DCP 1206: Probability

Lecture 10 — Special Continuous Random Variables (I)

Ping-Chun Hsieh

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#Fakenews? Fake Videos of Real People

(June, 2019)

<u>Mark</u>: …*Imagine this for a second: One man, with total control of billions of people's stolen data, all their secrets, their lives, their futures*…
(by Bill Posters @bill_posters_uk)

(July, 2017)

- Research from University of Washington:
  - Given only the audio, a video can be synthesized with accurate lip sync
- How to tell the fake from the true one?
  - Eye blinking? (arXiv:1806.02877)

TED talk: https://www.youtube.com/watch?v=o2DDU4g0PRo

Expected Value of a Continuous Random Variable Using CDF

Expected Value via CDF:

Let X be a continuous random variable with CDF $F_X(t)$.

The expected value of X is

$$E[X] = \int_0^\infty (1 - F_X(t))dt - \int_0^\infty F_X(-t)dt$$

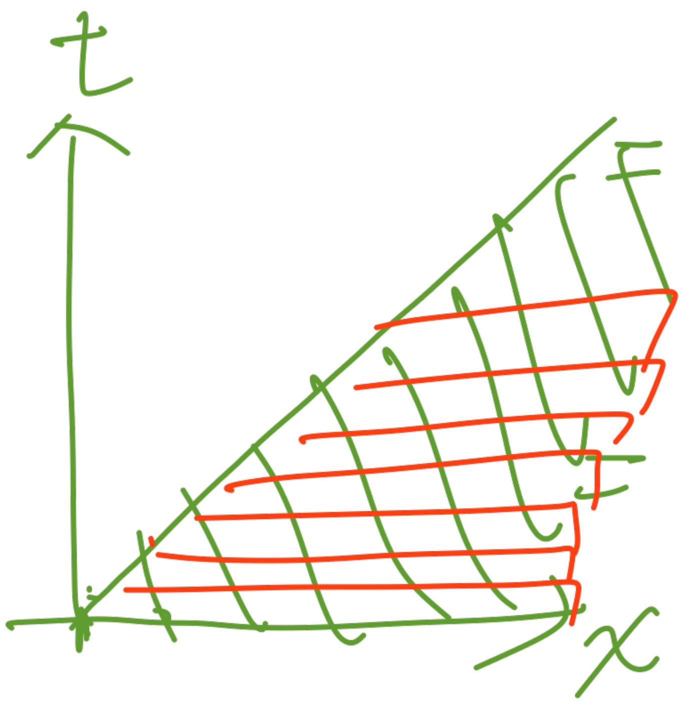
- What if X is a non-negative random variable?
- How to prove this?

Proof: Expected Value of a Continuous Random Variable Using CDF

$$E[X] = \int_0^{\infty} (1 - F_X(t))dt - \int_0^{\infty} F_X(-t)dt$$

1. dt
2. interchange of integration

Pf: $E[X] = \int_0^{\infty} \underline{x} f(x)dx + \int_{-\infty}^{0} x f(x) dx$



$$\int_0^{\infty} \underline{\int_0^{x} dt}\, f(x) dx + \int_{-\infty}^{0} \left(\int_0^{x} dt\right) f(x) dx$$

$$\int_0^{\infty} \int_0^{x} f(x) dt dx + \int_{-\infty}^{0} \int_0^{x} f(x) dt dx$$

$P(X > t)$

$$= \int_0^{\infty} \left(\int_t^{\infty} f(x) dx \right) dt +$$

Existence of Moments (Formally)

Existence of Moments:
Let X be a random variable. Then, the n-th moment of X (i.e. $E[X^n]$) is said to exist if $E[\,|X^n|\,] < \infty$

- Same definition for both discrete and continuous random variables

Example: Pareto Random Variable

- Example: Suppose $n \geq 1$ and the PDF of a r.v. X is

$$f_X(x) = \begin{cases} \frac{C}{x^{n+1}} & , x \geq C \\ 0 & \text{, otherwise} \end{cases}$$

- What is the value of C?
- For what values of m does $E[X^{m+1}]$ exist?

This Lecture

1. (Continuous) Uniform Random Variables

2. Normal Random Variables

- Reading material: Chapter 7.1-7.2

1. Uniform Random Variables

1. (Continuous) Uniform Random Variables

- Example: A bus arrive at a random time between 9:15am and 9:30am

- Example: Play wheel of fortune

- What are the common features?
  - <u>Principle of indifference</u>

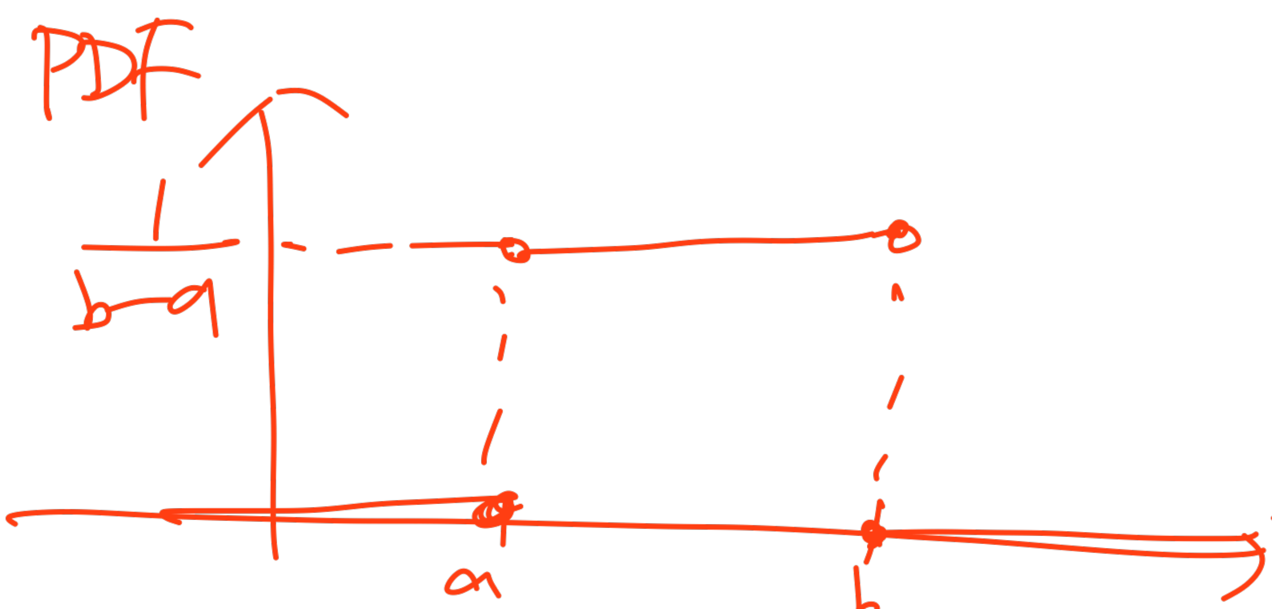
https://www.youtube.com/watch?v=h417SKtF9GY

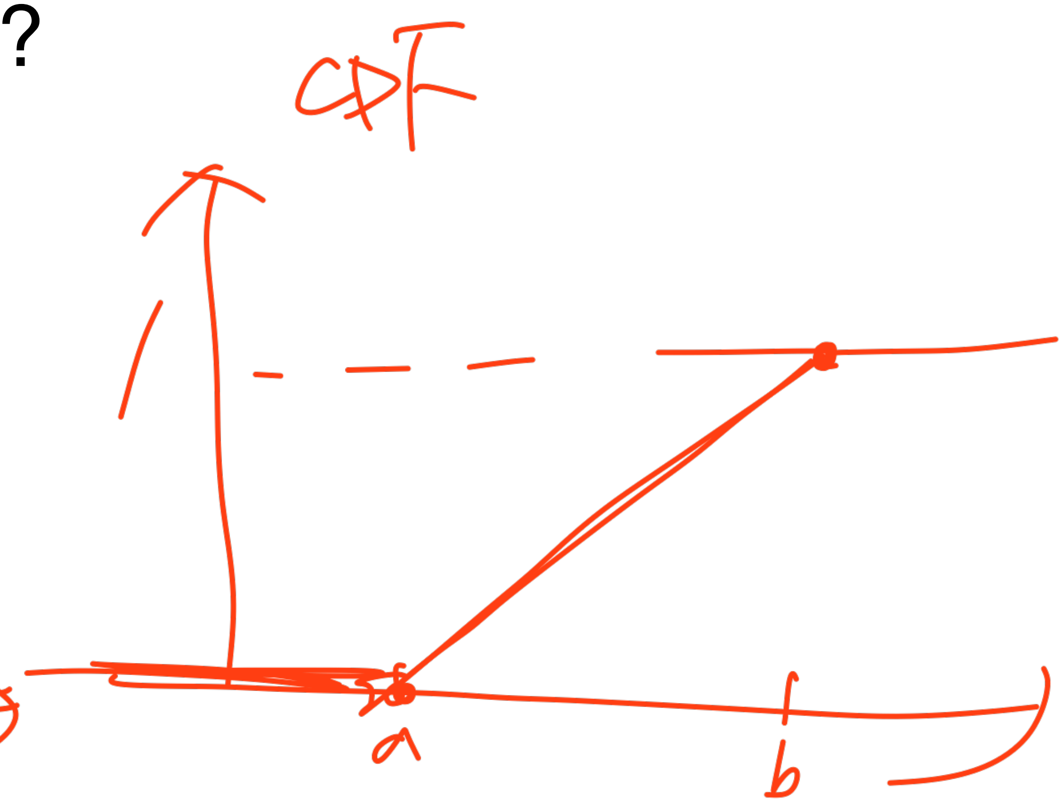
1. Uniform Random Variables (Formally)

Uniform Random Variables: A random variable X is uniform with parameters a, b ($a < b$) if its PDF is

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & \text{, if } a < x < b \\ 0 & \text{, otherwise} \end{cases}$$

- How to plot the PDF and CDF of X?





Example: Uniform Distribution

- Example: Let X be a random variable with CDF $F(t)$.
  - Define another random variable $Y = F(X)$
  - What type of random variable is Y?

$Y \sim Unif(0,1)$

$$P(Y \le t) = \begin{cases} 0, & t \le 0 \\ 1, & t \ge 1 \\ t, & 0 < t < 1 \end{cases}$$

$t = 0.3$

$P(Y \le 0.3)$

$=$

$P(F(X) \le 0.3)$

1. Uniform Random Variables: Mean and Variance

- Example: $X \sim \text{Unif}(a, b)$
  - What is $E[X]$?
  - How about $\text{Var}[X]$?

Why Are Uniform Random Variables Useful?

- Question: How to generate a customized random variable X with $\underbrace{\text{CDF } F(t)}$?

Inverse Transform Sampling (ITS): Generate any random variable with CDF $F(t)$ from a uniform random variable

1. Generate a random variable $U \sim \text{Unif}(0,1)$

2. Let $X = \underbrace{F^{-1}(U)}_{\text{inverse of } F(\cdot)}$, where $F^{-1}(u) := \inf\{z : F(z) \geq u\}$

Proof: Inverse Transform Sampling

Inverse Transform Sampling: Generate <u>any</u> random variable with CDF $F(t)$ from a uniform random variable

1. Generate a random variable $U \sim \text{Unif}(0,1)$
2. Let $X = F^{-1}(U)$, where $F^{-1}(u) := \inf\{z : F(z) \geq u\}$

- $P(\underbrace{F^{-1}(U)}_{= X} \leq x) = P(U \leq F(x)) = F(x)$

apply $F(\cdot)$ on both sides and use the fact: <u>$F(\cdot)$ is non-decreasing</u>

Example: Inverse Transform Sampling

- Example: Generate a random variable X with CDF

$$F_X(t) = \begin{cases} 1 - \exp(-x^2) & , x \geq 0 \\ 0 & \text{, otherwise} \end{cases}$$

Step 1: $U \sim \mathrm{Unif}(0,1)$

Unif (randomness) $\rightarrow$ $F^{-1}(U)$ (reshape CDF) $\rightarrow$ X

Step 2: inverse function $F_X^{-1}(U)$

$$U = 1 - e^{-x^2} \iff e^{-x^2} = 1 - U$$

$$\iff -x^2 = \ln(1-U)$$

$$\iff x = \sqrt{\ln\frac{1}{1-U}}$$

inverse function

2. Standard Normal Random Variables

Gaussian

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$

- Motivation: Consider $X \sim \text{Binomial}(n, \frac{1}{2})$

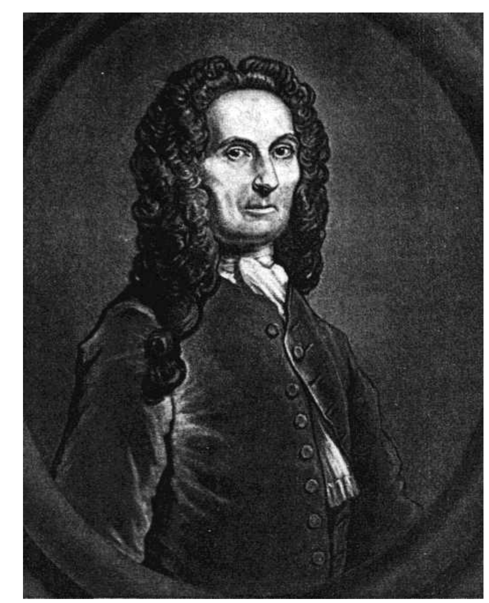
mean

- Define $Y = \dfrac{X - \frac{1}{2}n}{\frac{1}{2}\sqrt{n}}$. What is the CDF of Y vs n?

Euler

1730s

$n \to \infty$



Abraham de Moivre

2. Standard Normal Random Variables (Formally)

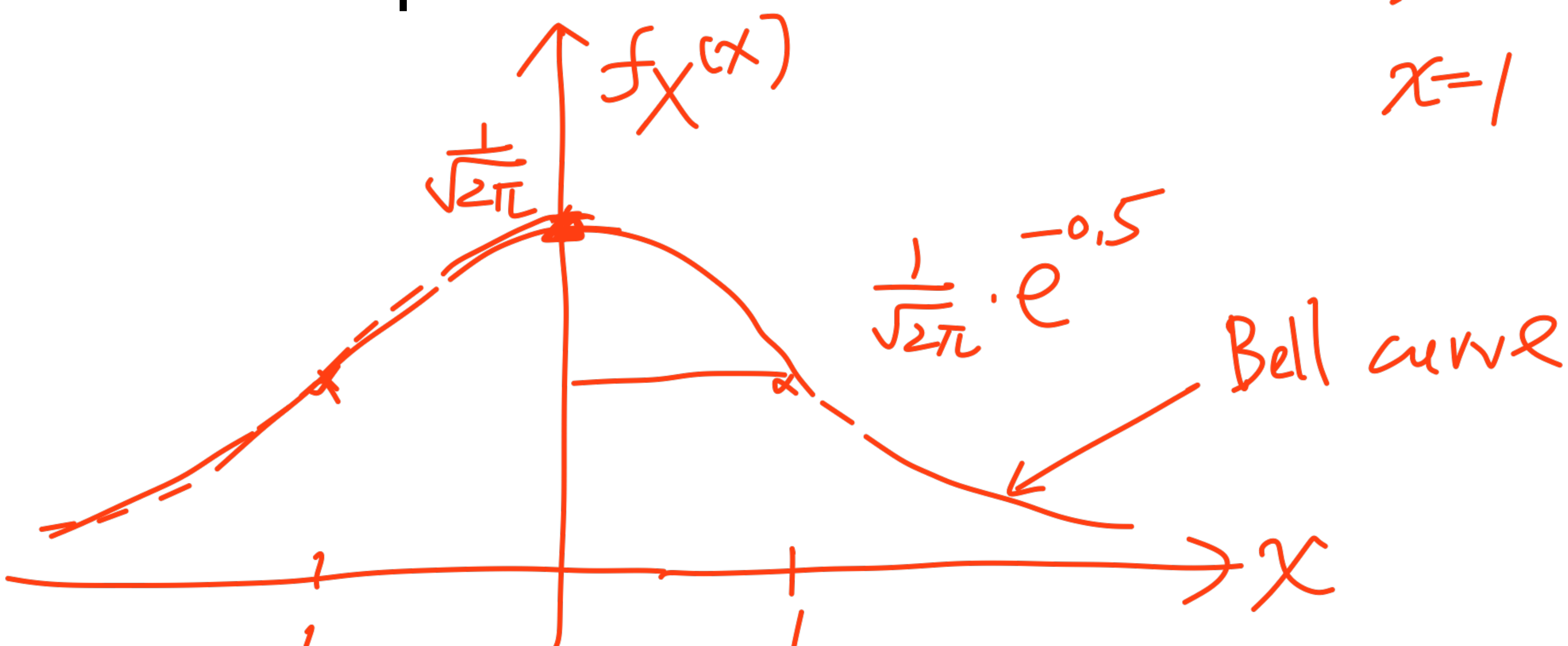
Standard Normal Random Variables: A random variable X is called standard normal if its PDF is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right), \text{ for all } x \in \mathbb{R}$$

- How to plot the PDF?

$x=0$

$x=1$



2. CDF of Standard Normal

phi

- As standard normal is widely applicable, we use a special notation $\Phi(\,\cdot\,)$ for its CDF

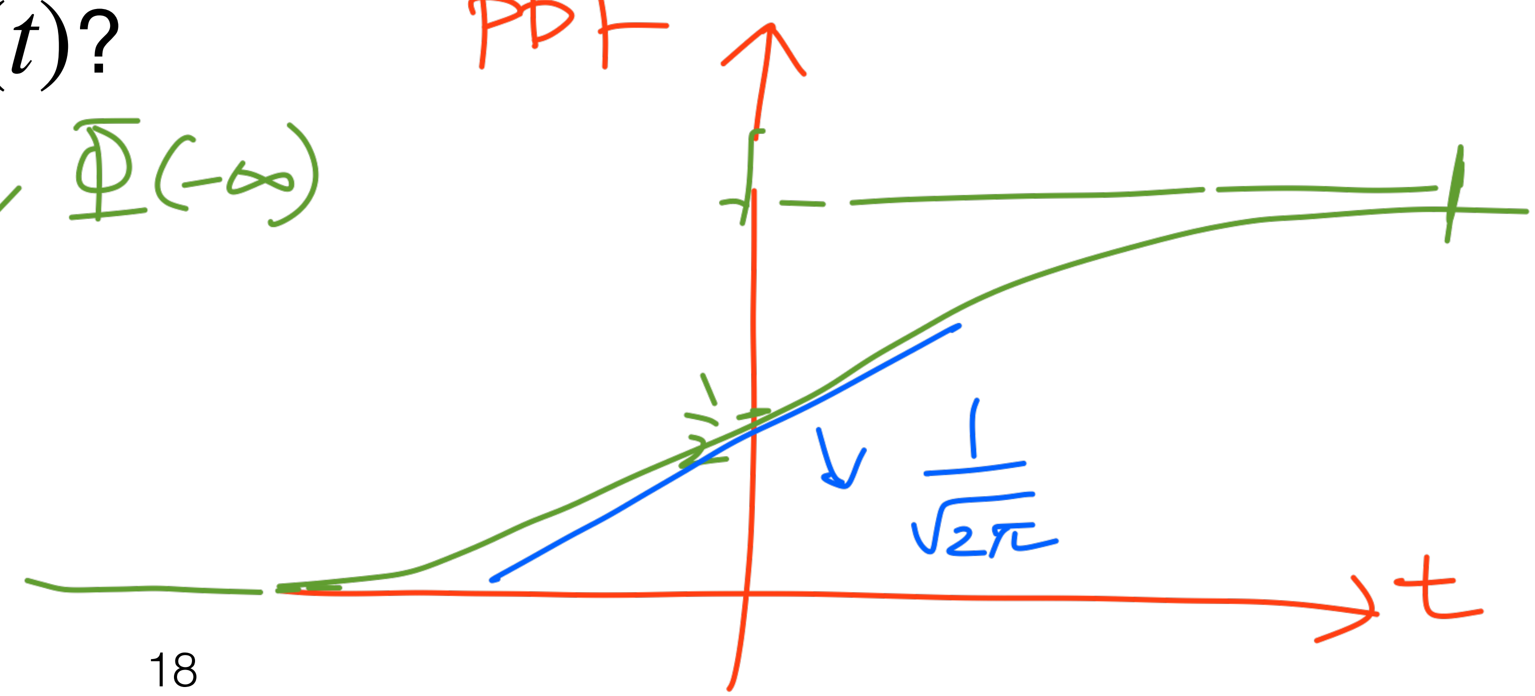
CDF of Standard Normal: The CDF of a standard normal random variable X is

$$\Phi(t) := P(X \leq t) = \int_{-\infty}^{t} \underbrace{\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)}_{=\ \text{PDF}} dx$$

- **Question**: How to plot $\Phi(t)$?
  - $\Phi(\infty) = ?$ $\Phi(0) = ?$, $\Phi(-\infty)$

$= P(X \leq \infty)$ $= P(X \leq 0)$

$\Phi'(0)$



2. Standard Normal: Mean and Variance

- Example: X is a standard normal random variable
  - What is $E[X]$?
  - How about $\text{Var}[X]$?

Normal Random Variables

mean

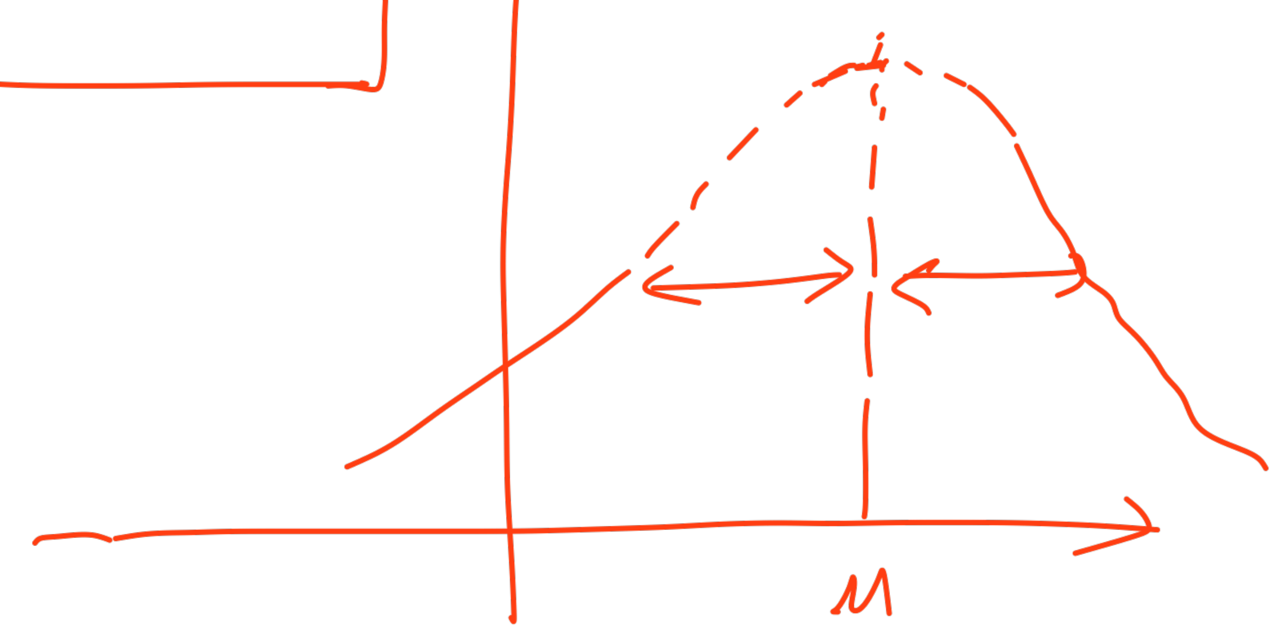
standard deviation

Normal Random Variables: A random variable X is called normal with parameters μ, σ if its PDF is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

- Notation: $X \sim \mathcal{N}(\mu, \cancel{\sigma}^2)$
- How to plot the PDF?

$\boxed{Y = X - \mu}$ $\Rightarrow$ Y is a standard normal variable.



Example: Normal Distribution

- Example: Let $X \sim \mathcal{N}(-2,5)$
  - What is $P(|X| < 4)$?

Why is Normal Distribution Useful?

1. **Central Limit Theorem**:

2. **Gaussian Process and Black-Box Optimization**:

Next Lecture

1. Special continuous random variables
   - Exponential
   - Gamma
   - Beta

1-Minute Summary

1. Special Continuous Random Variables

- Uniform and inverse transform sampling
- Standard normal and normal