Exercise in FYS-3002

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1 Solving the equation for the ISR spectrum

Following Farley and Hagfors (1999). Equations from Farley and Hagfors (1999) are written FH (xx).

We are going to derive the equation for the power spectral density of the electron number density, namely the expression

$$\langle |n_{\mathrm{e}}(\boldsymbol{k},\omega)|^2\rangle = \frac{n_{\mathrm{e},0}}{\pi\omega}\frac{\Im\{F_{\mathrm{e}}\}|1+\chi_{\mathrm{i}}|^2+\Im\{F_{\mathrm{i}}\}|\chi_{\mathrm{e}}|^2}{|1+\chi_{\mathrm{e}}+\chi_{\mathrm{i}}|^2}. \tag{1}$$

1.1 Vlasov's equation

Let us start from the Vlasov's equation. This is similar to the Boltzmann equation, but we omit the collision term. The Vlasov equation can be written as (FH 4.24)

$$\partial_t f_\alpha + \boldsymbol{v}\cdot\partial_{\boldsymbol{r}} f_\alpha + \mu_\alpha[\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}]\cdot\partial_{\boldsymbol{v}} f_\alpha = 0$$

where $f = f(\boldsymbol{r},\boldsymbol{v},t)$ describe the phase space, μ_α is the charge-to-mass ratio of particle species α and $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields, both functions of space and time.

Let us assume all parameters to consist of a linear term and a higher order term, that is, we assume all parameters are on the form $f = f_0[1+f_1]$ where f_0 is linear and f_1 is non-linear and that $f_1 \ll 1$. Let us also write up the Fourier transform and Laplace transform of f_1:

$$f_1(\boldsymbol{r},\boldsymbol{v},t) = \sum_{\boldsymbol{k}} f_1(\boldsymbol{k},\boldsymbol{v},t)\exp(-i\boldsymbol{k}\cdot\boldsymbol{r})$$

$$f_1(\boldsymbol{k},\boldsymbol{v},s) = \int_0^\infty f_1(\boldsymbol{k},\boldsymbol{v},t)\exp(-st)\mathrm{d}t$$

We linearize the Vlasov equation and obtain

$$sf_{\alpha,1}(\boldsymbol{k},\boldsymbol{v},s) - f_{\alpha,1}(\boldsymbol{k},\boldsymbol{v},t=0) - i\boldsymbol{k}\cdot\boldsymbol{v}f_{\alpha,1}(\boldsymbol{k},\boldsymbol{v},s) + \mu_\alpha\left[\frac{1}{f_{\alpha,0}(\boldsymbol{v})}\boldsymbol{E}\cdot\partial_{\boldsymbol{v}}f_{\alpha,0}(\boldsymbol{v}) - \boldsymbol{B}[\boldsymbol{v}\times\partial_{\boldsymbol{v}}f_{\alpha,1}(\boldsymbol{k},\boldsymbol{v},s)]\right] = 0$$

It can be shown that this has solution

$$f_{\alpha,1}(\boldsymbol{k},\boldsymbol{v},s) = \frac{1}{\mu_\alpha B}\int_{-\infty}^{\varphi} g_\alpha(\varphi,\varphi')\left\{f_{\alpha,1}(\boldsymbol{k},\boldsymbol{v}',t=0) \mp \frac{i2X_{\mathrm{p}}^2}{f_{\alpha,0}(\boldsymbol{v}')}\boldsymbol{k}\cdot\boldsymbol{v}'[Zn_{\mathrm{i}}(\boldsymbol{k},s) - n_{\mathrm{e}}(\boldsymbol{k},s)]\right\}\mathrm{d}\varphi' \tag{2}$$

The primes (e.g. on $\boldsymbol{v}'$) refer to terms on the *unperturbed* orbit. Specifically we have that the unperturbed velocity is

$$\boldsymbol{v}' = \mathbf{e}_1 w\cos\varphi' + \mathbf{e}_2 w\sin\varphi' + \mathbf{e}_3 u$$

and we see that the velocity is a function of the variable φ (i.e., $f_1(\boldsymbol{k},\boldsymbol{v},s) = f_1(\boldsymbol{k},w,u,\varphi,s)$). $g(\varphi,\varphi')$ can be seen as an integrating factor (Bernstein 1958).

Density perturbations can then be obtained by integration:

$$n_\alpha(\boldsymbol{k}, s) = \int f_{\alpha,0}(\boldsymbol{v}')f_{\alpha,1}(\boldsymbol{k}, \boldsymbol{v}, s)\mathrm{d}\boldsymbol{v} = n_{\alpha,0}\int f_{\alpha,1}(\boldsymbol{k}, \boldsymbol{v}, s)\mathrm{d}\boldsymbol{v}$$

1.2 Electron number density

The function F_α is defined as

$$\begin{aligned} F_\alpha(\boldsymbol{k}, \omega) &= 1 + i\omega\int_0^\infty \exp\left\{i\omega\tau - \frac{k_\mathrm{r}^2 T_\alpha k_\mathrm{B}\sin^2\theta}{m_\alpha\Omega_\alpha^2}[1-\cos(\omega\tau)] - \frac{1}{2}(k_\mathrm{r}\tau\cos\theta)^2\frac{T_\alpha k_\mathrm{B}}{m_\alpha}\right\}\mathrm{d}\tau \\ &= 1 + i\omega G_\alpha(\boldsymbol{k}, \omega) \end{aligned}$$

The integral $G_\alpha(\boldsymbol{k}, \omega)$ is often referred to as a Gordeyev integral. Another useful parameter is:

$$\chi_\alpha(\boldsymbol{k}, \omega) = \frac{1}{k^2\lambda_\mathrm{D}^2}F_\alpha(\boldsymbol{k}, \omega) = \frac{1}{(k\lambda_\mathrm{D})^2}[1 + i\omega G_\alpha(\boldsymbol{k}, \omega)] \tag{3}$$

Using our definitions of F_α and χ_α we can now readily compute the IS spectrum from eq. (1):

$$\langle|n_\mathrm{e}(\boldsymbol{k}, \omega)|^2\rangle = \frac{n_{\mathrm{e},0}}{\pi\omega}\frac{\Im\{F_\mathrm{e}\}|1+\chi_\mathrm{i}|^2 + \Im\{F_\mathrm{i}\}|\chi_\mathrm{e}|^2}{|1+\chi_\mathrm{e}+\chi_\mathrm{i}|^2}. \tag{4}$$

We may also rewrite the above using that $\Im\{F_\alpha\} = \omega G_\alpha$:

$$\langle|n_\mathrm{e}(\boldsymbol{k}, \omega)|^2\rangle = \frac{n_{\mathrm{e},0}}{\pi}\frac{G_\mathrm{e}|1+\chi_\mathrm{i}|^2 + G_\mathrm{i}|\chi_\mathrm{e}|^2}{|1+\chi_\mathrm{e}+\chi_\mathrm{i}|^2} = \frac{n_{\mathrm{e},0}}{\pi}\left[\frac{G_\mathrm{e}|1+\chi_\mathrm{i}|^2}{|1+\chi_\mathrm{e}+\chi_\mathrm{i}|^2} + \frac{G_\mathrm{i}|\chi_\mathrm{e}|^2}{|1+\chi_\mathrm{e}+\chi_\mathrm{i}|^2}\right]. \tag{5}$$

2 Problems

Problem 1

Task 1

What is the physical interpretation of χ in eq. (3)?

Task 2

Explain what the term $f_{\alpha,1}$ in eq. (2) describe.

Task 3

Explain what the two terms in the square bracket in eq. (5) describe.

Problem 2

Using any one of the expressions for $\langle|n_\mathrm{e}(\boldsymbol{k}, \omega)|^2\rangle$, write a program that calculates the power spectral density. The program should accept a number of input parameters:

- f_r Radar frequency
- n_e Electron number density
- B Magnetic field strength
- m_i Ion mass
- T_e Electron temperature
- T_i Ion temperature
- θ Aspect angle (the angle between the radar pointing direction and the magnetic field)

Explain the code. Some things to consider:

- Where in the code were the different equations solved?
- How was the numerical calculation implemented?

The code itself should be well commented and included as an appendix.

Hint: Before you integrate all the way to infinity: where along the axis of integration does most of the information lie?

Problem 3

We will now look at some specific parameters using our program. Run your program with the parameters given as:

Parameter	Unit	Value
f_r	[Hz]	430×10^6
n_e	[m^{-3}]	2×10^{10}
B	[T]	3.5×10^{-5}
m_i	[amu]	16
T_e	[K]	200
T_i	[K]	200
θ	[°]	135

for frequencies $f \in [-2 \times 10^6, 2 \times 10^6]$.

Task 1

Where could an experiment with these parameters be done? Make a sketch that include the radar beam and the magnetic field line, in addition to the radar's approximate position on earth. Assume that the radar points directly upwards, i.e., towards zenith.

Task 2

The spectrum is plotted for frequencies $f \in [-2 \times 10^6, 2 \times 10^6]$; relative to an observer at the radar location, which way does the features found at positive frequencies in the spectrum move?

Task 3

Plot the resulting power spectra calculated by the program and explain what the different peaks represent.

> ***Comment***: This cannot be solved without having already solved problem 2. Include figure?

Problem 4

In this exercise we will use the parameters:

Parameter	Unit	Value
f_r	[Hz]	933×10^6
n_e	[m^{-3}]	2×10^{11}
B	[T]	5×10^{-5}
m_i	[amu]	16
T_i	[K]	2000
θ	[°]	180

Task 1

Calculate the power spectral density on $f \in [3.5 \times 10^6, 7 \times 10^6]$ for $T_\mathrm{e} = 2000\,\mathrm{K}$, $4000\,\mathrm{K}$, $6000\,\mathrm{K}$, and $8000\,\mathrm{K}$ and plot the power spectra.

Task 2

Explain the changes that can be seen as the electron temperature changes.

> ***Comment***: This cannot be solved without having already solved problem 2. Include figure?

Task 3

Explain what the assumption $k^2\lambda_\mathrm{D}^2 \ll 1$ is referring to. Is this assumption valid for all temperatures? Why/why not?

Problem 5

You should now be able to experiment a bit for yourself.

Task 1

Explain which parameter(s) that needs to be changed to obtain a similar plot as is shown in fig. 1.

Task 2

Reproduce the plot in fig. 1 using your own program. Include values on the axis and labels to all spectra.

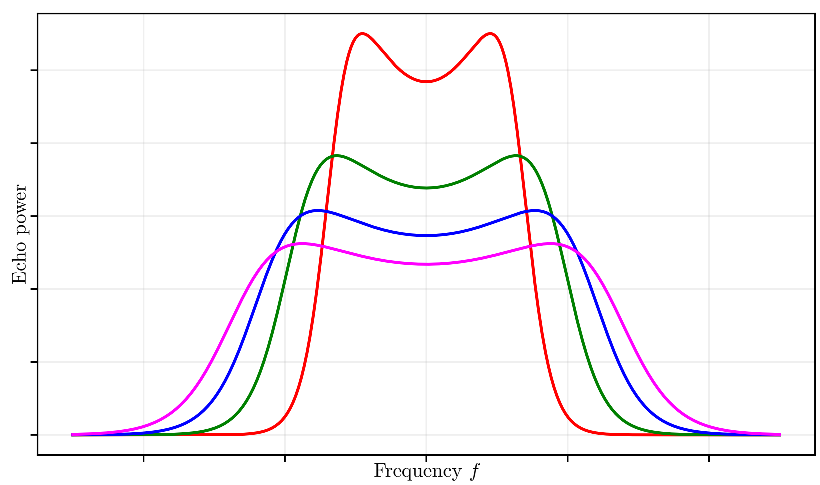

Figure 1: Power spectral density plot

References

Bernstein, Ira B. 1958. "Waves in a Plasma in a Magnetic Field." *Physical Review* 109 (1): 10–21. https://doi.org/10.1103/PhysRev.109.10.

Farley, D T, and Tor Hagfors. 1999. *AGF-304 textbook*.