DCP1206 Fall 2019: Probability **(Due: 2019/11/22 in class)**

Homework 4, Part I: Bivariate Distributions

Problem 1 (Independence of Two Random Variables) (10+10=20 points)

(a) Two points X and Y are selected at random and independently from the interval $(0, 2)$. Calculate $P(Y \leq X \text{ and } X^2 + Y^2 \leq 1)$. (Hint: To begin with, find out the joint PDF of X and Y by using independence)

(b) Let X and Y be two discrete random variables with given joint PMF $p_{XY}(x, y)$. We use $p_X(x)$ and $p_Y(y)$ to denote the marginal PMF of X and Y, respectively. Moreover, let $g(\cdot)$ and $h(\cdot)$ be two real-valued functions of X and Y, respectively. Show that if X and Y are independent, then $g(X)$ and $h(Y)$ are also independent. (Hint: As discussed in Lecture 13, to show that two random variables U, V are independent, we need to verify $P(U \in A, V \in B) = P(U \in A)P(V \in B)$, for any sets of real numbers A, B)

Problem 2 (Expected Value of Two Random Variables) (12+12=24 points)

Let the joint PDF of X and Y be given by

$$f(x, y) = \begin{cases} 1, & \text{if } |x| < y, 0 < y < 1 \\ 0, & \text{else} \end{cases}$$

(a) Show that $E[XY] = E[X]E[Y]$. (Hint: To find $E[X]$ and $E[Y]$, you need to first obtain the marginal PDF of X and Y. For more details, check Lecture 13)

(b) Show that X and Y are NOT independent. (Hint: Similar to Problem 2(b), to show that two random variables U, V are NOT independent, you may construct an example of two sets A, B such that $P(U \in A, V \in B) \neq P(U \in A)P(V \in B)$)

Problem 3 (Covariance and Correlation Coefficient) (10+12=22 points)

Suppose X is a standard normal random variable.

(a) Calculate $E[X^3]$ and $E[X^4]$.

(b) Define a new random variable $Y = aX^2 + bX + c$. Find the correlation coefficient $\rho(X, Y)$.