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Linear Equations in Linear Algebra

1.1

SYSTEMS OF LINEAR EQUATIONS

LINEAR EQUATION

- A **linear equation** in the variables $x_1, \dots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where b and the coefficients $a_1, \dots, a_n$ are real or complex numbers that are usually known in advance.

- A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables — say, $x_1, \dots, x_n$.

LINEAR EQUATION

- A **solution** of the system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively.

- The set of all possible solutions is called the **solution set** of the linear system.

- Two linear systems are called **equivalent** if they have the same solution set.

LINEAR EQUATION

- A system of linear equations has
  1. no solution, or
  2. exactly one solution, or
  3. infinitely many solutions.
- A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions.
- A system of linear equations is said to be **inconsistent** if it has no solution.

MATRIX NOTATION

- The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**.
- For the following system of equations,

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9, \end{aligned}$$

the matrix $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$

is called the **coefficient matrix** of the system.

MATRIX NOTATION

- An **augmented matrix** of a system consists of the coefficient matrix with an added column containing the constants from the right sides of the equations.
- For the given system of equations,

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

is called the augmented matrix.

MATRIX SIZE

- The size of a matrix tells how many rows and columns it has. If m and n are positive numbers, an $m \times n$ **matrix** is a rectangular array of numbers with m rows and n columns. (The number of rows always comes first.)

- The basic strategy for solving a linear system is to replace one system with an equivalent system (*i.e.*, one with the same solution set) that is easier to solve.

SOLVING SYSTEM OF EQUATIONS

- **Example 1:** Solve the given system of equations.

$$x_1 - 2x_2 + x_3 = 0 \quad \text{----(1)}$$
$$2x_2 - 8x_3 = 8 \quad \text{----(2)}$$
$$-4x_1 + 5x_2 + 9x_3 = -9 \quad \text{----(3)}$$

- **Solution:** The elimination procedure is shown here with and without matrix notation, and the results are placed side by side for comparison.

SOLVING SYSTEM OF EQUATIONS

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

- Keep x_1 in the first equation and eliminate it from the other equations. To do so, add 4 times equation 1 to equation 3.

$$\begin{aligned} 4x_1 - 8x_2 + 4x_3 &= 0 \\ \underline{-4x_1 + 5x_2 + 9x_3} &\underline{= -9} \\ -3x_2 + 13x_3 &= -9 \end{aligned}$$

SOLVING SYSTEM OF EQUATIONS

- The result of this calculation is written in place of the original third equation.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

- Use the $2x_2$ in equation 2 to eliminate the $-3x_2$ in equation 3 by multiplying equation 2 by $\frac{3}{2}$ and adding it to equation 3.

SOLVING SYSTEM OF EQUATIONS

old system

$$\begin{aligned} eq_1 &\quad x_1 - 2x_2 + x_3 = 0 \\ eq_2 &\quad 2x_2 - 8x_3 = 8 \\ eq_3 &\quad -3x_2 + 13x_3 = -9 \end{aligned}$$

$$\begin{array}{lr} \left(\frac{3}{2}\right)eq_2: & \left(\frac{3}{2}\right)2x_2 - \left(\frac{3}{2}\right)8x_3 = \left(\frac{3}{2}\right)8 \\ +\ eq_3: & -3x_2 + 13x_3 = -9 \\ \hline Eq\_3: & x_3 = 3 \end{array}$$

new system

$$\begin{aligned} Eq_1&: \quad x_1 - 2x_2 + x_3 = 0 \\ Eq_2&: \quad 2x_2 - 8x_3 = 8 \\ Eq_3&: \quad x_3 = 3 \end{aligned}$$

SOLVING SYSTEM OF EQUATIONS

- The new system has a triangular form.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

- Now, you may want to eliminate the $-2x_2$ term from equation 1, but it is more efficient to use the x_3 term in equation 3 first to eliminate the $-8x_3$ and x_3 terms in equations 2 and 1.

SOLVING SYSTEM OF EQUATIONS

$$\begin{array}{rl} (8)x_3 & = (8)3 \\ 2x_2 - 8x_3 & = 8 \\ \hline 2x_2 \qquad & = 32 \end{array} \qquad \begin{array}{rl} (-1)x_3 & = (-1)3 \\ x_1 - 2x_2 + x_3 & = 0 \\ \hline x_1 - 2x_2 \qquad & = -3 \end{array}$$

- Now, combine the results of these two operations.

$$\begin{array}{rcl} x_1 - 2x_2 & & = -3 \\ 2x_2 & & = 32 \\ & x_3 & = 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 2 & 0 & 32 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

SOLVING SYSTEM OF EQUATIONS

- Move back to the $2x_2$ in equation 2, and solve for it by times equation 2 by $\frac{1}{2}$.

$$\begin{array}{rcl} x_1 - 2x_2 & & = -3 \\ x_2 & & = 16 \\ & x_3 & = 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

SOLVING SYSTEM OF EQUATIONS

- Using now the x_2 in equation 2, and use it to eliminate the $-2x_2$ above it. Because of the previous work with x_3, there is now no arithmetic involving x_3 terms. Add 2 times equation 2 to equation 1 and obtain the system:

$$\begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

SOLVING SYSTEM OF EQUATIONS

- Thus, the only solution of the original system is (29,16,3). To verify that (29,16,3) is a solution, substitute these values into the left side of the original system, and compute.

$$(29) - 2(16) + (3) = 29 - 32 + 3 = 0$$

$$2(16) - 8(3) = 32 - 24 = 8$$

$$-4(29) + 5(16) + 9(3) = -116 + 80 + 27 = -9$$

- The results agree with the right side of the original system, so (29,16,3) is a solution of the system.

ELEMENTARY ROW OPERATIONS

- Elementary row operations include the following:
  1. (Replacement) Replace one row by the sum of itself and a multiple of another row.
  2. (Interchange) Interchange two rows.
  3. (Scaling) Multiply all entries in a row by a nonzero constant.

- Two matrices are called **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other.

ELEMENTARY ROW OPERATIONS

- It is important to note that row operations are reversible.
- If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.
- Two fundamental questions about a linear system are as follows:
  1. Is the system consistent; that is, does at least one solution *exist*?
  2. If a solution exists, is it the *only* one; that is, is the solution *unique*?

EXISTENCE AND UNIQUENESS OF SYSTEM OF EQUATIONS

- **Example 2:** Determine if the following system is consistent.

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned} \qquad \text{----(4)}$$

- **Solution:** The augmented matrix is

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

EXISTENCE AND UNIQUENESS OF SYSTEM OF EQUATIONS

- To obtain an x_1 in the first equation, interchange rows 1 and 2.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

- To eliminate the $5x_1$ term in the third equation, add $-5/2$ times row 1 to row 3.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{bmatrix} \qquad \text{----(5)}$$

EXISTENCE AND UNIQUENESS OF SYSTEM OF EQUATIONS

- Next, use the x_2 term in the second equation to eliminate the $-(1/2)x_2$ term from the third equation. Add $1/2$ times row 2 to row 3.

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix} \qquad \text{----(6)}$$

- The augmented matrix is now in triangular form. To interpret it correctly, go back to equation notation.

$$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= 8 \\ 0 &= 5/2 \end{aligned} \qquad \text{----(7)}$$

EXISTENCE AND UNIQUENESS OF SYSTEM OF EQUATIONS

- The equation $0 = 5/2$ is a short form of $0x_1 + 0x_2 + 0x_3 = 5/2.$
- There are no values of x_1, x_2, x_3 that satisfy (7) because the equation $0 = 5/2$ is never true.
- Since (7) and (4) have the same solution set, the original system is inconsistent (*i.e.*, has no solution).