Unit 13: Sides and Angles of Triangles

Overview

Theorem 13.1.1

If two sides of a triangle are unequal in length, the longer side has an angle of greater measure opposite to it.

Given:

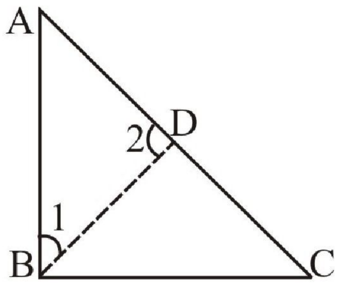
In ΔABC, $m\overline{AC} > m\overline{AB}$

To prove

$m\angle ABC > m\angle ACB$

Construction

On $\overline{AC}$ take a point D such that $\overline{AD} \cong \overline{AB}$. Join B to D so that ΔADB is an isosceles triangle. Label $\angle 1$ and $\angle 2$ as shown in the given figure.



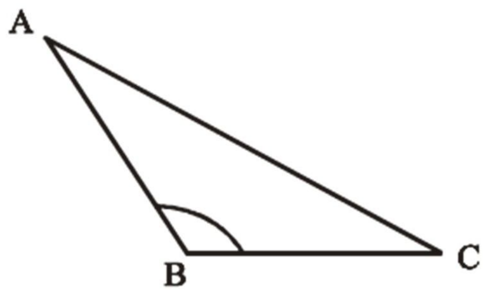
Proof

Statements	Reasons
In ΔABD	
$m\angle 1 = m\angle 2$ … (i)	Angles opposite to congruent sides (construction)
In ΔBCD, $m\angle ACB < m\angle 2$	
i.e. $m\angle 2 > m\angle ACB$ _____ (ii)	(An anterior angle of a triangle is greater than a non adjacent interior angle.)
$\therefore$ $m\angle 1 > m\angle ACB$ _____ (iii)	By (i) and (ii)
But $m\angle ABC = m\angle 1 + m\angle DBC$	Postulate of addition of angles
$\therefore$ $m\angle ABC > m\angle 1$ _____ (iv)	
$\therefore$ $m\angle ABC > m\angle 1 > m\angle ACB$	By (iii) and (iv)
Hence $m\angle ABC > m\angle ACB$	(Transitive property of inequality of real number)

Example 1

Prove that in a scalene triangle, the angle opposite to the largest side is of measure greater than $60°$.

(i.e., two-third of a right-angle)



Given

In $\Delta ABC, m\overline{AC} > m\overline{AB}, m\overline{AC} > m\overline{BC}$.

To prove

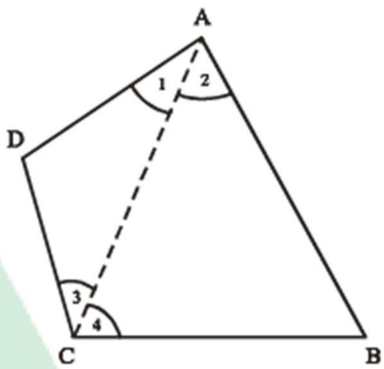
$m\angle B > 60°$

Proof

Statements	Reasons
In ΔABC	
$m\angle B > m\angle C$	$m\overline{AC} > m\overline{AB}$ (given)
$m\angle B > m\angle A$	$m\overline{AC} > m\overline{BC}$ (given)
But $m\angle A + m\angle B + m\angle C = 180°$	$\angle A, \angle B, \angle C$ are the angles of ΔABC
$\therefore m\angle B + m\angle B + m\angle B > 180°$	$m\angle B > m\angle C, m\angle B > m\angle A$ (proved)
Hence $m\angle B > 60°$	$\dfrac{\cancel{180°}^{60°}}{\cancel{3}} = 60°$

Example 2

In a quadrilateral ABCD, $\overline{AB}$ is the longest side and $\overline{CD}$ is the shortest side. Prove that $\Delta BCD > mBAD$



Given

In quad. ABCD, $\overline{AB}$ is the longest side and $\overline{CD}$ is the shortest side.

To prove

$m\angle BCD > m\angle BAD$

Construction

Joint A to C.

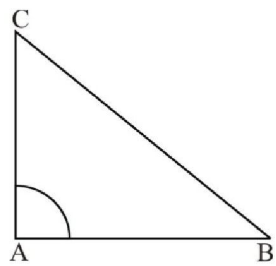
Name the angles $\angle 1, \angle 2, \angle 3$ and $\angle 4$ as shown in the figure.

Proof

Statements	Reasons
In $\Delta ABC, m\angle 4 > \angle 2 \ldots (\text{i})$	$m\overline{AB} > m\overline{BC}$ (given)
In $\Delta ACD, m\angle 3 > \angle 1 \ldots (\text{ii})$	$m\overline{AD} > m\overline{CD}$ (given)
$\therefore m\angle 4 + m\angle 3 > m\angle 2 + m\angle 1$	From (i) and (ii)
Hence $m\angle BCD > m\angle BAD$	$\therefore \begin{cases} m\angle 4 + m\angle 3 = m\angle BCD \\ m\angle 2 + m\angle 1 = m\angle BAD \end{cases}$

Theorem 13.1.2 (Converse of theorem 13.1.1)

If two angles of a triangle are unequal in measure, the side opposite to the greater angle is longer than the side opposite to the smaller angle.



Given:

In ΔABC, $m\angle A > m\angle B$

To prove

$m\overline{BC} > m\overline{AC}$

Proof

Statements	Reasons
If $m\overline{BC} \not> m\overline{AC}$, then Either (i) $m\overline{BC} = m\overline{AC}$ Or (ii) $m\overline{BC} < m\overline{AC}$	(Trichotomy property of real numbers)
From (i) if $m\overline{BC} = m\overline{AC}$, then	
$m\angle A = m\angle B$	(Angles opposite to congruent sides are congruent)
Which in not possible From (ii) if $m\overline{BC} < m\overline{AC}$, then	
$m\angle A < m\angle B$ This is also not possible	(The angle opposite to longer side is greater than angle opposite to smaller side? Contrary to the given
$\therefore$ $m\overline{BC} \neq m\overline{AC}$ And $m\overline{BC} \not< m\overline{AC}$ Thus $m\overline{BC} > m\overline{AC}$	Trichotomy property of real numbers.

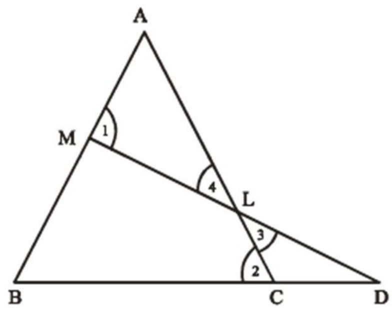
Corollaries

(i) The hypotenuse of a right triangle is longer than each of the other two sides.

(ii) In an obtuse angled triangle, the side opposite to the obtuse angle is longer than each of the other two sides.

Example

ABC is an isosceles triangle with base $\overline{BC}$. On $\overrightarrow{BC}$ a point D is taken away from C. A line segment through D cuts $\overline{AC}$ at L and $\overline{AB}$ at M. prove that $m\overline{AL} > m\overline{AM}$.



Given

In $\Delta ABC, \overline{AB} \cong \overline{AC}$

D is a point on $\overrightarrow{BC}$ away from C

A line segment through D cuts $\overline{AC}$ at L and $\overline{AB}$ at M.

To Prove

$m\overline{AL} > m\overline{AM}$

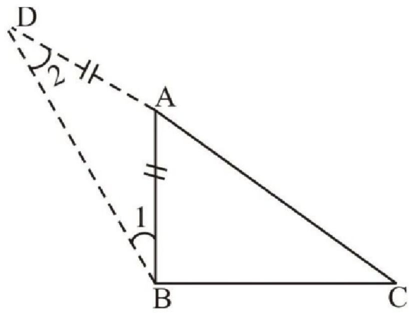
Proof

Statements	Reasons
In ΔABC $\angle B \cong \angle 2$..(i)	$\overline{AB} \cong \overline{AC}$ (given)
In ΔMBD $m\angle 1 > m\angle B$...(ii)	($\angle 2$ is an ext. $\angle$ and $\angle 3$ is its internal opposite $\angle$)
$\therefore m\angle 1 > m\angle 2$...(iii)	From (i) and (ii)
In ΔLCD $m\angle 2 > m\angle 3$	($\angle 1$ is an ext. $\angle$ and $\angle 3$ is its internal opposite $\angle$)
$\therefore m\angle 1 > m\angle 3$...(v)	From (iii) and (iv)

But $m\angle 3 \cong m\angle 4 ...(\text{vi})$	Vertical angles
$\therefore m\angle 1 > m\angle 4$	From (v) and (vi)
Hence $m\overline{AL} > m\overline{AM}$	In $\Delta ALM, m\angle 1 > m\angle 4(\text{proved})$

Theorem 13.1.3

The sum of the lengths of any two sides of a triangle is greater than the length of third side.



Given ΔABC

To prove

(i) $m\overline{AB} + m\overline{AC} > m\overline{BC}$

(ii) $m\overline{AB} + m\overline{BC} > m\overline{AC}$

(iii) $m\overline{BC} + m\overline{AC} > m\overline{AB}$

Construction

Take a point D on $\overrightarrow{CA}$ such that $\overline{AD} \cong \overline{AB}$ join B to D and name the angles $\angle 1$, $\angle 2$ as shown in the given figure.

Proof

Statements	Reasons
In ΔABD,	
$\angle 1 \cong \angle 2$ _____ (i)	$\overline{AD} \cong \overline{AB}$ (construction)
$m\angle DBC > m\angle 1$ _____ (ii)	$m\angle DBC = m\angle 1 + m\angle ABC$
$\because m\angle DBC > m\angle 2$ _____ (iii)	From (i) and (ii)
In ΔDBC	
$m\overline{CD} > m\overline{BC}$	By (iii)
i.e. $m\overline{AD} + m\overline{AC} > m\overline{BC}$	$m\overline{CD} = m\overline{AD} > m\overline{AC}$
Hence $m\overline{AB} + m\overline{AC} > m\overline{BC}$	$m\overline{AD} = m\overline{AB}$ (Construction)
Similarly	
$m\overline{AB} + m\overline{BC} > m\overline{AC}$	
And $m\overline{BC} + m\overline{CA} > m\overline{AB}$	

Example 1

Which of the following sets of lengths can be the lengths of the sides of a triangle?

(a) 2cm, 3cm, 5cm (b) 3cm, 4cm, 5cm, (c) 2cm, 4cm, 7cm,

(a) $\because 2+3=5$

$\therefore$ This set of lengths cannot be those of the sides of a triangle.

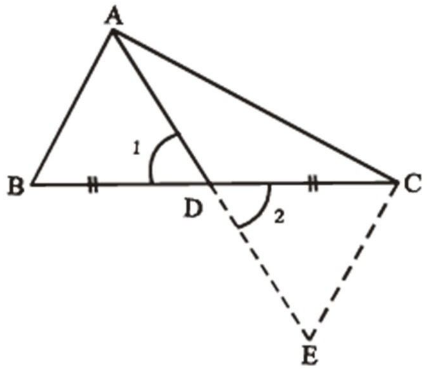
(b) $\because 3+4>5, 3+5>4, 4+5>3$

$\therefore$ This set can form a triangle

(c) $\because 2+4<7$

$\therefore$ This set of lengths cannot be the sides of a triangle.

Example 2



Prove that the sum of the measures of two sides of a triangle is greater than twice the measure of the median which bisects the third side.

Given

In $\Delta ABC,$ median AD bisects side $\overline{BC}$ at D.

To prove

$m\overline{BC}+\overline{AC}>2m\overline{AD}$.

Construction

On $\overrightarrow{AD}$, Take a point E, such that $\overline{DE}\cong\overline{AD}$.

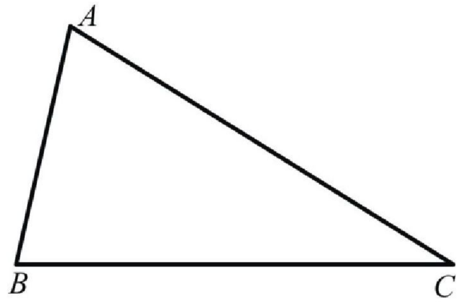
Join C to E. Name the angles $\angle 1, \angle 2$ as shown in the ________ figure.

Proof

Statements	Reasons
In $\Delta ABD \leftrightarrow \Delta ECD$	
$\overline{BD}\cong\overline{CD}$	Given
$\angle 1\cong\angle 2$	Vertical angles
$\overline{AD}\cong\overline{ED}$	Construction
$\Delta ABD\cong\Delta ECD$	S.A.S. Postulate
$\overline{AB}\cong\overline{EC}...(\text{i})$	Corresponding sides of $\cong\Delta$s
$m\overline{AC}+m\overline{EC}>m\overline{AE}...(\text{ii})$	ACE is a triangle
$m\overline{AC}+m\overline{AB}>m\overline{AE}...(\text{ii})$	From (i) and (ii)
Hence $m\overline{AC}+m\overline{AB}>2m\overline{AD}$	$m\overline{AE}=2m\overline{AD}$ (Construction)

Example 3

Prove that the difference of measures of two sides of a triangle is less than the measure of the third side.



Given

ΔABC

To Prove

$m\overline{AC}-m\overline{AB}<m\overline{BC}$

$m\overline{BC}-m\overline{AB}<m\overline{AC}$

$m\overline{BC}-m\overline{AC}<m\overline{AB}$

Proof

Statements	Reasons
$m\overline{AB}+m\overline{BC}>m\overline{AC}$	ABC is a triangle
$\left(\cancel{m\overline{AB}}+m\overline{BC}-\cancel{m\overline{AB}}\right)>\left(m\overline{AC}-m\overline{AB}\right)$	Subtracting $m\overline{AB}$ from both sides
$\therefore m\overline{BC}>\left(m\overline{AC}-m\overline{AB}\right)$	
or $m\overline{AC}-m\overline{AB}<m\overline{BC}...(\text{i})$	$a>b\Rightarrow b<a$
Similarly	
$\left.\begin{array}{l} m\overline{BC}-m\overline{AB}<m\overline{AC} \\ m\overline{BC}-m\overline{AC}<m\overline{AB}\end{array}\right\}$	Reason similar to (i)

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Report any mistake at freeilm786@gmail.com