F3b: modular arithmetic II

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modular arithmetic

- we have covered addition/multiplication and their inverses
- first, we dive into exponentiation in modular arithmetic

$$a^b \pmod{n}$$

a: base
b: exponent
n: modulus

let's forget modular arithmetic for a while

How do you compute…

5^8 using few multiplications?

First idea:

$5 \quad 5^2 \quad 5^3 \quad 5^4 \quad 5^5 \quad 5^6 \quad 5^7 \quad 5^8$

$= 5*5$ $= 5^2*5$

How do you compute…

5^8

Better idea:

$5 \quad 5^2 \quad 5^4 \quad 5^8$

$= 5*5 \quad = 5^2*5^2 \quad = 5^4*5^4$

Used only 3 mults instead of 7 !!!

Repeated squaring calculates
a^{2k}
in k multiply operations

compare with
$(2^k - 1)$ multiply operations
used by the naïve method

How do you compute…

5^{13}

Use repeated squaring again?

$5 \quad 5^2 \quad 5^4 \quad 5^8 \quad \cancel{5^{16}}$

too high! what now?

assume no divisions allowed…

How do you compute…

5^{13}

Use repeated squaring again?

$5 \quad 5^2 \quad 5^4 \quad 5^8$

Note that 13 = 8+4+1

$13_{10} = (1101)_2$

So $a^{13} = a^8 * a^4 * a^1$

Two more multiplies!

To compute a^m

Suppose $2^k \le m < 2^{k+1}$

$$a \quad a^2 \quad a^4 \quad a^8 \quad \ldots \quad a^{2k}$$

This takes k multiplies

Now write m as a sum of distinct powers of 2

say, $m = 2^k + 2^{i1} + 2^{i2} \ldots + 2^{it}$

$$a^m = a^{2k} * a^{2i1} * \ldots * a^{2it}$$

at most k more multiplies

Hence, we can compute a^m while performing at most $2 \lfloor \log_2 m \rfloor$ multiplies

Now come back to modular arithmetic

How do you compute…

5^{13} (mod 11)

First idea: Compute 5^{13} using 5 multiplies

$5 \quad 5^2 \quad 5^4 \quad 5^8 \quad 5^{12} \quad 5^{13} \qquad = 1{,}220{,}703{,}125$

$= 5^8*5^4$ $5^{12}*5$

then take the answer mod 11

1,220,703,125 (mod 11) = 4

How do you compute…

5^{13} (mod 11)

Better idea: keep reducing the answer mod 11

5 **5^2** **5^4** **5^8** **5^{12}** **5^{13}**

25%11 = 3

9%11=9

81%11=4

36%11=3

15%11=4

Hence, we can compute
$a^m \pmod{n}$
while performing at most
$2\lfloor \log_2 m \rfloor$ multiplies

where each time we multiply
together numbers
with $\lfloor \log_2 n \rfloor + 1$ bits

How do you compute…

$5^{121,242,653}$ **(mod 11)**

The current best idea would still need about 54 calculations

answer = 4

Can we exponentiate any faster?

OK, need a little more number theory for this one…

an example of modular exponentiation

- mod 7 case

Look at columns and rows!

$1^1 \equiv 1$	$1^2 \equiv 1$	$1^3 \equiv 1$	$1^4 \equiv 1$	$1^5 \equiv 1$	$1^6 \equiv 1$
$2^1 \equiv 2$	$2^2 \equiv 4$	$2^3 \equiv 1$	$2^4 \equiv 2$	$2^5 \equiv 4$	$2^6 \equiv 1$
$3^1 \equiv 3$	$3^2 \equiv 2$	$3^3 \equiv 6$	$3^4 \equiv 4$	$3^5 \equiv 5$	$3^6 \equiv 1$
$4^1 \equiv 4$	$4^2 \equiv 2$	$4^3 \equiv 1$	$4^4 \equiv 4$	$4^5 \equiv 2$	$4^6 \equiv 1$
$5^1 \equiv 5$	$5^2 \equiv 4$	$5^3 \equiv 6$	$5^4 \equiv 2$	$5^5 \equiv 3$	$5^6 \equiv 1$
$6^1 \equiv 6$	$6^2 \equiv 1$	$6^3 \equiv 6$	$6^4 \equiv 1$	$6^5 \equiv 6$	$6^6 \equiv 1$

Table 1.9: Powers of numbers modulo 7

Fermat's Little Theorem (FLT)

- $a^{p-1} = 1 \pmod{p}$
  - where p is prime and $\gcd(a,p)=1$
- also $a^p = a \pmod{p}$
- useful in public key and primality testing

if modulus is denoted by p, that implies p is a prime number

Fermat's Little Theorem (FLT)

(Fermat's Little Theorem) If p is a prime and $p \nmid a$, then

$$a^{p-1} \equiv 1 \pmod{p}.$$

Proof. Let $S = \{1, 2, 3, \ldots, p-1\}$. Consider the map $S \to S$: $m(x) \equiv ax \pmod{p}$. Clearly, $m(x) \not\equiv 0 \pmod{p}$. Now, suppose $x \neq y \in S$. We have $ax \not\equiv ay \pmod{p}$. Therefore, $m(1), m(2), \ldots, m(p-1)$ are distinct elements of S. It follows that $1 \cdot 2 \cdot 3 \cdots (p-1) \equiv m(1)m(2) \cdots m(p-1) \equiv (a \cdot 1) \cdot (a \cdot 2) \cdot (a \cdot 3) \cdots (a \cdot (p-1)) \equiv a^{p-1}(1 \cdot 2 \cdot 3 \cdots (p-1)) \pmod{p}$.

Since $\gcd(j, p) = 1$ for $j \in S$, we can divide this congruence by $1, 2, 3, \ldots, p-1$. What remains is $1 \equiv a^{p-1} \pmod{p}$.

FLT: proof (appendix)

- If $ax - ay = 0 \bmod p$, then $a(x-y) = 0 \bmod p$
  - as $a \neq 0$, $x-y=0 \bmod p$; however $-p < (x-y) < p$
  - Thus contradiction!
- There is a multiplicative inverse for all elements
  - 1..(p-1)
  - We can divide both sides by (p-1)!

FLT: examples

- 2^{10} (mod 11) ?
  - $2^{10} = 1024 = 1 \pmod{11}$
- 2^{53} (mod 11) ?
  - $(2^{10})^5\, 2^3 = 2^3 = 8 \pmod{11}$

Go back to page 13

what if modulus is not a prime?

- Euler's Theorem
  - A general version of FLT
- Let us start with Euler's totient/phi function $\phi(n)$
  - the number of the positive integers less than or equal to n that are relatively prime to n.

Euler's Totient Function $\phi(n)$

- when doing arithmetic modulo n
- **complete set of residues** is $Z_n = \{0, 1, \ldots, n-1\}$
- **reduced set of residues** is those numbers (residues) which are relatively prime to n
  - e.g. for n=10,
  - complete set of residues is $Z_n = \{0,1,2,3,4,5,6,7,8,9\}$
  - reduced set of residues is $Z_n^* = \{1,3,7,9\}$
- number of elements in reduced set of residues is called the Euler's Totient Function $\phi(n)$

Euler's Totient Function: $\phi(n)$

- $\phi(4) = 2$ (1, 3 are relatively prime to 4)
- $\phi(5) = 4$ (1, 2, 3, 4 are relatively prime to 5)
- $\phi(6) = 2$ (1, 5 are relatively prime to 6)
- $\phi(7) = 6$ (1, 2, 3, 4, 5, 6 are relatively prime to 7)

Euler's Totient Function $\phi(n)$

- to compute ϕ(n), we need to count the number of residues to be excluded
- in general we need prime factorization, but
  - for p (p: prime) $\phi(p) = p-1$
  - for p*q (p,q: prime) $\phi(pq) = (p-1)(q-1)$
  - for p^k (p: prime) $\phi(p^k) = p^{k-1}(p-1)$
- e.g.

  $\phi(37) = 36$

  $\phi(21) = (3-1)(7-1) = 2x6 = 12$

Euler's Theorem

- a generalisation of Fermat's Little Theorem
- $a^{\varphi(n)} = 1 \pmod{n}$
  - for any `a,n` where `gcd(a,n)=1`
  - if n is prime, this becomes FLT
- e.g.

  a=3;n=10; $\phi(10)=4$;

  hence $3^4 = 81 = 1 \bmod 10$

  a=2;n=11; $\phi(11)=10$;

  hence $2^{10} = 1024 = 1 \bmod 11$

Proof of Euler's theorem

Consider $x_1, x_2, \ldots x_{\varphi(n)} < n$ and coprime to n

Since a is also coprime to n, from previous result

$ax_1 \equiv x_i \pmod{n}$, $ax_2 \equiv x_j \pmod{n}$, … etc.

$\Rightarrow a^{\varphi(n)} x_1 x_2 x_3 \ldots x_{\varphi(n)} \equiv x_1 x_2 x_3 \ldots x_{\varphi(n)} \pmod{n}$

$\Rightarrow a^{\varphi(n)} x \equiv x \pmod{n}$ where $x = x_1 x_2 x_3 \ldots x_{\varphi(n)}$

$\Rightarrow n \mid x(a^{\varphi(n)} - 1)$

But n doesn't divide x

$\Rightarrow n \mid (a^{\varphi(n)} - 1)$

$\Rightarrow a^{\varphi(n)} \equiv 1 \pmod{n}$

Groups based on modular arithmetic

- group: a set of elements with a binary operation
  - the outcome of the operation should satisfy four properties below
- The group of positive integers modulo prime *p*

  $Z_p^* \equiv \{1, 2, 3, \ldots, p\text{-}1\}$

  $*_p \equiv$ multiplication modulo p

  Denoted as: $(Z_p^*, *_p)$
- **Required properties**
  1. Closure. Yes.
  2. Associativity. Yes.
  3. Identity. 1.
  4. Inverse. Yes.
- **Example:** $Z_7^* = \{1,2,3,4,5,6\}$

  $1^{-1} = 1,\ 2^{-1} = 4,\ 3^{-1} = 5,\ 6^{-1} = 6$

Other properties

- $|Z_p^*| = (p-1)$
- By Fermat's little theorem: $a^{(p-1)} = 1 \pmod p$
- Example of Z_7^*

Generators

x	x^2	x^3	x^4	x^5	x^6
1	1	1	1	1	1
2	4	1	2	4	1
3	**2**	**6**	**4**	**5**	**1**
4	2	1	4	2	1
5	**4**	**6**	**2**	**3**	**1**
6	1	6	1	6	1

For all p, the group is cyclic. A cyclic group is a group that can be generated by a single element (the generator)

What if n is not a prime?

- The group of positive integers modulo a non-prime n

  $Z_n \equiv \{1, 2, 3, \ldots, n\text{-}1\}$, n not prime

  $*_n \equiv$ multiplication modulo n

- **Required properties?**

  0. elements?
  1. Closure. ?
  2. Associativity. ?
  3. Identity. ?
  4. Inverse. ?

- How do we fix this?

Groups based on modular arithmetic

- The **<u>multiplicative group modulo n</u>**

  $Z_n^* \equiv \{m : 1 \leq m < n, \gcd(n,m) = 1\}$

  $*_n \equiv$ multiplication modulo n

  Denoted as $(Z_n^*, *_n)$
- **Required properties**:
  - Closure. Yes.
  - Associativity. Yes.
  - Identity. 1.
  - Inverse. Yes.
- **Example:** $Z_{15}^* = \{1,2,4,7,8,11,13,14\}$
- $1^{-1} = 1,\ 2^{-1} = 8,\ 4^{-1} = 4,\ 7^{-1} = 13,\ 11^{-1} = 11,\ 14^{-1} = 14$

Recap: Euler's Phi Function

$$\phi(n) = \left|Z_n^*\right| = n\prod_{p|n}(1-1/p)$$

If n is a product of two primes p and q, then

$$\phi(n) = pq(1-1/p)(1-1/q) = (p-1)(q-1)$$

Euler's Theorem:

$$a^{\phi(n)} = 1 \pmod{n} \;\; \text{for} \;\; a \in Z_n^*$$

Or for n = pq

$$a^{(p-1)(q-1)} = 1 \pmod{n} \;\; \text{for} \;\; a \in Z_{pq}^*$$

Law of exponentiation:

$$\text{if } a = b \bmod \phi(n) \text{ then } x^a = x^b \bmod n \quad x \in Z_n^*$$

This will be very important in RSA!

Generators in Groups

- Example of Z_{10}^{*}: {1, 3, 7, 9}

Generator = primitive element

x	x^2	x^3	x^4
1	1	1	1
3	**9**	**7**	**1**
7	**9**	**3**	**1**
9	1	9	1

Exponential Inverses

We want to encrypt and decrypt a message m. In RSA, an encryption performs *exponentiating* modulo N, i.e. $m^e \bmod N$.

We want to find a different exponent d based on e and N which will give us back m, i.e. we want $m^{ed} \bmod N = m$. In other words, we want an ***exponential inverse*** for e modulo N.

Exponential Inverses

To tackle the general problem, start first with the case of *N* a prime number. Exponentiation modulo a prime number is well understood.

e.g. exponentiating 3 modulo 7: (3 is generator)

1. $3^1 \bmod 7 = 3$
2. $3^2 \bmod 7 = 2$
3. $3^3 \bmod 7 = 6$
4. $3^4 \bmod 7 = 4$
5. $3^5 \bmod 7 = 5$
6. $3^6 \bmod 7 = 1$
7. $3^7 \bmod 7 = 3$
8. $3^8 \bmod 7 = 2$
9. $3^9 \bmod 7 = 6$
10. $3^{10} \bmod 7 = 4$
11. $3^{11} \bmod 7 = 5$
12. $3^{12} \bmod 7 = 1$

Exponential Inverses

Exponentiating to the *p* -1 power results in 1 (by FLT). Therefore, any further exponentiation results in a cycling, with repetitions occurring every 6 exponentiations.

Fermat's Little Theorem says that this effect happens for all relative-prime numbers under prime modulus:

1. $3^1 \bmod 7 = 3$
2. $3^2 \bmod 7 = 2$
3. $3^3 \bmod 7 = 6$
4. $3^4 \bmod 7 = 4$
5. $3^5 \bmod 7 = 5$
6. $3^6 \bmod 7 = 1$
7. $3^7 \bmod 7 = 3$
8. $3^8 \bmod 7 = 2$
9. $3^9 \bmod 7 = 6$
10. $3^{10} \bmod 7 = 4$
11. $3^{11} \bmod 7 = 5$
12. $3^{12} \bmod 7 = 1$

Exponential Inverses

Corollary: If e is relatively prime to p –1, where p is prime, then its *exponential* inverse modulo p exists and is the multiplicative inverse of e modulo p-1.

Proof. Supposing $ed \equiv 1 \pmod{p-1}$. Then for some k, $ed = 1+k(p-1)$. So if a is any number not divisible by p, FLT implies:

$$a^{ed} \equiv a^{1+k(p-1)} \pmod{p} \equiv a \pmod{p}$$

In other words, exponentiating by ed doesn't change numbers, modulo p, so by definition, d and e are exponential inverses. □

Exponential Inverses

E.g. Find the exponential inverse of 3 modulo 11.

p =11, so p-1 = 10. The inverse of 3 modulo 10 is **7**, which is the answer.

Exponential Inverses: Next Step

Q: Why don’t we just use a prime number as our base *N* since it’s so easy to find the decryptor *d* ?

Exponential Inverses: Next Step

A: *Because* it's so **easy** to find the decryptor *d!*

Recall, this is a *public cryptosystem*. The key (N,e) is available to all users including attackers. If a prime N were used, anybody can find the inverse of e modulo N-1.

RSA uses next simplest case: $N = pq$ where N is a product of two (different) primes.

N is publicly known
while p and q are hidden

Exponential Inverses: Next Step

If we know what *p* and *q* are, then we'll be able to find the exponential inverse.

But that's a big if.

Factoring large prime numbers is a surprisingly difficult problem.

No one knows how to do this in polynomial time.

RSA: one way function

- Multiplication of two prime numbers is believed to be a one-way function.
- We say “believed” because nobody has been able to prove that it is hard to factorise.

RSA Cryptosystem
Proof of Decryption

Proof that d is inverse of e mod $(p\text{-}1)(q\text{-}1)$:

We can therefore find k such that $ed = 1+k\,(p\text{-}1)(q\text{-}1)$.

Does m^{ed} equal itself modulo $N = pq$?

$$m^{ed} \equiv m^{1+k(p-1)(q-1)} \pmod{pq}.$$

$$\equiv m^{1} \cdot m^{k(p-1)(q-1)} \pmod{pq}$$

$$\equiv m \cdot m^{k(p-1)(q-1)} \pmod{pq}$$

$$\equiv m \cdot (m^{(p-1)(q-1)})^{k} \pmod{pq}$$

$$\equiv m \cdot (1)^{k} \pmod{pq}$$

$$\equiv m \pmod{pq}$$

RSA Cryptosystem: example

1. Select primes: p=17 & q=11
2. Compute $N = pq = 17\times11 = 187$
3. Compute $\varnothing(N)=(p-1)(q-1)=16\times10=160$
4. Select e: gcd(e,160)=1; choose e=7
5. Determine d: ed=1 mod 160 and d < 160
   Value is d=23 since 23×7=161=160+1
6. Publish public key KU={7,187}
7. Keep secret private key KR={23}

RSA Cryptosystem: example

- sample RSA encryption/decryption is:
- given message `M = 88` (nb. `88<187=N`)
- encryption:

  $C = 88^{7} \bmod 187 = 11$

- decryption:

  $M = 11^{23} \bmod 187 = 88$

usually, e is small, d is large!

Chinese Remainder Theorem (CRT)

For each $x \in Z_{15}$, write $x \bmod 3$ and $x \bmod 5$.

Each $x \in Z_{15}$ has a different $x \bmod 3$, $x \bmod 5$ pair.

Thus, the function $f(x) = (x \bmod 3, x \bmod 5)$ from Z_{15} to the 15 pairs (i, j) with $0 \leq i < 3$ and $0 \leq j < 5$ is one-to-one.

$\Rightarrow$ x is uniquely determined by its pair of remainders.

x	$x \bmod 3$	$x \bmod 5$
0	0	0
1	1	1
2	2	2
3	0	3
4	1	4
5	2	0
6	0	1
7	1	2
8	2	3
9	0	4
10	1	0
11	2	1
12	0	2
13	1	3
14	2	4

CRT

Let $m_1,\ldots,m_n > 0$ be relative prime. Then the system of equations $x \equiv a_i \pmod{m_i}$ (for i=1 to n) has a unique solution modulo $m = m_1 \cdot m_2 \cdot \ldots \cdot m_n$.

Proof: Let $M_i = m/m_i$. Thus $\gcd(m_i, M_i)=1$.

as multi. inverse exists, $\exists\, y_i$ such that $y_i M_i \equiv 1 \pmod{m_i}$.

Now let $x = \sum_i a_i y_i M_i$.

Since $m_i \mid M_k$ for $k \neq i$, *we have* $M_k \equiv 0 \pmod{m_i}$.

$\therefore$ $x \bmod m_i = \sum_l a_l y_l M_l \bmod m_i = a_i y_i M_i \bmod m_i = a_i \bmod m_i$.

Thus, $x \equiv a_i \pmod{m_i}$ holds for each i.

Thus, the congruences hold.

What's x such that:

$x \equiv 2 \pmod{3}$
$x \equiv 3 \pmod{5}$
$x \equiv 2 \pmod{7}$?

(So, $a_1 = 2$, etc. and $m_1 = 3$ etc.)

$$\boldsymbol{m = m_1 \cdot \ldots \cdot m_n}$$

$$\boldsymbol{M_i = m / m_i}$$

$$\boldsymbol{y_i M_i \equiv 1 \pmod{m_i}}$$

$$\boldsymbol{x = \sum_i a_i y_i M_i}$$

$m = 3 \times 5 \times 7 = 105$

$M_1 = m/3 = 105/3 = 35$

2 is an inverse of $M_1 = 35 \pmod{3}$ since $35 \times 2 \equiv 1 \pmod{3}$

$M_2 = m/5 = 105/5 = 21$

1 is an inverse of $M_2 = 21 \pmod{5}$ since $21 \times 1 \equiv 1 \pmod{5}$

$M_3 = m/7 = 15$

1 is an inverse of $M_3 = 15 \pmod{7}$ since $15 \times 1 \equiv 1 \pmod{7}$

$x \equiv 2 \times 2 \times 35 + 3 \times 1 \times 21 + 2 \times 1 \times 15 = 233 \equiv 23 \pmod{105}$

So answer: $x \equiv 23 \pmod{105}$

RSA with CRT (1/2)

- Assuming that M is not divisible by either p or q,
  - Fermat's Little Theorem tells us that

    $M^{p-1} \equiv 1 \pmod{p}$ and $M^{q-1} \equiv 1 \pmod{q}$

- Thus, we have that the following two congruences hold:

First: $C^d \equiv M \cdot (M^{p-1})^{k(q-1)} \equiv M \cdot 1^{k(q-1)} \equiv M \pmod{p}$

Second: $C^d \equiv M \cdot (M^{q-1})^{k(p-1)} \equiv M \cdot 1^{k(p-1)} \equiv M \pmod{q}$

RSA with CRT (2/2)

- For faster decryption, we use CRT

$\boxed{n = pq}$

- Decryption: $c^d \bmod n$.
- Instead of computing $c^d \bmod n$ directly, we
  - compute $c_1 = c \bmod p$ and $c_2 = c \bmod q$
  - compute $m_1 = (c_1)^d \bmod p$ and $m_2 = (c_2)^d \bmod q$
  - recover the plaintext by solving $\begin{cases} x \equiv m_1 \bmod p \\ x \equiv m_2 \bmod q \end{cases}$

RSA with CRT: example (1/2)

- Bob has a public/private key pair of RSA
  - N: 3293 = 37*89 (N=p*q)
  - e:35, d=2987
- Alice has a message to Bob, which is m=153
  - She sends $153^{35} \equiv 2494 \mod 3293$
- Bob is supposed to calculate
  - $2494^{2987} \mod 3293$
  - But he knows 3293 = 37*89; he can use CRT!

RSA with CRT: example (2/2)

- $X \equiv a \bmod pq$ iff $X \equiv a \bmod p$ and $X \equiv a \bmod q$
- So, rather than calculating $X \equiv c^d \bmod N$, we solve CRT
  - $X \equiv c^d \bmod p$, $X \equiv c^d \bmod q$
  - Since $c^{p-1} \equiv 1 \bmod p$, $c^{q-1} \equiv 1 \bmod q$
  - At worst, we just have to compute up to c^{p-2} or c^{q-2}
- Bob needs to solve
  - $X \equiv 2494^{2987} \bmod 37$ and $X \equiv 2494^{2987} \bmod 89$
  - He can reduce both bases by their respective moduli
    - $X \equiv 15^{2987} \bmod 37$ and $X \equiv 2^{2987} \bmod 89$
  - He can also reduce the exponents
    - $X \equiv 15^{36*82+35} \bmod 37$ and $X \equiv 2^{88*33+83} \bmod 89$
    - $X \equiv 5 \bmod 37$ and $X \equiv 64 \bmod 89$
- Finally, Bob simply calculates
  - $5*5*89 + 64*77*37 \equiv 153 \bmod (37*89)$

RSA: GCD(msg,modulus) is not 1

Let p be a prime.

FLT: for an integer a co-prime to p (i.e., $\gcd(a,p)=1$), $a^{p-1} = 1 \pmod{p}$

For RSA, we have $ed = 1 \pmod{p-1}$ and there is an integer k s.t. $ed=1+k(p-1)$

There are two cases:

1. if $\gcd(M,p)=1$ then $M^{ed}=M^{1+k(p-1)}=M \cdot (M^{p-1})^k = M \cdot 1^k = M \pmod{p}$ by FLT
2. if $\gcd(M,p)\neq 1$ then M is multiple of p (i.e., $M=0 \pmod{p}$),

   so $M^{ed}=0^{ed}=0=M \pmod{p}$

In both cases, we thus have $M^{ed}=M \pmod{p}$

For RSA, since we also have $ed = 1 \pmod{q-1}$, it can be shown in the same way that $M^{ed}=M \pmod{q}$

By CRT, we thus have $M^{ed}=M \pmod{N}$ where $N=pq$

Discrete Logarithms

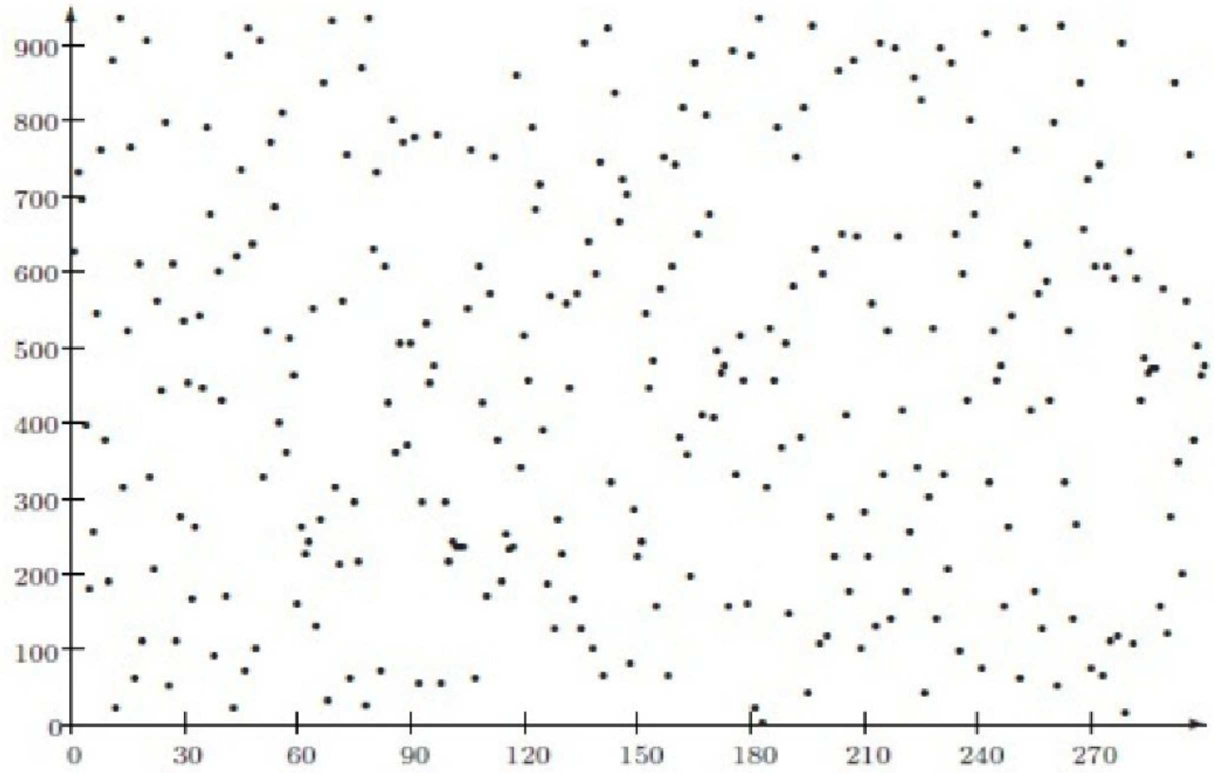
- the inverse problem to exponentiation is to find the **discrete logarithm** of a number modulo p
- that is to find $\mathtt{x}$ such that $\mathtt{y = g^x \ (mod\ p)}$
- this is written as $\mathtt{x = log_g\ y \ (mod\ p)}$
- if g is a generator then it always exists, otherwise it may not, e.g.

  $x = \log_3 4 \pmod{13}$ has no answer

  $x = \log_2 3 \pmod{13} = 4$ by trying successive powers
- whilst exponentiation is relatively easy, finding discrete logarithms is generally a **hard** problem

Discrete logarithm: plotting

A graph of $f(x) = 627^x \bmod 941$ for $x = 1, 2, 3, \ldots$



Source: Kaafarani@Oxford Univ.

Primitive Element Theorem

- $Z_p^* = \langle\alpha\rangle$, i.e. $ord(\alpha) = p-1$.

- Example
  - $Z_7^* = \langle 3\rangle$ $3^1=3, 3^2=2, 3^3=6, 3^4=4, 3^5=5, 3^6=1$
  - $Z_{13}^* = \langle 2\rangle$ $2^1=2, 2^2=4, 2^3=8, 2^4=3, 2^5=6, 2^6=12, 2^7=11, 2^8=9, 2^9=5, 2^{10}=10, 2^{11}=7, 2^{12}=1$

- Note. $ord(\alpha) = p-1 \Rightarrow \{\alpha, \alpha^2, \ldots, \alpha^{p-1}\}$ distinct.

* Primitive element = generator

Discrete Logarithms

- Discrete log problem
  - Given $Z_p^* = \langle\alpha\rangle$
  - $\mathrm{Log}_\alpha(y) = x$, if $y = \alpha^x$.

- Example
  - $Z_{13}^* = \langle 2\rangle$; $2^1=2$, $2^2=4$, $2^3=8$, $2^4=3$, $2^5=6$, $2^6=12$, $2^7=11$, $2^8=9$, $2^9=5$, $2^{10}=10$, $2^{11}=7$, $2^{12}=1$
  - $\mathrm{Log}_2(5) = 9$.

(mod 13)
is omitted

ElGamal algorithm

- One way function: modular exponentiation
  - Discrete logarithm is hard
- Solving $Log_{\alpha}(y)$ has some solutions, which are of high complexity
  - None of them run in polynomial time

Setting up ElGamal

- Let **p** be a large prime
  - By "large" we mean here a prime rather typical in length to that of an RSA modulus
- Select a special number **g**
  - The number g must be a **primitive element** modulo **p**.
- Choose a private key **x**
  - This can be any number bigger than 1 and smaller than **p**-1
- Compute public key **y** (from **x**, **p** and **g**)
  - The public key **y** is **g** raised to the power of the private key **x** modulo **p**. In other words:

$$y = g^x \bmod p$$

- Publicize p, g, y

y,p,g: known by everybody
C_1,C_2: seen by everybody
x: known only by receiver (who sets up parameters)

ElGamal encryption

The first job is to represent the plaintext M as a series of numbers modulo p. Then:

1. Generate a random number k (ephemeral key)
2. Compute two values C_1 and C_2, where

$$\mathbf{C_1 = g^k \bmod p} \quad \text{and} \quad \mathbf{C_2 = My^k \bmod p}$$

3. Send the ciphertext C, which consists of the two separate values C_1 and C_2.

M, k: known by sender
y,p,g: known by everybody
C_1,C_2: seen by everybody
x: known by receiver (who learns M after decryption)

ElGamal decryption

$C_1 = g^k \bmod p \quad C_2 = My^k \bmod p$

1 - The receiver begins by using their private key **x** to transform $\mathbf{C_1}$ into something more useful:

$$C_1^x = (g^k)^x \bmod p$$

NOTE: $C_1^x = (g^k)^x = (g^x)^k = (y)^k = y^k \bmod p$

2 - This is a very useful quantity because if you divide $\mathbf{C_2}$ by it you get **M**. In other words:

$$C_2 / y^k = (My^k) / y^k = M \bmod p$$

M,k: known by sender
y,p,g: known by everybody
C_1,C_2: seen by everybody
x: known by receiver (who learns M after decryption)

Setting up ElGamal: example

Step 1: Let p = **23**

Step 2: Select a primitive element g = **11**

Step 3: Choose a private key x = **6**

Step 4: Compute y = $\mathbf{11}^{\mathbf{6}}$ (mod **23**) = **9**

Public key is **9 (and 11, 23)**

Private key is **6**

M,k: known by sender

y,p,g: known by everybody

C_1,C_2: seen by everybody

x: known by receiver (who learns M after decryption)

ElGamal encryption: example

To encrypt M = 10 using Public key **9**

1 - Generate a random number k = **3**

2 - Compute $C_1 = \mathbf{11}^3 \bmod \mathbf{23} = \mathbf{20}$

$C_2 = 10 \times \mathbf{9}^3 \bmod \mathbf{23}$

$= 10 \times 16 = 160 \bmod \mathbf{23} = \mathbf{22}$

3 - Ciphertext C = (**20** , **22**)

M,k: known by sender
y,p,g: known by everybody
C_1,C_2: seen by everybody
x: known by receiver (who learns M after decryption)

ElGamal decryption: example

To decrypt C = (**20** , **22**)

1 - Compute $\mathbf{20}^{\mathbf{6}} = 16 \bmod \mathbf{23}$

2 - Compute $\mathbf{22} / 16 = 10 \bmod \mathbf{23}$

3 - Plaintext = 10

M,k: known by sender
y,p,g: known by everybody
C_1,C_2: seen by everybody
x: known by receiver (who learns M after decryption)