# TRANSFORMADA Z UNILATERAL

# DEFINIÇÃO E PROPRIEDADES

- É definida por

$$X^+(z) \equiv \sum_{n=0}^{\infty} x(n) z^{-n} \qquad Z^+\{x(n)\} \qquad x(n) \overset{z^+}{\leftrightarrow} X^+(z)$$

- Difere da bilateral no limite inferior da soma, que é sempre zero, se ou não o sinal x(n) é zero para n < 0.
- Possuí as seguintes características:
  - Não contém informação sobre o sinal x(n) para valores negativos de tempo ( n < 0 ).
  - É única somente para sinais causais, porque somente estes sinais são zero para n<0.
  - $X^+$ (z) de x(n) é identico a transformada Z bilateral do sinal x(n)u(n). Desde de que x(n)u(n) é causal, a ROC de sua transformada, e então a ROC de $X^+$ (z) , é sempre o exterior do circulo. Assim, quando nós lidamos com transformada Z unilateral não é necessário referir-se a sua ROC.

## EXAMPLE 3.6.1

Determine the one-sided $z$-transform of the signals in Example 3.1.1.

**Solution.** From the definition (3.6.1), we obtain

$$x_1(n) = \{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\} \overset{z^+}{\longleftrightarrow} X_1^+(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

$$x_2(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\} \overset{z^+}{\longleftrightarrow} X_2^+(z) = 5 + 7z^{-1} + z^{-3}$$

$$x_3(n) = \{\underset{\uparrow}{0}, 0, 1, 2, 5, 7, 0, 1\} \overset{z^+}{\longleftrightarrow} X_3^+(z) = z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}$$

$$x_4(n) = \{2, 4, \underset{\uparrow}{5}, 7, 0, 1\} \overset{z^+}{\longleftrightarrow} X_4^+(z) = 5 + 7z^{-1} + z^{-3}$$

$$x_5(n) = \delta(n) \overset{z^+}{\longleftrightarrow} X_5^+(z) = 1$$

$$x_6(n) = \delta(n-k), \qquad k > 0 \overset{z^+}{\longleftrightarrow} X_6^+(z) = z^{-k}$$

$$x_7(n) = \delta(n+k), \qquad k > 0 \overset{z^+}{\longleftrightarrow} X_7^+(z) = 0$$

Note that for a noncausal signal, the one-sided $z$-transform is not unique. Indeed, $X_2^+(z) = X_4^+(z)$ but $x_2(n) \neq x_4(n)$. Also for anticausal signals, $X^+(z)$ is always zero.

# DEFINIÇÃO E PROPRIEDADES

- Quase todas a propriedades que foram estudadas para transformada Z bilateral servem para a transformada Z unilateral com exceção da propriedade do deslocamento.
- Propriedade do Deslocamento
  - Caso 1: Tempo de atraso

$$x(n) \overset{z^+}{\leftrightarrow} X^+(z)$$

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$Z^+\{x(n-k)\} = \left[\sum_{n=0}^{\infty} x(n-k) z^{-n}\right] \quad l = n\text{-}k$$

$$Z^+\{x(n-k)\} = \left[\sum_{l=-k}^{\infty} x(l) z^{-l-k}\right]$$

$$Z^+\{x(n-k)\} = z^{-k}\left[\sum_{l=-k}^{-1} x(l) z^{-l} + \sum_{l=0}^{\infty} x(l) z^{-l}\right]$$

$$Z^+\{x(n-k)\} = z^{-k}\left[\sum_{n=1}^{k} x(-n) z^{n} + X^+(z)\right]$$

$$x(n-k) \overset{z^+}{\leftrightarrow} z^{-k} X^+(z) \rightarrow x(n) \text{ causal}$$

## EXAMPLE 3.6.2

Determine the one-sided $z$-transform of the signals

**(a)** $x(n) = a^n u(n)$

**(b)** $x_1(n) = x(n-2)$ where $x(n) = a^n$

**Solution.**

**(a)** From (3.6.1) we easily obtain

$$X^+(z) = \frac{1}{1 - az^{-1}}$$

**(b)** We will apply the shifting property for $k = 2$. Indeed, we have

$$\begin{aligned} Z^+\{x(n-2)\} &= z^{-2}[X^+(z) + x(-1)z + x(-2)z^2] \\ &= z^{-2}X^+(z) + x(-1)z^{-1} + x(-2) \end{aligned}$$

Since $x(-1) = a^{-1}$, $x(-2) = a^{-2}$, we obtain

$$X_1^+(z) = \frac{z^{-2}}{1 - az^{-1}} + a^{-1}z^{-1} + a^{-2}$$

# DEFINIÇÃO E PROPRIEDADES

- Propriedade do Deslocamento
  - Caso 2: Tempo de avanço

$$x(n) \overset{z^+}{\leftrightarrow} X^+(z)$$

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$Z^+\{x(n+k)\} = \left[ \sum_{n=0}^{\infty} x(n+k) z^{-n} \right] \quad l = n + k$$

$$Z^+\{x(n+k)\} = \left[ \sum_{l=k}^{\infty} x(l) z^{-l+k} \right] = z^k \sum_{l=k}^{\infty} x(l) z^{-l}$$

$$X^+(z) = \sum_{n=0}^{\infty} x(l) z^{-l} = \sum_{l=0}^{k-1} x(l) z^{-l} + \sum_{l=k}^{\infty} x(l) z^{-l}$$

$$\sum_{l=k}^{\infty} x(l) z^{-l} = X^+(z) - \sum_{l=0}^{k-1} x(l) z^{-l}$$

$$Z^+\{x(n+k)\} = z^k \left[ X^+(z) - \sum_{l=0}^{k-1} x(l) z^{-l} \right]$$

## EXAMPLE 3.6.3

With $x(n)$, as given in Example 3.6.2, determine the one-sided $z$-transform of the signal

$$x_2(n) = x(n+2)$$

**Solution.** We will apply the shifting theorem for $k = 2$. From (3.6.5), with $k = 2$,

$$Z^+\{x(n+2)\} = z^2 X^+(z) - x(0)z^2 - x(1)z$$

But $x(0) = 1$, $x(1) = a$, and $X^+(z) = 1/(1 - az^{-1})$. Thus

$$Z^+\{x(n+2)\} = \frac{z^2}{1 - az^{-1}} - z^2 - az$$

# DEFINIÇÃO E PROPRIEDADES

- Teorema do valor final

$$x(n) \overset{z^+}{\longleftrightarrow} X^+(z)$$

$$\lim_{n\to\infty} x(n) = \lim_{z\to 1}(z-1)X^+(z)$$

- O limite existe se a ROC de $(z-1)X^+(z)$ inclue o circulo unitário.

## EXAMPLE 3.6.4

The impulse response of a relaxed linear time-invariant system is $h(n) = \alpha^n u(n)$, $|\alpha| < 1$. Determine the value of the step response of the system as $n \to \infty$.

**Solution.** The step response of the system is

$$y(n) = h(n) * x(n)$$

where

$$x(n) = u(n)$$

Obviously, if we excite a causal system with a causal input the output will be causal. Since $h(n)$, $x(n)$, $y(n)$ are causal signals, the one-sided and two-sided $z$-transforms are identical. From the convolution property (3.2.17) we know that the $z$-transforms of $h(n)$ and $x(n)$ must be multiplied to yield the $z$-transform of the output. Thus

$$Y(z) = \frac{1}{1-\alpha z^{-1}} \frac{1}{1-z^{-1}} = \frac{z^2}{(z-1)(z-\alpha)}, \qquad \text{ROC: } |z| > |\alpha|$$

Now

$$(z-1)Y(z) = \frac{z^2}{z-\alpha}, \qquad \text{ROC: } |z| < |\alpha|$$

Since $|\alpha| < 1$, the ROC of $(z-1)Y(z)$ includes the unit circle. Consequently, we can apply (3.6.6) and obtain

$$\lim_{n\to\infty} y(n) = \lim_{z\to 1} \frac{z^2}{z-\alpha} = \frac{1}{1-\alpha}$$

# SOLUÇÃO DE EQUAÇÕES A DIFERENÇA

- A transformada z unilateral é uma ferramenta muito eficiente para a solução de equações a diferença com condições iniciais não-zero.
- Ela efetua isto reduzindo a equação a diferença relativa aos dois sinais no domínio do tempo para uma equação algébrica equivalente relativa a transformada z unilateral deles.

**EXAMPLE 3.6.5**

The well-known Fibonacci sequence of integer numbers is obtained by computing each term as the sum of the two previous ones. The first few terms of the sequence are

$$1, 1, 2, 3, 5, 8, \ldots$$

Determine a closed-form expression for the $n$th term of the Fibonacci sequence.

**Solution.** Let $y(n)$ be the $n$th term of the Fibonacci sequence. Clearly, $y(n)$ satisfies the difference equation

$$y(n) = y(n-1) + y(n-2) \tag{3.6.7}$$

with initial conditions

$$y(0) = y(-1) + y(-2) = 1 \tag{3.6.8a}$$

$$y(1) = y(0) + y(-1) = 1 \tag{3.6.8b}$$

From (3.6.8b) we have $y(-1) = 0$. Then (3.6.8a) gives $y(-2) = 1$. Thus we have to determine $y(n)$, $n \geq 0$, which satisfies (3.6.7), with initial conditions $y(-1) = 0$ and $y(-2) = 1$.

By taking the one-sided $z$-transform of (3.6.7) and using the shifting property (3.6.2), we obtain

$$Y^+(z) = [z^{-1}Y^+(z) + y(-1)] + [z^{-2}Y^+(z) + y(-2) + y(-1)z^{-1}]$$

or

$$Y^+(z) = \frac{1}{1 - z^{-1} - z^2} = \frac{z^2}{z^2 - z - 1} \tag{3.6.9}$$

where we have used the fact that $y(-1) = 0$ and $y(-2) = 1$.

We can invert $Y^+(z)$ by the partial-fraction expansion method. The poles of $Y^+(z)$ are

$$p_1 = \frac{1+\sqrt{5}}{2}, \qquad p_2 = \frac{1-\sqrt{5}}{2}$$

and the corresponding coefficients are $A_1 = p_1/\sqrt{5}$ and $A_2 = -p_2/\sqrt{5}$. Therefore,

$$y(n) = \left[\frac{1+\sqrt{5}}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1-\sqrt{5}}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n\right]u(n)$$

or, equivalently,

$$y(n) = \frac{1}{\sqrt{5}}\left(\frac{1}{2}\right)^{n+1}\left[\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}\right]u(n) \tag{3.6.10}$$

# RESPOSTA DE SISTEMAS POLO-ZERO COM CONDIÇÕES INICIAIS NÃO ZERO

**EXAMPLE 3.6.6**

Determine the step response of the system

$$y(n) = \alpha y(n-1) + x(n), \qquad -1 < \alpha < 1 \tag{3.6.11}$$

when the initial condition is $y(-1) = 1$.

**Solution.** By taking the one-sided $z$-transform of both sides of (3.6.11), we obtain

$$Y^+(z) = \alpha[z^{-1}Y^+(z) + y(-1)] + X^+(z)$$

Upon substitution for $y(-1)$ and $X^+(z)$ and solving for $Y^+(z)$, we obtain the result

$$Y^+(z) = \frac{\alpha}{1-\alpha z^{-1}} + \frac{1}{(1-\alpha z^{-1})(1-z^{-1})} \tag{3.6.12}$$

By performing a partial-fraction expansion and inverse transforming the result, we have

$$\begin{aligned} y(n) &= \alpha^{n+1}u(n) + \frac{1-\alpha^{n+1}}{1-\alpha}u(n) \\ &= \frac{1}{1-\alpha}(1-\alpha^{n+2})u(n) \end{aligned} \tag{3.6.13}$$

# RESPOSTA DE SISTEMAS POLO-ZERO COM CONDIÇÕES INICIAIS NÃO ZERO

- Suponha que o sinal $x(n)$ é aplicado para um sistema polo-zero em $n=0$. Assim, o sinal $x(n)$ é assumido ser causal.
- Os efeitos de todos sinais de entrada anteriores para o sistema são refletidos nas condições iniciais $y(-1), y(-2), \ldots, y(-N)$.
- Desde que o sinal de entrada $x(n)$ é causal e desde que estamos interessados em determinar a saída $y(n)$ para $n \geq 0$, podemos usar a transformada z unilateral, que permite-nos lidar com as condições iniciais.

# RESPOSTA DE SISTEMAS POLO-ZERO COM CONDIÇÕES INICIAIS NÃO ZERO

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$Y^+(z) = -\sum_{k=1}^{N} a_k z^{-k}\left[Y^+(z) + \sum_{k=1}^{N} y(-n) z^{n}\right] + \sum_{k=0}^{M} b_k z^{-k} X^+(z)$$

Lembrando que $x(n)$ é causal, portanto $x(-n) = 0$ e $X^+(z) = X(z)$

Isolando Y(z) tem - se :

$$Y^+(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} X(z) - \frac{\sum_{k=1}^{N} a_k z^{-k} \sum_{n=1}^{K} y(-n) z^{n}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$Y^+(z) = H(z) X(z) + \frac{N_0(z)}{A(z)} \qquad N_0(z) = -\sum_{k=1}^{N} a_k z^{-k} \sum_{n=1}^{K} y(-n) z^{n}$$

$$A(z) = 1 + \sum_{k=1}^{N} a_k z^{-k}$$

# RESPOSTA DE SISTEMAS POLO-ZERO COM CONDIÇÕES INICIAIS NÃO ZERO

- A equação $Y^{+}(z) = H(z)X(z) + \dfrac{N_0(z)}{A(z)}$ pode ser dividida em duas partes:

Resposta de estado zero:

$$Y_{zs}(z) = H(z)X(z)$$

Resposta de entrada zero, resultante das condições iniciais não - zero:

$$Y_{zi}^{+}(z) = \frac{N_0(z)}{A(z)}$$

$$Y^{+}(z) = Y_{zs}(z) + Y_{zi}^{+}(z) \qquad y(n) = y_{zs}(n) + y_{zi}(n)$$

O denominador de $Y_{zi}^{+}(z)$ é A(z) e seus polos são $p_1, p_2, \ldots, p_N$.

Consequentemente, $y_{zi}(n) = \sum_{k=1}^{N} D_k (p_k)^n u(n)$

$$y(n) = \sum_{k=1}^{N} A_k^{/} (p_k)^n u(n) + \sum_{k=1}^{L} Q_k (q_k)^n u(n) \qquad A_k^{/} = A_k + D_k$$

# RESPOSTA DE SISTEMAS POLO-ZERO COM CONDIÇÕES INICIAIS NÃO ZERO

- O efeito das condições iniciais é para alterar a resposta natural do sistema através da modificação do fator de escala $\{A_k\}$.
- Há nenhum novo polo introduzido pelas condições iniciais não-zero.
- Há nenhum efeito na resposta forçada do sistema.
- Estes pontos são reforçados no exemplo seguinte.

**EXAMPLE 3.6.7**

Determine the unit step response of the system described by the difference equation

$$y(n) = 0.9y(n-1) - 0.81y(n-2) + x(n)$$

under the following initial conditions $y(-1) = y(-2) = 1$.

**Solution.** The system function is

$$H(z) = \frac{1}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

This system has two complex-conjugate poles at

$$p_1 = 0.9e^{j\pi/3}, \qquad p_2 = 0.9e^{-j\pi/3}$$

The $z$-transform of the unit step sequence is

$$X(z) = \frac{1}{1 - z^{-1}}$$

Therefore,

$$\begin{aligned} Y_{zs}(z) &= \frac{1}{(1 - 0.9e^{j\pi/3}z^{-1})(1 - 0.9e^{-j\pi/3}z^{-1})(1 - z^{-1})} \\ &= \frac{0.0496 - j0.542}{1 - 0.9e^{j\pi/3}z^{-1}} + \frac{0.0496 + j0.542}{1 - 0.9e^{-j\pi/3}z^{-1}} + \frac{1.099}{1 - z^{-1}} \end{aligned}$$

and hence the zero-state response is

$$y_{zs}(n) = \left[1.099 + 1.088(0.9)^n \cos\left(\frac{\pi}{3}n - 5.2^\circ\right)\right] u(n)$$

For the initial conditions $y(-1) = y(-2) = 1$, the additional component in the $z$-transform is

$$Y_{\text{zi}}(z) = \frac{N_0(z)}{A(z)} = \frac{0.09 - 0.81z^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

$$= \frac{0.045 + j0.4936}{1 - 0.9e^{j\pi/3}z^{-1}} + \frac{0.045 - j0.4936}{1 - 0.9e^{-j\pi/3}z^{-1}}$$

Consequently, the zero-input response is

$$y_{\text{zi}}(n) = 0.988(0.9)^n \cos\left(\frac{\pi}{3}n + 87^\circ\right) u(n)$$

In this case the total response has the $z$-transform

$$Y(z) = Y_{\text{zs}}(z) + Y_{\text{zi}}(z)$$

$$= \frac{1.099}{1 - z^{-1}} + \frac{0.568 + j0.445}{1 - 0.9e^{j\pi/3}z^{-1}} + \frac{0.568 - j0.445}{1 - 0.9e^{-j\pi/3}z^{-1}}$$

The inverse transform yields the total response in the form

$$y(n) = 1.099u(n) + 1.44(0.9)^n \cos\left(\frac{\pi}{3}n + 38^\circ\right) u(n)$$

Exemplo 3.6.7 $y(n) = 0{,}9\,y(n-1) - 0{,}81\,y(n-2) + x(n)$

$Y(z) = 0{,}9\,z^{-1}.Y(z) - 0{,}81.z^{-2}.Y(z) + X(z)$ $y(-1) = y(-2) = 1$

$x(n) = u(n)$

$Y(z)\left[1 - 0{,}9\,z^{-1} + 0{,}82\,z^{-2}\right] = X(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 0{,}9.z^{-1} + 0{,}81.z^{-2}}$$

$z^2 - 0{,}9\,z + 0{,}81 = 0$ $p_1 = 0{,}9\angle 60^\circ$ $p_2 = 0{,}9\angle -60^\circ$

$p_1 = 0{,}9\,e^{j\frac{\pi}{3}}$ $p_2 = 0{,}9\,e^{-j\frac{\pi}{3}}$

$$X(z) = \frac{1}{1 - z^{-1}} \Longleftrightarrow x(n) = u(n)$$

$$Y(z) = H(z).X(z) = \frac{z^2}{z^2 - 0{,}9z + 0{,}82} \cdot \frac{z}{z-1}$$

$$Y_{zs}(z) = \frac{z^3}{(z - 0{,}9\,e^{j\frac{\pi}{3}})(z - 0{,}9\,e^{-j\frac{\pi}{3}})(z-1)} = \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} + \frac{A_3}{1 - p_3 z^{-1}}$$

$$\frac{Y_{zs}(z)}{z} = \frac{z^{\cancel{3}\,2}}{(z-p_1)(z-p_2)(z-p_3)} = \frac{z\,A_1}{z-p_1} + \frac{z\,A_2}{z-p_2} + \frac{z\,A_3}{z-p_3}$$

$$A_1 = \left.\frac{z^2\,\cancel{(z-p_1)}}{\cancel{(z-p_1)}(z-p_2)(z-p_3)}\right|_{z=p_1} = \frac{(0{,}9.\angle 60^\circ)^2}{(0{,}9\angle 60^\circ - 0{,}9\angle -60^\circ)(0{,}9\angle 60^\circ - 1)}$$

$A_1 = 0{,}5447\angle -95{,}2^\circ = -0{,}0496 - j\,0{,}542$

$$A_2 = \left.\frac{z^2\,\cancel{(z-p_2)}}{(z-p_1)\cancel{(z-p_2)}(z-p_3)}\right|_{z=p_2} = \frac{(0{,}9\angle{-60^\circ})^2}{(0{,}9\angle{-60^\circ}-0{,}9\angle{60^\circ})(0{,}9\angle{-60^\circ}-1)}$$

$$A_2 = 0{,}5447\angle{95{,}2^\circ} = -0{,}0496 + j\,0{,}542$$

$$A_3 = \left.\frac{z^2\,\cancel{(z-p_3)}}{(z-p_1)(z-p_2)\cancel{(z-p_3)}}\right|_{z=p_3} = \frac{1^2}{(1-0{,}9\angle{60^\circ})(1-0{,}9\angle{-60^\circ})}$$

$$A_3 = 1{,}099 \qquad a^n u(n) \longleftrightarrow \frac{1}{1-az^{-1}}$$

$$Y_{zs}(z) = \frac{0{,}5447\angle{-95{,}2^\circ}}{1-0{,}9\,e^{j\frac{\pi}{3}}\cdot z^{-1}} + \frac{0{,}5447\angle{95{,}2^\circ}}{1-0{,}9\,e^{-j\frac{\pi}{3}}z^{-1}} + \frac{1{,}099}{1-z^{-1}}$$

$$y_{zs}(n) = \left[0{,}5447\angle{-95{,}2^\circ}\,(0{,}9\,e^{j\frac{\pi}{3}})^n + 0{,}5447\angle{95{,}2^\circ}\,(0{,}9\,e^{-j\frac{\pi}{3}})^n\right] u(n) + \frac{z}{z}$$

$$+\,1{,}099\,u(n) \qquad \cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$$

$$y_{zs}(n) = \frac{1{,}08\cdot 0{,}9^n\, e^{j(\frac{\pi}{3}n-95{,}2^\circ)} + 1{,}08\cdot 0{,}9^n\cdot e^{-j(\frac{\pi}{3}n-95{,}2^\circ)}}{2} +$$

$$+\,1{,}099\,u(n)$$

$$\boxed{y_{zs}(n) = \left[1{,}099 + 1{,}08\cdot 0{,}9^n\cos\left(\frac{\pi}{3}n-95{,}2^\circ\right)\right]}$$

$$Y_{zi}(z) = \frac{N_0(z)}{A(z)} \qquad N_0(z) = -\sum_{k=1}^{N} a_k z^{-k}\sum_{n=1}^{k} y(-n)\cdot z^{n}$$

$$N_0(z) = -\sum_{k=1}^{2} a_k\cdot z^{-k}\sum_{n=1}^{k} y(-n)\cdot z^{n} \qquad y(-1)=y(-2)=1$$

$$N_0(z) = -\left(a_1 z^{-1}\cdot y(-1)\,z^{1} + a_2\cdot z^{-2}y(-1)z^{1} + a_2\cdot z^{-2}y(-2)\,z^{2}\right)$$

$$N_0(z) = -\left(a_1 + a_2 + a_2 z^{-1}\right) = -\left(0{,}9 - 0{,}81 - 0{,}81z^{-1}\right)$$

$$N_0(z) = -\left(0{,}09 - 0{,}81z^{-1}\right)$$

$$Y_{zi}(z) = \frac{N_0(z)}{A(z)} = \frac{-0{,}09 + 0{,}8\,z^{-1}}{1 - 0{,}9\,z^{-1} + 0{,}81\,z^{-2}} \times \frac{z^2}{z^2}$$

$$Y_{zi}(z) = \frac{\cancel{z}(-0{,}09z + 0{,}8)}{z^2 - 0{,}9z + 0{,}81} = \frac{\cancel{z}A_1}{z - p1} + \frac{\cancel{z}A_2}{z - p2}$$

$$A_1 = \frac{-0{,}09z + 0{,}8}{\cancel{(z-p1)}(z-p2)}\cancel{(z-p1)}\Bigg|_{z=p1} = \frac{-0{,}09 \cdot 0{,}9\angle 60^\circ + 0{,}8}{(0{,}9\angle 60^\circ - 0{,}9\angle -60^\circ)}$$

$$A_1 = 0{,}489\angle -95{,}27^\circ = -0{,}045 - j0{,}49$$

$$A_2 = \frac{-0{,}09z + 0{,}8}{(z-p1)\cancel{(z-p2)}}\cancel{(z-p2)}\Bigg|_{z=p2} = \frac{-0{,}09 \cdot 0{,}9\angle -60^\circ + 0{,}8}{(0{,}9\angle -60^\circ - 0{,}9\angle 60^\circ)}$$

$$A_2 = 0{,}489\angle 95{,}27^\circ = -0{,}045 + j0{,}49$$

$$Y_{zi}(z) = \frac{0{,}489\,e^{-j95{,}27^\circ}}{1 - 0{,}9\,e^{j\frac{\pi}{3}} \cdot z^{-1}} + \frac{0{,}489\,e^{+j95{,}27^\circ}}{1 - 0{,}9 \cdot e^{-j\frac{\pi}{3}} \cdot z^{-1}}$$

$$a^n u(n) \leftrightarrow \frac{1}{1 - a z^{-1}}$$

$$y_{zi}(n) = 0{,}489\,e^{-j95{,}2^\circ} \cdot 0{,}9^n\,e^{j\frac{\pi}{3}n} + 0{,}489 \cdot e^{j95{,}2^\circ} \cdot 0{,}9^n \cdot e^{-j\frac{\pi}{3}} \frac{2}{2}$$

$$y_{zi}(n) = 0{,}97 \cdot 0{,}9^n \left[\frac{e^{j(\frac{\pi}{3}n - 95{,}2^\circ)} + e^{-j(\frac{\pi}{3}n - 95{,}2^\circ)}}{2}\right]$$

$$\boxed{y_{zi}(n) = 0{,}97 \cdot 0{,}9^n \cos\left(\frac{\pi}{3}n - 95{,}2^\circ\right)}$$

$$Y(z) = Y_{zs}(z) + Y_{zi}(z)$$

$$Y(z) = \frac{-0{,}0496 - j0{,}542}{1 - 0{,}9e^{j\frac{\pi}{3}}.z^{-1}} + \frac{-0{,}0496 + j0{,}542}{1 - 0{,}9e^{-j\frac{\pi}{3}}.z^{-1}} + \frac{1{,}099}{1 - z^{-1}}$$

$$+ \frac{-0{,}045 - j0{,}49}{1 - 0{,}9e^{j\frac{\pi}{3}}.z^{-1}} + \frac{-0{,}045 + j0{,}49}{1 - 0{,}9e^{-j\frac{\pi}{3}}.z^{-1}}$$

$$Y(z) = \frac{1{,}099}{1 - z^{-1}} + \frac{1{,}033e^{-j95{,}2^\circ}}{1 - 0{,}9e^{j\frac{\pi}{3}}.z^{-1}} + \frac{1{,}033e^{j95{,}2^\circ}}{1 - 0{,}9e^{-j\frac{\pi}{3}}.z^{-1}}$$

$$\boxed{Y(z) = 1{,}099\,u(n) + 2{,}067.(0{,}9)^n \cos\left(\frac{\pi}{3}n - 95{,}2^\circ\right)}$$