UNIVERSAL COMBINATORIAL STRUCTURES

By

MICHAEL ENGEN

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Abstract of Dissertation Presented to the Graduate School
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Michael Engen

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A combinatorial structure is said to be _m-universal_ if it contains all other structures of the same kind of size m. Rado first introduced the notion of universality for infinite graphs in 1964, and in the same year, Moon first studied the problem of seeking smallest universal graphs under the induced subgraph order. Since then, universality has been studied in a variety of other contexts, with connections to information theory, probability theory, and more.

This work exhibits the problem of seeking smallest universal structures for words, integer partitions, compositions, graphs, and permutations. For partitions and compositions, we show that the smallest m-universal structures have size $\Theta(m \log m)$. In the context of graphs, we present the state of the art, culminating in Alon's 2017 proof that the smallest m-universal graphs have $2^{(m-1)/2}$ vertices asymptotically. The problem for permutations was first studied in 1999 by Arratia, who observed that the smallest m-universal permutations have size $\Theta(m^2)$, but researchers hold conflicting conjectures about their precise smallest size. We present the best-known lower and upper bounds on this quantity, including a recent improvement on the lower bound by Chroman, Kwan, and Singhal as well as a recent improvement on the upper bound in joint work with Vatter.

In each of these contexts, we also present results old and new for structures that are universal for proper classes of structures. We show that for both partitions and compositions, any proper class admits universal structures of linear size, whereas for both graphs and permutations, the answers vary widely.

INTRODUCTION

In 1964, Rado [45] introduced the concept of a <u>universal</u> graph, which he defined as a countably infinite graph that contained all finite graphs as induced subgraphs. Later that same year, Moon [42] studied finite graphs that contain all finite graphs on m vertices, a property that he called m-universality, initiating a line of research that continues to enjoy scholarly attention. In this work, we provide a brief survey of universality for various combinatorial structures—words, integer partitions, compositions, graphs, and permutations—under induced containment orders.

We say that the structure σ is <u>m-universal</u> if it contains all structures of size m of the same kind. More generally, if S is a family of structures, we say that σ is <u>S_m-universal</u> if it contains every structure in S of size m.

1.1   Words over a Finite Alphabet

As a simple example that will be prove to be a useful reference, consider the collection of words over a k-letter alphabet, where we say a word u is contained in the word v if u is a subsequence of of v, and the <u>size</u> of a word is its number of letters.

Proposition 1-1. *Let $\mathcal{W}$ denote the collection of words over a k-letter alphabet under the subword order. The smallest $\mathcal{W}_m$-universal words have size km.*

Proof. To begin, observe that any $\mathcal{W}_m$-universal word must have at least m occurrences of each letter, as for each letter ℓ, it must contain the word ℓ^m, ie. the letter ℓ repeated m times. Thus any $\mathcal{W}_m$-universal word must have at least km letters.

To show that this lower bound is tight, we recursively construct a $\mathcal{W}_m$-universal word of size km. Let our alphabet be $\{1, 2, \ldots, k\}$. If v is a $\mathcal{W}_{m-1}$-universal word, then the concatenation $v12\cdots k$ is $\mathcal{W}_m$-universal, as the first $m-1$ letters of an abritrary word u of size m embed into v, and the final letter embeds into $12\cdots k$. This fact, together with the $\mathcal{W}_0$-universal empty word, shows that the word $(12\cdots k)^m$ of size km is $\mathcal{W}_m$-universal, completing the proof. $\qquad\square$

1.2   Keeping it in the Family: Proper Universal Structures

Not all kinds of structures admit m-universal structures for all m. Consider, for example, the family $\mathcal{C} = \{\varepsilon, \mathsf{a}, \mathsf{b}, \mathsf{aa}, \mathsf{bb}, \cdots\}$ of constant words over the two-letter alphabet $\{\mathsf{a}, \mathsf{b}\}$. No word in

$\mathcal{C}$ contains both a and b, so no word in $\mathcal{C}$ is $\mathcal{C}_m$-universal for any $m \geq 1$. Nevertheless, one may still seek universal words for $\mathcal{C}$ in the collection of all words over $\{\mathsf{a},\mathsf{b}\}$.

A set of structures forms a <u>class</u> if it is closed downward under the induced-substructure order, and a class is called <u>proper</u> if it does not contain every structure of the given kind. Given a family of structures $\mathcal{F}$, we say that a structure is <u>$\mathcal{F}_m$-universal</u> or <u>m-universal for $\mathcal{F}$</u> if it contains every structure in $\mathcal{F}_m$. If an $\mathcal{F}_m$-universal structure itself lies in $\mathcal{F}$, we say it is a <u>proper</u> $\mathcal{F}_m$-universal structure. The classes that admit proper universal objects of all sizes are precisely those that satisfy the joint-embedding property, where we say that a set $\mathcal{S}$ of structures satisfies the <u>joint-embedding property</u> if for any $\rho, \sigma \in \mathcal{S}$, there is some $\tau \in \mathcal{S}$ so that $\rho \leq \tau$ and $\sigma \leq \tau$.

CHAPTER 2
INTEGER PARTITIONS

An (integer) partition $p = p(1) \cdots p(n)$ is a weakly decreasing sequence of positive integers, which are called the <u>parts</u> of p. The <u>size</u> of a partition, denoted $|p|$, is the sum of its parts. For convenience, we say that for $i > n$ the ith part of p is 0 and write $p(i) = 0$. We say that a partition $p = p(1) \cdots p(m)$ is contained in another partition $q = q(1) \cdots q(n)$ if $p(i) \leq q(i)$ for all i. This partial order on partitions is simply the one of Young's lattice, namely containment of Ferrers diagrams. The <u>Ferrers diagram</u> of a partition $p = p(1) \cdots p(m)$ is a visual representation of p consisting of an arrangement of $p(1)$ cells in the first (topmost) row, $p(2)$ cells in the second row, and so on. An example of partition containment displayed through Ferrers diagrams is presented in Figure 2-1.

$$31 \quad \leq \quad 4211$$
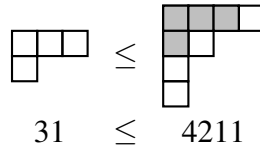
Figure 2-1. An example partition containment presented by way of Ferrers diagrams.

The <u>conjugate</u> of a partition p is the partition whose Ferrers diagram is the reflection of the Ferrers diagram of p reflected along the anti-diagonal $y = -x$. An example of a partition and its conjugate is displayed in Figure 2-2.
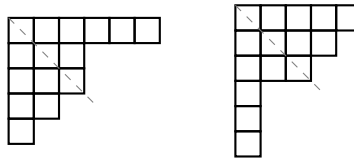
Figure 2-2. The Ferrers diagram of a partition and its conjugate,
both displayed with the anti-diagonal.

Given a finite set of partitions S, the unique smallest partition that contains each partition in S is the partition whose i^{th} part is equal to the largest i^{th} part among all members of S.

Observation 2-1. *If S is a finite set of partitions, then the smallest partition p that is S-universal*

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has ith part given by

$$p(i) = \max\{q(i) \,:\, q \in S\}.$$

Thus, to construct the smallest *m*-universal partition for the set of all partitions, one must only observe that the largest i^{th} part among all partitions of size *m* is $\lfloor m/i \rfloor$, meaning the unique smallest *m*-universal partition has i^{th} part equal to $\lfloor m/i \rfloor$ for all *i*.

Theorem 2-2. *The unique smallest m-universal partition has size*

$$\phi(m) = \lfloor m/1 \rfloor + \lfloor m/2 \rfloor + \cdots + \lfloor m/m \rfloor^{1}.$$

Figure 2-3 displays the 24-universal partition of size $\phi(24) = 84$. In [20], Dirichlet shows that $\phi(m) = m(\log(m) + 2\gamma - 1) + \Delta(m)$, where $\gamma \approx 0.5772$ is the Euler–Mascheroni constant and $\Delta(m) = O(\sqrt{m})$, and thus $\phi(m) \sim m \log m$. The asymptotics of $\Delta(m)$ are the subject of the <u>Dirichlet Divisor Problem</u>, which aims to find the smallest value θ so that $\Delta(m) = O(m^{\theta})$. The best known bound is due to Huxley [33], who showed that $\inf \theta \le 131/416 \approx 0.3149$.
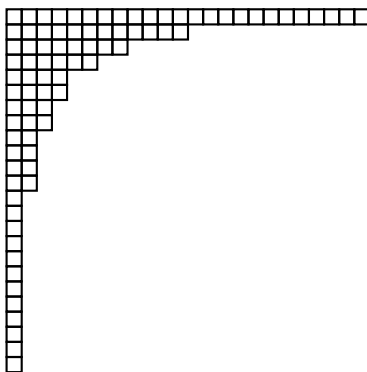
Figure 2-3. The unique 24-universal partition of size $\phi(24) = 84$.

2.1 Proper Classes of Partitions

In this section, we show that any proper class of partitions admits universal partitions of linear size. For reference, some known or computed values of minimum sizes for universal partitions for various classes are presented in Appendix A.

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[1]The sequence ϕ appears as sequence A006218 in the OEIS [49]

We require one more definition to discuss proper classes of partitions. We say that an potentially infinite partition p is an infinite weakly decreasing sequence over the infinite ordered alphabet $\{0 < 1 < \cdots < \omega\}$, where ω represents an infinite part. If $p(n) = \ell > 0$ for each $n > N$, then we abbreviate p as $p(1) \cdots p(N)\ell^\omega$, where ℓ^ω representing an infinite number of parts equal to ℓ. If $p(n) = 0$ for each $n > N$, then we abbreviate p as $p(1) \cdots p(N)$. Infinite partitions give way to infinite Ferrers diagrams, and we say that the age of a potentially infinite partition p, denoted Age(p), is the set of finite partitions whose Ferrers diagrams embed into the Ferrers diagram of p (the term age dates to Fraïssé [28]). An example of containment of a finite Ferrers diagram into an infinite one, together with the corresponding set containment, is presented in Figure 2-4.

$$4211 \quad \in \quad \text{Age}(\omega\omega41^\omega)$$
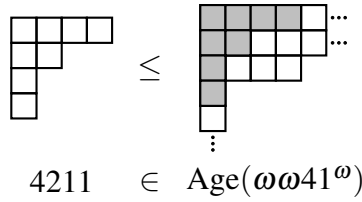
Figure 2-4. An example of containment of a finite Ferrers digram into an infinite one.

We show that every proper class of partitions admits linear-size universal partitions in two parts. First, we show that every proper class of partitions is contained in an age of the form Age$(\omega^k\ell^\omega)$ in Proposition 2-3, and then that every such age admits linear-size universal partitions in Theorem 2-4.

Proposition 2-3. *For any proper partition class $\mathcal{P}$, there are nonnegative integers k, ℓ such that $\mathcal{P} \subseteq \text{Age}(\omega^k\ell^\omega)$.*

Proof. Let ℓ be the largest integer such that $\mathcal{P}$ contains the partition ℓ^n for all n and let k be the largest integer such that $\mathcal{P}$ contains $(\ell+1)^k$. We claim that $\mathcal{P} \subseteq \text{Age}(\omega^k\ell^\omega)$. For any partition $p \in \mathcal{P}$, we must have $p(n) \leq \ell$ for each $n > k$, as otherwise p contains $(\ell+1)^n$, and as $\mathcal{P}$ is closed downward, we would have $(\ell+1)^n \in \mathcal{P}$, a contradiction. The condition that $p(n) \leq \ell$ for each $n > k$ is precisely the condition that defines containment in Age$(\omega^k\ell^\omega)$, and thus $p \in \text{Age}(\omega^k\ell^\omega)$, completing the proof. □

Theorem 2-4. *Let $\mathcal{P} = \mathrm{Age}(\omega^k \ell^\omega)$ be a class of partitions. Then for $m \geq k\ell$, the smallest $\mathcal{P}_m$-universal partitions have size*

$$\mathrm{u}_\mathcal{P}(m) = \sum_{i=1}^{k} \left\lfloor \frac{m}{i} \right\rfloor + \sum_{i=1}^{\ell} \left\lfloor \frac{m}{i} \right\rfloor - k\ell$$

and thus $\mathrm{u}_\mathcal{P}(m) = \Theta(m)$.

Before proving Theorem 2-4, observe that if $P \subseteq Q$ are sets of partitions and q is a Q-universal partition, then q is P-universal as well, as q necessarily contains each partition in P. Thus, by proving that each age of the form $\mathrm{Age}(\omega^k \ell^\omega)$ admits universal partitions of linear size, Proposition 2-3 implies that every proper partition class does as well.

Proof of Theorem 2-4. By Observation 2-1, the unique smallest $\mathcal{P}_m$-universal partition P has its ith part equal to the largest i^{th} part among all partitions in $\mathcal{P}_m$. The only restriction on partitions in $\mathcal{P}$ is that their i^{th} part is at most ℓ for $i > k$, so we have

$$P(i) = \begin{cases} \left\lfloor \dfrac{m}{i} \right\rfloor & \text{if } 1 \leq i \leq k, \\ \min\left\{ \left\lfloor \dfrac{m}{i} \right\rfloor, \ell \right\} & \text{if } k < i \leq m. \end{cases}$$

As $\left\lfloor \frac{m}{i} \right\rfloor \geq \ell$ if and only if $\left\lfloor \frac{m}{\ell} \right\rfloor \geq i$, this may be simplified to

$$P(i) = \begin{cases} \left\lfloor \dfrac{m}{i} \right\rfloor & \text{if } 1 \leq i \leq k, \\ \ell & \text{if } k < i \leq \left\lfloor \dfrac{m}{\ell} \right\rfloor, \\ \left\lfloor \dfrac{m}{i} \right\rfloor & \text{if } \left\lfloor \dfrac{m}{\ell} \right\rfloor < i \leq m. \end{cases}$$

Thus, for $m \geq k\ell$, the size of P is

$$|P| = \underbrace{\left\lfloor \frac{m}{1} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + \cdots + \left\lfloor \frac{m}{k} \right\rfloor + \ell + \ell + \cdots + \ell}_{\left\lfloor \frac{m}{\ell} \right\rfloor \text{ terms}} + \left\lfloor \frac{m}{\left\lfloor \frac{m}{\ell} \right\rfloor + 1} \right\rfloor + \cdots + \left\lfloor \frac{m}{m} \right\rfloor.$$

The sum of the parts $P(k+1), \cdots, P(m)$ is equal to $\sum_{i=1}^{\ell} \left\lfloor \frac{m}{i} \right\rfloor - k\ell$, as these correspond to the first ℓ parts of the conjugate of P, which are $\left\lfloor \frac{m}{i} \right\rfloor$ for $1 \leq i \leq \ell$, and we need to subtract those

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cells that overlap with the first k parts of P. As $m \geq k\ell$, we have both $\lfloor \frac{m}{k} \rfloor \geq \ell$ and $\lfloor \frac{m}{\ell} \rfloor \geq k$, and thus

$$|P| = \sum_{i=1}^{k} \left\lfloor \frac{m}{i} \right\rfloor + \sum_{i=1}^{\ell} \left\lfloor \frac{m}{i} \right\rfloor - k\ell.$$

For all n and j, the inequality

$$\log(j+1)n - j \leq \sum_{i=1}^{j} \left\lfloor \frac{n}{j} \right\rfloor \leq (\log(j)+1)n,$$

holds, and thus $|P| = \Theta(m)$, as desired. $\qquad\square$

2.2 Concluding Remarks

Observation 2-1 establishes that the smallest m-universal partition is unique, and Theorem 2-2 establishes its precise size for all m. For any proper class of partitions $\mathcal{P}$, there are proper m-universal partitions for $\mathcal{P}$ for all m if and only if $\mathcal{P}$ satisfies the joint-embedding property, which is equivalent to $\mathcal{P}$ being the age of some potentially infinite partition u, as the following result of Fraïssé (which we have specialized to our context here) shows:

Theorem 2-5 (Fraïssé [28]; see also Hodges [32, Section 7.1]). *The following are equivalent for a class $\mathcal{P}$ of integer partitions or compositions:*

1. *$\mathcal{P}$ cannot be expressed as the union of two proper subclasses,*

2. *$\mathcal{P}$ satisfies the <u>joint embedding property</u>, meaning that for every $a, b \in \mathcal{P}$ there is some $c \in \mathcal{P}$ such that $a, b \leq c$, and*

3. *$\mathcal{P} = \mathrm{Age}(u)$ for some word $u \in (\mathbb{P} \cup \{n^\omega : n \in \mathbb{P}\} \cup \{\omega, \omega^\omega\})^*$.*

CHAPTER 3
COMPOSITIONS

A composition $c = c(1) \cdots c(n)$ is a finite sequence of positive integers, which are called the parts of c. The length of c, denoted $\text{len}(c)$ is the length of the sequence, the size of a composition, denoted $|c|$, is the sum of its parts. We say a composition $c = c(1)c(2) \cdots c(m)$ is contained in another composition $d = d(1)d(2) \cdots d(n)$ if there is a subsequence $d(i_1)d(i_2) \cdots d(i_m)$ of d such that $c(j) \leq d(i_j)$ for all j, and such a sequence of indices is called an embedding of c into d. Note that with this definition, when determining whether c is contained in d, it suffices to take a greedy approach: embed the first part of c as far left as possible in d, then embed the second part of c as far left as possible after that, and so on.

This partial order may be exhibited visually by way of skyline diagrams, which are simply a symmetry of the Ferrers diagrams of Chapter 2. The skyline diagram of the composition $c = c(1) \cdots c(n)$ consists of n columns of cells, with the ith column (from the left) having $c(i)$ cells. For compositions c and d, we have $c \leq d$ if the skyline diagram of c can be embedded into that of d. An example of composition containment displayed through skyline diagrams is shown in Figure 3-1.

$$3\ 4\ 1\ 3 \quad \leq \quad 1\ \underline{4}\ 1\ \underline{4}\ \underline{2}\ 1\ 1\ \underline{4}\ 3$$
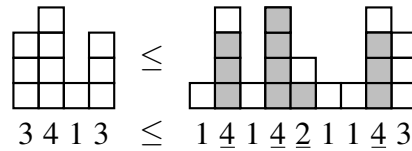
Figure 3-1. An example of composition containment presented by way of skyline diagrams.

This partial order on compositions is called the generalized subword order for words over the positive integers $\mathbb{P}$. It was first considered by Bergeron, Bousquet-Mélou, and Dulucq [11], who studied the saturated chains in this poset. Later, Sagan and Vatter [46] determined the Möbius function of this poset, and Björner and Sagan [13] showed that this Möbius function has a rational generating function. Finally, Vatter [50] considered the analogue of the reconstruction conjecture in this poset, and Engen and Vatter [21] characterized the finite-dimensional classes in this poset with respect to Dushnik–Miller dimension.

The following result shows that smallest universal compositions can be constructed recursively, leading to an explicit formula on their size, and has been adapted from [3].

Theorem 3-1 (Albert, Engen, Pantone, and Vatter [3]). *Let $\ell(m)$ be the size of a smallest m-universal composition. Then $\ell(0) = 0$ and for $m \geq 1$,*

$$\ell(m) = m + \min\{\ell(k) + \ell(m-k-1) : 0 \leq k \leq m-1\}^1. \tag{3-1}$$

Proof. We prove the theorem by induction on m. As the base case of $m = 0$ is trivial, suppose that the statement is true for all values less than m.

We begin by showing that

$$\ell(m) \leq m + \min\{\ell(k) + \ell(m-k-1) : 0 \leq k \leq m-1\}$$

by constructing an m-universal composition of this size. To this end, choose k so that $\ell(k) + \ell(m-k-1)$ is minimized and then choose a composition c of size $\ell(k)$ that is k-universal and a composition d of size $\ell(m-k-1)$ that is $(m-k-1)$-universal. We claim that the composition cmd is m-universal.

Consider any composition $e = e(1)e(2)\cdots e(n)$ of size m, and choose j such that

$$
\begin{aligned}
e(1) \quad + \quad \cdots \quad + \quad e(j-1) & \quad & \leq \quad k, \\
e(1) \quad + \quad \cdots \quad + \quad e(j-1) \quad + \quad e(j) & \quad & > \quad k.
\end{aligned}
$$

Since c is k-universal and $|e(1)\cdots e(j-1)| \leq k$, we have that c contains the composition $e(1)\cdots e(j-1)$. Moreover, the part $e(j)$ embeds into the part m. Finally, since d is $(m-k-1)$-universal, it necessarily contains $e(j+1)\cdots e(n)$ as $|e(j+1)\cdots e(n)| \leq m-k-1$. Thus $e(1)\cdots e(n)$ is contained in the composition cmd, and as e was an arbitrary composition of size m, we have that cmd is m-universal.

---

[1]Up to shifting indices by 1, the sequence $\ell(m)$ in Theorem 3-1 is sequence A001855 in the OEIS [49].

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To establish the reverse inequality

$$\ell(m) \geq m + \min\{\ell(k) + \ell(m-k-1) : 0 \leq k \leq m-1\},$$

we prove that every m-universal composition must have size at least $m + \ell(k) + \ell(m-k-1)$ for some $0 \leq k \leq m-1$. Let c be an m-universal composition. As c contains the composition m, we may choose some index j so that $c(j) \geq m$. Choose k so that

$$\ell(k) \leq c(1) + \cdots + c(j-1) < \ell(k+1),$$

meaning that there is some composition d of size $k+1$ that does not embed into $c(1) \cdots c(j-1)$, and thus the earliest that d may embed into c is into $c(1) \cdots c(j)$.

We claim that $c(j+1) \cdots c(n)$ must be $(m-k-1)$-universal. Let e be an arbitrary composition of size $m-k-1$. As c is m-universal, it must contain the composition de. Since d can embed no earlier than into $c(1) \cdots c(j)$, e must embed entirely within $c(j+1) \cdots c(n)$. As e was an arbitrary composition of size $m-k-1$, the composition $c(j+1) \cdots c(n)$ is therefore $(m-k-1)$-universal, and thus $c(j+1) + \cdots + c(n) \geq \ell(m-k-1)$. Together, this shows

$$|c| = |c(1) \cdots c(j-1)| + |c(j)| + |c(j+1) \cdots c(n)|$$
$$\geq \ell(k) + m + \ell(m-k-1),$$

completing the proof. $\qquad\square$

By showing that ℓ is a convex sequence, we may obtain an explicit formula for $\ell(m)$. In [43], Morris proves that the sequence defined by $G(1) = 0$ and, for $m > 1$

$$G(m) = n + \min\{G(k) + G(m-k) : 1 \leq k \leq m-1\}$$

is convex, and it is elementary to show that $G(m+1) = \ell(m) + m$. For completeness, we present a proof of the convexity of ℓ, adapted from Morris's proof of the convexity of G.

Proposition 3-2. *The sequence $\ell(m)$ is convex, ie. $2\ell(m) \leq \ell(m-1) + \ell(m+1)$ for all $m \geq 1$.*

Proof. We prove that ℓ is convex at m, ie. $2\ell(m) \leq \ell(m-1) + \ell(m+1)$ by induction on m. Our base case of $m = 1$ follows from computing $\ell(0) = 0$, $\ell(1) = 1$, and $\ell(2) = 3$, so assume that the statement holds for all values less than m. We proceed in two cases based on the parity of m.

First, assume that m is even, with $m = 2n$. We may compute the formulas for $\ell(m-1)$, $\ell(m)$, and $\ell(m+1)$ using that convexity of ℓ at values $m' < m$ implies that the minimum of $\ell(k) + \ell(m'-k)$ over all values of k occurs when $k = \lfloor m'/2 \rfloor$.

$$\ell(2n-1) = 2n - 1 + \min\{\ell(k) + \ell(2n-2-k) : 0 \leq k \leq 2n-2\}$$

$$= 2n - 1 + \ell(n-1) + \ell(n-1)$$

$$\ell(2n) = 2n + \min\{\ell(k) + \ell(2n-1-k) : 0 \leq k \leq 2n-1\}$$

$$= 2n + \ell(n-1) + \ell(n)$$

$$\ell(2n+1) = 2n + 1 + \min\{\ell(k) + \ell(2n-k) : 0 \leq k \leq 2n\}$$

$$= 2n + 1 + \ell(n) + \ell(n)$$

These formulas show that in this case, we in fact have equality: $2\ell(2n) = \ell(2n-1) + \ell(2n+1)$.

On the other hand, assume that m is odd, with $m = 2n + 1$. As before, the convexity of ℓ at values less than m allows us to write explicit recursive formulas for $\ell(2n)$, $\ell(2n+1)$, and $\ell(2n+2)$.

$$\ell(2n) = 2n + \min\{\ell(k) + \ell(2n-1-k) : 0 \leq k \leq 2n-1\}$$

$$= 2n + \ell(n-1) + \ell(n)$$

$$\ell(2n+1) = 2n + 1 + \min\{\ell(k) + \ell(2n-k) : 0 \leq k \leq 2n\}$$

$$= 2n + 1 + \ell(n) + \ell(n)$$

$$\ell(2n+2) = 2n + 2 + \min\{\ell(k) + \ell(2n+1-k) : 0 \leq k \leq 2n+1\}$$

$$= 2n + 2 + \ell(n) + \ell(n+1)$$

Some arithmetic reveals that the inequality $2\ell(2n+1) \leq \ell(2n) + \ell(2n+2)$ is equivalent to the inequality $2\ell(n) \leq \ell(n-1) + \ell(n+1)$, which is true by induction, completing the proof. $\square$

Proposition 3-2 allows us to recursively define an explicit sequence of smallest m-universal compositions. Let w_0 be the empty composition, and for $m \geq 1$, define w_m as the concatenation $w_{\lfloor (m-1)/2 \rfloor} m w_{\lceil (m-1)/2 \rceil}$. Then, as $\ell(k) + \ell(m-1-k)$ is minimized at $k = \lfloor (m-1)/2 \rfloor$, w_m is m-universal by the proof of Theorem 3-1 and has length m and size $\ell(m)$. Figure 3-2 shows the skyline diagram of w_{63}.

Knuth gave an explicit formula for $\ell(m)$ in The Art of Computer Programming, Volume 3 [38, Section 5.3.1, Eq. (3)], where it is related to sorting by binary insertion. Knuth shows there that,

$$\ell(m) = (m+1)\lceil \log_2(m+1) \rceil - 2^{\lceil \log_2(m+1) \rceil} + 1,$$

so $\ell(m) \sim m \log_2(m)$.
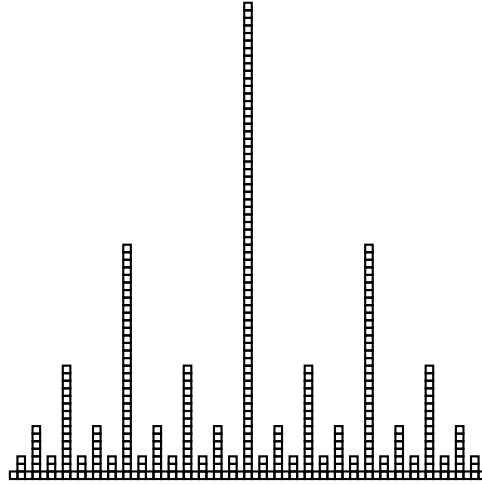
Figure 3-2. The 63-universal composition w_{63} of length 63 and size $\ell(63) = 321$.

3.1   Proper Classes of Compositions

In this section, we examine the problem of universality for proper classes of compositions. As we will see, every proper composition class admits universal compositions of linear size. Moreover, we study universality of two natural families of compositions as well as for the class of compositions with at most one large part. For reference, some known or computed values of minimum sizes for universal compositions for various classes are presented in Appendix B.

As in Chapter 2, we define infinite structures to aid discussion of proper classes. We define a potentially infinite composition to be a word over the alphabet $\mathbb{P} \cup \{n^\omega : n \in \mathbb{P}\} \cup \{\omega, \omega^\omega\}$. As in the case of partitions, the symbol ω stands for an infinite part, the symbol n^ω stands for an infinite number of parts equal to n, and the symbol ω^ω stands for an infinite number of infinite parts. Given a potentially infinite composition c, the age of c, denoted $\mathrm{Age}(c)$, is the set of all compositions whose skyline diagrams embed into the skyline diagram of c (the term age dates to Fraïssé [28].) An example of containment of a finite skyline diagram in an infinite one, together with the corresponding set containment, is presented in Figure 3-3.

$$1\ 3\ 2\ 3\ 1 \quad \in \quad \mathrm{Age}(1^\omega \omega 2131^\omega)$$

Figure 3-3. Containment of a finite skyline diagram into an infinite one.

The following proposition shows that any proper composition class admits universal compositions of linear size.

Theorem 3-3. *For any proper class of compositions $\mathcal{C}$, there is a positive constant c so that there are $\mathcal{C}_m$-universal compositions of size cm for all positive integers m.*

Proof. We show that $\mathcal{C}$ admits linear-size universal compositions by first showing that $\mathcal{C}$ is contained in an age of the form $\mathrm{Age}(\ell^\omega(\omega\ell^\omega)^k)$, which in turn admits linear-size universal compositions.

Let ℓ be the largest integer such that $\mathcal{C}$ contains ℓ^n for all n, and let k be the largest integer such that $\mathcal{C}$ contains $(\ell+1)^k$. We claim that $\mathcal{C} \subseteq \mathrm{Age}(\ell^\omega(\omega\ell^\omega)^k)$. If $c \in \mathcal{C}$, then c does not contain $(\ell+1)^k$, and thus c has at most k parts of size larger than ℓ. The class $\mathrm{Age}(\ell^\omega(\omega\ell^\omega)^k)$ is precisely the class of compositions that have at most k parts larger than ℓ, and thus we have $c \in \mathrm{Age}(\ell^\omega(\omega\ell^\omega)^k)$.

To construct an m-universal composition for $\mathrm{Age}(\ell^{\omega}(\omega\ell^{\omega})^k)$, note that if c and d are m-universal compositions for $\mathrm{Age}(u)$ and $\mathrm{Age}(v)$ respectively, then cd is m-universal for $\mathrm{Age}(uv)$. Thus, as the composition ℓ^m is m-universal for $\mathrm{Age}(\ell^{\omega})$ and the composition m is m-universal for $\mathrm{Age}(\omega)$, we may conclude that $\ell^m(m\ell^m)^k$ is m-universal for $\mathrm{Age}(\ell^{\omega}(\omega\ell^{\omega})^k)$. $\quad\square$

3.1.1 Compositions with Bounded Part Sizes

Fix a positive integer t, and consider the class $\mathrm{Age}(t^{\omega})$ of compositions whose parts are all of size at most t. In a composition c, each part of size larger than t may be replaced one of size t without disturbing any of the embeddings of compositions in $\mathrm{Age}(t^{\omega})$.

Observation 3-4. *Let c be a composition of size n. For each positive integer t, there is composition c' of size at most n in $\mathrm{Age}(t^{\omega})$ that contains every composition in $\mathrm{Age}(t^{\omega})$ that c does.*

In particular, if c is any $\mathrm{Age}(t^{\omega})_m$-universal composition, then there is a weakly smaller proper $\mathrm{Age}(t^{\omega})_m$-universal composition, and thus the smallest $\mathrm{Age}(t^{\omega})_m$-universal compositions and the smallest proper $\mathrm{Age}(t^{\omega})_m$-universal compositions have the same size.

Smallest proper universal compositions for $\mathrm{Age}(t^{\omega})$ can be constructed in a manner similar that of the proof of Theorem 3-1. The construction only differs in that the "large" part of size m used in the proof of Theorem 3-1 needs to only have size $\min\{t,m\}$ in this case.

Theorem 3-5. *Let $\ell_t(m)$ denote the size of the smallest (proper) $\mathrm{Age}(t^{\omega})_m$-universal compositions. Then $\ell_t(0) = 0$ and for $m \geq 1$,*

$$\ell_t(m) = \min\{t,m\} + \min\{\ell_t(k) + \ell_t(m-k-1) \, : \, 0 \leq k \leq m-1\}.$$

Proof. We prove the theorem by induction on m. As the base case of $m = 0$ is trivial, suppose that the statement is true for all values less than m.

We begin by showing that

$$\ell_t(m) \leq \min\{t,m\} + \min\{\ell_t(k) + \ell_t(m-k-1) \, : \, 0 \leq k \leq m-1\}.$$

by constructing an $\text{Age}(t^\omega)_m$-universal composition of this size. To this end, choose k so that $\ell_t(k) + \ell_t(m-k-1)$ is minimized and then choose a proper $\text{Age}(t^\omega)_k$-universal composition c of size $\ell_t(k)$ and a proper $\text{Age}(t^\omega)_{m-k-1}$-universal composition d of size $\ell_t(m-k-1)$. We claim that the composition $c(\min\{t,m\})d \in \text{Age}(t^\omega)$ is $\text{Age}(t^\omega)_m$-universal.

Consider any composition $e = e(1)e(2)\cdots e(n) \in \text{Age}(t^\omega)$ of size m, and choose j such that

$$e(1) \;\; + \;\; \cdots \;\; + \;\; e(j-1) \qquad\qquad \leq \;\; k,$$

$$e(1) \;\; + \;\; \cdots \;\; + \;\; e(j-1) \;\; + \;\; e(j) \;\; > \;\; k.$$

Since c is $\text{Age}(t^\omega)_k$-universal and $|e(1)\cdots e(j-1)| \leq k$, we have that c contains the composition $e(1)\cdots e(j-1)$. Moreover, the part $e(j)$ embeds in the part $\min\{t,m\}$. Finally, since d is $\text{Age}(t^\omega)_{m-k-1}$-universal, it contains the composition $e(j+1)\cdots e(n)$ as $|e(j+1)\cdots e(n)| \leq m-k-1$. Thus $e(1)\cdots e(n)$ is contained in the composition $c(\min\{t,m\})d$, and as e was an arbitrary composition of size m, we have that $c(\min\{t,m\})d$ is $\text{Age}(t^\omega)_m$-universal.

To establish the reverse inequality

$$\ell_t(m) \geq \min\{t,m\} + \min\{\ell_t(k) + \ell_t(m-k-1) \;:\; 0 \leq k \leq m-1\},$$

we prove that every proper $\text{Age}(t^\omega)_m$-universal composition must have size at least $\min\{t,m\} + \ell_t(k) + \ell_t(m-k-1)$ for some k. Let c be an $\text{Age}(t^\omega)_m$-universal composition. As c contains the composition $\min\{t,m\}$, so there must be some index j so that $c(j) \geq \min\{t,m\}$. Choose k so that

$$\ell_t(k) \leq c(1) + \cdots + c(j-1) < \ell_t(k+1),$$

meaning that there is some composition d of size $k+1$ that does not embed into $c(1)\cdots c(j)$, and thus the earliest d may embed into c is into $c(1)\cdots c(j)$.

We claim that $c(j+1)\cdots c(n)$ must be $(m-k-1)$-universal for $\text{Age}(t^\omega)$. Let e be an arbitrary composition in $\text{Age}(t^\omega)$ of size $m-k-1$. As c is $\text{Age}(t^\omega)_m$-universal, it must contain

the composition de. Since d can embed no earlier than into $c(1)\cdots c(j)$, e must embed entirely within $c(j+1)\cdots c(n)$. As e was an abitrary composition of size $m-k-1$, the composition $c(j+1)\cdots c(n)$ is therefore $(m-k-1)$-universal for $\mathrm{Age}(t^\omega)$, and thus $c(j+1)+\cdots+c(n) \geq \ell_t(m-k-1)$. Finally, we have

$$|c| = |c(1)\cdots c(j-1)| + |c(j)| + |c(j+1)\cdots c(n)|$$
$$\geq \ell_t(k) + \min\{t,m\} + \ell_t(m-k-1),$$

completing the proof. $\qquad\square$

Figure 3-4. A proper $\mathrm{Age}(7^\omega)_{63}$-universal composition of length 63 and size $\ell_7(63) = 185$.

Figure 3-4 shows a smallest (proper) $\mathrm{Age}(7^\omega)_{63}$-universal composition of the above construction. Unlike the sequence $\ell(m)$, the sequences $\ell_t(m)$ do not appear to be convex. Through tedious casework, the explicit formulas for the sequences $\ell_t(m)$ may be determined for small values of t, and computation supports the following conjecture:

Conjecture 3-6. *Let $t = 2^k + r$ with $0 \leq r < 2^k$. Then for $m \geq k$ we have*

$$\ell_t(m) = km + k - 2^k + 1 + \sum_{i=0}^{r-1} \left\lfloor \frac{m-i}{2^k} \right\rfloor.$$

In particular, if $t = 2^k - 1$, then for $m \geq k$ we conjecture

$$\ell_t(m) = km + k - 2^k + 1.$$

3.1.2 Compositions with a Bounded Number of Parts

Fix a positive integer t, and consider the class $\mathrm{Age}(\omega^t)$ of compositions that have at most t parts. For $m \leq t$, $\mathrm{Age}(\omega^t)$ contains all compositions of size m, and the composition w_m is in $\mathrm{Age}(\omega^t)$, so the smallest proper $\mathrm{Age}(\omega^t)_m$-universal compositions are smallest m-universal compositions for $m \leq t$.

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Proposition 3-7. *For all $m \leq t$, the smallest proper $\mathrm{Age}(\omega^t)_m$-universal compositions have size $\ell(m)$.*

Given an $(m-1)$-universal composition c for $\mathrm{Age}(\omega^t)$, we may form an m-universal composition for $\mathrm{Age}(\omega^t)$ of the same length by incrementing each of its parts, as the next result shows.

Proposition 3-8. *For all $m > t$, if c is a $\mathrm{Age}(\omega^t)_{m-1}$-universal composition, then the composition c^{+1} formed by incrementing each part of c is m-universal for $\mathrm{Age}(\omega^t)$.*

Before presenting the proof of Proposition 3-8, we note that it implies the existence of proper $\mathrm{Age}(\omega^t)_m$-universal compositions of size $t(m-t) + \ell(t)$ by simply incrementing each part of w_t a total of $m-t$ times. For any composition c, let c^{+n} denote the composition formed by adding n to each part of c. A proper universal composition for $\mathrm{Age}(\omega^7)$ formed in this way is drawn in Figure 3-5.

Corollary 3-9. *For all $m > t$, $w_t^{+(m-t)}$ is a proper $\mathrm{Age}(\omega^t)_m$-universal of size $tm - t^2 + \ell(t)$.*

Proof of Proposition 3-8. Suppose that $m > t$ and that c is a proper $\mathrm{Age}(\omega^t)_{m-1}$-universal composition. Let c^{+1} be the composition formed by incrementing each part of c and let $d = d(1) \cdots d(n)$ be an arbitrary composition in $\mathrm{Age}(\omega^t)$ of size m. As $m > t$, there is some part $d(j)$ of d such that $d(j) > 1$. Thus, the composition d_j^{-1} formed from d by decrementing its jth part is a composition of size $m-1$ in $\mathrm{Age}(\omega^t)$, and thus embeds into c, ie. there is a subsequence $c(i_1)c(i_2) \cdots c(i_n)$ of c such that $d_j^{-1}(k) \leq c(i_k)$ for all k. Then, as $d(k) \leq d_j^{-1}(k) + 1$ for all k, we have that $d(k) \leq c(i_k) + 1$ for all k, so d embeds into c^{+1}, completing the proof. $\qquad\square$

In the case that $t = 2$, Corollary 3-9 shows that there are proper $\mathrm{Age}(\omega^2)_m$-universal compositions of size $2m - 1$. In fact, the smallest $\mathrm{Age}(\omega^2)_m$-universal compositions of arbitrary length have this size as well:

Proposition 3-10. *The smallest (proper) $\mathrm{Age}(\omega^2)_m$-universal compositions have size $2m - 1$.*
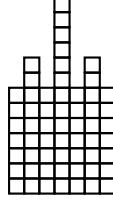
Figure 3-5. The proper $\mathrm{Age}(\omega^7)_{13}$-universal composition w_7^{+6} of length 7 and size 59.

Proof. Let $c = c(1) \cdots c(\ell)$ be an $\mathrm{Age}(\omega^2)_m$-universal composition for $m \geq 2$. As c must contain the composition m, there must be some j such that $c(j) \geq m$. Let a be the size of the largest part before $c(i)$ and 0 if $i = 1$. Similarly, let b be the size of the largest part after $c(i)$ and 0 if $i = \ell$. There are now two cases to consider: if $a < m - 1$ and if $a \geq m - 1$. If $a < m - 1$, then c contains the composition $(a+1)(m-a-1)$. As a is the largest part to the left of $c(i)$, we must have $b \geq m - a - 1$. Thus

$$
\begin{aligned}
|c| &\geq a + m + b \\
&\geq a + m + (m - a - 1) \\
&= 2m - 1.
\end{aligned}
$$

On the other hand, if $a \geq m - 1$, then we have

$$
\begin{aligned}
|c| &\geq a + m + b \\
&\geq (m - 1) + m + 0 \\
&= 2m - 1,
\end{aligned}
$$

as well, completing the proof. $\qquad\square$

Computational evidence indicates that each smallest m-universal composition for $\mathrm{Age}(\omega^t)$ for $m > t$ can be constructed with Proposition 3-8, and thus we make the following conjecture.

Conjecture 3-11. *For $m > t$, the smallest proper m-universal compositions for* $\mathrm{Age}(\omega^t)$ *have size* $tm - t^2 + \ell(t)$.

In the case that $t = 3$, Corollary 3-9 shows that there are proper $\mathrm{Age}(\omega^3)_m$-universal

compositions of size $3m-4$, and empirical evidence suggests—and we conjecture—that the smallest $\text{Age}(\omega^3)_m$-universal compositions of arbitrary length have this size as well. However, for $t \geq 4$, computations reveal that the smallest $\text{Age}(\omega^t)_m$-universal compositions are strictly smaller than the proper $\text{Age}(\omega^t)_m$-universal composition presented in Corollary 3-9, and thus the size of smallest $\text{Age}(\omega^t)_m$-universal compositions for $t > 4$ is unknown.

Conjecture 3-12. *For all $m \geq 3$, there are smallest $\text{Age}(\omega^3)_m$-universal compositions of length* 3.

3.1.3 Compositions with at Most One Large Part

As one final case study, consider the class $\mathcal{C} = \text{Age}(1^\omega \omega 1^\omega)$ of compositions with at most one part of size greater than 1.

Theorem 3-13. *For $m \geq 3$, the smallest (proper) $\mathcal{C}_m$-universal compositions have size $3m-4$.*

Proof. To begin, we claim that the composition $c = 1^{m-2}m1^{m-2}$ is a proper $\mathcal{C}_m$-universal composition. Let d be an arbitrary composition in $\mathcal{C}_m$. If d has a part that is at least 2, then it must decompose as $1^i j 1^k$, where $2 \leq j \leq m$ and both i and k are at most $m-2$. Thus d embeds into $1^{m-2}m1^{m-2}$. Otherwise, d must be the composition 1^m, and as the c has length $2m-3 \geq m$, we have that d embeds into c, and thus c is $\mathcal{C}_m$-universal.

Now, suppose c is any $\mathcal{C}_m$-universal composition. We claim that c must have size at least $3m-4$. As c contains the composition m, there must some index j so that $c(j) \geq m$. Let a denote the prefix of c before the part $c(j)$ and let b denote the suffix of c after the part $c(j)$ so that $c = ac(j)b$. Let $\text{h}(a)$ and $\text{h}(b)$ denote the largest parts of a and b, respectively. If $\text{h}(a) < m$, then c must contain the composition $(\text{h}(a)+1)1^{m-\text{h}(a)-1}$, and as the earliest that the part $\text{h}(a)+1$ may embed into c is into $c(i)$, the composition $1^{m-\text{h}(a)-1}$ must embed into b, and thus $\text{len}(b) \geq m - \text{h}(a) - 1$. If $\text{h}(a) \geq m$, then $\text{len}(b) \geq 0 > m - \text{h}(a) - 1$ anyways. Similarly, we must have $\text{len}(a) \geq m - \text{h}(b) - 1$ in any case. For any composition d, we have $|d| \geq \text{len}(d) + \text{h}(d) - 1$,

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and thus

$$|c| = c(i) + |a| + |b|$$
$$\geq m + \text{len}(a) + \text{h}(a) - 1 + \text{len}(b) + \text{h}(b) - 1$$
$$= m + (\text{len}(a) + \text{h}(b)) + (\text{len}(b) + \text{h}(a)) - 2$$
$$\geq m + (m-1) + (m-1) - 2$$
$$= 3m - 4,$$

as desired. □

3.2 Concluding Remarks

In its proof, Theorem 3-1 provides a recursive strategy for constructing smallest m-universal compositions, leading to a precise formula for their size asymptotic to $m \log m$. As in the case of integer partitions discussed in Chapter 2, any proper class of compositions $\mathcal{C}$ admits proper $\mathcal{C}_m$-universal compositions if and only if $\mathcal{C}$ satisfies the joint-embedding property. This property is equivalent to $\mathcal{C}$ being the age of some potentially infinite composition u, as the following result of Fraïssé shows:

Theorem 3-14 (Fraïssé [28]; see also Hodges [32, Section 7.1]). *The following are equivalent for a class $\mathcal{P}$ of integer partitions or compositions:*

1. *$\mathcal{P}$ cannot be expressed as the union of two proper subclasses,*

2. *$\mathcal{P}$ satisfies the <u>joint embedding property</u>, meaning that for every $a, b \in \mathcal{P}$ there is some $c \in \mathcal{P}$ such that $a, b \leq c$, and*

3. *$\mathcal{P} = \text{Age}(u)$ for some word $u \in (\mathbb{P} \cup \{n^\omega : n \in \mathbb{P}\} \cup \{\omega, \omega^\omega\})^*$.*

While Theorem 3-5 provides a recursive formula for the size of the smallest (proper) universal compositions for the classes of compositions whose part sizes are bounded, an explicit formula exists only in Conjecture 3-6. For the classes of compositions of bounded size, the both the sizes of the smallest proper and not-necessarily-proper universal compositions remain unknown, but Conjecture 3-11 provides a guess in the former case.

CHAPTER 4
GRAPHS

In this chapter, we consider the problem of universality for simple undirected graphs without loops. Given a graph G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. We call the number of vertices of G its <u>size</u>, denoted $|G|$, and denote its number of edges by $e(G)$. For a set S of vertices of G, the <u>subgraph of G induced by S</u> is the graph on vertex set S containing each edge in $E(G)$ in which both endpoints lie in S. We consider graphs up to isomorphism, where two graphs G and H are isomorphic if there is a bijection $\varphi : V(G) \to V(H)$ such that u is adjacent to v in G if and only if $\varphi(u)$ is adjacent to $\varphi(v)$ in H for all $u, v \in V(G)$. Finally, we say that a graph G <u>contains</u> another graph H if there is some set S of vertices of G that induces the graph H in G, and otherwise we say that G is H-free.

In [45], Rado introduced the concept of a <u>universal</u> graph, which he defined as a countably infinite graph that contained all finite graphs as induced subgraphs. Building on this idea, Moon [42] defined a graph to be m-universal if it contained all graps on m finite graphs that contain all finite graphs on m vertices. Moon observed that there are at least $2^{\binom{m}{2}}/m!$ graphs on m vertices, as there are $2^{\binom{m}{2}}$ labeled graphs on m vertices and at most $m!$ of these correspond to each unlabeled graph. Thus, any m-universal graph of size n must have at least $2^{\binom{m}{2}}/m!$ subsets of size m, so

$$\binom{n}{m} \geq \frac{2^{\binom{m}{2}}}{m!}.$$

Using the inequality $\binom{n}{m} \leq n^m/m!$, it follows that $n \geq 2^{(m-1)/2}$. Moreover, Moon constructed an m-universal graph $\gamma(m)$, where the size of $\gamma(m)$ is given by

$$|\gamma(m)| \leq \begin{cases} m \cdot 2^{(m-1)/2} & \text{if } m \text{ is odd,} \\ \dfrac{3}{2\sqrt{2}} \, m \cdot 2^{(m-1)/2} & \text{if } m \text{ is even} \end{cases}$$

Building on decades of intermediate results, Alon [5] proved that the smallest m-universal graphs have size asymptotic to $2^{(m-1)/2}$.

4.1 Proper Classes of Graphs

In the remainder of this chapter, we examine the problem of universality for proper classes of graphs. Before proceeding, we introduce some definitions that will prove necessary. A clique is a graph with all possible edges. Given graphs G and H, the complement of G, denoted co-G, is the graph on the same vertex set that contains precisely those edges that are not present in G. The union of G and H, denoted $G + H$, is the graph whose vertex set is the disjoint union of the vertex sets of G and H and whose edge set consists of all the edges from both G and H. Similarly, the join of G and H, denoted $G \times H$, is the graph whose vertex set is the disjoint union of the vertex sets of G and H and whose edge set consists of all the edges from both G and H as well as every edge connecting a vertex from G to a vertex from H.

In [6], Alstrup and Rauhe present a simple argument lower bounding the size of a universal graph for any set of graphs.

Observation 4-1 (Alstrup and Rauhe [6])**.** *For any set of graphs $\mathcal{G}$, the size of a $\mathcal{G}_m$-universal graph must be at least $\dfrac{m}{e} |\mathcal{G}_m|^{1/m}$.*

Proof. Let H be a $\mathcal{G}_m$-universal graph, with our aim being to show that $|H| \geq \frac{m}{e} |\mathcal{G}_m|^{1/m}$. The number of induced subgraphs of size m in G is bounded below by $\binom{|H|}{m}$, as each subgraph of H of size m corresponds to at least one subset of m vertices of H, and thus $\binom{|H|}{m} \geq |\mathcal{G}_m|$. As $\left(\frac{e \cdot |H|}{m}\right)^m \geq \binom{|H|}{m}$, we have $|H| \geq \frac{m}{e} |\mathcal{G}_m|^{1/m}$, as desired. $\qquad\square$

In [39], Lozin and Rudolf define a sense of optimality for universal graphs, which we now present. To this end, they begin with the same observation of Alstrup and Rauhe above, noting that if $\mathcal{G}$ is a class of graphs and H is a $\mathcal{G}_m$-universal graph, then we must have

$$|\mathcal{G}_m| \leq \binom{|H|}{m}$$

as each graph in $\mathcal{G}_m$ must correspond to at least one subset of m vertices of G. As $\binom{|H|}{m} \leq |H|^m$, we have

$$|\mathcal{G}_m| \leq |H|^m.$$

This inequality, together with $|H| \geq m$, which is true if $\mathcal{G}$ is infinite and thus each $\mathcal{G}_m$ is nonempty, gives the pair of inequalities

$$\log|\mathcal{G}_m| \leq m\log|H|$$

$$m\log m \leq m\log|H|.$$

Finally, we say that a sequence of graphs H_m that are $\mathcal{G}_m$-universal is <u>asymptotically optimal</u> if

$$\lim_{m\to\infty} \frac{m\log|H_m|}{\max\{\log|\mathcal{G}_m|, m\log m\}} = 1$$

and <u>optimal in order</u> (<u>order-optimal</u>) if there is a constant c such that for any $m \geq 1$,

$$\frac{m\log|H_m|}{\max\{\log|\mathcal{G}_m|, m\log m\}} \leq c.$$

In [35], Kannan, Naor, and Rudich ask if every graph class $\mathcal{G}$ with $\log|\mathcal{G}_m| = O(m\log m)$ admits universal graphs of polynomial size. Spinrad [48] called the (unproven) affirmation to this question the <u>implicit graph conjecture</u> and provided a generalization to classes of all sizes:

Generalized implicit graph conjecture. For every graph class $\mathcal{G}$, there are $\mathcal{G}_m$-universal graphs H_m with $\log|H_m| = O(\log|\mathcal{G}_m|/m)$.

In the language of Lozin and Rudolf above, the generalized implicit graph conjecture may be restated:

Generalized implicit graph conjecture (equivalent). Every graph class admits an order-optimal sequence of universal graphs.

4.1.1 Cluster Graphs

Let $\mathcal{C}$ be the class of <u>cluster</u> graphs, which are precisely those graphs that are disjoint unions of cliques. Equivalently, $\mathcal{C}$ is the class of P_3-free graphs. We begin by noting that $\mathcal{C}$ is isomorphic to the poset of integer partitions. This means that the unique smallest universal partition constructed in Theorem 2-2 translates to a smallest proper universal graph for $\mathcal{C}$ of the same size:

Proposition 4-2. *For all integers $m \geq 0$, the smallest proper $\mathcal{C}_m$-universal graphs have size*

$$\mathrm{u}_{\mathcal{C}}^p(m) = \phi(m) = \left\lfloor \frac{m}{1} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + \cdots + \left\lfloor \frac{m}{m} \right\rfloor.$$

We will show that no smaller $\mathcal{C}_m$-universal graph may be found by searching outside the class by showing that any graph may be transformed into a cluster graph of the same size without losing any of its contained cluster graphs.

Proposition 4-3. *Given any graph G of size n, there is a cluster graph of size n that contains every cluster graph contained in G.*

Proof. Let $S \subseteq V(G)$ be a set of vertices that induce a clique of maximum size in G, and let H be the graph formed from G by deleting each edge in which precisely one endpoint lies in S. We claim that H contains every cluster graph contained in G. Let g be a cluster graph contained in G, and fix some embedding ι of g into G. Note that at most one clique of g uses vertices of S in G. If (the fixed embedding of) g uses no vertices of S, then the same ι may be used to embed g into H. Otherwise, whichever clique of g uses vertices of S may be embedded <u>entirely</u> within S in G. Let ι' be this embedding. Then ι' may be used to embed g into H as well, and thus H contains each cluster graph contained in G.

By induction, we can transform the graph H_1 induced by remaining vertices $V(H) \setminus S$ into a cluster graph C_1 while retaining each of the cluster graphs contained in H_1. The union of C_1 with the clique S is thus a cluster graph that contains each cluster graph that G does, completing the proof. $\qquad\square$

In particular, Proposition 4-3 implies that any $\mathcal{C}_m$-universal graph may be transformed into a proper $\mathcal{C}_m$-universal graph of the same size.

Corollary 4-4. *For all integers $m \geq 0$, the smallest $\mathcal{C}_m$-universal graphs have size*

$$\mathrm{u}_{\mathcal{C}}(m) = \phi(m) = \left\lfloor \frac{m}{1} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + \cdots + \left\lfloor \frac{m}{m} \right\rfloor.$$

The number of cluster graphs of size n is equal to the number of partitions of size n, and thus $|\mathcal{C}_n| \sim \dfrac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3})$ by a result of Hardy and Ramanujan [31]. From this, it follows that the unique sequence of smallest proper $\mathcal{C}_m$-universal graphs of size $\phi(m) \sim m\log m$ is asymptotically optimal.

Proposition 4-3 also implies two linear lower bounds on the size of universal graphs for classes that contain certain disjoint unions of cliques:

Corollary 4-5. *If the graph G contains $kK_{\lfloor m/k\rfloor}$ for $1 \le k \le \ell$, then*

$$|G| \ge \left\lfloor \frac{m}{1} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + \cdots + \left\lfloor \frac{m}{\ell} \right\rfloor.$$

Proof. Suppose that the graph G contains the graph $\left\lfloor \frac{m}{k} \right\rfloor K_k$ for $1 \le k \le \ell$. By Proposition 4-3, we may assume that G is a cluster graph. As the poset of cluster graphs is isomorphic to the poset of integer partitions, it suffices to show that if $p = p(1)\cdots p(t)$ is a partition that contains $\left\lfloor \frac{m}{k} \right\rfloor^k$ for all $1 \le k \le \ell$, then $|p| \ge \lfloor m/1 \rfloor + \lfloor m/2 \rfloor + \cdots + \lfloor m/\ell \rfloor$. For each k, the fact that p contains $\left\lfloor \frac{m}{k} \right\rfloor^k$ is equivalent to $p(k) \ge \left\lfloor \frac{m}{k} \right\rfloor$, and thus

$$
\begin{aligned}
|p| &= p(1) + p(2) + \cdots + p(n) \\
&\ge p(1) + p(2) + \cdots + p(\ell) \\
&\ge \left\lfloor \frac{m}{1} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + \cdots + \left\lfloor \frac{m}{\ell} \right\rfloor,
\end{aligned}
$$

as desired. $\qquad\square$

Corollary 4-6. *If the graph G contains $\left\lfloor \frac{m}{k} \right\rfloor K_k$ for $1 \le k \le \ell$, then*

$$|G| \ge \left\lfloor \frac{m}{1} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + \cdots + \left\lfloor \frac{m}{\ell} \right\rfloor.$$

Proof. Suppose that the graph G contains the graph $\left\lfloor \frac{m}{k} \right\rfloor K_k$ for $1 \le k \le \ell$. By Proposition 4-3, we may assume that G is a cluster graph. As the poset of cluster graphs is isomorphic to the poset of integer partitions, it suffices to show that if $p = p(1)\cdots p(t)$ is a partition that contains $k^{\lfloor m/k \rfloor}$ for all $1 \le k \le \ell$, then $|p| \ge \lfloor m/1 \rfloor + \lfloor m/2 \rfloor + \cdots + \lfloor m/\ell \rfloor$.

For any partitions q, r, we have $q \leq r$ if and only if $q^{\text{conj}} \leq r^{\text{conj}}$. The conjugate of $k^{\lfloor m/k \rfloor}$ is $\lfloor \frac{m}{k} \rfloor^k$, so if $p^{\text{conj}} = p'(1)p'(2)\cdots p'(s)$, then p^{conj} contains $\lfloor \frac{m}{k} \rfloor^k$ for each $1 \leq k \leq \ell$, and thus $p'(k) \geq \lfloor \frac{m}{k} \rfloor$ for each $1 \leq k \leq \ell$. Finally, we have

$$
\begin{aligned}
|p| = \left| p^{\text{conj}} \right| \\
= p'(1) + p'(2) + \cdots + p'(n) \\
\geq p'(1) + p'(2) + \cdots + p'(\ell) \\
\geq \left\lfloor \frac{m}{1} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + \cdots + \left\lfloor \frac{m}{\ell} \right\rfloor,
\end{aligned}
$$

as desired. $\qquad\square$

In the case that $\ell = 2$, Corollary 4-6 shows that any graph that contains mK_1, $\lfloor \frac{m}{2} \rfloor K_2$, and $\lfloor \frac{m}{3} \rfloor K_3$ has size at least $m + \lfloor \frac{m}{2} \rfloor + \lfloor \frac{m}{3} \rfloor$, improving upon and generalizing the following result, which appears in [26].

Claim 4-1 (Esperet, Labourel, and Ochem [26, Claim 1]). *Let $\mathcal{F}$ denote the class of graphs with maximum degree 2. Every $\mathcal{F}_m$-universal graph has size at least $11\lfloor m/6 \rfloor$.*

4.1.2 Bipartite Permutation Graphs

Let $\mathcal{G}$ be the class of bipartite permutation graphs. Then $\mathcal{G}$ is the collection of $\{S_{2,2,2}, \text{Sun}_3, \Phi, C_3, C_5, C_6, \cdots\}$-free graphs, where the graphs $S_{2,2,2}$, Sun_3, and Φ are drawn in Figure 4-1. In [39], Lozin and Rudolf explicitly construct a proper $\mathcal{G}_m$-universal of size m^2. It is

$S_{2,2,2}$ Sun_3 Φ
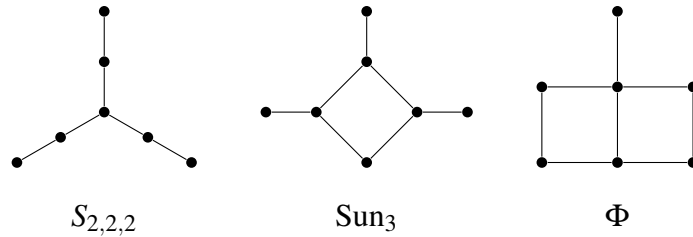
Figure 4-1. The graphs that, in addition to the cycles $\{C_3, C_5, C_6, \dots\}$, comprise the set of minimal forbidden subgraphs of the class of bipartite permutation graphs.

well known that $\log|\mathcal{G}_m| \sim \frac{m}{2} \log m$[1], so their construction is order-optimal. The proper

---

[1]The number of bipartite permutation graphs of size n is sequence A000085 in the OEIS [49].

$\mathcal{G}_5$-universal graph of their construction is drawn in Figure 4-2.
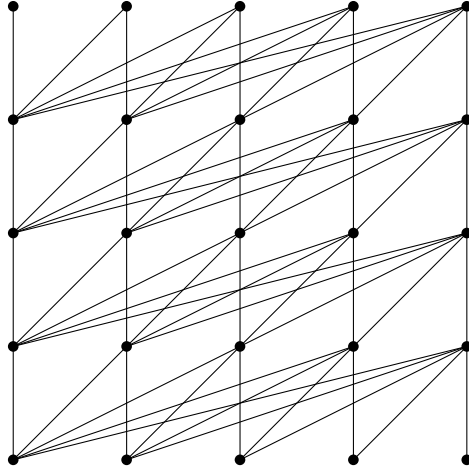

Figure 4-2. The $\mathcal{G}_5$-universal graph constructued of Lozin and Rudolf's construction [39].

In [4], Alecu, Lozin, and Malyshev prove that every proper $\mathcal{G}_m$-universal graph has $\Omega(m^\alpha)$ vertices for all $\alpha < 2$. Moreover, they conjecture that the minimum size of a proper $\mathcal{G}_m$-universal graph is $\Omega(m^2)$.

4.1.3   Caterpillar Forests

A <u>tree</u> is a connected acyclic graph, and a <u>forest</u> is a disjoint union of trees, ie. an acyclic graph. A <u>leaf</u> is a vertex of degree 1. A connected graph is called a <u>caterpillar</u> if its set of non-leaf vertices vertices induces a path. Every caterpillar is a tree, and a disjoint union of caterpillar graphs is called a <u>cateprillar forest</u>. Let $\mathcal{G}$ be the class of caterpillar forests, or equivalently, the class of acyclic $S_{2,2,2}$-free graphs, where the graph $S_{2,2,2}$ is drawn in Figure 4-1.

In [15], Bonichon, Gavoille, and Labourel construct graphs on at most $64m$ vertices that contain all caterpillars of size m. There are precisely $2^{m-4} + 2^{\lfloor (m-4)/2 \rfloor}$ caterpillars of size m, so their construction is asymptotically optimal.

4.1.4   Linear Forests

Let $\mathcal{G}$ be the class of graphs that are disjoint unions of paths. Equivalently, $\mathcal{G}$ is the the set of acyclic $K_{1,3}$-free graphs. In [1], Abrahamsen, Alstrup, Holm, Knudsen, and Stöckel construct a $\mathcal{G}_m$-universal graph of size $\lfloor 3m/2 \rfloor$, which they note is best-possible by extracting a lower bound of $\lfloor 3m/2 \rfloor$ from the proof of Claim 4-1 in [26]. Alternatively, this same lower bound is implied by

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Corollary [4-6], as any $\mathcal{G}_m$-universal graphs contains both mK_1 and $\lfloor \frac{m}{2} \rfloor K_2$.
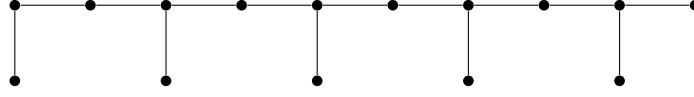

Figure 4-3. The $\mathcal{G}_{10}$-universal graph of size 15 of Abrahamsen,
Alstrup, Holm, Knudsen, and Stöckel's construction [1].

Linear forests of size n are in bijection with partitions of size n, and thus
$|\mathcal{G}_n| \sim \frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3})$ by a result of Hardy and Ramanujan [31]. From this, it follows that
the sequence of $\mathcal{G}_m$-universal graphs constructed by Abrahamsen, Alstrup, Holm, Knudsen, and
Stöckel is asymptotically optimal.

4.1.5 Star Forests

We call complete bipartite graphs of the form $K_{1,n}$ <u>star</u> graphs or just <u>stars</u>. Disjoint unions
of star graphs are called <u>star forests</u>. Let $\mathcal{SF}$ be the class of star forests. Equivalently, $\mathcal{SF}$ is the
collection of acyclic P_4-free graphs.

In [4], Alecu, Lozin, and Malyshev show that the minimum size of an $\mathcal{SF}_m$-universal
bipartite permutation graph is $\Theta(m\log m)$. This implies that a proper $\mathcal{SF}_m$-universal graph must
have size $\Omega(m\log m)$ and that any $\mathcal{SF}_m$-universal graph must have size $O(m\log m)$. As the set of
star forests of size n is in bijection with the set of partitions of size n, we have that
$|\mathcal{SF}_n| \sim \frac{1}{4n\sqrt{3}} \exp(\pi\sqrt{2n/3})$, and thus the graphs constructed by Alecu, Lozin, and Malyshev
are asymptotically optimal.

4.1.6 Threshold Graphs

A <u>threshold graph</u> is a graph that may be constructed from the empty graph through a
sequence of unions and joins with a single vertex. Threshold graphs have been widely studied, so
much so that they serve as the subject of a 2003 book [40] by Spinrad. Let $\mathcal{T}$ denote the class of
non-empty threshold graphs. Below, we show that $\mathcal{T}$ is isomorphic to the poset of words over a
two-letter alphabet, discussed in generality in Section [1.1].

Proposition 4-7. *Define the map $\gamma \colon \{u,j\}^* \to \mathcal{T}$ recusively as follows:*

    1. $G(\varepsilon) = g$ is the graph on one vertex.

2. $G(vu) = G(v) + g$ is the union of $G(v)$ with a vertex.

3. $G(vj) = G(v) \times g$ is the join of $G(v)$ with a vertex.

Then γ is a poset isomorphism.

The map γ sends words of size m to graphs of size $m+1$, so we omit the 0-vertex graph from $\mathcal{T}$. Before proving Proposition 4-7, we present a useful lemma.

Lemma 4-1. *If G and H are graphs and g is the one-vertex graph, then*

1. $G + g \le H + g$ if and only if $G \le H$.

2. $G + g \le H \times g$ if and only if $G + g \le H$.

3. $G \times g \le H + g$ if and only if $G \times g \le H$.

4. $G \times g \le H \times g$ if and only if $G \le H$.

We now present the proof of Proposition 4-7.

Proof of Proposition 4-7. We begin by proving that if $v, w \in \{u, j\}^*$ with $v \le w$, then $\gamma(v) \le \gamma(w)$.

Leveraging the recursive nature of the map γ, we proceed by induction on the size v. As a base case, observe that $\gamma(\varepsilon)$ is the one-vertex graph, which is contained in every non-empty graph. Thus, assume that v is a non-empty word, let the final letter of v be ℓ, and write $v = v_0 \ell$. Let $w \in \{u, j\}^*$ be any word with $v \le w$. As v is a subsequence of w, we may write $w = w_0 \ell w_1$, where $v_0 \le w_0$. By induction, we have $\gamma(v_0) \le \gamma(w_0)$. By Lemma 4-1, we have $\gamma(v_0 \ell) \le \gamma(w_0 \ell)$ no matter the value of ℓ. Finally, as $\gamma(w_0 \ell) \le \gamma(w_0 \ell w_1)$, so

$$\gamma(v_0 \ell) \le \gamma(w_0 \ell w_1),$$

as desired.

For the converse, we aim to show that if $\gamma(v)$ and $\gamma(w)$ are graphs with $\gamma(v) \le \gamma(w)$, then $v \le w$. Again, we proceed by induction on the size of $\gamma(v)$ and begin by noting that the one-vertex

graph $\gamma(\varepsilon)$ is contained in every graph in $\mathcal{T}$, and the empty word ε is contained in every word in $\{u, j\}^*$, so the base case is satisfied.

Let $\gamma(v)$ and $\gamma(w)$ be graphs with $\gamma(v) \leq \gamma(w)$ and $|\gamma(v)| \geq 2$, or equivalently, $|v| \geq 1$. Without loss of generality, assume that $v = v_0 u$ and $w = w_0 u j^k$ for some $k \geq 0$. By repeated applications of Lemma 4-1, the containment $\gamma(v_0 u) \leq \gamma(w_0 u j^k)$ is equivalent to $\gamma(v_0) \leq \gamma(w_0)$, and by induction we have $v_0 \leq w_0$, and thus $v_0 u \leq w_0 u j^k$, completing the proof. $\qquad\square$

One consequence of this isomorphism is that the smallest universal words constructed in Proposition 1-1 translate into smallest proper universal threshold graphs of size $2m - 1$. Proper $\mathcal{T}_m$-universal graphs of this size were first constructed by Hammer and Kelmans in [30].

Corollary 4-8 (Hammer & Kelmans [30])**.** *The smallest proper $\mathcal{T}_m$-universal graphs have size* $2m - 1$.

Any $\mathcal{T}_m$-universal graph contains both a clique and a co-clique of size m, and these subgraphs must not intersect in more than one vertex. Thus any $\mathcal{T}_m$-universal graph must contain at least $2m - 1$ vertices, meaning the smallest proper $\mathcal{T}_m$-universal graphs derived from Proposition 1-1 are smallest $\mathcal{T}_m$-universal graphs as well:

Corollary 4-9 (Hammer & Kelmans [30])**.** *The smallest (proper) $\mathcal{T}_m$-universal graphs have size* $2m - 1$.

4.1.7   Split Permutation Graphs

A split graph is a graph whose vertex set may be partitioned into a clique and a co-clique. Let $\mathcal{G}$ be the class of split permutation graphs. Equivalently, $\mathcal{G}$ is also the class of $\{2K_2, C_4, C_5, \text{net}, \text{co-net}, \text{rising sun}, \text{co-rising sun}\}$-free graphs, where the net and rising sun graphs as well as their complements, the co-net and co-rising sun graphs, are drawn in Figure 4-4. In [8], Atminas, Kitaev, Lozin, and Valyuzhenich construct a proper $\mathcal{G}_m$-universal graph of size $4m^3$, and in [17], Brignall, Lozin, and Stacho construct a $\mathcal{G}_m$-universal graph of size m^2.

4.2   Concluding Remarks

Just two years after Moon introduced m-universal graphs, Breur [16] introduced adjacency labeling schemes, which aim to efficiently encode the adjacencies of graphs. Informally, an
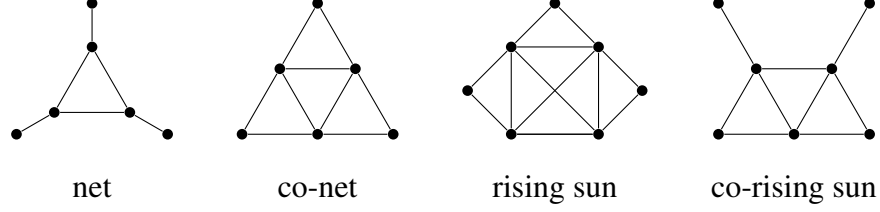
Figure 4-4. The forbidden subgraphs for the class of split permutation graphs.

adjacency labeling scheme for a graph G is a labeling of the vertices so that the adjacency of any two vertices may be recovered by knowing only their labels.

More formally, a *b-bit adjacency labeling scheme* for a family of graphs $\mathcal{F}$ is (1) an assignment of a length-b bitstring to each vertex in each graph in F and (2) some method so that, given two vertices u, v of a graph G in F, we can determine the adjacency between u and v in G given only their labels[2]. More than three decades after universal graphs and adjacency labelings schemes were introduced, Kannan, Naor, and Rudich [34] proved their equivalence: a family of graphs F admits a b-bit labeling scheme if an only if there is an F-universal graph with 2^b vertices.

To conclude, we would be remiss if we did not remark upon the substantial efforts to characterize *m-universal* graphs. We say that a set of vertices in a graph G is homogenous if it induces either a clique or a co-clique in G. The following sequence of results show that, with various interpretations, graphs that contain only small homogenous sets are universal. Let $\hom(G)$ denote the size of the largest homogenous subset of G, and define

$$\hom(n,m) = \min\{\hom(G) : G \text{ of size } n \text{ not } m\text{-universal}\}$$

Equivalently, $\hom(n,m)$ is the largest integer so that for all graphs G on n vertices, the inequality $\hom(G) < \hom(n,m)$ implies that G is m-universal.

In [23], Erdős and Hajnal show that for all integers m, for all positive constants c, and for all n sufficiently large, $\hom(n,m) \geq c\log(n)$. In [24], they go on to improve this result, proving the

---

[2]Some scholars insist that the algorithm that recovers the adjacency of *u* and *v* be computable in polynomial time, but we do not.

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following theorem.

Theorem 4-10 (Erdős and Hajnal [24]). *Let m be a positive integer, and let $0 < c < 1/m$. There there is some $n_0 = n_0(m, c)$ such that for all $n > n_0$, we have*

$$\text{hom}(n, m) \geq e^{c\sqrt{\log n}/2}.$$

In [44], Prömel and Rödl allowed the "level" of universality to vary with the size of the graph, proving the following.

Theorem 4-11 (Prömel and Rödl [44]). *For any $c_1 > 0$, there is some $c_2 > 0$ such that for all n, we have*

$$\text{hom}(n, c_2 \log n) \geq c_1 \log n.$$

In [27], Fox and Sudakov generalize Theorem 4-11:

Theorem 4-12 (Fox and Sudakov [27]). *There are constants $c_1, c_2 > 1$ such that for all n, m, we have*

$$\text{hom}(n, m) \geq c_1 2^{c_2 \sqrt{\frac{\log n}{m}}} \log n.$$

Another structural description of *m*-universal graphs came in [19], wherein Chung and Graham give alternative sufficient conditions for a graph to be *m*-universal:

Theorem 4-13 (Chung and Graham [19]). *Let m be a positive integer. If the graph G of size n is not m-universal, then there is some induced subgraph $H \subseteq G$ of size $\lfloor n/2 \rfloor$ so that*

$$\left| e(H) - \frac{n^2}{16} \right| > 2^{-2(m^2+27)} n^2.$$

Restated, Theorem 4-13 says that graphs that are not universal must contain a subgraph that has an edge density that strongly deviates from what is "typical". Let $D(n, m)$ denote the largest integer such that every graph G on n vertices that is not m-universal contains a subgraph H of size $\lfloor n/2 \rfloor$ with $\left| e(H) - \frac{1}{16} n^2 \right| > D(n, m)$. The above result shows that $D(n, m) \geq 2^{-2(m^2+27)} n^2$ for sufficiently large n. In [27], Fox and Sudakov use Theorem 4-12 to show the following result.

Proposition 4-14. *There are constants $c_3, c_4 > 1$ such that $c_3^{-m} n^2 < D(n,m) < c_4^{-m} n^2$ for all positive integers n, m.*

CHAPTER 5
PERMUTATIONS

From our perspective, a permutation of size n is a sequence of length n taken from the alphabet $\{1, 2, \ldots, n\}$ in which each letter occurs precisely once. To define containment, we say that two sequences u and v of the same length are order-isomorphic if for all indices i and j we have

$$u(i) > u(j) \iff v(i) > v(j).$$

Finally, the permutation π contains another permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(m)$ as a pattern if there is a subsequence $\pi(i_1)\pi(i_2)\cdots\pi(i_m)$ of π that is order-isomorphic to σ. If π does not contain σ, then we say that π avoids σ.

It is helpful to identify a permutation $\pi = \pi(1)\cdots\pi(n)$ with its plot. A plot of π is a set of points in the plane, no two with the same x- or y-value, so that the sequence of y-values read from left to right are order-isomorphic to π, and the (canonical) plot of π is the set of points $\{(i, \pi(i))\}_{i=1}^{n}$. This graphical representation of π helps to visualize the pattern-containment order: π contains σ if a subset of the plot of π constitutes a plot of σ. An example of containment displayed through plots is exhibited in Figure 5-1.

$$2\,3 \quad 1 \qquad \leq \qquad \underline{2}\,\underline{5}\,3\,\underline{1}\,4$$
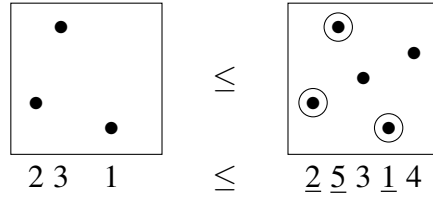
Figure 5-1. An example of permutation containment displayed through their plots.

Before introducing the universality problem for permutations, we give a few definitions that will be helpful throughout this chapter. Given permutations π and σ of respective sizes m and n, their direct sum is the permutation $\pi \oplus \sigma$ of size $m+n$ defined by

$$(\pi \oplus \sigma)(i) = \begin{cases} \pi(i) & \text{if } 1 \leq i \leq m, \\[2mm] \sigma(j-m) + m & \text{if } m+1 \leq i \leq m+n. \end{cases}$$

Pictorially, the plot of $\pi \oplus \sigma$ consists of the plot of σ placed above and to the right of the plot of π, as shown on the left of Figure 5-2. Similarly, we define the skew sum of π and σ by

$$(\pi \ominus \sigma)(i) = \begin{cases} \pi(i) + n & \text{if } 1 \le i \le m, \\ \\ \sigma(j - m) & \text{if } m + 1 \le i \le m + n. \end{cases}$$

Pictorially, the plot of $\pi \ominus \sigma$ consists of the plot of σ placed below and to the right of the plot of π, as shown on the right of Figure 5-2. A permutation that cannot be written as a direct sum (resp. skew sum) is said to be sum-indecomposable (resp. skew-indecomposable.)

As usual, we say that a permutation is m-universal if it contains all permutations of size m as patterns. Such a permutation is sometimes called an m-superpattern (for example, by Bóna [14, Chapter 5, Exercises 19–22 and Problems Plus 9–12].) The first result about universal permutations was obtained by Simion and Schmidt in 1985 [47, Section 5], who computed the number of 3-universal permutations of size $n \ge 5$ to be

$$n! - 6C_n + 5 \cdot 2^n + 4\binom{n}{2} - 2F_n - 14n + 20.$$

(Here C_n denotes the nth Catalan number and F_n denotes the nth combinatorial Fibonacci number, so $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$.) Simion and Schmidt did not use the term "universal" and received their formula as a corollary to a sequence of results enumerating those permutation classes where membership is defined by avoiding sets of 3-patterns. The first to study the universal permutation problem for general m was Arratia [7] in 1999.

Let $u(m)$ denote the size of the smallest m-universal permutation. Arratia observed that for any permutation of size n to be m-universal, it must have at least $m!$ length-m subsequences, and
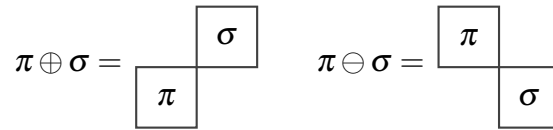

Figure 5-2. The plot of the direct sum of π and σ on the left and the plot of the skew sum of π and σ on the right.

thus

$$\binom{n}{m} \geq m!.$$

This simple inequality, together with the inequalities $\left(\frac{ne}{m}\right)^m \geq \binom{n}{m}$ and $m! \geq \left(\frac{m}{e}\right)^m$, implies that $n \geq m^2/e^2$. This also follows from Observation 4-1, again using that $m! \geq \left(\frac{m}{e}\right)^m$.

Arratia also provided a simple construction of an m-universal permutation of size m^2, proving that $u(m) = \Theta(m^2)$. Consider the permutation whose plot consists of an $m \times m$ grid of points, with sides initially parallel to the axes, then rotated slightly clockwise. More precisely, partition the integers 1 to m^2 into congruence classes modulo m, write each congruence class in increasing order, and then concatenate these sequences in decreasing order according to their first element. The left panel of Figure 5-3 displays the plot of the 4-universal permutation constructed in this manner. Arratia observed that such a permutation is m-universal, as the point $(i, \pi(i))$ of the pattern π of size m can simply embed into the point ith column (from the left) and $\pi(i)$th row (from the bottom). This kind of embedding actually allows for much more freedom in the universal permutations formed: dividing the square $(0, m^2] \times (0, m^2]$ into m^2 many $m \times m$ squares, this construction demonstrates that any permutation whose canonical plot places precisely 1 point in each sqaure is m-universal. We exhibit another 4-universal permutation of size 16 that meets this criteria in the right panel of Figure 5-3.
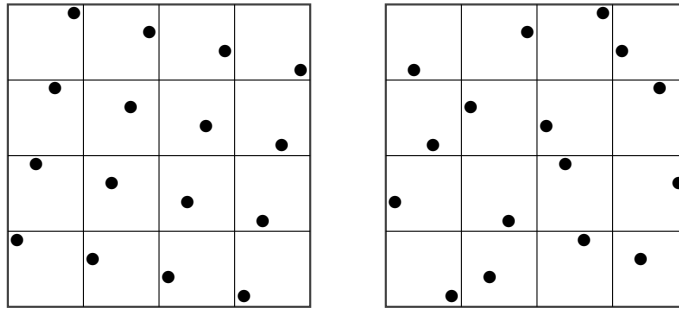

Figure 5-3. The 4×4 tilted-square permutation and another 4-universal permutation of size 16.

The first to improve upon these trivial bounds provided by Arratia were Eriksson, Eriksson, Linusson, and Wästlund [25], who used probabilistic methods to construct an m-universal permutation of size $(2/3 + o(1))m^2$. The next improvement came in [41], wherein Miller proves

that there is a word over the alphabet $[m+1]$ of length $(m^2+m)/2$ that contains subsequences order-isomorphic to every permutation of size m. Miller noted that by "breaking ties" between the letters of such a word, one obtains an m-universal permutation of size $(m^2+m)/2$.

Theorem 5-1 (Miller [41]). *For all $m \geq 1$, there is a word over the alphabet $[m+1]$ of length $(m^2+m)/2$ containing subsequences order-isomorphic to every permutation of size m.*

To establish this result, define the <u>infinite zigzag word</u> to be the word formed by alternating between ascending <u>runs</u> of the odd positive integers $1357\cdots$ and descending <u>runs</u> of the even positive integers $\cdots 8642$,

$$(1357\cdots)\ (\cdots 8642)\ (1357\cdots)\ (\cdots 8642)\ (1357\cdots)\ (\cdots 8642)\ \cdots.$$

While this object does not conform to most definitions of the word <u>word</u> in combinatorics, we hope the reader forgives the slight expansion of the definition adopted here. We are interested in the leftmost embeddings of words over $\mathbb{P}$ into the infinite zigzag word.

We also need two definitions. First, given a word $p \in \mathbb{P}^*$, we define the word $p^{+1} \in \mathbb{P}^*$ to be the word formed by adding 1 to each letter of p, so $p^{+1}(i) = p(i)+1$ for all indices i of p. Next we say that the word $p \in \mathbb{P}^*$ has an <u>immediate repetition</u> if there is an index i with $p(i) = p(i+1)$, i.e., if p contains a factor equal to $\ell\ell$ for some letter $\ell \in \mathbb{P}$.

Proposition 5-2 (Miller [41]). *If the word $p \in \mathbb{P}^m$ has no immediate repetitions, then either p or p^{+1} occurs as a subsequence of the first m runs of the infinite zigzag word.*

Before proving Proposition 5-2, note that permutations do not have immediate repetitions. Thus if π is a permutation of size m, Proposition 5-2 implies that either π or π^{+1} occurs as a subsequence in the first m runs of the infinite zigzag word. Since π^{+1} is order-isomorphic to π and both π and π^{+1} are words over $[m+1]$, this implies that the restriction of the first m runs of the infinite zigzag word to the alphabet $[m+1]$ contains every permutation of size m. For example, in the case of $m = 5$ we obtain the universal word

$$135\ 642\ 135\ 642\ 135$$

of length 15 over the alphabet $[6]$.

The restriction of the infinite zigzag word described above consists of m runs of average length $(m+1)/2$: if m is odd, then all runs are of this length, while if m is even, then half are of length $m/2$ and half are of length $(m+2)/2$. Thus Proposition 5-2 implies Theorem 5-1. While Proposition 5-2 does not appear explicitly in [41], its proof, presented below, is adapted from Miller's proof of Theorem 5-1.

Proof of Proposition 5-2. We define the <u>score</u> of the word $p \in \mathbb{P}^*$, denoted by $s(p)$, as the minimum number of runs that an initial segment of the infinite zigzag word must have in order to contain p, minus the length of p. Thus our goal is to show that for every word $p \in \mathbb{P}^*$ without immediate repetitions, either $s(p) \leq 0$ or $s(p^{+1}) \leq 0$. In fact, we show that for such words we have $s(p) + s(p^{+1}) = 1$, which implies this.

We prove this claim by induction on the length of p. For the base case, we see that words consisting of a single odd letter are contained in the first run of the infinite zigzag word (thus corresponding to scores of 0) while words consisting of a single even letter are contained in the second run (corresponding to scores of 1). Thus for every $\ell \in \mathbb{P}^1$ we have $s(\ell) + s(\ell^{+1}) = 1$, as desired. Now suppose that the claim is true for all words $p \in \mathbb{P}^m$ without immediate repetitions and let $\ell \in \mathbb{P}$ denote a letter. We see that, for any $p \in \mathbb{P}^m$,

$$
s(p\ell) - s(p) = \begin{cases} -1 & \begin{aligned} &\text{if } p(n) < \ell \text{ and both entries are odd or} \\[4pt] &\text{if } p(n) > \ell \text{ and both entries are even;} \end{aligned} \\[12pt] 0 & \begin{aligned} &\text{if } p(n) \text{ and } \ell \text{ are of different parity; or} \\[4pt] &\text{if } p(n) < \ell \text{ and both entries are even,} \end{aligned} \\[12pt] +1 & \begin{aligned} &\text{if } p(n) = \ell, \text{ or} \\[4pt] &\text{if } p(n) > \ell \text{ and both entries are odd.} \end{aligned} \end{cases}
$$

Because our words do not have immediate repetitions, we can ignore the possibility that $\ell = p(m)$. In the other cases, it can be seen by inspection that

$$\left(s(p\ell) - s(p)\right) + \left(s\left((p\ell)^{+1}\right) - s\left(p^{+1}\right)\right) = 0.$$

By rearranging these terms, we see that

$$s(p\ell) + s\left((p\ell)^{+1}\right) = s(p) + s\left(p^{+1}\right).$$

Since $s(p) + s\left(p^{+1}\right) = 1$ by induction, this completes the proof of the inductive claim, and thus also of the proposition. $\qquad\square$

Here, we give a new improvement to Miller's upper bound. In order to do so, we further restrict the infinite zigzag word, and then break ties between its letters to obtain a specific permutation ζ_m. To this end, we define the word z_m to be the restriction of the first m runs of the infinite zigzag word to the alphabet $[m]$. When m is even, each run of z_m has length $m/2$. When m is odd, z_m consists of $(m+1)/2$ ascending odd runs, each of length $(m+1)/2$, and $(m-1)/2$ descending even runs, each of length $(m-1)/2$. Thus we have

$$|z_m| = \begin{cases} \dfrac{m^2}{2} & \text{if } m \text{ is even,} \\[2ex] \dfrac{m^2 + 1}{2} & \text{if } m \text{ is odd.} \end{cases}$$

As one additional definition, we say that a word u is <u>order-homomorphic</u> to another word v of the same length if for all indices i and j, we have

$$u(i) > u(j) \implies v(i) > v(j).$$

Next, we choose a specific permutation, ζ_m, such that z_m is order-homomorphic to ζ_m. In constructing ζ_m, we have the freedom to break ties between equal letters of z_m. That is to say, if $z_m(i) = z_m(j)$ for $i \neq j$, then in constructing ζ_m we may choose whether $\zeta_m(i) < \zeta_m(j)$ or $\zeta_m(i) > \zeta_m(j)$ arbitrarily without affecting any other pair of comparisons and thus without losing

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any occurrences of permutations. We choose to break these ties by replacing all instances of a given letter $k \in [m]$ in z_m by a decreasing subsequence in ζ_m. Thus for indices $i < j$, we have

$$z_m(i) = z_m(j) \implies \zeta_m(i) > \zeta_m(j).$$

This choice uniquely determines ζ_m (up to order-isomorphism), as all comparisons between its letters are determined either in z_m, if the corresponding letters of z_m differ, or by the rule above, if the corresponding letters of z_m are the same. Figure 5-4 shows the plots of z_5 and ζ_5, where the plot of a word w over $\mathbb{P}$ is the set $\{(i, w(i))\}$ of points in the plane.
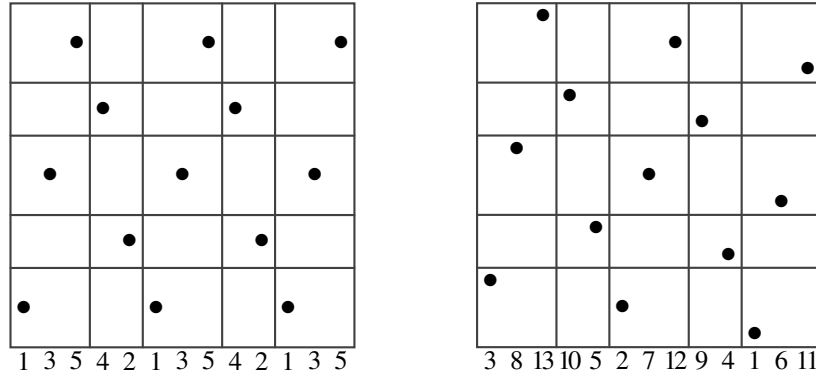

Figure 5-4. On the left, the further restriction we define of the infinite zigzag word, z_5. On the right, the normalized permutation formed by breaking ties, ζ_5.

In the following sequence of results, we show that ζ_m is almost universal. In fact, we show that ζ_m fails to be universal only for even m, and in that case, the only missing permutation is the decreasing permutation $m \cdots 21$. The first of these results, Proposition 5-3, covers almost all permutations. (In fact, Proposition 5-4 shows that Proposition 5-3 handles all but 2^{m-1} permutations of size m.)

We say that two entries $\pi(j)$ and $\pi(k)$ form an inverse-descent if $j < k$ and $\pi(j) = \pi(k) + 1$. (As the name is meant to indicate, if a pair of entries forms an inverse-descent in π, then the corresponding entries of π^{-1} form a descent.) If $\pi(j)$ and $\pi(k)$ form an inverse-descent and they are not adjacent in π (so $k \geq j + 2$), then we say that they form a distant inverse-descent.

Proposition 5-3. *If the permutation π of size m has a distant inverse-descent, then ζ_m contains a*

subsequence order-isomorphic to π.

Proof. Suppose that the entries $\pi(a)$ and $\pi(b)$ form a distant inverse-descent in π, meaning that $\pi(a) = \pi(b) + 1$ and $b \geq a + 2$. We define the word $p \in [m-1]^m$ by

$$
p(i) = \begin{cases} \pi(i) & \text{if } \pi(i) \leq \pi(b), \\[2ex] \pi(i) - 1 & \text{if } \pi(i) \geq \pi(a) = \pi(b) + 1. \end{cases}
$$

The word p has two occurrences of the letter $\pi(b)$, but because $\pi(a)$ and $\pi(b)$ form a distant inverse-descent, these two occurrences of $\pi(b)$ in p do not constitute an immediate repetition. Thus Proposition 5-2 shows that either p or p^{+1} occurs as a subsequence in the first n runs of the infinite zigzag word. As p and p^{+1} are both words over $[m]$, whichever of these words occurs in the first n runs of the infinite zigzag word also occurs as a subsequence of z_m. Suppose that this subsequence occurs in the indices $1 \leq i_1 < i_2 < \cdots < i_m \leq |z_m|$, so $z_m(i_1)z_m(i_2)\cdots z_m(i_m)$ is equal to either p or p^{+1}, and thus for $j, k \in [m]$ we have

$$
z_m(i_j) > z_m(i_k) \iff p(j) > p(k).
$$

Because z_m is order-homomorphic to ζ_m, this implies that for all pairs of indices $j, k \in [m]$ except the pair $\{a, b\}$, we have

$$
\zeta_m(i_j) > \zeta_m(i_k) \iff p(j) > p(k) \iff \pi(j) > \pi(k).
$$

Furthermore, since $p(a) = p(b)$, we have $z_m(a) = z_m(b)$, and so by our construction of ζ_m it follows that $\zeta_m(a) > \zeta_m(b)$, while we know that $\pi(a) > \pi(b)$ because those entries form an inverse-descent. This verifies that $\zeta_m(i_1)\zeta_m(i_2)\cdots\zeta_m(i_m)$ is order-isomorphic to π, completing the proof. $\qquad\square$

To describe the permutations that Proposition 5-3 does not apply to, we need the notion of a layered permutation. A permutation is said to be layered if it can be expressed as a sum of decreasing permutations, and in this case, these decreasing permutations are themselves called the
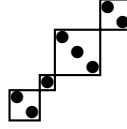
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Figure 5-5. The plot of the layered permutation 21 3 654 87 with layer lengths 2, 1, 3, 2.

layers. An example of a layered permutation is shown in Figure 5-5.

Proposition 5-4. *The permutation π is layered if and only if it does not have a distant inverse-descent.*

Proof. One direction is completely trivial: if π is layered then all of its inverse-descents are between consecutive entries, so it does not have a distant inverse-descent. For the other direction, we use induction on the size of π. The empty permutation is layered, so the base case holds. If π is a nonempty permutation without distant inverse-descents, then it must begin with the entries $\pi(1), \pi(1) - 1, \ldots, 2, 1$ in that order. This means that $\pi = \delta \oplus \sigma$ where δ is a nonempty decreasing permutation and σ is a permutation smaller than π that also does not have any distant inverse-descents. By induction, σ is layered, and thus π is as well, completing the proof. $\square$

Having characterized the permutations to which Proposition 5-3 does not apply, we now show that almost all of them are nevertheless contained in ζ_m.

Proposition 5-5. *If the permutation π of size n is layered and not a decreasing permutation of even size, then ζ_m contains a subsequence order-isomorphic to π.*

Proof. Let π denote an arbitrary layered permutation of size *n*. To prove the result, we compute the score of π as in the proof of Proposition 5-2, show that this score can only take on the values 0 or ± 1, and then describe an alternative embedding of π in ζ_m in the case where the score of π is 1, except when π is a decreasing permutation of even size.

Recall that the score of any word π, $s(\pi)$, is defined as the number of initial runs of the infinite zigzag word necessary to contain π minus the size of π. As observed in the proof of Proposition 5-2, the score of a word does not change upon reading a letter of opposite parity. This

implies that, while reading a layered permutation, the score changes only when transitioning from one layer to the next, and thus we compute the score of π layer-by-layer.
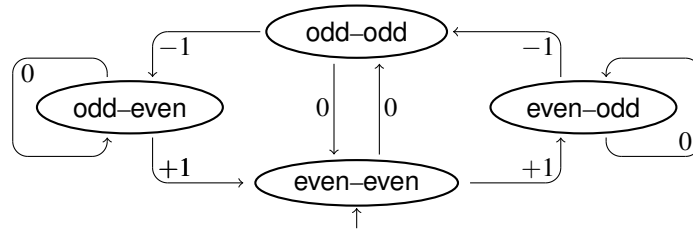

Figure 5-6. A directed graph describing the scoring of a layered permutation.

The change in score when moving from one layer of π to the next is determined by the parity of the last entry of the layer we are leaving and the first entry of the layer we are entering. Specifically, the score changes by -1 if both of these entries are odd and $+1$ if both are even. This shows that in order to compute the score of the layered permutation π, we simply need to know the parities of the first and last entries of each of its layers. This information is represented by the labels of the nodes of the directed graph shown in Figure 5-6.

Moreover, not all transitions between these nodes are possible, because the last entry of a layer is precisely 1 greater than the first entry of the preceding layer. This is why there are only eight edges shown in Figure 5-6. In this figure, each of those edges is labeled by the change in the score function. Note that the first layer must end with 1 (an odd entry), and its first entry must be either odd (for a score of 0) or even (for a score of 1); this is equivalent to starting our walk on the graph in Figure 5-6 at the node labeled (even–even) before any layers are read.

From this graphical interpretation of the scoring process, it is apparent that the score of a layered permutation can take on only three values: -1 if it ends at the node (odd–even); 0 if it ends at either node (even–even) or (odd–odd); or 1 if it ends at the node (even–odd). Except in this final case, we are done.

Now suppose that we are in the final case, so the ultimate layer of π is of (even–odd) type. The first entry of this layer is the greatest entry of π, so we know that π has even size. If π were a decreasing permutation then there would be nothing to prove (as we have not claimed anything in this case), so let us further suppose that π is not a decreasing permutation, and thus that π has at

least two layers. We further divide this case into two cases. In both cases, as in the proof of Proposition 5-3, we construct a word $p \in [m-1]^m$ such that if z_m contains p, then ζ_m contains π.

First, suppose that the penultimate layer of π is of (even–odd) type and that this layer begins with the entry $\pi(b)$. This implies that the penultimate layer of π has at least two entries (because its first and last entries have different parities). In this case, we define p by

$$
p(i) = \begin{cases} \pi(i) & \text{if } \pi(i) < \pi(b), \\[2ex] \pi(i) - 1 & \text{if } \pi(i) \geq \pi(b). \end{cases}
$$

In other words, to form p from π we decrement the first entry of the penultimate layer and all entries of the ultimate layer. Because the penultimate layer of π has at least two entries, performing this operation creates an immediate repetition (of the entry $\pi(b) - 1$) at the beginning of this layer. For example, if $\pi = 21\ 6543\ 87$ then $\pi(b) = 6$ and we decrement the 6, 8, and 7 to obtain the word $p = 21\ 5543\ 76$.

As with our previous constructions, if z_m contains an occurrence of p, then ζ_m will contain a copy of π. We establish that z_m contains p by showing that $s(p) = 0$, which requires a further bifurcation into subcases. In both subcases, the scoring of p is computed by considering its score in the antepenultimate layer (the layer immediately before the penultimate layer), the score change when reading the newly decremented first entry of the penultimate layer, the score penalty of $+1$ because p contains an immediate repetition (namely, $\pi(b) - 1$ occurs twice in a row), and finally the score change between the penultimate and ultimate layers. We label these cases by the final three nodes of the directed graph from Figure 5-6 visited while computing the score of π.

- The final three layers are of type (even–even)(even–odd)(even–odd). Note that this case includes the possibility that π has only two layers. If p has an antepenultimate layer, then the score while reading that layer is 0 and the ascent between its last entry and the newly decremented first entry of the penultimate layer is of different parity (even to odd), contributing 0 to the score. If p does not have an antepenultimate layer, then p begins with the newly decremented first entry of its penultimate layer, which contributes 0 to the score. In either case, the score of p is 0 upon reading the first entry of the penultimate layer. The immediate repetition in the penultimate layer contributes $+1$ to the score, while the ascent between the last entry of the penultimate layer and the newly decremented first entry of the

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ultimate layer is odd and thus contributes -1, so $s(p) = 0$.

- The final three layers are of type (even–odd)(even–odd)(even–odd). The score while reading the antepenultimate layer is $+1$. The ascent between the last entry of the antepenultimate layer and the newly decremented first entry of the penultimate layer is odd, so it contributes -1 to the score, the immediate repetition in the penultimate layer contributes $+1$, and the ascent between the last entry of the penultimate layer and the newly decremented first entry of the ultimate layer is odd and thus contributes -1, so $s(p) = 0$.

It remains to treat the case where the penultimate layer is of (even–even) type. Note that this case includes the possibility that the penultimate layer consists of a single entry. Suppose that the penultimate layer ends with the entry $\pi(a)$. We define p by

$$
p(i) = \begin{cases}
\pi(i) & \text{if } \pi(i) < \pi(a) \text{ or } \pi(i) = m, \\[2ex]
\pi(i) + 1 & \text{if } \pi(i) \geq \pi(a) \text{ and } \pi(i) \neq m.
\end{cases}
$$

Thus in forming p from π we increment all entries of the penultimate layer and all but the first entry of the ultimate layer. For example, if $\pi = 21\ 3\ 654\ 87$, then we increment the 6, 5, 4, and 7 to obtain the word $p = 21\ 3\ 765\ 88$.

As before, if z_m contains an occurrence of p then ζ_m will contain a copy of π. Thus we need only show that $s(p) = 0$, which we do, as in the previous case, by considering the scoring of the final three layers. As in that case, we identify two subcases.

- The final three layers are of type (odd–even)(even–even)(even–odd). The score while reading the antepenultimate layer is -1. The ascent between the last entry of the antepenultimate layer and the newly incremented first entry of the penultimate layer is of different parity (even to odd) and thus contributes 0 to the score. The ascent between the newly incremented last entry of the penultimate and the first entry of the ultimate layer (which is m) is of different parity (odd to even) and thus contributes 0 to the score. Finally, the immediate repetition at the beginning of the ultimate layer (the two entries equal to m) contributes $+1$ to the score, so $s(p) = 0$.

- The final three layers are of type (odd–odd)(even–even)(even–odd). The score while reading the antepenultimate layer is 0. The ascent between the last entry of the antepenultimate layer and the newly incremented first entry of the penultimate layer contributes -1 to the score (as both entries are now odd). The ascent between the newly incremented last entry of the penultimate layer and the first entry of the ultimate layer (which is n) is of different parity (odd to even) and thus contributes 0 to the score. Finally,

the immediate repetition at the beginning of the ultimate layer contributes $+1$ to the score, so $s(p) = 0$.

As we have considered all of the cases, the proof is complete. □

It remains only to conclude. The size of ζ_m is $(m^2+1)/2$ when m is odd and $m^2/2$ when m is even. When m is odd, we have established that ζ_m is m-universal. However, Proposition 5-5 shows that ζ_m need not be universal when m is even. (Indeed, it can be checked that ζ_m is <u>not</u> m-universal when m is even.) However, in this case, we know that ζ_m contains the decreasing permutation $(m-1)\cdots 21$ (for instance because it contains the permutation $(m-1)\cdots 21 \oplus 1$). Thus we obtain an m-universal permutation by prepending a new maximum entry to ζ_m, giving us the following bound.

Theorem 5-6 (Engen and Vatter [22]). *There is an m-universal permutation of size $\lceil (m^2+1)/2 \rceil$.*

A computer search reveals that the bound in Theorem 5-6 is best-possible for $m \le 5$. Alas, for $m = 6$ the 6-universal permutation of Theorem 5-6's construction has size 19, but Arnar Arnarson [private communication] has found that the permutation

$$6\ 14\ 10\ 2\ 13\ 17\ 5\ 8\ 3\ 12\ 9\ 16\ 1\ 7\ 11\ 4\ 15$$

of size 17 is 6-universal, and computations have shown that no smaller permutation suffices.

Arratia [7, Conjecture 2] conjectured that the size of the smallest m-universal permutations is asymptotic to m^2/e^2. In [18], Chroman, Kwan, and Singhal prove that any m-universal permutation must have size at least $(1.000076/e^2)m^2$ asymptotically, refuting Arratia's conjecture. In [25], Eriksson, Eriksson, Linusson, and Wästlund conjecture that the smallest m-universal permutations have size asymptotic to $m^2/2$, and in [7], Arratia presents Alon's conjecture that, asymptotically, most permutations of size $m^2/4$ are m-universal.

5.1 Proper Permutation Classes

In this section, we consider the problem of universality for proper permutation classes. For reference, some known or computed values of minimum sizes for universal permutations for

various classes are presented in Appendix C. Every class may be described as the set of permutations that avoid each element of a set of permutations, and the minimal such set is called the <u>basis</u> of the class. If $\mathcal{P}$ is the set of permutations that avoid the set of permutations S, then we write $\mathcal{P} = \text{Av}(S)$.

The pattern containment order is closely related to the induced subgraph order, a connection that frequently proves useful. If π is a permutation of size n, define the <u>inversion graph</u> of π, denoted $G(\pi)$, to be the graph with vertex set $[n]$ and edges $i \sim j$ if and only if $i < j$ and $\pi(i) > \pi(j)$. Then, if π contains τ as a pattern, then $G(\pi)$ contains $G(\tau)$ as an induced subgraph.

For any collection of permutations $\mathcal{P}$, let $G(\mathcal{P}) = \{G(\pi) : \pi \in \mathcal{P}\}$. Suppose that $\mathcal{G}$ is a class of graphs with $\mathcal{G} = G(\mathcal{P})$ for some permutation class $\mathcal{P}$. If the permutation π is $\mathcal{P}_m$-universal, then $G(\pi)$ is $\mathcal{G}(\mathcal{P})_m$-universal. Moreover, if π is a proper $\mathcal{P}_m$-universal permutation, then $G(\pi)$ lies in $\mathcal{G}$, so $G(\pi)$ is a proper $\mathcal{G}_m$-universal graph. This shows that $u_{\mathcal{G}}^p(m) \le u_{\mathcal{P}}^p(m)$, giving us the following rhombus of inequalities.

$$
\begin{array}{ccc}
u_{\mathcal{G}}(m) & \le & u_{\mathcal{G}}^p(m) \\[2mm]
|\wedge & & |\wedge \\[2mm]
u_{\mathcal{P}}(m) & \le & u_{\mathcal{P}}^p(m)
\end{array}
$$

5.1.1 Layered Permutations

Recall that a permutation is layered if it is the direct sum of decreasing permutations. Equivalently, a permutation is layered if it avoid the patterns 231 and 312. Let $\mathcal{L} = \text{Av}(231, 312)$ denote the class of layered permutations. Proposition 4-3 shows that any graph may be transformed into a cluster graph of the same size that contains each cluster graph that the original graph contained. Similarly, the following proposition, which appears in [3], shows that any permutation may be transformed into a layered permutation of the same size while retaining each of its layered containments.

Proposition 5-7 (Albert, Engen, Pantone, and Vatter [3])**.** *Given any permutation π of size n, there is a layered permutation of size n that contains every layered permutation contained in π.*

Proof. We prove the claim by induction on *n*. Note that the base case is trivial, and let π be a permutation of size $n \geq 1$. Let *D* denote a decreasing subsequence of π of maximum possible size. Because *D* is a maximal decreasing subsequence, every entry of π that is not in *D* must either lie to the southwest of an entry of *D* or to the northeast of such an entry, but not both. Let D^- denote the set of entries that lie to the southwest of an entry of *D* and let D^+ denote the set of entries that lie to the northeast of such an entry, so that D, D^-, and D^+ together constitute a partition of the entries of π. An example of this decomposition is shown on the leftmost panel of Figure 5-7.

Define π^- (resp., π^+) to be the permutation in the same relative order as the entries of D^- (resp., D^+). Let δ be the decreasing permutation of size $|D|$, and define $\pi^* = \pi^- \oplus \delta \oplus \pi^+$. Thus, in some sense, π^* is a "straightened-out" version of π, and an example is shown in the central panel of Figure 5-7.
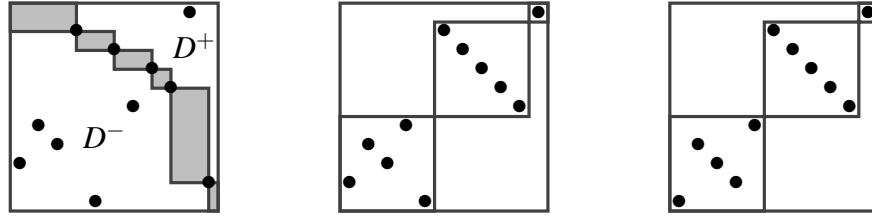

Figure 5-7. The steps in the proof of Proposition 5-7. From left to right, the drawings show an example of π, of π^*, and of the layered permutation $\tau^- \oplus \delta \oplus \tau^+$.

We claim that every layered permutation contained in π is also contained in π^*. Suppose $\lambda = \lambda_1 \oplus \cdots \oplus \lambda_\ell$ is a layered permutation contained in π, where each λ_i is a decreasing permutation, and fix an embedding of λ into π. Choose *j* maximally so that in this embedding of λ into π, the layers $\lambda_1 \oplus \cdots \oplus \lambda_{j-1}$ are embedded entirely using entries in D^-. It follows that the entries of $\lambda_{j+1} \oplus \cdots \oplus \lambda_\ell$ are embedded entirely using entries of D^+. Since λ_j certainly embeds into D and consequently into δ, we have that $\lambda \leq \pi^*$.

Finally, by induction we see that there are layered permutations μ^- and μ^+ that contain all of the layered permutations contained in π^- and π^+, respectively. It follows that $\mu^- \oplus \delta \oplus \mu^+$ is

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layered and contains all of the layered permutations contained in π^*, which in turn contains all of the layered permutations contained in π, proving the proposition. An example of this final construction is shown in the rightmost panel of Figure 5-7. □

In particular, any $\mathcal{L}_m$-universal permutation may be transformed into a layered permutation of the same size that is still $\mathcal{L}_m$-universal, affirming a conjecture of Gray [29]. Thus, among the smallest $\mathcal{L}_m$-universal permutations there are proper $\mathcal{L}_m$-universal permutations, establishing following corollary, which appears in [3].

Corollary 5-8 (Albert, Engen, Pantone, and Vatter [3]). *For all integers $m \geq 0$, the smallest proper $\mathcal{L}_m$-universal permutations are also smallest $\mathcal{L}_m$-universal permutations.*

We note that if λ and μ are layered permutations with layer lengths $\lambda_1, \cdots, \lambda_m$ and $\mu_1, \cdots, \mu_n$ respectively, then we have $\lambda \leq \mu$ if and only if there are indices $i_1 < i_2 < \cdots < i_m$ so that $\lambda_j \leq \mu_{i_j}$ for all j. Therefore the poset of layered permutations is isomorphic to the poset of compositions under the generalized subword order discussed in Chapter 3. As in any question of universality for classes of layered permutations we may assume that the containing permutation is layered, we may reduce each such question to the corresponding question of universality for composition classes. Theorem 3-1, adapted from [3], gives a precise formula for the size of the smallest m-universal compositions, and thus a formula for the size of the smallest $\mathcal{L}_m$-universal permutations.

Corollary 5-9 (Albert, Engen, Pantone, and Vatter [3]). *For all integers $m \geq 0$, the smallest (proper) $\mathcal{L}_m$-universal permutations have size $\ell(m) = (m+1)\lceil \log_2(m+1) \rceil - 2^{\lceil \log_2(m+1) \rceil} + 1$.*

Corollary 5-9 improves upon lower and upper bounds of $\Omega(m \log m)$ and $m \log_2 m + m$ respectively proven by Bannister, Cheng, Devanny, and Eppstein [9] as well as lower and upper bounds of $m \log m - m$ and $m\lfloor \log_2 m \rfloor + m$ respectively proven by Gray [29].

The recursive construction used to prove Theorem 3-1 can be repurposed to construct universal permutations for classes using direct sums. Let $\mathcal{P}$ be a permutation class and let $\mathcal{P}^{\boxtimes}$ denote the set of those permutations in $\mathcal{P}$ that are sum-indecomposable. For each permutation

$\pi \in \mathcal{P}$, decompose π as $\pi_1 \oplus \cdots \oplus \pi_m$, where each π_i is a nonempty sum-indecomposable permutation. Define $\mathrm{comp}(\pi)$ to be the composition $|\pi_1||\pi_2|\cdots|\pi_m|$ and let $\mathrm{comp}(\mathcal{P}) = \{\mathrm{comp}(\pi) \,:\, \pi \in \mathcal{P}\}$.

Proposition 5-10. *Suppose that for each $m \leq n$ the permutation τ_m is $\mathcal{P}_m^{\varnothing}$-universal. If the composition $c = c(1)\cdots c(n)$ is $\mathrm{comp}(\mathcal{P})_m$-universal, then the permutation $\tau_{c(1)} \oplus \tau_{c(2)} \cdots \oplus \tau_{c(n)}$ is $\mathcal{P}_m$-universal.*

Proof. Let $\pi \in \mathcal{P}_n$, and write $\pi = \pi_1 \oplus \cdots \oplus \pi_k$, where each π_i is a nonempty sum-indecomposable permutation. Then as c is $\mathrm{comp}(\mathcal{P})_n$-universal, there must be some sequence of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $|\pi_j| \leq c(i_j)$ for all j. As $\tau_{c(i_j)}$ is $c(i_j)$-universal, π_j embeds into $\tau_{c(i_j)}$, and we have that $\tau_{c(1)} \oplus \tau_{c(2)} \cdots \oplus \tau_{c(n)}$ contains π, as desired. $\square$

We now present a two brief applications of Proposition 5-10. As our first example, let $\mathcal{Q} = \mathrm{Av}(132, 231, 321) = \{\varepsilon\} \cup \{(1 \ominus \iota_a) \oplus \iota_b \,:\, a, b \geq 0\}$. Then, using vocabulary introduced in Chapter 3, $\mathrm{comp}(\mathcal{P}) = \mathrm{Age}(\omega 1^\omega)$ is the class of compositions in which every part after the first is 1. The composition $m1^{m-1}$ is $\mathrm{Age}(\omega 1^\omega)_m$-universal for each m, and the permutation $1 \ominus \iota_{m-1}$ is m-universal for the set $\mathcal{Q}^{\varnothing} = \{1 \ominus \iota_a \,:\, a \geq 0\}$ of sum-indecomposable permutations in $\mathcal{Q}$. Thus, by Proposition 5-10, the permutation $(1 \ominus \iota_{m-1}) \oplus \iota_{m-1}$ is $\mathcal{Q}_m$-universal.

Proposition 5-11. *The permutation $(1 \ominus \iota_{m-1}) \oplus \iota_{m-1}$ is m-universal for $\mathrm{Av}(132, 231, 321)$.*

For our second example, let $\mathcal{P} = \mathrm{Av}(231, 321)$. Then $\mathcal{P}^{\varnothing} = \{1 \ominus \iota_k \,:\, k \geq 0\}$ contains permutations of all sizes, and as $\mathcal{P}$ is closed under direct sums, $\mathrm{comp}(\mathcal{P})$ is the class of all compositions. The composition w_m defined by $w_0 = \varepsilon$ and, for $m \geq 1$,

$$w_m = w_{\lfloor (m-1)/2 \rfloor} m w_{\lceil (m-1)/2 \rceil}$$

of length m and size $\ell(m)$ defined in Chapter 3 is m-universal, and the permutation $\tau_m = 1 \ominus \iota_{m-1}$ is $\mathcal{P}_m^{\varnothing}$-unviersal. Thus the permutation σ_m defined as $\sigma_0 = \varepsilon$ and for $m \geq 1$

$$\sigma_m = \sigma_{\lfloor (m-1)/2 \rfloor} \oplus \tau_m \oplus \sigma_{\lceil (m-1)/2 \rceil}$$

of size $\ell(m)$ is $\mathcal{P}_m$-universal by Proposition 5-10. As $\mathcal{P}$ is closed under direct sums and $\tau_m \in \mathcal{P}$ for each m, each σ_m lies in $\mathcal{P}$ and is thus a proper $\mathcal{P}_m$-universal permutation.

Proposition 5-12. *Let* $\sigma_0 = \varepsilon$ *and, for* $m \geq 1$,

$$\sigma_m = \sigma_{\lfloor (m-1)/2 \rfloor} \oplus (m12\cdots(m-1)) \oplus \sigma_{\lceil (m-1)/2 \rceil}.$$

Then σ_m of size $\ell(m)$ is m-universal for $\mathrm{Av}(231,321)_m$-universal for all m.

The permutations σ_m are in fact asymptotically optimal, in the sense that their size is asymptotic to the size of the smallest proper $\mathcal{P}_m$-universal permutations as the next result shows.

Theorem 5-13. *Let* $\mathcal{P} = \mathrm{Av}(231,321)$, *and let* $S = \{\pi \in \mathcal{P} : \pi(1) \neq 1\}$. *Let $s(m)$ denote the minimum size of an S_m-universal permutation in $\mathcal{P}$. Then $s(0) = s(1) = 0$, and for $m \geq 2$,*

$$s(m) \geq m + \min\{s(k) + s(m-k-1) : 1 \leq k \leq m-1\}.$$

The proof of Theorem 5-13 is similar to that of the lower bound of Theorem 3-1, so it may serve the reader well to "warm up" with that proof first.

Proof. We proceed by induction on m. Each of S_0 and S_1 are empty, so the empty pattern is both S_0- and S_1-universal, and thus $s(0) = s(1) = 0$.

Let $m \geq 2$. Suppose that $\pi \in \mathcal{P}$ is an S_m-universal permutation and write $\pi = \pi_1 \oplus \cdots \oplus \pi_\ell$, where each π_j is sum-indecomposable. Then at least one π_j has size at least m, as π must contain the pattern $m12\cdots(m-1)$ and must do so entirely within one summand π_j. Suppose $|\pi_j| \geq m$, and choose $k \geq 1$ so that

$$s(k) \leq |\pi_1 \oplus \cdots \oplus \pi_{j-1}| < s(k+1).$$

As $|\pi_1 \oplus \cdots \oplus \pi_{j-1}| < s(k+1)$, there must be some $\sigma \in S_k$ that does not embed into $\pi_1 \oplus \cdots \oplus \pi_{j-1}$. Therefore the earliest that σ may embed into π is into $\pi_1 \oplus \cdots \oplus \pi_j$. In particular, the earliest that the last summand of σ may embed into π is into π_j.

For any pair of sum-indecomposable permutations $\tau, \rho \in \mathcal{P}$, the permutation $\tau \oplus \rho$ necessarily contains the pattern 132 as ρ necessarily contains the pattern 12. Thus π_j must avoid

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$\tau \oplus \rho$, as every sum-indecomposable permutation in $\mathcal{P}$ is of the form $1 \ominus \iota_n$ for some n, and thus must avoid the pattern 132. Thus, if $\rho^* \in S_{m-k-1}$, then its first sum-component ρ has size at least 2 (as its first sum-component cannot by 1 by definition,) and so any embedding of $\sigma \oplus \rho^*$ into π embeds ρ^* entirely within $\pi_{j+1} \oplus \cdots \oplus \pi_\ell$. As ρ^* is an abitrary permutation in S_{m-k-1}, it follows that $\pi_{j+1} \oplus \cdots \oplus \pi_\ell$ is a S_{m-k-1}-universal permutation in $\mathcal{P}$, and thus $\left| \pi_{j+1} \oplus \cdots \oplus \pi_\ell \right| \geq s(m-k-1)$. Together, we have

$$
\begin{aligned}
|\pi| &= \left| \pi_1 \oplus \cdots \oplus \pi_{j-1} \oplus \pi_j \oplus \pi_{j+1} \oplus \cdots \oplus \pi_\ell \right| \\
&= \left| \pi_1 \oplus \cdots \oplus \pi_{j-1} \right| + \left| \pi_j \right| + \left| \pi_{j+1} \oplus \cdots \oplus \pi_\ell \right| \\
&\geq s(k) + m + s(m-k-1) \\
&\geq m + \min\{s(k) + s(m-k-1) : 1 \leq k \leq m-1\},
\end{aligned}
$$

as desired. □

Let $t(0) = t(1) = 0$ and $t(m) = m + \min\{t(k) + t(m-k-1) : 1 \leq k \leq m-1\}$, then one can show that $t(m) \geq \ell(m) - m$ by induction, and Theorem 5-13 implies that every proper $\mathcal{P}_m$-universal permutations has size at least $t(m)$. Together with the upper bound of $\ell(m)$ provided by Proposition 5-12, we have have that the smallest proper $\mathcal{P}_m$-universal permutations have size between $\ell(m) - m$ and $\ell(m)$, meaning they have size asymptotic to $\ell(m) \sim m\log_2 m$.

Supported by computational evidence, we conjecture that the size of the smallest (proper) $\mathcal{P}_m$-universal permutations is in fact equal to $\ell(m)$.

Conjecture 5-14. *For all $m \geq 0$, the smallest (proper) m-universal permutations for* $\mathrm{Av}(231,321)$ *have size $\ell(m)$.*

5.1.2 Grid Classes

One well-studied family of permutation classes are the monotone grid classes. Roughly speaking, the grid class of a matrix M is the collection of permutations whose plots may be divided, in a manner prescribed by M, into blocks each containing a monotone pattern. More formally, let M be a $0/\pm 1$ matrix with t columns and u rows. An $\underline{M\text{-gridding}}$ of the permutation

π of size n is a choice of column divisions $0 = c_0 \leq c_1 \leq \cdots \leq c_t = n$ and row divisions $0 = r_0 \leq r_1 \leq \cdots \leq r_u = n$ such that for all i and j, the subsequence of π with indices in the real interval $(c_{i-1}, c_i]$ and values in the real interval $(r_{j-1}, r_j]$ is increasing if $M_{i,j} = 1$, decreasing if $M_{i,j} = -1$, and empty if $M_{i,j} = 0$. The <u>grid class</u> of M, denote $\mathrm{Grid}(M)$, is the collection of all permutations that admit an M-gridding. An example of an M-gridding of a permutation for some matrix M is drawn in Figure 5-8.

To aid comprehension, we denote grid classes by their <u>cell diagrams</u> rather than by their matrices. For example, we abbreviate $\mathrm{Grid}\left(\begin{smallmatrix} 0 & -1 & 1 \\ 1 & 1 & 0 \end{smallmatrix}\right)$ as . Occasionally, where it is convenient, we use a cell diagram to stand for the matrix itself.

We begin the results portion of this subsection with an observation useful for providing lower bounds on the size of proper universal permutations for grid classes.

Observation 5-15. *Let $\mathcal{P} = \mathrm{Grid}(M)$ be a grid class, and let $\sigma, \pi \in \mathcal{P}$. If in every M-gridding of σ there are k points in cell $M_{i,j}$ and $\sigma \leq \pi$, then in every M-gridding of π there are at least k points in the cell $M_{i,j}$.*

As one simple example where Observation 5-15 is useful, let $M = $ and consider the grid class $\mathcal{P} = \mathrm{Grid}(M)$. In the unique M-gridding of the permutation $m12\cdots(m-1)$, the lower cell of M contains $m-1$ points, and thus any M-gridding of any proper $\mathcal{P}_m$-universal permutation must place at least $m-1$ points the lower cell of M. Likewise, the unique M-gridding of the permutation $2\cdots m1$ places $m-1$ points in the upper cell of M, and thus any M-gridding of any proper $\mathcal{P}_m$-universal permutation must place at least $m-1$ points in the upper cell of M. Together,
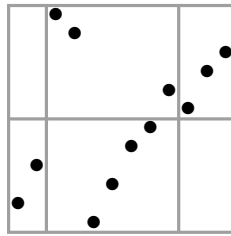

Figure 5-8. A $\left(\begin{smallmatrix} 0 & -1 & 1 \\ 1 & 1 & 0 \end{smallmatrix}\right)$-gridding of a permutation π. As π admits such a gridding, we write $\pi \in$.

these observations show that any proper $\mathcal{P}_m$-universal permutation must have at least $2(m-1)$ points, which is nearly optimal as we will see shortly in Theorem 5-19.

5.1.3 Wedge Permutations and Riffle Shuffle Permutations

Up to symmetry, the two simplest non-trivial grid classes are ⧄ and ⧅. In this section, we show that both classes have smallest (proper) m-universal permutations of size $2m-1$. To begin, let $\mathcal{W} = \mathrm{Av}(132,312) = ⧄$ denote the class of non-empty <u>wedge</u> permutations. In [9], Bannister, Cheng, Devanny, and Eppstein prove that $\mathcal{W}$ has smallest (proper) universal permutations of size $2m-1$, and below we provide a proof of this fact. To begin, we show that the poset of nonempty wedge permutations is isomorphic to the poset of words over a two-letter alphabet, discussed in generality in Section 1.1.

Proposition 5-16. *Define the map $P : \{p,m\}^* \to \mathcal{W}$ recursively as follows:*

1. *$P(\varepsilon) = 1$ is the permutation of size 1.*

2. *$P(vp) = P(v) \oplus 1$ is the direct sum of $P(v)$ with the pattern 1.*

3. *$P(vm) = P(v) \ominus 1$ is the skew sum of $P(v)$ with the pattern 1.*

Then P is a poset isomorphism.

The map P sends words of size m to permutations of size $m+1$, so we omit the 0-pattern from $\mathcal{W}$. Before proving Proposition 5-16, we present a useful lemma.

Lemma 5-1. *If π and σ are permutations, then*

1. *$\pi \oplus 1 \le \sigma \oplus 1$ if and only if $\pi \le \sigma$.*

2. *$\pi \oplus 1 \le \sigma \ominus 1$ if and only if $\pi \oplus 1 \le \sigma$.*

3. *$\pi \ominus 1 \le \sigma \oplus 1$ if and only if $\pi \ominus 1 \le \sigma$.*

4. *$\pi \ominus 1 \le \sigma \ominus 1$ if and only if $\pi \le \sigma$.*

We now present the proof of Proposition 5-16.

Proof of Proposition 5-16. We begin by proving that if $v, w \in \{p, m\}^*$ with $v \leq w$, then $P(v) \leq P(w)$.

Leveraging the recursive nature of the map P, we proceed by induction on the size of v. For our base case, note that if v is the empty word, then $P(v) = 1$, which is contained in every non-empty permutation. Thus, assume that v is a non-empty word, let ℓ be the final letter of v, and write $v = v_0 \ell$. Let $w \in \{p, m\}^*$ be any word with $v \leq w$. As v is a subsequence of w, we may write $w = w_0 \ell v_1$, where $v_0 \leq w_0$. By induction, we have $P(v_0) \leq P(w_0)$. By Lemma 5-1, we have $P(v_0 \ell) \leq P(w_0 \ell)$ no matter the value of ℓ. Finally, as $P(w_0 \ell) \leq P(w_0 \ell w_1)$, we have

$$P(v) = P(v_0 \ell) \leq P(w_0 \ell) \leq P(w_0 \ell w_1) = P(w),$$

as desired.

For the converse, we show that if $P(v)$ and $P(w)$ are permutations with $P(v) \leq P(w)$, then $v \leq w$. Again, we proceed by induction on the size of v and begin by noting that the permutation $P(\varepsilon) = 1$ is contained in every permutation in $\mathcal{W}$, and the empty word ε is contained in every word in $\{p, m\}^*$, so the base case is satisfied.

Let $P(v)$ and $P(w)$ be permutations with $P(v) \leq P(w)$ and $|P(v)| \geq 2$, or equivalently, $|v| \geq 1$. Without loss of generality, assume that $v = v_0 p$ and $w = w_0 p m^k$ for some $k \geq 0$. Repeated applications of Lemma 5-1 mean that $P(v_0 p) \leq P(w_0 p m^k)$ implies that $P(v_0) \leq P(w_0)$. By induction we must have $v_0 \leq w_0$, and thus $v_0 p \leq w_0 p m^k$, completing the proof. □

One consequence of this isomorphism is that the smallest universal words constructed in Section 1.1 translate into smallest proper universal permutations for $\mathcal{W}$.

Corollary 5-17. *The smallest proper $\mathcal{W}_m$-universal permutations have size $2m - 1$.*

As $\mathcal{W}_m$ contains both $12 \cdots m$ and $m \cdots 21$, any $\mathcal{W}_m$-universal permutation must contain a length-m increasing subsequence as well as a length-m decreasing subsequence. As these subsequences may not intersect in more than one entry, any $\mathcal{W}_m$-universal permutation must have

size at least $2m - 1$, so no smaller $\mathcal{W}_m$-universal permutation exists outside $\mathcal{W}$.

Corollary 5-18. *The smallest (proper) $\mathcal{W}_m$-universal permutations have size $2m - 1$.*

The other 2×1 grid class is $\mathcal{R} = \mathrm{Av}(123, 3412, 3142) = $, the class of <u>riffle shuffle</u> permutations, so called as they are precisely those patterns that may be formed by a single riffle shuffle of a deck of n cards. In [10], Bannister, Devanny, and Eppstein prove that the permutation $(m+1)1(m+2)2\cdots(2m-1)(m-1) \in \mathcal{R}$ is $\mathcal{R}_m$-universal. This construction may be seen to be optimal, as we now show. Consider the layered permutation class $\mathcal{P} = $. By Proposition 5-7, the minimum size of a $\mathcal{P}_m$-universal permutation is the same as the minimum size of an m-universal composition for the composition class $\mathrm{Age}(\omega^2)$, which is $2m - 1$ by Proposition 3-10. As $\mathcal{P} \subseteq \mathcal{R}$, the smallest $\mathcal{R}_m$-universal permutation must have size at least $2m - 1$, and thus the construction by Bannister, Devanny, and Eppstein is optimal.

Theorem 5-19. *Let $\mathcal{R} = $. The smallest (proper) $\mathcal{R}_m$-universal permutations have size $2m - 1$ for $m \geq 1$.*

5.1.4 $\{123, 312\}$-avoiding Permutations

One immediate consequence of Theorem 5-19 is that any class of permutations that contains and is contained in must have smallest m-universal permutations of size $2m - 1$. The class meets this criteria, and thus the corollary below follows.

Corollary 5-20. *Let $\mathcal{P} = \mathrm{Av}(123, 312) = $. For each $m \geq 1$, the smallest $\mathcal{P}_m$-universal permutations have size $2m - 1$.*

The size of the smallest proper $\mathrm{Av}(123, 312)_m$-universal permutations is slightly larger, as the following theorem shows.

Theorem 5-21. *Let $\mathcal{P} = \mathrm{Av}(123, 312) = $. For each $m \geq 3$, the smallest proper $\mathcal{P}_m$-universal permutations have size $3m - 4$.*

Proof. To begin, we claim that the permutation $\pi = (\delta_{m-1} \oplus \delta_{m-1}) \ominus \delta_{m-2}$ is $\mathcal{P}_m$-universal. Let $\sigma \in \mathcal{P}_m$. If σ avoids 12, then σ is the decresing permutation of size m, and as π contains

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$\delta_{m-2} \ominus \delta_{m-1} = \delta_{2m-3}$ and $2m - 3 \geq m$, we have that π contains σ. Otherwise, σ contains 12, and we may decompose σ as $\sigma = (\delta_a \oplus \delta_b) \ominus \delta_c$ for nonnegative integers a, b, c with $a, b \geq 1$ and $a + b + c = m$. In this case, both a and b must be at most $m - 1$, and c must be at most $m - 2$, which implies that σ is contained in $(\delta_{m-1} \oplus \delta_{m-1}) \ominus \delta_{m-2} = \pi$.

Let M be the matrix ⊞. To establish a lower bound, we show that in any M-gridding of a proper $\mathcal{P}_m$-universal permutation, the lower-right cell must have at least $m - 2$ points and both the leftmost cell and topmost cell must each have at least $m - 1$ points.

Consider the permutations $\sigma_1 = 1 \oplus \delta_{m-1}$, $\sigma_2 = \delta_{m-1} \oplus 1$, and $\sigma_3 = 12 \ominus \delta_{m-2}$.

1. The permutation $\sigma_1 = 1m \cdots 2$ has a unique M-gridding, where the $m - 1$ entries $2, 3, \ldots, m$ are placed in topmost cell and the entry 1 is placed in the leftmost cell.

2. The permutation $\sigma_2 = (m - 1) \cdots 2m$ has a unique M-gridding, where the entry m is placed in the topmost cell and the $m - 1$ entries $1, 2, \ldots, (m - 1)$ are placed in the leftmost cell.

3. The permutation $\sigma_3 = (m - 1)m(m - 2) \cdots 21$ has a unique gridding in M, where the $m - 2$ entries $1, 2, \ldots, (m - 2)$ are placed in the lower-right cell, the entries $m - 1$ and m are placed in the leftmost and topmost cells, respectively.

By Observation 5-15, any M-gridding of any proper $\mathcal{P}_m$-universal permutation must contain at least $m - 1$ points in the topmost cell, at least $m - 1$ points in the leftmost cell, and at least $m - 2$ points in the lower-left cell, and therefore must have size at least

$(m - 2) + (m - 1) + (m - 1) = 3m - 4$, as desired. $\qquad \square$

5.1.5 {231, 2143}-avoiding Permutations

Let $\mathcal{P} = \text{Av}(231, 2143) = $ ⊠ be the class of {231, 2143}-avoiding permutations. In [9], Bannister, Cheng, Devanny, and Eppstein prove that there are $\mathcal{P}_m$-universal permutations of size $3m - 4$ for $m \geq 3$. We provide a simple proof of this construction.

Theorem 5-22 (Bannister, Cheng, Devanny, and Eppstein [9]). *For each $m \geq 3$, there is a proper $\mathcal{P}_m$-universal permutation of size $3m - 4$.*

Proof. Let $v_3 = 51324$, and for each $m > 3$, let $v_{m+1} = 1 \ominus (1 \oplus v_m \oplus 1)$. We claim that, for each $m \geq 3$, v_m is (1) in $\mathcal{P}$ and (2) $\mathcal{P}_m$-universal. To begin, note that v_3 is in $\mathcal{P}$ and that for any permutation $\pi \in \mathcal{P}$, the permutations $1 \ominus \pi$, $1 \oplus \pi$, and $\pi \oplus 1$ all lie in $\mathcal{P}$. As v_{m+1} is constructed from v_m using only these operations, we have that $v_m \in \mathcal{P}$ for all $m \geq 3$ by induction.

To show that v_m is m-universal for all $m \geq 3$, we begin by noting that $v_3 = 51324$ is $\mathcal{P}_3$-universal, meaning it contains the pattern 123, 132, 213, 312 and 321, which we may check by hand. Given any permutation $\pi \in \mathcal{P}$, we may decompose π as at least one of $\pi = 1 \oplus \pi^*$, $\pi = 1 \ominus \pi^*$, or $\pi = \pi^* \oplus 1$, where $\pi^* \in \mathcal{P}_{m-1}$. As v_m contains each of $1 \oplus v_{m-1}$, $1 \ominus v_{m-1}$, and $v_{m-1} \oplus 1$, and v_{m-1} contains π^* necessarily by induction, we have that v_m contains π, completing the proof. $\qquad\square$

Moreover, Bannister, Cheng, Devanny, and Eppstein prove that, with $\mathcal{Q} = \mathrm{Av}(231, 312, 2143, 1324) = \boxed{\diagup}\!\boxed{} \cup \boxed{}\!\boxed{\diagdown}$, any $\mathcal{Q}_m$-universal permutation has size at least $3m - 4$. As $\mathcal{Q} \subseteq \mathcal{P}$, any $\mathcal{P}_m$-universal permutation must have size at least $3m - 4$, which together with the $\mathcal{P}_m$-universal permutations of size $3m - 4$ of Theorem 5-22 are the best-possible even outside the class.

Theorem 5-23 (Bannister, Cheng, Devanny, and Eppstein [9])**.** *The smallest m-universal permutations for* $\mathrm{Av}(231, 2143)$ *have size* $3m - 4$.

As a corollary, Bannister, Cheng, Devanny, and Eppstein note that any class contained in $\mathcal{P}$ that contains $\mathcal{Q}$ has smallest universal permutations of this size as well:

Corollary 5-24 (Bannister, Cheng, Devanny, and Eppstein [9])**.** *If $\mathcal{R}$ is a permutation class with $\mathcal{Q} \subseteq \mathcal{R} \subseteq \mathcal{P}$, then the smallest $\mathcal{R}_m$-universal permutations have size* $3m - 4$.

5.1.6 231-**avoiding Permutations**

In [37], Knuth shows that the permutations that may be sorted with a stack are precisely those that avoid the permutation 231. Let $\mathcal{P} = \mathrm{Av}(231)$ denote the class of 231-avoiding permutations. In [9], Bannister, Cheng, Devanny, and Eppstein construct $\mathcal{P}_m$-universal permutations of size $\lfloor m^2/4 \rfloor + m$ and construct, for every proper subclass of $\mathcal{P}$, m-universal

permutations of size $O\left(m\log^{O(1)} m\right)$. The $\mathcal{P}_{10}$-universal permutation of their construction is presented in the left panel of Figure .

To present a construction of proper $\mathcal{P}_m$-universal permutations, we say the reverse-inverse-reverse of a permutation π, denoted π^{RIR}, is defined as $\pi^{\text{RIR}} = ((\pi^r)^{-1})^r$, where π^r is the permutation formed by reversing π, and π^{-1} is the function inverse of π. Pictorially, a plot of π^{RIR} may be obtained by reflecting the plot of π across the line $y = -x$. The reverse and inverse symmetries preserve containment, so the reverse-inverse-reverse symmetry does as well, meaning that $\sigma \leq \pi$ if and only if $\sigma^{\text{RIR}} \leq \pi^{\text{RIR}}$. As $231 = 231^{\text{RIR}}$, the class $\mathcal{P}$ is closed under the reverse-inverse-reverse symmetry, and thus π is $\mathcal{P}_m$-universal if and only if π^{RIR} is.

Proposition 5-25. *Let $\Pi_0 = \varepsilon$, and for $m \geq 1$, define*

$$\Pi_m = \left(1 \ominus \Pi_{m-1}^{\text{RIR}}\right) \oplus \Pi_{\lfloor m/2 \rfloor}.$$

Then Π_m is a proper $\mathcal{P}_m$-universal permutation for all m.

Proof. Before showing that Π_m is $\mathcal{P}_m$-universal for each m, we show that each $\mathcal{P}_m$ lies in $\mathcal{P}$ by a brief inductive arguemnt. The permutation $\Pi_0 = \varepsilon$ lies in every class of permutations, so our base case is satisfied. In addition to being closed under the reverse-inverse-reverse symmetry, the class $\mathcal{P}$ is closed under prepending a new largest element and under direct sums, ie. if $\pi, \sigma \in \mathcal{P}$, then both $1 \ominus \pi$ and $\pi \oplus \sigma$ lie in $\mathcal{P}$. Thus, by induction, $\Pi_m = (1 \ominus \Pi_{m-1}^{\text{RIR}}) \oplus \Pi_{\lfloor m/2 \rfloor} \in \mathcal{P}$ for all m.

To show that Π_m is $\mathcal{P}_m$-universal for all m, we again proceed by induction on m. The permutation $\Pi_0 = \varepsilon$ is clearly 0-universal for every class, so assume that $\Pi_{m'}$ is m'-universal for $\mathcal{P}$ for all $m' < m$.

Let $\pi \in \mathcal{P}_m$. If π is sum-indecomposable, then $\pi = 1 \ominus \tau$ for some $\tau \in \mathcal{P}_{m-1}$. As Π_{m-1} is $\mathcal{P}_{m-1}$-universal, Π_{m-1}^{RIR} is as well. Thus, as Π_m contains $1 \ominus \Pi_{m-1}^{\text{RIR}}$ and $\pi = 1 \ominus \sigma$ for $\sigma \in \mathcal{P}_{m-1}$, we have that $\pi \leq \Pi_m$.

Otherwise, we may write $\pi = \pi_1 \oplus \pi_2$, where π_2 is a non-empty sum-indecomposable permutation, and we complete our analysis in two cases: when $|\pi_2| \leq \lfloor m/2 \rfloor$ and when
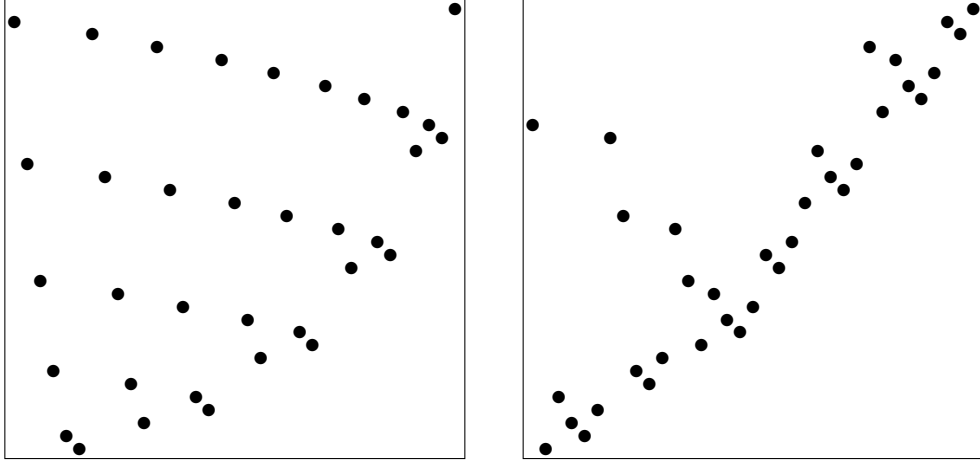
Figure 5-9. On the left, the $\mathcal{P}_{10}$-universal permutation of size 35 of Bannister, Chung, Devanny, and Eppstein's construction [9]. On the right, the proper $\mathcal{P}_8$-universal permutation Π_8 of size 35.

$|\pi_2| \geq \lfloor m/2 \rfloor + 1$. In the first case, assume that $|\pi_2| \leq \lfloor m/2 \rfloor$. As $|\pi_1| \leq m - 1$, we have that π_1 embeds into Π_{m-1}^{RIR}, and as $|\pi_2| \leq \lfloor m/2 \rfloor$, we have that π_2 embeds into $\Pi_{\lfloor m/2 \rfloor}$. Since Π_m contains $\Pi_{m-1}^{\mathrm{RIR}} \oplus \Pi_{\lfloor m/2 \rfloor}$, we may conclude that $\pi = \pi_1 \oplus \pi_2 \leq \Pi_m$. In the second case, assume that $|\pi_2| \geq \lfloor m/2 \rfloor + 1$, and thus $|\pi_1| \leq \lceil m/2 \rceil - 1$. We claim that π embeds into $\Pi_{m-1}^{\mathrm{RIR}} = \Pi_{k-1}^{\mathrm{RIR}} \oplus (1 \ominus \Pi_{m-2})$. As π_2 is sum-indecomposable, we may write $\pi_2 = 1 \ominus \tau$ for some τ with $|\tau| \leq m - 2$. Thus we have $\pi = \pi_1 \oplus (1 \ominus \tau)$, where $|\pi_1| \leq k - 1$ and $|\tau| \leq 2k - 2$, and so by induction, π embeds into $\Pi_{k-1}^{\mathrm{RIR}} \oplus (1 \ominus \Pi_{2k-2}) = \Pi_{2k-1}^{\mathrm{RIR}}$, completing the proof. $\qquad\square$

Corollary 5-26. *The sequence* $|\Pi_m| = p(m)$ *is given by* $p(0) = 0$ *and, for* $m \geq 1$,

$$p(m) = p(m-1) + p(\lfloor m/2 \rfloor) + 1,$$

and thus the smallest proper $\mathcal{P}_m$-*universal permutations have size at most* $p(m)$.

The proper $\mathcal{P}_m$-universal permutations constructed in Proposition 5-25 are much larger than the $\mathcal{P}_m$-universal permutations of size $\lfloor m^2/4 \rfloor + m$ constructed by Bannister, Cheng, Devanny, and Eppstein in [9], as the next result shows.

Theorem 5-27 (Knuth [36])**.** *Let* $a(1) = 1$ *and* $a(n) = a(n-1) + a(\lfloor n/2 \rfloor)$ *for* $n \geq 2$. *Then* $a(n)$ *grows faster than any polynomial. That is, for any power* k, *there is some* N_k *such that for all* $n \geq N_k$ *we have* $a(n) > n^k$.

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Before presenting the proof of Theorem 5-27, note that $s(n) \geq a(n)$, and thus s grows faster than any polynomial.

Proof of Proposition 5-27. Fix k. We aim to show that for sufficiently large n, we have $a(n) > n^k$. Let N be such that

$$2^{k+1} + 1 \geq \left(2 + \frac{1}{N}\right)^{k+1}$$

and let

$$c = \min\left\{\frac{a(n)}{n^{k+1}} : N \leq n \leq 2N\right\}.$$

We claim that for all $n \geq N$, we have $a(n) \geq cn^{k+1}$. For $N \leq n \leq 2N$, this is clear as $c \leq a(n)/n^{k+1}$. If $n > 2N$, induction shows that

$$
\begin{aligned}
a(n) &= a(n-1) + a(\lfloor n/2 \rfloor) \\
&\geq c \cdot (n-1)^{k+1} + c \cdot \lfloor n/2 \rfloor^{k+1} \\
&\geq c\left((n-1)^{k+1} + \left(\frac{n-1}{2}\right)^{k+1}\right) \\
&= c\left(\frac{2^{k+1}+1}{2^{k+1}}\right)(n-1)^{k+1} \\
&\geq c\left(\frac{2+\frac{1}{N}}{2}\right)^{k+1}(n-1)^{k+1} \\
&= c\left(1 + \frac{1}{2N}\right)^{k+1}(n-1)^{k+1} \\
&\geq c\left(1 + \frac{1}{n-1}\right)^{k+1}(n-1)^{k+1} \\
&= cn^{k+1}.
\end{aligned}
$$

To conclude, choose N_k such that $N_k \geq N$ and $N_k \geq 1/c$ (so that $cn \geq cN_k \geq 1$), and the proof is complete. □

5.1.7 321-avoiding Permutations

Let $\mathcal{S}$ be the class of 321-avoiding permutations. Equivalently, $\mathcal{S}$ is the class of permutations that may be partitioned into two increasing subsequences. In [10], Bannister,

Devanny, and Eppstein construct $\mathcal{S}_m$-universal permutations of size $22m^{3/2} + \Theta(m)$.

In [8], Atminas, Kitaev, Lozin, and Valyuzhenich construct a proper $\mathcal{S}_m$-universal permutation of size m^2. The corresponding permutation graph is thus a proper $G(\mathcal{S})_m$-universal graph, where in this case $G(\mathcal{S})$ is the class of bipartite permutation graphs. This corresponding $G(\mathcal{S})_m$-universal permutation graph is in fact equal to the size-m^2 graph that Lozin and Rudolf construct in [39]. The question of whether $\mathcal{S}$ admits sub-quadratic sized proper universal permutations remains open, but evidence points towards "no". In [4], Alecu, Lozin, and Malyshev prove that any proper m-universal graph for the class of bipartite permutation graphs must have size $\Omega(m^\alpha)$ for all $\alpha < 2$, implying that any proper $\mathcal{S}_m$-universal permutation must have size $\Omega(m^\alpha)$ for all $\alpha < 2$. They conjecture that the smallest m-universal graphs for the class of bipartite permutation graphs have size $\Omega(m^2)$, which would imply that the smallest proper $\mathcal{S}_m$-universal permutations have size $\Theta(m^2)$.

5.1.8 Subclasses of $\mathrm{Av}(321)$: Truncated Staircases

The class $\mathrm{Av}(321)$ may be represented as the grid class of an infinite matrix. First shown by Albert, Atkinson, Brignall, Ruškuc, Smith, and West [2], we have
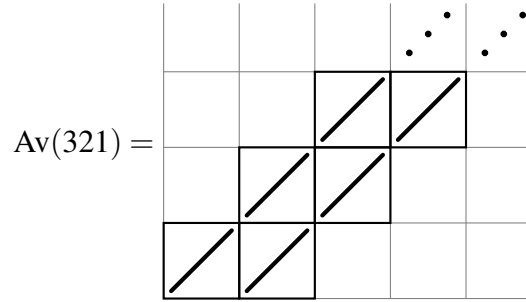
The figure on the right is known as the infinite positive staircase. One natural family of subclasses of $\mathrm{Av}(321)$ are the finite positive staircases. Let $\mathcal{S}^k$ be the positive staircase consisting of the first k nonempty cells of the infinite staircase, e.g., $\mathcal{S}^1 = \boxslash$, $\mathcal{S}^2 = \boxslash\boxslash$, $\mathcal{S}^3 = \boxslash\boxslash$, and so on.

Then, as

$$\mathcal{S}^1 \subseteq \mathcal{S}^2 \subseteq \mathcal{S}^3 \subseteq \cdots \subseteq \mathcal{S},$$

we have

$$\mathrm{u}_{\mathcal{S}^1}(m) \leq \mathrm{u}_{\mathcal{S}^2}(m) \leq \mathrm{u}_{\mathcal{S}^3}(m) \leq \cdots \leq \mathrm{u}_{\mathcal{S}}(m).$$

For all m, every permutation in $\mathcal{S}_m$ may be gridded using the first $\lceil(m+1)/2\rceil$ cells of the infinite positive, and there is some permutation in $\mathcal{S}_m$ that cannot be gridded using $\lceil(m-1)/2\rceil$ cells. Thus, any proper $\mathcal{S}_m$-universal permutation may not use fewer than the first $\lceil(m+1)/2\rceil$ cells, and computation suggests that among all smallest proper $\mathcal{S}_m$-universal permutations, there is at least one that may itself be gridded in $\lceil(m+1)/2\rceil$ cells.

Conjecture 5-28. *For all $m \geq 0$, there is a smallest proper $\mathcal{S}_m$-universal permutation that lies in $\mathcal{S}^{\lceil(m+1)/2\rceil}$.*

We now demonstrate, through Observation 5-15, a lower bound on the size of proper $\mathcal{S}_m^k$-universal permutations. For $k \geq 2$, let M be the matrix corresponding to the the finite positive staircase with k nonempty cells. The permutation $\pi = 2143 \cdots (2k-2)(2k-3)$ has a unique M-gridding, placing one point in each of the first and last cells and two points in every other cell. Consider some entry $\pi(i)$ of π that is placed in the cell C in the unique M-gridding of π. Replacing the entry $\pi(i)$ of π with a contiguous increasing sequence of entries of size $m - 2k + 3$ and yields a permutation of size m that has a unique M-gridding, placing $m - 2k + 3$ points in the cell C if C is the first or last cell of M or $m - 2k + 4$ points in the cell C otherwise.

Thus for $m \geq 2k - 2$, any M-gridding of a proper $\mathcal{S}_m^k$-universal permutation must have at least $m - 2k + 3$ points in its first and last cells, and $m - 2k + 4$ points in each of its other $k - 2$ cells. This shows that any proper $\mathcal{S}_m^k$-universal permutation must have size at least

$$2(m - 2k + 3) + (k - 2)(m - 2k + 4) = km - 2(k-1)^2.$$

Proposition 5-29. *For $m \geq 2k - 2$, any proper $\mathcal{S}_m^k$-universal permutation has size at least $km - 2(k-1)^2$.*

Using Conjecture 5-28 and Proposition 5-29, we have the conjollary[1] below.

Conjollary 5-30. *For all $m \geq 4$, any proper $\mathcal{S}_m$-universal permutation must have size at least $m^2/8$.*

---

[1] A conjollary is a corollary that follows from a conjecture.

Proof. By Conjecture 5-28, we have that the size of the smallest proper $\mathcal{S}_m$-universal permutations and the size of the smallest proper $\mathcal{S}_m^k$-universal permutations are the same for all $k \leq (m+2)/2$. By Proposition 5-29, every proper $\mathcal{S}_m^k$ universal permutation has size at least $km - 2(k-1)^2$, and thus

$$
\begin{aligned}
u_{\mathcal{S}}^p(m) &\geq \max\{mk - 2(k-1)^2 : k \leq (m+2)/2\} \\
&\geq m\left\lfloor \frac{m}{4} + 1 \right\rfloor - 2\left\lfloor \frac{m}{4} \right\rfloor^2 \\
&> \frac{m^2}{8},
\end{aligned}
$$

as desired. $\qquad\square$

5.1.9 Skew riffle Permutations

We say that π is a <u>skew riffle</u> permutation if it is the sum of riffle and antiriffle permutations. Let $\mathcal{SR}$ be the class of skew riffle permutations:

$$
\mathcal{P} = \bigoplus \left(\boxed{\diagup\!\!\diagup\!\!\diagup} \cup \boxed{\diagup} \right).
$$

In [10], Bannister, Devanny, and Eppstein construct $\mathcal{P}_m$-universal permutations of size $16m\log m + \Theta(m)$. Let $\mathcal{R} = \boxed{\diagup\!\!\diagup} \cup \boxed{\diagup}$. The permutation

$$
\pi_m = 135\cdots(2m-1)2(2m)4(2m+1)6\cdots(3m-3)(2m-2)(3m-2)
$$

of size $3m-2$ is $\mathcal{R}_m$-universal, as the lowest $2m-1$ values form the permutation $135\cdots(2m-1)246\cdots(2m-2)$, which is m-universal for $\boxed{\diagup\!\!\diagup}$, and the rightmost $2m-1$ values form the permutation $m1(m+1)2\cdots(m-1)(2m-1)$, which is m-universal for $\boxed{\diagup}$. In the language of Proposition 5-10, $\mathrm{comp}(\mathcal{R})$ is the class of all compositions, and the recursively-defined composition $w_m = w_{\lfloor(m-1)/2\rfloor}mw_{\lceil(m-1)/2\rceil}$ (with $w_0 = \varepsilon$) of length m and size $\ell(m)$ is m-universal. Thus the permutation $\tau_m = \oplus_{i=1}^m \pi_{w_m(i)}$ is $\mathcal{R}_m$-universal and has size at most $3\ell(m) \sim 3m\log_2 m$. For example, Figure 5-10 shows the plots of π_4 and τ_3.
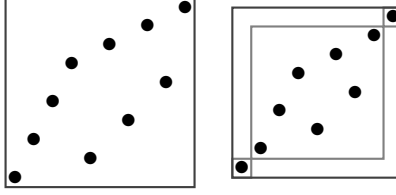
Figure 5-10. On the left, the permutation π_4, which is 4-universal for ▨▨∪◪. On the right, the $\mathcal{P}_3$-universal permutation τ_3 with its summands π_1, π_3, and π_1 outlined.

5.2   Concluding Remarks

Unlike in the context of graphs, there is no clear guess about the "optimal" size of an m-universal permutation for a class based on the number of m-patterns in the class. Both the class of wedge permutations and the class of layered permutations contain precisely 2^{m-1} patterns of size m, but their smallest (proper) universal permutations have sizes $\ell(m) \sim m\log_2 m$ and $2m-1$, respectively. One immediately desirable characterization would be of those maximal permutation classes that admit linear-size universal permutations.

Question 5-31. What are the maximal permutations classes that admits linear-size universal permutations?

In this chapter, we present some additional problems related to universality. As remarked in Chapter 1, each n-structure considered in this work contains at most $\binom{n}{m}$ many m-structures, as any substructure of size m corresponds to a subset of entries of size m. One may ask more generally, "what is the maximum number of m-structures that an n-structure may contain?" In [12], Bevan, Homberger, and Tenner prove there are n-permutations that attain the above binomial bound, meaning they contain $\binom{n}{m}$ many m-permutations, if and only if $n \geq \lceil (n-m)^2/2 + 2(n-m) + 1 \rceil$. Focusing on permutations, let $C(n,m)$ denote the maximum number of m-permutations that any n-permutation contains. Figure 6-1 displays known values of $C(m+k,m)$ for small m and k. The cells with dark gray backgrounds contain values where $C(m+k,m) = m!$, meaning there are m-universal permutations of size $m+k$. The cells with white backgrounds contain values so that $C(m+k,m) = \binom{m+k}{m}$. We are grateful to Axel Bacher [private communication] for computing many of these values.

k													
11	1	2	6	24	120	720	4383						
10	1	2	6	24	120	710	3931	16215					
9	1	2	6	24	120	683	3212	11391	30728				
8	1	2	6	24	120	618	2484	7320	17514	35849			
7	1	2	6	24	119	526	1728	4402	9070	17150	29422		
6	1	2	6	24	112	408	1094	2356	4368	7405	11944	18109	
5	1	2	6	24	94	273	614	1127	1867	2885	4282	6150	8531
4	1	2	6	23	71	156	291	477	699	988	1355	1815	2380
3	1	2	6	19	41	76	114	162	220	286	364	455	560
2	1	2	6	12	21	28	36	45	55	66	78	91	105
1	1	2	4	5	6	7	8	9	10	11	12	13	14
m	1	2	3	4	5	6	7	8	9	10	11	12	13

Figure 6-1. An array of known values of $C(m+k,m)$ for small values of m,k.

Given a set of structures $\mathcal{S}$, and let $C_{\mathcal{S}}(n,m)$ denote the maximum number of $\mathcal{S}_m$-structures that any n-structure may contain. Likewise, let $C_{\mathcal{S}}^p(n,m)$ denote the maximum number of

$\mathcal{S}_m$-structures that any $\mathcal{S}_n$-structure may contain. As a case study, consider the poset $\mathcal{T}$ of words over the two-letter alphabet $\{a, b\}$ under the subword ordering. In this case, we may provide an exact formula for the maximum number of words of size m that any word in $\mathcal{T}_n$ may contain.

Proposition 6-1. *For all n, m, the maximum number of m-words contained in a $\mathcal{T}_n$-word is*

$$C_{\mathcal{T}}(n, m) = \sum_{\ell=0}^{n-m} \binom{m}{\ell}.$$

Before proving Proposition 6-1, we note that the formula provided equals 2^m precisely when $n \geq 2m$, which is implied by Theorem 1-1.

Proof. To begin, we obtain an recursive formula for the number of m-words contained in a word w of size n for $n > m$. If w contains only a or only b, then w contains only the constant word of size m. Otherwise, w contains both a and b, and without loss of generality we can assume that w ends in a and write $w = w_0 ba^k$.

For each word v that w contains that ends in b, there is some embedding of v into w that places the final letter of v into the final b of w, and thus the set of m-words contained in w that end in b is in bijection with the set of $(m-1)$-words contained in w_0. Likewise, the set of m-words contained in w that end in a is in bijection with the set of $(m-1)$-words contained in $w_0 ba^{k-1}$. We have shown that

$$\binom{\text{\# of } m\text{-words}}{\text{contained in } w} = \binom{\text{\# of } (m-1)\text{-words}}{\text{contained in } w_0} + \binom{\text{\# of } (m-1)\text{-words}}{\text{contained in } w_0 ba^{k-1}}.$$

This formula implies an upper bound on the quantity $C_{\mathcal{T}}(n, m)$. As w_0 has size at most $n - 2$ and $w_0 ba^{k-1}$ has size at most $n - 1$, we have

$$\binom{\text{\# of } m\text{-words}}{\text{contained in } w} = \binom{\text{\# of } (m-1)\text{-words}}{\text{contained in } w_0} + \binom{\text{\# of } (m-1)\text{-words}}{\text{contained in } w_0 ba^{k-1}}$$
$$\leq C_{\mathcal{T}}(n-2, m-1) + C_{\mathcal{T}}(n-1, m-1)$$

for all words w. Thus the maximum number of m-words contained among all words in $\mathcal{T}_n$ obeys

this bound as well, and thus we have

$$C_{\mathcal{T}}(n,m) \leq C_{\mathcal{T}}(n-2,m-1) + C_{\mathcal{T}}(n-1,m-1).$$

To establish equality, we exhibit a word w_n of size n that attains this upper bound. For each $n \geq 0$, let w_n be the word of size n wherein each letter in an odd position is a and each letter in an even position is b, e.g. $w_0 = \varepsilon$, $w_1 = \mathsf{a}$, $w_2 = \mathsf{ab}$, $w_3 = \mathsf{aba}$, and so on. Let $c(n,m)$ denote the number of m-words contained in w_n.

We claim that for all n, $c(n,m) = C_{\mathcal{T}}(n,m)$, and proceed by induction on m. For our base case of $m = 0$, every word contains the unique empty word, so $c(n,0) = C_{\mathcal{T}}(n,0) = 1$. Now, assume that for all n we have $c(n,m') = C_{\mathcal{T}}(n,m')$ for all $m' < m$. By the recursive formula at the beginning of this proof, we have

$$c(n,m) = c(n-1,m-1) + c(n-2,m-1)$$
$$= C_{\mathcal{T}}(n-1,m-1) + C_{\mathcal{T}}(n-2,m-1)$$
$$\geq C_{\mathcal{T}}(n,m).$$

As w_n is a word of size n, we have that $c(n,m) \leq C_{\mathcal{T}}(n,m)$, and thus we have equality:

$$c(n,m) = C_{\mathcal{T}}(n,m) = C_{\mathcal{T}}(n-1,m-1) + C_{\mathcal{T}}(n-1,m-1).$$

It only remains to show that for all n,

$$C_{\mathcal{T}}(n,m) = \sum_{\ell=0}^{n-m} \binom{m}{\ell},$$

which we show by induction on m.

For all n, we have $C_{\mathcal{T}}(n,0) = \binom{0}{0} + \binom{0}{1} + \cdots + \binom{0}{n-m} = 1 + 0 + \cdots + 0 = 1$. Fix $m \geq 1$, and

assume that the statement holds for all values less than m. Then we have

$$C_{\mathcal{T}}(n,m) = C_{\mathcal{T}}(n-1,m-1) + C_{\mathcal{T}}(n-2,m-1)$$

$$= \sum_{\ell=0}^{n-m} \binom{m-1}{\ell} + \sum_{\ell=0}^{n-m-1} \binom{m-1}{\ell}$$

$$= \binom{m-1}{0} + \sum_{\ell=0}^{n-m-1} \left[\binom{m-1}{\ell+1} + \binom{m-1}{\ell} \right]$$

$$= \binom{m}{0} + \sum_{\ell=0}^{n-m-1} \binom{m}{\ell+1}$$

$$= \binom{m}{0} + \sum_{\ell=1}^{n-m} \binom{m}{\ell}$$

$$= \sum_{\ell=0}^{n-m} \binom{m}{\ell},$$

as desired. $\square$

Values of $C_{\mathcal{T}}(m+k,m)$ for small values of m and k are displayed in Figure 6-2. The recurrence proven in the proof of Proposition 6-1 may be interpreted in this context as "the value in a cell is equal to the sum of the values in the cells immediately to its west and to its south-west". As in Figure 6-1, the cells with dark gray backgrounds correspond to universality, and the cells with white backgrounds contain values so that $C_{\mathcal{T}}(m+k,m) = \binom{m+k}{m}$.

k										
10	2	4	8	16	32	64	128	256	512	1024
9	2	4	8	16	32	64	128	256	512	1023
8	2	4	8	16	32	64	128	256	511	1013
7	2	4	8	16	32	64	128	255	502	968
6	2	4	8	16	32	64	127	247	466	848
5	2	4	8	16	32	63	120	219	382	638
4	2	4	8	16	31	57	99	163	256	386
3	2	4	8	15	26	42	64	93	130	176
2	2	4	7	11	16	22	29	37	46	56
1	2	3	4	5	6	7	8	9	10	11
m	1	2	3	4	5	6	7	8	9	10

Figure 6-2. An array containing the values $C_{\mathcal{T}}(m+k,m)$ for small values of m and k.

By the isomorphism between the poset of words over a two-letter alphabet and the poset $\mathcal{W}$ of non-empty wedge permutations in Proposition 5-16, the maximum number of $\mathcal{W}_m$-permutations that any $\mathcal{W}_n$-permutation may contain is $\sum_{\ell=0}^{n-m} \binom{m-1}{\ell}$. Computation suggests that the maximum number of $\mathcal{W}_m$-permutations contained by <u>any</u> permutation of size n is $\sum_{\ell=0}^{n-m} \binom{m-1}{\ell}$ as well.

Conjecture 6-2. *For all m and n, the maximum number of $\mathcal{W}_m$-permutations contained by any permutation of size n is*

$$\sum_{\ell=0}^{n-m} \binom{m-1}{\ell}.$$

UNIVERSAL PARTITION SIZES

Table A-1 presents the known sizes for the smallest universal partitions for various classes of partitions. As remarked upon in Section 2.2, every class of partitions $\mathcal{P}$ that satisfies the joint-embedding property contains its smallest universal partitions, ie. $u_{\mathcal{P}}(m) = u_{\mathcal{P}}^p(m)$, and every other class $\mathcal{P}$ does not admit proper $\mathcal{P}_m$-universal partitions for all m, so in each case only $u_{\mathcal{P}}(m)$ is provided. As a slight abuse of notation, classes that are a union of ages $\text{Age}(u) \cup \text{Age}(v)$, which are precisely those classes that do not satisfy the joint-embedding property discussed at the end of Chapter 1, are represented as being the age of $u \cup v$. Every class of partitions may be represented as the collection of partitions that do not contain any of a finite set of "forbidden" partitions, so each class is presented both as an age representation as well as with a minimal finite set of forbidden partitions.

Forbidden partitions	Age of	Smallest universal sizes $m \geq 1$	Asymptotic formula
None	ω^ω	1, 3, 5, 8, 10, 14, 16, 20, 23, . . .	$m \log m$
2	1^ω	1, 2, 3, 4, 5, 6, 7, 8, 9, . . .	m
3	2^ω	1, 3, 4, 6, 7, 9, 10, 12, 13, . . .	$\frac{3}{2} m$
4	3^ω	1, 3, 5, 7, 8, 11, 12, 14, 16, . . .	$\frac{11}{6} m$
5	4^ω	1, 3, 5, 8, 9, 12, 13, 16, 18, . . .	$\frac{25}{12} m$
6	5^ω	1, 3, 5, 8, 10, 13, 14, 17, 19, . . .	$\frac{137}{60} m$
21	$1^\omega \cup \omega$	1, 3, 5, 7, 9, 11, 13, 15, 17, . . .	$2m$
31	$2^\omega \cup \omega$	1, 3, 5, 8, 10, 13, 15, 18, 20, . . .	$\frac{5}{2} m$
41	$3^\omega \cup \omega$	1, 3, 5, 8, 10, 14, 16, 19, 22, . . .	$\frac{17}{6} m$
51	$4^\omega \cup \omega$	1, 3, 5, 8, 10, 14, 16, 20, 23, . . .	$\frac{37}{12} m$
22	$\omega 1^\omega$	1, 3, 5, 7, 9, 11, 13, 15, 17, . . .	$2m$
32	$2^\omega \cup \omega 1^\omega$	1, 3, 5, 8, 10, 13, 15, 18, 20, . . .	$\frac{5}{2} m$
42	$3^\omega \cup \omega 1^\omega$	1, 3, 5, 8, 10, 14, 16, 19, 22, . . .	$\frac{17}{6} m$
33	$\omega 2^\omega$	1, 3, 5, 8, 10, 13, 15, 18, 20, . . .	$\frac{5}{2} m$
311	$2^\omega \cup \omega\omega$	1, 3, 5, 8, 10, 14, 16, 20, 22, . . .	$3m$

Table A-1. Sizes of smallest (proper) universal partitions for various partition classes.

411	$3^{\omega} \cup \omega\omega$	1, 3, 5, 8, 10, 14, 16, 20, 23, ...	$\frac{43}{12}m$
221	$\omega 1^{\omega} \cup \omega\omega$	1, 3, 5, 8, 10, 13, 15, 18, 20, ...	$\frac{5}{2}m$
321	$2^{\omega} \cup \omega 1^{\omega} \cup \omega\omega$	1, 3, 5, 8, 10, 14, 16, 20, 22, ...	$3m$

Table A-1. Continued

Table B-1 presents the known sizes for the smallest universal compositions for various classes of compositions. If a composition class $\mathcal{C}$ is known to satisfy $u_{\mathcal{C}}^p(m) = u_{\mathcal{C}}(m)$, then only values of $u_{\mathcal{C}}(m)$ are provided, and otherwise both sequences are displayed. If a formula is known for a given sequence, it is provided as well. Every class of compositions may be represented as the collection of compositions that do not contain any of a finite set of "forbidden" compositions, so each class is presented as both the age of an infinite composition and through a minimal finite set of forbidden compositions.

Forbidden compositions	Age of	Smallest universal sizes $m \geq 1$	Formula
None	ω^ω	$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 11, 14, 17, 21, 25,\ldots$	$\ell(m)$
21	$1^\omega\omega$	$u_{\mathcal{C}}(m) = 1, 3, 5, 7, \ 9, 11, 13, 15, 17,\ldots$	$2m-1$
111	$\omega\omega$	$u_{\mathcal{C}}(m) = 1, 3, 5, 7, \ 9, 11, 13, 15, 17,\ldots$	$2m-1$
111, 21	1ω	$u_{\mathcal{C}}(m) = 1, 3, 4, 5, \ 6, \ 7, \ 8, \ 9, 10,\ldots$	$m+1$
21, 3	$1^\omega 2$	$u_{\mathcal{C}}(m) = 1, 3, 4, 5, \ 6, \ 7, \ 8, \ 9, 10,\ldots$	$m+1$
121	$\omega 1^\omega \omega$	$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 11, 14, 17, 20, 23,\ldots$ $u_{\mathcal{C}}^p(m) = 1, 3, 6, 9, 12, 15, 18, 21, 24,\ldots$	
13	$\omega 2^\omega$	$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 10, 13, 15, 18, 20,\ldots$ $u_{\mathcal{C}}^p(m) = 1, 3, 6, 8, 11, 13, 16, 18, 21,\ldots$	
211	$1^\omega \omega\omega$	$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 11, 14, 17, 20, 23,\ldots$ $u_{\mathcal{C}}^p(m) = 1, 3, 5, 8, 11, 14, 17, 20, 23,\ldots$	
22	$1^\omega \omega 1^\omega$	$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 11, 14, 17, 20, 23,\ldots$	$3m-4$
1^3	ω^2	$u_{\mathcal{C}}(m) = 1, 3, 5, 7, \ 9, 11, 13, 15, 17,\ldots$	$2m-1$
1^4	ω^3	$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 11, 14, 17, 20, 23,\ldots$ $u_{\mathcal{C}}^p(m) = 1, 3, 5, 8, 11, 14, 17, 20, 23,\ldots$	$3m-4$
1^5	ω^4	$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 11, 14, 17, 21, 24,\ldots$ $u_{\mathcal{C}}^p(m) = 1, 3, 5, 8, 12, 16, 20, 24, 28,\ldots$	$4m-8$
1^6	ω^5	$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 11, 14, 17, 21, 25,\ldots$ $u_{\mathcal{C}}^p(m) = 1, 3, 5, 8, 11, 16, 21, 26, 31,\ldots$	$5m-14$

Table B-1. Sizes of smallest (proper) universal compositions for various composition classes.

1^7	ω^6	$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 11, 14, 17, 21, 25, \ldots$	
		$u_{\mathcal{C}}^p(m) = 1, 3, 5, 8, 11, 14, 20, 26, 32, \ldots$	$6m - 22$
3	2^ω	$u_{\mathcal{C}}(m) = 1, 3, 4, 6, \ \ 7, \ \ 9, 10, 12, 13, \ldots$	$\lfloor 3m/2 \rfloor$
4	3^ω	$u_{\mathcal{C}}(m) = 1, 3, 5, 7, \ \ 9, 11, 13, 15, 17, \ldots$	$2m - 1$
5	4^ω	$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 10, 12, 14, 17, 19, \ldots$	$\lfloor 9m/4 \rfloor - 1$
6	5^ω	$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 11, 13, 15, 18, 21, \ldots$	$\lfloor 5m/4 \rfloor + \lceil 5m/4 \rceil - 2$
7	6^ω	$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 11, 14, 16, 19, 22, \ldots$	$\lfloor 6m/4 \rfloor + \lceil 5m/4 \rceil - 3$

Table B-1. Continued

Table C-1 presents the known sizes of smallest universal permutations for various classes of permutations. If a permutation class $\mathcal{P}$ is known to satisfy $u^p_{\mathcal{P}}(m) = u_{\mathcal{P}}(m)$, then only one sequence is provided, and otherwise both sequences are displayed. If a formula is known for a given sequence, it is provided as well. Where possible, the basis of the class and a concise description of the class are provided.
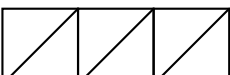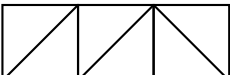
Basis elements	Description	Smallest universal sizes $m \geq 1$	Formula
$\varnothing$		$u_C(m) = 1, 3, 5, 9, 13, 17, \quad\quad\quad \ldots$	
21		$u_C(m) = 1, 2, 3, 4, \ 5, \ 6, \ 7, \ 8, \ 9, \ldots$	m
$231, 132$		$u_C(m) = 1, 3, 5, 7, \ 9, 11, 13, 15, 17, \ldots$	$2m-1$
$231, 312$		$u_C(m) = 1, 3, 5, 8, 11, 14, 17, 21, 25, \ldots$	$\ell(m)$
$231, 132, 321$		$u_C(m) = 1, 3, 4, 6, \ 7, \ 9, 10, 12, 13, \ldots$ $u^p_C(m) = 1, 3, 4, 6, \ 8, 10, 12, 14, 16, \ldots$	$\lfloor 3m/2 \rfloor$ $2m-2$
$213, 321$		$u_C(m) = 1, 3, 5, 7, \ 9, 11, 13, 15, 17, \ldots$ $u^p_C(m) = 1, 3, 5, 8, 11, 14, 17, 20, 23, \ldots$	$2m-1$ $3m-4$
$231, 321$		$u_C(m) = 1, 3, 5, 7, \ 9, 12, 14, \quad\quad \ldots$ $u^p_C(m) = 1, 3, 5, 8, 11, 14, 17, 21, 25, \ldots$	
$2143, 3412$		$u_C(m) = 1, 3, 5, 8, 12, \quad\quad\quad\quad \ldots$ $u^p_C(m) = 1, 3, 5, 8, 17, \quad\quad\quad\quad \ldots$	
finite		$u_C(m) = 1, 3, 5, 8, 11, 14, \quad\quad \ldots$ $u^p_C(m) = 1, 3, 5, 8, 11, 14, \quad\quad \ldots$	
finite		$u_C(m) = 1, 3, 5, 8, 11, 14, \quad\quad \ldots$ $u^p_C(m) = 1, 3, 5, 8, 11, 15, \quad\quad \ldots$	
finite		$u_C(m) = 1, 3, 5, 8, 11, 14, \quad\quad \ldots$ $u^p_C(m) = 1, 3, 5, 8, 11, 14, \quad\quad \ldots$	

Table C-1. Sizes of smallest (proper) universal permutations for various permutation classes.

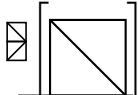| Basis elements	Description	Smallest universal sizes $m \geq 1$	Formula
321		$u_{\mathcal{C}}(m) = 1, 3, 5, 7, 10, 13, \qquad \ldots$ $u_{\mathcal{C}}^{p}(m) = 1, 3, 5, 7, 10, 13, 16, \qquad \ldots$	
321, 3142, 2143		$u_{\mathcal{C}}(m) = 1, 3, 5, 7, \ 9, 11, 13, 15, 17, \ldots$	$2m - 1$
finite		$u_{\mathcal{C}}(m) = 1, 3, 5, 7, 10, 13, \qquad \ldots$ $u_{\mathcal{C}}^{p}(m) = 1, 3, 5, 7, 10, 13, 16, 19, \qquad \ldots$	
321, 3412		$u_{\mathcal{C}}(m) = 1, 3, 5, 7, 10, 13, \qquad \ldots$ $u_{\mathcal{C}}^{p}(m) = 1, 3, 5, 7, 10, 14, 18, \qquad \ldots$	
321, 3142		$u_{\mathcal{C}}(m) = 1, 3, 5, 7, 10, 13, \qquad \ldots$ $u_{\mathcal{C}}^{p}(m) = 1, 3, 5, 8, 12, 16, 20, \qquad \ldots$	
321, 1342		$u_{\mathcal{C}}(m) = 1, 3, 5, 7, 10, 13, \qquad \ldots$ $u_{\mathcal{C}}^{p}(m) = 1, 3, 5, 8, 13, 19, \qquad \ldots$	
321, 1324		$u_{\mathcal{C}}(m) = 1, 3, 5, 7, 10, 13, \qquad \ldots$ $u_{\mathcal{C}}^{p}(m) = 1, 3, 5, 8, \qquad \ldots$	
321, 2143		$u_{\mathcal{C}}(m) = 1, 3, 5, 7, 10, 13, \qquad \ldots$ $u_{\mathcal{C}}^{p}(m) = 1, 3, 5, \qquad \ldots$	
321, 2134		$u_{\mathcal{C}}(m) = 1, 3, 5, 7, \ 9, 12, \qquad \ldots$ $u_{\mathcal{C}}^{p}(m) = 1, 3, 5, 7, 10, 15, 20, \qquad \ldots$	
321, 2341		$u_{\mathcal{C}}(m) = 1, 3, 5, 7, \ 9, 12, \qquad \ldots$ $u_{\mathcal{C}}^{p}(m) = 1, 3, 5, 7, 10, 14, 19, \qquad \ldots$	

Table C-1. Continued
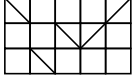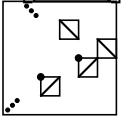
Basis elements	Description	Smallest universal sizes $m \geq 1$	Formula
231		$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 11, 15, \qquad \ldots$ $u_{\mathcal{C}}^{p}(m) = 1, 3, 5, 8, 12, 15, 20, \qquad \ldots$	
231, 1234	non-atomic	$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 11, 14, \qquad \ldots$ $u_{\mathcal{C}}^{p}(m) = 1, 3, \; 5$	
231, 1243		$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 11, 14, \qquad \ldots$ $u_{\mathcal{C}}^{p}(m) = 1, 3, 5, 8, 12, 17, 22, \qquad \ldots$	
231, 1324		$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 11, 14, \qquad \ldots$ $u_{\mathcal{C}}^{p}(m) = 1, 3, 6, 9, 13, 18, \qquad \ldots$	
231, 2143		$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 11, 14, \qquad \ldots$ $u_{\mathcal{C}}^{p}(m) = 1, 3, 5, 8, 11, 14, 17, 20, 23, \ldots$	$3m - 4$
231, 3124		$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 11, 15, \qquad \ldots$ $u_{\mathcal{C}}^{p}(m) = 1, 3, 5, 8, 12, 16, 20, \qquad \ldots$	
231, 3214		$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 11, 14, \qquad \ldots$ $u_{\mathcal{C}}^{p}(m) = 1, 3, 5, 8, 12, 16, 20, \qquad \ldots$	
231, 4123		$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 11, 14, \qquad \ldots$ $u_{\mathcal{C}}^{p}(m) = 1, 3, 5, 8, 13, 18, \qquad \ldots$	
231, 4213		$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 11, 15, \qquad \ldots$ $u_{\mathcal{C}}^{p}(m) = 1, 3, 5, 9, 13, 17, 21, \qquad \ldots$	
231, 4312		$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 11, 14, \qquad \ldots$ $u_{\mathcal{C}}^{p}(m) = 1, 3, 5, 9, 13, 18, 24, \qquad \ldots$	
231, 4321		$u_{\mathcal{C}}(m) = 1, 3, 5, 8, 11, 14, \qquad \ldots$ $u_{\mathcal{C}}^{p}(m) = 1, 3, 5, 8, 14, 19, \qquad \ldots$	

Table C-1. Continued

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BIOGRAPHICAL SKETCH

Michael Engen was born in George Town, Cayman Islands. Shortly thereafter, he moved to Florida, attending a sequence of schools, each farther North than the one that preceded it, before finally enrolling as an undergraduate student at the University of Florida in 2013.

He completed a Bachelor of Science degree in mathematics in 2015 and began his graduate studies later that year. He earned a Master of Science degree in mathematics in 2017 and finally a Doctor of Philosophy degree in mathematics in 2021.