Consider $Z \sim N(0,1)$ and $W \sim N(0,1)$.

- Z, W are independent $\Rightarrow f_{ZW}(z,w) = \frac{1}{2\pi} \exp\left(-\frac{z^2}{2} - \frac{w^2}{2}\right)$

- Define $\begin{cases} X_1 = \sigma_1 Z \\ X_2 = \sigma_2\left(\rho Z + \sqrt{1-\rho^2}\, W\right) \end{cases}$

$$\Leftrightarrow \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sigma_2\sqrt{1-\rho^2} \end{bmatrix}}_{= A} \begin{bmatrix} Z \\ W \end{bmatrix}$$

$\det(A) = \sigma_1 \cdot \sigma_2 \cdot \sqrt{1-\rho^2}$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} \sigma_2\sqrt{1-\rho^2} & 0 \\ -\rho\sigma_2 & \sigma_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma_1} & 0 \\ \frac{-\rho}{\sigma_1 \cdot \sqrt{1-\rho^2}} & \frac{1}{\sigma_2\sqrt{1-\rho^2}} \end{bmatrix}$$

By linear transformation, we know:

$$f_{X_1X_2}(x_1,x_2) = \frac{1}{|\det(A)|} f_{ZW}\left(A^{-1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

$$= \frac{1}{\sigma_1\sigma_2 \cdot \sqrt{1-\rho^2}} \cdot f_{ZW}\left(\frac{x_1}{\sigma_1}, \frac{-\rho x_1}{\sigma_1\sqrt{1-\rho^2}} + \frac{x_2}{\sigma_2\sqrt{1-\rho^2}}\right)$$

$$= \frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}} \left(\frac{1}{2\pi} \cdot \exp\left(-\frac{1}{2}\left(\frac{x_1}{\sigma_1}\right)^2 - \frac{1}{2}\left(\frac{-\rho x_1}{\sigma_1\sqrt{1-\rho^2}} + \frac{x_2}{\sigma_2\sqrt{1-\rho^2}}\right)^2\right)\right)$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \cdot \exp\left(-\frac{(\sigma_2^2 x_1^2 - 2\rho\sigma_1\sigma_2 x_1 x_2 + \sigma_1^2 x_2^2)}{2\sigma_1^2\sigma_2^2 \cdot (1-\rho^2)}\right)$$