**DCP1206 Fall 2019: Probability** **(Due: 2019/10/23 in class)**

# Homework 2: Random variables, Expected Value, and Variance

**Problem 1 (Special Discrete Random Variables)** (6+6+6+6+6=30 points)

**(a)** Consider a binomial random variable $X$ with parameters $n$ and $p$. Let $k^*$ be the largest integer that is less than or equal to $(n+1)p$. Show that the PMF $p_X(k)$ of $X$ is monotonically non-decreasing with $k$ in the range from 0 to $k^*$, and is monotonically decreasing with $k$ for $k \geq k^*$.

**(b)** For certain software, independently of other users, the probability that a user encounters a fault is 0.05. Suppose the users arrive sequentially. Let $X$ be the number of users who do not encounter a fault before the 7th user who encounters a fault. What is the PMF of $X$?

**(c)** The Houston Astros and the Tampa Bay Rays are set to play a playoff of $n$ baseball games, with $n$ being odd. The Astros have a probability $q$ of winning any one game (and independently of other games). Find the values of $q$ for which $n = 5$ is better than $n = 3$ for the Astros.

**(d)** Let $X_1$ and $X_2$ be two independent Geometric random variables with parameters $p_1$ and $p_2$, respectively. Define $X = \min(X_1, X_2)$. What is the PMF of $X$?

**(e)** Consider a close election in a small city between two candidates: Amy and Bill. Currently we know that Amy wins the election with 1236 votes and an extremely small winning margin of 10 votes. However, 57 votes are illegal and have to be thrown out. Suppose that the illegal votes are not biased in any particular way and the count is reliable. What is the probability that the removal of the illegal votes changes the result of the election? (Hint: You might need to write a short MATLAB or python script to compute the probability)

**Problem 2 (Communication Over a Binary Channel and Poisson)** (8+8+8=24 points)

At each time, the transmitter sends out either a '1' with probability $p$, or a '0' with probability $1-p$. Given that the transmitted bit is '1', the receiver successfully receives the '1' with probability $\alpha_1$ or it receives an incorrect message of '0' with probability $1-\alpha_1$. Similarly, given that the transmitted bit is '0', the receiver either receives a '0' with probability $\alpha_0$, or a '1' with probability $1-\alpha_0$, respectively.

**(a)** Suppose the number of transmissions within a given observation window $T$ has a Poisson PMF with average rate $\lambda$. Define a random variable $X$ to be the number of 1's *transmitted* in that time interval. Show that $X$ has a Poisson PMF with average rate $\lambda p$. (Hint: Define $V$ = total transmitted bits in the given interval. Try to use the total probability theorem $P(X = k) = \sum_{n=0}^{\infty} P(X = k|V = k+n) \cdot P(V = k+n)$)

**(b)** Same as the setting in (a), we suppose that the number of transmissions within a given observation window $T$ has a Poisson PMF with average rate $\lambda$. Define a random variable $Y$ to be the total number of 1's *received* in that time interval. What is the PMF of $Y$?

**(c)** Suppose now there are 2 transmitters $T_1$ and $T_2$ that keep sending bits to the same receiver. The numbers of bits transmitted from $T_1$ and $T_2$ within a given observation window $T$ are assumed to be *independent* and have a Poisson PMF with average rates $\lambda_1$ and $\lambda_2$, respectively. For the other system parameters (i.e. $p, \alpha_0$, and $\alpha_1$), the two transmitters have exactly the same configuration. Define $Z$ to be the total number of 1's *received* by the receiver in that time interval $T$. What is the PMF of $Z$?

**Problem 3 (PMF and Entropy)** (6+6+8=20 points)

**(a)** Consider a discrete random variable $X$ with range $\{1, 2, \cdots, n\}$. Define $p_i := P(X = i)$, for all $i = 1, \cdots, n$. Next, we define a metric called *entropy*:

$$H(X) := -\sum_{i=1}^{n} p_i \ln p_i.$$

(Note: "ln" means the base of the logarithm is $e$). What is the maximum possible value of $H(X)$? Find the PMF $\{p_i\}_{i=1}^n$ that achieves this maximum.

**(b)** Given the same setting of (a), what is the minimum possible value of $H(X)$? Please find all the PMFs $\{p_i\}_{i=1}^n$ that achieve this minimum.

**(c)** Let $q_1, q_2, \cdots, q_n$ be non-negative real numbers such that $\sum_{i=1}^n q_i = 1$. Show that: Regardless of the PMF of $X$, we always have

$$H(X) \leq -\sum_{i=1}^{n} p_i \ln q_i.$$

Under what condition do we have equality (i.e. $H(X) = -\sum_{i=1}^n p_i \ln q_i$)?

**Problem 4 (Expected Value and Variance of Discrete Random Variables)** (6+6+6=18 points)

Please use the definitions of expected value and variance to show the following properties:

**(a)** Suppose $X \sim \text{Unif}(a, b)$. Show that $E[X] = (a + b)/2$ and $\text{Var}[X] = (b - a + 1)^2 - 1/12$.

**(b)** Suppose $X \sim \text{NB}(p, r)$. Show that $E[X] = r/p$ and $\text{Var}[X] = r(1 - p)/p^2$. (Hint: Write down the PMF and try to reuse the fact that the total probability is 1.)

**(c)** Suppose $X \sim \text{HyperGeometric}(N, D, n)$. Show that $E[X] = nD/N$.

**Problem 5 (Expected Value and Variance)** (8 points)

Let $X$ and $Y$ be two discrete random variables with the identical set of possible values $\{a_1, a_2, a_3\}$ ($a_1$, $a_2$, and $a_3$ are different). Show that if $E[X] = E[Y]$ and $\text{Var}[X] = \text{Var}[Y]$, then $X$ and $Y$ are identically distributed.