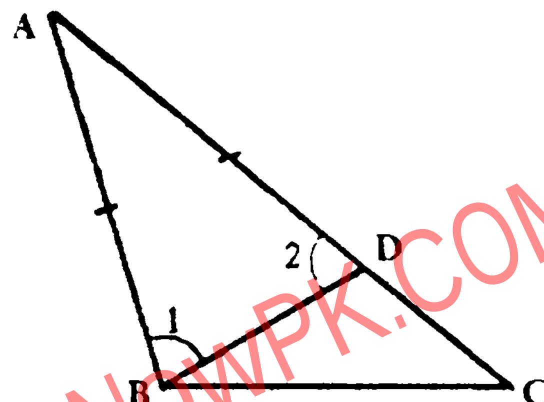
Unit 13
Sides And Angles Of A Triangle

THEOREM 13.1.1

If two sides of a triangle are unequal in length, the longer side has an angle of greater measure opposite to it.



Solution:

Given:

In ΔABC,

$m\overline{AC} > m\overline{AB}$

To prove:

$m\angle ABC > m\angle ACB$

Construction:

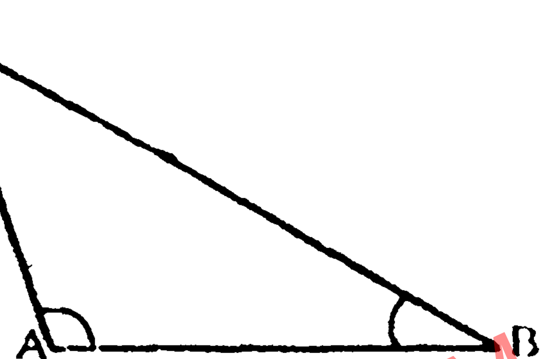
On $\overline{AC}$ taking $\overline{AD} = \overline{AB}$. Join B and D so that ΔADB is an isosceles triangle.

Proof:

Statements	Reasons
In ΔABD $m\angle 1 = m\angle 2$ (i)	Angles opposite to congruent sides;
In ΔBCD $m\angle 2 > m\angle ACB$ .... (ii)	An exterior angle of triangle is greater than every non adjacent interior angle.
$\therefore$ $m\angle 1 > m\angle ACB$ ...... (iii)	By (i) and (ii)
$m\angle ABC = m\angle 1 + m\angle DBC$	Postulate of addition of measure of angles.
$\therefore$ $m\angle ABC > m\angle 1$ .... (iv) or $m\angle ABC > m\angle ACB$	By (iii) and (iv) Transitive property of inequality of real numbers.

THEOREM 13.1.2
Converse of THEOREM 13.1.1

If two angles of triangle are unequal in measure, the side opposite to the greater angle is longer than the side opposite to the smaller angle.



Given: In ΔABC

$m\angle A > m\angle B$

To prove:

$\overline{BC} > m\,\overline{AC}$

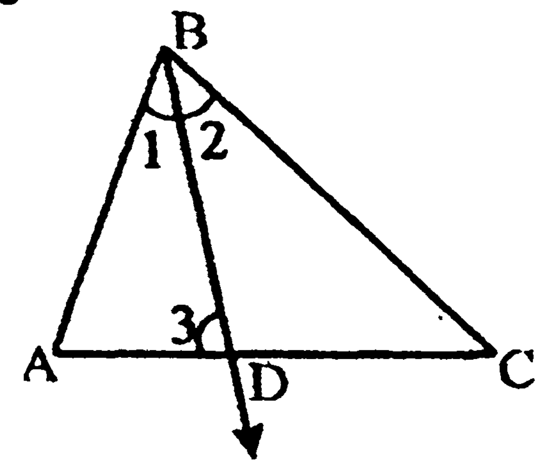
Proof:

Statements	Reasons
If $m\overline{BC} \not> m\overline{AC}$, then either (i) $m\,\overline{BC} = m\,\overline{AC}$ or (ii) $m\,\overline{BC} < m\,\overline{AC}$	Trichotomy property of real numbers.
From (i) If $m\overline{BC} = m\overline{AC}$ $m\angle A = m\angle B$	Angles opposite to congruent sides are congruent.
which is impossible	Contrary to what is given.
From (ii) if $m\overline{BC} < m\overline{AC}$, then $m\angle A < m\angle B$	The angle opposite to longer side is greater than angle opposite to smaller side
This is also impossible to contrary to what is given	
$\therefore\ m\,\overline{BC} \neq m\overline{AC}$ and $m\overline{BC} \not< m\overline{AC}$ Hence $m\,\overline{BC} > m\,\overline{AC}$	Trichotomy property of real numbers.

THEOREM 13.1.3

The sum of lengths of any two sides of a triangle is greater than the length of the third side.

Solution:



Given:

A triangle ABC

To prove:

(i) $m\overline{AB} + m\overline{BC} > m\overline{AC}$

(ii) $m\overline{AC} + m\overline{AB} > m\overline{BC}$

(iii) $m\overline{AC} + m\overline{BC} > m\overline{AB}$

Construction:

Draw the bisector of $\angle B$ to meet the side $\overline{AC}$ at the point D.

Proof:

Statements	Reasons
In $\Delta\, CBD$	
$m\angle 3 > m\angle 2$ ……….. (i)	Exterior angle is greater than non adjacent interior angle
$m\angle 2 = m\angle 1$ ……….. (ii)	Construction
$\therefore \quad m\angle 3 > m\angle 1$	By (i) and(ii)
and $m\overline{AB} > m\overline{AD}$ …… (iii)	In Δ ABD the side opposite to the larger angle is greater than that of the side opposite to the smaller angle.
similarly $m\overline{BC} > m\overline{DC}$	
$m\overline{AB} + m\overline{BC} > m\overline{AD} + m\overline{DC}$	Adding (iii) and (iv)
or $m\overline{AB} + m\overline{BC} > m\overline{AC}$	$\therefore \quad m\overline{AD} + m\overline{DC} = m\overline{AC}$
Similarly by drawing angle bisector of $\angle A$ and $\angle C$ it can be proved that $m\overline{AC} + m\overline{AB} > m\overline{BC}$ and $m\overline{AC} + m\overline{BC} > m\overline{AB}$	

$$m\angle BCD > \frac{1}{2} m\angle CBD$$

$\therefore \quad \overline{BD} > \overline{DC}$

Opposite sides of the angles.

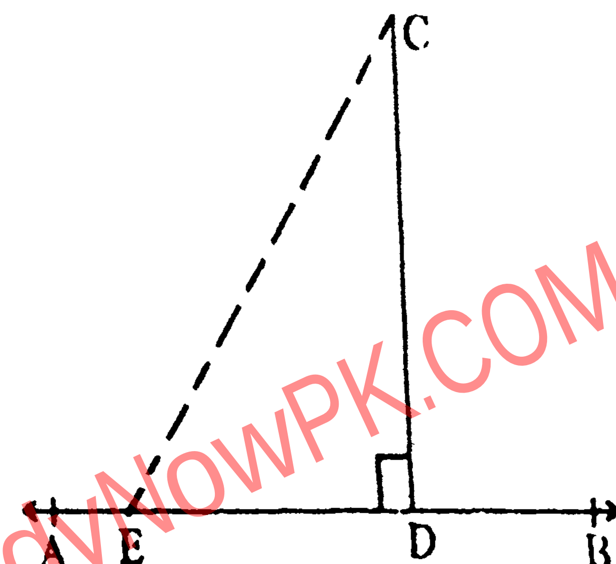
THEOREM 13.1.4

From a point, outside a line, the perpendicular is the shortest distance from the point to the line.

Solution:

Given:

A line $\overleftrightarrow{AB}$ and a point C (not lying on $\overleftrightarrow{AB}$) and a point D on $\overleftrightarrow{AB}$ such that $\overline{CD}$ is perpendicular to $\overleftrightarrow{AB}$.



To prove:

$m\overline{CD}$ is the shortest distance from point C to the line $\overleftrightarrow{AB}$.

Construction:

Take a point E on $\overleftrightarrow{AB}$. Join C and E to get a Δ CDE.

Proof:

Statements	Reasons
If Δ CDE	
$m\angle CDB > m\angle CED$	An exterior angle of a triangle is greater than every non adjacent interior angle.
But $m\angle CDB = m\angle CDE$	Supplement of right angle.
$\therefore \quad m\angle CDE > m\angle CED$	
or $m CED < m\angle CDE$	Reflexive property of inequality
or $m\overline{CD} < m\overline{CE}$	Side opposite to greater angle is greater.
BUT E was any point on AB	
Hence $m\overline{CD}$ is the shortest distance from C to $\overleftrightarrow{AB}$.	

EXERCISE 13.1

Q1. Two sides of a triangle measure 10 cm and 15 cm. Which of the following measure is possible for the third side?

(a) 5 cm (b) 20 cm (c) 25 cm (d) 30 cm

Solution:

(a) Measure of sides are 10 cm, 15 cm and 5 cm

As $10 + 5 = 15$

Since the sum of two sides is equal to the third side therefore:

So **5 cm is not possible.**

(b) Sides are 10 cm, 15 cm and 20 cm

$$10 + 15 > 20$$
$$10 + 20 > 15$$
$$15 + 20 > 10$$

Since the sum of two sides is greater than third side therefore:

20 cm is possible for third side.

(c) Sides are 10 cm, 15 cm and 25 cm

As $10 + 15 = 25$

Since the sum of two sides is equal to the third side therefore:

So **25 cm is not possible.**

(d) Sides are 10 cm, 15 cm and 30 cm

$$10 + 15 < 30$$

Since the sum of two sides is equal to the third side therefore:

So **30 cm is not possible.**

Q2. O is interior point of the ΔABC. Show that

$$m\overline{OA} + m\overline{OB} + m\overline{OC} > \frac{1}{2}\left(m\overline{AB} + m\overline{BC} + m\overline{CA}\right)$$

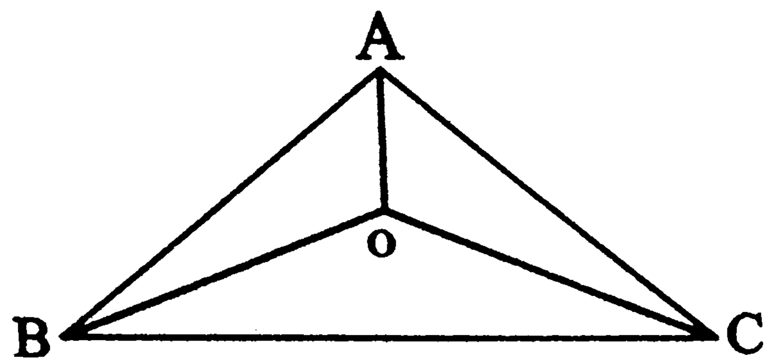
Solution:

Given:

O is a point in side a triangle ABC. O is joined with A, B and C.

To prove:

$$m\overline{OA} + m\overline{OB} + m\overline{OC} > \frac{1}{2}\left(m\overline{AB} + m\overline{BC} + m\overline{CA}\right)$$



Proof:

Statements	Reasons
In ΔOAB	
$m\overline{OA} + m\overline{OB} > m\overline{AB}$	In a triangle the sum of measure of two dies is greater than measure of third side
In ΔOBC	
$m\overline{OB} + m\overline{OC} > m\overline{BC}$	
In ΔCOA	
$m\overline{OC} + m\overline{OA} > m\overline{AC}$	
Adding (i), (ii), (iii)	
$2(m\overline{OA} + m\overline{OB} + m\overline{OC}) > m\overline{AB} + m\overline{BC} + m\overline{AC}$	
or $m\overline{OA} + m\overline{OB} + m\overline{OC} > \frac{1}{2}(m\overline{AB} + m\overline{BC} + m\overline{CA})$	Dividing both sides by 2

Q3. In the $\Delta ABC, m\angle B = 70^\circ$ and $m\angle C = 45^\circ$. Which of the sides of the triangle is longest and which is the shortest?

Solution:

$m\angle B = 70^\circ$

$m\angle C = 45^\circ$

$m\angle A + m\angle B + m\angle C = 180^\circ$

$m\angle A + 70^\circ + 45^\circ = 180^\circ$

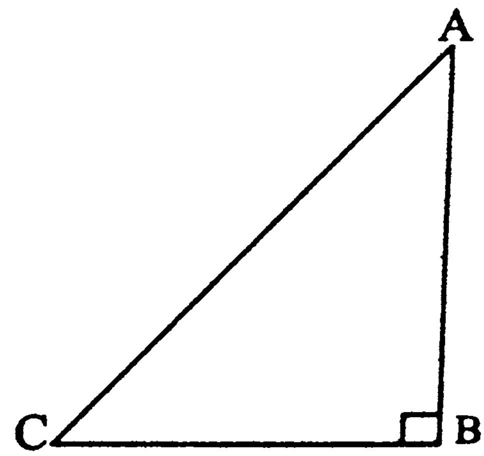
$m\angle A + 115^\circ = 180^\circ$

$m\angle A = 180^\circ - 115^\circ = 65^\circ$

Since the largest angle is B. So the longest side is opposite to B is $\overline{AC}$ **(Longest)**

Since the smallest angle is C. So the shortest side is opposite to C is $\overline{AB}$ **(Shortest)**

Q4. Prove that in a right-angled triangle, the hypotenuse is longer than each of the other two sides.



Solution:

In the right angled triangle ABC, $m\angle B = 90^\circ$ and AC is hypotenuse.

$\angle A$ and $\angle C$ both are acute

$\therefore$ $m\angle B > m\angle A$

And $m\angle B\ m\angle C$

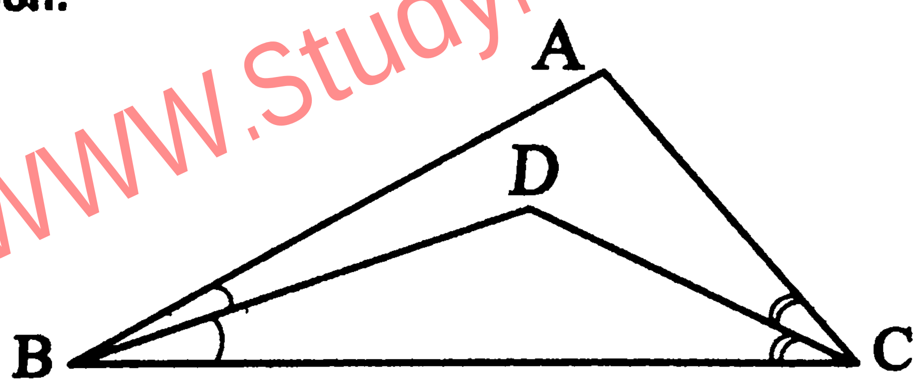
$\therefore$ $\angle B$ is the largest angle

$\therefore$ Side opposite to $\angle B$

Which is hypotenuse is longer than each of the other two sides.

Q5. In the triangle figure, $\overline{AB} > \overline{AC}$. $\overline{BD}$ and $\overline{CD}$ are the bisectors of $\angle B$ and $\angle C$ respectively. Prove that $\overline{BD} > \overline{DC}$.

Solution:



$\overline{BD}$ is bisector of $\angle B$

$\therefore$ $m\angle ABD = m\angle CBD$

So $m\angle CBD = \frac{1}{2} m\angle ABC$

Similarly as $\overline{CD}$ is bisector of $\angle C$.

So $m\angle BCD = \frac{1}{2} m\angle ACB$

It is given that $\overline{AB} > \overline{AC}$

$\therefore$ $ACB < ABC$ Opposite angles

$\Rightarrow$ $\frac{1}{2} mACB > \frac{1}{2} mABC$

$ACB < ABC$ Opposite angles

$$m\angle BCD > \frac{1}{2} m\angle CBD$$

$\therefore \quad \overline{BD} > \overline{DC}$

Opposite sides of the angles.

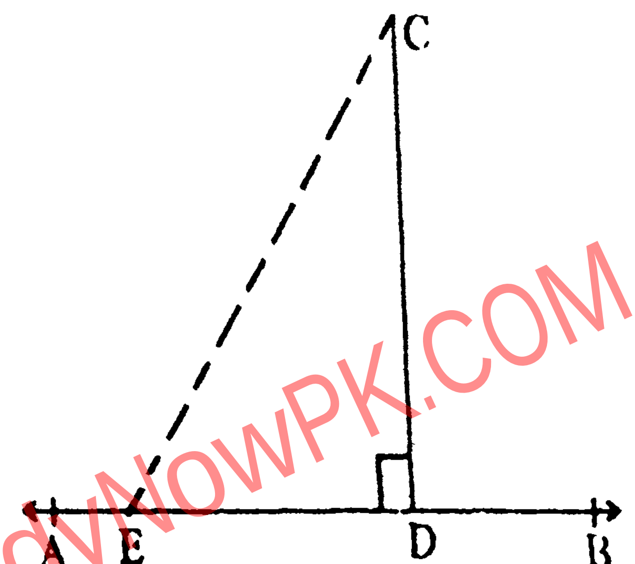
THEOREM 13.1.4

From a point, outside a line, the perpendicular is the shortest distance from the point to the line.

Solution:

Given:

A line $\overleftrightarrow{AB}$ and a point C (not lying on $\overleftrightarrow{AB}$) and a point D on $\overleftrightarrow{AB}$ such that $\overline{CD}$ is perpendicular to $\overleftrightarrow{AB}$.



To prove:

$m\overline{CD}$ is the shortest distance from point C to the line $\overleftrightarrow{AB}$.

Construction:

Take a point E on $\overleftrightarrow{AB}$. Join C and E to get a Δ CDE.

Proof:

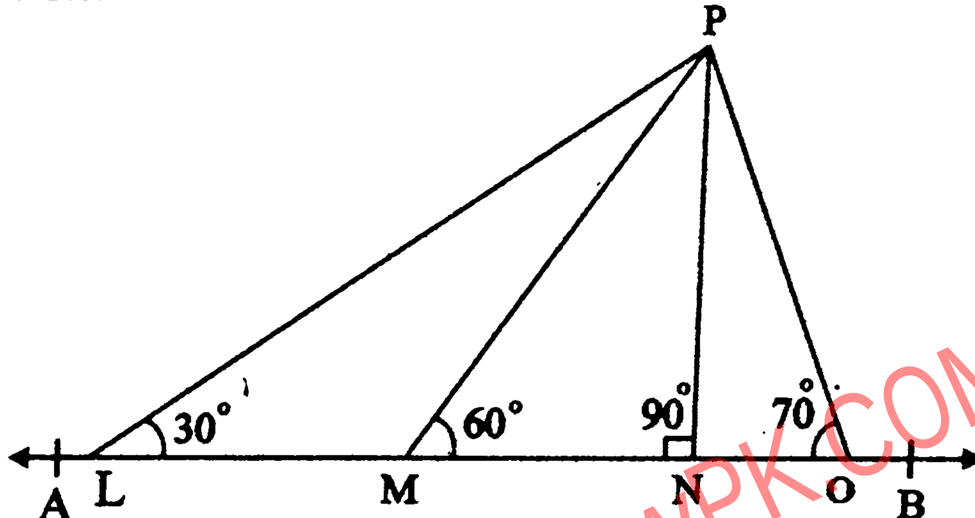
Statements	Reasons
If Δ CDE	
$m\angle CDB > m\angle CED$	An exterior angle of a triangle is greater than every non adjacent interior angle.
But $m\angle CDB = m\angle CDE$	Supplement of right angle.
$\therefore \quad m\angle CDE > m\angle CED$	
or $\quad m CED < m\angle CDE$	Reflexive property of inequality
or $\quad m\overline{CD} < m\overline{CE}$	Side opposite to greater angle is greater.
BUT E was any point on AB	
Hence $m\overline{CD}$ is the shortest distance from C to $\overleftrightarrow{AB}$.	

EXERCISE 13.2

Q1. In the figure, P is any point and AB is a line. Which of the following is the shortest distance between the point P and the line AB.

(a) $m\overline{PL}$ (b) $m\overline{PM}$ (c) $m\overline{PN}$ (d) $m\overline{PO}$

Solution:



We know that form a point outside a line, the perpendicular is the shortest distance from the point to the line.

As $\overline{PN}$ is perpendicular to AB

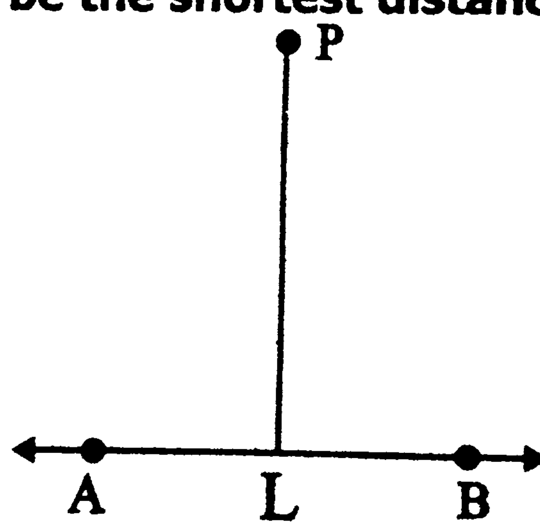
So $\overline{PN}$ is the shortest distance.

Q2. In the figure, P is any point lying away from the line AB. Then $m\overline{PL}$ will be the shortest distance if

(a) $m\angle PLA = 80^\circ$

(b) $m\angle PLB = 100^\circ$

(c) $m\angle PLA = 90^\circ$



Solution:

We know that for a point outside a line, the shortest distance from the point to the line is perpendicular to the line.

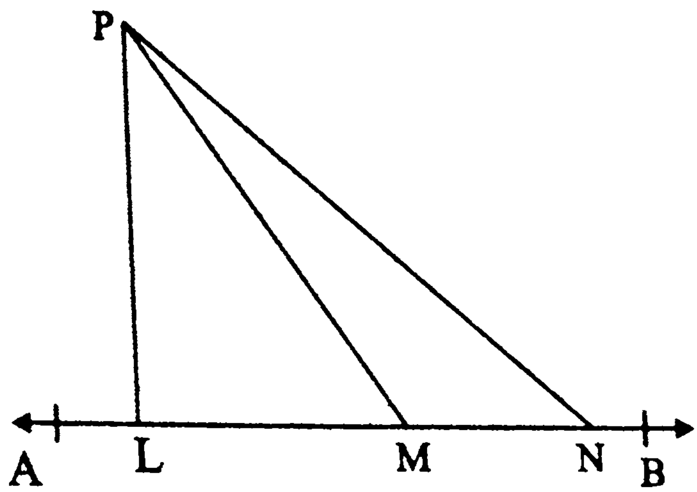
As $m\overline{PL}$ is shortest,

So $\overline{PL}$ is perpendicular to $\overline{AB}$.

So $m\angle PLA = 90^\circ$

Q3. In the figure, $\overline{PL}$ is perpendicular to the line AB and $m\overline{LN} > m\overline{LM}$. Prove that $m\overline{PN} > m\overline{PM}$.

Solution:



Given:

$\overline{PL}$ is perpendicular to $\overleftrightarrow{AB}$ and $m\overline{LN} > m\overline{LM}$.

To prove:

$m\overline{PN} > m\overline{PM}$

Proof:

Statements	Reasons
In ΔLPN	
$m\angle PLN = 90^\circ$	Given
$\therefore$ $m\angle PLN < 90^\circ$ (i)	Angle of a triangle
In ΔPLM	
$m\angle PMN > m\angle PLM$	Exterior angle
$\therefore$ $m\angle PMN < 90^\circ$ (ii)	$\angle PLM = 90^\circ$
In ΔPMN	
$m\angle PMN > m\angle PNL$	from (i) and (ii)
$m\overline{PN} > m\overline{PM}$	Opposite sides

REVIEW EXERCISE 13

Q1. Which of the following are true and which are false?

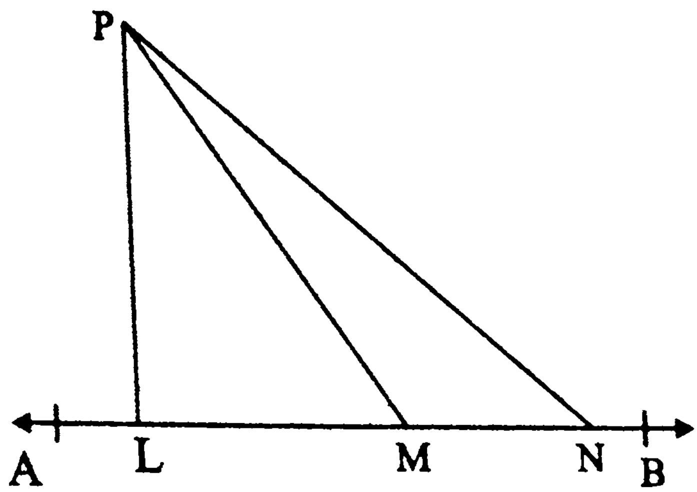
(i) The angle opposite to the longer side is greater.

(ii) In a right-angled triangle greater angle is of 60°.

(iii) In an isosceles right-angled triangle, angles other than right angle are each of 60°.

Q3. In the figure, $\overline{PL}$ is perpendicular to the line AB and $m\overline{LN} > m\overline{LM}$. Prove that $m\overline{PN} > m\overline{PM}$.

Solution:



Given:

$\overline{PL}$ is perpendicular to $\overleftrightarrow{AB}$ and $m\overline{LN} > m\overline{LM}$.

To prove:

$m\overline{PN} > m\overline{PM}$

Proof:

Statements	Reasons
In ΔLPN	
$m\angle PLN = 90^\circ$	Given
$\therefore$ $m\angle PLN < 90^\circ$ (i)	Angle of a triangle
In ΔPLM	
$m\angle PMN > m\angle PLM$	Exterior angle
$\therefore$ $m\angle PMN < 90^\circ$ (ii)	$\angle PLM = 90^\circ$
In ΔPMN	
$m\angle PMN > m\angle PNL$	from (i) and (ii)
$m\overline{PN} > m\overline{PM}$	Opposite sides

REVIEW EXERCISE 13

Q1. Which of the following are true and which are false?

(i) The angle opposite to the longer side is greater.

(ii) In a right-angled triangle greater angle is of 60°.

(iii) In an isosceles right-angled triangle, angles other than right angle are each of 60°.

(iv) A triangle having two congruent sides is called equilateral triangle.
(v) A perpendicular from a point to line is shortest distance.
(vi) Perpendicular to line form an angle of 60°.
(vii) A point outside the line is collinear.
(viii) Sum of two sides of triangle is greater than the third.
(ix) The distance between a line a point on it is zero.
(x) Triangle can be formed of lengths 2 cm, 3 cm and 5 cm.

Solution:

(i) T	**(ii) F**	**(iii) T**	**(iv) F**	**(v) T**
(vi) T	**(vii) F**	**(viii) T**	**(ix) T**	**(x) F**

Q2. What will be angle for shortest distance from an outside point to the line?

Solution:

The shortest distance between a point and a line is perpendicular from the point to the line.
So the angle for shortest distance from an outside point **is 90°.**

Q3. If 13 cm, 12 cm and 5 cm are the lengths of a triangle, then verify that difference of measures of any two sides of a triangle is less than measure of the third side.

Solution:

Let the three sides be
$a = 13\text{ cm}, b = 12\text{ cm}, c = 5\text{ cm}$
$a - b = 13 - 12 = 1 < 5 = c$
i.e. $a - b < c$ .... (i)
$12 - 5 = 7 < 13$
i.e. $b - c < a$ .... (ii)
and $13 - 5 = 8 < 12$
i.e. $a - c < b$

From (i), (ii) and (iii) we find that the difference of measures of any two sides of a triangle is less than the measure of the third side.

Q4. If 10 cm, 6 cm and 8 cm are the lengths of a triangle, then verify that sum of measures of two sides of a triangle is greater than the third side.

Solution:

Let the measure of the sides of the triangle be

$a = 10\text{ cm}, \quad b = 6\text{ cm}, \quad c = 8\text{ cm}$

$10 + 6 = 16 > 8$

$a + b < c$ .... (i)

$6 + 8 = 14 > 10$

$b + c < a$ .... (ii)

$8 + 10 = 18 > 6$

$c + a > b$

From (i), (ii) and (iii) we conclude that the sum of measures of two sides of a triangle is greater than the third side.

Q5. 3 cm, 4 cm and 7 cm are not the lengths of a triangle. Give the reason.

Solution:

Let $a = 3\text{ cm}, \quad b = 4\text{ cm}, \quad c = 7\text{ cm}$

$a + b = 3 + 4 = 7 = c$

i.e. $a + b = c$

Since we know that for a triangle sum of measures of two sides should be greater than measure of the third side.

$3 + 4 \ngtr 7$

So 3 cm, 4 cm and 7 cm are not the lengths of the triangle.

Q6. If 3 cm and 4 cm are the lengths two sides of a right angle triangle then what should be the third length of the triangle.

Solution:

Let the three sides be

$\therefore \quad 3^2 + 4^2 = a^2$ (Pythagoras theorem)

$9 + 16 = a^2$

$a^2 = 25$

$\Rightarrow \quad a = 5$

The third length is 5 cm.