2\sigma_1^2}(\mu_1+\mu_2)z+\cancel{\sigma_1^2\sigma_2^2}(\mu_1^2+2\mu_1\mu_2+\mu_2^2)}{2\cancel{\sigma_1^2\sigma_2^2}(\sigma_1^2+\sigma_2^2)}\right]$$

$$= \exp\left[-\frac{\left(z-(\mu_1+\mu_2)\right)^2}{2(\sigma_1^2+\sigma_2^2)}\right]$$

Now, we can rewrite (1) as:

$$f_Z(z) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}} \cdot \frac{1}{\sqrt{2\pi}\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}} \cdot \exp\left[- \frac{\left(x - \frac{\mu_1\sigma_2^2 + \sigma_1^2(z-\mu_2)}{\sigma_1^2+\sigma_2^2} \right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2} \right] \cdot$$

$$\underbrace{\exp\left[- \frac{(z-(\mu_1+\mu_2))^2}{2(\sigma_1^2+\sigma_2^2)} \right]}_{\text{no "x" involved}} \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}} \exp\left[- \frac{(z-(\mu_1+\mu_2))^2}{2(\sigma_1^2+\sigma_2^2)} \right] \cdot \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}} \cdot \exp\left[- \frac{\left(x - \frac{\mu_1\sigma_2^2 + \sigma_1^2(z-\mu_2)}{\sigma_1^2+\sigma_2^2} \right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\right)^2} \right] dx$$

PDF of $\mathcal{N}\left(\frac{\mu_1\sigma_2^2 + \sigma_1^2(z-\mu_2)}{\sigma_1^2+\sigma_2^2}, \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} \right)$

Therefore, this integration should end up being 1

$$= \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}} \exp\left[- \frac{(z-(\mu_1+\mu_2))^2}{2(\sigma_1^2+\sigma_2^2)} \right]$$

PDF of $\mathcal{N}(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$.