1 Linear Equations in Linear Algebra

1.3

VECTOR EQUATIONS

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Vectors in $\mathbb{R}^2$

- A matrix with only one column is called a **column vector**, or simply a **vector**.

- An example of a vector with two entries is

$$\mathrm{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

where w_1 and w_2 are any real numbers.

- The set of all vectors with 2 entries is denoted by $\mathbb{R}^2$ (read "r-two").

VECTOR EQUATIONS

- The $\mathbb{R}$ stands for the real numbers that appear as entries in the vector, and the exponent 2 indicates that each vector contains 2 entries.
- Two vectors in $\mathbb{R}^2$ are **equal** if and only if their corresponding entries are equal.
- Given two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$, their **sum** is the vector $\mathbf{u}+\mathbf{v}$ obtained by adding corresponding entries of $\mathbf{u}$ and $\mathbf{v}$.
- Given a vector $\mathbf{u}$ and a real number c, the **scalar multiple** of $\mathbf{u}$ by c is the vector $c\mathbf{u}$ obtained by multiplying each entry in $\mathbf{u}$ by c.

VECTOR EQUATIONS

- **Example 1:** Given $\mathbf{u}=\begin{bmatrix}1\\-2\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}2\\-5\end{bmatrix}$, find $4\mathbf{u}$, $(-3)\mathbf{v}$, and $4\mathbf{u}+(-3)\mathbf{v}$.

Solution: $4\mathbf{u}=4\begin{bmatrix}1\\-2\end{bmatrix}=\begin{bmatrix}(4)(1)\\(4)(-2)\end{bmatrix}=\begin{bmatrix}4\\-8\end{bmatrix}$,

$$(-3)\mathbf{v}=(-3)\begin{bmatrix}2\\-5\end{bmatrix}=\begin{bmatrix}(-3)(2)\\(-3)(-5)\end{bmatrix}=\begin{bmatrix}-6\\15\end{bmatrix}, \quad \text{and}$$

$$4\mathbf{u}+(-3)\mathbf{v}=\begin{bmatrix}4\\-8\end{bmatrix}+\begin{bmatrix}-6\\15\end{bmatrix}=\begin{bmatrix}(4)+(-6)\\(-8)+(15)\end{bmatrix}=\begin{bmatrix}-2\\7\end{bmatrix}$$

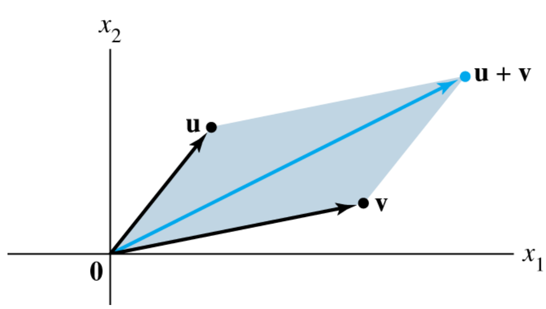
GEOMETRIC DESCRIPTIONS OF $\mathbb{R}^2$

- Consider a rectangular coordinate system in the plane. Because each point in the plane is determined by an ordered pair of numbers, *we can identify a geometric point (a, b) with the column vector* $\begin{bmatrix} a \\ b \end{bmatrix}$.

- So we may regard $\mathbb{R}^2$ as the set of all points in the plane.

PARALLELOGRAM RULE FOR ADDITION

- If **u** and **v** in $\mathbb{R}^2$ are represented as points in the plane, then $\mathbf{u} + \mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are **u**, **0**, and **v**. See the figure below.



VECTORS IN $\mathbb{R}^3$ and $\mathbb{R}^n$

- Vectors in $\mathbb{R}^3$ are 3×1 column matrices with three entries.
- They are represented geometrically by points in a three-dimensional coordinate space, with arrows from the origin.
- If n is a positive integer, $\mathbb{R}^n$ (read "r-n") denotes the collection of all lists (or *ordered n-tuples*) of n real numbers, usually written as $n \times 1$ column matrices, such as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.$$

ALGEBRAIC PROPERTIES OF $\mathbb{R}^n$

- The vector whose entries are all zero is called the **zero vector** and is denoted by $\mathbf{0}$.
- For all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:
  - (i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  - (ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  - (iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
  - (iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$, where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$
  - (v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
  - (vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

ALGEBRAIC PROPERTIES OF $\mathbb{R}^n$

(vii) $c(d\mathbf{u}) = (cd)(\mathbf{u})$

(viii) $1\mathbf{u} = \mathbf{u}$

LINEAR COMBINATIONS

- Given vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, ..., c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

  is called a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with **weights** $c_1, \ldots, c_p$.

- The weights in a linear combination can be any real numbers, including zero.
  - example: $\underset{\mathbb{R}^n}{\mathbf{0}} = \underset{\mathbb{R}}{0}\mathbf{v}_1 + \cdots + \underset{\mathbb{R}}{0}\mathbf{v}_p$

LINEAR COMBINATIONS

- **Example 2:** Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$.

  Determine whether $\mathbf{b}$ can be generated (or written) as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. That is, determine whether weights x_1 and x_2 exist such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \qquad \text{----(1)}$$

  If vector equation (1) has a solution, find it.

Slide 1.3- 11

LINEAR COMBINATIONS

Solution: Use the definitions of scalar multiplication and vector addition to rewrite the vector equation

$$x_1 \underbrace{\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}}_{\mathbf{a}_1} + x_2 \underbrace{\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}}_{\mathbf{a}_2} = \underbrace{\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}}_{\mathbf{b}},$$

which is same as $\begin{bmatrix} x_1 \\ -2x_1 \\ -5x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$

Slide 1.3- 12

LINEAR COMBINATIONS

and $\begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$. ----(2)

- The vectors on the left and right sides of (2) are equal if and only if their corresponding entries are both equal. That is, x_1 and x_2 make the vector equation (1) true if and only if x_1 and x_2 satisfy the following system.

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ -2x_1 + 5x_2 &= 4 \qquad \text{----(3)} \\ -5x_1 + 6x_2 &= -3 \end{aligned}$$

LINEAR COMBINATIONS

- To solve this system, row reduce the augmented matrix of the system as follows.

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

- The solution of (3) is $x_1 = 3$ and $x_2 = 2$. Hence $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$, with weights $x_1 = 3$ and $x_2 = 2$. That is,

$$3\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}.$$

LINEAR COMBINATIONS

- Now, observe that the original vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{b}$ are the columns of the augmented matrix that we row reduced:

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b} \end{array} \end{array}$$

- Write this matrix in a way that identifies its columns.

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{b}] \qquad \text{----(4)}$$

LINEAR COMBINATIONS

- A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}\,]. \qquad \text{----(5)}$$

- In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (5).

LINEAR COMBINATIONS

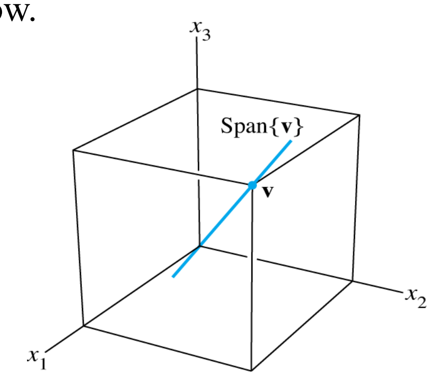
- **Definition:** If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset of** $\mathbb{R}^n$ **spanned** (or **generated**) **by** $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

  That is,
  Span $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form
  $$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$
  with $c_1, \ldots, c_p$ scalars.

Slide 1.3- 17

A GEOMETRIC DESCRIPTION OF Span $\{\mathbf{v}\}$

- Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^3$. Then Span $\{\mathbf{v}\}$ is the set of all scalar multiples of $\mathbf{v}$, which is the set of points on the line in $\mathbb{R}^3$ through $\mathbf{v}$ and $\mathbf{0}$. See the figure below.



Slide 1.3- 18

A GEOMETRIC DESCRIPTION OF Span $\{\mathbf{u}, \mathbf{v}\}$

- If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^3$, with $\mathbf{v}$ not a multiple of $\mathbf{u}$, then Span $\{\mathbf{u}, \mathbf{v}\}$ is the plane in $\mathbb{R}^3$ that contains $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{0}$.
- In particular, Span $\{\mathbf{u}, \mathbf{v}\}$ contains the line in $\mathbb{R}^3$ through $\mathbf{u}$ and $\mathbf{0}$ and the line through $\mathbf{v}$ and $\mathbf{0}$. See the figure below.