DCP 1206: Probability
Lecture 7 — Discrete Random Variables (II)

Ping-Chun Hsieh

October 4, 2019

About the Makeup Class

- Plan A: 10-minute extension for classes on
  - 10/9, 10/16, 10/23, 10/30 (Wednesdays)
  - 10:10am-**12:10**pm
- Plan B: A makeup class tomorrow 3:30pm-4:20pm

What to Think of About Canada?

Justin Bieber

Avril Lavigne

Maple syrup

- Anything else?

Canada and Machine Learning

Jeff Hinton
(University of Toronto)

Yoshua Bengio
(University of Montreal)

Richard Sutton
(University of Alberta)

- Why so many top researchers in Canada?
- 80-20 rule in the academia?

This Lecture

1. Special Discrete Random Variables

- Reading material: Chapter 5.3

1. Special Discrete Random Variables

4. Geometric Random Variables

- Example: Play with a claw machine, and each trial is successful with probability 0.7. What is $P(\text{get 1st toy at 10-th trial})$?

- Example: Po-Jung Wang makes a hit with probability 0.28 at each at-bat. What is $P(\text{he makes his 1st hit at 5-th at-bat})$?

- What are the common features?
  - Repetitions of the same Bernoulli experiment
  - Want: how many trials needed until the 1st success?

4. PMF of Geometric Random Variables

- Example: Play with a claw machine, and each trial is successful with probability 0.7. All trials are independent.
  - X = the number of trials until we get the first toy $\in \{1, 2, 3, \cdots\}$
  - What is the PMF of X?

$X=1$: $P(X=1) = 0.7$

$X=2$: F S $P = 0.3 \times (0.7)$

$X=3$: F F S $P = (0.3)^2 \times 0.7$

$X=k$: $P = (0.3)^{k-1} \cdot (0.7)$

4. Geometric Random Variables (Formally)

Geometric Random Variables: A random variable X is Geometric with parameters p if its PMF is given by

$\in [0,1]$

$$\underbrace{P(X = k) = (1-p)^{k-1}p,}\ \underbrace{k = 1,2,3,\cdots}$$

$$\sum_{k=1}^{\infty} \underbrace{(1-p)^{k-1} \cdot p} = 1$$

$$= \frac{p}{1-(1-p)}$$

- Do we have $\sum_{k=1}^{\infty} P(X = k) = 1$?

CDF of Geometric Random Variables

PMF: $P(X = k) = (1 - p)^{k-1}p, \ k = 1,2,3,\cdots$

- CDF: $F_X(t) = P(X \le t)$ 3.5 $\Rightarrow$ 1, 2, 3

$t \in \mathbb{N}$: $\sum_{k=1}^{t} (1-p)^{k-1} p = \frac{p(1-(1-p)^t)}{1-(1-p)} = 1-(1-p)^t$

t is not an integer, $t>0$:

$\cdots = 1-(1-p)^{\lfloor t \rfloor}$ ← floor function

- How about $P(X > t)$? $= 1 - P(X \le t) = (1-p)^t$

Geometric r.v.: Memoryless Property

first m trials

- Example: Suppose $X \sim \text{Geometric}(p)$, $p \in (0,1)$
  - What is $P(X = n + m \mid X > m)$? $(n, m \in \mathbb{N})$
  - What is $P(X > n + m \mid X > m)$? $(n, m \in \mathbb{N})$ ← Your job!

$$= \frac{P(X=n+m)}{P(X>m)} = \frac{(1-p)^{\cancel{n+m}-1} \cdot p}{(1-p)^{\cancel{m}}}$$

$$= (1-p)^{n-1} \cdot p = P(X=n)$$

5. Negative Binomial Random Variables

Dark Knight

- Example: Play with a claw machine, and each trial is successful with probability 0.7. What is P(get 3rd toy at 10-th trial)?

- Example: Po-Jung Wang makes a hit with probability 0.28 at each at-bat. What is P(he makes his 3rd hit at 5-th at-bat)?

- What are the common features?
  - Repetitions of the same Bernoulli experiment
  - Want: how many trials needed until the r-th success?

$r = 1, 2, 3, \cdots$

5. PMF of Negative Binomial Random Variables

- Example: Play with a claw machine, and each trial is successful with probability 0.7. All trials are independent.
  - r.v. X = the number of trials until we get the 3rd toy $\in \{3, 4, 5, \cdots\}$
  - What is the PMF of X?

$X=3$: S S S $\Rightarrow P(X=3) = (0.7)^3$

$X=4$: $\begin{cases} F\ S\ S\ S \\ S\ F\ S\ S \\ S\ S\ F\ S \end{cases} \Rightarrow P(X=4) = 3 \times (0.7)^3 \times (0.3)$, $3 = C^{4-1}_{3-1}$

$X=k$: $(k \geq 4)$ $\underbrace{\text{2 successes, } k-3 \text{ failures}}_{k-1}$ S $\Rightarrow P(X=k) = C^{k-1}_{3-1} (0.7)^3 \times (0.3)^{k-3}$

5. Negative Binomial Random Variables (Formally)

$X \sim NB(p, r)$

success prob. # of successes

Negative Binomial Random Variables: A random variable X is Negative Binomial with parameters (p, r) if its PMF is given by

$r=3$

$$P(X = n) = C_{r-1}^{n-1} p^r (1-p)^{n-r}, \; n = r, r+1, \cdots$$

- What if $r = 1$? $NB(p, r=1) \equiv \text{Geometric}(p)$.

- Why is it called 'Negative Binomial'?

Why Is It Called Negative Binomial? or Pascal

PMF = $P(X = n) = C_{r-1}^{n-1} p^r (1-p)^{n-r}, \; n = r, r+1, \cdots$

- Suppose: let $k = n - r$

$$C_{r-1}^{n-1} p^r (1-p)^{n-r} = C_k^{k+(r-1)} p^r (1-p)^k$$

$$= (-1)^k C_k^{-r} p^r (1-p)^k$$

Binomial expansion

K terms

$$\frac{\overbrace{(-r)\cdot(-r-1)\cdot(-r-2)\cdots(-r-(k-1))}^{}}{k!}$$

Example: Modified Gambler's Ruin

σ-Total probability theorem

- Example: Suppose two gamblers A and B play a game.
  - In each play, A wins with probability $p \in (0,1)$
  - In each play, the loser loses \$1, but the winner wins \$0
  - Initially, A has a dollars and B has b dollars
  - What is probability that B loses all the money first? ; E

4   3

$r=b$

$X \sim NB(p, \cancel{r=3})$

2 losses B

0, 1, 2, 3 losses from A   B

B B B

A B B B

B A A B B

X = # of trials until the b-th ~~3rd~~ loss from B

PMF

$b \Rightarrow a+b-1$

$$P(E) = \sum_{k=3}^{} P(X=k)$$

$k=3 = b$

6. Discrete Uniform Random Variables

- Example: Roll a 4-sided die, and the numbers 1, 2, 3, 4 are equally likely to occur

- Example: The correct answer to an exam question: A, B, C, D are equally likely

- What are the common features?
  - 1 experiment trial (no repetition) with n equally-likely outcomes
  - Want: Whether a specific outcome occurs

6. Discrete Uniform Random Variables (Formally)

Discrete Uniform Random Variables: A random variable X is discrete uniform with parameters (a, b) $(a, b \in \mathbb{Z}$ with $a \leq b)$, if its PMF is given by

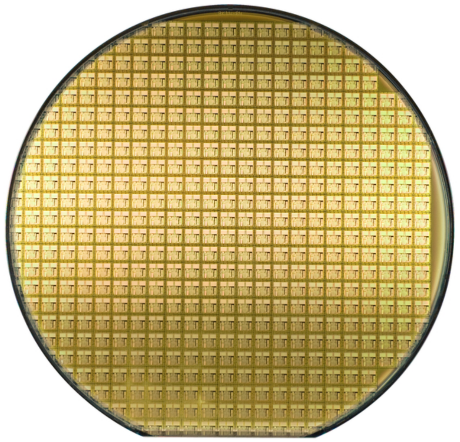
$$P(X = k) = \frac{1}{b - a + 1}, \quad k = \underbrace{a, a + 1, \cdots, b}_{(b-a+1) \text{ possible values}}$$

-2 3

7. Hypergeometric Random Variables

- Example: There are 1000 students voting for 交清小徑 and 200 students voting for 清交小徑.
  - Randomly sample 5 students from 1200 students.
  - What is P(2 out of 5 vote for 交清小徑)?

- Example: On a silicon wafer, there are 800 good ICs and 200 defective ICs. What is P(3 out of 10 samples are defective)?



- What are the common features?
  - 2 types of objects (A and B)
  - Want: how many type A objects in n samples?

PMF of Hypergeometric Random Variables

- Example: There are 7 students voting for 交清小徑 and 3 students voting for 清交小徑.
  - Randomly sample 3 students from 10 students.
  - X = the number of students voting for 交清小徑
  - What is the PMF of X?

(from the 3 samples)

$X=0$:

$$P(X=0) = \frac{1}{C^{10}_3}, \quad 1 = C^7_0 \cdot C^3_3$$

$X=1$:

$$P(X=1) = \frac{(C^7_1)(C^3_2)}{C^{10}_3}$$

$X=2$:

$X=3$:

Your turn!!

7. Hypergeometric Random Variables (Formally)

all the students (1200)

交清小径

10

Hypergeometric Random Variables: A random variable X is hypergeometric with parameters (N, D, n) ($N, D, n \in \mathbb{N}$ with $D < N$ and $n \le \min(D, N-D)$), if its PMF is given by

we want.

$$P(X = k) = \frac{C_k^D C_{n-k}^{N-D}}{C_n^N}, \; k = 0,1,2,\cdots,n$$

total comb.

- Do we have $\sum_{k=0}^{n} P(X = k) = 1$?

Why Is It Called "Hypergeometric"?

$$P(X=k)=\frac{C_k^D C_{n-k}^{N-D}}{C_n^N},\ k=0,1,2,\cdots,n$$

- Can we find $\dfrac{P(X=k+1)}{P(X=k)}$?

Next Lecture

1. Expectation and variance

2. Higher moments

1-Minute Summary

1. Special Discrete Random Variables

1st success — r-th success

- Geometric / Negative Binomial
- Discrete Uniform
- Hypergeometric