Analytic Inverse Kinematics

(Date : 2020/08/31)

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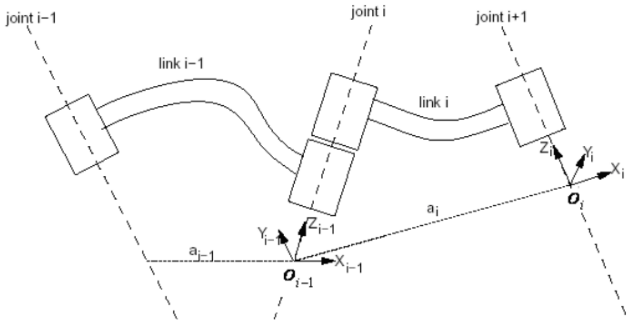
Analytic Inverse Kinematics

Inverse kinematics described in this technical paper uses an analytic method and is calculated based on standard DH parameters.

Standard D-H Parameter

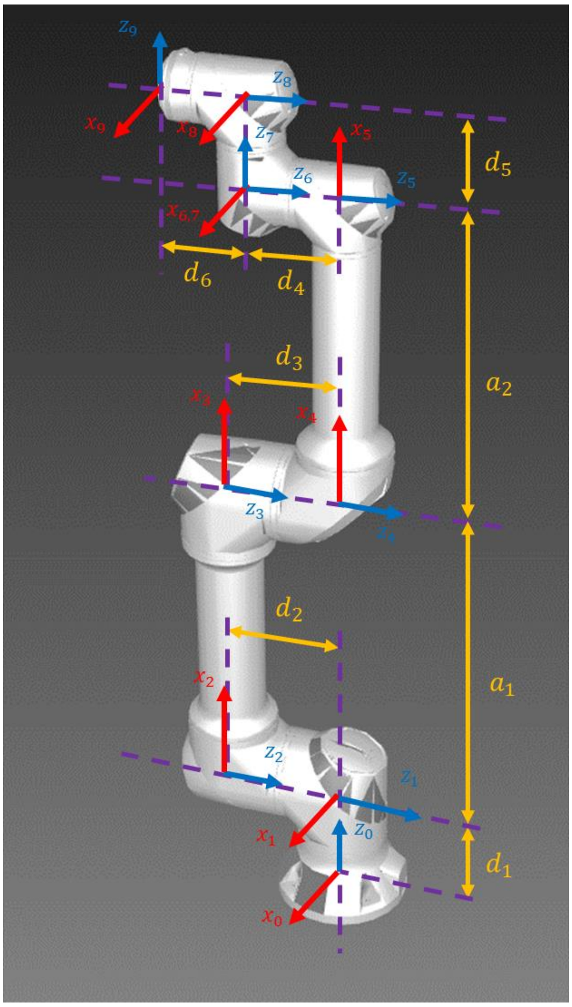


- A rotation θ_i about z_{i-1} axis;
- A translation d_i along the z_{i-1} axis;
- A translation a_i along the x_{i-1} axis;
- A rotation α_i about x_{i-1} axis;

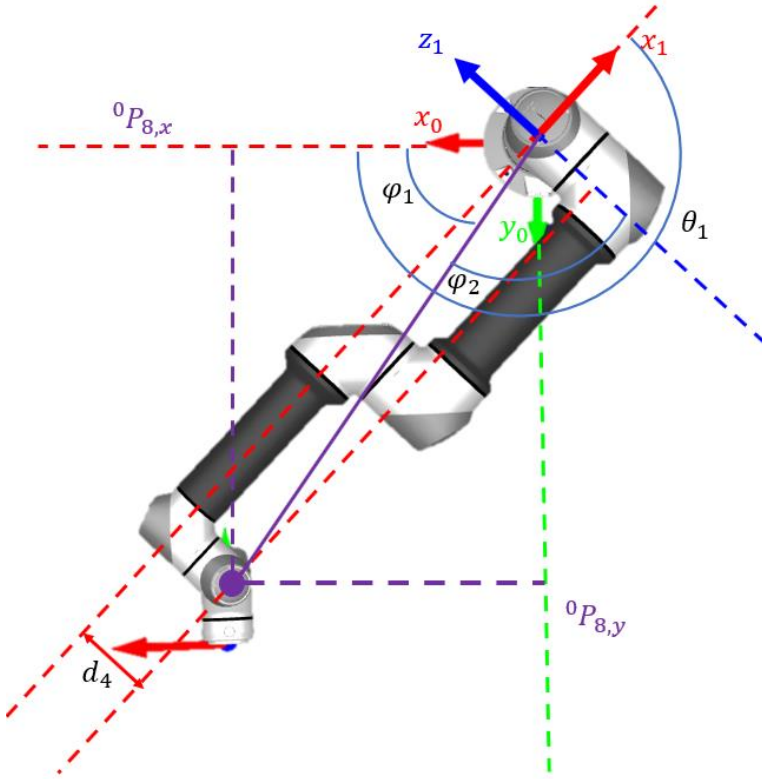
$$^{i-1}T_i = \begin{bmatrix} C\theta_i & -C\alpha_i S\theta_i & S\alpha_i S\theta_i & a_i C\theta_i \\ S\theta_i & C\alpha_i C\theta_i & -S\alpha_i C\theta_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_n = {}^0T_1\,{}^1T_2\,{}^2T_3 \cdots {}^{n-1}T_n$$

	d1	d2	d3	d4	d5	d6	a1	a2	
RB5-850	169.2	148.4	148.4	110.7	110.7	96.7	425.0	392.0	
RB3-1200	169.2	148.4	148.4	110.7	110.7	96.7	566.9	522.4	
RB10-1300	197.0	187.5	148.4	117.15	117.15	115.3	612.7	570.15	



Standard(Spong)				
link i	θi	di	ai	αi
L1	θ1	d1	0	-90
L2	θ2 - 90	-d2	0	0
L3	0	0	a1	0
L4	θ3	d3	0	0
L5	0	0	a2	0
L6	θ4+90	-d4	0	0
L7	0	0	0	90
L8	θ5	d5	0	-90
L9	θ6	-d6	0	90



$$ {}^0P_8 = {}^0P_9 + d_4\hat{Y} $$

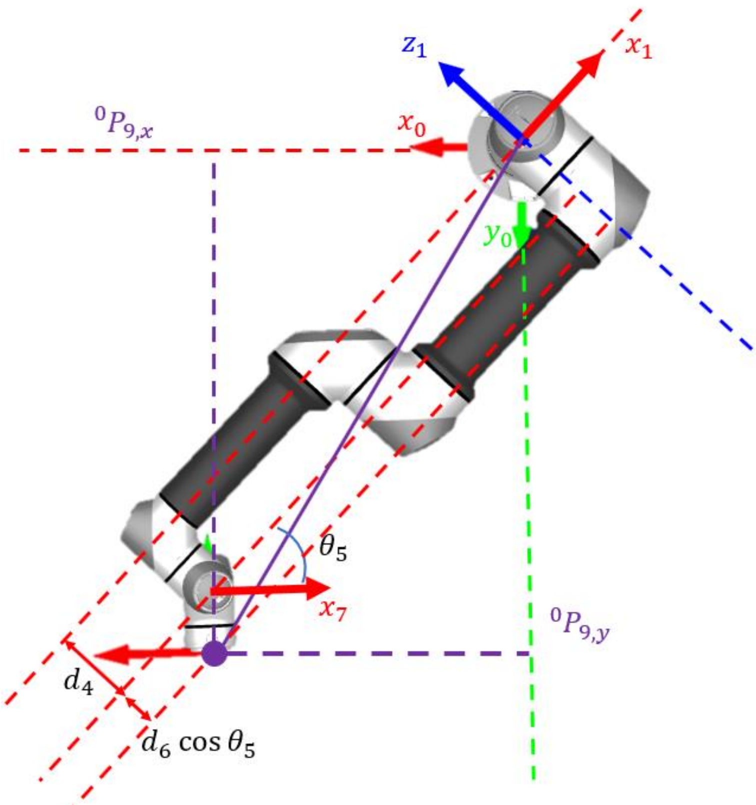
$$ \theta_1 = \varphi_1 + \varphi_2 + \frac{\pi}{2} $$

$$ \varphi_1 = atan2({}^0P_{8,y},\ {}^0P_{8,x}) $$

$$ \cos\varphi_2 = \frac{d_4}{\left|{}^0P_{8,xy}\right|} = \frac{d_4}{\sqrt{{}^0P_{8,x}{}^2 + {}^0P_{8,y}{}^2}} $$

$$ \varphi_2 = \pm\mathrm{acos}\left(\frac{d_4}{\sqrt{{}^0P_{8,x}{}^2 + {}^0P_{8,y}{}^2}}\right) $$

$$ \theta_1 = atan2({}^0P_{8,y},\ {}^0P_{8,x} \pm \mathrm{acos}\left(\frac{d_4}{\sqrt{{}^0P_{8,x}{}^2 + {}^0P_{8,y}{}^2}}\right) + \frac{\pi}{2} $$



$$d_4 + d_6 \cos\theta_5 = -{}^1P_{9,z}$$

$${}^1P_9 = {}^0R_1{}^T\,{}^0P_9$$

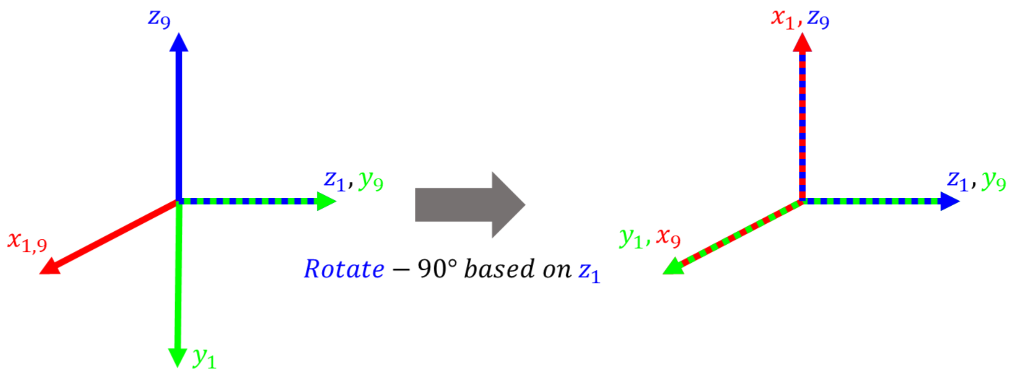
$$= \begin{bmatrix} c_1 & s_1 & 0 \\ 0 & 0 & -1 \\ -s_1 & c_1 & 0 \end{bmatrix} \begin{bmatrix} {}^0P_{9,x} \\ {}^0P_{9,y} \\ {}^0P_{9,z} \end{bmatrix}$$

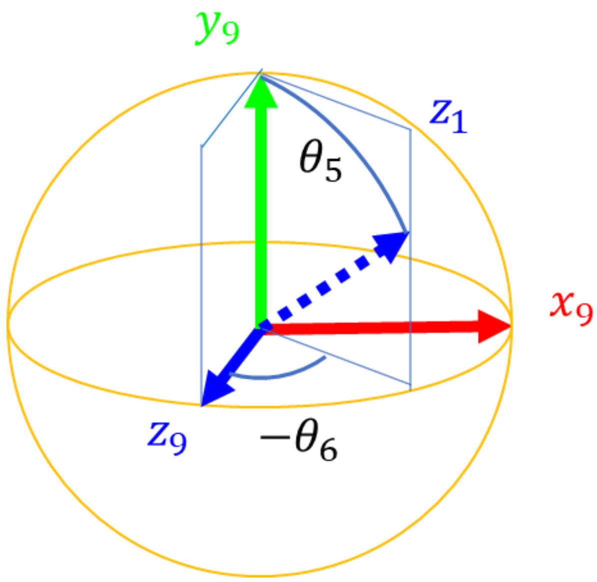
$$= \begin{bmatrix} c_1\,{}^0P_{9,x} + s_1\,{}^0P_{9,y} \\ -{}^0P_{6,z} \\ -s_1\,{}^0P_{9,x} + c_1\,{}^0P_{9,y} \end{bmatrix}$$

$$d_4 + d_6 \cos\theta_5 = -\left(-s_1\,{}^0P_{9,x} + c_1\,{}^0P_{9,y}\right)$$

$$\cos\theta_5 = \frac{s_1\,{}^0P_{9,x} - c_1\,{}^0P_{9,y} - d_4}{d_6}$$

$$\theta_5 = \pm acos\left(\frac{s_1\,{}^0P_{9,x} - c_1\,{}^0P_{9,y} - d_4}{d_6}\right)$$





$$
{}^1\hat{Z}_9 = \begin{bmatrix} -\sin\theta_5 \sin\left(-\theta_6 + \frac{\pi}{2}\right) \\ \cos\theta_5 \\ \sin\theta_5 \cos\left(-\theta_6 + \frac{\pi}{2}\right) \end{bmatrix}
$$

$$
= \begin{bmatrix} \sin\theta_5 \sin\left(\theta_6 - \frac{\pi}{2}\right) \\ \cos\theta_5 \\ \sin\theta_5 \cos\left(\theta_6 - \frac{\pi}{2}\right) \end{bmatrix}
$$

$$ {}^{9}R_1 = {}^{1}R_9{}^{T} = \left[{}^{0}R_1{}^{-1}\,{}^{0}R_9\right]^T = \left[{}^{0}R_1{}^{T}\,{}^{0}R_9\right]^T $$

$$ = \left[\begin{bmatrix} c_1 & s_1 & 0 \\ 0 & 0 & -1 \\ -s_1 & c_1 & 0 \end{bmatrix}\begin{bmatrix} \hat{X}_x & \hat{Y}_x & \hat{Z}_x \\ \hat{X}_y & \hat{Y}_y & \hat{Z}_y \\ \hat{X}_z & \hat{Y}_z & \hat{Z}_z \end{bmatrix}\right]^T $$

$$ = \begin{bmatrix} c_1\hat{X}_x + s_1\hat{X}_y & -\hat{X}_z & -s_1\hat{X}_x + c_1\hat{X}_y \\ c_1\hat{Y}_x + s_1\hat{Y}_y & -\hat{Y}_z & -s_1\hat{Y}_x + c_1\hat{Y}_y \\ c_1\hat{Z}_x + s_1\hat{Z}_y & -\hat{Z}_z & -s_1\hat{Z}_x + c_1\hat{Z}_y \end{bmatrix}^T $$

$$ = \begin{bmatrix} c_1\hat{X}_x + s_1\hat{X}_y & -\hat{X} & -s_1\hat{X}_x + c_1\hat{X}_y \\ c_1\hat{Y}_x + s_1\hat{Y}_{y_z} & -\hat{Y}_z & -s_1\hat{Y}_x + c_1\hat{Y}_y \\ c_1\hat{Z}_x + s_1\hat{Z}_y & -\hat{Z}_z & -s_1\hat{Z}_x + c_1\hat{Z}_y \end{bmatrix} $$

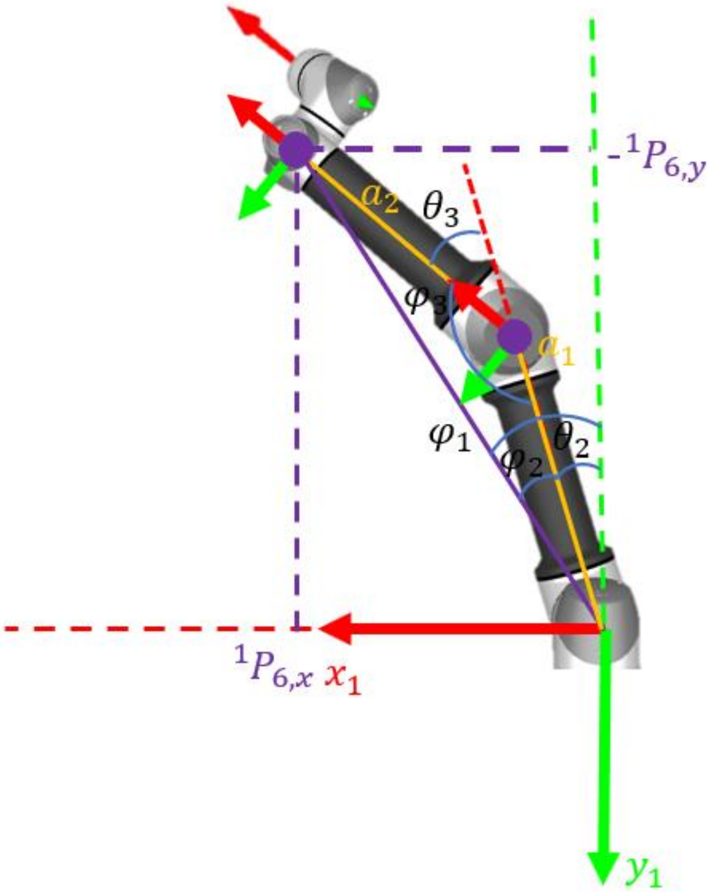
$$ {}^{1}\hat{Z}_9 = \begin{bmatrix} -s_1\hat{X}_x + c_1\hat{X}_y \\ -s_1\hat{Y}_x + c_1\hat{Y}_y \\ -s_1\hat{Z}_x + c_1\hat{Z}_y \end{bmatrix} = \begin{bmatrix} \sin\theta_5 \sin\left(\theta_6 - \frac{\pi}{2}\right) \\ \cos\theta_5 \\ \sin\theta_5 \cos\left(\theta_6 - \frac{\pi}{2}\right) \end{bmatrix} $$

$$ \sin\left(\theta_6 - \frac{\pi}{2}\right) = \frac{s_1\hat{X}_x - c_1\hat{X}_y}{\sin\theta_5} $$

$$ \cos\left(\theta_6 - \frac{\pi}{2}\right) = \frac{-s_1\hat{Z}_x + c_1\hat{Z}_y}{\sin\theta_5} $$

$$ \theta_6 - \frac{\pi}{2} = atan2\left(\frac{s_1\hat{X}_x - c_1\hat{X}_y}{\sin\theta_5}, \frac{-s_1\hat{Z}_x + c_1\hat{Z}_y}{\sin\theta_5}\right) $$

$$ \theta_6 = atan2\left(\frac{s_1\hat{X}_x - c_1\hat{X}_y}{\sin\theta_5}, \frac{-s_1\hat{Z}_x + c_1\hat{Z}_y}{\sin\theta_5}\right) + \frac{\pi}{2} $$



$$\cos\varphi_3 = \frac{{a_1}^2 + {a_2}^2 - \left|{}^{1}P_{6,xy}\right|^2}{2a_1a_2}$$

$$\cos\theta_3 = \cos(\pi - \varphi_3) = -\cos\varphi_3 = -\frac{{a_1}^2 + {a_2}^2 - \left|{}^{1}P_{6,xy}\right|^2}{2a_1a_2}$$

$$\theta_3 = \pm acos\left(-\frac{{a_1}^2 + {a_2}^2 - \left|{}^{1}P_{6,xy}\right|^2}{2a_1a_2}\right)$$

$$\varphi_1 = \varphi_2 + \theta_2 = atan2({}^{1}P_{6,x}, -{}^{1}P_{6,y})$$

$$\frac{\sin\varphi_2}{a_2} = \frac{\sin\varphi_3}{\left|{}^{1}P_{6,xy}\right|}$$

$$\varphi_2 = \text{asin}\left(\frac{a_2\sin(\pi - \theta_3)}{\left|{}^{1}P_{6,xy}\right|}\right) = \text{asin}\left(\frac{a_2\sin\theta_3}{\left|{}^{1}P_{6,xy}\right|}\right)$$

$$\theta_2 = \varphi_1 - \varphi_2 = atan2({}^{1}P_{6,x}, -{}^{1}P_{6,y})\ \text{asin}\left(\frac{a_2\sin\theta_3}{\left|{}^{1}P_{6,xy}\right|}\right)$$

Because coordinate system 5 and coordinate system 6 differ by 90 degrees,

$$^{5}R_6 = \begin{bmatrix} \cos\left(\theta_4 + \frac{\pi}{2}\right) & -\sin\left(\theta_4 + \frac{\pi}{2}\right) & 0 \\ \sin\left(\theta_4 + \frac{\pi}{2}\right) & \cos\left(\theta_4 + \frac{\pi}{2}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\theta_4 + \frac{\pi}{2} = atan2\left({}^{5}\hat{X}_{6,y}, {}^{5}\hat{X}_{6,x}\right)$$

$$\theta_4 = atan2\left({}^{5}\hat{X}_{6,y}, {}^{5}\hat{X}_{6,x}\right) - \frac{\pi}{2}$$

```
% -------------------------------------------------
% Rainbow Robotics
% Analytic Inverse Kinematics Example Code (based on Octave)
% All Right Reserved.
% -------------------------------------------------

clc; clear all; close all;
% -------------------------------------------------
D2R = pi/180.;
R2D = 180./pi;
% -------------------------------------------------
disp('----------------------------------');
disp('Input Cartesian Value (mm & deg)');
disp('----------------------------------');
% Rainbow Robotics use ZYX euler notation
% Z -> Y' -> X"
input_x = -156.76
input_y = -155.15
input_z = 814.96
input_rx = -43.47
input_ry = 80.56
input_rz = -60.88
% -------------------------------------------------
x = input_x;
y = input_y;
z = input_z;
rx = input_rx*D2R;
ry = input_ry*D2R;
rz = input_rz*D2R;
% -------------------------------------------------
% Link Length parameter (RB5 - 850)
d1 = 169.2;
d2 = 148.4;
d3 = 148.4;
d4 = 110.7;
d5 = 110.7;
d6 = 96.7;

a1 = 425.0;
a2 = 392.0;
% -------------------------------------------------

Rz = [cos(rz) -sin(rz) 0;
    sin(rz) cos(rz) 0;
    0 0 1;];
```

```
Ry = [cos(ry) 0 sin(ry);
     0 1 0;
     -sin(ry) 0 cos(ry);];

Rx = [1 0 0;
     0 cos(rx) -sin(rx);
     0 sin(rx) cos(rx);];

R = Rz * Ry * Rx;
% -----------------------------------------------

Y06 = R(:, 2);
P06 = [x;y;z];


P05 = P06 + d6*Y06;

th1 = atan2(P05(2), P05(1)) - acos(d4/sqrt(P05(2)^2 + P05(1)^2))+0.5*pi;
th5 = +acos((sin(th1)*P06(1)-cos(th1)*P06(2)-d4)/d6);
th6 = atan2(-(-sin(th1)*R(1,1)+cos(th1)*R(2,1))/sin(th5), (-
sin(th1)*R(1,3)+cos(th1)*R(2,3))/sin(th5))+0.5*pi;


A01 = [cos(th1) -cos(-pi*0.5)*sin(th1) sin(-pi*0.5)*sin(th1) 0;
     sin(th1) cos(-pi*0.5)*cos(th1) -sin(-pi*0.5)*cos(th1) 0;
     0 sin(-pi*0.5) cos(-pi*0.5) d1;
     0 0 0 1];

A67 = [cos(0) -cos(pi*0.5)*sin(0) sin(pi*0.5)*sin(0) 0;
     sin(0) cos(pi*0.5)*cos(0) -sin(pi*0.5)*cos(0) 0;
     0 sin(pi*0.5) cos(pi*0.5) 0;
     0 0 0 1];

A78 = [cos(th5) -cos(-pi*0.5)*sin(th5) sin(-pi*0.5)*sin(th5) 0;
     sin(th5) cos(-pi*0.5)*cos(th5) -sin(-pi*0.5)*cos(th5) 0;
     0 sin(-pi*0.5) cos(-pi*0.5) d5;
     0 0 0 1];

A89 = [cos(th6) -cos(pi*0.5)*sin(th6) sin(pi*0.5)*sin(th6) 0;
     sin(th6) cos(pi*0.5)*cos(th6) -sin(pi*0.5)*cos(th6) 0;
     0 sin(pi*0.5) cos(pi*0.5) -d6;
     0 0 0 1];

A17 = inv(A01)*[R P06;0 0 0 1]*inv(A89)*inv(A78)*inv(A67);

P14 = [A17(1,4);
     A17(2,4);
     A17(3,4)];

th3 = +acos((P14(1)^2+P14(2)^2-a1^2-a2^2)/(2*a1*a2));
th2 = atan2(P14(1), -P14(2))-asin(a2*sin(th3)/sqrt(P14(1)^2+P14(2)^2));

A12 = [cos(th2-pi*0.5) -cos(0)*sin(th2-pi*0.5) sin(0)*sin(th2-pi*0.5) 0;
     sin(th2-pi*0.5) cos(0)*cos(th2-pi*0.5) -sin(0)*cos(th2-pi*0.5) 0;
     0 sin(0) cos(0) -d2;
     0 0 0 1];

A23 = [cos(0) -cos(0)*sin(0) sin(0)*sin(0) a1*cos(0);
     sin(0) cos(0)*cos(0) -sin(0)*cos(0) a1*sin(0);
```

```
    0 sin(0) cos(0) 0;
    0 0 0 1];

A34 = [cos(th3) -cos(0)*sin(th3) sin(0)*sin(th3) 0;
    sin(th3) cos(0)*cos(th3) -sin(0)*cos(th3) 0;
    0 sin(0) cos(0) d3;
    0 0 0 1];

A45 = [cos(0) -cos(0)*sin(0) sin(0)*sin(0) a2*cos(0);
    sin(0) cos(0)*cos(0) -sin(0)*cos(0) a2*sin(0);
    0 sin(0) cos(0) 0;
    0 0 0 1];

A56_cal = inv(A45)*inv(A34)*inv(A23)*inv(A12)*inv(A01)*[R P06;0 0 0 1]*inv(A89)*inv(A78)*inv(A67);

th4 = atan2(A56_cal(2,1), A56_cal(1,1))-0.5*pi;
% -------------------------------------------------
disp('---------------------------------');
disp('Inverse Kinematics Result (deg)');
disp('---------------------------------');
th1 = th1 * R2D
th2 = th2 * R2D
th3 = th3 * R2D
th4 = th4 * R2D
th5 = th5 * R2D
th6 = th6 * R2D
```

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