FIBER-BASED OPTICAL RESONATORS

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Cavity QED, Resonator Design and Technological Applications

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University of Y

LOGO

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To my parents

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Chapter 1

Basics of Electromagnetic Waves

In the 19^{th} century, one of the most important discoveries in the field of physics took place which was the unification of electrostatic and magnetic effects. This established a direct link between the laws of electromagnetism and the laws governing the propagation of light in terms of Maxwells equation. These are the four differential equations that underpin the summary of almost all electromagnetic (EM) phenomena and EM-wave propagation including light. For this reason, Maxwells equations are still the essential knowledge resource for electronics; optics, and communication sciences. In this chapter, a brief description of Maxwell's equations and their application for understanding electromagnetic wave propagation under boundary conditions are presented. This includes waveguide systems and fiber optic waveguides.

1.1. Electromagnetism

The science of unified electromagnetism evolved from the spectacular advances made by many different scientists while discovering the electrostatic and magnetic effects. To name a few which directly influenced the work of Scottish physicist James Clerk Maxwell, were the pioneering works of Carl Friedrich Gauss (1775-1855), Andere Ampere (1775-1836), and Michael Faraday (1791-1867). They established the fact that

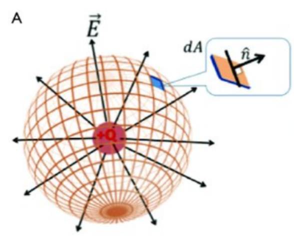

Figure 1.1: Depiction of Guass Law for electric field lines due to an enclosed charge inside a sphere.

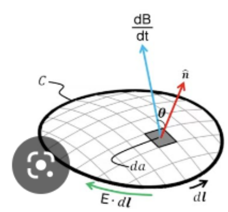

Figure 1.2: Line integral of the electric field is related to the rate of change of magnetic field.

- The electric flux through an enclosed surface is related to the volumetric charge density, ρ (Fig. 1.1).

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1.1}$$

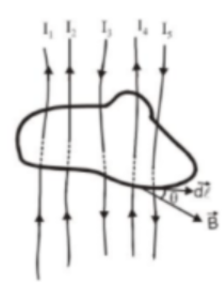

Figure 1.3: Curl of the magnetic field is related to the enclosed current sources.

- A changing magnetic field produces an electric field and therefore induces electricity in a metallic coil through which the magnetic flux transverse (Fig. 1.1).

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.2}$$

- The magnetic field in a closed loop circuit depends on the current through the loop. In its differential form, the curl of the magnetic field depends on the associated current density J (Fig. 1.1)

$$\nabla \times \mathbf{B} = \mu_0 \, \mathbf{J} \tag{1.3}$$

  Here ϵ_0 and μ_0 are the electric permittivity and magnetic permeability in a vacuum. $\nabla \cdot ()$ and $\nabla \times ()$ are the divergence and the curl operators.

These discoveries made it clear that the two seemingly different phenomena, electricity, and magnetism, both are produced by electric charges depending on whether they are stationary or in motion. With the realization of electromagnetism as a combined study, a clear mathematical description of electromagnetism was introduced by the detailed theoretical work from Maxwell. Simple differential equations emerging from Maxwell's work contained all the essence of the electromagnetic phenomenon. He also put forward a theory where he added an additional term of displacement current to Amperes law. After further research and simplification by British physicist Oliver Heaviside, these equations were finally condensed into the currently famous four Maxwell electromagnetic wave equations, which present a concise and elegant description of light sources and associated phenomena in wave optics.

1.2. Electromagnetic Waves

Light traveling through a medium or vacuum can be described in terms of oscillating electric and magnetic field vectors at each point of time and space. These sinusoidal fields are an exact analogy of the sinusoidal displacement of particles as a mechanical wave propagates through a medium.

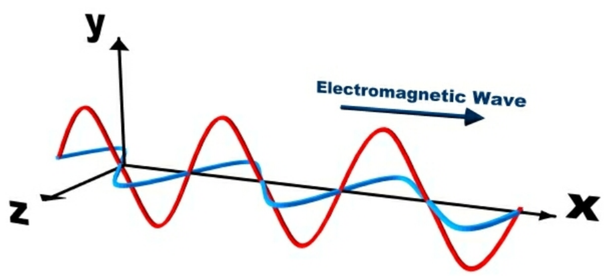

Figure 1.4: Electromagnetic wave propagation in a vacuum can be considered as oscillating electric and magnetic field amplitude in two orthogonal axes and perpendicular to the propagation axis.

As shown in Fig. 1.4, the electric field vector $\mathbf{E}$ and magnetic field vector $\mathbf{B}$ are orthogonal to the wave propagation direction and their amplitude sinusoidally vary in space and time. This description of EM-wave directly arises from the four basic laws of electromagnetism. They describe a deeper connection between the following four fundamental quantities of electromagnetism.

- Displacement vector inside a medium $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$
- Magnetization vector $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$
- Charge density ρ
- Current density $\mathbf{J} = \sigma \mathbf{E}$.

where $\mathbf{P}$ and $\mathbf{M}$ are defined as the polarizability and the magnetization and are related to electric and magnetic fields via electric (χ_e) and magnetic susceptibility (χ_m) respectively.

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \; ; \; \mathbf{M} = \chi_m \mathbf{H} \tag{1.4}$$

Note that with Eq. 1.4, Displacement and Magnetization vectors can also be written in terms of vacuum and relative permittivity and permeability respectively.

$$\mathbf{D} = \epsilon_0 \, (1 + \chi_e) \, \mathbf{E} = \epsilon_0 \, \epsilon_r \, \mathbf{E} \tag{1.5}$$

$$\mathbf{B} = \mu_0 \, (1 + \chi_m) \, \mathbf{H} = \mu_0 \, \mu_r \, \mathbf{E} \tag{1.6}$$

where $\epsilon_r = 1 + \chi_e$ is the relative permittivity and $\mu_r = 1 + \chi_m$ is the realtive permeability.

Maxwell's equation establishes relations between these quantities. We will see in the next section that the solution of those differential equations produces classical wave equations with electric and magnetic field vectors recognized as oscillating variables and speed of light as a fundamental constant defined in terms of the permittivity of the medium, producing a velocity constant $\frac{1}{\sqrt{\mu_0 \epsilon_0}}$ in a vacuum, equal to the speed of light.

1.3. Maxwell's Equations

Maxwell's equation concisely describes the relation between electromagnetic quantities such that they provide the complete essence of electromagnetic wave propagation under various conditions. The four Maxwell's equations in the differential form relating the fundamental quantities of electromagnetism are:

$$\nabla \cdot \mathbf{D} = \rho \tag{1.7}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.8}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.9}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{1.10}$$

The most intriguing feature of Maxwell's equations is revealed by obtaining a wave equation with $\mathbf{E}$ and $\mathbf{B}$ being the oscillating terms of the propagating wave as follows.

By applying $\nabla\times$ operator to Eq. 1.8:

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \frac{\partial(\nabla \times \mathbf{H})}{\partial t}. \tag{1.11}$$

Using the following vector identity

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}, \tag{1.12}$$

which in combination with Eq. 1.9, produces the following wave equation, note $\nabla \cdot \mathbf{D} = 0$:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \tag{1.13}$$

and similarly for the magnetic field

$$\nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = 0. \tag{1.14}$$

For vacuum, $c_0 = \frac{c}{n} = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$, where c_0 is the speed of light in vacuum and n is the refractive index of the medium. The wave equation for both electric and magnetic fields in a vacuum reads as follows:

$$\nabla^2 \mathbf{E} - \frac{1}{c_0{}^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \;\; ; \;\; \nabla^2 \mathbf{B} - \frac{1}{c_0{}^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} = 0. \tag{1.15}$$

1.3.1. Energy transfer and Poynting vector

The energy transported by EM-wave per unit time per unit area is an important characteristic of the EM-field. This is represented by the Poynting vector as follows

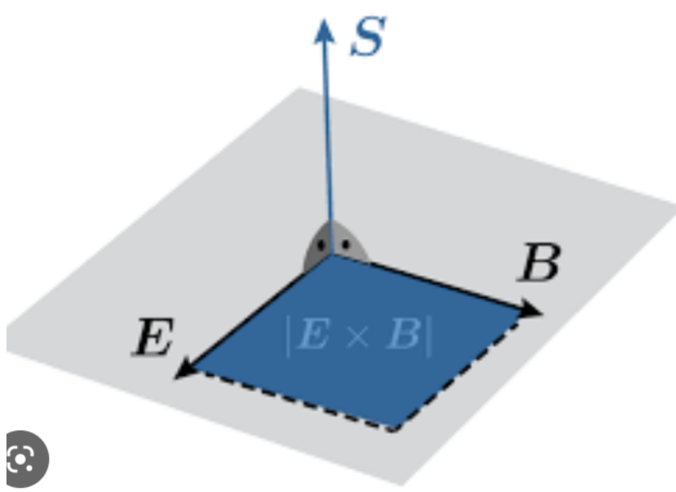

Figure 1.5: Poynting vector S represents the energy transported per unit time per unit area in a plane orthogonal to the plane containing E and B vectors.

$$\mathbf{S} = \frac{1}{\mu_0}\left(\, \mathbf{E} \times \mathbf{B} \right). \tag{1.16}$$

The above expression can be understood by calculating the work done by the EM-wave on some charge configuration inside volume V, which is as follows

$$\frac{\partial W}{\partial t} = \int_V \left(\mathbf{E} \cdot \mathbf{J}\right) d\tau \tag{1.17}$$

Using Maxwell's equations, the above expression takes the following form

$$\frac{\partial W}{\partial t} = -\frac{d}{dt}\left(\int_V (\frac{1}{2}\epsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0}\mathbf{B}^2) d\tau\right) - \frac{1}{\mu_0}\int_S (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{a} \tag{1.18}$$

Here $\int_V$ and $\int_S$ denotes the volume and surface integral over a selected region, within which the charges are enclosed.

Therefore, the work done on a charge configuration by EM-field interaction is equal to the reduction in the rate of stored energy, and the energy flown out of a given surface. The last expression in Eq. 1.18, for this reason, represents the energy transported by the EM-fields per unit area, per unit time. The Poynting theorem is an energy conservation theorem for a time-dependent electromagnetic field.

1.3.2. Maxwell's equations under boundary condition

One of the interesting applications of Maxwell's equation is for the theoretical analysis of electromagnetic (EM) wave propagation in a bounded medium. For example, EM-wave traveling inside a hollow metallic waveguide or a coaxial cable. This leads to the discussion of waveguide and resonator modes. These systems resemble very closely to the optical fiber systems, used for guiding light. Guided propagation of the EM-wave is the backbone of the communication system.

In the following section, a brief introduction to the waveguide modes is given as follows:

1.3.3. Waveguide modes

Guiding electromagnetic waves with appropriate boundary conditions in a confined geometry is the basis of the current electromagnetic and fiber optic communication. Maxwell's equation provides the framework for reflection and refraction within the guiding medium. In the following section, an example of waveguide propagation provides a direct application of Maxwell's equation. The specific case of the light propagation in the fiber optic and confinement of light in an optical resonator is discussed in further sections.

For an EM-wave propagation through a metallic waveguide, assuming it as a perfect conductor, the typical boundary conditions are:
1) The tangential component of the electric field should be equal to zero.
2) The normal derivative of the tangential component of the magnetic field should be equal to zero.
For a typical hollow waveguide, the wave propagation involves guiding the EM fields inside a hollow metal pipe. In an ideal case, for a monochromatic wave propagating inside the waveguide and considering the above boundary conditions, the following general electric and magnetic field vectors can be written

$$\mathbf{E}(\mathbf{x{,}y{,}z{,}t} = \mathbf{E}_0(\mathbf{x{,}y{,}z}) \exp i(kz - \omega t) \tag{1.19}$$

$$\mathbf{B}(\mathbf{x{,}y{,}z{,}t} = \mathbf{B}_0(\mathbf{x{,}y{,}z}) \exp i(kz - \omega t) \tag{1.20}$$

where $\mathbf{E}_0(\mathbf{x{,}y{,}z})$ and $\mathbf{B}_0(\mathbf{x{,}y{,}z})$ are three dimensional vectors with components (E_x, E_y, E_z) and (B_x, B_y, B_z).

As there are no free charges present inside the wave-guide, Maxwell's equations are as follows

$$\nabla \cdot \mathbf{E} = 0 \tag{1.21}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.22}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.23}$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{1.24}$$

Now one can use the last two of Maxwell's equations (Eq. 1.23 and Eq. 1.24) and do further simplification to obtain x- and y- field components in terms of E_z and B_z.

$$\mathbf{E}_x = \frac{i}{{k_w}^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right) \; ; \; \mathbf{E}_y = \frac{i}{{k_w}^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right), \tag{1.25}$$

$$\mathbf{B}_x = \frac{i}{{k_w}^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right) \; ; \; \mathbf{B}_y = \frac{i}{{k_w}^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right), \tag{1.26}$$

where $k_w = (\omega/c)^2 - k^2$.

Using these relations with the first two Maxwell's equations (Eq. 1.21 and Eq. 1.22) one obtains two uncoupled differential equations for z-components.

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right] E_z = 0 \tag{1.27}$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right] B_z = 0 \tag{1.28}$$

Therefore, solving the above differential equations for z-components with the boundary conditions allows obtaining the complete description of the field components for wave-guide propagation.

The two specific cases of wave-guide propagation are for

- $E_z = 0$: This is called as Transverse electric (TE-wave): E= (Ex, Ey,0) and

B=(Bx, By, Bz)

- $B_z = 0$: This is called as Transverse magnetic (TM-wave): E=(Ex, Ey, Ez) and B=(Bx, By,0).

It can be established from Maxwell's equation that both $E_z = 0$ and $B_z = 0$ (which is known as TEM_{00}) mode is not possible inside a hollow metallic waveguide. However, a coaxial cable with one inner and outer conductor configuration and the in-between space filled with a dielectric medium allows for the propagation of the lowest fundamental mode i.e. TEM_{00} mode.

1.3.4. Rectangular Waveguide

Figure 1.6: A rectangular metal waveguide.

Typical waveguides used for guided EM-wave transmission are usually rectangular or mostly cylindrical in shape. The design considerations of these waveguides are based on solving Maxwell's differential equations with boundary conditions.

The boundary condition for an ideal hollow conductor is:

- There are no currents on the surface of the conductor

$$\mathbf{n} \cdot \mathbf{B} = 0 \tag{1.29}$$

- There are no free charges on the surface of the conductor

$$\mathbf{n} \times \mathbf{E} = 0 \tag{1.30}$$

This means that the following EM-field components vanish at the boundaries

$$[E_x, E_z, B_y] = 0 \tag{1.31}$$

at boundaries y=0 and y=b.

And

$$[E_y, E_z, B_x] = 0 \tag{1.32}$$

at boundaries x=0 and x=a.

It is clear that the z-components of the fields are always zero and only components orthogonal to the propagation direction contribute. One can then write the following general forms for the EM-fields

$$\mathbf{E}(x, y, z, t) = \mathbf{E}_0(x, y) \exp -i(kz - \omega t) \ , \ \ \mathbf{B}(x, y, z, t) = \mathbf{B}_0(x, y) \exp -i(kz - \omega t) \tag{1.33}$$

With the wave, Eq. 1.25 & 1.26, Maxwells equation, and above boundary conditions, one can express the field components in terms of geometric

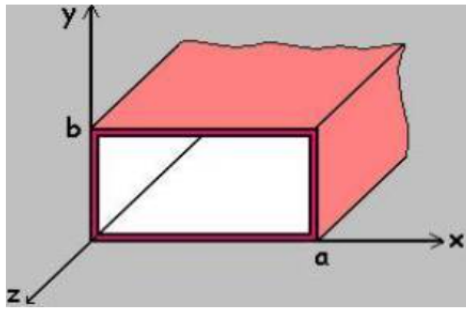

Figure 1.7: Hollow metallic waveguides are used for guided EM-wave with microwave frequencies. For example, a rectangular waveguide with dimensions a and b along the two orthogonal x- and y-axes and extension along the propagation direction (z-axis) can be solved for Maxwells differential equation to obtain the limits on the frequencies which can propagate with minimum losses.

parameters of the waveguide

$$E_x = \alpha_x cos(\frac{m\pi x}{a}) sin(\frac{n\pi y}{b}) \tag{1.34}$$

$$E_y = \alpha_y sin(\frac{m\pi x}{a}) cos(\frac{n\pi y}{b}) \tag{1.35}$$

$$E_z = \alpha_z sin(\frac{m\pi x}{a}) sin(\frac{n\pi y}{b}) \tag{1.36}$$

$$B_x = \beta_x sin(\frac{m\pi x}{a}) cos(\frac{n\pi y}{b}) \tag{1.37}$$

$$B_y = \beta_y cos(\frac{m\pi x}{a}) sin(\frac{n\pi y}{b}) \tag{1.38}$$

$$B_z = \beta_z cos(\frac{m\pi x}{a}) cos(\frac{n\pi y}{b}) \tag{1.39}$$

The coefficients $[\alpha_x, \alpha_y, \alpha_z]$ and $[\beta_x, \beta_y, \beta_z]$ indicates many possible solution for the fields. Relation between these coefficients can be easily found by using Eq. 1.20 and Eq. 1.22.

Another important result of substituting EM-field components into the wave equation is that the wave propagation constant is constrained by the geometry of the waveguide as follows

$$k^2 = \left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \tag{1.40}$$

Therefore, for proper guiding of the EM-wave, which corresponds to having real values of k, the frequency must be above the following critical frequency

$$\omega_c = c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \tag{1.41}$$

The lowest cutoff frequency for a given waveguide can be calculated for the mode TE_{10}:

$$\omega_{10} = c\pi/a. \tag{1.42}$$

Similarly, the lowest frequency which can be guided is for the mode TM_{11}:

$$\omega_{11} = c\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2} \tag{1.43}$$

It can be seen that the hollow conductor waveguides allow for the transmission of EM-wave in microwave frequencies. For the transmission of light waves having frequencies of a few hundred tera-hertz, optical fibers are used. They are cylindrical dielectric tubes of very small diameter which can guide optical signals to several hundreds of kilometers with very low losses. Optical fiber can be considered as dielectric waveguides. The light propagation inside optical fiber can be simplified to the exact analytic solution of the Maxwells equation for some special cases as explained in the next sections.

1.4. Optical modes inside an optical fiber

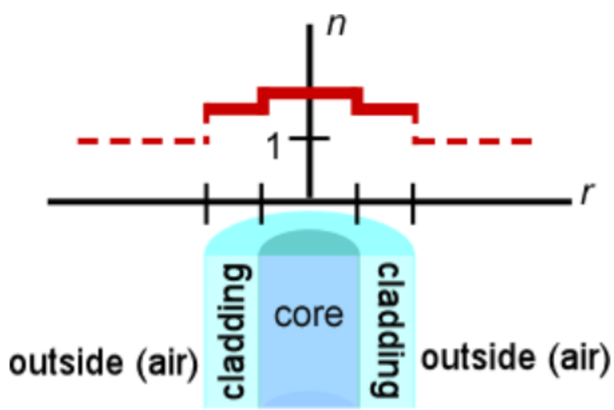

Figure 1.8: Figure shows the refractive index profile of a typical step-index single-mode fiber. The core has a slightly higher refractive index in comparison to the cladding due to small Germanium doping.

Optical fibers are very widely used in telecommunications systems, optical sensors, and precision optical systems. These are cylindrically symmetric hair-thin dielectric materials with a few hundred-micrometer diameters. They allow low-loss transmission of light beams via the mechanism of total internal reflection. In principle, they can have various cross-sectional shapes with a varying profiles of the refractive index along the cross-section. However, standard optical fibers, which are widely used for optical communication, are step-index fibers. They have a central core region with a diameter of a few microns and an outer cladding region with an overall fiber diameter of a few hundred of microns. The core region is Germanium doped to have a slightly higher refractive index compared to the cladding region. For a very small core diameter typically one can have a single optical mode propagation inside the fiber. The light propagation inside a dielectric waveguide fiber can be analyzed by solving Maxwell's equations. In the following sections, mode analysis of the guided light in a fiber is described below and single-mode fiber propagation conditions are explained

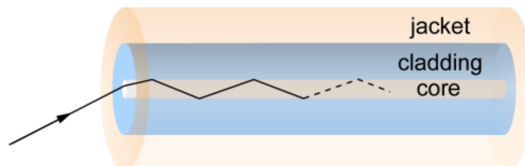

Figure 1.9: Figure shows a ray propagation scheme of light traveling through the fiber. The slightly lower refractive index of the cladding region allows total internal reflection of light for a specific input incident angle on the fiber.

1.4.1. EM-wave equation and its solution for an SM-fiber

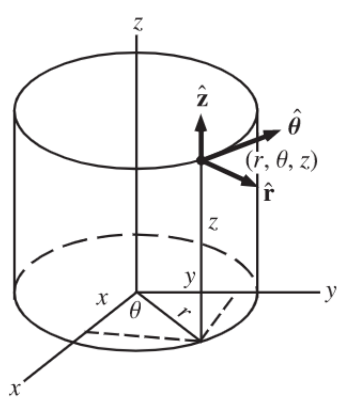

Figure 1.10: Cylindrical coordinate system for solving Maxwell's equations. The z-axis is chosen along the symmetry axis of the fiber .

Light propagation inside a step-index fiber can be solved by first solving Maxwell's equations for isotropic dielectrics in cylindrical coordinates. After that, the solutions for the core and cladding region of the fiber are applied to satisfy the boundary conditions at the core-cladding interface. Here the main focus is on the natural modes which are purely guided modes close to the core-cladding interface and decay exponentially in the transverse direction from the core of the fiber. For this reason, these modes are sometimes also referred to as core-guided modes.

The electromagnetic wave inside an isotropic charge-free dielectric has zero charge and current density.

$$\rho = 0, J = 0. \tag{1.44}$$

The wave equation Eq. 1.13 then has the following form

$$\Delta E(r,t) - \frac{1}{v(r)^2}\frac{\partial^2 E(r,t)}{\partial t^2} = 0 \tag{1.45}$$

where $v(r) = \frac{c}{n(r)}$ with c is the speed of light in vacuum and the refractive index of the medium is defined as $n(r) = \sqrt{\mu_r(r)\epsilon_r(r)}$. The wave equation for the H components is written by simply replacing E with H.

Assuming the cylindrical axis of symmetry of the fiber along the z-axis, the Laplace operator Δ in the cylindrical coordinates (r,ϕ,z), can be defined as follows

$$\Delta = \partial_r{}^2 + \frac{1}{r^2}\partial_\phi{}^2 + \partial_z{}^2 \tag{1.46}$$

where $[\partial_r, \partial_\phi, \partial_z] = \left[\frac{\partial}{\partial r}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial z}\right]$.

The light propagates along the z-axis in an isotropic charge-free medium of the fiber. The temporal evolution of the electric and magnetic field components for the angular frequency ω of light and the fiber propagation constant β can be written as.

$$\begin{bmatrix} E(r,t) \\ H(r,t) \end{bmatrix} = \begin{bmatrix} E(r,\phi) \\ H(r,\phi) \end{bmatrix} \exp\left(-i(\omega t + \beta z)\right) \tag{1.47}$$

Using Maxwell's equations Eq. 1.8 and Eq. 1.10, one can express the transversal components in terms of the axial field components E_z and H_z as follows

$$E_r = \frac{i\beta}{\omega^2\mu\epsilon - \beta^2}\left(\partial_r E_z + \frac{\omega\mu}{\beta}\frac{\partial_\phi}{r}H_z\right) \tag{1.48}$$

$$E_\phi = \frac{i\beta}{\omega^2\mu\epsilon - \beta^2}\left(\frac{1}{r}\partial_\phi E_z - \frac{\omega\mu}{\beta}\frac{\partial_r}{r}H_z\right) \tag{1.49}$$

$$H_r = \frac{i\beta}{\omega^2\mu\epsilon - \beta^2}\left(\partial_r H_z - \frac{\omega\epsilon}{\beta}\frac{\partial_\phi}{r}E_z\right) \tag{1.50}$$

$$H_\phi = \frac{i\beta}{\omega^2\mu\epsilon - \beta^2}\left(\frac{1}{r}\partial_\phi H_z + \frac{\omega\epsilon}{\beta}\frac{\partial_r}{r}E_z\right) \tag{1.51}$$

Therefore the wave equations need to be solved only for the axial components (E_z and H_z). The transversal components can be derived from the above equations.

The wave equation for the axial component in cylindrical coordinates is as follows

$$\left[{\partial_r}^2 + \frac{1}{r}\partial_r + (k^2 - \beta^2)\right]\begin{bmatrix} E(r,\phi) \\ H(r,\phi) \end{bmatrix} = 0 \tag{1.52}$$

where $k = n\frac{\omega}{c}$ is the wave number inside the fiber with refractive index n. This is a standard differential equation that can be separated into radial and azimuthal parts with the following ansatz.

$$\begin{bmatrix} E(z,\phi) \\ H(z,\phi) \end{bmatrix} = \begin{bmatrix} E_0 e_z(r) \\ H_0 h_z(r) \end{bmatrix} \exp\left(\pm il\phi\right) \tag{1.53}$$

here l is a positive integer and $e_z(r), h_z(r)$ are the electric and magnetic field functions defining the field configuration in the radial plane.

The differential equation arising from the wave equation takes the following form

$$\left[\partial_r{}^2 + \frac{1}{r}\partial_r + (k^2 - \beta^2 - \frac{l^2}{r^2})\right] \begin{bmatrix} e_z(r) \\ h_z(r) \end{bmatrix} = 0 \tag{1.54}$$

This differential equation is well known and its solutions are known as the Bessel function or the modified Bessel function of order l. This depends on the coefficient of the last term of the differential equation.

When $h^2 = k^2 - \beta^2 > 0$, the solutions in terms of the Bessel functions of the first and second kind is as follows

$$\begin{bmatrix} e_z(r) \\ h_z(r) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_1{}^x \end{bmatrix} J_l(hr) + \begin{bmatrix} c_2 \\ c_2{}^x \end{bmatrix} Y_l(hr) \tag{1.55}$$

on the other hand if $q^2 = k^2 - \beta^2 < 0$, the solutions in terms of the modified Bessel functions of the first and second kind is as follows

$$\begin{bmatrix} e_z(r) \\ h_z(r) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_1{}^x \end{bmatrix} I_l(qr) + \begin{bmatrix} c_2 \\ c_2{}^x \end{bmatrix} K_l(qr) \tag{1.56}$$

The complex coefficients $c_1, c_1{}^x, c_2, c_2{}^x$ are determined by the boundary conditions at the core-cladding interfaces.

For core region $r < a$, $k^2 - \beta^2 > 0$ and the Eq. 1.55 is applicable. As the Bessel function of the second kind $Y_l(hr)$, diverges for $r \to 0$. Therefore to have meaningful modes at the central region of the fiber core, one needs to set the coefficient corresponding to this function equal to zero.

$$\begin{bmatrix} c_2 \\ c_2{}^x \end{bmatrix} = 0 \tag{1.57}$$

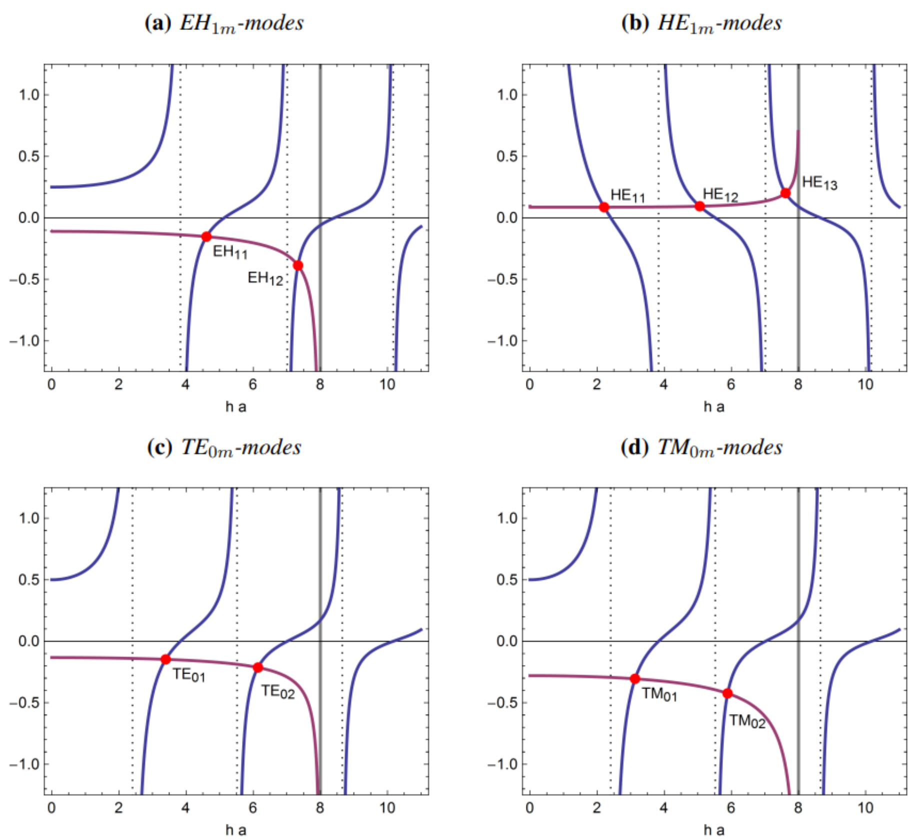

Figure 1.11: Solution to the transcendental equation for light propagation inside the fiber i.e. Eq. 1.61. The intersection between the left and right-hand side functions in the equation indicates the allowed propagation modes inside the fiber.

Consequently, the field equation in the core region $r < a$ is as follows

$$\begin{bmatrix} e_z(r) \\ h_z(r) \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} J_l(hr) \tag{1.58}$$

For cladding region $r > a$, $\beta^2 - k^2 = 0$ therefore Eq. 1.56 is valid. However, except at the core-cladding interface, the electric field far away in the radial direction should vanish i.e. as $r \to \infty$. As $J_l(hr)$ diverges in this region. One can set

$$\begin{bmatrix} c_1 \\ {c_1}^x \end{bmatrix} = 0 \tag{1.59}$$

The remaining modified Bessel function describes the field evolution in the cladding region. This represents the radially decaying evanescent field. Therefore no power propagates along the radial direction. The electric field equation is the cladding region is therefore as follows

$$\begin{bmatrix} e_z(r) \\ h_z(r) \end{bmatrix} = \begin{bmatrix} C \\ D \end{bmatrix} K_l(qr) \tag{1.60}$$

Finding the expression for the core-guided mode propagating inside the fiber core involves estimating the complex coefficients A, B, C, D. This is done by satisfying the boundary conditions at the core-cladding interface such that the field components E_ϕ, E_z, H_ϕ, and H_z are continuous at the core-cladding interface. Substituting Eq. 1.58,1.60 into the radial and azimuthal components of the fields in Eq. 1.48-1.51, one gets radial and azimuthal components in relation to z-components. For the continuity of the perpendicular field components at the core-cladding interface, a set of equations relating the four coefficients A, B, C, D, and propagation constant are obtained. For the existence of these non-trivial sets of equations, the determinant of the coefficients is set to zero. This generates the following relation which allows finding the value of the propagation constant for a given set of values of l and ω.

$$\left[\frac{J_l'(ha)}{ha\, J_l(ha)} + \frac{K_l'(qa)}{qa\, K_l(qa)}\right] \left[{n_1}^2 \frac{J_l'(ha)}{ha\, J_l(ha)} + {n_2}^2 \frac{K'_l(qa)}{qa\, K_l(qa)}\right] = \left[\frac{1}{(qa)^2} + \frac{1}{(ha)^2}\right]^2 \left[\frac{l\beta}{k_0}\right]^2 \tag{1.61}$$

The relation between the field coefficients is as follows

$$\frac{B}{A} = i\left(\frac{\beta sl}{\mu\omega}\right) \tag{1.62}$$

$$\frac{C}{A} = \zeta \tag{1.63}$$

$$\frac{D}{B} = \zeta \tag{1.64}$$

where

$$s = \left(\frac{1}{(qa)^2} + \frac{1}{(ha)^2}\right)\left(\frac{J_l'(ha)}{ha\, J_l(ha)} + \frac{K_l'(qa)}{qa\, K_l(qa)}\right) \tag{1.65}$$

and

$$\zeta = \frac{J_l'(ha)}{K_l'(qa)} \tag{1.66}$$

The axial propagation constant β in has a value in a range given by the following relation

$$n_2 k_0 \leq \beta \leq n_1 k_0 \tag{1.67}$$

The vector components of the field function inside the core and cladding are represented as follows.

For $r < a$ inside the fiber core

$$e_r(r) = i\frac{q}{h\zeta}\left[(1-sl)J_{l-1}(hr) - (1+sl)J_{l+1}(hr)\right] \tag{1.68}$$

$$e_\phi(r) = -\frac{q}{h\zeta}\left[(1-sl)J_{l-1}(hr) + (1+sl)J_{l+1}(hr)\right] \tag{1.69}$$

$$e_z(r) = -\frac{2q}{v\beta}J_l(hr) \tag{1.70}$$

$$h_r(r) = -\frac{\omega\epsilon_0 {n_1}^2 q}{h\beta v}\left[(1-s_1 l)J_{l-1}(hr) + (1+s_1 l)J_{l+1}(hr)\right] \tag{1.71}$$

$$h_{phi}(r) = -i\frac{\omega\epsilon_0 {n_1}^2 q}{h\beta v}\left[(1-s_1 l)J_{l-1}(hr) - (1+s_1 l)J_{l+1}(hr)\right] \tag{1.72}$$

$$h_z(r) = i\frac{2qsl}{\omega\mu v}J_l(hr) \tag{1.73}$$

For $r > a$ in the cladding region

$$e_r(r) = i\left[(1-sl)K_{l-1}(qr) - (1+sl)K_{l+1}(qr)\right] \tag{1.74}$$

$$e_\phi(r) = -\left[(1-sl)K_{l-1}(qr) - (1+sl)K_{l+1}(qr)\right] \tag{1.75}$$

$$e_z(r) = \frac{2q}{\beta}K_l(qr) \tag{1.76}$$

$$h_r(r) = -\frac{\omega\epsilon_0 {n_2}^2}{\beta}\left[(1-s_2 l)K_{l-1}(qr) - (1+s_2 l)K_{l+1}(qr)\right] \tag{1.77}$$

$$h_\phi(r) = i\frac{\omega\epsilon_0 {n_2}^2}{\beta}\left[(1-s_2 l)K_{l-1}(qr) - (1+s_2 l)K_{l+1}(qr)\right] \tag{1.78}$$

$$h_z(r) = i\frac{2qsl}{\omega\mu}K_l(qr) \tag{1.79}$$

where $s_i = \frac{\beta^2}{{k_0}^2 {n_i}^2}s$, i = 1, 2.

$$J'(x) = \frac{\partial J(x)}{\partial x},\ K'(x) = \frac{\partial K(x)}{\partial x} \tag{1.80}$$

The field components inside the fiber can be transformed into Cartesian coordinates also.

1.4.2. Propagation constant and fiber modes

By numerically solving Eq. 1.61, one can obtain a discrete set of values for propagation constant β and they represent a different propagation mode inside the fiber. The quadratic nature of Eq. 1.61 results in two different sets of modes.

HE-modes for which $E_z > H_z$ and EH-modes for which $E_z < H_z$. Depending on the value of l there exists a set of solutions characterizing the different fiber propagating modes. For this reason, EH_{lm}/HE_{lm} symbols are used for denoting various fiber propagation modes. A special case of the above general solution, for $l = 0$, is useful to define the mode in single-mode fibers.

TM-modes which correspond solution of EH_{0m}.

TE-modes which correspond solution of EH_{0m}.

The transverse magnetic (TM) and the transverse electric (TE) modes have null magnetic and electric fields respectively towards the propagation direction. In other words, the field components are pure transversal with no longitudinal components along the propagation direction.

Fundamental Fiber Parameter: The fundamental fiber parameter depends on the fiber properties which are the refractive indices of core (n_1) and cladding (n_2), the core radius and the wavelength of the light used, λ.

$$V = \frac{2\pi a}{\lambda}\sqrt{{n_1}^2 - {n_2}^2} \tag{1.81}$$

Parameter V defines the meaningful solution of Eq.xxxx (1.12). By plotting both the left and right-hand sides of the equations, a graphical solution can be obtained. For example, for l=1 the solutions are plotted in Fig.xxx. The function is plotted against variable ha. The intersections of the plotted curves represent the propagation modes in the fiber. There is a lower limit in V for which a mode E_{lm} can propagate in the fiber. The cut-off value for E_{lm} mode is given by the m roots of $J_l(ha) = 0$. For example, ha = 3.832, 7.016, 10.713 for E_{11}, E_{12}, E_{13}. The general EH-modes have six non-vanishing components of the fields. Therefore the propagation inside the fiber is hybrid modes, something similar to skew rays (in the geometric optics concept) spiraling inside the fiber.

For HE-modes, the graphical solution can be obtained by plotting again the similar to EH-modes. HE_{11} mode has no cut-off, which means that this mode can always propagate. Therefore, HE_{11} mode is also called the fundamental mode of the fiber.

1.4.2.1. Single-mode fiber condition

A single-mode (SM) optical fiber allows only a single transverse mode of light to propagate through. The mode allowed is the fundamental Gaussian mode due to the design properties of the fiber. In fiber optics, single-mode fibers are one of the most used fibers. The importance of SM-fibers is reflected by the 2009 Nobel Prize in Physics awarded to Charles K. Kao for his extensive theoretical contribution to SM-fibers. In the following, a very brief introduction to SM-mode propagation conditions in the fibers is introduced.

As discussed in the previous section 1.4.2, while solving Maxwell's equations for the core-guided modes inside a fiber, one naturally obtains a very useful parameter. This parameter is called as "V-parameter" or the "normalized frequency" and depends on the guided wavelength, core-radius, and the refractive index profile of the fiber. V-parameter determines whether for a given optical frequency there is a single core-guided mode or multiple of them.

- For V $\leq$ 2.405, there exists only a single core-guided mode and therefore a fiber for a given wavelength satisfying the above inequality with its refractive indices is called a single-mode fiber.
- For V $>$ 2.405, more than one mode exists which can be guided in the core and therefore fiber is called multi-mode fiber.

Another way to visualize the single-mode fiber propagation condition is as follows:

First one can define the number of guided modes in terms of a normalized propagation constant b which can be defined as follows

$$b = \frac{a^2 w^2}{V^2} = \frac{(\beta/k)^2 - {n_2}^2}{{n_1}^2 - {n_2}^2}. \tag{1.82}$$

Now from the plots in Fig. 1.11, normalized modes (b) vs V-parameter, one can see that there exists a minimum cut-off value above which a specific guided mode exists. However, for HE_{11} mode can exist if the core diameter is above zero. Therefore, the single-mode fiber condition exists for the fundamental mode above a certain V-parameter value as defined above. The lowest non-fundamental mode inside the fiber is TM_{01}. The cut-off value of the fiber parameter V then sets the lowest cut-off below which the single-mode propagation inside the fiber occurs. V= 2.405 is the lowest cut-off for this mode. Therefore the single-mode fiber condition is $V < 2.405$.

Bibliography