DCP 1206: Probability
Lecture 13 — Independent Random Variables and Joint PDF

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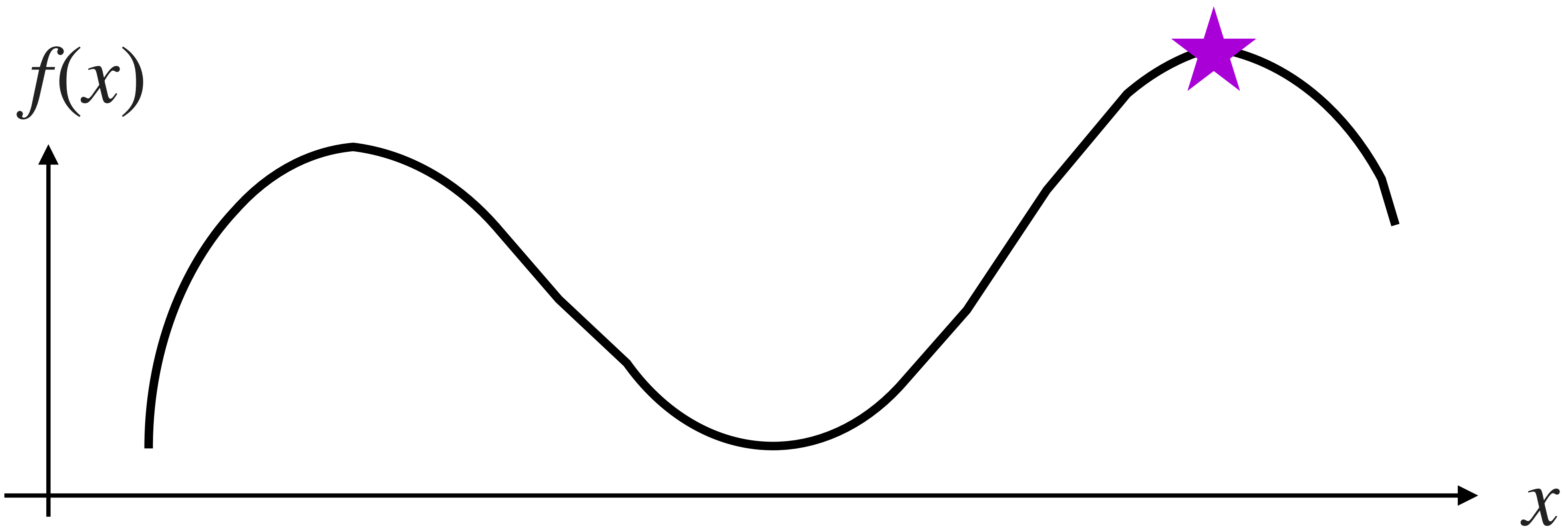
October 30, 2019

Announcements

- HW3 is on E3 (Due: 11/1 in class)
- TA hours
  - 10/31 (Thursday), 7pm-8pm @ EC500A
  - 11/4 (Monday), 7pm-8pm @ EC345

Any CS application that require knowledge of (joint) PDF or CDF?

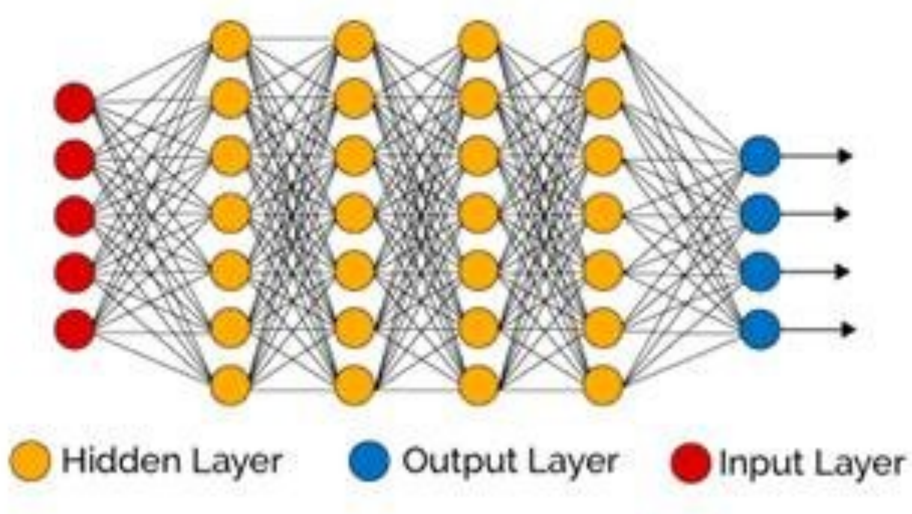
Black-Box Optimization (BBO)



- Question： How to find the max or min of $f(x)$ in general?
- Black-box function: A function that you don't know the form and cannot take derivatives

What shall we do? **Sampling!**

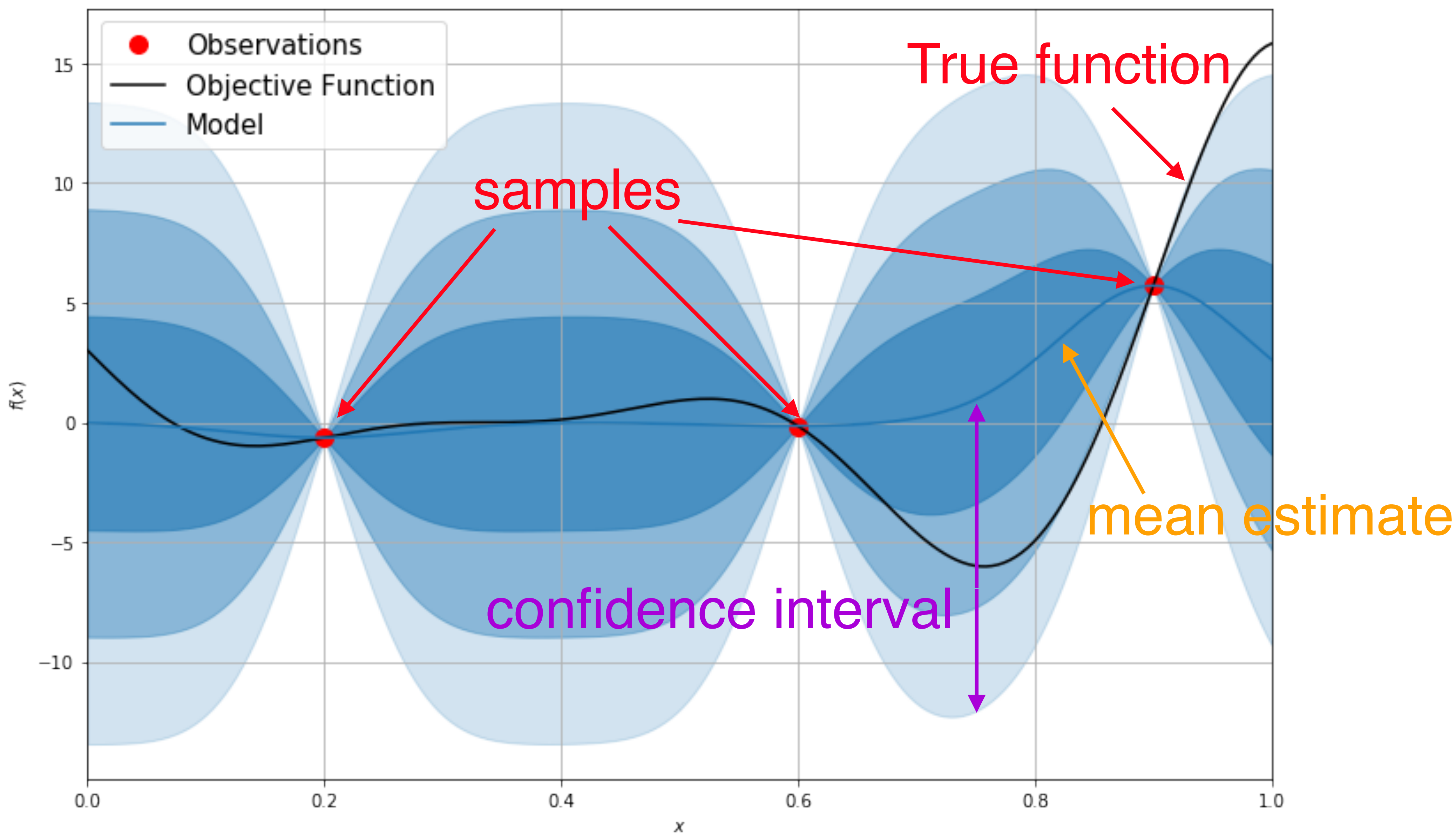
Applications of BBO: Hyperparameter Tuning for DNN



- Hyperparameters = parameters that are predetermined and not trained
  - Number of layers?
  - Number of neurons for each layer?
  - Learning rate?

- Target black-box function: Accuracy under a set of hyperparameters
  - Input: hyperparameters
  - Output: accuracy
  - Black-box: We are not able to take derivatives of this function
  - Goal: Find the set of hyperparameters that maximize accuracy

Sampling for Black-Box Optimization



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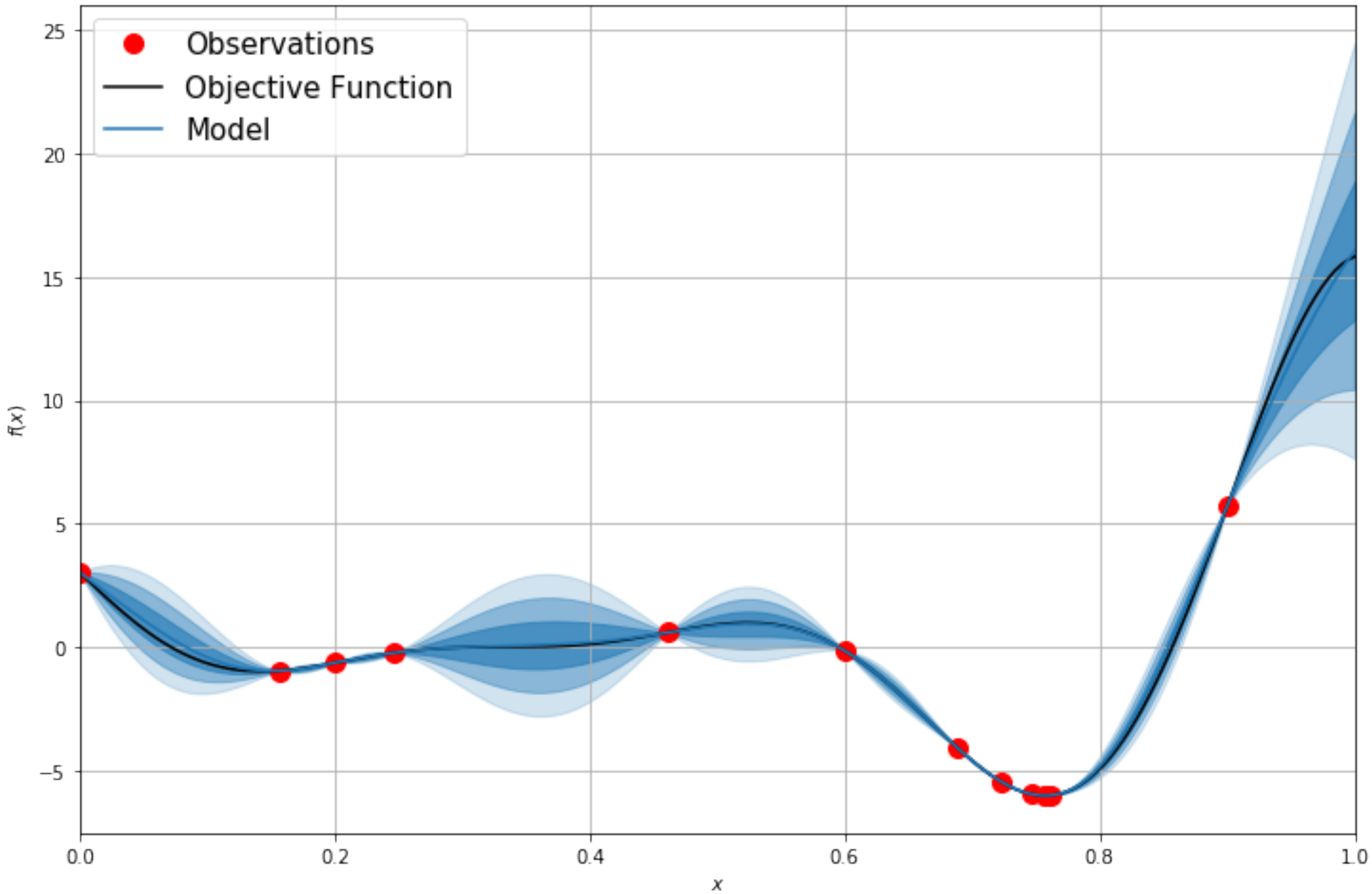
Sampling for Black-Box Optimization (Cont.)



Key Idea: Joint distributions of Gaussian RVs

https://nbviewer.jupyter.org/github/amzn/emukit/blob/master/notebooks/index.ipynb

A Quick Review (I)

Joint PMF
$P(X = x, Y = y)$

Marginal PMF
$P(X = x), P(Y = y)$

Joint CDF
$F_{XY}(x, y)$

$P(X \le x, Y \le y)$

Marginal CDF
$F_X(x), F_Y(y)$

$F_{XY}(\infty, y)$
$=$
$P(X \le x) = P(X \le x, Y \in \mathbb{R}) = F_{XY}(x, \infty)$, $\underbrace{P(Y \le y)}$

A Quick Review (II)

$$
\begin{tikzcd}
\begin{array}{c}\text{Joint PMF}\\ P(X = x, Y = y)\end{array} \arrow[r] & \begin{array}{c}\text{Marginal PMF}\\ P(X = x), P(Y = y)\end{array} \arrow[d, shift right] \\
\begin{array}{c}\text{Joint CDF}\\ F_{XY}(x, y)\end{array} \arrow[u] & \begin{array}{c}\text{Marginal CDF}\\ F_X(x), F_Y(y)\end{array} \arrow[u, shift right]
\end{tikzcd}
$$

Marginal CDF → Joint CDF?

Joint PMF
$P(X = x, Y = y)$

Marginal PMF
$P(X = x), P(Y = y)$

Joint CDF
$F_{XY}(x, y)$ $\longleftarrow$ Marginal CDF
$F_X(x), F_Y(y)$

- Question: Could we get joint CDF from marginal CDF?

In general: NO
"Independence": Yes

This Lecture

1. Independent Random Variables

2. Joint PDF and Marginal PDF

- Reading material: Chapter 8.1-8.2

1. Independent Random Variables

Review: Independence of 2 Events

Definition: Two events A and B are said to be **independent** if $P(A \cap B) = P(A)P(B)$

Moreover, if $P(B) > 0$, then independence is equivalent to the condition $P(A \mid B) = P(A)$

- **Question:** How to extend the notion of **independence** to 2 random variables?

2 coins: (fair) H/T → Prob $= \frac{1}{2}$

2nd \ 1st	H	T
H	$\frac{1}{4}$	$\frac{1}{4}$
T	$\frac{1}{4}$	$\frac{1}{4}$

Independence of 2 Random Variables (Formally)

Definition: Two random variables X, Y are said to be **independent** if for arbitrary sets of real numbers A, B, the events $\{X \in A\}$ and $\{Y \in B\}$ are independent, i.e.

$$P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B)$$

- Remark: Same definition for both discrete and continuous random variables
- Question: What if we choose the sets as $A = (-\infty, t]$ and $B = (-\infty, u]$?

$F_{XY}(t,u)$

$\parallel$

$P(X \in (-\infty, t], Y \in (-\infty, u])$

$F_Y(u)$

$\parallel$

$F_X(t) = P(X \in (-\infty, t]) \, P(Y \in (-\infty, u])$

Property: Independence of 2 Random Variables

Independence $\equiv$ joint CDF is the product of the marginal CDFs:

Two random variables X, Y are **independent** <u>if and only if</u>

$$\underbrace{F_{XY}(t, u) = F_X(t) \cdot F_Y(u)}, \quad \forall\, t \in \mathbb{R},\ u \in \mathbb{R}.$$

- Remark: This property holds for both discrete and continuous random variables

Example: 2 Discrete Uniform Random Variables

$F_{XY}(t,u) = F_X(t) \cdot F_Y(u)$

- Example: X, Y are two independent discrete uniform random variables with the same range $\{0,1,2\}$.
  - CDF of X? How about Y?
  - Joint CDF of X and Y?

$$F_X(t) = \begin{cases} 0, & t<0 \\ \frac{1}{3}, & 0 \le t < 1 \\ \frac{2}{3}, & 1 \le t < 2 \\ 1, & t \ge 2 \end{cases}$$

$$F_Y(u) = \begin{cases} 0, & u<0 \\ \frac{1}{3}, & 0 \le u < 1 \\ \frac{2}{3}, & 1 \le u < 2 \\ 1, & u \ge 2 \end{cases}$$

X \ Y	$u<0$	$0 \le u < 1$	$1 \le u < 2$	$u \ge 2$
$t<0$	$0 \times 0 = 0$			
$0 \le t < 1$			$\frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$	
$1 \le t < 2$			$\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$	
$t \ge 2$				$1 \times 1 = 1$

Example: Continuous Uniform and Exponential

- Example: $X \sim$ Unif$(0,1)$ and $Y \sim$ Exp$(\lambda = 1)$ be two <u>independent</u> continuous random variables.
  - Joint CDF of X and Y?

$X \backslash Y$	$u \geq 0$	$u < 0$
$t > 1$	$1 \cdot (1 - e^{-t})$	
$0 \leq t \leq 1$		$t \cdot 0 = 0$
$t < 0$	$0 \cdot (1 - e^{-t}) = 0$	

$$F_X(t) = \begin{cases} 1 & , t > 1 \\ t & , 0 \leq t \leq 1 \\ 0 & , t < 0 \end{cases}$$

$$F_Y(u) = \begin{cases} 1 - e^{-t} & , u \geq 0 \\ 0 & , u < 0 \end{cases}$$

Marginal PMF → Joint CDF

Joint PMF
$P(X = x, Y = y)$

Marginal PMF
$P(X = x), P(Y = y)$

Joint CDF
$F_{XY}(x, y)$

Marginal CDF
$F_X(x), F_Y(y)$

$$F_{XY}(x, y) \xleftarrow{\text{"Independence of X and Y"}} F_X(x), F_Y(y)$$

"Independence of X and Y"

(check: use $F_{XY} = F_X \cdot F_Y$)

2. Joint PDF and Marginal PDF

What We Will Learn

Done!

Joint PMF
$P(X = x, Y = y)$

Marginal PMF
$P(X = x), P(Y = y)$

Joint CDF
$F_{XY}(x, y)$

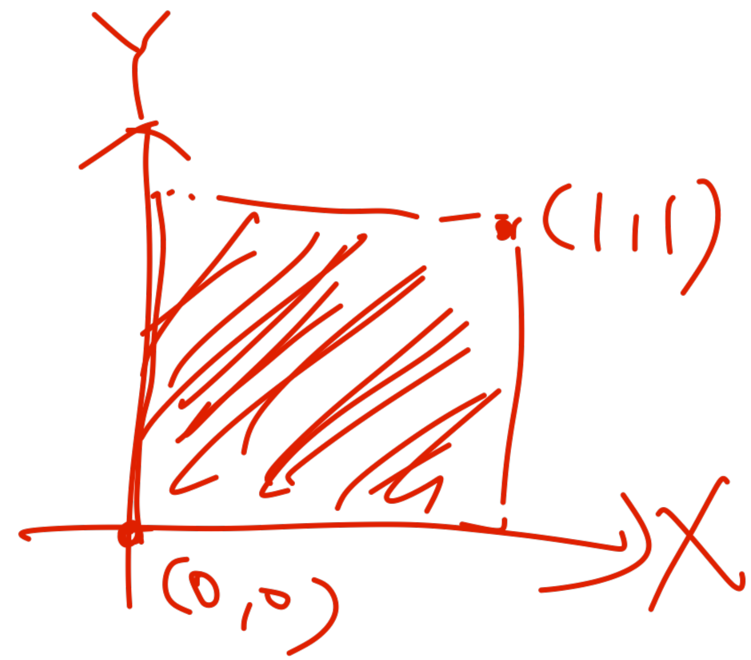
Marginal CDF
$F_X(x), F_Y(y)$

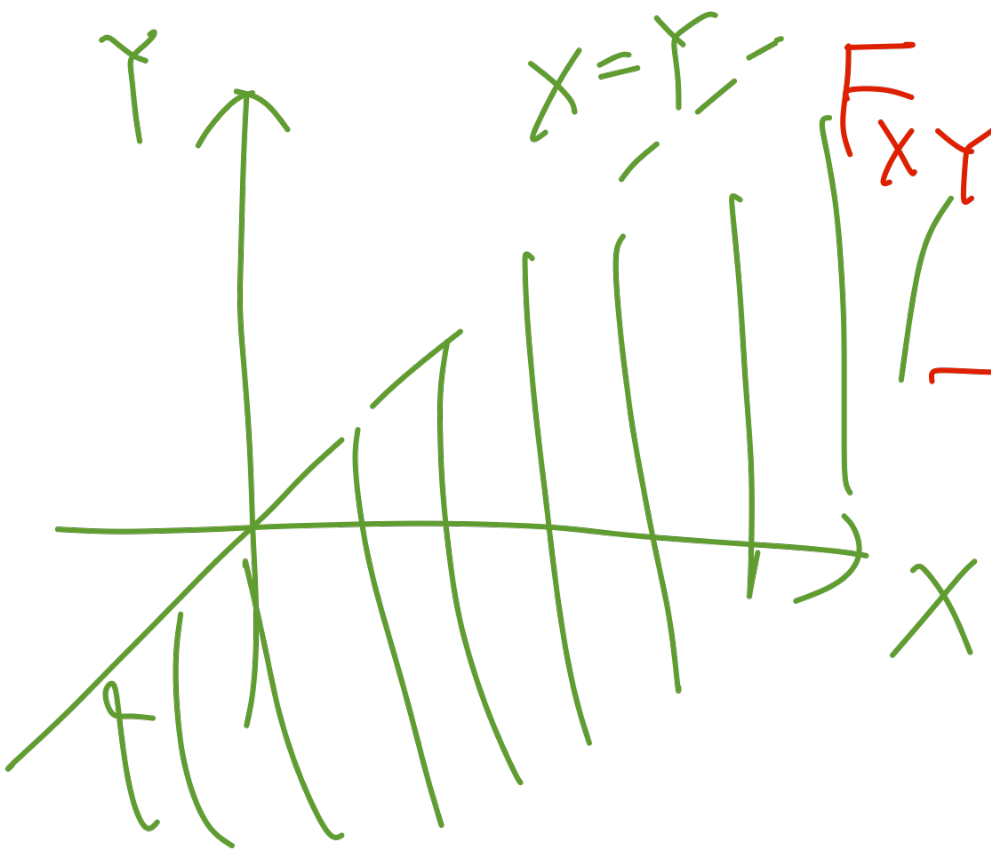
Joint PDF
$f_{XY}(x, y)$

Marginal PDF
$f_X(x), f_Y(y)$

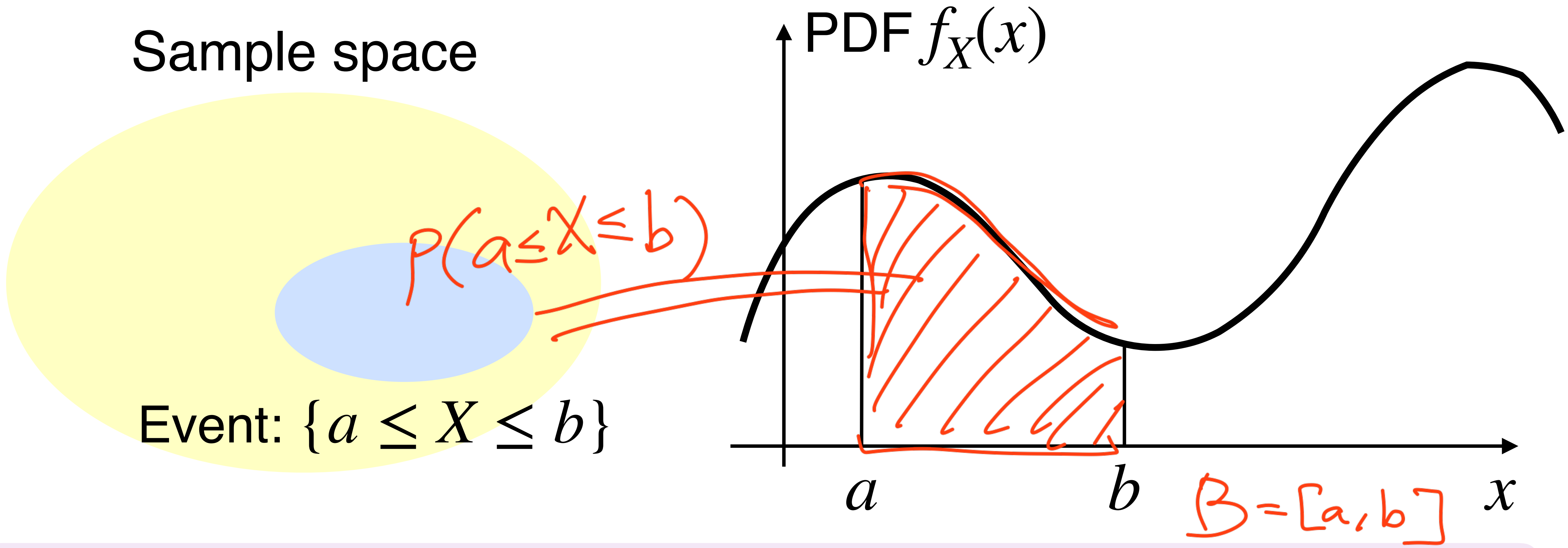
Why Studying Joint PDF?

- Question: X, Y are continuous random variables with the joint CDF $F_{XY}(x, y)$
  - $P(X \leq 0, Y \leq 1)$? $= F_{XY}(0,1)$
  - $P(0 < X \leq 1, 0 < Y \leq 1)$? $= F_{XY}(1,1) - F_{XY}(0,1) - F_{XY}(1,0) + F_{XY}(0,0)$
  - $P(X \geq Y)$?





Review: Probability Density Function (PDF)



Probability Density Function (PDF): valid

Let X be a random variable. Then, $f_X(x)$ is the PDF of X if for every subset B of the real line, we have

$$\underline{P(X \in B)} = \int_B f_X(x)dx$$

Joint PDF

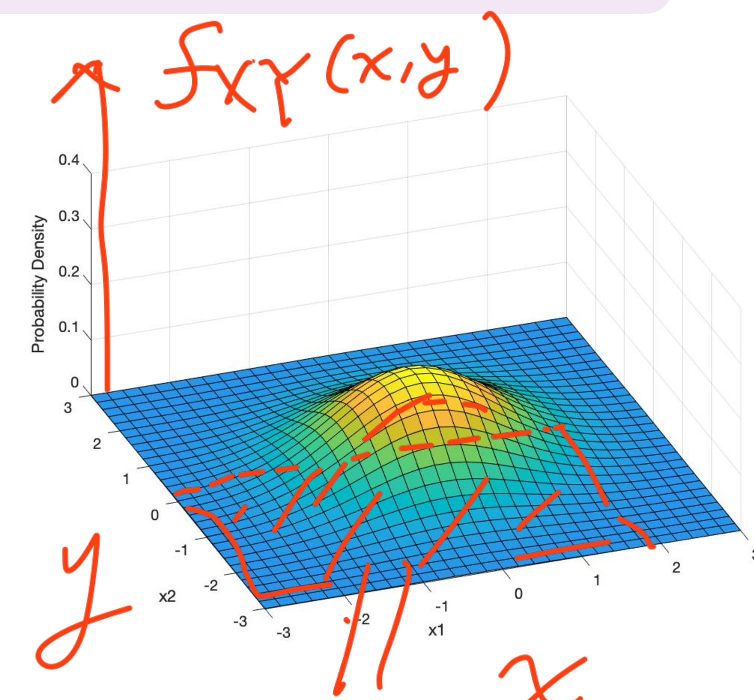
Joint PDF: Let X and Y be two continuous random variables. Then, $f_{XY}(x,y)$ is the joint PDF of X and Y if for every subset B of $\mathbb{R}^2$, we have

$$P((X,Y) \in B) = \underbrace{\iint_B f_{XY}(x,y)\,dx\,dy}$$

- $P(X \in B_X, Y \in B_Y) = \int_{B_Y}\int_{B_X} \underbrace{f_{XY}(x,y)}\,dx\,dy$

  B_X: $[0,1]$, B_Y: $[0,2]$

- $P(X \in \mathbb{R}, Y \in \mathbb{R}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underbrace{f_{XY}(x,y)}\,dx\,dy = 1$



$P((X,Y) \in B)$

Joint PDF and Event Probabilities

$f_{XY}(x, y)$ is the joint PDF of X and Y

- $P(X \leq t) =$ $\int_{-\infty}^{t} \int_{-\infty}^{+\infty} \underbrace{f_{XY}(x,y)}\, dy\, dx$

  $= P(X \leq t, Y \in \mathbb{R})$

- $\underbrace{f_X(x)}_{\text{marginal PDF}} = \dfrac{d F_X(x)}{dx} = \dfrac{d\,P(X \leq x)}{dx} = \int_{-\infty}^{+\infty} \underbrace{f_{XY}(x,y)}_{\text{joint PDF}}\, dy$

Joint PDF and Event Probabilities

$f_{XY}(x, y)$ is the joint PDF of X and Y

- $\underbrace{P(Y \le u)}_{\shortparallel} = \int_{-\infty}^{u} \int_{-\infty}^{+\infty} \underbrace{f_{XY}(x,y)}\, dx\, dy$

  $P(X \in \mathbb{R}, Y \le u)$

- $\underbrace{f_Y(y)}_{\parallel} = \int_{-\infty}^{+\infty} f_{XY}(x,y)\, \underline{\underline{dX}}$

  marginal PDF

Marginal PDF

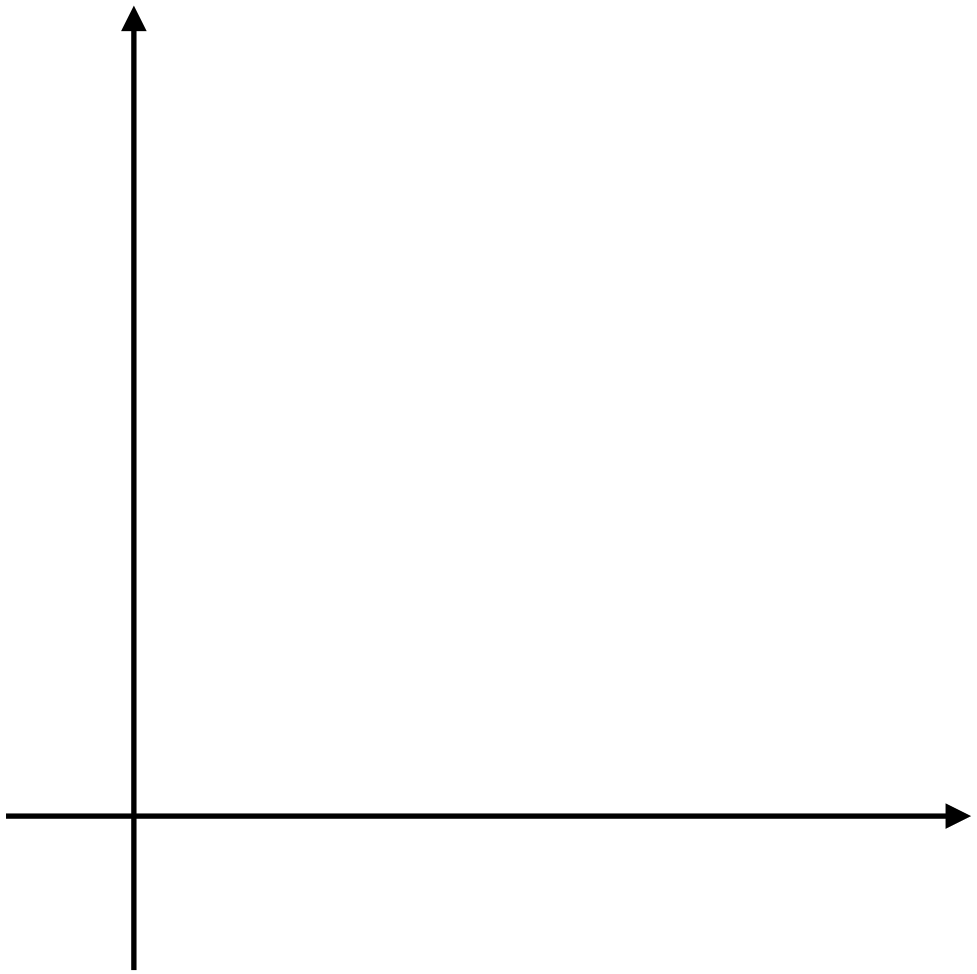
Marginal PDF: Let X and Y be two continuous random variables, and $f_{XY}(x, y)$ is the joint PDF of X and Y. The marginal PDF of X and Y are defined as

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx$$

Interpret the Marginal PDF

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy$$



Example: (1) Joint PDF → Joint CDF

- Example: The joint PDF of X and Y is

$$f(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2) & \text{, if } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{, otherwise} \end{cases}$$

- Joint CDF of X and Y?

$F_{XY}(1.5, 1.5) = 1$

$F_{XY}(t,u)$

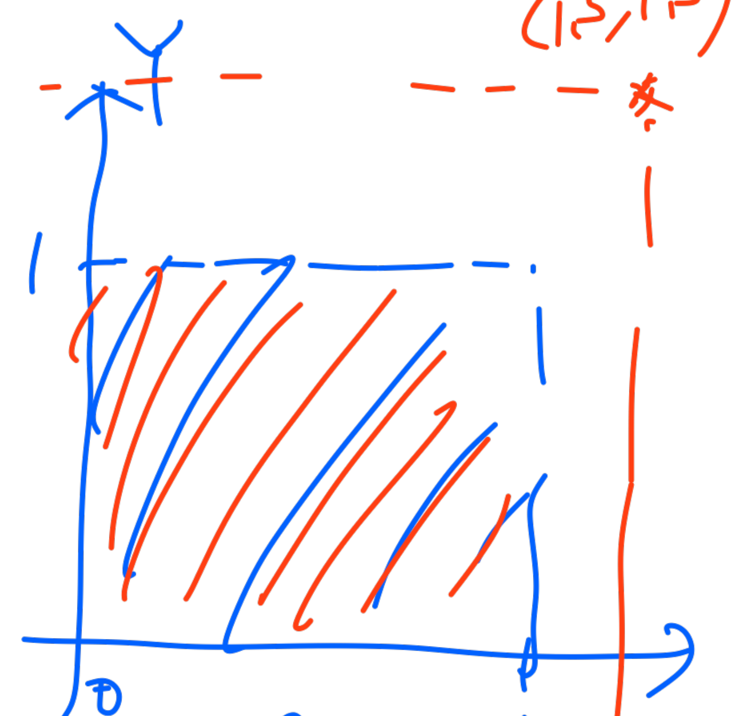
$t \backslash u$	$u<0$	$0 \le u \le 1$	$u>1$
$t<0$	0		
$0 \le t \le 1$		$\frac{t^3u}{2} + \frac{tu^3}{2}$	
$t>1$			1

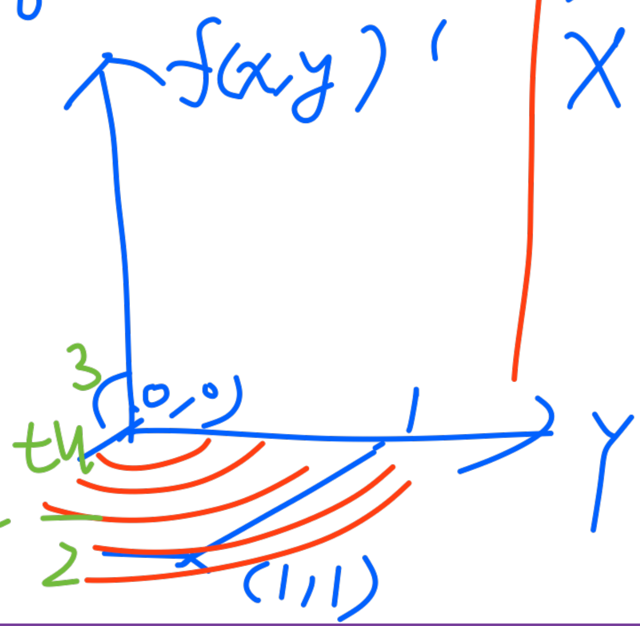
$$\int_0^u \left(\int_0^t \frac{3}{2}(x^2+y^2)\,dx \right) dy$$

$$= \int_0^u \left(\frac{3}{2}\left(\frac{x^3}{3} + xy^2\right) \Big|_0^t \right) dy$$

$$= \int_0^u \left(\frac{t^3}{2} + \frac{3ty^2}{2} \right) dy$$

$$= \left(\frac{t^3}{2} y + \frac{ty^3}{2} \right) \Big|_0^u = \frac{t^3u}{2} + \frac{tu^3}{2}$$





Example: (2) Joint PDF → Marginal CDF

- Example: X and Y be

$$f(x, y) = \begin{cases} \frac{3}{2}(x^2 + y^2) & \text{, if } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{, otherwise} \end{cases}$$

- The marginal CDF of X?

Example: (3) Joint PDF → Marginal PDF

- Example: X and Y be

$$f(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2) & \text{, if } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{, otherwise} \end{cases}$$

$f_X(x)$	$x<0$	$0 \le x \le 1$	$x>1$
	0	$\frac{3}{2}(x^2+\frac{1}{3})$	0

- Marginal PDF of X?

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy$$

$$= \int_0^1 \frac{3}{2}(x^2+y^2)\,dy = \frac{3}{2}\left(x^2 y + \frac{y^3}{3}\right)\Big|_0^1$$

$$= \frac{3}{2}\left(x^2 + \frac{1}{3}\right)$$

Review: From CDF to PDF (Formally)

Derivative of CDF is PDF:
Let X be a random variable with a CDF $F_X(\cdot)$ and a PDF $f_X(\cdot)$. If $f_X(\cdot)$ is continuous at x_0, then

$$F'_X(x_0) = f_X(x_0)$$

- Question: Do we have any similar property regarding joint CDF and joint PDF?

Given Joint CDF: Find Joint PDF

Partial Derivative of Joint CDF is Joint PDF:
X and Y are two continuous random variables.
Let $F_{XY}(x, y)$ be the joint CDF of X and Y.
Assume the partial derivatives of $F_{XY}(x, y)$ exist. Then, one valid choice of PDF can be

$$f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$$

Joint PDF: Interpret "Density" Using Limits

prob. in $\square$

Joint CDF

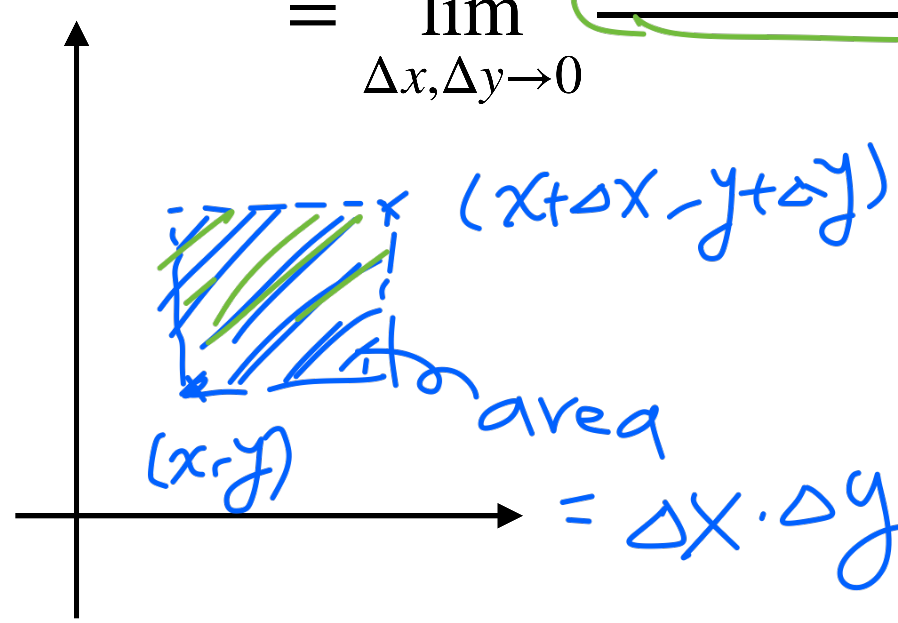
$\downarrow$

Joint PDF

$$f_{XY}(x,y) \equiv \lim_{\Delta x,\Delta y\to 0} \frac{P(x<X\le x+\Delta x, y<Y\le y+\Delta y)}{\Delta x\Delta y}$$

area

$$= \lim_{\Delta x,\Delta y\to 0} \frac{F_{XY}(x+\Delta x, y+\Delta y) - F_{XY}(x, y+\Delta y) - F_{XY}(x+\Delta x, y) + F_{XY}(x,y)}{\Delta x\Delta y}$$



area $= \Delta x \cdot \Delta y$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{1}{\Delta x}\left[\frac{F_{XY}(x+\Delta x, y+\Delta y) - F_{XY}(x+\Delta x, y)}{\Delta y}\right]$$

$$\Downarrow$$

$$\frac{\partial F_{XY}(x+\Delta x, y)}{\partial y}$$

$$\text{Density} = \lim_{\square \to 0} \frac{\text{Probability of } \square}{\text{Area of } \square}$$

Technical Issues With Joint PDF

1. Given joint CDF $F_{XY}(x, y)$, the joint PDF is <u>NOT unique</u>

2. Suppose the partial derivatives of $F_{XY}(x, y)$ exist, then $\frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$ is a valid joint PDF

3. In this class, we usually assume (unless stated otherwise):
   1. The partial derivatives of $F_{XY}(x, y)$ exist
   2. $\frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$ is the joint PDF to operate with

A Quick Summary (I)

$$\int_0^u \int_0^t f_{XY}\,dx\,dy \qquad F_X(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{t} f_{XY}\,dx\,dy$$

$$\|$$

$$\begin{tikzcd}
\begin{array}{c}\text{Joint CDF}\\ F_{XY}(\cancel{x}\,t, \cancel{y}\,u)\end{array} & \begin{array}{c}\text{Marginal CDF}\\ F_X(\cancel{x}\,t), F_Y(\cancel{y}\,u)\end{array} \\
\begin{array}{c}\text{Joint PDF}\\ f_{XY}(x, y)\end{array} \arrow[u] \arrow[ur] \arrow[r] & \begin{array}{c}\text{Marginal PDF}\\ f_X(x), f_Y(y)\end{array}
\end{tikzcd}$$

$$f_X(x) = \int f_{XY}\,dy$$

$$f_Y(y) = \int f_{XY}\,dx$$

A Quick Summary (II)

Joint CDF
$F_{XY}(x, y)$

$$\frac{\partial^2 F_{XY}(x,y)}{\partial x \, \partial y}$$

Joint PDF
$f_{XY}(x, y)$

Marginal CDF
$F_X(x), F_Y(y)$

Marginal PDF
$f_X(x), f_Y(y)$

Marginal PDF to Joint PDF?

- Question: Could we get joint PDF from marginal PDF?

Joint CDF
$F_{XY}(x, y)$

Marginal CDF
$F_X(x), F_Y(y)$

Joint PDF
$f_{XY}(x, y)$

Marginal PDF
$f_X(x), f_Y(y)$

Property: Independence of 2 Continuous Random Variables

Joint PDF is the product of the marginal PDFs under independence:
If two continuous random variables X, Y are **independent**, then the joint PDF satisfies that

$$f_{XY}(t, u) = f_X(t) \cdot f_Y(u)$$

- Proof:

Example: Uniform and Exponential

- Example: $X \sim$ Unif$(0,1)$ and $Y \sim$ Exp$(\lambda = 1)$ be two independent continuous uniform random variables.
  - What is the joint PDF?

Example: 2 Uniform Random Variables

- Example: $X \sim \text{Unif}(0,3)$ and $Y \sim \text{Unif}(0,3)$ be two independent continuous uniform random variables.
  - Joint PDF?
  - Find the probability that $X \geq Y$?

1-Minute Summary

1. Independent Random Variables

$F_{XY} = F_X \cdot F_Y$

- Joint CDF is the product of marginal CDFs

2. Joint PDF and Marginal PDF

- Joint PDF ↔ Joint CDF / Marginal CDF / Marginal PDF
- Independence ⇒ Joint PDF is the product of marginal PDFs

15-Minutes Brain Workout

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\begin{array}{c}\text{Joint PMF}\\ P(X=x, Y=y)\end{array} \arrow[r] & \begin{array}{c}\text{Marginal PMF}\\ P(X=x), P(Y=y)\end{array} \arrow[d, shift right] \\
\begin{array}{c}\text{Joint CDF}\\ F_{XY}(x,y)\end{array} \arrow[r, shift left] \arrow[d, shift right] & \begin{array}{c}\text{Marginal CDF}\\ F_X(x), F_Y(y)\end{array} \arrow[u, shift right] \arrow[l, shift left] \arrow[d, shift right] \\
\begin{array}{c}\text{Joint PDF}\\ f_{XY}(x,y)\end{array} \arrow[u, shift right] \arrow[ur] \arrow[r] & \begin{array}{c}\text{Marginal PDF}\\ f_X(x), f_Y(y)\end{array} \arrow[u, shift right]
\end{tikzcd}
$$